الجمهوريـــة الجزائريـــة الديمقراطيـــة الشعبيـــة

وزارة التعليـم العالـي والبحـث العلمـي

جامعة محمد بوضياف بالمسيلـــة

كليـــة العلـــــوم

رقم:430/ك ع/2025

جامعة محمد بوضياف - المسيلة
Université Mohamed Boudiaf - M'sila

المسيـــلة فى: 2025/09/30

شهادة موافقة علمية على مطبوعة بيداغوجية

للأستاذ صابرساعد السعود - أستاذ محاضر أ -

يشهد رئيس المجلس العلمي لكلية العلوم بجامعة محمد بوضياف بالمسيلة، أنه بعد الإطـــلاع على تقاريـــر الخبـــــرة الواردة من طرف الخبــــراء من صف الأستاذيـــة:

- السيد **علالي جمال** ، أستاذ التعليم العالي بجامعة محمد بوضياف- المسيلة.
- السيد **معوش جمال** ، أستاذ التعليم العالي بجامعة سطيف 1

والمعينين من طرف المجلس العلمي لكلية العلوم في الاجتماع المنعقد في دورته العادية يوم2025/06/11 لإجراء الخبرة للمطبوعة البيداغوجيـة الخاصة بالأستاذ: صابر ساعد السعود - أستاذ محاضر- أ- بقسم بالجذع المشترك علوم المادة والمتعلقة بخبرة للمطبوعة البيداغوجية للمـــادة المعنونة بـــ:

PHYSICS 2: Electricity and magnetism

والمقررة في برنامج التكوين ليسانس، تخصص: « **L1 (SM et MI)** » و المفتوح بالجذع المشترك علوم المادة

تـمت الموافقـة عليها شكـلا ومضمونـا.

رئيس المجلس العلمي لكلية العلوم

بعزيز حكيم

People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

Mohamed Boudiaf University of M'sila

جامعة محمد بوضياف - المسيلة

Université Mohamed Boudiaf - M'sila

PEDAGOGICAL HAND-OUT IN:

PHYSICS 2: Electricity and magnetism

For 1st year LMD students

Prepared by : Dr. Saber Saad Essaoud

Academic Year 2024-2025

Preface:

This educational academic manuscript includes lessons and solved exercises for students of scientific disciplines: First year university level. These lessons are designed according official curriculum issued by the Algerian Ministry of Higher Education and Scientific Research. This manuscript will contribute to achieving pedagogical objectives and enhancing students' understanding.

The manuscript includes a number of main themes, which are listed as follows:

Chapter 1*: Electrostatics*

Chapter 2*: Conductors*

Chapter 3*: Electric Circuits (Electrokinetics)*

Chapter 4*: Magnetostatics*

Detailed Contents

Chapter 1: Electrostatics

Chapter 2: Conductors

Chapter 2: Conductors

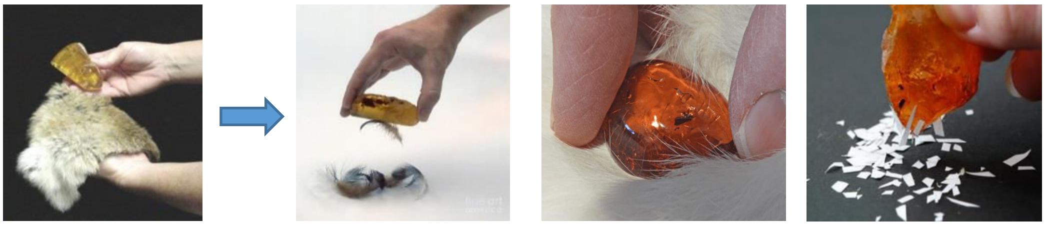
Chapter 3: Electric Circuits (Electrokinetics)

Chapter 1: Electrostatics

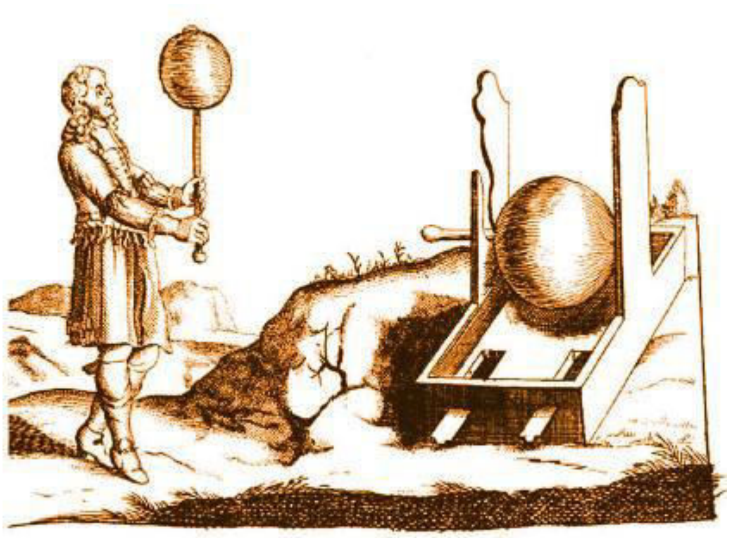
I-1) Introduction :(History of electricity and magnetism)

The history of static electricity in ancient times can be summarized in the following points:

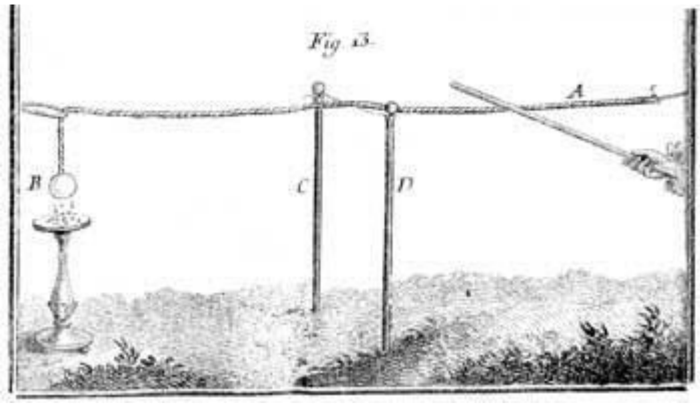
- *The history of electricity can be traced back to ancient times, particularly with the observations of the Greek philosopher **Thales**. He noted that rubbing amber with fur could cause it to attract lightweight objects, an early observation of static electricity. However, Thales did not propose a scientific explanation.*



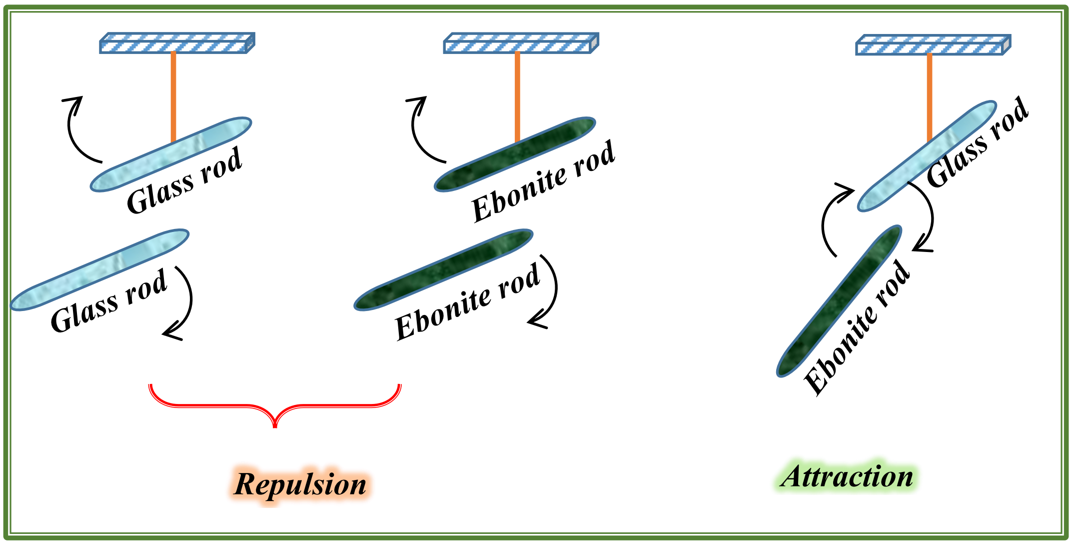
- *The first to use the Greek word "**ēlektron**" (**ἤλεκτρον**) were the ancient Greeks, and it meant "amber," which is known for its ability to attract small objects when rubbed. The term was later used to refer to electrical phenomena by **William Gilbert** (1600), who was the first to derive the Latin term "**electricus**" to describe materials that exhibit effects similar to amber when rubbed. This usage eventually led to the emergence of the word "electricity".*
- *In 1663, **Otto von Guericke** was able to manufacture the first charge generator, where he used a sulfur ball fixed on an axis, which was then rotated, and its surface was covered with a piece of cloth while rotating, which enabled it to attract light objects as well as produce weak electric sparks. After that, both **von Marum** and **Franklin** were able to manufacture improved charge generators.*



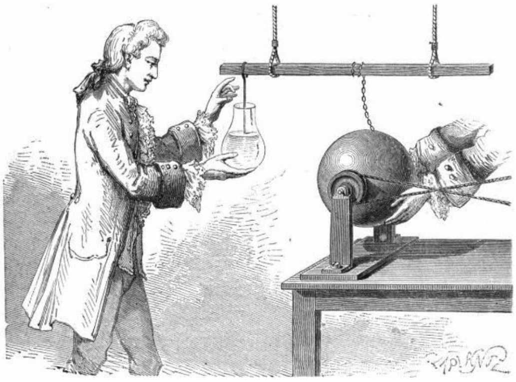
- *In 1729,* ***Stephen Gray*** *was able to obtain the concept of insulating and conducting materials through his experiments, as he rubbed a glass tube and touched one end of long threads (up to 80 meters long) made of different materials. He found that some of these materials could transfer electricity to the opposite end (as light objects were attracted to this end).*



- ***Du Fay*** *(1745) discovered two types of electricity, where rubbing two ebonite (plastic) rods with fur causes them to repel each other. Two glass rods brushed with silk resist each other. Fur-rubbed ebonite (plastic) rods attract silk-rubbed glass rods. These empirical observations led to the adoption of two types of charges: positive charges and negative charges.*

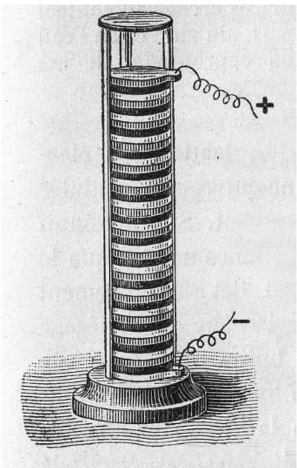


- *The Leiden Bottle Experiment was first presented in 1745 by* ***Pieter van Musschenbroek*** *and* ***Ewald Georg von Kleist****, who each inserted a metal rod into a water-filled bottle, then contacted the metal rod with a charged object, accumulating charges on the inner surface of the bottle and opposite charges on the outer surface, and when the metal rod was touched* *by hand, it was shocked as a result of discharging the charge. The Leiden bottle experiment is regarded as the first electrolytic capacitor.*

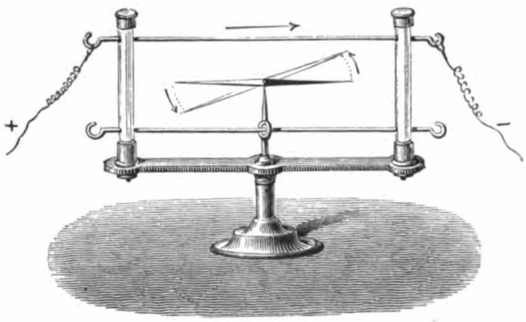


- *In 1752, the scientist* ***Benjamin Franklin*** *was able to explain the phenomenon of lightning as a form of electricity, as he explained it as an electric fluid (liquid) through the experiment of a kite made of silk with a metal wire attached to its top. He tied it with a silk thread soaked in water and attached a metal key to the thread (as shown in the picture below). Franklin flew the kite in the sky during a thunderstorm and noticed that when he brought his hand close to the key, he got an electric shock. This experiment enabled an explanation of lightning as well as the discovery of lightning arrestors installed on buildings.*

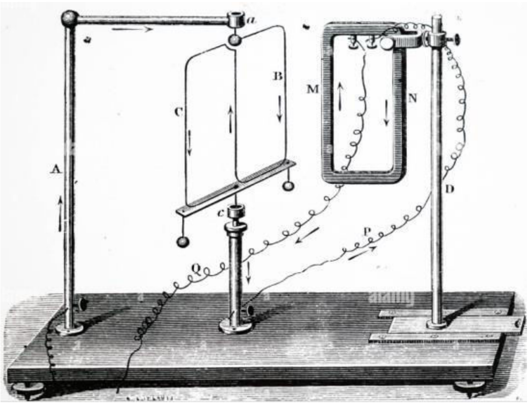
- *Electricity (direct electric current) was first produced chemically by the scientist* ***Alessandro Volta*** *in 1800, who created a column containing discs of copper and zinc arranged alternately, separated by pieces of cloth soaked in an acidic solution.*



- *In 1820, the scientist* ***Oersted*** *discovered the relationship between electricity and magnetism when he noticed the deflection of a compass placed next to a conducting wire carrying an electric current.*

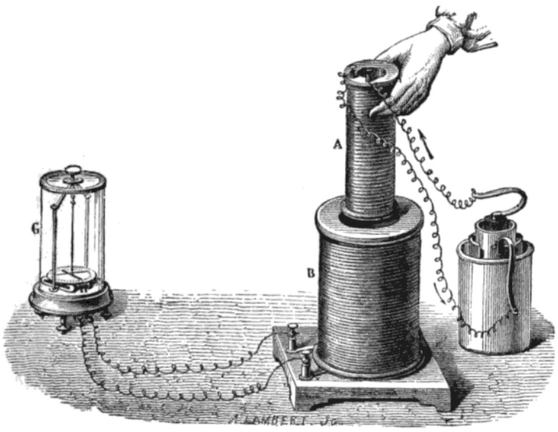


- *In 1822, the scientist* ***Ampere*** *was able to establish a law describing the relationship between magnets and electric currents after he conducted a set of experiments on the effect of electric currents on magnets as well as the effect of two electric currents on each other.*

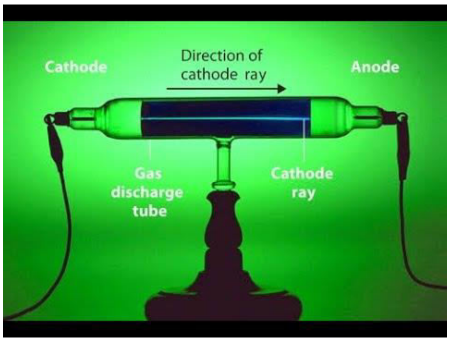


- *In 1831,* ***Michel Faraday*** *discovered electromagnetic induction, when he observed that the passage of an electric current in a coil would induce an electric current in a neighboring coil.*

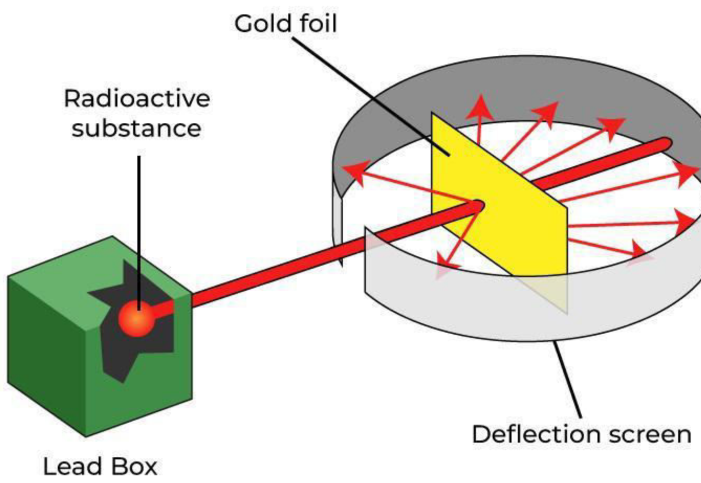
- *In 1831, the scientist* ***Michael Faraday*** *created the first electric generator that depended on the phenomenon of electromagnetic induction when he noticed that moving a magnet near a copper ring leads to the production of an electric current.*



- *In 1873* ***Maxwell*** *presented his four equations describing how electric and magnetic fields affect each other and electric charges. Many scientists relied on these equations to develop their experiments and research, such as the German scientist* ***Hertz****, who produced electromagnetic waves in the air in 1887. Then several other researches followed, such as the researches of* ***Thomas Edison*** *and* ***Tesla****.*
- *The discovery of the components of matter, or rather atoms, is the most important event that concerned electrical phenomena.*
    - ✓ ***Thomson*** *observed the deflection of cathode rays inside an evacuated tube when exposed to an electric or magnetic field during his experiments in* ***1897****, which led him to the conclusion that these rays, known as Lenard rays, are a collection of negatively charged particles he referred to as electrons. During this experiment, he was also able to determine the ratio of an electron's charge to its mass.*



