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Mohamed Boudiaf University of Msila
Faculty of Mathematics and Computer Sciences
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Some properties of Modulation spaces

Presented by :

Name and surname : Benyahia Imane

Dahmane Achour
Douadi Drihem
Abdelhamid Tallab

Prof, University of Msila
Prof, University of Msila
Prof, University of Msila

President.
Supervisor.
Examiner.

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Dedication

I dedicate the fruits of my humble efforts to those who gave me life and a knowledgeable upbringing, and who taught me to climb the ladder of life with wisdom, patience, kindness, and loyalty : my dear father and mother, who have been my source of support and strength. To those whom God has blessed me with in my life, to the strong bond of support and assistance in my research journey, my brothers and sisters, and Finally, to my esteemed professors who have never held back from offering us their knowledge and guidance.

[Benyahia imane]

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Notation

- $\mathbb{R}^n$ is the n -dimensional real Euclidean space.
- $\mathbb{N}$ is the collection of all natural numbers.
- $\mathbb{Z}$ is the set of all integer numbers.
- For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we write $|\alpha| = \alpha_1 + \dots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$.
- The Euclidean scalar product of $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ is given by $x \cdot y = x_1 y_1 + \dots + x_n y_n$.
- The expression $f \lesssim g$ means that $f \leq c g$ for some independent constant c (and non-negative functions f and g).
- $f \approx g$ means $f \lesssim g \lesssim f$ for non-negative functions f and g .
- For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball in $\mathbb{R}^n$ with center x .

Introduction

function spaces have been a central topic in modern analysis, and are now of increasing applications in many fields of mathematics especially harmonic analysis and partial differential equations. The most known general scales of function spaces are the scales of Lebesgue spaces, Sobolev spaces and Besov spaces. See [9] for more details about for Besov spaces.

Modulation spaces, which were initiated by Feichtinger in 1983, were investigated for the purpose of measuring smoothness of functions and distributions in a way other than Besov spaces.

Besov spaces are very close to Hölder-Zygmund spaces and are used in partial differential equations. These spaces are defined by way of dilations. Feichtinger took full advantage of the group structure of $\mathbb{R}^n$. Recall that $\mathbb{R}^n$ carries the structure of a Lie group not with dilation but with addition. Therefore, it seems natural that we consider the translation.

Modulation spaces are applicable to various partial differential equations. Recently modulation spaces can be applied even to the modeling of wireless channels and the quantum mechanics.

The aim of this master thesis is to present some important properties of Modulation spaces.

Our thesis consists of three chapters.

In the first one, we give some basic properties of Lebesgue spaces. We also give some key technical lemmas that we will use later.

In the second chapter, we present the definition of modulation spaces and prove their basic properties. Also, we prove some elementary embeddings..

In the third chapter, we study the duality of modulation spaces.

Basic spaces

In this chapter, we presented some fundamental proprieties to our work. We also give some key technical lemmas needed in the proofs of the main statements of this thesis such as Nikol'skij-Triebel types inequalities.

1.1 Lebesgue spaces

In this section we recall and present some properties of Lebesgue spaces.

Definition 1.1. *Let $0 < p \leq \infty$. The Lebesgue space L^p is the set of all measurable functions $f : \mathbb{R}^n \rightarrow \mathbb{C}$ such that*

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} < \infty,$$

with $0 < p < \infty$ and

$$\|f\|_\infty = \operatorname{ess-sup}_{x \in \mathbb{R}^n} |f(x)| < \infty.$$

Remark 1.2. *Let $1 \leq p \leq \infty$. The L^p spaces are Banach spaces. For $0 < p < 1$, they are only quasi-Banach spaces, that is, they are complete but the triangle inequality is now in the following form*

$$\|f + g\|_p \leq 2^{\frac{1-p}{p}} (\|f\|_p + \|g\|_p), \quad f, g \in L^p.$$

Now, we recall the Hölder inequality in Lebesgue spaces.

Theorem 1.3. *Let $0 < p, q, r \leq \infty$, $f \in L^p$ and $g \in L^q$. Then $fg \in L^r$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. In addition, we have*

$$\|fg\|_r \leq \|f\|_p \|g\|_q.$$

The notion of convolution of two functions is defined as follows.

Definition 1.4. Let $f, g \in L^1$. The convolution $f * g$ is defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy, \quad x \in \mathbb{R}^n.$$

Let us recall the following Riesz-Thorin interpolation theorem, see [5].

Theorem 1.5. Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and $\theta \in (0, 1)$. Define $1 \leq p < \infty, 1 \leq q \leq \infty$ by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

If T is a linear operator with

$$T : L^{p_i} \rightarrow L^{q_i}, \quad \|T(f)\|_{q_i} \leq M_i \|f\|_{p_i}, \quad f \in L^{p_i}, i \in \{0, 1\}.$$

Then

$$\|T(f)\|_q \leq M_0^{1-\theta} M_1^\theta \|f\|_p, \quad f \in L^p.$$

Theorem 1.6. Let $1 \leq p, r, q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Assume that $f \in L^p$ and $g \in L^q$. Then $f * g \in L^r$ and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Proof. Fix a function g in L^q . Let f be a measurable function. We set

$$T_g(f) = f * g.$$

Let $f \in L^{q'}$ and $x \in \mathbb{R}^n$. By Hölder's inequality, we obtain

$$\begin{aligned} |f * g(x)| &\leq \int_{\mathbb{R}^n} |f(x-y)| |g(y)| dy \\ &\leq \|g\|_{L^q} \|f\|_{L^{q'}}. \end{aligned}$$

Then

$$T_g : L^{q'} \rightarrow L^\infty, \quad \|T_g(f)\|_{L^\infty} \leq \|g\|_{L^q} \|f\|_{L^{q'}}, \quad f \in L^{q'}.$$

Let $f \in L^1$. We have

$$\|T_g(f)\|_q = \sup_{\|h\|_{q'} \leq 1} \left| \int_{\mathbb{R}^n} T_g(f)(x) h(x) dx \right|, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

Observe that

$$\begin{aligned} \int_{\mathbb{R}^n} T_g(f)(x) h(x) dx &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y) g(x-y) dy h(x) dx \\ &= \int_{\mathbb{R}^n} f(y) \int_{\mathbb{R}^n} g(x-y) h(x) dx dy \end{aligned}$$

Let $y \in \mathbb{R}^n$ and $h \in L^{q'}$ with $\|h\|_{q'} \leq 1$. By Hölder's inequality, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |g(x-y)| |h(x)| dx &\leq \|g(\cdot - y)\|_q \|h\|_{q'} \\ &\leq \|g\|_q \|h\|_{q'} \\ &\leq \|g\|_q. \end{aligned}$$

This leads to

$$\int_{\mathbb{R}^n} |T_g(f)(x)| |h(x)| dx \leq \|g\|_q \|f\|_1.$$

Then

$$T_g : L^1 \rightarrow L^q, \quad \|T_g(f)\|_q \leq \|g\|_q \|f\|_1, \quad f \in L^1.$$

By Riesz-Thorin interpolation theorem, we get

$$T_g : L^p \rightarrow L^r, \quad \|T_g(f)\|_r \leq \|g\|_q \|f\|_p, \quad f \in L^p.$$

with

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{q'}, \quad \frac{1}{r} = \frac{1-\theta}{q} + \frac{\theta}{\infty}. \quad (1.1)$$

Finally, observe that equations (1.1) give $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. The proof is complete. $\square$

1.2 Tempered distributions

In this section, we recall some definitions and classical properties about tempered distributions, the translation, dilation, and modulation operators.

