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GENERATING FUNCTIONS

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الإهداء

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نسير في دروب الحياة، ويبقى من يُسيطر على أذهاننا في كل مسلك نسلكه
صاحب الوجه الطيب، والأفعال الحسنة.
فلم يبخل عليّ طيلة حياته
(والدي العزيز).
إلى أصدقائي، وجميع من وقفوا بجواري وساعدوني بكل ما يملكون، وفي أصعدة كثيرة
أقدّم لكم هذا البحث، وأتمنّى أن يحوز على رضاكم.

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General Introduction

The subject of this thesis is memorandum about generating functions. This is a very important tool in combinatorics and probability, because it is used formally (algebraically) without taking in consideration the topological and analytic properties of convergence (except in some cases, where we are interested to asymptotics).

It seems that generating functions were used , for the first time by the french mathematician Abraham De Moivre in the 17 th century. The general properties of the generating functions are stated and examples to illustrate their importance are provided.

The thesis is organized as follows. In the first chapter, we outline some basic concepts of generating functions and operations on them. In the second chapter, we will see how to use the generating functions to solve recurrence relations. Many examples many are included in this chapter.

In the third chapter, we look at their use and manipulation in solving and proving some combinatorial identities, and then we end our work with a general conclusion.

CHAPTER 1

INTRODUCTION

In this chapter we collect all the results we will need in the sequel. Let $\mathbb{K}$ be a field. The set

$$K[[x]] = \left\{ \sum_{i=0}^{\infty} a_i x^n, a_i \in \mathbb{K} \right\}$$

will be defined. Equipped with the two laws $(+, \cdot\,)$ it is a commutative ring and is called the ring of power series, or generating functions.

1.1 Preliminairies

The field $\mathbb{K}$ will be always the field of real numbers $\mathbb{R}$, or the field of the complex numbers $\mathbb{C}$. In what follows, we give the definition of generating functions.

1.1.1 Generating functions

Definition 1.1.

Let $(a_n)_{n\geq 0}$ *be a sequence* $\in \mathbb{K}$. *The ordinary generating function* ***(O.G.F)*** *(or the formal power series) of the sequence* $(a_n)_{n\geq 0}$ *is the series*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Another type of generating functions is

Definition 1.2.

The exponential generating function ***(E.G.F)*** *.*

$$\sum_{n=0}^{\infty} a_n x^n/n! = a_0 + a_1 x/1! + a_2 x^2/2! + \cdots$$

Definition 1.3.

The set of all generating functions on $\in \mathbb{K}$ *is denoted by* $\mathbb{K}[[x]]$.

Later on, we will define many operations on the set $\mathbb{K}[[x]]$. Endowed with the two operations (addition +, and multiplication .), it is known that the set $\mathbb{K}[[x]]$ is an abelian ring.

Example 1.1.

Consider the following generating function

$$\sum_{n=0}^{\infty} a^n x^n = 1 + ax + (ax)^2 + ... + (ax)^n + ... = \frac{1}{1-ax}, a \in \mathbb{C}.$$

Another example deduced from the previous one, by putting $a = 2x$, is

$$1 + 2x + 4x^2 + \cdots + 2^n x^n + \cdots = \frac{1}{1-2x}.$$

Example 1.2.

The simplest exponential G.F is certainly

$$e^{ax} = \sum_{n=0}^{\infty} a^n x^n / n!.$$

1.2 Operations on generating functions

Let $f, g, h \in \mathbb{K}[[x]]$. Let us write, once for all

$$f := f(x) = \sum_{n=0}^{\infty} a_n x^n, g := g(x) = \sum_{n=0}^{\infty} b_n x^n, h := h(x) = \sum_{n=0}^{\infty} c_n x^n.$$

We define some operations on $\mathbb{K}\,[[x]]$.

Definition 1.4. (Equality)

The generating functions f and g are equal $\iff$ $a_n = b_n, \ \forall n \in \mathbb{N}$.

Definition 1.5. (Multiplication by a scalar)

Let $f \in \mathbb{K}[[x]]$ *and* $\lambda \in \mathbb{K}$, *then:*

$$\lambda f(x) = \lambda \left(\sum_{n=0}^{\infty} a_n x^n \right) = \sum_{n=0}^{\infty} \lambda a_n x^n.$$

Definition 1.6. (Sum)

The sum $(f+g)$ *of* f *and* g, *is defined by*

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n.$$

Remark 1.1.

$$f(x) = 0 \iff \sum_{i=0}^{\infty} a_i x^n = 0 \iff a_0 + a_1 x + \cdots + a_n x^n + \cdots = 0.$$

This means that $a_i = 0, \quad \forall \quad i \in \mathbb{N}$

Example 1.3.

Consider the two generating functions :

$$f(x) = \sum_{n=0}^{\infty} (3)^n x^n = 1 + 3x + 9x^2 + 27x^3 + \cdots = \frac{1}{1-3x},$$

and

$$g(x) = \sum_{n=0}^{\infty} (-3)^n x^n = 1 - 3x + 9x^2 - 27x^3 + \cdots = \frac{1}{1+3x}.$$

The sum of f and g is then

$$f(x)+g(x)=2+18x^2+162x^4+\cdots=\frac{1}{1-3x}+\frac{1}{1+3x}.$$

After an easy calculation, we obtain

$$f(x)+g(x)=2\left(\sum_{n=0}^{\infty}(9)^n x^{2n}\right).$$

Definition 1.7.

The subtraction of f and g is defined by:

$$f(x)-g(x)=\sum_{n=0}^{\infty}(a_n-b_n)x^n.$$

Example 1.4.

Let g be defined by:

$$g(x)=1+3x+6x^2+10x^3+\cdots=\sum_{n=0}^{\infty}\frac{(n+1)\,(n+2)}{2}x^n=\frac{1}{(1-x)^3},$$

and h defined by

$$h(x)=1+4x+9x^2+16x^3+25x^4+\cdots=\sum_{n=0}^{\infty}(1+n)^2\,x^n=\frac{(1+x)}{(1-x)^3}.$$

We have:

$$h(x)-g(x)=0+x+3x^2+6x^3+\cdots=\sum_{n=0}^{\infty}(1+n)\,x^n=\frac{x}{(1-x)^3}.$$

Definition 1.8. (Product)

The product of f and g is defined by

$$f(x)\cdot g(x) = \left(\sum_{j=0}^{\infty} a_j x^j\right)\left(\sum_{k=0}^{\infty} b_k x^k\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n.$$

Remark 1.2.

Let

$$f(x) = 1 \iff \sum_{i=0}^{\infty} a_i x^n = 1 \iff a_0 + a_1 x + \cdots + a_n x^n + \cdots = 1.$$

This means that $a_0 = 1$ and $a_i = 0$, for all $i \geq 1$.

Example 1.5.

Let the two generating functions be given by

$$f(x) = \sum_{n=0}^{\infty} 2n x^{n-1} \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} 5x^n.$$

The product of these two generating functions is then:

$$h(x) = f(x)\cdot g(x) = 2\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots\right) \times \left(1 + 5x + 25x^2 + 125x^3 + 625x^4 \cdots\right).$$

We obtain

$$(8 \times 1) + (6 \times 5)\, x + (4 \times 25)\, x^2 + (2 \times 125)\, x^3 + \cdots =$$

$$8 + 30x + 100x^2 + 250x^3 + \cdots =$$

$$\frac{2}{(1-x)^2(1-5x)}.$$

Definition 1.9. Shifting (right)

Shifting (right) a generating function by t terms means to multiply it by x^t, we then obtain a generating function, such that $a_i = 0$ for $0 \leq i \leq t-1$.

For example if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

Then the shifted series by l terms is

$$\hat{f}(x) = x^t f(x) = \sum_{n=0}^{\infty} a_n x^{n+t},$$

1.3 Important operations with generating functions

Now, we want to prove some useful properties of sums of generating functions. In particular it is very useful that like other mathematical objects as integers or polynomials, generating functions form also a ring with addition and multiplication.

Theorem 1.1.

Generating functions are closed under multiplication and addition .

Proof:

If we take the definitions of addition and multiplication of generating functions we can see that their output gives another infinite series.Therefore it is another generating function and they are closed under both addition and multiplication.

Theorem 1.2.

The multiplication of generating functions is distributive with respect to the addition that is;

$$f(x)\cdot (g(x)+h(x)) = (f(x)\cdot g(x) + f(x)\cdot h(x) = (g(x)+h(x))\cdot f(x).$$

Proof:

$$f(x)\cdot (g(x)+h(x)) = \sum_{n=0}^{\infty} a_n x^n \cdot \left(\sum_{n=0}^{\infty} b_n + \sum_{n=0}^{\infty} c_n \right).$$

Now we apply the summation we defined before:

$$\sum_{n=0}^{\infty} a_n \cdot \sum_{n=0}^{\infty} (b_n + c_n).$$

Now we apply the multiplication defined above:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k (b_{n-k} + c_{n-k}).$$

The distributive property of integers gives

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} + a_k c_{n-k} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} + \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k c_{n-k}.$$

This is nothing than:

$$f(x) \cdot g(x) + f(x) \cdot h(x))$$

Remark 1.3.