- ✓ *In 1909, scientist* ***Rutherford*** *introduced a new atomic model, building upon Thomson's depiction of the atom as a uniform sphere of positive charge filled with* *negative electrons. He conducted an experiment by bombarding a thin gold foil with alpha particles, observing that a significant proportion of the particles traversed the foil without deflection, a small fraction were deflected at minor angles indicating repulsive forces, and some alpha particles were deflected at angles surpassing 90^0. Rutherford's experiments concluded that the atom is not a uniform sphere; instead, the majority of the atom's mass is concentrated in a tiny, positively charged region known as the nucleus, with negatively charged electrons orbiting around it.*

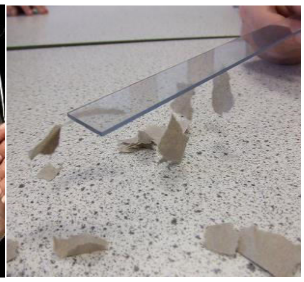


- ✓ *In 1917, **Rutherford** performed an important experiment by bombarding nitrogen gas with alpha particles and observed the emission of unexpected particles that possessed characteristics identical to those of the hydrogen nucleus and exhibited a positive charge, afterward he named them protons.*
- ✓ *In 1932, physicist **James Chadwick** discovered the neutron as a particle with no charge and a mass equal to that of the proton as a consequence of an experiment he performed on a slice of beryllium. Chadwick bombarded slice of beryllium with high-energy alpha particles and noticed the emission of uncharged particles he named neutrons.*

I-2) The Phenomenon of Electrification and Its Relationship to the Structure of Matter

The following phenomena are the most important examples from daily life of the phenomenon of electrocution:

1- *Feeling an electric shock when bringing your hand close to some things or shaking hands with someone*
2- *Hair being attracted to the comb when combing it*
3- *Hearing a small electric crackle when removing woolen or polyester clothes*
4- *Seeing small sparks when removing a woolen jacket in the dark*
5- *Small paper scraps being attracted to the rubbed edge of a plastic ruler.*

Electricity is a fundamental aspect of nature, often explained by the presence of electric charges within materials. When materials become electrified, they possess an amount of electricity, known as electrical charge. This charge is carried by tiny particles called electrons, which were discovered by physicist Joseph John Thomson in 1897. Further advancements in atomic theory came with the discoveries of the proton and neutron. Physicist Ernest Rutherford identified the proton in 1919, elucidating its role as a positively charged particle within the atomic nucleus. This discovery provided crucial insights into the composition of atomic nuclei and their interactions.

In 1932, scientists James discovered the neutron, a neutral particle found within atomic nuclei. The existence of neutrons helped reconcile discrepancies in atomic mass and contributed to the development of nuclear physics.

These discoveries collectively expanded our understanding of electricity and paved the way for advancements in fields such as particle physics, nuclear energy, and electronics.

Several models have been adopted to describe the atom, such as the Bohr model. The quantum model is the most recent model to describe the atom, which considers the atom to consist of a nucleus that includes positive protons and charge less neutrons, and electrons orbiting around the nucleus in orbits (s, p, d, f) described by mathematical functions derived from the solution of the Schrödinger equation. The charge and mass of an electron, proton and neutron are given as follows:

- ✓ ***Electron (e)*** *: charge* $q_e = -1.60219\ 10^{-19}$ **C** *mass* $m_e = 9.1095\ 10^{-31}$ **Kg**

- ✓ ***Proton (p)*** *: charge* $q_p = +1.60219\ 10^{-19}$ **C** *mass* $m_p = 1.67264\ 10^{-27}$ **Kg**
- ✓ ***Neutron (n)*** *: charge* $q_n = 0$ **C** *mass* $m_p = 1.67492\ 10^{-27}$ **Kg**

Note:

The mass of electrons is very small compared to protons, as a proton is about 1836 times heavier than an electron.

-Electrons, especially those in the outer shells of an atom, are attached to the nucleus by an electrical force (Coulomb force) resulting from the attraction between the negative charge of the electron and the positive charge of the nucleus. In contrast, protons inside the nucleus are attached by a very strong nuclear force, much greater than the electrical force, which makes protons unable to move easily between atoms.

As a result of these differences, during the process of electrical charging, electrons move between atoms and materials, while protons remain in their places within the nucleus.

Therefore, when an object is electrically charged (whether by rubbing, contact, or induction charging), the main cause is the gain or loss of electrons rather than protons, which leads to a change in the charge of the object.

I-3) Properties of electric charge

I-3-1) Millikan Oil Drop Experiment (for quantization of charge):

The Millikan Oil Drop Experiment, conducted by physicist Robert A. Millikan in 1909, aimed to quantify the charge of an electron. In this experiment, tiny oil droplets suspended in a chamber were observed under the influence of an electric field. By carefully measuring the motion of these droplets and analyzing their behavior, Millikan determined that electric charge is quantized, meaning it exists in discrete units. His precise measurements led to the determination of the charge of a single electron, providing fundamental insights into the nature of atomic and subatomic particles.

As a result of Millikan's experiment, any positive or negative charge q that can be detected is expressed as:

$$q = n\varepsilon$$

Where n is an integer (positive or negative), representing the number of elementary charges, and ϵ is the smallest unit charge in nature. This smallest unit charge, denoted as ε, has a value of approximately 1.60219×10^{-19} coulombs.

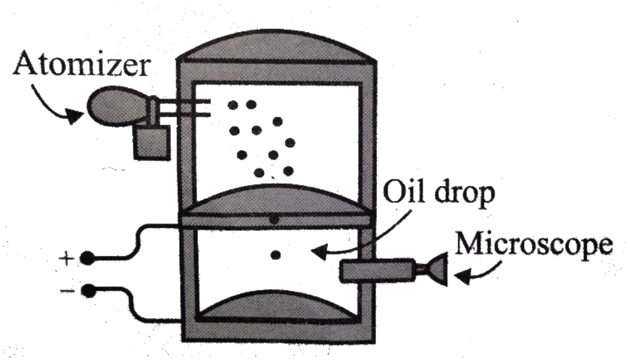


Figure 02: Millikan Oil Drop Experiment

I-3-2) Conservation of Charge

Electric charge is neither created nor destroyed, but is transferred from one body to another. This means that during the electrification process, only a process of redistributing charges between the two electrified bodies, so one body appears to have an excess of charges and the other body has a deficiency of charges.

I-3-3) Electrification Mechanism Explained

The figure below illustrates the mechanism of electrification. Initially, the material is neutral (uncharged). However, when we rub a piece of amber with a cloth, a transfer of charges occurs from the surface of one body to the surface of the other. This transfer causes a redistribution of charges, resulting in one surface becoming positively charged and the other becoming negatively charged.

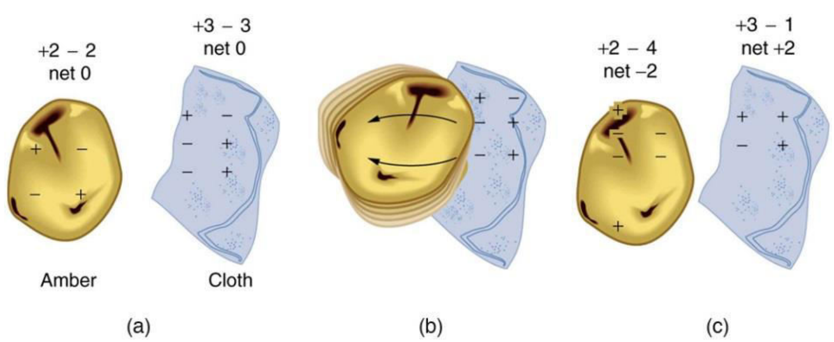


Figure 03: Electrification Mechanism

I-3-4) Methods of electrification

As depicted in the figure below, there are various methods of electrification:

I-3-4-1) Charging by rubbing

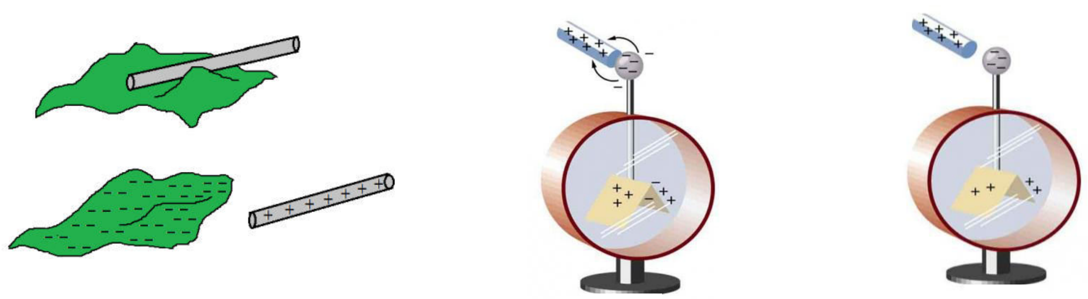
This occurs when two objects are rubbed together, resulting in the transfer of electrons from one object to the other. The object that gains electrons becomes negatively charged, while the other becomes positively charged. An illustrative example is the act of rubbing a plastic rod with a piece of cloth.

I-3-4-2) Charging by contact

Contact charging takes place when two objects make physical contact, leading to the exchange of electrons between them. If a negatively charged object touches a neutral object, the charge is redistributed between them. This phenomenon occurs frequently in everyday scenarios when different objects come into contact with each other.

I-3-4-3) Charging by Induction

Induction charging occurs when a charged object influences the charge distribution in another object without direct contact. The presence of the charged object alters the distribution of charges in the nearby object. A classic demonstration of this is bringing a charged rod close to a piece of paper and observing how the charge distribution in the paper is affected by the proximity of the charged rod

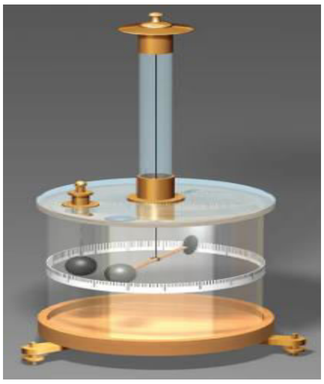


(a) **(b)** **(c)**

Figure 04: Electrification (charging) methods

I-4) Electrostatic forces

The physicist Coulomb was able to measure the electric force exchanged between two charged balls using the following experimental setup.



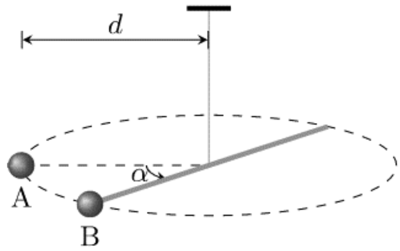


Figure 05: Coulomb's experimental

Coulomb's law describes the electrostatic force between two charged objects. It states that the force between two point charges is directly proportional to the product of their charges and inversely proportional to the square of the distance between them. Mathematically, Coulomb's law is expressed as:

$$\overrightarrow{F_{A/B}} = \frac{1}{4\pi\varepsilon_0}\frac{q_A q_B}{r^2}\overrightarrow{u_{AB}} = k\,\frac{q_A q_B}{\left\|\overrightarrow{AB}\right\|^3}\overrightarrow{AB}$$

Where k *is Coulomb's constant (*$k \approx 8.9875\times10^9\ N.m^2/C = 9\times10^9\ N.m^2/C$*, in vacuum).*

$r = \left\|\overrightarrow{AB}\right\|$ *is the distance between the two charges A and B.*

ε_0 *denotes the vacuum permittivity (*$\varepsilon_0 \approx 8.854\times10^{-12}\ C^2N^{-1}m^{-2}$ *)*

- *The electric force applied by charge A to the point charge B is represented by a vector characterized by:*

Magnitude: *The magnitude of the electric force between two charges* q_A *and* q_B *separated by a distance r is indeed given by Coulomb's law as* $\left\|\overrightarrow{F_{A/B}}\right\| = k\,\frac{q_A q_B}{r^2}$

Support: *refers to the line connecting the centers of the two point charges A and B*

Direction: *The direction of the electric force depends on the types of charges involved. The force vector points along the line connecting the charges. If the charges are of opposite signs, the force is attractive, pulling the charges together. If the charges are of the same sign, the force is repulsive, pushing the charges apart.*

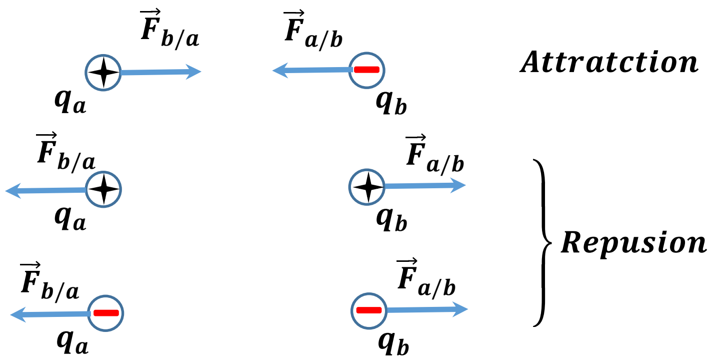


Figure 06: attractive (between Opposite Charges (One Positive, One Negative)) and repulsive forces (between the same Charges (Both Positive or Both Negative))

In the case where a set of electric charges $\boldsymbol{q_i}$ *located at the* $\boldsymbol{P_i}$ *positions with respect to a given reference frame exert a force on an electric charge* $\boldsymbol{Q}$ *positioned at the* $\boldsymbol{M}$ *position, the net forces acting on it is written as follows:*

$$\overrightarrow{F_{Net}} = \sum_i \overrightarrow{F_{q_\iota/Q}} = \sum_i K \frac{Q\, q_i}{\left\|\overrightarrow{P_\iota M}\right\|^3} \overrightarrow{P_\iota M}$$

I-5) The Electric Field

Understanding the concept of the electric field requires recalling the following points:

1- *Coulomb's forces arise between an electric charge (considered as a charge source Q) and an affected charge q are action at a distance (do not require contact between charges) forces.*
2- *The strength of the electric force (Coulomb force) increases as the distance between the two charges decreases and the magnitude of the two charges increases.*
3- *If we place (a charged point body) a test electric charge q somewhere and notice that this test charge was affected by its motion in a certain direction, this is evidence that it was affected by another charge Q, which is considered the source of this effect.*

- *By moving the test charge q, we can define the place of the source charge Q and whether is negative or positive.*
- *The closer the test charge gets to the source charge Q, the stronger the force influenced it.*

- *As a conclusion to this experiment: when the test charge experiences an electric force, it implies that it is within the influence of the source charge, meaning it is inside the electric field generated by Q.*

So we define the electric field as the property of the electric effect created by an electric charge in the space surrounding it.

Every electric charge gives the space around it the property of the electric Coulomb effect, i.e. it creates an electric field around it. Therefore, any charge present within this space has been affected by the field resulting from the source charge.

The Electric field, denoted by the vector $\vec{E}$ and is expressed in N/C or V/m. The electric field extends outward from positive charges and inward towards negative charges, causing positive charges to experience a force in the same direction and negative charges to experience a force in the opposite direction.

The electric field is q vector field generated (induced) by an electric charge. It represents a spatial electrical property, its magnitude at specific point in the space depends on the value of source charge and the distance of that point from the source, and its direction at that point is depends on the type (sign : positive (+) or negative(-)) of the charge.