Definition 1.7. Let f be a continuous function. By $\text{supp } f$ we denote the support of the function f , i.e., the closure of its non-zero set;

$$\text{supp } f = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}.$$

Let us recall some of properties of the support of the function.

(i) $f = 0 \iff \text{supp } f = \emptyset$

(ii) $\text{supp } f$ is closed in $\mathbb{R}^n$

(iii) $\text{supp}(fg) \subset \text{supp}(f) \cap \text{supp}(g)$

(iv) $\text{supp}(f)$ is the complement of the largest open subset of $\mathbb{R}^n$ where f does not vanish.

Definition 1.8. We set

$$\mathcal{D}(\mathbb{R}^n) = \{\varphi \in \mathcal{C}^\infty(\mathbb{R}^n) : \text{supp}(\varphi) \text{ is compact}\}.$$

$\mathcal{D}(\mathbb{R}^n)$ is called space of test functions.

Now, we define the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing smooth functions as follows.

Definition 1.9. We say that $\varphi \in \mathcal{S}(\mathbb{R}^n)$ if $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$ and

$$\|\varphi\|_{M,\beta} = \sup_{x \in \mathbb{R}^n} |\partial^\beta \varphi(x)| (1 + |x|)^M < \infty$$

for all nonnegative integers M and multi-indices β .

Equivalently, the complete topology of $\mathcal{S}(\mathbb{R}^n)$ can be given through the family of seminorms

$$p_{\lambda,\beta}(f) = \sup_{x \in \mathbb{R}^n} |x^\lambda \partial^\beta \varphi(x)| < \infty.$$

Definition 1.10. A temperate distribution f on $\mathbb{R}^n$ is a continuous linear functional $f : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathbb{C}$, i.e., $\varphi_k \rightarrow 0$ in $\mathcal{S}(\mathbb{R}^n) \implies \langle f, \varphi_k \rangle \rightarrow 0$.

Remark 1.11. The space of temperate distributions on $\mathbb{R}^n$ is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

Theorem 1.12. Let $f : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathbb{C}$ be linear. $f \in \mathcal{S}'(\mathbb{R}^n)$ if and only if there exist $N, M \in \mathbb{N}$ such that

$$|\langle f, \varphi \rangle| \leq c \sum_{|\lambda| \leq N, |\beta| \leq M} p_{\lambda, \beta}(\varphi)$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

The usual operations on a smooth function have a counterpart in the realm of distributions. We can talk, for example, about the translation, dilation, or derivative of a distribution. Let $x, y, \xi \in \mathbb{R}^n, \varepsilon > 0$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Then, the translation, dilation, and modulation operators are defined as follows:

- (a) Translation, or time-shift, by y $T_y \varphi(x) = \varphi(x - y)$;
- (b) (L^1 -normalized -dilation) $D_\varepsilon \varphi(x) = \varepsilon^{-n} \varphi(\varepsilon^{-1} x)$;
- (c) Modulation, or frequency-shift, by ξ $M_\xi \varphi(x) = e^{2\pi i \xi \cdot x} \varphi(x)$.

We can define the operations of translation, dilation, and modulation, respectively, of a distribution u in a natural way. Let $u \in \mathcal{D}'(\mathbb{R}^n)$.

- (a) $\langle T_y u, \varphi \rangle = \langle u, T_{-y} \varphi \rangle$;
- (b) $\langle D_\varepsilon u, \varphi \rangle = \langle u, \varphi(\varepsilon \cdot) \rangle$;
- (c) $\langle M_\xi u, \varphi \rangle = \langle u, M_{-\xi} \varphi \rangle$.

Definition 1.13. The convolution between a distribution f and a test function φ is a function, defined as follows:

$$f * \varphi(x) = \langle f, \varphi(x - \cdot) \rangle = \langle f, T \varphi(x) \rangle, \quad x \in \mathbb{R}^n.$$

Remark 1.14. Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The following properties of convolution follow straightforwardly from the definition:

- (i) $T_y(f * \varphi) = T_y f * \varphi = f * T_y \varphi$.
- (ii) $\partial^\alpha(f * \varphi) = \partial^\alpha f * \varphi = f * \partial^\alpha \varphi, \quad \forall \alpha \in \mathbb{N}^n$.
- (iii) $f * \varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$.

Definition 1.15. We define the Fourier transform of a function $f \in L^1$ by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx, \quad \xi \in \mathbb{R}^n.$$

Its inverse is denoted by $\mathcal{F}^{-1}f$.

Proposition 1.16. *Let $f, g \in L^1$, $y, \xi \in \mathbb{R}^n$, $\varepsilon > 0$ and $\alpha \in \mathbb{N}^n$. Then*

- (i) $\mathcal{F}(T_y f) = M_{-y} \mathcal{F} f$.
- (ii) $\mathcal{F}(M_y f) = T_y \mathcal{F} f$.
- (iii) $\mathcal{F}(D_\varepsilon f)(\xi) = \mathcal{F} f(\varepsilon \xi)$.
- (iv) $\mathcal{F}(\partial^\alpha f)(\xi) = (2\pi i \xi)^\alpha \mathcal{F}(f)(\xi)$.
- (v) $\mathcal{F}(f * g) = \mathcal{F}(f) \mathcal{F}(g)$.

1.3 Nikol'skij-Triebel types inequalities

Let $\xi_0 \in \mathbb{R}^n$ and the compact set $\mathbb{K} \subset \mathbb{R}^n$. We recall that

$$\text{diam} \mathbb{K} = \sup\{|x - y| : x, y \in \mathbb{K}\}.$$

Definition 1.17. *Let $1 \leq p \leq \infty$. We say that $f \in PW^p(\mathbb{K}, \xi_0)$ if $f \in L^p$ and $\text{supp} \mathcal{F} f \subset \mathbb{K} + \xi_0$.*

Remark 1.18. *If $\xi_0 = 0$, we simply write $f \in PW^p(\mathbb{K})$ for $f \in PW^p(\mathbb{K}, 0)$.*

In the following we present Nikol'skij-Triebel inequality, see [9].