Finally, by the commutativity property of generating functions with multiplication we have:

$$(g+h) \cdot f = f \cdot (g+h) = f \cdot g + f \cdot h.$$

1.3.1 Identities and inverses

Many elements in group theory or number theory, have a special importance. For example units, or the identity elements. We know that the identity of addition is 0 and the identity of multiplication for the integers is 1. The set of the generating functions, also have identity elements. First we want to find $I(x)$ such that :

$$f(x) + I(x) = f(x) = I(x) + f(x).$$

Remark 1.4.

For every generating function $v(x)$ there exists another generating function $v_r(x)$ such that

$$v(x) + v_r(x) = 0.$$

In fact, if

$$v(x) = \sum_{n=0}^{\infty} a_n x^n, \text{ then } v_r(x) = \sum_{n=0}^{\infty} (-a_n) x^n.$$

Remark 1.5.

There exists an identity for the multiplication of generating functions.

Remark 1.6.

For the product the identity element $I(x)$ is such that:

$$f(x)I(x) = I(x)f(x) = f(x).$$

Here, obviously, $I(x) = 1$.

Example 1.6.

$$f(x) = -(1-x) - \frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} - \cdots = \sum_{n=1}^{\infty} -\frac{(1-x)^n}{n}.$$

$$g(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n} = e^x.$$

Finally, we have

$$f\left(g(x)\right) =,$$

$$-\left(1-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\right) - \frac{\left(1-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\right)^2}{2} - \frac{\left(1-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\right)^3}{3} - \cdots$$

$$\ln\left(e^x\right) = x.$$

Let us define the inverse of f, if it exists by g such that

$$f.g(x) = g.f(x) = 1.$$

The element g is denoted by f^{-1}.

Theorem 1.3.

The generating function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

has an inverse if and only if $a_0 \neq 0$.

Proof:

Let f be a generating function and assume that $g(x) = \sum_{n=0}^{\infty} b_n x^n$ is the inverse of f. Multiply both series:

$$f(x)g(x) = a_0b_0 + (a_0b_1 + a_1b_0)\,x + (a_0b_2 + a_1b_1 + a_2b_0)\,x^2 + \cdots = 1.$$

From the first equation $a_0b_0 = 1$, and then necessarily $a_0 \neq 0$. Conversely, if $a_0 \neq 0$. Then the coefficients of the generating function g satisfying $f.g = 1$ are such that

$$a_0b_0 = 1,\ a_0b_1 + a_1b_0 = 0,\ a_0b_2 + a_1b_1 + a_2b_0 = 0,\ etc.$$

So, the the coefficients of g may be determined by the formulas:

$$b_0 = \frac{1}{a_0},\ b_1 = \frac{-a_1b_0}{a_0},\ b_2 = \frac{-\left(a_1b_0 + a_1b_0\right)}{a_0},\ etc.$$

1.3.2 Derivatives and integration

Definition 1.10.

Derivative of a generating function $f(x)$: We define the derivative of the generating function f by

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty} n a_n \cdot x^{n-1}.$$

The second derivative of f is defined similarly, by considering the generating series $\sum_{n=0}^{\infty} b_n x^n$, where $b_i = i a_i$. The higher derivatives are defined by an obvious way.

We use the notation $\frac{d^n}{dx^k}$ for the k^{th} derivative in respect to x.

Example 1.7.

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}.$$

Then

$$\frac{df(x)}{dx} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots = \sum_{n=0}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}.$$

Definition 1.11.

We define the integral of the generating function f as

$$\int f(x)dx = \int \sum_{n=0}^{\infty} a_n x^n dx = C + \sum_{n=0}^{\infty} \frac{a}{n+1} x^{n+1},$$

where C is an arbitrary constant. Note that the integral of generating function, as with a normal function, is defined up to a constant term.

Almost all the rules known in Calculus for derivatives and integrals are valid with generating functions too.

Example 1.8.

We have $g(x)$ the generating functions:

$$g(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = \cos(x).$$

So,

$$\int g(x)dx = x - \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{5040}x^7 - \cdots + C.$$

Now note that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)}x^{2n+1} = \sin(x) + C.$$

Theorem 1.4.

Leibnitz rule for differentiation:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

Proof:

First we evaluate the left side of the equation.

$$f(x)g(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}(a_k \cdot b_{m-k}x^m)$$

$$(f(x)g(x))' = \sum_{n=0}^{\infty}\sum_{k=0}^{n} m(a_m\cdot b_{m-k})x^{m-1}$$

We first have that:

$$f'(x)g(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}(k\cdot a_k\cdot b_{m-k})x^{m-1}.$$

The term

$$f(x)g'(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}(a_k\cdot (m-k)\cdot b_{m-k})x^{m-1}.$$

Therefore,

$$f'(x)g(x)+f(x)g'(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}(k\cdot a_k\cdot b_{m-k})x^{m-1}+\sum_{n=0}^{\infty}\sum_{k=0}^{n}(a_k\cdot (m-k)\cdot b_{m-k})x^{m-1}.$$

This may be written

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}(k\cdot a_k\cdot b_{m-k})x^{m-1}+((m-k)\cdot a_k\cdot b_{m-k})x^{m-1} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}(a_k\cdot b_{m-k})\cdot (k+(m-k))x^{m-1}.$$

Finally

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}m\cdot (a_k\cdot b_{m-k})x^{m-1} = (f(x)g(x))'.$$

Theorem 1.5.

We have $\frac{d^n}{dx^n}f(x)=0$ *if and only if* $f(x)$ *is a polynomial of degree* $\leq n$.

Proof:

Let us prove the left sense first:

Let $f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots c_nx^n$, then

$$\frac{d^n}{dx^n}f(x) = \sum_{d=0}^{n} c_d \prod_{i=0}^{n}(d-i)x^{d-n}.$$

Now, since $n > d$ for all d's , there exists an i for every d such that $(d-i) = 0$. This implies that the whole product must be zero and thus the summation of the products must be also 0.

Now let us prove the right sense:

$$\int \frac{dx^n}{d} 0 = \sum_{d=0}^{n} \frac{c_d x^d}{d!}$$

Which as it shown, it has only degree at most n (depending on what $c_i \neq 0$).

1.3.3 Compositions

Definition 1.12.

Let be f and g the two generating functions :

$$f(x) = \sum_{n=0}^{\infty} a_nx^n \text{ and } g(x) = \sum_{n=0}^{\infty} b_nx^n.$$

The composition of f and g is defined by

$$f(g(x)) = \sum_{n=0}^{\infty} a_n \left(\sum_{m=0}^{\infty} b_m x^m \right)^n .$$

Example 1.9.

Consider the two generating functions:

$$f(x) = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

and

$$g(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x.$$

So,

$$f\left(g(x)\right) = 1 + \sum_{n=0}^{\infty} \frac{x^n}{n!} + \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)^2 + \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)^3 + \cdots = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \frac{x^m}{m!}\right)^n.$$

This summation is just

$$\frac{1}{1-e^x}.$$

Remark 1.7.

Note that the identity function for the composition is simple $I(x) = x$ since it returns its input:

$$I(f(x)) = f(I(x)) = f(x).$$

TABLE 1.1-Table elementary ordinary generating functions

G(z)	**Summation Notation**	**Expanded Notation**
$\frac{1}{1-x}$	$\sum_{k=0}^{\infty} x^k$	$1+x+x^2+x^3\cdots$
$\frac{1}{1+x}$	$\sum_{k=0}^{\infty}(-1)^k x^k$	$1-x+x^2-x^3\cdots$
$\frac{1}{1-x^n}$	$\sum_{k=0}^{\infty} x^{nk}$	$1+x^n+x^{2n}+x^{3n}\cdots$
$\frac{1}{1-cx^n}$	$\sum_{k=0}^{\infty} c^k x^k$	$1+cx^n+c^2x^{2n}+c^3x^{3n}\cdots$
$\frac{1}{(1-x)^n}$	$\sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$	$1+nx+\binom{n+1}{2}x^2+\binom{n+2}{3}x^3\cdots$
$\frac{1}{(1-x)^2}$	$\sum_{k=0}^{\infty} kx^k$	$0+x+2x^2+3x^3\cdots$
$(1+x)^c$	$\sum_{k=0}^{\infty} \binom{c}{k} x^k$	$1+cx+\binom{c}{2}x^2+\binom{c}{3}x^3\cdots$
e^x	$\sum_{k=0}^{\infty} \frac{1}{k!}x^k$	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\cdots$