The electric field generated by a point charge Q (located at the point P and considered as the source) at a specific point in space M is characterized by three main features:

- ♣ *Support: It is the line connecting the source charge to the point M.*
- ♣ *Direction: The direction of the field depends on the polarity of the source charge, pointing outward for positive charges and inward for negative charges.*

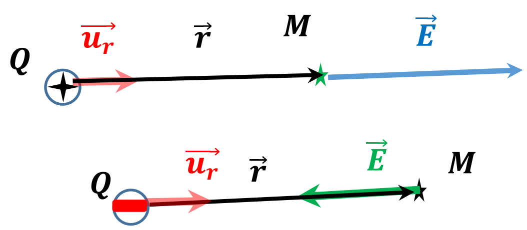


Figure 07: the direction of both the electric field at different points

- ♣ *Magnitude (Length): The length of the field's vector is influenced by both the magnitude of the source charge and the distance between the source charge and the point in question.*

The mathematical expression for the electric field vector produced by a point charge Q (placed at P and considered as the source) at a point in space is as follows:

$$\overrightarrow{E_Q}(M) = KQ\frac{\overrightarrow{PM}}{\left\|\overrightarrow{PM}\right\|^3}$$

$$\overrightarrow{PM} = \vec{r} = \left\|\overrightarrow{PM}\right\| \overrightarrow{u_r} = r\,\overrightarrow{u_r} \text{ where} \begin{cases} r \text{ is the magnitude of the } \overrightarrow{PM} \text{ vector} (r = \left\|\overrightarrow{PM}\right\|) \\ \overrightarrow{u_r} \text{ is the unit vector} \end{cases}$$

$$\overrightarrow{E_Q}(M) = KQ\frac{\vec{r}}{\|\vec{r}\|^3} = \frac{KQ}{r^2}\,\overrightarrow{u_r}$$

$\overrightarrow{PM}$ *The vector connecting the source charge location P to the point M in space where the field is being calculated.*

The relationship between the electric field induced by the electric charge Q (located at P point) and the electric force applied by Q on q charge located at M point is given as follows:

$$\overrightarrow{F_{Q/q}} = \frac{KqQ}{\left\|\overrightarrow{PM}\right\|^3}\overrightarrow{PM}$$

$$\overrightarrow{F_{Q/q}}(M) = q\,\overrightarrow{E_Q}$$

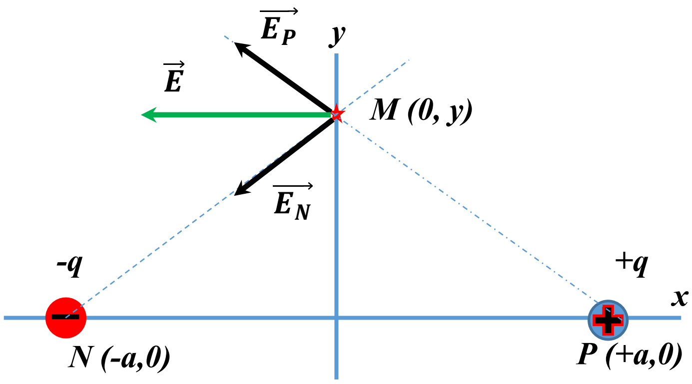
The electric field at a specific point in space ***M****, generated by a collection of charges* $\boldsymbol{q_i}$ *located at the positions* $\boldsymbol{P_i}$ *with respect to a given reference frame, is the vector sum of the individual partial fields. The electric field is written as follows:*

$$\overrightarrow{E_{Net}}(M) = \sum_i \overrightarrow{E_i}(M) = \sum_i K\frac{q_i}{\left\|\overrightarrow{P_iM}\right\|^3}\overrightarrow{P_iM}$$

I-5-1) Electric field produced by these two charges

The electric dipole introduced in the figure below consists of a positive charge $\boldsymbol{q_P} = +\boldsymbol{q}$ *and a negative charge* $\boldsymbol{q_N} = -\boldsymbol{q}$ *separated by a distance* ***2a***

♣ *Electric field produced by these two charges* ***at point M, which lies on the midpoint between the ends of NP****, can be calculated as follows:*



$$\vec{E}\,(M) = \overrightarrow{E_P}(M) + \overrightarrow{E_N}(M)$$

$$\overrightarrow{E_P}(M) = \ Kq\frac{\overrightarrow{PM}}{\left\|\overrightarrow{PM}\right\|^3}$$

$$\overrightarrow{E_N}(M) = \ -Kq\frac{\overrightarrow{NM}}{\left\|\overrightarrow{NM}\right\|^3}$$

$$\overrightarrow{PM} = (\,x_M - x_P)\vec{\imath} + (\,y_M - y_P) = (-a)\vec{\imath} + (\,y)\vec{\jmath}$$

$$\overrightarrow{NM} = (\,x_M - x_N)\vec{\imath} + (\,y_M - y_N)\vec{\jmath} = \ (\,a)\vec{\imath} + (\,y)\vec{\jmath}$$

$$\left\|\overrightarrow{PM}\right\| = \ \sqrt{a^2 + y^2}$$

$$\left\|\overrightarrow{NM}\right\| = \ \sqrt{a^2 + y^2}$$

$$\overrightarrow{E_P}(M) = \ Kq\frac{(-a)\vec{\imath} + (\,y)\vec{\jmath}}{(\sqrt{a^2 + y^2})^3}$$

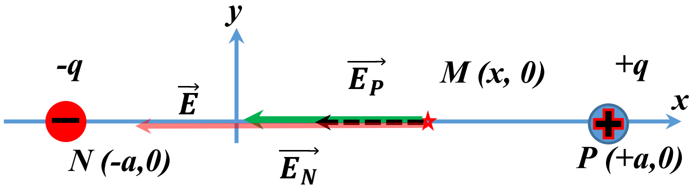
$$\overrightarrow{E_N}(M) = \ -Kq\frac{(+a)\vec{\imath} + (\,y)\vec{\jmath}}{(\sqrt{a^2 + y^2})^3}$$

$$\vec{E}\,(M) = Kq\frac{(-a)\vec{\imath} + (\,y)\vec{\jmath}}{\left(\sqrt{a^2 + y^2}\right)^3} - Kq\frac{(a)\vec{\imath} + (\,y)\vec{\jmath}}{(\sqrt{a^2 + y^2})^3}$$

$$\vec{E}\,(M) = \frac{-2Kq\,a\,\vec{\imath}}{\left(\sqrt{a^2 + y^2}\right)^3}$$

The electric field generated along the dipole axis, can be calculated as follows*:*

1st case : for M point located inside the segment $[PN]$



$$\vec{E}\,(M) = \overrightarrow{E_P}(M) + \overrightarrow{E_N}(M)$$

$$\overrightarrow{E_P}(M) = Kq\frac{\overrightarrow{PM}}{\|\overrightarrow{PM}\|^3}$$

$$\overrightarrow{E_N}(M) = -Kq\frac{\overrightarrow{NM}}{\|\overrightarrow{NM}\|^3}$$

$$\overrightarrow{PM} = (\,x_M - x_P)\vec{\imath} + (\,y_M - y_P) = (x-a)\vec{\imath}$$

$$\overrightarrow{NM} = (\,x_M - x_N)\vec{\imath} + (\,y_M - y_N)\vec{\jmath} = (\,x+a)\vec{\imath}$$

$$\|\overrightarrow{PM}\| = \sqrt{(x-a)^2} = (a-x)$$

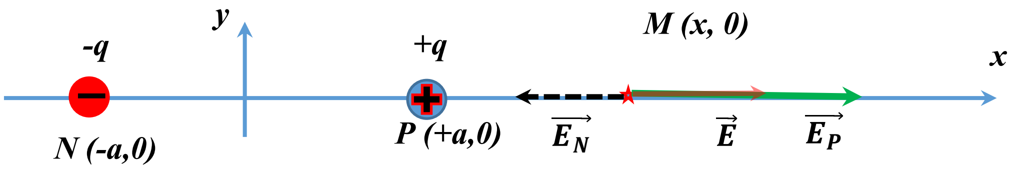
$$\|\overrightarrow{NM}\| = \sqrt{(x+a)^2} = (x+a)$$

$$\overrightarrow{E_P}(M) = Kq\frac{(x-a)\vec{\imath}}{(a-x)^3} = \frac{-Kq\,\vec{\imath}}{(a-x)^2}$$

$$\overrightarrow{E_N}(M) = -Kq\frac{(x+a)\vec{\imath}}{(x+a)^3} = \frac{-Kq\,\vec{\imath}}{(x+a)^2}$$

$$\vec{E}\,(M) = -Kq\left(\frac{1}{(a-x)^2} + \frac{1}{(x+a)^2}\right)\vec{\imath}$$

2nd case : for M point located outside the segment $[PN]$ *i.e., on one of its endpoints*



$$\vec{E}\,(M) = \overrightarrow{E_P}(M) + \overrightarrow{E_N}(M)$$

$$\overrightarrow{E_P}(M) = Kq\frac{\overrightarrow{PM}}{\|\overrightarrow{PM}\|^3}$$

$$\overrightarrow{E_N}(M) = -Kq\frac{\overrightarrow{NM}}{\|\overrightarrow{NM}\|^3}$$

$$\|\overrightarrow{PM}\| = \sqrt{(x-a)^2} = (x-a)$$

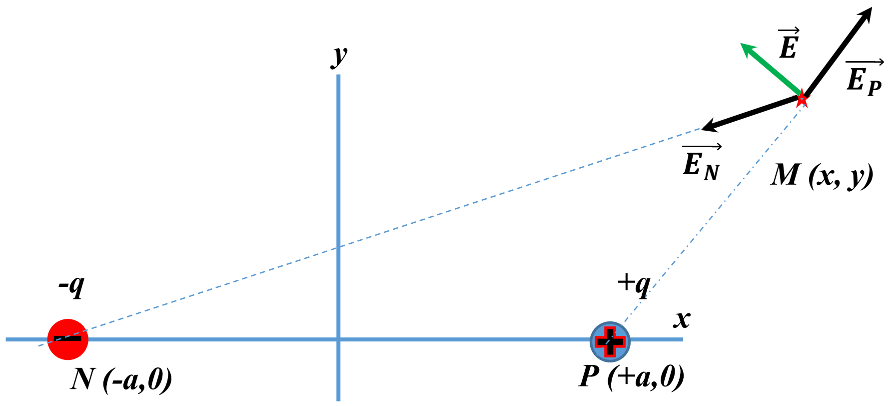
$$\|\overrightarrow{NM}\| = \sqrt{(x+a)^2} = (x+a)$$

$$\overrightarrow{E_P}(M) = Kq\frac{(x-a)\vec{\imath}}{(x-a)^3} = \frac{Kq\,\vec{\imath}}{(x-a)^2}$$

$$\overrightarrow{E_N}(M) = -Kq\frac{(x+a)\vec{\imath}}{(x+a)^3} = \frac{-Kq\,\vec{\imath}}{(x+a)^2}$$

$$\vec{E}\,(M) = Kq\left(\frac{1}{(x-a)^2} - \frac{1}{(x+a)^2}\right)\vec{\imath}$$

♣ *Electric field produced by these two charges* ***at point M (general case)*** *can be*



calculated as follows:

$$\vec{E}\,(M) = \overrightarrow{E_P}(M) + \overrightarrow{E_N}(M)$$

$$\overrightarrow{E_P}(M) = Kq\frac{\overrightarrow{PM}}{\|\overrightarrow{PM}\|^3}$$

$$\overrightarrow{E_N}(M) = -Kq\frac{\overrightarrow{NM}}{\|\overrightarrow{NM}\|^3}$$

$$\overrightarrow{PM} = (\,x_M - x_P)\vec{\imath} + (\,y_M - y_P) = (x-a)\vec{\imath} + (\,y)\vec{\jmath}$$

$$\overrightarrow{NM} = (\,x_M - x_N)\vec{\imath} + (\,y_M - y_N)\vec{\jmath} = (\,x+a)\vec{\imath} + (\,y)\vec{\jmath}$$

$$\|\overrightarrow{PM}\| = \sqrt{(x-a)^2 + y^2}$$

$$\|\overrightarrow{NM}\| = \sqrt{(x+a)^2 + y^2}$$

$$\overrightarrow{E_P}(M) = Kq\frac{(x-a)\vec{\imath} + (\,y)\vec{\jmath}}{(\sqrt{(x-a)^2 + y^2})^3}$$

$$\overrightarrow{E_N}(M) = -Kq\frac{(x+a)\vec{\imath}+(y)\vec{\jmath}}{(\sqrt{(x+a)^2+y^2})^3}$$

$$\vec{E}\,(M) = Kq\left(\frac{(x-a)\vec{\imath}+(y)\vec{\jmath}}{(\sqrt{(x-a)^2+y^2})^3}-\frac{(x+a)\vec{\imath}+(y)\vec{\jmath}}{(\sqrt{(x+a)^2+y^2})^3}\right)$$

I-6) The electric potential generated by a point charge

In order to move a charge q within an electric field E, work must be done by applying a force that opposes the electric force (overcoming the electric field).

$$\overrightarrow{F_{applied}} = -q\,\vec{E}$$

$$W_{AB}^{ext} = \int_A^B \overrightarrow{F_{applied}}\,d\vec{r} = -q\int_A^B \vec{E}\,d\vec{r}$$

The difference in potential between two points A and B within the field is defined as the result of dividing the applied work by the value of the electric charge

$$V_{BA} = \frac{W_{AB}^{ext}}{q} = -\int_A^B \vec{E}.d\vec{r}$$

The relationship between electric potential and electric field is expressed as follows:

$$dV = -\vec{E}.d\vec{r}$$

By substituting the expressions for the electric field and the differential displacement in different coordinate systems, we find:

- ✓ *In Cartesian coordinates system*

$$dV = dV_x + dV_y + dV_z = -\vec{E}.d\vec{r} \;\; where \begin{cases}\vec{E} = E_x\,\vec{\imath} + E_y\,\vec{\jmath} + E_z\,\vec{k}\\ d\vec{r} = dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k}\end{cases}$$

$$dV = dV_x + dV_y + dV_z = -E_x\,dx - E_y\,dy - E_z\,dz$$

$$\begin{cases}dV_x = -E_x\,dx\\ dV_y = -E_y\,dy\\ dV_z = -E_z\,dz\end{cases}$$

- ✓ *In Polar coordinates system*

$$dV = dV_r + dV_\theta = -\vec{E}.d\vec{r} \ \ where \begin{cases} \vec{E} = E_r\,\overrightarrow{u_r} + E_\theta\ \overrightarrow{u_\theta} \\ d\vec{r} = dr\,\overrightarrow{u_r} + r\,d\theta\ \ \overrightarrow{u_\theta} \end{cases}$$

$$dV = dV_r + dV_\theta = -E_r\,dr - E_\theta\,r\,d\theta$$

$$\begin{cases} dV_r = -E_r\,dr \\ dV_\theta = -\,E_\theta\,r\,d\theta \end{cases}$$

- *In Cylindrical coordinates system*

$$dV = dV_r + dV_\theta + dV_z = -\vec{E}.d\vec{r} \ \ where \begin{cases} \vec{E} = E_r\,\overrightarrow{u_r} + E_\theta\ \overrightarrow{u_\theta} + E_z\ \vec{k} \\ d\vec{r} = dr\,\overrightarrow{u_r} + r\,d\theta\ \ \overrightarrow{u_\theta} + dz\ \vec{k} \end{cases}$$

$$dV = dV_r + dV_\theta + dV_z = -E_r\,dr - E_\theta\,r\,d\theta - E_z\,dz$$

$$\begin{cases} dV_r = -E_r\,dr \\ dV_\theta = -\,E_\theta\,r\,d\theta \\ dV_z = -E_z\,dz \end{cases}$$

- *In Spherical coordinates system*

$$V = dV_r + dV_\theta + dV_\varphi = -\vec{E}.d\vec{r} \ \ where \begin{cases} \vec{E} = E_r\,\overrightarrow{u_r} + E_\theta\ \overrightarrow{u_\theta} + E_\varphi\ \overrightarrow{u_\varphi} \\ d\vec{r} = dr\,\overrightarrow{u_r} + r\,d\theta\ \ \overrightarrow{u_\theta} + r\sin\theta\,d\varphi\ \ \overrightarrow{u_\varphi} \end{cases}$$

$$dV = dV_r + dV_\theta + dV_\varphi = -E_r\,dr - E_\theta\,r\,d\theta - E_\varphi\,r\sin\theta\,d\varphi$$

$$\begin{cases} dV_r = -E_r\,dr \\ dV_\theta = -\,E_\theta\,r\,d\theta \\ dV_\varphi = -E_\varphi\,r\sin\theta \end{cases}$$

To calculate the electric field from the electric potential expression, we use the following gradient relationship:

$$\vec{E} = -\,\overrightarrow{grad}\,V$$

The gradient operator ($\overrightarrow{grad}$) expression in different coordinate systems is given as follows:

- *In Cartesian coordinates system*

$$\overrightarrow{grad} = \vec{\nabla} = \frac{\partial}{\partial x}\,\vec{\imath} + \frac{\partial}{\partial y}\,\vec{\jmath} + \frac{\partial}{\partial z}\,\vec{k}$$

- *In Polar coordinates system*

$$\overrightarrow{grad} = \vec{\nabla} = \frac{\partial}{\partial r}\overrightarrow{u_r} + \frac{1}{r}\frac{\partial}{\partial\theta}\overrightarrow{u_\theta}$$

- ✓ *In Cylindrical coordinates system*

$$\overrightarrow{grad} = \vec{\nabla} = \frac{\partial}{\partial r}\overrightarrow{u_r} + \frac{1}{r}\frac{\partial}{\partial\theta}\overrightarrow{u_\theta} + \frac{\partial}{\partial z}\vec{k}$$

- ✓ *In Spherical coordinates system*

$$\overrightarrow{grad} = \vec{\nabla} = \frac{\partial}{\partial r}\overrightarrow{u_r} + \frac{1}{r}\frac{\partial}{\partial\theta}\overrightarrow{u_\theta} + \frac{1}{r\sin\varphi}\frac{\partial}{\partial\varphi}\overrightarrow{u_\varphi}$$