Proposition 1.19. *Let $\mathbb{K} \subset \mathbb{R}^n$ be a compact set and $0 < p \leq q \leq \infty$. Let $f \in PW^p(\mathbb{K})$. Then there exists a constant $C = C(\text{diam} \mathbb{K}, p)$ such that*

$$\|f\|_q \leq C \|f\|_p.$$

An immediate consequence of Nikol'skij-Triebel's inequality is the following convolution inequality

Proposition 1.20. *Let $\mathbb{K}_1, \mathbb{K}_2 \subset \mathbb{R}^n$ be compact sets and $0 < p \leq 1$. Let $f \in PW^p(\mathbb{K}_1)$ and $g \in PW^p(\mathbb{K}_2)$. Then there exists a constant $C = C(\text{diam} \mathbb{K}_1, \text{diam} \mathbb{K}_2, p)$ such that*

$$\| |f| * |g| \|_p \leq C \|f\|_p \|g\|_p.$$

Proof. Recall that $\mathbb{K}_1 - \mathbb{K}_2 = \{x_1 - x_2 : x_i \in \mathbb{K}_i, i \in \{1, 2\}\}$. Let $x \in \mathbb{R}^n$. We set $h_x(y) = f(y) \widetilde{T_x g}(y) = f(y)g(x-y)$, $y \in \mathbb{R}^n$. By Proposition 1.16, we get

$$\mathcal{F} h_x = \mathcal{F} f * \widetilde{M_{-x} \mathcal{F} g}.$$

Observe that $\text{supp } \mathcal{F} h_x \subset \mathbb{K}_1 - \mathbb{K}_2$. Hence, by Proposition 1.19, we obtain

$$\begin{aligned} |f| * |g|(x) &= \|h_x\|_1 \\ &\leq C \|h_x\|_p \\ &= C \left(\int_{\mathbb{R}^n} |f(y)|^p |g(x-y)|^p dy \right)^{1/p}. \end{aligned}$$

Therefore

$$\begin{aligned} \| |f| * |g| \|_p &= \left(\int_{\mathbb{R}^n} (|f| * |g|)^p(x) dx \right)^{1/p} \\ &\leq C \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(y)|^p |g(x-y)|^p dy dx \right)^{1/p} \\ &\leq C \|f\|_p \|g\|_p, \end{aligned}$$

where, we have used Tonelli's Theorem. The proof is complete. $\square$

Proposition 1.21. *Let $\mathbb{K} \subset \mathbb{R}^n$ be compact sets, $\xi_0 \in \mathbb{R}^n$ and $0 < p \leq q \leq \infty$. If $f \in PW^p(\mathbb{K}_1, \xi_0)$, then there exists a positive constant $C = C(\text{diam } \mathbb{K}, p)$ such that*

$$\|f\|_q \leq C \|f\|_p.$$

Proof. Observe that $f \in L^p \Leftrightarrow M_{-\xi_0} f \in L^p$ and $\mathcal{F}(M_{-\xi_0} f) = T_{-\xi_0} \mathcal{F}(f)$. Proposition 1.19 gives the desired estimate. $\square$

Proceeds similar to the proof of Proposition 1.21, but now using the statement of Proposition 1.20, we get the following result.

Proposition 1.22. *Let $\mathbb{K}_1, \mathbb{K}_2 \subset \mathbb{R}^n$ be compact sets, $\xi_1, \xi_2 \in \mathbb{R}^n$ and $0 < p \leq 1$. Let $f \in PW^p(\mathbb{K}_1, \xi_1)$ and $g \in PW^p(\mathbb{K}_2, \xi_2)$. Then there exists a constant $C = C(\text{diam } \mathbb{K}_1, \text{diam } \mathbb{K}_2, p)$ such that*

$$\| |f| * |g| \|_p \leq C \|f\|_p \|g\|_p.$$

Proposition 1.23. *Let $\mathbb{K} \subset \mathbb{R}^n$ be compact sets, $\xi_0 \in \mathbb{R}^n, 0 < p \leq \infty$ and $\alpha = \alpha(\mathbb{K}) > 0$ sufficiently small. let $f \in PW^p(\mathbb{K}, \xi_0)$. Then*

$$\|f\|_p \approx \left(\sum_{k \in \mathbb{Z}^n} |f(\alpha(k))|^p \right)^{1/p}.$$

The mixed sequence Lebesgue space is defined in the following way.

Definition 1.24. $0 < p, q \leq \infty$. The space $\ell^{p,q}$ is the set of doubly indexed sequence $s = (s_{kl})_{k,l \in \mathbb{Z}^n}$ such that

$$\|s\|_{\ell^{p,q}} = \left(\sum_{k \in \mathbb{Z}^n} \left(\sum_{l \in \mathbb{Z}^n} |s_{kl}|^p \right)^{q/p} \right)^{1/q} < \infty$$

with the usual modification if $p = \infty$ and or $q = \infty$.

We recall the definition of the Bessel potential space H_p^s , see [6] for more details about this function space.

Definition 1.25. Let $1 < p < \infty, 0 < q < \infty$ and $s \geq 0$. Then the Bessel potential space H_p^s is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{H_p^s} = \left\| \mathcal{F}^{-1}((1 + 4\pi^2|\xi|^2)^{\frac{s}{2}} \mathcal{F}f) \right\|_p < \infty.$$

Remark 1.26. (i) We will write $H^s = H_2^s$.

(ii) Note that by virtue of Plancherel's theorem,

$$f \in H^s \Leftrightarrow \int_{\mathbb{R}^n} (1 + 4\pi^2|\xi|^2)^s |\mathcal{F}f(\xi)|^2 d\xi < \infty.$$

The following estimate will play an import tools in this thesis, which called Bernstein's multiplier theorem.

Theorem 1.27. Let $\mathbb{K} \subset \mathbb{R}^n$ be a compact set and $0 < p \leq \infty$. Let $s > s(p, n) := n(\frac{1}{\min(p, 1)} - \frac{1}{2})$. If $f \in PW^p(\mathbb{K})$ and $\psi \in H^s$, then there exists a constant $C > 0$ such that

$$\left\| \mathcal{F}^{-1} \psi * f \right\|_p \leq C \left\| \psi \right\|_{H^s} \left\| f \right\|_p \quad (1.2)$$

Moreover, if $p \geq 1$, the estimate (1.2) holds under the weaker condition $f \in L^p$.

Definition 1.28. $0 < p, q \leq \infty$. The space $L^p(\ell^q)$ is the collection of all sequences of measurable functions $f = (f_k)_{k \in \mathbb{Z}^n}$ in $\mathbb{R}^n$ such that

$$\|f\|_{L^p(\ell^q)} = \left(\sum_{k \in \mathbb{Z}^n} \|f_k\|_p^q \right)^{1/q} < \infty,$$

with the usual modification if $q = \infty$.

Let $1 \leq p < \infty$. We define the dual exponent p' in the usual way, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. The following result will play an import tools in this thesis.

Proposition 1.29. Let $1 \leq p, q < \infty$. Then $(L^p(\ell^q))' = L^{p'}(\ell^{q'})$. More precisely, for all $\mathbf{f} = (f_k)_{k \in \mathbb{Z}^n}$, any $F \in (L^p(\ell^q))'$ can be expressed as

$$F(\mathbf{f}) = \sum_{k \in \mathbb{Z}^n} \int_{\mathbb{R}^n} g_k(x) f_k(x) dx,$$

where $\mathbf{g} = (g_k)_{k \in \mathbb{Z}^n} \in L^{p'}(\ell^{q'})$ and

$$\|\mathbf{F}\|_{(L^p(\ell^q))'} = \|\mathbf{g}\|_{L^{p'}(\ell^{q'})}.$$

Proof. For the proof, see [9]. □

Modulation spaces

In, this chapter, we present the definition of modulation spaces and prove their basic properties. Also, we prove some elementary embeddings.