TABLE 1.2-Operation on ordinary generating functions

Operatios OGF	Squence
$f(x) = \sum_{n\geq 0} a_n x^n$	$a_0, a_1, a_2, ..., a_n, ...$
$g(x) = \sum_{n\geq 0} b_n x^n$	$b_0, b_1, b_2, ..., b_n, ...$
$xf(x) = \sum_{n\geq 1} a_{n-1} x^n$	$0, a_0, a_1, a_2, ..., a_{n-1}, ...$
$\dfrac{f(x) - a_0}{x} = \sum_{n\geq 0} a_{n+1} x^n$	$a_1, a_2, a_3, ..., a_{n+1}, ...$
$f'(x) = \sum_{n\geq 0}(n+1)a_{n+1}x^n$	$a_1, 2a_2, ..., (n+1)a_{n+1}, ...$
$\int_0^x f(x)dx = \sum_{n\geq 1} \dfrac{a_{n+1}}{n} x^n$	$0, a_0, \frac{a_1}{2}, \frac{a_2}{3}, ..., \frac{a_{n-1}}{n}, ...$
$f(\lambda x) = \sum_{n\geq 0} \lambda^n a_n x^n$	$a_0, \lambda a_1, \lambda^2 a_2, ..., \lambda^n a_n, ...$
$f(x) + g(x) = \sum_{n\geq 0}(a_n + b_n)x^n$	$a_0 + b_0, a_1 + b_1, ..., a_n + b_n, ...$
$(1-x)f(x) = a_0 + \sum_{n\geq 1}(a_n - a_{n-1})x^n$	$a_0, a_1 - a_0, ..., a_n - a_{n-1}, ...$
$f(x)g(x) = \sum_{n\geq 0} \left(\sum_{0\geq k\geq n} a_k b_{n-k}\right) x^n$	$a_0 b_0, a_1 b_0, a_0 b_1, ..., \sum_{0\geq k\geq n} a_k b_{n-k}, ...$

CHAPTER 2

GENERATING FUNCTIONS AND RECURRENCE RELATION

After defining the generating functions and highlighting their operations, we will in this chapter show their role in solving the iterative relationship, after highlighting the concept of the recurring relationship, in addition to solving various examples that prove the wonderful role of the generative functions in dealing with the recurring relationships

2.1 Recurrence relation

Definition 2.1.

A recurrence relation for the sequence $(a_n)_{n\geq 0}$ *is an equation relating the term* a_n *to certain of its preceding terms* $a_i,\ i<n$*, for each* $n \geqslant n_0$*.*

Example 2.1.

Fibonacci sequence: The recurrence relation for Fibonacci sequence $(0,1,1,2,3,5,8,13,\cdots)$ is given by: $F_{n+2}=F_{n+2}+F_n$ for$n\geq 0, F_0=0, F_1=1$ This formula arises in the 13^{th} century, when the Italian mathematician **Leonardo Fibonacci** proposed and solved the problem of "rabbits multiplying".

Definition 2.2.

The recurrence relation is linear if it expresses an as a linear function of a fixed number of preceding terms. Otherwise the relation is non linear.

A recurrence relation defining a sequence $(a_n)_{n=m}^{\infty}$ *is said to be of* ***order*** d *if the recurrence relation can be written in the form:*

$$a_{n+d}=f(a_n,a_{n+1},\cdots,a_{n+d-1}) \text{ for all } n\geq 0,$$

where f *is a function with* d *variables and* $f(a_n,a_{n+1},\cdots,a_{n+d-1})$ *is not identical to* $f(c,a_{n+1},\cdots,a_{n+d-1})$ *for some constant* c*.*

Definition 2.3.

The recurrence relation is k-th order if a_n *can be expressed in terms of* $a_{n-1},a_{n-2},\cdots,a_{n-k}$*.*

Example 2.2.

$a_n = 3a_{n-1} - 4a_{n-2}\ n \geqslant 1$. The recurrence relation is linear of the 2th-order.

Definition 2.4. NON HOMOGENEOUS

The nonhomogeneous recurrence relation has the form

$$C_0 a_n + C_1 a_{n-1} + \cdots + C_k a_{n-k} = f(n),\ n \geqslant k,$$

where $C_0 \neq 0, C_k \neq 0$, *and* $f(n) \neq 0$ *for at least one value of* $n\ if\ n \geqslant k$. *If* $f(n) = 0$, *then the recurrence relation is homogeneous .*

Example 2.3.

$a_n - 3a_{n-1} - 4a_{n-2} = n + 1\ n \geqslant 1$

The recurrence relation is linear, non homogeneous and of the 2-th order.

We know that recurrence relations, may be solved using the the fact that the vector space is generated by the exponential functions. So,let us remainder this fact:

Theorem 2.1.

If the characteristic equation $x^2 - r_1 x - r_2 = 0$ *of the recurrence equation* $a_n = r_1 a_{n-1} + r_2 a_{n-2}$ *has two distinct roots* S_1 *and* S_2 *then*

$$a_n = uS_1^n + vS_2^n$$

is the closed form formula for the sequence where u *and* v *depend on the initial conditions.*

Theorem 2.2.

If $\left\{ a_n^{(p)} \right\}$ *is a particular solution of the non-homogeneous linear recurrence relation with constant coefficients*

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} + F(n),$$

then every solution of the recurrence relation has the form

$$\left\{a_n^{(P)} + a_n^{(h)}\right\},$$

where $\left\{a_n^{(h)}\right\}$ *is a solution of the associated homogeneous recurrence relation*

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k}.$$

Example 2.4.

Solve the recurrence relation given by:

$$\begin{cases} a_n = a_{n-1} + 2a_{n-2} \\ a_0 = 2, a_1 = 7 \end{cases}$$

Solution.

The characteristic equation of this relation is

$$r^2 - r - 2 = 0.$$

So, we have two distinct solutions $r = 2$ and $r = -1$, and then the solutions are given by

$$a_n = \alpha 2^n + \beta(-1)^n.$$

Using the initial conditions, we obtain

$$\begin{cases} a_0 = 2 = \alpha + \beta \\ a_1 = 7 = \alpha \cdot 2 + \beta \cdot (-1) \end{cases}$$

This gives the values of α and β:

$$\alpha = 3, \ \beta = -1,$$

and then

$$a_n = 3 \cdot 2n - (-1)^n.$$

Example 2.5.

Solve the recurrence relation

$$\begin{cases} a_n = 6a_{n-1} - 9a_{n-2} \\ a_0 = 1, a_1 = 6 \end{cases}$$

Solution.

The characteristic equation is given by

$$r^2 - 6r + 9 = 0,$$

and then the solution is $r = 3$. The general solutions of the recurrence relation is given by

$$a_n = \alpha 3^n + \beta n 3^n.$$

Using the initial condition, we obtain

$$\begin{cases} a_0 = 1 = \alpha \\ a_1 = 6 = \alpha \cdot 3 + \beta \cdot 3 \end{cases}$$

So, $\alpha = 1,\ \beta = 1$, and

$$a_n = 3^n + n3^n$$

Example 2.6.

Consider another example, with a non-homogeneous example

$$\begin{cases} a_n - a_{n-1} - a_{n-2} = 2n \\ a_0 = 0, a_1 = 1 \end{cases}$$

Solution.

First, consider the homogeneous equation:

$$a_n - a_{n-1} - a_{n-2} = 0.$$

Its characteristic equation is

$$x^2 - x - 1 = 0.$$

The solutions are given by

$$x_1 = \frac{1+\sqrt{5}}{2} \text{ and } x_2 = \frac{1-\sqrt{5}}{2}.$$

The general solution is then

$$a_n = a\left[\frac{1+\sqrt{5}}{2}\right]^n + b\left[\frac{1-\sqrt{5}}{2}\right]^n.$$

$$a_n = Cn + D$$

$$Cn + D - C(n-1) - D - C(n-2) - D = 2n$$

or

$$(Cn - Cn - Cn) + D + C - D + 2C - D = 2n$$

$$-Cn + 3C - D = 2n$$

$$-C = 2, 3C - D = 0$$

$$C = -2, D = -6$$

$$a_n = -2n - 6$$

$$a_n = a\left[\frac{1+\sqrt{5}}{2}\right]^n + b\left[\frac{1-\sqrt{5}}{2}\right]^n - 2n - 6$$

$$\left\{\begin{array}{l} a_0 = a\left[\frac{1+\sqrt{5}}{2}\right]^0 + b\left[\frac{1-\sqrt{5}}{2}\right]^0 - 2\times(0) - 6 \\ a_1 = a\left[\frac{1+\sqrt{5}}{2}\right]^1 + b\left[\frac{1-\sqrt{5}}{2}\right]^1 - 2\times(1) - 6 \end{array}\right. \Longrightarrow \left\{\begin{array}{c} 0 = a + b \\ 1 = a\left[\frac{1+\sqrt{5}}{2}\right] + b\left[\frac{1-\sqrt{5}}{2}\right] - 8 \end{array}\right.$$

$$a = \frac{3(\sqrt{5}+2)}{\sqrt{5}} \text{ and } b = \frac{3(\sqrt{5}-2)}{\sqrt{5}}$$

$$a_n = \frac{3(\sqrt{5}+2)}{\sqrt{5}} \left[\frac{1+\sqrt{5}}{2}\right]^n + \frac{3(\sqrt{5}-2)}{\sqrt{5}} \left[\frac{1-\sqrt{5}}{2}\right]^n - 2n - 6$$

2.2 Solving recurrence relation with generating functions

In this section, we will see how to use the generating functions to solve recurrence relations. Let us start with the non-homogeneous case.