Based on this foundation, we can express the electric potential components as follows:

- ✓ *In Cartesian coordinates system*

$$\vec{E} = -\overrightarrow{grad}V(x,y,z) = -\frac{\partial V(x,y,z)}{\partial x}\vec{\imath} - \frac{\partial V(x,y,z)}{\partial y}\vec{\jmath} - \frac{\partial V(x,y,z)}{\partial z}\vec{k}$$

$$\vec{E} = E_x\vec{\imath} + E_y\vec{\jmath} + E_z\vec{k} \text{ where } \begin{cases} E_x = -\frac{\partial V(x,y,z)}{\partial x} \\ E_y = -\frac{\partial V(x,y,z)}{\partial y} \\ E_z = -\frac{\partial V(x,y,z)}{\partial z} \end{cases}$$

- ✓ *In Polar coordinates system*

$$\vec{E} = -\overrightarrow{grad}\,V(r,\theta) = -\frac{\partial V(r,\theta)}{\partial r}\overrightarrow{u_r} - \frac{1}{r}\frac{\partial V(r,\theta)}{\partial\theta}\overrightarrow{u_\theta}$$

$$\vec{E} = E_r\overrightarrow{u_r} + E_\theta\overrightarrow{u_\theta} \text{ where } \begin{cases} E_r = -\frac{\partial V(r,\theta)}{\partial r} \\ E_\theta = -\frac{1}{r}\frac{\partial V(r,\theta)}{\partial\theta} \end{cases}$$

- ✓ *In Cylindrical coordinates system*

$$\vec{E} = -\overrightarrow{grad}\,V(r,\theta,z) = -\frac{\partial V(r,\theta,z)}{\partial r}\overrightarrow{u_r} - \frac{1}{r}\frac{\partial V(r,\theta,z)}{\partial\theta}\overrightarrow{u_\theta} - \frac{\partial V(r,\theta,z)}{\partial z}\vec{k}$$

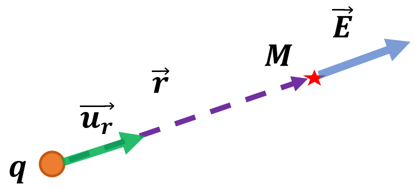
$$\vec{E} = E_r\overrightarrow{u_r} + E_u\overrightarrow{u_\theta} + +E_z\vec{k} \text{ where } \begin{cases} E_r = -\frac{\partial V(r,\theta,z)}{\partial r} \\ E_\theta = -\frac{1}{r}\frac{\partial V(r,\theta,z)}{\partial\theta} \\ E_z = -\frac{\partial V(r,\theta,z)}{\partial z} \end{cases}$$

- ✓ *In Spherical coordinates system*

$$\vec{E} = -\overrightarrow{grad}\,V(r,\theta,\varphi) = -\frac{\partial V(r,\theta,\varphi)}{\partial r}\overrightarrow{u_r} - \frac{1}{r}\frac{\partial V(r,\theta,\varphi)}{\partial\theta}\overrightarrow{u_\theta} - \frac{1}{r\sin\varphi}\frac{\partial V(r,\theta,\varphi)}{\partial\varphi}\overrightarrow{u_\varphi}$$

$$\vec{E} = E_r\,\overrightarrow{u_r} + E_u\,\overrightarrow{u_\theta} + +E_\varphi\overrightarrow{u_\varphi}\quad where\ \begin{cases} E_r = -\dfrac{\partial V(r,\theta,\varphi)}{\partial r} \\ E_\theta = -\dfrac{1}{r}\dfrac{\partial V(r,\theta,\varphi)}{\partial \theta} \\ E_\varphi = -\dfrac{1}{r\sin\varphi}\dfrac{\partial V(r,\theta,\varphi)}{\partial \varphi} \end{cases}$$

The electric field generated by a point charge (assumed to be positive) at a point m in the space surrounding the charge q is given as follows:



$$\vec{E}(M) = K\frac{q}{\|\vec{r}\|^3}\vec{r}\ \ where\ \vec{r} = \|\vec{r}\|\,\overrightarrow{\boldsymbol{u_r}} = r\,\overrightarrow{\boldsymbol{u_r}}\ \Rightarrow \vec{E}(M) = K\frac{q}{r^2}\overrightarrow{\boldsymbol{u_r}}$$

$$\overrightarrow{\boldsymbol{dr}} = \boldsymbol{dr}\,\overrightarrow{\boldsymbol{u_r}}$$

$$V(M) = \int dV = \int -\vec{E}.d\vec{r} = \int -(K\frac{q}{r^2}\overrightarrow{\boldsymbol{u_r}}).(\boldsymbol{dr}\,\overrightarrow{\boldsymbol{u_r}}) = \int -K\frac{q}{r^2}\boldsymbol{dr}$$

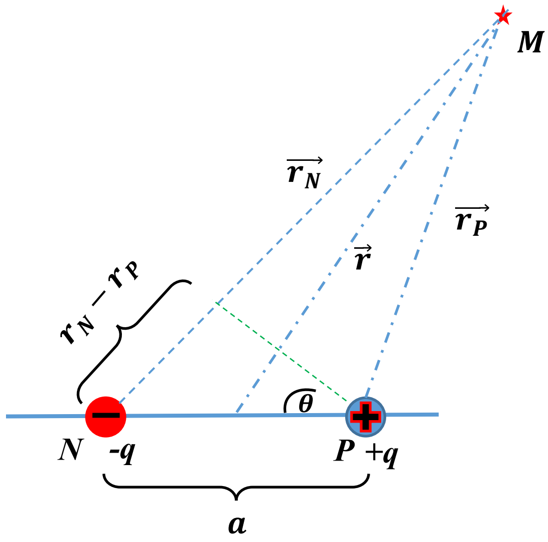
$$V(M) = K\frac{q}{r} + C$$

The total electric potential $\boldsymbol{V}$ *arising from a collection of point charges* $\boldsymbol{q_i}$ *located at* $\boldsymbol{P_i}$ *at a specific point M in space is equal to the sum of the potentials arising from each individual point charge at that point.*

$$V(M) = \sum_{i=1}^{N} V_i\,(M) = k\sum_{i=1}^{N}\frac{q_i}{\|\overrightarrow{\boldsymbol{P_i}M}\|} + C$$

I-6-1) The electric potential of an electric dipole

*We want to calculate the electric potential generated by a dipole at a distant point **M** and then derive the electric field term at this point.*



$$V_{net}(M) = \sum_i V_i = V_P(M) + V_N(M) = \sum_i \frac{Kq_i}{\|\overrightarrow{P_iM}\|} = \frac{Kq_P}{\|\overrightarrow{P_PM}\|} + \frac{Kq_N}{\|\overrightarrow{P_NM}\|}$$

$$\|\overrightarrow{P_PM}\| = \|\vec{r_P}\| = r_P \text{ and } \|\overrightarrow{P_NM}\| = \|\vec{r_N}\| = r_N$$

$$V_{net}(M) = \frac{Kq}{r_P} - \frac{Kq}{r_N} = Kq(\frac{r_N - r_P}{r_N r_P})$$

For a point M located at a great distance from the dipole, the distances r_N and r_P are approximately equal ($r_N = r_P = r$) and the difference $r_N - r_P$ is expressed as follows:

$$\cos\theta = \frac{r_N - r_P}{a} \Rightarrow r_N - r_P = a\cos\theta$$

$$V_{net}(M) = Kq(\frac{a\cos\theta}{r^2})$$

We can also find the equation of the equipotential lines as follows:

$$V_{net}(M) = Kq\left(\frac{a\cos\theta}{r^2}\right) = \text{constant potential} = V_C$$

$$r^2 = Kq\left(\frac{a\cos\theta}{V_C}\right) \Rightarrow r = \sqrt{Kq\left(\frac{a\cos\theta}{V_C}\right)}$$

Therefore, the surface of the equipotential is in the form of spheres with radius r.

The equation of the field lines can be obtained after finding the expression for the electric field in polar coordinates as follows:

$$\vec{E} = -\overrightarrow{grad}\, V(r,\theta) = -\frac{\partial V(r,\theta)}{\partial r}\,\overrightarrow{u_r} - \frac{1}{r}\frac{\partial V(r,\theta)}{\partial \theta}\,\overrightarrow{u_\theta}$$

$$\vec{E} = E_r\,\overrightarrow{u_r} + E_\theta\,\overrightarrow{u_\theta} \quad where \begin{cases} E_r = -\dfrac{\partial V(r,\theta)}{\partial r} \\ E_\theta = -\dfrac{1}{r}\dfrac{\partial V(r,\theta)}{\partial \theta} \end{cases}$$

$$\begin{cases} E_r = -\dfrac{\partial}{\partial r}\left(Kq\left(\dfrac{a\cos\theta}{r^2}\right)\right) = \dfrac{2Kq\, a\cos\theta}{r^3} \\ E_\theta = -\dfrac{1}{r}\dfrac{\partial}{\partial \theta}\left(Kq\left(\dfrac{a\cos\theta}{r^2}\right)\right) = \dfrac{Kq\, a\sin\theta}{r^3} \end{cases}$$

$$\vec{\boldsymbol{E}}\wedge\overrightarrow{\boldsymbol{dl}} = \vec{0} \Rightarrow \begin{cases} \vec{E} = \dfrac{2Kq\, a\cos\theta}{r^3}\,\overrightarrow{u_r} + \dfrac{Kq\, a\sin\theta}{r^3}\,\overrightarrow{u_\theta} \\ d\vec{l} = dr\,\overrightarrow{u_r} + r\, d\theta\,\overrightarrow{u_\theta} \end{cases}$$

$$\frac{2Kq\, a\cos\theta}{r^3} r\, d\theta - dr\frac{Kq\, a\sin\theta}{r^3} = 0 \Rightarrow \frac{dr}{r} = \frac{2\cos\theta}{\sin\theta}\, d\theta$$

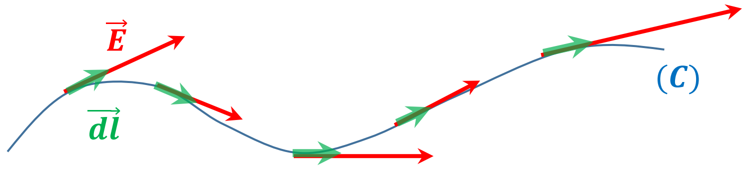
We integrate both sides of the equation and obtain:

$$r = c\sin^2\theta$$

I-7) Electric field lines, equipotential surfaces and electric potential energy

*An electric field line is an **imaginary curve (C)** to which the electric field vector is tangent at every point on the curve, and it specifies the direction of the force acting on a test charge (assuming a positive charge) placed in the field.*

Electric field lines are continuous and do not intersect, perpendicular to equipotential surfaces, and extend from positive charges to negative charges.



The differential (infinitesimal) length vector $\overrightarrow{\boldsymbol{dl}}$ *shown in the figure above is parallel to the electric field* $\vec{\boldsymbol{E}}$ *vector at every point on the curve (C).*

By writing the expression for each of the differential length vector and the electric field vector in the Cartesian, cylindrical and spherical coordinate systems:

- ✓ *In Cartesian coordinates system* $\begin{cases} \vec{E} = E_x\,\vec{\imath} + E_y\,\vec{\jmath} + E_z\,\vec{k} \\ d\vec{l} = dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k} \end{cases}$

- ✓ *In Cylindrical coordinates system* $\begin{cases} \vec{E} = E_r\,\overrightarrow{u_r} + E_\theta\,\overrightarrow{u_\theta} + E_z\,\vec{k} \\ d\vec{l} = dr\,\overrightarrow{u_r} + r\,d\theta\,\overrightarrow{u_\theta} + dz\,\vec{k} \end{cases}$

- ✓ *In Spherical coordinates system* $\begin{cases} \vec{E} = E_r\,\overrightarrow{u_r} + E_\theta\,\overrightarrow{u_\theta} + E_\varphi\,\overrightarrow{u_\varphi} \\ d\vec{l} = dr\,\overrightarrow{u_r} + r\,d\theta\,\overrightarrow{u_\theta} + r\sin\theta\,d\varphi\,\overrightarrow{u_\varphi} \end{cases}$

Applying the condition of parallelism between the differential length $\overrightarrow{\boldsymbol{dl}}$ *vectors and the electric field* $\vec{\boldsymbol{E}}$*, we obtain the analytical equation of lines of force:*

$$\vec{\boldsymbol{E}} \wedge \overrightarrow{\boldsymbol{dl}} = \vec{\boldsymbol{0}}$$

Where it is written according to the approved coordinate system as follows:

- ¤ *In Cartesian coordinates system*

$$\vec{\boldsymbol{E}} \wedge \overrightarrow{\boldsymbol{dl}} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ E_x & E_y & E_z \\ dx & dy & dz \end{vmatrix} = \left(E_y dz - dyE_z\right)\vec{\imath} - \left(E_x dz - dxE_z\right)\vec{\jmath} + \left(E_x dy - dxE_y\right)\vec{k}$$

$$\vec{\boldsymbol{E}} \wedge \overrightarrow{\boldsymbol{dl}} = \vec{\boldsymbol{0}} \Rightarrow \begin{cases} E_y dz - dyE_z = 0 \\ E_x dz - dxE_z = 0 \\ E_x dy - dxE_y = 0 \end{cases} \Rightarrow \begin{cases} E_y dz = dyE_z \\ E_x dz = dxE_z \\ E_x dy = dxE_y \end{cases} \Rightarrow \boldsymbol{\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z}}$$

- ¤ *In Cylindrical coordinates system*

$$\vec{\boldsymbol{E}} \wedge \overrightarrow{\boldsymbol{dl}} = \begin{vmatrix} \overrightarrow{u_r} & \overrightarrow{u_\theta} & \vec{k} \\ E_r & E_\theta & E_z \\ dr & rd\theta & dz \end{vmatrix} = \left(E_\theta dz - rd\theta E_z\right)\overrightarrow{u_r} - \left(E_r dz - drE_z\right)\overrightarrow{u_\theta} + \left(E_r rd\theta - drE_\theta\right)\vec{k}$$

$$\vec{\boldsymbol{E}} \wedge \overrightarrow{\boldsymbol{dl}} = \vec{\boldsymbol{0}} \Rightarrow \begin{cases} E_\theta dz - rd\theta E_z = 0 \\ E_r dz - drE_z = 0 \\ E_r rd\theta - drE_\theta = 0 \end{cases} \Rightarrow \begin{cases} E_\theta dz = rd\theta E_z \\ E_r dz = drE_z \\ E_r rd\theta = drE_\theta \end{cases} \Rightarrow \boldsymbol{\frac{dr}{E_r} = \frac{rd\theta}{E_\theta} = \frac{dz}{E_z}}$$

- ¤ *In spherical coordinates system*

$$\vec{E}\wedge\vec{dl} = \begin{vmatrix} \vec{u_r} & \vec{u_\theta} & \vec{u_\varphi} \\ E_r & E_\theta & E_\varphi \\ dr & rd\theta & r\sin\theta\, d\varphi \end{vmatrix}$$

$$= \left(E_\theta r\sin\theta\, d\varphi - rd\theta E_\varphi\right)\vec{u_r} - \left(E_r r\sin\theta\, d\varphi - drE_\varphi\right)\vec{u_\theta} + \left(E_r rd\theta - drE_\theta\right)\vec{u_\varphi}$$

$$\Rightarrow \begin{cases} E_\theta r\sin\theta\, d\varphi - rd\theta E_\varphi = 0 \\ E_r r\sin\theta\, d\varphi - drE_\varphi = 0 \\ E_r rd\theta - drE_\theta = 0 \end{cases} \Rightarrow \begin{cases} E_\theta r\sin\theta\, d\varphi = rd\theta E_\varphi \\ E_r r\sin\theta\, d\varphi = drE_\varphi \\ E_r rd\theta = drE_\theta \end{cases} \Rightarrow \boldsymbol{\frac{dr}{E_r} = \frac{rd\theta}{E_\theta} = \frac{r\sin\theta\, d\varphi}{E_\varphi}}$$

*An **equipotential surface** is defined as a surface in space where the potential is equal at all points, and this surface is perpendicular to the field lines. The equation describing an equipotential surface is given by the following relation:*

$$V(r) = V(x, y, z) = constant$$

Example of field lines and equipotential surfaces of a point charge

The expression for the electric field and electric potential due to a positive point charge is given as follows:

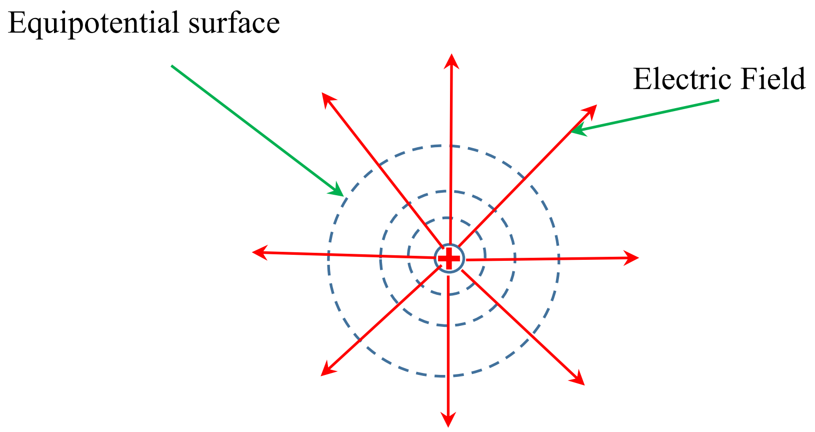
$$\vec{E} = K\frac{q}{r^2}\vec{u_r}$$

$$V(r) = K\frac{q}{r}$$

Equipotential surfaces can be found by the relation:

$$V(r) = K\frac{q}{r} = constant \Rightarrow K\frac{q}{constant} = K\frac{q}{V} = r$$

This equation is for the surface of a sphere with radius r.