2.1 Definition and basic properties

To present the definition of Modulation spaces, We first need the concept of a smooth dyadic resolution of unity.

Definition 2.1. Fix $\gamma > 0$ to be sufficiently small and let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. We say that $\varphi \in \mathcal{S}_\gamma(\mathbb{R}^n)$ if $\text{supp } \mathcal{F}\varphi \subset \{x : |x| \leq 1\}$ and

$$\sum_{k \in \mathbb{Z}^n} \mathcal{F}\varphi(\xi - \gamma k) = 1, \quad \xi \in \mathbb{R}^n.$$

We are now in a position to state the definition of Herz-type modulation spaces.

Definition 2.2. Let $0 < p, q \leq \infty$, and $\varphi \in \mathcal{S}_\gamma(\mathbb{R}^n)$, with $\gamma > 0$ sufficiently small, be fixed. The modulation space $M^{p,q}$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{M^{p,q}} = \left(\int_{\mathbb{R}^n} \|f * M_\xi \varphi\|_p^q d\xi \right)^{1/q} < \infty,$$

with the obvious modification if $\beta = \infty$.

Let $0 < p, q \leq \infty$, and $\varphi \in \mathcal{S}_\gamma(\mathbb{R}^n)$, with $\gamma > 0$ sufficiently small, be fixed. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. We set

$$\|f\|_{M^{p,q}}^* = \left(\sum_{k \in \mathbb{Z}^n} \|f * M_{\gamma k} \varphi\|_p^q \right)^{1/q} < \infty.$$

Theorem 2.3. *Let $0 < p, q \leq \infty$ and $\varphi \in \mathcal{S}_\gamma(\mathbb{R}^n)$, with $\gamma > 0$ sufficiently small, be fixed. Then $\|\cdot\|_{M^{p,q}}^*$ is an equivalent quasi-norm on $M^{p,q}$.*

Proof. Let $f \in M^{p,q}$. *Step 1.* In this step we prove that

$$\|f\|_{M^{p,q}} \lesssim \|f\|_{M^{p,q}}^*. \quad (2.1)$$

Let $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ and $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ be such that $\gamma k_i \leq \xi_i < \gamma(k_i + 1)$, $i \in \{1, \dots, n\}$. As in Lemma ??, we can find an $N = N(\text{supp } \mathcal{F}\varphi, \gamma, n) \in \mathbb{N}$ such that

$$f * M_\xi \varphi = \sum_{|l| \leq N} f * M_{\gamma(k+l)} \varphi * M_\xi \varphi.$$

Assume that $p \geq 1$. By Young's inequality, see Lemma 1.6, we obtain

$$\begin{aligned} \|f * M_{\gamma(k+l)} \varphi * M_\xi \varphi\|_p &\leq \|f * M_{\gamma(k+l)} \varphi\|_p \|M_\xi \varphi\|_1 \\ &\lesssim \|f * M_{\gamma(k+l)} \varphi\|_p. \end{aligned}$$

Now, assume that $0 < p < 1$. Observe that

$$\|f * M_{\gamma(k+l)} \varphi * M_\xi \varphi\|_p \leq \| |f * M_{\gamma(k+l)} \varphi| * |M_\xi \varphi| \|_p.$$

Then by Proposition 1.22, we get

$$\begin{aligned} \|f * M_{\gamma(k+l)} \varphi * M_\xi \varphi\|_p &\leq \|f * M_{\gamma(k+l)} \varphi\|_p \|M_\xi \varphi\|_p \\ &\lesssim \|f * M_{\gamma(k+l)} \varphi\|_p. \end{aligned}$$

Hence

$$\begin{aligned}
\|f * M_\xi \varphi\|_p^p &\leq \int_{\mathbb{R}^n} \left(\sum_{|l| \leq N} |f * M_{\gamma(k+l)} \varphi * M_\xi \varphi(x)| \right)^p dx \\
&\lesssim \sum_{|l| \leq N} \int_{\mathbb{R}^n} |f * M_{\gamma(k+l)} \varphi * M_\xi \varphi(x)|^p dx \\
&= c \sum_{|l| \leq N} \|f * M_{\gamma(k+l)} \varphi * M_\xi \varphi\|_p^p \\
&\lesssim \sum_{|l| \leq N} \|f * M_{\gamma(k+l)} \varphi\|_p^p.
\end{aligned}$$

Let $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$. We set

$$A_k = \{\xi \in \mathbb{R}^n : \gamma k_i \leq \xi_i < \gamma(k_i + 1)\}.$$

Now, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \|f * M_\xi \varphi\|_p^q d\xi &\lesssim \sum_{k \in \mathbb{Z}^n} \int_{A_k} \left(\sum_{|l| \leq N} \|f * M_{\gamma(k+l)} \varphi\|_p^p \right)^{q/p} d\xi \\
&\lesssim \sum_{k \in \mathbb{Z}^n} \int_{A_k} \sum_{|l| \leq N} \|f * M_{\gamma(k+l)} \varphi\|_p^q d\xi \\
&\lesssim \sum_{k \in \mathbb{Z}^n} \|f * M_{\gamma k} \varphi\|_p^q.
\end{aligned}$$

Therefore, we obtain (2.1). *Step 2.* In this step we prove that

$$\|f\|_{M^{p,q}}^* \lesssim \|f\|_{M^{p,q}}. \quad (2.2)$$

Let $k \in \mathbb{Z}^n$ and let $\xi \in [\alpha k, \alpha(k+1)]$ be arbitrary. Then we have

$$f * M_{\gamma k} \varphi = \sum_{|l| \leq N} f * M_{\xi + \gamma l} \varphi * M_{\gamma k} \varphi.$$

As before, we can prove that we have the following estimate:

$$\|f * M_{\xi + \gamma l} \varphi * M_{\gamma k} \varphi\|_p \lesssim \|f * M_{\xi + \gamma l} \varphi\|_p$$

and

$$\|f * M_{\gamma k} \varphi\|_p^p \lesssim \sum_{|l| \leq N} \|f * M_{\xi + \gamma l} \varphi\|_p^p.$$

This leads to

$$\begin{aligned} \|f * M_{\gamma k} \varphi\|_p^q &\lesssim \left(\sum_{|l| \leq N} \|f * M_{\xi + \gamma l} \varphi\|_p \right)^{q/p} \\ &\lesssim \int_{A_k} \left(\sum_{|l| \leq N} \|f * M_{\xi + \gamma l} \varphi\|_p \right)^{q/p} d\xi \\ &\lesssim \sum_{|l| \leq N} \int_{A_k} \|f * M_{\xi + \gamma l} \varphi\|_p^q d\xi. \end{aligned}$$

Therefore, we obtain (2.2). □

From Theorem 2.3 and Proposition 1.23, we obtain the following statement.