2.2.1 Linear non homogeneous recurrence relation (NHCR)

The following theorem gives the genral solution of the NHCR

Theorem 2.3.

Let the recurrence relation be

$$c_n a_n + c_{n-1} a_{n-1} + \cdots + c_{n-k} a_{n-k} = p(n).$$

Then the sequence a_n is given by

$$\sum_{n=0}^{\infty} a_n x^n = \frac{-(c_{n-1}a_0 + \cdots + c_{n-k}a_0 + c_{n-k}a_1 x + \cdots + c_{n-k}a_{n-k-1}x^{k-1}) + P(x)}{c_n + c_{n-1}x + \cdots + c_{n-k}x^k},$$

where

$$P(x) = \sum_{n=0}^{\infty} p(n) x^n.$$

The a_n is obtained after the decomposition of the fraction into simple elements.

Proof:

Multiply both sides of the relation by x^n. Then:

$$c_n a_n x^n + c_{n-1} a_{n-1} x^n + \cdots + c_{n-k} a_{n-k} x^n = p(n) x^n.$$

Summing up both sides of the relation term by term we get

$$c_n \sum_{n=0}^{\infty} a_n x^n + c_{n-1} a_0 + c_{n-1} \sum_{n=1}^{\infty} a_{n-1} x^n + \cdots + c_{n-k} a_0 + c_{n-k} a_1 x + \cdots + c_{n-k} a_k x^k + c_{n-k} \sum_{n=k}^{\infty} a_{n-k} x^n = \sum_{n=0}^{\infty} p(n) x^n,$$

then

$$c_{n-1} a_0 + c_n \sum_{n=0}^{\infty} a_n x^n + c_{n-1} \sum_{n=0}^{\infty} a_{n-1} x^{n-1} + \cdots + c_{n-k} a_0 + c_{n-k} a_1 x + \cdots + c_{n-k} a_{k-1} x^{k-1} + c_{n-k} x^k \sum_{n=0}^{\infty} a_{n-k} x^{n-k} =$$

$$\sum_{n=0}^{\infty} p(n) x^n.$$

Let:

$$\sum_{n=0}^{\infty} a_n x^n = f(x) \quad \text{and} \quad \sum_{n=0}^{\infty} p(n) x^n = P(x).$$

Letting $f(x)$ be the generating function of the sequence in relation.

We have:

$$\begin{cases} c_n f(x) + c_{n-1} x f(x) + \cdots + c_{n-k} x^k f(x) + c_{n-1} a_0 + \cdots + c_{n-k} a_0 + c_{n-k} a_1 x + \cdots + c_{n-k} a_{k-1} x^{k-1} = P(x) \\ (c_n + c_{n-1} x + \cdots + c_{n-k} x^k) f(x) = -(c_{n-1} a_0 + \cdots + c_{n-k} a_0 + c_{n-k} a_1 x + \cdots + c_{n-k-1} a_{k-1} x^{k-1}) + P(x) \end{cases}$$

So:

$$f(x) = \frac{-(c_{n-1} a_0 + \cdots + c_{n-k} a_0 + c_{n-k} a_1 x + \cdots + c_{n-k} a_{n-k-1} x^{k-1}) + P(x)}{c_n + c_{n-1} x + \cdots + c_{n-k} x^k},$$

We have

$$\frac{-(c_{n-1}a_0+\cdots+c_{n-k}a_0+c_{n-k}a_1x+\cdots+c_{n-k}a_{n-k-1}x^{k-1})+P(x)}{c_n+c_{n-1}x+\cdots+c_{n-k}x^k}=\sum_{n=0}^{\infty}\theta_n x^n$$

Hence $a_n=\theta_n$ is the desired solution

2.2.2 Linear homogeneous recurrence relation

This case was almost settled, in fact, we just let $p(n)=)$, to obatin the following theorem

Theorem 2.4.

Let the recurrence relation $c_na_n+c_{n-1}a_{n-1}+\cdots+c_{n-k}a_{n-k}=0$*. Then, the general solution of this relation is given by*

$$f(x)=\frac{-(c_{n-1}a_0+\cdots+c_{n-k}a_0+c_{n-k}a_1x+\cdots c_{n-k}a_{n-k-1}x^{k-1})}{c_n+c_{n-1}x+\cdots+c_{n-k}x^k}=\sum_{n=0}^{\infty}\omega_n x^n,$$

and $a_n=\omega_n$ *is obtained after decomposition of the fraction.*

$$(c_n+c_{n-1}x+\cdots c_{n-k}x^k)f(x)=-(c_{n-1}a_0+\cdots+c_{n-k}a_0+c_{n-k}a_1x+\cdots c_{n-k-1}a_{k-1}x^{k-1})$$

$$f(x)=\frac{-(c_{n-1}a_0+\cdots+c_{n-k}a_0+c_{n-k}a_1x+\cdots c_{n-k}a_{n-k-1}x^{k-1})}{c_n+c_{n-1}x+\cdots+c_{n-k}x^k}$$

$$=\sum_{n=0}^{\infty}\omega_n x^n.$$

Now a_n *is the coefficient of* x^n*, the term in the square brackets. Note that this solution could also be written as:*

$$a_n=\omega_n.$$

TABLE 2.1-The following table gives the most known generating functions Table of Generating Functions

Sequence a_n	**Generating Function** $A(x)$
$C(k,n)$	$(1+x)^k$
1	$\frac{1}{1-x}$
a^n	$\frac{1}{1-ax}$
$(-1)^n$	$\frac{1}{1+x}$
$(-1)^n a^n = (-a)^n$	$\frac{1}{1+ax}$
$C(k-1+n,n)$	$\frac{1}{(1-x)^k}$
$C(k-1+n,n)a^n$	$\frac{1}{(1-ax)^k}$
$C(k-1+n,n)(-a)^n$	$\frac{1}{(1+ax)^k}$
$n+1$	$\frac{1}{(1-x)^2}$
n	$\frac{1}{(1-x)^2}$
$(n+2)(n+1)$	$\frac{2}{(1-x)^3}$
$(n+1)(n)$	$\frac{2x}{(1-x)^3}$
n^2	$\frac{x(1+x)}{(1-x)^3}$
$(n+3)(n+2)(n+1)$	$\frac{6}{(1-x)^4}$
$(n+2)(n+1)(n)$	$\frac{6x}{(1-x)^4}$
n^3	$\frac{x(1+4x+x^2)}{(1-x)^2}$
$(n+1)a^n$	$\frac{1}{(1-ax)^2}$
na^n	$\frac{ax}{(1-ax)^2}$
n^2a^n	$\frac{ax(1+ax)}{(1-ax)^3}$
n^3a^n	$\frac{ax(1+4ax+a^2x^2)}{(1-ax)^3}$

Example 2.7.

Let us start with a simple example: solve the following recursion, using the generating function technique

$$\begin{cases} a_n = 2a_{n-1} \\ a_0 = 1 \end{cases}$$

Solution.

First, put

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Write $f(x)$ as

$$f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n.$$

Now use the identity $a_n = 2a_{n-1}$ to get

$$f(x) = 1 + \sum_{n=1}^{\infty} 2a_{n-1} x^n$$

or, which is the same

$$f(x) = 1 + 2x \sum_{n=1}^{\infty} a_{n-1} x^{n-1}$$

$$f(x) = 1 + 2x \sum_{n=0}^{\infty} a_n x^n = 1 + 2xf(x).$$

Solving for $f(x)$ to get

$$f(x) = \frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} a_n x^n.$$

Finally

$$a_n = 2^n.$$

Example 2.8.

Solve the recursion using the generating functions

$$\begin{cases} a_n = 2a_{n-1} - a_{n-2} & n \geqslant 2 \\ a_0 = 3, a_1 = -2. \end{cases}$$

Solution.