*To simplify the concept of **potential energy** of a system of charges, we must address some of the following important points:*

- *The potential energy of a point charge q located at position M within a potential arising from another point charge Q (the source) is equal to the work done by an electric force to move the charge q from infinity to point M(* $dE_p = dW = \vec{F}.\overrightarrow{dr}$*)*
- *The force of work to be exerted on the charge q is equal in intensity to the force between the charge q and the charge* $\boldsymbol{q}$ *and opposite in direction(* $\vec{F} = -q\vec{E}(M)$*)*
- *By integrating the term* $\boldsymbol{dE_p}$*, we find the expression for potential energy:*

$$E_p = \int_{\infty}^{M} dE_p = E_p(M) - E_p(\infty) = \int_{\infty}^{M} -q(\vec{E}.\overrightarrow{dr}) = \int_{\infty}^{M} -q(-dV) = q\int_{\infty}^{M} dV$$

$$= qV(M)$$

$$here\ E_p(\infty) = 0\ and\ V(\infty) = 0$$

$$and\ thus\ E_p(M) = q\, V_Q(M)$$

We denote that $V_Q(M)$ *is the potential created by Q at the point M*

❖ *If we bring a first test charge "*$\boldsymbol{q_1}$*" from infinity and want to place it in position* $\boldsymbol{p_1}$ *free of charge, we do not exert any energy (work).*

❖ *If we bring a second test charge "*$\boldsymbol{q_2}$*" from infinity and want to place it in position* $\boldsymbol{p_2}$ *near the first test charge "*$\boldsymbol{q_1}$*", We have to apply potential energy (work) that opposes and cancels the mutual effect between the two charges* $\boldsymbol{q_1}$ *and* $\boldsymbol{q_2}$*.*

$$Ep_{q_1/q_2} = q_2\, V_{q_1}(p_2) = Ep_{q_2/q_1} = q_1\, V_{q_2}(p_1)\ where \begin{cases} V_{q_1}(p_2) = \dfrac{Kq_1}{\|\overrightarrow{p_1p_2}\|} \\ V_{q_2}(p_1) = \dfrac{Kq_2}{\|\overrightarrow{p_2p_1}\|} \end{cases}$$

Therefore, the internal potential energy (the total potential energy) of the two charges $\boldsymbol{q_1}$ *and* $\boldsymbol{q_2}$ *in positions* $\boldsymbol{p_1}$ *and* $\boldsymbol{p_2}$*, respectively, is equal to:*

$$U = Ep_{q_1/q_2} = Ep_{q_2/q_1} = q_2\, V_{q_1}(p_2) = q_1\, V_{q_2}(p_1) = \frac{1}{2}\left(q_2\, V_{q_1}(p_2) + q_1\, V_{q_2}(p_1)\right)$$

$$= \frac{Kq_1q_2}{\|\overrightarrow{p_1p_2}\|}$$

❖ *If we bring a third test charge "*q_3*" from infinity and want to place it in position* p_3 *near the first test charge "*q_1*" and the second test charge "*q_2*", then We have to apply three potential energies (works): the first to oppose and cancel the mutual effect between the two charges* q_1

and q_2,the second to oppose and cancel the mutual effect between the two charges q_1 and q_3, and the third to oppose and cancel the mutual effect between the two charges q_2 and q_3.

$$\begin{cases} Ep_{q_1/q_2} = q_2\, V_{q_1}(p_2) = Ep_{q_2/q_1} = q_1\, V_{q_2}(p_1) \text{ where } \begin{cases} V_{q_1}(p_2) = \dfrac{Kq_1}{\|\overrightarrow{p_1p_2}\|} \\ V_{q_2}(p_1) = \dfrac{Kq_2}{\|\overrightarrow{p_2p_1}\|} \end{cases} \\ Ep_{q_1/q_3} = q_3\, V_{q_1}(p_3) = Ep_{q_3/q_1} = q_1\, V_{q_3}(p_1) \text{ where } \begin{cases} V_{q_3}(p_1) = \dfrac{Kq_3}{\|\overrightarrow{p_3p_1}\|} \\ V_{q_1}(p_3) = \dfrac{Kq_1}{\|\overrightarrow{p_1p_3}\|} \end{cases} \\ Ep_{q_2/q_3} = q_3\, V_{q_2}(p_3) = Ep_{q_3/q_2} = q_2\, V_{q_3}(p_2) \text{ where } \begin{cases} V_{q_2}(p_3) = \dfrac{Kq_2}{\|\overrightarrow{p_2p_3}\|} \\ V_{q_3}(p_2) = \dfrac{Kq_3}{\|\overrightarrow{p_3p_2}\|} \end{cases} \end{cases}$$

Where $\boldsymbol{V_{q_i}(p_j)}$: represents the potential generated by the charge $\boldsymbol{q_i}$ at position $\boldsymbol{p_j}$.

$\boldsymbol{Ep_{q_i/q_j}}$: represents the potential energy resulting from the presence of charge $\boldsymbol{q_j}$ at position $\boldsymbol{q_j}$ within the potential resulting from charge $\boldsymbol{q_i}$.

Therefore, the internal potential energy (the total potential energy) the three charges $\boldsymbol{q_1}$, $\boldsymbol{q_2}$ and $\boldsymbol{q_3}$ in positions $\boldsymbol{p_1}$, $\boldsymbol{p_2}$, and $\boldsymbol{p_3}$ respectively, is equal to:

$$U = Ep_{q_1/q_2} + Ep_{q_1/q_3} + Ep_{q_2/q_3} = \frac{Kq_1q_2}{\|\overrightarrow{p_1p_2}\|} + \frac{Kq_1q_3}{\|\overrightarrow{p_1p_3}\|} + \frac{Kq_2q_3}{\|\overrightarrow{p_2p_3}\|}$$

*For a system of charges consisting of **N** point charges $\boldsymbol{q_i}$ located at $\boldsymbol{p_i}$, the total potential energy (internal energy) of this system is given by:*

$$U = \frac{1}{2}\sum_{i=1}^{N} K\boldsymbol{q_i} \sum_{j=1, i\neq j}^{N} \frac{\boldsymbol{q_j}}{\|\overrightarrow{p_ip_j}\|} = \sum_{i=1}^{N}\sum_{j=1, i\neq j}^{N} \frac{1}{2}\frac{K\boldsymbol{q_i}\boldsymbol{q_j}}{\|\overrightarrow{p_ip_j}\|}$$

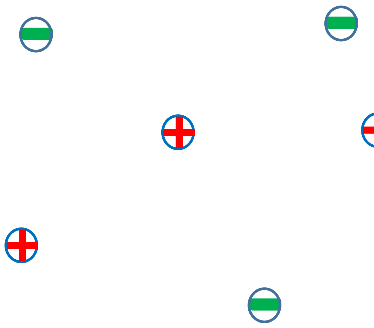
$$U = \sum_{i=1}^{N}\sum_{j=i+1}^{N} \frac{K\boldsymbol{q_i}\boldsymbol{q_j}}{\|\overrightarrow{\boldsymbol{p_ip_j}}\|}$$

I-8) The electric field and electric potential for a continuous Charge distribution

*We previously studied various cases of **discrete** charge distribution, where the studied system consists of a finite number of charged particles, which are considered as point charges since their dimensions are extremely small compared to the distances between the charges.*

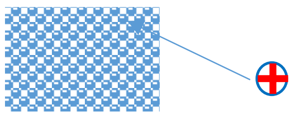
During this study, we calculated we calculated the electric potential and the electric field as fundamental electrical parameters.

In this section, we will explore the case of continuous charge distribution. ***Continuous charge distribution*** *can be considered a special case in which a very large number of charges is very large are distributed over a very small volume, surface, or distance (length). The distance between any two adjacent charges is extremely small, i.e., infinitesimal. As a result, it becomes impractical to treat charges as individual particles. Instead, we are compelled to use the concept of* ***charge density*** *to describe how charges are distributed within the system.*



Discrete charge distribution:

- ✓ *finite number of charged particles*
- ✓ *the dimensions of any particle charge are extremely small compared to the distances between the charges*



Continuous charge distribution:

- ✓ *Large number of charged particles The distance between any two adjacent charges is extremely small, i.e., infinitesimal*

I-8-1) Charge density distribution:

In general, the electric charge density $\boldsymbol{C_d}$ *in the case of a uniform continuous distribution is defined as the ratio of the total charge* $\boldsymbol{Q}$ *to the quantity* $\boldsymbol{A}$ *over which this charge is distributed, whether it is volume (*$\boldsymbol{V}$*), surface area (*$\boldsymbol{S}$*) or linear distance (*$\boldsymbol{L}$*). It can also be defined in terms of the differential (infinitesimal) charge* $\boldsymbol{dq}$ *and the differential (infinitesimal) quantity over which this charge is distributed* $\boldsymbol{dA}$*, so that it is expressed as follows:*

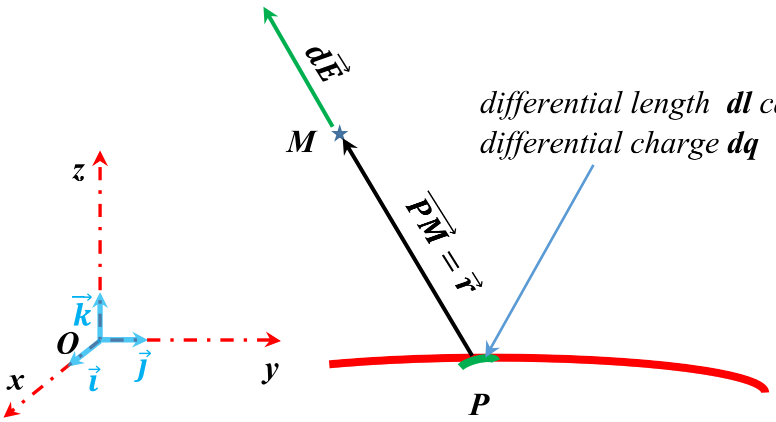
$$\boldsymbol{C_d = \frac{Q}{A} = \frac{dq}{dA}}$$

We can distinguish three types of charge distribution: linear charge density:

Linear charge density λ: *If the charge is uniformly distributed along the length or perimeter of an object. The expression for charge density, electric field, and electric potential at point M is written as follows:*

$$\lambda = \frac{Q}{L} = \frac{dq}{dl} \Rightarrow dq = \lambda\, dl$$

In this case, each differential charge dq carried by the differential length dl generates at the point M a differential electric field $d\vec{E}$ and a differential potential dV. To obtain the total field $\vec{E}$ and the total electric potential, we integrate the differential electric field $d\vec{E}$ and the differential electric potential dV.



The electric field

$$\vec{E}(M) = \int d\vec{E}(M) = \int \frac{Kdq}{\|\overrightarrow{PM}\|^3}\,\overrightarrow{PM} = \int \frac{Kdq}{r^3}\,\vec{r}\ ;\text{where } \overrightarrow{PM} = \vec{r} = r\,\vec{u_r}\ ;\ r = \|\overrightarrow{PM}\|$$

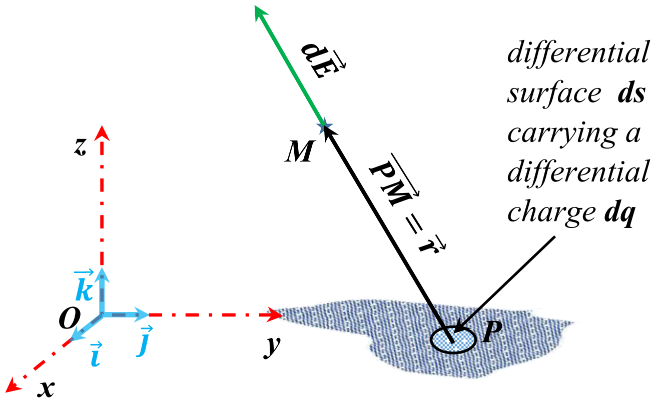
$$\vec{E}(M) = \int \frac{Kdq}{r^3}\,\vec{r} = \int \frac{Kdq}{r^2}\,\vec{u_r} = \int \frac{K\boldsymbol{\lambda}\,\boldsymbol{dl}}{r^2}\,\vec{u_r}$$

The electric potential

$$V(M) = \int dV(M) = \int \frac{Kdq}{\|\overrightarrow{PM}\|} = \int \frac{Kdq}{r} = \int \frac{K\boldsymbol{\lambda}\,\boldsymbol{dl}}{r}$$

Surface charge density σ: *If the charge is uniformly distributed over the surface area of an object. The expression for charge density, electric field, and electric potential at point M is written as follows:*

$$\sigma = \frac{Q}{S} = \frac{dq}{ds} \Rightarrow dq = \sigma\, ds$$



The electric field

$$\vec{E}(M) = \int d\vec{E}(M) = \int \frac{Kdq}{\|\overrightarrow{PM}\|^3}\,\overrightarrow{PM} = \int \frac{Kdq}{r^3}\,\vec{r}\ ;where\ \overrightarrow{PM} = \vec{r} = r\,\vec{u_r}\ ;\ r = \|\overrightarrow{PM}\|$$

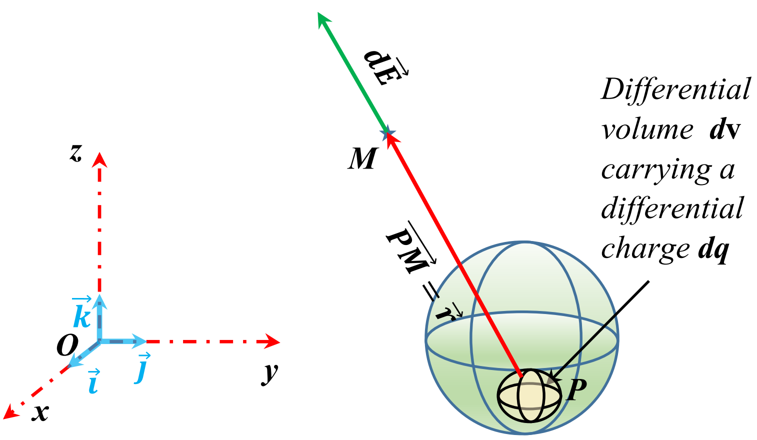
$$\vec{E}(M) = \int \frac{Kdq}{r^3}\,\vec{r} = \int \frac{Kdq}{r^2}\,\vec{u_r} = \int \frac{K\boldsymbol{\sigma}\,\mathbf{ds}}{r^2}\,\vec{u_r}$$

The electric potential

$$V(M) = \int dV(M) = \int \frac{Kdq}{\|\overrightarrow{PM}\|} = \int \frac{Kdq}{r} = \int \frac{K\boldsymbol{\sigma}\,\mathbf{ds}}{r}$$

Volume charge density ρ : *If the charge is uniformly distributed throughout the volume of an object. The expression for charge density, electric field, and electric potential at point M is written as follows:*

$$\boldsymbol{\rho} = \frac{\mathbf{Q}}{\mathbf{V}} = \frac{\mathbf{dq}}{\mathbf{dv}} \Rightarrow \mathbf{dq} = \boldsymbol{\rho}\,\mathbf{dv}$$



The electric field

$$\vec{E}(M) = \int d\vec{E}(M) = \int \frac{Kdq}{\|\overrightarrow{PM}\|^3}\,\overrightarrow{PM} = \int \frac{Kdq}{r^3}\,\vec{r}\ ;where\ \overrightarrow{PM} = \vec{r} = r\,\overrightarrow{u_r}\,;\ r = \|\overrightarrow{PM}\|$$

$$\vec{E}(M) = \int \frac{Kdq}{r^3}\,\vec{r} = \int \frac{Kdq}{r^2}\,\overrightarrow{u_r} = \int \frac{K\boldsymbol{\rho}\,\mathbf{dv}}{r^2}\,\overrightarrow{u_r}$$

The electric potential

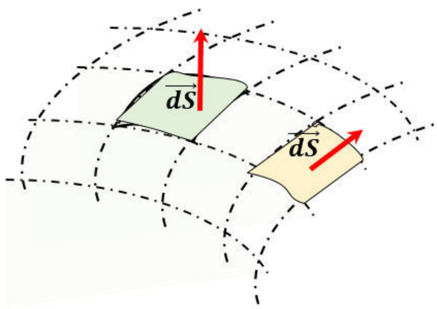
$$V(M) = \int dV(M) = \int \frac{Kdq}{\|\overrightarrow{PM}\|} = \int \frac{Kdq}{r} = \int \frac{K\boldsymbol{\rho}\,\mathbf{dv}}{r}$$

I-9) Gauss's theorem

I-9-1) Electric Flux

Electric field flux Φ is a scalar physical quantity that expresses ***the number of field lines penetrating a given surface****. It can be calculated by dividing the surface into infinitesimal surfaces dS (differential surfaces), such that the electric field vector $\vec{E}$ is considered uniform (constant in both magnitude and direction) on each differential surface.*

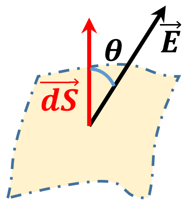
The differential surface vector $\overrightarrow{dS}$ is a vector whose magnitude is equal to the area of the differential surface and whose direction is perpendicular to the surface and directed outward.