Theorem 2.4. *Let $0 < p, q \leq \infty, \gamma, \beta > 0$ be sufficiently small and $\varphi \in \mathcal{S}_\gamma(\mathbb{R}^n)$ be fixed. Then*

$$\|f\|_{M^{p,q}}^\bullet = \left(\sum_{k \in \mathbb{Z}^n} \left(\sum_{l \in \mathbb{Z}^n} |f * M_{\gamma k} \varphi(\beta l)|^p \right)^{q/p} \right)^{1/q},$$

is an equivalent quasi-norm on $M^{p,q}$, with the usual modification is $p = \infty$ and/or $q = \infty$.

Let $0 < p, q \leq \infty$ and $\varphi \in \mathcal{S}_\gamma(\mathbb{R}^n)$, with $\gamma > 0$ sufficiently small, be fixed. We set

$$\|f\|_{M^{p,q}}^{*,\varphi} = \left(\sum_{k \in \mathbb{Z}^n} \|f * M_{\gamma k} \varphi\|_p^q \right)^{1/q} < \infty.$$

Proposition 2.5. *Let $0 < p, q \leq \infty$ and $\varphi \in \mathcal{S}_\gamma(\mathbb{R}^n)$, with $\gamma > 0$ sufficiently small, be fixed. The definition of $M^{p,q}$ is independent of the window φ .*

Proof. Let $\varphi_1, \varphi_2 \in \mathcal{S}_\gamma(\mathbb{R}^n)$, with $\gamma > 0$ sufficiently small, be fixed. It is sufficient to show that there exists a constant $c > 0$ such that for all $f \in M^{p,q}$ we have

$$\|f\|_{M^{p,q}}^{*,\varphi_1} \leq c \|f\|_{M^{p,q}}^{*,\varphi_2}. \quad (2.3)$$

Interchanging the roles of φ_1 and φ_2 we obtain the desired result. As in Theorem 2.3, we

can find an $N = N(\gamma, n) \in \mathbb{N}$ such that

$$f * M_{\gamma k} \varphi_1 = \sum_{|l| \leq N} f * M_{\gamma k} \varphi_1 * M_{\gamma(k+l)} \varphi_2, \quad k = (k_1, \dots, k_n) \in \mathbb{Z}^n.$$

The argument that used to prove (2.1) yields

$$\|f * M_{\gamma k} \varphi_1\|_p^q \lesssim \sum_{|l| \leq N} \|M_{\gamma(k+l)} \varphi_2\|_p^q. \quad (2.4)$$

Summing over $k \in \mathbb{Z}^n$ in (2.4) and then raising to the $1/q$ power give (2.3). $\square$

2.2 Embeddings

The following lemma and proposition give some embedding results for modulation spaces.

Lemma 2.6. (i) *Let $0 < p \leq \infty$ and $0 < q_1 \leq q_2 \leq \infty$. Then*

$$M^{p, q_1} \hookrightarrow M^{p, q_2},$$

holds..

(ii) *Let $0 < q \leq \infty$. Suppose $0 < p_2 \leq p_1 \leq \infty$. Then*

$$M^{p_1, q} \hookrightarrow M^{p_2, q}. \quad (2.5)$$

Proof. The proof of (i) follows by embedding $\ell^{q_1}(\mathbb{Z}^n) \hookrightarrow \ell^{q_2}(\mathbb{Z}^n)$. Let $\varphi \in \mathcal{S}_\gamma(\mathbb{R}^n)$, with $\gamma > 0$ sufficiently small, be fixed. Let $f \in M^{p_1, q}$. We see that $\text{supp } \mathcal{F}(f * M_{\gamma k} \varphi) \subset \{\xi \in \mathbb{R}^n : |\xi - \gamma k| \leq 1\}$. By Lemma 1.21

$$\|f * M_{\gamma k} \varphi\|_{p_1} \leq c \|f * M_{\gamma k} \varphi\|_{p_2}, \quad (2.6)$$

for any $k \in \mathbb{Z}$, where the implicit constant is independent of k . The estimate (2.6) yields the desired embedding (2.5). $\square$

Theorem 2.7. *Let $0 < p, q \leq \infty$.*

(i) *We have the embedding*

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow M^{p, q}. \quad (2.7)$$

In addition if $0 < p, q < \infty$, then $\mathcal{S}(\mathbb{R}^n)$ is dense in $M^{p,q}$.

(ii) We have the embedding

$$M^{p,q} \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Proof. Step 1. Proof of (i). Let $f \in \mathcal{S}(\mathbb{R}^n)$, $\varphi \in \mathcal{S}_\gamma(\mathbb{R}^n)$, with $\gamma > 0$ sufficiently small, be fixed and $M, N \in \mathbb{N}$ be such that $M > n/p$ and $N > n/q$. We have

$$\begin{aligned} f * M_\xi \varphi(x) &= \int_{\mathbb{R}^n} f(x-y) \varphi(y) e^{2\pi i \xi y} dy \\ &= (1 + 4\pi^2 |\xi|^2)^{-N/2} \int_{\mathbb{R}^n} f(x-y) \varphi(y) (1 + 4\pi^2 |\xi|^2)^{N/2} e^{2\pi i \xi y} dy. \end{aligned}$$

Denote $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial_j^2}$. Then

$$\begin{aligned} f * M_\xi \varphi(x) &= \int_{\mathbb{R}^n} f(x-y) \varphi(y) e^{2\pi i \xi y} dy \\ &= (1 + 4\pi^2 |\xi|^2)^{-N/2} \int_{\mathbb{R}^n} f(x-y) \varphi(y) (I - \Delta)^{N/2} (e^{2\pi i \xi y}) dy. \end{aligned}$$

By using integration by parts repeatedly and the product rule, we obtain the following sequence of equalities;

$$\begin{aligned} f * M_\xi \varphi(x) &= (1 + 4\pi^2 |\xi|^2)^{-N/2} \int_{\mathbb{R}^n} (I - \Delta)^{N/2} (f(x-y) \varphi(y)) e^{2\pi i \xi y} dy \\ &= (1 + 4\pi^2 |\xi|^2)^{-N/2} \sum_{|\alpha| \leq N} C_{N,\alpha} \int_{\mathbb{R}^n} \partial^\alpha f(x-y) \partial^{N-\alpha} \varphi(y) e^{2\pi i \xi y} dy. \end{aligned}$$

Observe that

$$(1 + |x|) \leq (1 + |x-y|)(1 + |y|), \quad x, y \in \mathbb{R}^n.$$

This together with the fact that $f, \varphi \in \mathcal{S}(\mathbb{R}^n)$, we obtain

$$\begin{aligned} |f * M_\xi \varphi(x)| &\lesssim (1 + |x|)^{-M} (1 + |\xi|)^{-N} \sum_{|\alpha| \leq N} C_{N,\alpha} \int_{\mathbb{R}^n} (1 + |y|)^M |\partial^{N-\alpha} \varphi(y)| dy \\ &\lesssim (1 + |x|)^{-M} (1 + |\xi|)^{-N} \sum_{|\alpha| \leq N} C_{N,\alpha} \|f\|_{N,\alpha}. \end{aligned}$$