We have

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots .$$

$$2xf(x) = 2a_0 x + 2a_1 x^2 + \cdots + 2a_{n-1} x^n + \cdots .$$

$$x^2 f(x) = a_0 x^2 + \cdots + a_{n-2} x^n + \cdots .$$

Therefore,

$$f(x) - 2xf(x) + x2f(x)$$

$$= a_0 + (a_1 - 2a_0)x + (a_2 - 2a_1 + a_0)x^2 + \cdots + (a_n - 2a_{n-1} + a_{n-2})x^n + \cdots$$

$$= 3 - 8x.$$

Since

$$a_0 = 3, a_1 = -2 \text{ and } a_n - 2a_{n-1} + a_{n-2} = 0 \text{ for } n \geq 2.$$

So,

$$(1 - 2x + x^2)f(x) = 3 - 8x$$

$$(1 - x)2f(x) = 3 - 8x$$

$$\Longrightarrow f(x) = \frac{1}{(1-x)^2}(3 - 8x)$$

$$= (1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots)(3 - 8x)$$

$$= 3 - 2x - 7x^2 - 12x^3 + \cdots + [3(n+1) - 8n]x^n + \cdots$$

$$= 3 - 2x - 7x^2 - 12x^3 + \cdots + (-5n + 3)x^n + \cdots$$

Therefore a $n = 3 - 5x$ is the desired solution.

Example 2.9.

Solve the recurrence relation

$$\begin{cases} a_n = -3a_{n-1} + 10a_{n-2}, & n \geqslant 2 \\ a_0 = 1, a_1 = 4. \end{cases}$$

Solution.

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the a_n Multiply by x^n both sides of the previous relation, we get

$$\sum_{n=2}^{\infty} a_n x^n = -3\sum_{n=2} a_{n-1}x^n + 10\sum_{n=2} a_{n-2}x^n.$$

Rewrite the previous identity as

$$f(x) - 1 - 4x = -3x(f(x) - 1) + 10x^2 f(x),$$

or

$$f(x)(1 + 3x - 10x^2) = 1 + 7x$$

Therefore

$$f(x) = \frac{1 + 7x}{1 + 3x - 10x^2} = \frac{1 + 7x}{(1 + 5x)(1 - 2x)}.$$

Write

$$f(x) = \frac{1}{(1 + 5x)(1 - 2x)} = \frac{A}{1 + 5x} + \frac{B}{1 - 2x}.$$

Multiply both side by $(1+5x)$ and then let $x=-1/5$, we get A. To get B, just let $x=0$. Indeed

$$A=\frac{5}{7} \text{ and } B=\frac{2}{7}.$$

Therefore,

$$f(x)=\frac{5}{7}\left(\frac{1}{1+5x}\right)+\frac{2}{7}\left(\frac{1}{1-2x}\right).$$

Expanding the right hand side to get

$$f(x)=\frac{5}{7}\sum_{i=0}^{\infty}(-5x)^n+\frac{2}{7}\sum_{i=0}^{\infty}(2x)^n.$$

The solution is then

$$a_n=-\frac{-(-5)^{n+1}}{7}+\frac{2^{n+1}}{7}.$$

Example 2.10.

Solve the recurrence relation

$$\begin{cases} a_n=-a_{n-1}+2n-3, & n\geqslant 1 \\ a_0=1. \end{cases}$$

Solution.

Let

$$f(x)=\sum_{n=0}a_nx^n$$

be the generating function of (a_n). We have, after multiplying by x^n and then summing:

$$f(x) - 1 = -xf(x) + \sum_{n=1}(2n-3)x^n.$$

Using the table of the generating functions, we obtain

$$f(x) = -xf(x) + \frac{1}{(1-x)^2} - \frac{3x}{1-x}.$$

Therefore,

$$f(x) + xf(x) = \frac{1}{(1-x)^2} - \frac{3x}{1-x},$$

and then

$$f(x) = \frac{1}{(1+x)(1-x)^2} - \frac{3x}{1-x^2}.$$

More explicitly

$$\sum_{k=0}^{n-1}(-1)^k k = \begin{cases} -\frac{n}{2} \text{ if } n \text{ is even} \\ \frac{n-1}{2} \text{ if } n \text{ is odd} \end{cases}$$

If n is even,

$$a_n = 0 - 2\left(-\frac{n}{2}\right) + 1 = n+1,$$

and if n is odd, then

$$a_n = (2n-3) - 2\left(\frac{n-1}{2}\right) - 1 = 2n - 3 - n + 1 - 1 = n - 3.$$

Finally

$$\text{The solution is } a_n = \begin{cases} n+1 \text{ if } n \text{ is even} \\ n-3 \text{ if } n \text{ is odd} \end{cases}$$

The solution may be written in the closed formula

$$a_n = 2(-1)^n + n - 1.$$

Example 2.11.

Solve the recurrence relation

$$\begin{cases} a_n = a_{n-1} + 2(n-1) \\ a_0 = 1. \end{cases}$$

Solution.

By the same method, we obtain after calculations and simplifications

$$f(x) = \frac{1}{(1-x)^3} - \frac{2x}{(1-x)^3} + \frac{3x^2}{(1-x)^3}.$$

The value of a_n is given by the coefficient of x^n in $f(x)$. Since the generating function $\frac{1}{(1-x)^3}$ is very known, it easy to determine the coefficient of x^n in

$$\frac{1}{(1-x)^3} - \frac{2x}{(1-x)^3} + \frac{3x^2}{(1-x)^3},$$

this is coeff of x^n in $\frac{1}{(1-x)^3}$-2 coeff of x^{n-1} in $\frac{1}{(1-x)^3}$+3 coeff of x^{n-2} in $\frac{1}{(1-x)^3}$. Thus,

$$a_n = \frac{n^2+3n+2}{2} - (n^2+n) + \frac{3(n^2-n)}{2} = n^2 - n + 1.$$

Example 2.12.

This example is very famous, in fact, using the generating function, we can find an explicit formula of the Fibonacci numbers. Solve the recurrence relation defined by

$$\begin{cases} g_n = g_{n-1} + g_{n-2} \ \ n \geqslant 2 \\ g_0 = 0, g_1 = 1. \end{cases}$$

Solution.

Put

$$f(x) = \sum_{n=0}^{\infty} g_n x^n.$$

Then we have

$$f(x) = g_0 + g_1 x + \sum_{n=2}^{\infty} g_n x^n.$$

Write $f(x)$ as follows

$$f(x) = g_0 + g_1 x + \sum_{n=2}^{\infty} [g_{n-1} + g_{n-2}] x^n.$$

So, we have

$$f(x) = g_1 x + \sum_{n=2}^{\infty} g_{n-1} x^n + \sum_{n=2}^{\infty} g_{n-2} x^n$$

$$f(x) = x + \sum_{n=2}^{\infty} g_{n-1} x^n + \sum_{n=2}^{\infty} g_{n-2} x^n.$$

$$f(x) = x + x \sum_{n=1}^{\infty} g_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} g_{n-2} x^{n-2}.$$

$$x = 1 + xf(x) + x^2 f(x)$$

$$f(x) = \frac{x}{(1 - x - x^2)}.$$

To find (g_n), we decompose the fraction $\frac{x}{1-x-x^2}$ into simple fractions

$$f(x) = \frac{1}{\sqrt{5}} \left[\frac{A}{1 - y_1 x} - \frac{B}{1 - y_2 x} \right]$$

$$y_1 = \frac{1+\sqrt{5}}{2} \text{ and } y_2 = \frac{1-\sqrt{5}}{2}$$

$$A = \frac{1+\sqrt{5}}{2} \text{ and } B = \frac{1-\sqrt{5}}{2}.$$

The coefficient of x^n is then

$$\frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

$$g_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

In the next example, we use the exponential generating functions to find the explicit formula of a sequence (a_n), satisfying a recursion formula.

Example 2.13.

Solve the following recurrence equation using exponential generating functions

$$\begin{cases} d_n = (n-1)(d_{n-1} + d_{n-2}) \ \ n \geqslant 3 \\ d_1 = 0, d_2 = 1. \end{cases}$$

Solution.

Put

$$g(x) = \sum_{n=0}^{\infty} d_n \frac{x^n}{n!}.$$

Differentiate g with respect with x:

$$g'(x) = \sum_{n=0}^{\infty} d_n \frac{x^{n-1}}{n-1!}.$$

Now replace d_n by its values:

$$g'(x) = \sum_{n=0}^{\infty} (n-1)(d_{n-1} + d_{n-2}) \frac{x^{n-1}}{n-1!}.$$

We get

$$g'(x) = \sum_{n=0}^{\infty} (n-1) d_{n-1} \frac{x^{n-1}}{n-1!} + \sum_{n=0}^{\infty} (n-1) d_{n-2} \frac{x^{n-1}}{n-1!}.$$

or,

$$g'(x) = \sum_{n=0}^{\infty} d_{n-1} \frac{x^{n-1}}{n-2!} + \sum_{n=0}^{\infty} d_{n-2} \frac{x^{n-1}}{n-2!}.$$

This is exactly

$$g'(x) = x\sum_{n=0}^{\infty} d_{n-1}\frac{x^{n-2}}{n-2!} + x\sum_{n=0}^{\infty} d_{n-2}\frac{x^{n-2}}{n-2!}.$$

We obtain the differential equation:

$$g'(x) = xg'(x) + xg(x).$$

Which may be written

$$(1-x)g'(x) = xg(x),$$

or

$$\frac{g'(x)}{g(x)} = \frac{x}{(1-x)}.$$

Let us transform it to

$$\frac{g'(x)}{g(x)} = \frac{1}{(1-x)} - 1.$$

Integrate it, to obtain

$$\log g(x) = -\log(1-x) - x \longrightarrow \log g(x)(1-x) = -x,$$

Then

$$g(x)(1-x) = e^{-x} \text{ or } g(x) = \frac{1}{(1-x)} \cdot e^{-x}$$

The term d_n of this generating function is the coefficient of $\frac{x^n}{n!}$ that is:

$$d_n = n!\left(1 - \frac{1}{1!} + \frac{2}{2!} - \cdots (-1)^n \frac{1}{n!}\right).$$

Note that d_n is the number of derangements of an n-element set

Theorem 2.5.