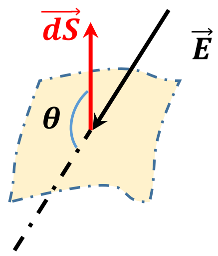


The differential electric field flux $d\Phi$ is expressed by the scalar product of the electric field vector $\vec{E}$ and the differential surface vector $\overrightarrow{dS}$.

$$d\Phi = \vec{E}.\overrightarrow{dS} = \|\vec{E}\|.\|\overrightarrow{dS}\|\cos\theta \Rightarrow \Phi = \oint E\ dS\cos\theta$$

Where θ is the angle between the vectors $\vec{E}$ and $\overrightarrow{dS}$, and $\oint$ symbol means an integration over a closed surface



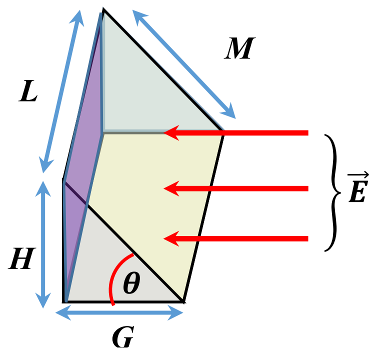


Depending on the value of the angle $\boldsymbol{\theta}$*, we can distinguish three states of electric field flux* $\boldsymbol{\Phi}$*:*

$$\boldsymbol{\Phi} = \begin{cases} > 0;\ if\ \theta < \pi \ \Rightarrow \vec{E}\ crosses\ the\ surface\ from\ the\ inside\ to\ the\ outside \\ zero; if\ \ \theta = \dfrac{\pi}{2} \quad \Rightarrow \vec{E}\ grazes\ the\ surface \\ n < 0;\ if\ \dfrac{\pi}{2} < \theta < 2\pi \Rightarrow \vec{E}\ crosses\ the\ surface\ from\ the\ outside\ to\ the\ inside \end{cases}$$

Example

Calculate the electric field flux through the surfaces of the geometric shown in the opposite figure, assuming that the uniform electric field (constant $\boldsymbol{E} = \boldsymbol{constant}$*).*



Solution:

The electric flux through this geometric soli dis equal to the sum of the fluxes through its five surfaces, each of them is calculated as follows:

$$\Phi_{Tot} = \sum_i \Phi_i = \Phi_{\vec{E}/S_1} + \Phi_{\vec{E}/S_2} + \Phi_{\vec{E}/S_3} + \Phi_{\vec{E}/S_4} + \Phi_{\vec{E}/S_5}$$

Surface 1

$$\Phi_{\vec{E}/S_1} = \int \vec{E}.\overrightarrow{dS_1} = E\ \times S_1 \times \cos\theta_1\ ;\begin{cases} S_1 = M \times L \\ \theta_1 = \dfrac{\pi}{2} + \theta \end{cases} \Rightarrow \Phi_{\vec{E}/S_1} = E\ \times M \times L \times \cos\left(\frac{\pi}{2} + \theta\right)$$

Surface 2

$$\Phi_{\vec{E}/S_2} = \int \vec{E}.\overrightarrow{dS_2} = E\ \times S_2 \times \cos\theta_2\ ;\begin{cases} S_2 = H \times L \\ \theta_2 = 0 \end{cases} \Rightarrow \Phi_{\vec{E}/S_2} = E\ \times H \times L$$

Surface 3

$$\Phi_{\vec{E}/S_3} = \int \vec{E}.\overrightarrow{dS_3} = E\ \times S_3 \times \cos\theta_3\ ;\begin{cases} S_3 = G \times L \\ \theta_3 = \dfrac{\pi}{2} \end{cases} \Rightarrow \Phi_{\vec{E}/S_3} = 0$$

Surface 4

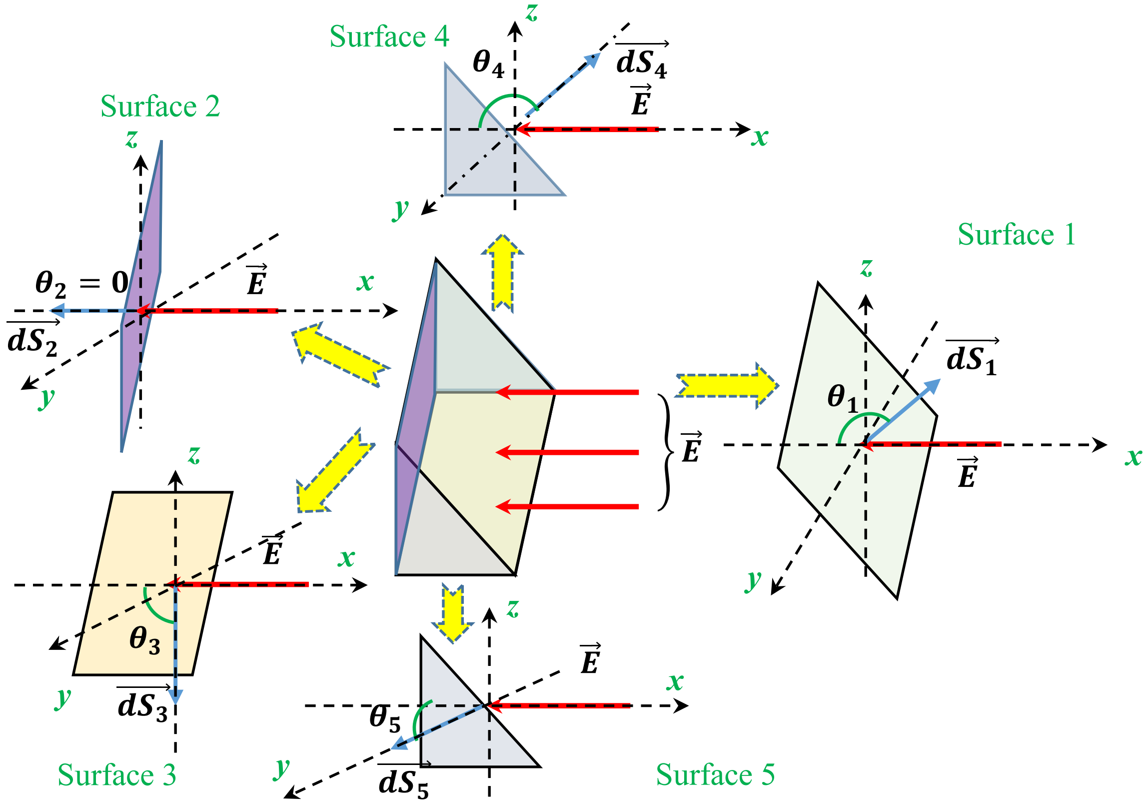
$$\Phi_{\vec{E}/S_4} = \int \vec{E}.\overrightarrow{dS_4} = E \times S_4 \times \cos\theta_4 \ ; \begin{cases} S_4 = (G \times H)/2 \\ \theta_4 = \dfrac{\pi}{2} \end{cases} \Rightarrow \Phi_{\vec{E}/S_4} = 0$$

Surface 5

$$\Phi_{\vec{E}/S_5} = \int \vec{E}.\overrightarrow{dS_5} = E \times S_5 \times \cos\theta_5 \ ; \begin{cases} S_5 = (G \times H)/2 \\ \theta_5 = \dfrac{\pi}{2} \end{cases} \Rightarrow \Phi_{\vec{E}/S_5} = 0$$

$$\Phi_{Tot} = \sum_i \Phi_i = E \times M \times L \times \cos\left(\frac{\pi}{2} + \theta\right) + E \times H \times L + 0 + 0 + 0$$

$$\Phi_{Tot} = \sum_i \Phi_i = E \times L \times \left(M \times \cos\left(\frac{\pi}{2} + \theta\right) + H\right)$$



I-9-2) Gauss's Law

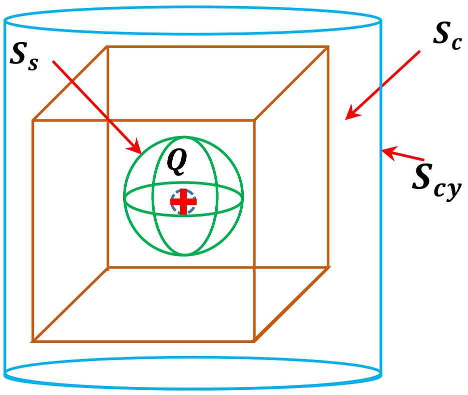
Gauss's theorem states that the electric field flux $\Phi_{\vec{E}}$ through a closed surface S (the imaginary Gauss surface) is equal to the algebraic sum of the electric charges Q_{in} inside this surface divided by the permittivity of free space ε_0. Gauss's theorem is expressed by the following mathematical expression:

$$\Phi_{\vec{E}} = \oint \vec{E}.\overrightarrow{dS} = \frac{\sum Q_{in}}{\varepsilon_0}$$

Applying this mathematical expression requires considering several important points, such as the optimal choice of the imaginary Gaussian surface, ensuring that it enclosed the point where the electric field is to be calculated, so that the value of the electric field over the surface remains constant.

Example:

We place a positive charge inside a sphere, then we place the sphere inside a cube, and finally place the cube inside a cylinder as shown in the figure bellow. Compare the electric field flux through the surface of the sphere, the electric field flux through the surface of the cube, and the electric field flux through the surface of the cylinder. What do you conclude and how do you explain it?



- *The electric field flux through the surface of the sphere* $\boldsymbol{S_s}$

$$\Phi_{\vec{E}/S_s} = \oint \vec{E}.\overrightarrow{dS_s} = \sum_i \frac{Q_{in}}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$$

- *The electric field flux through the surface of the cube* $\boldsymbol{S_c}$

$$\Phi_{\vec{E}/S_c} = \oint \vec{E}.\overrightarrow{dS_c} = \sum_i \frac{Q_{in}}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$$

♣ *The electric field flux through the surface of the cylinder* $\boldsymbol{S_{cy}}$

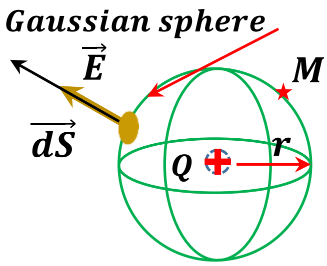
$$\Phi_{\vec{E}/S_{cy}} = \oint \vec{E}.\overrightarrow{dS_{cy}} = \sum_i \frac{Q_{in}}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$$

We note that the field flux through these different closed surfaces is equal ($\Phi_{\vec{E}/S_s} = \Phi_{\vec{E}/S_c} = \Phi_{\vec{E}/S_{cy}}$*), and we conclude that the field flux through the closed surface is related to the total charge inside within it. This can be explained by the fact that the electric field lines passing through the first closed surface are the same in number as those passing through the second and third closed surfaces.*

I-9-3) Some applications of Gauss's theorem

a) Application of Gauss's theorem to calculate the electric field and potential of a positive point charge q

To calculate the electric field generated at a point M in the space surrounding a positive point charge, we assume the existence of an imaginary Gaussian surface in the shape of sphere with radius r, where r represents the distance between the point M and the point charge Q (i.e., point M belongs (lies) to the imaginary Gaussian surface).



By applying Gauss's theorem and calculating the electric field flux through this imaginary Gaussian surface, we get:

$$\Phi_{\vec{E}/S_s} = \oint \vec{E}.\overrightarrow{dS} = \sum_i \frac{Q_{in}}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$$

$$\begin{cases} \vec{E}.\overrightarrow{dS} = E.S = E.4\pi r^2 \\ \sum_i \frac{Q_{in}}{\varepsilon_0} = \frac{+Q}{\varepsilon_0} \end{cases} \Rightarrow E.4\pi r^2 = \frac{+Q}{\varepsilon_0} \Rightarrow E = \frac{+Q}{4\pi r^2 \varepsilon_0}$$

$$K = \frac{1}{4\pi\varepsilon_0} \Rightarrow E = \frac{KQ}{r^2}$$

The value of the electric field obtained by Gauss's theorem is the same as the value obtained by applying Coulomb's law.

After that, we proceed to calculate the electric potential based on the electric field expression in spherical coordinates as follows:

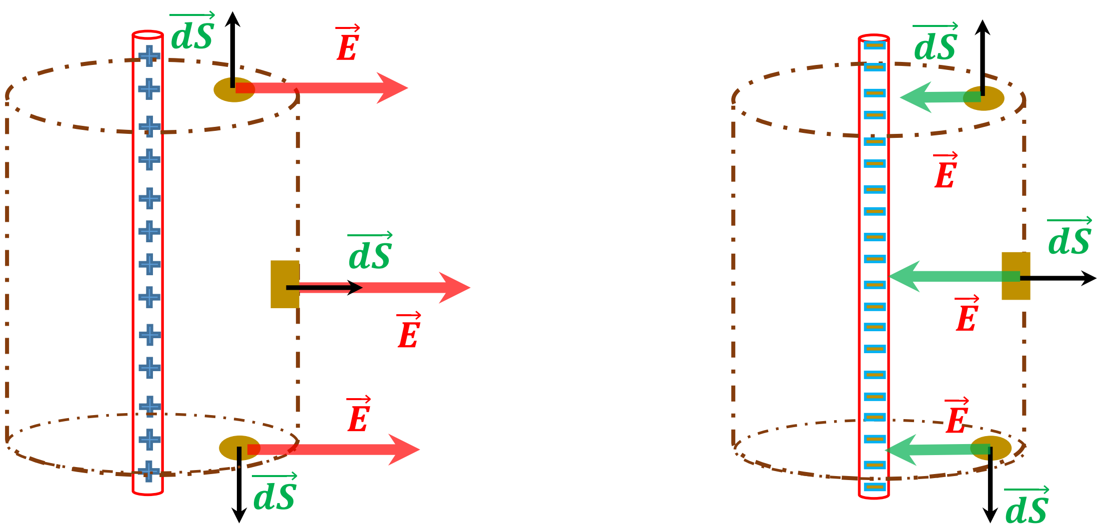
$$dV = -\vec{E}.d\vec{r} \Rightarrow V = -\int \frac{KQ}{r^2}\,dr = \frac{KQ}{r} + C$$

$$\lim_{r\to\infty} V\,(r) = 0 \;\Rightarrow C = 0$$

$$V = \frac{KQ}{r}$$

b) Application of Gauss's theorem to determine the electric field and the electric potential of an infinitely long wire with uniform linear charge density

Let us consider an infinitely long thin rod with a uniform positive or negative linear charge λ, as illustrated in the figure bellow. To calculate the electric field at a point M in the space surrounding this wire, we employ Gauss's theorem. We assume an imaginary Gaussian surface in the form of a cylinder with a height L and base radius r. The total electric field flux through this cylinder $\Phi_{\vec{E}/Cy}$ is the sum of the electric field flux through the lateral surface $\Phi_{\vec{E}/L}$, the electric field flux through the bottom base surface $\Phi_{\vec{E}/S_B}$, and the electric field flux through the top base surface $\Phi_{\vec{E}/S_T}$, and is calculated as follows:



$for\ \lambda > 0$ $\qquad$ $for\ \lambda < 0$

for positive chage denisty (λ)

$$\Phi_{\vec{E}/S_L} = \int \vec{E}.\overrightarrow{dS_L} = \int E.S_L \cos 0 = E.S_L$$

$$S_L = 2\pi\, r\, L$$

$$\Phi_{\vec{E}/S_L} = E\, 2\pi\, r\, L$$

$$\Phi_{\vec{E}/S_B} = \int \vec{E}.\overrightarrow{dS_B} = \int E.S_B \cos\frac{\pi}{2} = 0$$

$$\Phi_{\vec{E}/T} = \int \vec{E}.\overrightarrow{dS_T} = \int E.S_T \cos\frac{\pi}{2} = 0$$

$$\Phi_{\vec{E}/Cy} = \Phi_{\vec{E}/S_L} + \Phi_{\vec{E}/S_B} + \Phi_{\vec{E}/S_T}$$

$$\Phi_{\vec{E}/Cy} = E\, 2\pi\, r\, L$$

$$\Phi_{\vec{E}/Cy} = \int \vec{E}.\overrightarrow{dS} = \frac{1}{\varepsilon_0}\sum_i Q_{in}$$

$$\sum_i Q_{in} = \boldsymbol{Q} \Rightarrow \boldsymbol{\lambda} = \frac{\boldsymbol{Q}}{\boldsymbol{L}} \Rightarrow \boldsymbol{Q} = \boldsymbol{\lambda}.\boldsymbol{L}$$

$$E\, 2\pi\, r\, L = \frac{\boldsymbol{\lambda}.\boldsymbol{L}}{\varepsilon_0}$$

$$E = \frac{\boldsymbol{\lambda}}{2\pi\, r\, \varepsilon_0}$$

for nagative chage denisty $(-\lambda)$

$$\Phi_{\vec{E}/S_L} = \int \vec{E}.\overrightarrow{dS_L} = \int E.S_L \cos\pi = -E.S_L$$

$$S_L = 2\pi\, r\, L$$

$$\Phi_{\vec{E}/S_L} = -E\, 2\pi\, r\, L$$

$$\Phi_{\vec{E}/S_B} = \int \vec{E}.\overrightarrow{dS_B} = \int E.S_B \cos\frac{\pi}{2} = 0$$

$$\Phi_{\vec{E}/T} = \int \vec{E}.\overrightarrow{dS_T} = \int E.S_T \cos\frac{\pi}{2} = 0$$

$$\Phi_{\vec{E}/Cy} = \Phi_{\vec{E}/S_L} + \Phi_{\vec{E}/S_B} + \Phi_{\vec{E}/S_T}$$

$$\Phi_{\vec{E}/Cy} = E\, 2\pi\, r\, L$$

$$\Phi_{\vec{E}/Cy} = \int \vec{E}.\overrightarrow{dS} = \frac{1}{\varepsilon_0}\sum_i Q_{in}$$

$$\sum_i Q_{in} = \boldsymbol{Q} \Rightarrow -\boldsymbol{\lambda} = \frac{\boldsymbol{Q}}{\boldsymbol{L}} \Rightarrow \boldsymbol{Q} = -\boldsymbol{\lambda}.\boldsymbol{L}$$

$$-E\, 2\pi\, r\, L = \frac{-\boldsymbol{\lambda}.\boldsymbol{L}}{\varepsilon_0}$$

$$E = \frac{\boldsymbol{\lambda}}{2\pi\, r\, \varepsilon_0}$$

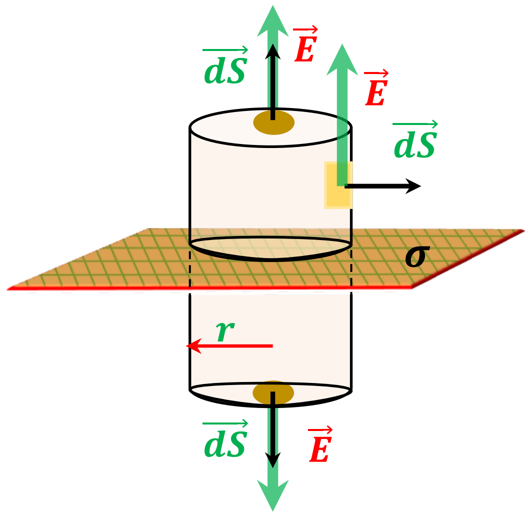
The electric potential V(r) at any point M in the space surrounding the wire is derived as follows from the expression electric field:

$$dV = -\vec{E}.d\vec{r} \Rightarrow V(r) = -\int \frac{\boldsymbol{\lambda}}{2\pi\, r\, \varepsilon_0}\, dr = \frac{-\boldsymbol{\lambda}}{2\pi\ \varepsilon_0}\ln r + C = \frac{\boldsymbol{\lambda}}{2\pi\ \varepsilon_0}\ln\frac{1}{r} + C$$

Where the constant C is determined according to a known reference potential at a given point.

c) Application of Gauss's theorem to determine the electric field and the electric potential of an infinitely large uniformly charged plane.

Deriving the expression for the electric field generated by an infinitely large, uniformly charged plane is done by choosing a Gaussian surface in the form of a cylinder with a base radius r and height L.



The total electric field flux is calculated by considering the contributions from the bottom $\Phi_{\vec{E}/B}$ and top $\Phi_{\vec{E}/T}$ bases of the cylinder, as well as through the lateral surface $\Phi_{\vec{E}/L}$. By applying Gauss's theorem, we obtain the following results:

$$\Phi_{\vec{E}/S_L} = \int \vec{E}.\overrightarrow{dS_L} = \int E.S_L \cos\frac{\pi}{2} = 0$$

$$\Phi_{\vec{E}/S_B} = \int \vec{E}.\overrightarrow{dS_B} = \int E.S_B \cos 0 = E.S_B$$

$$\Phi_{\vec{E}/T} = \int \vec{E}.\overrightarrow{dS_T} = \int E.S_T \cos 0 = E.S_T$$

$$\Phi_{\vec{E}/Cy} = \Phi_{\vec{E}/S_L} + \Phi_{\vec{E}/S_B} + \Phi_{\vec{E}/S_T}$$

$$\Phi_{\vec{E}/Cy} = E.S_B + E.S_T + 0 = 2\,E.S_B = E.2\,\pi r^2 \quad \textit{where } S_B = S_T = \pi r^2$$

$$\Phi_{\vec{E}/Cy} = \oint \vec{E}.\vec{dS} = \frac{1}{\varepsilon_0}\sum_i Q_{in}$$

$$\sum_i Q_{in} = \boldsymbol{Q} \Rightarrow \boldsymbol{\sigma} = \frac{\boldsymbol{Q}}{\boldsymbol{S}} \Rightarrow \boldsymbol{Q} = \boldsymbol{\sigma}.\boldsymbol{S} = \boldsymbol{\sigma}.\pi r^2$$

$$E.2\,\pi r^2 = \frac{\boldsymbol{\sigma}.\pi r^2}{\varepsilon_0} \Rightarrow E = \frac{\boldsymbol{\sigma}}{\boldsymbol{2}\,\varepsilon_0}$$

The electric potential V(r) at any point M in the space surrounding the wire is derived as follows from the expression electric field:

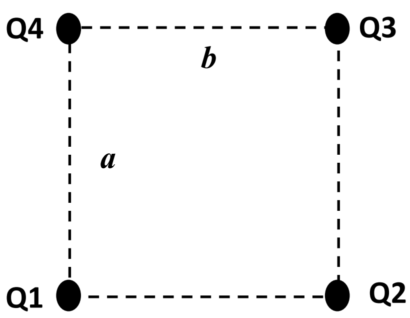
$$dV = -\vec{E}.d\vec{r} \Rightarrow V(r) = -\int \frac{\sigma}{2\,\varepsilon_0}\,dz = \frac{\sigma\,z}{2\,\varepsilon_0} + C$$

The constant c is determined depending on certain condition.

I-10) Solved exercises

Exercise 01:

Four charges Q_1, Q_2, Q_3, and Q_4 ($Q_1 = Q_2 = Q_3 = -1$ µC, $Q_4 = 2$µC) are arranged as shown in the opposite figure.



1) *Determine the type of interaction between each pair of charges.*
2) *Represent on the figure the electric forces applied to the electric charge Q_3*
3) *Write the analytical expression for each electric force acting on the electric charge Q_3*
4) *Calculate the net force applied to charge Q_3. ($Q_1 = Q_2 = Q_3 = 1$ µC, $Q_4 = 2$µC)*

Solution:

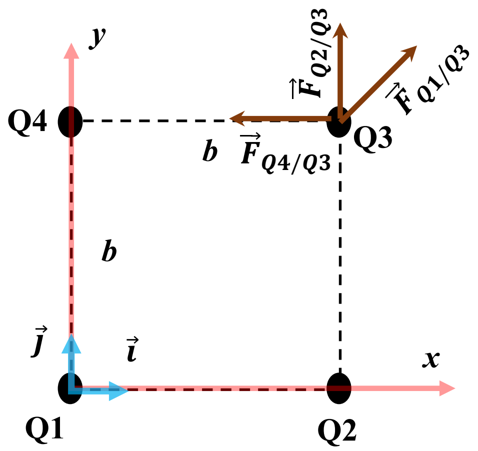
1- *Determine the type of interaction between each pair of charges*

Like charges repel each other, while opposite charges attract each other

$$\begin{cases} Q_1 \\ Q_2 \end{cases} \text{Repulsive} \quad \begin{cases} Q_1 \\ Q_3 \end{cases} \text{Repulsive} \quad \begin{cases} Q_1 \\ Q_4 \end{cases} \text{Attractive}$$

$$\begin{cases} Q_2 \\ Q_3 \end{cases} \text{Repulsive} \quad \begin{cases} Q_2 \\ Q_4 \end{cases} \text{Attractive} \quad \begin{cases} Q_3 \\ Q_4 \end{cases} \text{Attractive}$$

2- Represent on the figure the electric forces applied to the electric charge Q_3



3- Write the analytical expression for each electric force acting on the electric charge Q_3

$$\begin{cases} \overrightarrow{F_{Q1/Q3}} = k \dfrac{Q_1 Q_3}{\left\|\overrightarrow{M_1M_3}\right\|^3} \overrightarrow{M_1M_3} \\ \overrightarrow{F_{Q2/Q3}} = k \dfrac{Q_2 Q_3}{\left\|\overrightarrow{M_2M_3}\right\|^3} \overrightarrow{M_2M_3} \\ \overrightarrow{F_{Q4/Q3}} = k \dfrac{Q_4 Q_3}{\left\|\overrightarrow{M_4M_3}\right\|^3} \overrightarrow{M_4M_3} \end{cases}$$

$$M_1(0\,;\,0) \quad M_2(b\,;\,0) \quad M_3(b\,;\,b) \quad M_4(0\,;\,b)$$

$$\overrightarrow{M_1M_3} = (x_3 - x_1)\vec{\imath} + (y_3 - y_1)\vec{\jmath} = b\,\vec{\imath} + b\,\vec{\jmath} \;\; ; \left\|\overrightarrow{M_1M_3}\right\| = \sqrt{b^2 + (b)^2} = b\sqrt{2}$$

$$\overrightarrow{M_2M_3} = (x_3 - x_2)\vec{\imath} + (y_3 - y_2)\vec{\jmath} = b\,\vec{\jmath} \;\; ; \left\|\overrightarrow{M_2M_3}\right\| = b$$

$$\overrightarrow{M_4M_3} = (x_3 - x_4)\vec{\imath} + (y_3 - y_4)\vec{\jmath} = b\,\vec{\imath} \;\; ; \left\|\overrightarrow{M_4M_3}\right\| = b$$

$$Q1 = Q2 = Q3 = -1\,\mu C, \quad Q4 = 2\mu C$$

$$\begin{cases} \overrightarrow{F_{Q1/Q3}} = k \dfrac{Q_1 Q_3}{\left(b\sqrt{2}\right)^3} (b\,\vec{\imath} + b\,\vec{\jmath}) = 9 \times 10^9 \dfrac{(-1 \times 10^{-6})(-1 \times 10^{-6})}{2\sqrt{2}\,b^2} (\vec{\imath} + \vec{\jmath}) \\ \overrightarrow{F_{Q2/Q3}} = k \dfrac{Q_2 Q_3}{(b)^3} (b\,\vec{\jmath}) = 9 \times 10^9 \dfrac{(-1 \times 10^{-6})(-1 \times 10^{-6})}{b^2}\,\vec{\jmath} \\ \overrightarrow{F_{Q4/Q3}} = k \dfrac{Q_4 Q_3}{(b)^3} (b\,\vec{\imath}) = 9 \times 10^9 \dfrac{(-1 \times 10^{-6})(2 \times 10^{-6})}{b^2}\,\vec{\imath} \end{cases}$$

$$\begin{cases} \overrightarrow{F_{Q1/Q3}} = \dfrac{3.182 \times 10^{-3}}{b^2} (\vec{\imath} + \vec{\jmath}) \\ \overrightarrow{F_{Q2/Q3}} = \dfrac{9 \times 10^{-3}}{b^2} \vec{\jmath} \\ \overrightarrow{F_{Q4/Q3}} = \dfrac{-18 \times 10^{-3}}{b^2} \vec{\imath} \end{cases}$$

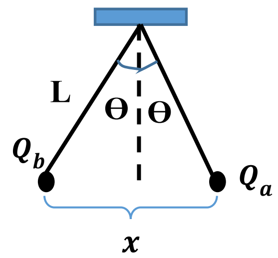
4- *The net force acting on the charge* $\boldsymbol{Q_3}$

$$\overrightarrow{F_{net/\boldsymbol{Q_3}}} = \overrightarrow{F_{Q1/Q3}} + \overrightarrow{F_{Q2/Q3}} + \overrightarrow{F_{Q4/Q3}} = \frac{3.182 \times 10^{-3}}{b^2} (\vec{\imath} + \vec{\jmath}) + \frac{9 \times 10^{-3}}{b^2} \vec{\jmath} + \frac{-18 \times 10^{-3}}{b^2} \vec{\imath}$$

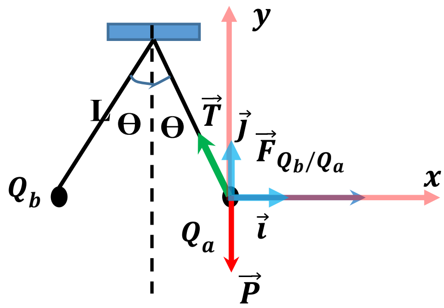
$$= \frac{10^{-3}}{b^2}[(3.182 - 18)\vec{\imath} + (3.182 + 9)\vec{\jmath}] = \frac{10^{-3}}{b^2}[(-14.818)\vec{\imath} + (12.182)\vec{\jmath}]$$

Exercise 02:

Two tiny conducting balls of identical mass m and identical charge $\boldsymbol{Q_a = Q_b = q}$ *hang from non-conducting threads of length L. Assume that θ is so small that tan θ can be replaced by sin θ; show that, for equilibrium:* $x = (q^2 L / mg 2\pi\varepsilon_0)^{1/3}$



Exercise 03:



$$\sum \vec{F}_{ext} = m\,\vec{a} = \vec{o} \Rightarrow \vec{P} + \vec{T} + \vec{F}_{Q_b/Q_a} = \vec{o}$$

$$\begin{cases} \vec{P} = -m\,g\,\vec{\jmath} \\ \vec{T} = -T\,\sin\theta\,\,\vec{\imath} + T\cos\theta\,\vec{\jmath} \\ \vec{F}_{Q_b/Q_a} = k\,\dfrac{Q_b Q_a}{\left\|\overrightarrow{M_b M_a}\right\|^3}\,\overrightarrow{M_b M_a} \end{cases}$$

$$M_a(0\,;\,0) \quad M_b(-x\,;\,0)$$

$$\overrightarrow{M_b M_a} = (x_a - x_b)\vec{\imath} + (y_a - y_b)\vec{\jmath} = x\,\vec{\imath} \qquad ; \left\|\overrightarrow{M_b M_a}\right\| = x$$

$$\vec{F}_{Q_b/Q_a} = k\,\frac{Q_b Q_a}{\left\|\overrightarrow{M_b M_a}\right\|^3}\,\overrightarrow{M_b M_a} = k\,\frac{q\,q}{x^3}\,x\,\vec{\imath} = k\,\frac{q^2}{x^2}\,\vec{\imath}$$

$$\vec{P} + \vec{T} + \vec{F}_{Q_b/Q_a} = \vec{o} \Rightarrow -m\,g\,\vec{\jmath} - T\,\sin\theta\,\vec{\imath} + T\cos\theta\,\vec{\jmath} + k\,\frac{q^2}{x^2}\,\vec{\imath} = \vec{o}$$

$$(-m\,g + T\cos\theta)\vec{\jmath} + \left(-T\,\sin\theta + k\,\frac{q^2}{x^2}\right)\vec{\imath} = \vec{o} \Rightarrow \begin{cases} -T\,\sin\theta + k\,\dfrac{q^2}{x^2} = 0 \\ -m\,g + T\cos\theta = 0 \end{cases}$$

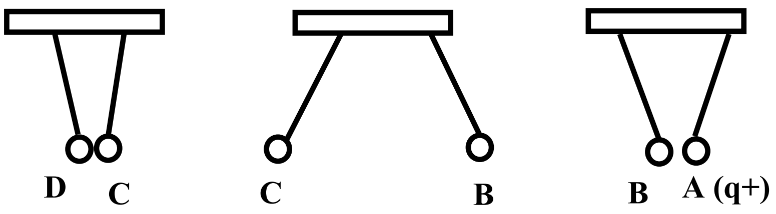
$$\begin{cases} T\,\sin\theta = k\,\dfrac{q^2}{x^2} \\ T\cos\theta = m\,g \end{cases} \Rightarrow \tan\theta = \frac{k\,\frac{q^2}{x^2}}{mg} = \frac{k\,q^2}{mg\,x^2}$$

For small angles: $\tan\theta \approx \sin\theta = \frac{x}{2L}$

$$\frac{k\,q^2}{mg\,x^2} = \frac{x}{2L} \Rightarrow x^3 = \frac{2\,L\,k\,q^2}{mg} \Rightarrow x = \sqrt[3]{\frac{2\,L\,k\,q^2}{mg}} = \left(\frac{2\,L\,k\,q^2}{mg}\right)^{\frac{1}{3}}$$

Exercise 03:

What is the type of the electrical charges carried by the balls shown in the figure?