Hence

$$\|f * M_{\xi} \varphi\|_p \lesssim (1 + |\xi|)^{-N} \sum_{|\alpha| \leq N} C_{N,\alpha} \|f\|_{N,\alpha}$$

and

$$\begin{aligned} \|f\|_{M^{p,q}} &\lesssim \sum_{|\alpha| \leq N} C_{N,\alpha} \|f\|_{M,\alpha} \left(\int_{\mathbb{R}^n} (1 + |\xi|)^{-Nq} d\xi \right)^{1/q} \\ &\lesssim \sum_{|\alpha| \leq N} C_{N,\alpha} \|f\|_{M,\alpha}. \end{aligned}$$

For the proof of the density of $\mathcal{S}(\mathbb{R}^n)$ is dense in $M^{p,q}$, see [8]. *Step 2. Proof of (ii).* From Lemma 2.6 it suffices to show that $M^{\infty,\infty} \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. Let $f \in M^{\infty,\infty}$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$, then $\langle f, \psi \rangle$ denotes the value of the functional f of $\mathcal{S}'(\mathbb{R}^n)$ for the test function ψ . Recall that, we can find an $N = N(\text{supp } \mathcal{F} \varphi, \gamma, n) \in \mathbb{N}$ such that

$$f * M_{\gamma k} \varphi = \sum_{|l| \leq N} f * M_{\gamma(k+l)} \varphi * M_{\gamma k} \varphi.$$

We obtain

$$\begin{aligned} |\langle f, \psi \rangle| &\leq \sum_{k \in \mathbb{Z}^n} \left| \sum_{|l| \leq N} \langle f * M_{\gamma(k+l)} \varphi * M_{\gamma k} \varphi, \psi \rangle \right| \\ &= \sum_{k \in \mathbb{Z}^n} \left| \langle f * M_{\gamma k} \varphi, \sum_{|l| \leq N} M_{\gamma(k+l)} \varphi * \psi \rangle \right| \\ &\leq \sum_{k \in \mathbb{Z}^n} \left\| (f * M_{\gamma k} \varphi) \left(\sum_{|l| \leq N} M_{\gamma(k+l)} \varphi * \psi \right) \right\|_1 \\ &= \sum_{k \in \mathbb{Z}^n} \|f * M_{\gamma k} \varphi\|_{\infty} \left\| \left(\sum_{|l| \leq N} M_{\gamma(k+l)} \varphi * \psi \right) \right\|_1. \end{aligned}$$

The last term clearly bounded by

$$c \|f\|_{M^{\infty,\infty}} \|\psi\|_{M^{1,1}}.$$

Observe that

$$\|\psi\|_{M^{1,1}} \lesssim \sum_{|\alpha| \leq N} C_{N,\alpha} \|\psi\|_{N,\alpha}.$$

Then $f \in \mathcal{S}'(\mathbb{R}^n)$. The proof is complete. $\square$

Theorem 2.8. *Let $0 < p, q \leq \infty$. Then $M^{p,q}$ is a quasi-Banach space.*

Proof. Let $\beta, \gamma > 0$ be sufficiently small, and $\varphi \in \mathcal{S}'(\mathbb{R}^n)$. We have

$$\|f\|_{M^{p,q}} \approx \|f\|_{M^{p,q}}^\bullet.$$

We set

$$c_{kl}(f) = f * M_{\gamma k} \varphi(\beta l) \quad \text{and} \quad c(f) = (c_{kl}(f))_{k,l \in \mathbb{Z}^n}.$$

We have $c : M^{p,q} \rightarrow \mathbb{C}$ is a linear mapping. Observe that

$$\|f\|_{M^{p,q}}^\bullet = \|c(f)\|_{\ell^{p,q}}. \quad (2.8)$$

Let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $M^{p,q}$. From (2.8) we get that $(c(f_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in $\ell^{p,q}$. Since $\ell^{p,q}$ is a quasi-Banach space, there exists a sequence $c = (c_{kl})_{k,l \in \mathbb{Z}^n} \in \ell^{p,q}$ such that

$$\lim_{n \rightarrow \infty} \|c - c(f_n)\|_{\ell^{p,q}} = 0.$$

Define $f \in \mathcal{S}'(\mathbb{R}^n)$ so that $\langle t, T_{\beta l} M_{\gamma k} \varphi \rangle = c_{kl}$. This implies that $f \in M^{p,q}$ and

$$\lim_{n \rightarrow \infty} \|f - f_n\|_{M^{p,q}} = 0.$$

This is the desired conclusion. □

Definition 2.9. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a fixed, nonzero window function and $u \in \mathcal{S}'(\mathbb{R}^n)$. The short-time Fourier transform (STFT) of u with respect to φ is defined as*

$$V_\varphi u(x, \xi) = \langle u, M_\xi T_x \varphi \rangle, \quad x, \xi \in \mathbb{R}^n.$$

Remark 2.10. *We observe that*

$$V_\varphi u(x, \xi) = e^{-2\pi i x \cdot \xi} (u * M_\xi \overline{\varphi}), \quad f \in L^2(\mathbb{R}^n).$$

Duality

3.1 The Wiener Decomposition

Let $k \in \mathbb{Z}^n$. We set $Q_k = k + Q_0$, where the $Q_0 = [-\frac{1}{2}, \frac{1}{2}]^n$. The set $\{Q_k : k \in \mathbb{Z}^n\}$ is called the Wiener decomposition of $\mathbb{R}^n$.

Definition 3.1. We say that $\psi \in \mathcal{S}_*(\mathbb{R}^n)$ if $\psi \in \mathcal{S}(\mathbb{R}^n)$, $0 \leq \psi \leq 1$, $\text{supp } \psi \subset [-1, 1]^n$, $\psi(x) = 1$, $x \in Q_0$ and

$$\sum_{k \in \mathbb{Z}^n} \psi(\xi - k) = 1, \quad \xi \in \mathbb{R}^n.$$

Let us now fix $\psi \in \mathcal{S}_*(\mathbb{R}^n)$. We define next the smooth operators that are Q_k .

Definition 3.2. Denote by $\psi_k = T_k \psi = \psi(\cdot - k)$ the translation of ψ at k . We set

$$P_k^\psi = \mathcal{F}^{-1} \psi_k \mathcal{F}.$$

We say that $\{P_k^\psi : k \in \mathbb{Z}^n\}$ is the set of smooth frequency-uniform decomposition operators associated with $\psi \in \mathcal{S}_*(\mathbb{R}^n)$.

Remark 3.3. Notice that

$$P_k^\psi(f)(x) = \mathcal{F}^{-1} \psi_k * f(x), \quad f \in \mathcal{S}'(\mathbb{R}^n).$$

Definition 3.4. Let $0 < p, q \leq \infty$ and $\psi \in \mathcal{S}_*(\mathbb{R}^n)$ be fixed. The modulation space $M_{p,q}$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{M_{p,q}} = \left(\sum_{k \in \mathbb{Z}^n} \|P_k^\psi(f)\|_p^q \right)^{1/q} < \infty,$$

with the usual modification if $p = \infty$ or $q = \infty$.