If $\{a_n\}_{n=0}^{\infty}$ *is a sequence of numbers which satisfy the linear recurrence relation with constant coefficients,* $a_n + C_1 a_{n-1} + \cdots + C_k a_{n-k} = 0$ *where* $C_k \neq 0$ *, and* $n \geqslant k$ *then the generating function* $A(x) = \sum_{n=0}^{\infty} a_n x^n$ *equals* $\frac{P(x)}{Q(x)}$ *,where* $P(x) = a_0 + (a_1 + C_1 a_0)x + \cdots + (a_{k-1} + C_1 a_{k-2} + \cdots + C_{k-1} a_0)x^{k-1}$ *and* $Q(x) = 1 + C_1 x + \cdots + C_k x^k$*.*

Conversely, given polynomials $P(x)$ *and* $Q(x)$*, where* $P(x)$ *has degree less than* k*, and* $Q(x)$ *has degree* k*, there is a sequence* $\{a_n\}_{n=0}^{\infty}$ *whose generating function is* $A = \frac{P(x)}{Q(x)}$*.*

Example 2.14.

To illustrate the previous theorem, let us solve the following recursion

$$a_n - 7a_{n-1} + 10a_{n-2} = 0$$

for

$$n \geqslant 2.$$

Solution.

Put

$$A(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Write $A(x)$ as follows

$$A(x) = \sum_{n=2}^{\infty} a_n x^n - 7\sum_{n=2}^{\infty} a_{n-1} x^n + 10\sum_{n=2}^{\infty} a_{n-2} x^n = 0,$$

Now

$$[A(x) - a_0 - a_1x] - 7x[A(x) - a_0] + 10x^2[A(x)] = 0.$$

Rearrange to get

$$A(x)(1 - 7x + 10x^2) = a_0 + a_1x - 7a_0x$$

or

$$A(x) = \frac{a_0 + (a_1 - 7a_0)x}{1 - 7x + 10x^2} = \frac{a_0 + (a_1 - 7a_0)x}{(1 - 2x)(1 - 5x)}.$$

After decompostion of $A(x)$:

$$A(x) = \frac{C_1}{1 - 2x} + \frac{C_2}{1 - 5x}.$$

We get

$$A(x) = \frac{C_1}{1 - 2x} + \frac{C_2}{1 - 5x} = C_1 \sum_{n=0}^{\infty} 2^n x^n + C_2 \sum_{n=0}^{\infty} 5^n x^n$$

$$a_n = C_1 2^n + C_2 5^n$$

Now the constants C_1 and C_2 are uniquely determined once the inital conditions are given, i.e; the values for a_0 and a_1. For example, if $a_0 = 10$, and $a_1 = 41$, we replace n by 0 and 1 in the solution $a_n = C_1 2^n + C_2 5^n$ to obtain the two equations.

$C_1 + C_2 = 10$ and $2C_1 + 5C_2 = 41$ which give the values $C_1 = 3$ and $C_2 = 7$. Thus, in this case the unique solution of the recurrence relation is $a_n = (3)2^n + (7)5^n$.

Consider another example

Example 2.15.

Solve the recursion

$$a_n - 9a_{n-1} + 26a_{n-2} - 24a_{n-3} = 0, \quad \text{for} \quad n \geqslant 3.$$

Solution.

As usual, let

$$A(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Now, write

$$A(x) = \sum_{n=3}^{\infty} a_n x^n - 9\sum_{n=3}^{\infty} a_{n-1}x^n + 26\sum_{n=3}^{\infty} a_{n-2}x^n - 24\sum_{n=3}^{\infty} a_{n-3}x^n = 0,$$

or which is equivalent

$$(A(x) - a_0 - a_1x - a_2x^2) - 9x(A(x) - a_0 - a_1x) + 26x^2(A(x) - a_0) - 24x^3A(x) = 0,$$

Taking $A(x)$ from one side yields

$$A(x)(1 - 9x + 26x^2 - 24x^3) = a_0 + a_1x + a_2x^2 - 9a_0x - 9a_1x^2 + 26a_0x^2.$$

or

$$A(x) = \frac{a_0 + (a_1 - 9a_0)x + (a_2 - 9a_1 + 26a_0)x^2}{1 - 9x + 26x^2 - 24x^3}.$$

We need the factorization:

$$1 - 9x + 26x^2 - 24x^3 = (1 - 2x)(1 - 3x)(1 - 4x).$$

Now $A(x)$ may be written

$$A(x) = \frac{C_1}{(1-2x)} + \frac{C_2}{(1-3x)} + \frac{C_3}{(1-4x)}.$$

So,

$$A(x) = C_1 \sum_{n=0}^{\infty} 2^n x^n + C_2 \sum_{n=0}^{\infty} 3^n x^n + C_3 \sum_{n=0}^{\infty} 4^n x^n,$$

and then

$$A(x) = \sum_{n=0}^{\infty} (C_1 2^n + C_2 3^n + C_3 4^n) x^n.$$

Since $a_0 = 0, a_1 = 1$ and $a_2 = 10$,

$$A(x) = \frac{x + x^2}{(1-2x)(1-3x)(1-4x)} = \frac{C_1}{(1-2x)} + \frac{C_2}{(1-3x)} + \frac{C_3}{(1-4x)}.$$

Now, multiply both sides by $(1-2x)(1-3x)(1-4x)$, we get

$$C_1(1-3x)(1-4x) + C_2(1-2x)(1-4x) + C_3(1-2x)(1-3x) = x + x^2.$$

To get C_1 let $x = \frac{1}{2}$ in the previous identity we get $C_1 = \frac{3}{2}$

$C_2, \ C_3$ are obtained similarly: by setting $x = \frac{1}{3}$ we obtain $C_2 = -4$

For

$$x = \frac{1}{4}, \ C_3 = \frac{5}{2}.$$

Finally we obtain the wanted value of (a_n)

$$a_n = \frac{3}{2}(2^n) - 4(3^n) + \frac{5}{2}(4^n).$$

Example 2.16.

Solve the recurrence relation by the generating functions

$$\begin{cases} a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0 \;\; n \geqslant 3 \\ a_0 = 0 \\ a_1 = 1 \\ a_2 = 2. \end{cases}$$

Solution.

Set

$$A(x) = \sum_{n=0}^{\infty} a_n x^n.$$

According the previous theorem

$$A(x) = \frac{P(x)}{Q(x)},$$

where $deg\ p = 2,\ deg\ Q = 3$ We have

$$= \frac{a_0 + (a_1 - 6a_0)x + (a_2 - 6a_1 12a_0)x^2}{1 - 6x + 12x - 8x}.$$

But $41 - 6x + 12x^2 - 8x^3 = (1 - 2x)^3$. So,

$$A(x) = \frac{\alpha}{(1-2x)} + \frac{\beta}{(1-2x)^2} + \frac{\gamma}{(1-2x)^3}.$$

To find γ, multiply both sides by $(1 - 2x)^3$ and let $x = 1/2$ gives

$$\gamma = -\frac{1}{2}.$$

Using the system

$$\begin{cases} \alpha+\beta+\gamma=0 \\ 4\alpha=-4 \\ -4\alpha-2\beta=1 \end{cases}$$

We deduce the other values:

$$\alpha=-1, \quad \beta=\frac{3}{2}.$$

So

$$A(x)=-1\sum_{n=0}^{\infty}2^nx^n+\frac{3}{2}\sum_{n=0}^{\infty}\tbinom{n+1}{n}2^nx^n+\frac{-1}{2}\sum_{n=0}^{\infty}\tbinom{n+1}{n}2^nx^n$$

$$=\sum_{n=0}^{\infty}\left[(-1)2^n+\frac{3}{2}(n+1)2^n+\frac{-1}{2}\left(\frac{(n+2)(n+1)}{2}\right)2^n\right]x^n,$$

and the wanted value of a_n :

$$a_n=(-1)2^n+\frac{3}{2}(n+1)2^n+\frac{-1}{2}\left(\frac{(n+2)(n+1)}{2}\right)2^n.$$

Example 2.17.

Solve the recursion relation

$$\begin{cases} a_n-8a_{n-1}+21a_{n-2}-18a_{n-3}=0 \ \ n\geqslant 3 \\ a_0=1, a_1=3, a_2=4. \end{cases}$$

Solution.

Let

$$A(x)=\sum_{n=0}^{\infty}a_nx^n.$$

Write $A(x)$ as

$$A(x) = \sum_{n=3}^{\infty} a_n x^n - 8\sum_{n=3}^{\infty} a_{n-1}x^n + 21\sum_{n=3}^{\infty} a_{n-2}x^n - 18\sum_{n=3}^{\infty} a_{n-3}x^n = 0.$$

Replace the summations by $A(x)$, we get

$$(A(x) - a_0 - a_1x - a_2x^2) - 8x(A(x) - a_0 - a_1x) + 21x^2(A(x) - a_0) - 18x^2A(x) = 0.$$

This yields

$$A(x) = \frac{a_0 + (a_1 - 8a_0)x + (a_2 - 8a_1 + 21a_0)x^n}{1 - 8x + 21x^2 - 18x^3}$$

$$A(x) = \frac{1 - 5x + x^2}{1 - 8x + 21x^2 - 18x^3}.$$

Note that

$$1 - 8x + 21x^2 - 18x^3 = (1 - 2x)(1 - 3x)^2.$$

Decompose this fraction to

$$A(x) = \frac{\alpha}{1 - 2x} + \frac{\beta}{1 - 3x} + \frac{\gamma}{(1 - 3x)^2},$$

where α, β, γ are constants. By identification, we obtain the system

$$\begin{cases} 9\alpha + 6\beta = 1 \\ -6\alpha - 5\beta - 2\gamma = -5 \\ \alpha + \beta + \gamma = 1 \end{cases}$$

, its solution is given by

$$\begin{cases} \alpha = -5 \\ \beta = \frac{23}{3} \\ \gamma = \frac{-5}{3} \end{cases}$$

So,

$$A(x) = \sum_{n=0}^{\infty} \left[(-5)2^n + \left(\frac{23}{3}\right) 3^n + \left(\frac{-5}{3}\right) n^3 \, C(n+1, n) \right] x^n,$$

and then the values of a_n;

$$a_n = (-5)2^n + \left(\frac{23}{3}\right) 3^n + \left(\frac{-5}{3}\right) (n+1)3^n.$$

CHAPTER 3

APPLICATIONS OF GENERATING FUNCTIONS

In this chapter we will see many applications of the generating functions in dealing with combinatorial identities. First, we will use the famous "Snake oil method" to prove some identities. This method and the examples are taken from the excellent book "Generatingfunctionology" of Herbert Wilf.

3.1 Combinatorial Identities

3.1.1 The Snake Oil Method for Doing Combinatorial Sums

This method, as its name suggestes, may work for almost all of the caes, were a summation is involved. We apply it in four steps:

1. Identify the free variable say n, and call wanted sum $f(n)$.
2. Consider the generating function for $\sum_n f(n)z^n$.
3. Solve the new inner summation and the outer one.
4. Equate the coefficients to find $f(n)$.

First let us recall the following basic and classic generating functions

Proposition 3.1.

$$\sum_k \binom{n}{k} x^k = (1+x)^n \tag{3.1}$$

$$\sum_{n\geqslant 0} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}} \qquad (k \geq 0). \tag{3.2}$$

$$\sum_n \frac{1}{1+n} \binom{2n}{n} x^n = \frac{1}{2x}\left(1-\sqrt{1-4x}\right) \tag{3.3}$$

Now, let us start with some examples.

Example 3.1. Fiboancci numbers

Suppose, we are seeking for a closed formula for the following sum

$$\sum_{k\geq 0} \binom{k}{n-k} \qquad (n = 0, 1, 2, ...).$$

The free variable is n. Call the considered sum $f(n)$. Since

$$\binom{n}{k} = 0 \quad \text{for } k < 0 \text{ and } k > n,$$

we can sum over $\mathbb{Z}$. Write it

$$f(n) = \sum_{k\geq 0} \binom{k}{n-k}.$$

So,

$$F(x) = \sum_n f(n)x^n,$$

is nothing but

$$F(x) = \sum_n x^n \sum_{k\geq 0} \binom{k}{n-k}.$$

So, we can write it again as

$$F(x) = \sum_{k\geq 0} \sum_n \binom{k}{n-k} x^n,$$

In order to apply (3.1), write $x^n = x^k.x^{n-k}$, so

$$F(x) = \sum_{k\geq 0} x^k \sum_n \binom{k}{n-k} x^{n-k},$$

Now make the change $r = n - k$ to obtain

$$F(x) = \sum_{k\geq 0} x^k \sum_n \binom{k}{r} x^r,$$

The second summation is $(1+x)^k$, so

$$F(x) = \sum_{k\geq 0} x^k(1+x)^k = \sum_{k\geq 0}(x+x^2)^k = \frac{1}{1-x-x^2}.$$

This is the Fibonacci generating function:

$$F(x) = \sum_{n\geq 0} F_n x^n = \frac{1}{1-x-x^2}.$$

We have proved that:

$$\sum_{k\geq 0} \binom{k}{n-k} = F_{n+1} \quad (n = 0, 1, 2, ...).$$

Example 3.2. Consider the sum

$$\sum_k \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1} \qquad (m, n \geq 0).$$

$$F(x) = \sum_{n\geq 0} x^n \sum_k \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1}$$

$$= \sum_k \binom{2k}{k} \frac{(-1)^k}{k+1} x^{-k} \sum_{k\geq 0} \binom{n+k}{m+2k} x^{n+k}$$

$$= \sum_k \left(\begin{array}{c} 2k \\ k \end{array} \right) \frac{(-1)^k}{k+1} x^{-k} \sum_{r \geq k} \left(\begin{array}{c} r \\ m+2k \end{array} \right) x^r$$

$$= \sum_k \left(\begin{array}{c} 2k \\ k \end{array} \right) \frac{(-1)^k}{k+1} x^{-k} \frac{x^{m+2k}}{(1-k)^{m+2k+1}}$$

$$= \frac{x^m}{(1-x)^{m+1}} \sum_k \left(\begin{array}{c} 2k \\ k \end{array} \right) \frac{1}{k+1} \left\{ \frac{-x}{(1-x)^2} \right\}^k$$

$$= \frac{-x^{m-1}}{2(1-x)^{m-1}} \left\{ 1 - \sqrt{1 + \frac{4x}{(1-x)^2}} \right\}$$

$$= \frac{-x^{m-1}}{2(1-x)^{m-1}} \left\{ 1 - \frac{1+x}{1-x} \right\}$$

$$= \frac{x^m}{(1-x)^m}.$$

So, our sum is just the coefficient in the expansion of $\frac{x^m}{(1-x)^m}$ which is $\binom{n-1}{m-1}$.

Example 3.3. Consider the sum to evaluate:

$$f_n = \sum_k \binom{n+k}{2k} 2^{n-k}, \quad (n \geq 0).$$

Consider the generating function

$$F = \sum_n f_n x^n.$$

So

$$\begin{aligned} F &= \sum_k 2^{-k} \sum_{n\geq 0} \binom{n+k}{2k} 2^n x^n \\ &= \sum_k 2^{-k}(2x)^{-k} \sum_{n\geq 0} \binom{n+k}{2k} (2x)^{n+k} \\ &= \sum_k 2^{-k}(2x)^{-k} \frac{(2x)^{2k}}{(1-2x)^{2k+1}} \\ &= \frac{1}{1-2x} \sum_{n\geq 0} \left\{ \frac{x}{(1-2x)^2} \right\}^k \end{aligned}$$

Now, we have a geometric series, so

$$\begin{aligned} F &= \frac{1}{1-2x} \frac{1}{1-\frac{x}{(1-2x)^2}} \\ &= \frac{1-2x}{(1-4x)(1-x)} \\ &= \frac{2}{3(1-4x)} + \frac{1}{3(1-x)}. \end{aligned}$$

Equate the coefficients of x^n in both sides we get

$$\sum_k \binom{n+k}{2k} 2^{n-k} = \frac{2^{2n+1}+1}{3}, \quad (n \geq 0).$$

Example 3.4. Prove the identity

$$\sum_k \binom{m}{k}\binom{n+k}{m} = \sum_k \binom{m}{k}\binom{n}{k}.$$

Multiply the sum in the left side by x^k, and write $x^n = x^k.x^{n-k}$:

$$\begin{aligned} \sum_k \binom{m}{k} x^{-k} \sum_{n\geq 0} \binom{n+k}{m} x^{n+k} &= \sum_k \binom{m}{k} x^{-k} \frac{x^m}{(1-x)^{m+1}}. \\ &= \frac{x^m}{(1-x)^{m+1}} \left(1+\frac{1}{x}\right)^m \\ &= \frac{(1+x)^m}{(1-x)^{m+1}}. \end{aligned}$$

Do the same thing for the sum in the right side, we obtain

$$\begin{aligned}\sum_k \sum_k \binom{m}{k} 2^k \sum_{n\geq 0} \sum_k \binom{n}{k} x^n &= \frac{1}{(1-x)} \sum_k \binom{m}{k} \left(\frac{2x}{(1-x)}\right)^k \\ &= \frac{1}{(1-x)} \left(1+\frac{2x}{1-x}\right)^m \\ &= \frac{(1+x)^m}{(1-x)^{m+1}}.\end{aligned}$$

This means that we have the same sum.