Proposition 3.5. *Let $0 < p, q \leq \infty$ and $\psi \in \mathcal{S}_*(\mathbb{R}^n)$ be fixed. The definition of $M_{p,q}$ is independent of the window ψ .*

Proof. Let $\psi, \phi \in \mathcal{S}_*(\mathbb{R}^n)$ be fixed. It is sufficient to show that there exists a constant $c > 0$ such that for all $f \in M^{p,q}$ we have

$$\|f\|_{M_{p,q}}^{*,\psi} \leq c \|f\|_{M_{p,q}}^{*,\phi}. \quad (3.1)$$

Interchanging the roles of ψ and ϕ we obtain the desired result. Let us write Λ for the set of all n -tuples $k \in \mathbb{Z}^n$ which satisfy

$$B(k, \sqrt{n}) \cap B(0, \sqrt{n}) \neq \emptyset.$$

Λ has at most $O(\sqrt{n})$ elements. In particular, this also implies that

$$\|P_k^\psi(f)\|_p \lesssim \sum_{l \in \Lambda} \|P_{k+l}^\phi P_k^\psi(f)\|_p.$$

Observe that

$$P_0^\psi(f) = M_k(P_{-k}^\psi(M_{-k}f)), \quad k \in \mathbb{Z}^n.$$

Then

$$\|P_{k+l}^\phi P_k^\psi(f)\|_p = \|P_{-k}^{\phi_{k+l}\psi_k}(M_{-k}f)\|_p. \quad (3.2)$$

Let $\mathbb{K} = \cup_{k \in \Lambda} \overline{B(k, \sqrt{n})}$. We have $\mathbb{K}$ is a compact set. Then by Theorem 1.27 and (3.2), we get

$$\|P_{k+l}^\phi P_k^\psi(f)\|_p \lesssim \|\psi_k\|_{H^s} \|P_{k+l}^\phi\|_p, \quad s > n\left(\frac{1}{\min(1, p)} - \frac{1}{2}\right).$$

We know that $\|\psi_k\|_{H^s} \lesssim 1, k \in \mathbb{Z}^n$. Hence

$$\|P_k^\psi(f)\|_p \lesssim \sum_{l \in \Lambda} \|P_{k+l}^\phi f\|_p.$$

This leads to (3.1). The proof is complete. $\square$

Proposition 3.6. *Let $1 \leq p, q < \infty$ and $f \in M_{p,q}$. Then, for a fixed $\psi \in \mathcal{S}_*(\mathbb{R}^n)$ there exists*

$\mathbf{f}=(f_k) \in L^p(\ell^q)$ such that

$$f = \sum_{k \in \mathbb{Z}^n} P_k^\psi(f_k). \quad (3.3)$$

Moreover,

$$\|f\|_{M_{p,q}}^\Delta = \inf\{\|\mathbf{f}\|_{L^p(\ell^q)} : f = \sum_{k \in \mathbb{Z}^n} P_k^\psi(f_k), \mathbf{f}=(f_k) \in L^p(\ell^q)\} \quad (3.4)$$

is an equivalent norm on $M_{p,q}$. Conversely, if an $f \in \mathcal{S}'(\mathbb{R}^n)$ can be represented as in (3.3) such that the norm in (3.4) is finite, then $f \in M_{p,q}$.

Proof. Let $f \in M_{p,q}$. We set

$$f_k = \sum_{l \in \Lambda} P_{k+l}^\psi f, \quad \mathbf{f}=(f_k),$$

where Λ is the set obtained in Proposition 3.5. We have

$$f = \sum_{k \in \mathbb{Z}^n} P_k^\psi(f_k)$$

and

$$\|\mathbf{f}\|_{L^p(\ell^q)} \lesssim \|f\|_{M_{p,q}}.$$

Using now Bernstein's multiplier theorem, Proposition 1.27, we see that

$$\|P_k^\psi(f)\|_p \lesssim \sum_{l \in \Lambda} \|P_k^\psi P_{k+l}^\psi(f_{k+l})\|_p \lesssim \sum_{l \in \Lambda} \|f_{k+l}\|_p.$$

This leads to

$$\|f\|_{M_{p,q}} \lesssim \|\mathbf{f}\|_{L^p(\ell^q)}.$$

The proof is complete. □

Proposition 3.7. *Let $0 < p, q \leq \infty$. Then*

$$M_{p,q} = M^{p,q},$$

in the sense of equivalent quasi-norms.

Proof. Let $\psi \in \mathcal{S}_*(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$. Let $\mathcal{F}\phi$ be a smooth bump function compactly

supported in $B(0, R)$ with an $R \approx 10\sqrt{n}$ and $\mathcal{F}\phi(x) = 1, x \in B(0, 3\sqrt{n})$. We have

$$\begin{aligned} V_\phi f(x, \xi) &= e^{-2\pi i x \cdot \xi} V_{\mathcal{F}\phi} \mathcal{F}f(\xi, -x) \\ &= e^{-2\pi i x \cdot \xi} \mathcal{F}^{-1}(\mathcal{F}\phi(\cdot - \xi) \mathcal{F}f)(x). \end{aligned}$$

Using the mean value theorem, we will be able to find $\xi_k \in \mathbb{Q}_k$ such that

$$\|f\|_{\mathcal{M}^{p,q}}^\phi = \left(\int_{\mathbb{R}^n} \|V_\phi f\|_p^q d\xi \right)^{1/q} \approx \sum_{k \in \mathbb{Z}^n} \|P_{\xi_k}^{\mathcal{F}\phi}(f)\|_p^q)^{1/q}.$$

By the choice of ϕ and Proposition 1.27 we have the following:

$$\|P_k(f)\|_p = \|\mathcal{F}^{-1}\psi_k * f\|_p = \|\mathcal{F}^{-1}\psi_k * \mathcal{F}^{-1}\phi_{\xi_k} * f\|_p \lesssim \|P_{\xi_k}^{\mathcal{F}\phi}(f)\|_p.$$

This leads to $\|f\|_{M_{p,q}}^\psi \lesssim \|f\|_{\mathcal{M}^{p,q}}^\phi$. Let us observe that the support of $\mathcal{F}\phi_{\xi_k} = T_{\xi_k} \mathcal{F}\phi$ intersects at most $O(\sqrt{n})$ many supports of $\psi_k = T_k \psi$. Using again Proposition 1.27, and recalling the notation for the set Λ of size $O(\sqrt{n})$ utilized in the proof of Proposition 3.5, we obtain the following:

$$\|P_{\xi_k}^{\mathcal{F}\phi}(f)\|_p = \|\mathcal{F}^{-1}(\sum_{l \in \Lambda} \psi_{k+l} \phi_{\xi_k} \mathcal{F}f)\|_p \lesssim \sum_{l \in \Lambda} \|P_{k+l}(f)\|_p.$$

Hence

$$\|f\|_{\mathcal{M}^{p,q}}^\phi \lesssim \|f\|_{M_{p,q}}^\psi.$$

The proof is complete. □

3.2 Duality of Modulation spaces

Let X be a normed space over $\mathbb{R}$ or $\mathbb{C}$. The dual of X is the space of continuous linear maps from X into $\mathbb{R}$ or $\mathbb{C}$. The notation for the dual of X is X' .