Example 3.5.

Evaluate the sum

$$\sum_{k=m}^{n} (-1)^k \binom{n}{k} \binom{k}{m}.$$

Let

$$f(m) = \sum_{k=m}^{n} (-1)^k \binom{n}{k} \binom{k}{m},$$

and

$$F(m) = \sum_m f(m) x^m.$$

Then we have :

$$\begin{aligned}F(x) &= \sum_m f(m) x^m = \sum_m x^m \sum_{k=m}^{n} (-1)^k \left(\begin{array}{c} n \\ k \end{array} \right) \left(\begin{array}{c} k \\ m \end{array} \right). \\ &= \sum_{k\leqslant n} (-1)^k \left(\begin{array}{c} n \\ k \end{array} \right) \sum_{m\leqslant k} \left(\begin{array}{c} k \\ m \end{array} \right) x^m \\ &= \sum_{k\leqslant n} (-1)^k \left(\begin{array}{c} n \\ k \end{array} \right) (1+x)^k.\end{aligned}$$

Here we have used

$$\sum_{k\leqslant n} \left(\begin{array}{c} k \\ n \end{array} \right) x^m = (1+x)^k.$$

Dalje je

$$F(x) = (-1)^n \sum_{k\leqslant n} \left(\begin{array}{c} n \\ k \end{array} \right) (-1)^{n-k} (1+x)^k = (-1)^n \left(\ (1+x)-1\ \right)^n = (-1)^n x^n$$

Therefore we obtained $F(x) = (-1)^n x^n$ and since this is a generating function of the sequence $f(m)$ we have

$$f(m) = \begin{cases} (-1)^n & , n = m, \\ 0 & , n < m. \end{cases}$$

Example 3.6. Prove the identity

$$\sum_k \binom{2n+1}{k}\binom{m+k}{2n} = \binom{2m+1}{2n}.$$

Let

$$F(x) = \sum_m x^n \sum_k \binom{2n+1}{k}\binom{m+k}{2n},$$

and

$$G(x) = \sum_m x^m \binom{2m+1}{2n}.$$

We will prove that $F(x) = G(x)$. We have

$$\begin{aligned} F(x) &= \sum_m x^m \sum_k \binom{2n+1}{k}\binom{m+k}{2n} \\ &= \sum_m \binom{m+k}{2n} x^m \\ &= \sum_k \binom{2n+1}{2k}\binom{m+k}{2n} x^m \\ &= \sum_k \binom{2n+1}{2k} x^{-k} \sum_m \binom{m+k}{2n} x^{m+k} \\ &= \sum_k \binom{2n+1}{2k} x^{-k} \frac{x^{2n}}{(1-x)^{2n+1}} \\ &= \frac{x^{2n}}{(1-x)^{2n+1}} \sum_k \binom{2n+1}{2k} (x^{-\frac{1}{2}})^{2k} \end{aligned}$$

We know that

$$\sum_k \binom{2n+1}{2k} \left(x^{-\frac{1}{2}}\right)^{2k} = \frac{1}{2}\left(\left(1+\frac{1}{\sqrt{x}}\right)^{2n+1} + \left(1-\frac{1}{\sqrt{x}}\right)^{2n+1}\right).$$

So

$$F(x)=\frac{1}{2}(\sqrt{x})^{2n-1}\left(\frac{1}{(1-\sqrt{x})^{2n+1}}-\frac{1}{(1+\sqrt{x})^{2n+1}}\right).$$

On the other hand;

$$G(x)=\sum_{m}\binom{2m+1}{2n}x^{m}=\left(x^{-\frac{1}{2}}\right)\sum_{m}\binom{2m+1}{2n}\left((x^{\frac{1}{2}}\right)^{2m+1}.$$

This means that

$$G(x)=\left(x^{-\frac{1}{2}}\right)\left[\frac{(x^{\frac{1}{2}})^{2n}}{2}\left(\frac{1}{(1-x^{\frac{1}{2}})^{2n+1}}-(-1)^{2n}\frac{1}{(1+x^{\frac{1}{2}})^{2n+1}}\right)\right],$$

or, which is the same

$$G(x)=\frac{1}{2}(\sqrt{x})^{2n-1}\left(\frac{1}{(1-\sqrt{x})^{2n+1}}-\frac{1}{(1+\sqrt{x})^{2n+1}}\right),$$

and this completes the proof of the identity.

Now, let us prove another identity in this last example.

Example 3.7. Prove that

$$\sum_{k=0}^{n}\binom{2n}{2k}\binom{2k}{k}2^{2n-2k}=\binom{4n}{2n}.$$

Let n be the free variable on the left and right side of $F(x)$ and $G(x)$, where as usual

$$F(x)=\sum_{n}x^{n}\sum_{0\leqslant k\leqslant n}^{n}\binom{2n}{2k}\binom{2k}{2}^{2n-2k}\quad\text{and } G(x)=\sum_{n}\binom{4n}{2n}x^{n}.$$

We want to prove that $F(x)=G(x)$. We have

$$\begin{aligned}F(x) &= \sum_{n}x^{n}\sum_{0\leqslant k\leqslant n}^{n}\binom{2n}{2k}\binom{2k}{k}2^{2n-2k}\\ &= \sum_{0\leqslant k}\binom{2k}{k}2^{-2k}\sum_{n}\binom{2n}{2k}x^{n}2^{2n}.\end{aligned}$$

The last expression may be written as

$$F(x)=\sum_{0\leqslant k}\binom{2k}{k}2^{-2k}\sum_{n}\binom{2n}{2k}(2\sqrt{x})^{2n}.$$

Now we use the formula for summation of even powers and get

$$\sum_{n}\binom{2n}{2k}(2\sqrt{x})^{2n}=\frac{1}{2}(2\sqrt{x})^{2k}\left(\frac{1}{(1-2\sqrt{x})^{2k+1}}+\frac{1}{(1+2\sqrt{x})^{2k+1}}\right),$$

we get then

$$F(x) = \frac{1}{2(1-2\sqrt{x})}\sum_k \binom{2k}{k}\left(\frac{x}{(1-2\sqrt{x})^2}\right)^k + \frac{1}{2(1+2\sqrt{x})}\sum_k \binom{2k}{k}\left(\frac{x}{(1+2\sqrt{x})^2}\right)^k.$$

Since $\sum_n \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$, we get

$$F(x) = \frac{1}{2(1-2\sqrt{x})}\cdot\frac{1}{\sqrt{1-4\frac{x}{(1-2\sqrt{x})^2}}} + \frac{1}{2(1+2\sqrt{x})}\cdot\frac{1}{\sqrt{1-4\frac{x}{(1+2\sqrt{x})^2}}}$$

Which implies

$$F(x) = \frac{1}{2\sqrt{1-4\sqrt{x}}} + \frac{1}{2\sqrt{1+4\sqrt{x}}}.$$

For $G(x)$, note that we have

$$\sum_n \binom{2n}{n}(-x)^n = \frac{1}{\sqrt{1+4x}}.$$

So, we obatain

$$G(x) = \frac{1}{2}\left(\frac{1}{\sqrt{1-4\sqrt{x}}} + \frac{1}{\sqrt{1+4\sqrt{x}}}\right),$$

and $F(x) = G(x)$.

Conclusion

In this work, we have seen many properties of the generating functions. Their definition, properties and some of their applications, , especially in solving recurrence problems and finding combinatorial identities. Many examples were supplied to illustrate these applications. Applications of generating functions in Probability and asymptotics were not investigated in this work, and this may be another master subject, since it may be considered to belong to analysis rather than combinatorics.

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Résumé

Dans ce mémoire, nous étudions les fonction génératrices. On donnera la définition et les principales propriétés. En plus, nous donnons quelques applications liés aux relations de récurrences et à la preuve des identitées combinatoires.

ملخص

في هذه المذكرة نهتم بدراسة الدوال المولدة. بعد تعريفها وعرض خواصها نتطرق لتطبيقاتها لحل العلاقات التكرارية وتطبيقات اخرى اضافية كاثبات بعض المتطابقات.

Abstract

In this thesis, we are interested in generating functions, and their applications. We give their definition and properties. These properties are applied to solve recurrence relations and to prove some combinatorial identities.