The aim of this section is to determinate the duality of the modulation spaces. Recall that if $p \geq 1$, we define the dual exponent p' in the usual way, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 3.8. *Let $1 \leq p, q < \infty$. Then $(M^{p,q})' = M^{p',q'}$.*

Proof. We see that to show that $(M_{p,q})' = M_{p',q'}$. We show first that

$$M_{p',q'} \hookrightarrow (M_{p,q})'. \quad (3.5)$$

Let $g \in M_{p',q'}$, and $f \in \mathcal{S}(\mathbb{R}^n)$. Note that we have the decomposition

$$\langle g, f \rangle = \sum_{k \in \mathbb{Z}^n} \sum_{l \in \Lambda} \langle P_{k+l}(g), P_k(f) \rangle.$$

Using Hölder's inequality, we get

$$\begin{aligned} \sum_{k \in \mathbb{Z}^n} \sum_{l \in \Lambda} |\langle P_{k+l}(g), P_k(f) \rangle| &\leq \sum_{l \in \Lambda} \sum_{k \in \mathbb{Z}^n} \|P_{k+l}(g)\|_{p'} \|P_k(f)\|_p \\ &\leq \sum_{l \in \Lambda} \left(\sum_{k \in \mathbb{Z}^n} \|P_{k+l}(g)\|_{p'}^{q'} \right)^{1/q'} \sum_{k \in \mathbb{Z}^n} \|P_k(f)\|_p^q \Big)^{1/q} \\ &\lesssim \|g\|_{M_{p',q'}} \|f\|_{M_{p,q}}. \end{aligned}$$

The density of $\mathcal{S}(\mathbb{R}^n)$ in $M_{p,q}$ yields (3.5).

Now we prove that $(M_{p,q})' \hookrightarrow M_{p',q'}$. Let $f \in M_{p,q}$. The mapping $f \mapsto (P_k(f))_{k \in \mathbb{Z}^n}$ isometric from $M_{p,q}$ into the subspace $\mathcal{X}$ of $L^p(\ell^q)$. Therefore, if $F \in (M_{p,q})'$, we can think of F as belonging to $\mathcal{X}'$ and then extending F to a continuous functional in $(L^p(\ell^q))'$. By some abuse of notation, we call this extended functional also F . By Proposition 1.29, for all $\mathbf{f} = (f_k)_{k \in \mathbb{Z}^n}$ we can represent F as

$$F(\mathbf{f}) = \sum_{k \in \mathbb{Z}^n} \int_{\mathbb{R}^n} g_k(x) f_k(x) dx,$$

where $\mathbf{g} = (g_k)_{k \in \mathbb{Z}^n} \in L^{p'}(\ell^{q'})$ and

$$\|\mathbf{F}\|_{(L^p(\ell^q))'} = \|\mathbf{g}\|_{L^{p'}(\ell^{q'})}.$$

Let now $f \in \mathcal{S}(\mathbb{R}^n)$. The inclusion $\mathcal{S}(\mathbb{R}^n) \hookrightarrow M_{p,q}$ implies that $\mathbf{f} = (P_k(f))_{k \in \mathbb{Z}^n} \in L^p(\ell^q)$.

Hence

$$\begin{aligned} F(\mathbf{f}) &= \sum_{k \in \mathbb{Z}^n} \int_{\mathbb{R}^n} g_k(x) P_k(f)(x) dx \\ &= \sum_{k \in \mathbb{Z}^n} \int_{\mathbb{R}^n} P_k g_k(x) f(x) dx. \end{aligned}$$

Consequently, we have the identity

$$\mathbf{F} = \sum_{k \in \mathbb{Z}^n} P_k g_k$$

in a distributional sense. Using now Proposition 3.6, we see that

$$\|\mathbf{F}\|_{M_{p',q'}} \lesssim \|\mathbf{g}\|_{L^{p'}(\ell^{q'})} = \|\mathbf{F}\|_{(L^p(\ell^q))'}. \quad \square$$

This yields the desired result. □

conclusion

In this note, we discussed some properties of modification spaces, which play a role in mathematical analysis, especially differential equations. To study these spaces, we provided a summary of Lebesgue spaces and some important theorems that helped us complete the thesis, such as Nikolsky-Tribal inequalities. In this note, we demonstrated some equivalent systems, containments, and duality in these spaces. We also discussed some properties of modification spaces, which play a role in mathematical analysis, especially differential equations. To study these spaces, we provided a summary of Lebesgue spaces and some important theorems that helped us complete the thesis, such as Nikolsky-Tribal inequalities. In this note, we have demonstrated some equivalent systems, containments, and dualities in these spaces. The future of this research lies in expanding it to practical applications, particularly in the analysis of regular solutions to differential equations within modulation spaces, which opens the way for the use of these spaces in practical problems of an applied nature.

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ملخص

في هذه المذكرة تطرقنا الى بعض خصائص فضاءات التعديل والتي لها اهمية بالغة في التحليل الرياضى خاصة المعادلات التفاضلية. لدراسة هذه الفضاءات قدمنا ملخصا لفضاءات لوبيغ وبعض التوطئات و النظريات التي تساعدنا في انجاز المذكرة كمترجمات نيكولسى تريبال. اظهرنا في هذه المذكرة بعض النظميات المكافئة والاحتواءات والازدواجية في هذه الفضاءات.

كلمات مفتاحية :

فضاءات التعديل ، تحويل فورييه قصير المدى ، فضاءات لوبيغ ، دالة النافذة ، ثنائية الفضاء ، الفضاءات الدالية.

Résumé

Dans cette thèse, nous avons présenté certaines propriétés des espaces de modulation, qui sont très importants en analyse mathématique, notamment pour les équations différentielles. Pour étudier ces espaces, nous avons présenté un résumé sur les espaces de Lebesgue ainsi que quelques introductions et théorèmes qui nous ont aidés pour compléter la thèse, comme les inégalités de Nikol'skij-Triebel. Dans cette thèse, nous avons démontré quelques normes équivalentes et dualités dans ces espaces .

Mots-clés :

Espaces de modulation, transformée de Fourier à court terme, espaces de Lebesgue, fonction fenêtre, dualité, espaces fonctionnels.

Abstract

In this thesis, we present some of the properties of modulation spaces, which are used in mathematical analysis, especially differential equations. To study these spaces, we give some key technical results needed in the proof of the main results. We present a summary of Lebesgue spaces and some inequalities, such as Nikolskiy-Triebel inequalities. In this thesis, we give some equivalent norms and duality in these spaces.

Key words : Modulation spaces, short-term Fourier transform, Lebesgue spaces, window function, duality, functional spaces.