Solution:

The type of the electrical charges carried by the balls

Like charges repel each other, while opposite charges attract each other

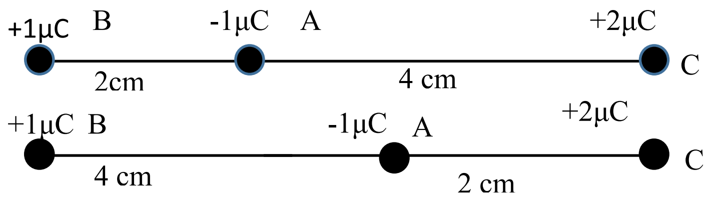
$$\begin{cases} A\,(+) \\ B \end{cases} \quad Attraction \Rightarrow B(-)$$

$$\begin{cases} B(-) \\ C \end{cases} \quad Repulsive \Rightarrow C(-)$$

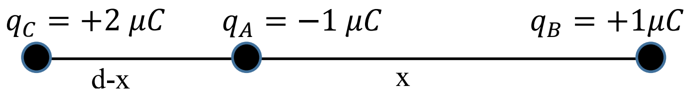
$$\begin{cases} C(-) \\ D \end{cases} \quad Attractive \Rightarrow D(+)$$

Exercise 04:

1) *What is the net force on charge A in each configuration shown in the opposite picture?*



2) *We want to put the charge $q_a = -1\ \mu C$ between two charges q_c and q_b (on the line joining*

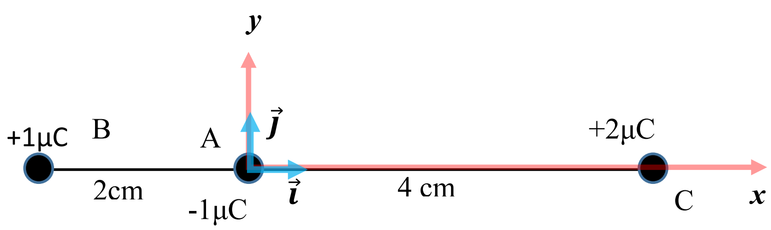


the two charges as shown figure below) where the applied force from $q_b = +1\mu C$ charge on the charge q_a is cancelled by the force from the charge $q_c = +2\ \mu C$ charge. Since forces are vectors,. We will assume that the 1 μC charge is some distance x from the +1μC. Calculate the distance x between q_a and q_b

Solution:

1- *The net force on charge A*

1st case



$$\overrightarrow{F_{net/A}} = \overrightarrow{F_{B/A}} + \overrightarrow{F_{C/A}}$$

$$\begin{cases} \overrightarrow{F_{B/A}} = k \dfrac{q_B q_A}{\left\|\overrightarrow{BA}\right\|^3} \overrightarrow{BA} \\ \overrightarrow{F_{C/A}} = k \dfrac{q_C q_A}{\left\|\overrightarrow{CA}\right\|^3} \overrightarrow{CA} \end{cases}$$

$$A(0\,;\ 0) \quad B(-0.02\,;\ 0) \quad C(0.04\,;\ 0)$$

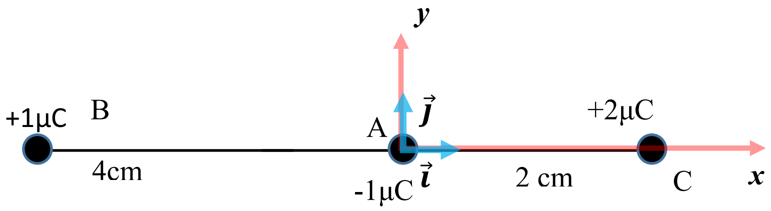
$$\overrightarrow{BA} = 0.02\,\vec{\imath} \ \ ; \left\|\overrightarrow{BA}\right\| = 0.02$$

$$\overrightarrow{CA} = -0.04\,\vec{\imath} \ \ ; \left\|\overrightarrow{CA}\right\| = 0.04$$

$$\begin{cases} \overrightarrow{F_{B/A}} = 9 \times 10^9 \dfrac{(1 \times 10^{-6})(-1 \times 10^{-6})}{(0.02)^3} 0.02\,\vec{\imath} = -22.5\,\vec{\imath} \\ \overrightarrow{F_{C/A}} = 9 \times 10^9 \dfrac{(2 \times 10^{-6})(-1 \times 10^{-6})}{(0.04)^3} (-0.04\,\vec{\imath}) = 11.25\,\vec{\imath} \end{cases}$$

$$\overrightarrow{F_{net/A}} = \overrightarrow{F_{B/A}} + \overrightarrow{F_{C/A}} = -22.5\,\vec{\imath} + 11.25\,\vec{\imath} = -11.25\,\vec{\imath}$$

2nd case



$$\overrightarrow{F_{net/A}} = \overrightarrow{F_{B/A}} + \overrightarrow{F_{C/A}}$$

$$\begin{cases} \overrightarrow{F_{B/A}} = k\,\dfrac{q_B q_A}{\|\overrightarrow{BA}\|^3}\,\overrightarrow{BA} \\ \overrightarrow{F_{C/A}} = k\,\dfrac{q_C q_A}{\|\overrightarrow{CA}\|^3}\,\overrightarrow{CA} \end{cases}$$

$$A(0\,;\,0) \quad B(-0.04\,;\,0) \quad C(0.02\,;\,0)$$

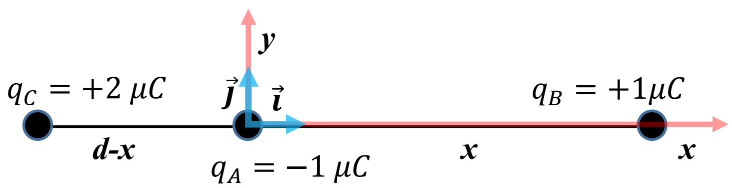
$$\overrightarrow{BA} = 0.04\,\vec{\imath} \;\; ; \|\overrightarrow{BA}\| = 0.04$$

$$\overrightarrow{CA} = -0.02\,\vec{\imath} \;\; ; \|\overrightarrow{CA}\| = 0.02$$

$$\begin{cases} \overrightarrow{F_{B/A}} = 9\times10^9\,\dfrac{(1\times10^{-6})(-1\times10^{-6})}{(0.04)^3}\,0.04\,\vec{\imath} = -5.625\,\vec{\imath} \\ \overrightarrow{F_{C/A}} = 9\times10^9\,\dfrac{(2\times10^{-6})(-1\times10^{-6})}{(0.02)^3}\,(-0.02\,\vec{\imath}) = 45\,\vec{\imath} \end{cases}$$

$$\overrightarrow{F_{net/A}} = \overrightarrow{F_{B/A}} + \overrightarrow{F_{C/A}} = -5.625\,\vec{\imath} + 45\,\vec{\imath} = 39.375\,\vec{\imath}$$

2- *Calculate the distance x between* q_a *and* q_b



$$\begin{cases} \overrightarrow{F_{B/A}} = k\,\dfrac{q_B q_A}{\|\overrightarrow{BA}\|^3}\,\overrightarrow{BA} \\ \overrightarrow{F_{C/A}} = k\,\dfrac{q_C q_A}{\|\overrightarrow{CA}\|^3}\,\overrightarrow{CA} \end{cases}$$

$$A(0\,;\,0) \quad B(x\,;\,0) \quad C(-d+x\,;\,0)$$

$$\overrightarrow{BA} = (-x)\,\vec{\imath} \;\; ; \|\overrightarrow{BA}\| = x$$

$$\overrightarrow{CA} = (d-x)\,\vec{\imath} \;\; ; \|\overrightarrow{CA}\| = d-x$$

$$\begin{cases} \overrightarrow{F_{B/A}} = 9\times10^9 \dfrac{(1\times10^{-6})(-1\times10^{-6})}{(x)^3}(-x)\,\vec{\imath} = \dfrac{9\times10^9}{x^2}\,\vec{\imath} \\ \overrightarrow{F_{C/A}} = 9\times10^9 \dfrac{(2\times10^{-6})(-1\times10^{-6})}{(d-x)^3}(d-x)\,\vec{\imath} = \dfrac{-18\times10^9}{(d-x)^2}\,\vec{\imath} \end{cases}$$

$$\overrightarrow{F_{net/A}} = \vec{0} \Rightarrow \overrightarrow{F_{B/A}} = -\overrightarrow{F_{C/A}}$$

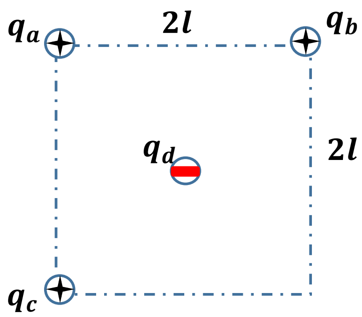
$$\frac{9\times10^9}{x^2}\,\vec{\imath} = -\left(\frac{-18\times10^9}{(d-x)^2}\,\vec{\imath}\right) \Rightarrow \frac{1}{x^2} = \frac{2}{(d-x)^2}$$

$$\left(\frac{d-x}{x}\right)^2 = 2 \Rightarrow \begin{cases} \dfrac{d-x}{x} = \sqrt{2} \\ \dfrac{d-x}{x} = -\sqrt{2} \end{cases}$$

$$\begin{cases} x = 0.41d \text{ This value is acceptable } \text{ wher } d > x \\ x = 2.38\,d \text{ This value is not acceptable } \text{ where } d < x \end{cases}$$

Exercise 05:

Three identical positive charges ($q_a = q_b = q_c = q$) are placed at the vertices of a square as shown in the adjacent figure opposite.

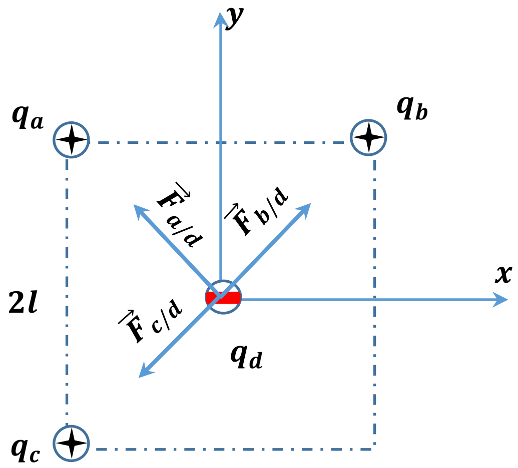


1- *Represent the forces acting on the charge $q_d = -q$ at the center of the square.*

2- *Write the analytical expression for each force, then deduce the resultant (net) force acting on the charge q_d.*

Solution :

1- *Represent the forces acting on the charge q_d at the center of the square*



2- *Write the analytical expression for each force, then deduce the resultant (net) force acting on the charge* $\boldsymbol{q_d}$.

$$\begin{cases} \overrightarrow{F_{a/d}} = k\,\dfrac{q_a q_d}{\left\|\overrightarrow{ad}\right\|^3}\,\overrightarrow{ad} \\ \overrightarrow{F_{b/d}} = k\,\dfrac{q_b q_d}{\left\|\overrightarrow{bd}\right\|^3}\,\overrightarrow{bd} \\ \overrightarrow{F_{c/d}} = k\,\dfrac{q_c q_d}{\left\|\overrightarrow{cd}\right\|^3}\,\overrightarrow{cd} \end{cases}$$

$$a(-l\,;\,l)\quad b(l\,;\,l)\quad c(-l\,;\,-l)\qquad d(0\,;\,0)$$

$$\overrightarrow{ad} = (x_d - x_a)\vec{\imath} + (y_d - y_a)\vec{\jmath} = l\,\vec{\imath} - l\,\vec{\jmath}\quad ;\left\|\overrightarrow{ad}\right\| = \sqrt{l^2 + (-l)^2} = l\sqrt{2}$$

$$\overrightarrow{bd} = (x_d - x_b)\vec{\imath} + (y_d - y_b)\vec{\jmath} = -l\,\vec{\imath} - l\,\vec{\jmath}\quad ;\left\|\overrightarrow{ad}\right\| = \sqrt{(-l)^2 + (-l)^2} = l\sqrt{2}$$

$$\overrightarrow{cd} = (x_d - x_c)\vec{\imath} + (y_d - y_c)\vec{\jmath} = l\,\vec{\imath} + l\,\vec{\jmath}\quad ;\left\|\overrightarrow{ad}\right\| = \sqrt{l^2 + l^2} = l\sqrt{2}$$

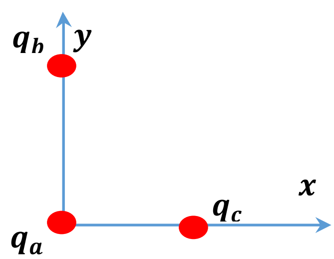
$$\begin{cases} \overrightarrow{F_{a/d}} = k\,\dfrac{q(-q)}{\left(l\sqrt{2}\right)^3}\,(l\,\vec{\imath} - l\,\vec{\jmath}) = -k\,\dfrac{q^2}{2\sqrt{2}\,l^2}\,(\vec{\imath} - \vec{\jmath}) \\ \overrightarrow{F_{b/d}} = k\,\dfrac{q(-q)}{\left(l\sqrt{2}\right)^3}\,(-l\,\vec{\imath} - l\,\vec{\jmath}) = -k\,\dfrac{q^2}{2\sqrt{2}\,l^2}\,(-\vec{\imath} - \vec{\jmath}) \\ \overrightarrow{F_{c/d}} = k\,\dfrac{q(-q)}{\left(l\sqrt{2}\right)^3}\,(l\,\vec{\imath} + l\,\vec{\jmath}) = -k\,\dfrac{q^2}{2\sqrt{2}\,l^2}\,(\vec{\imath} + \vec{\jmath}) \end{cases}$$

The net force acting on the charge $\boldsymbol{q_d}$

$$\overrightarrow{F_{net/d}} = \overrightarrow{F_{a/d}} + \overrightarrow{F_{b/d}} + \overrightarrow{F_{c/d}} = -k\,\dfrac{q^2}{2\sqrt{2}\,l^2}\,(\vec{\imath} - \vec{\jmath})$$

Exercise 06:

Three charges ($\boldsymbol{q_a = -q}$; $\boldsymbol{q_b = +2q}$ **and** $\boldsymbol{q_c = -q}$) *are placed at the points (0;0) , (0; 2d) and (d;0) respectively as shown in the opposite figure*

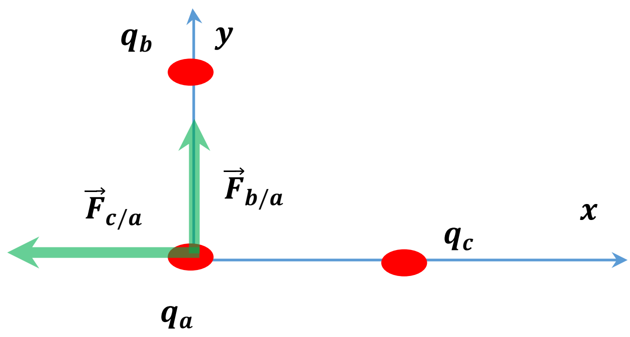


1- *Represent the forces acting on the charge* $\boldsymbol{q_a}$

2- *Calculate the resultant (net) force acting on the charge* $\boldsymbol{q_a}$.

Solution:

1. *Represent the forces acting on the charge* $\boldsymbol{q_a}$



2. *Calculate the resultant (net) force acting on the charge* $\boldsymbol{q_a}$.

$$\begin{cases} \overrightarrow{F_{b/a}} = k \dfrac{q_b q_a}{\|\overrightarrow{ba}\|^3} \overrightarrow{ba} \\ \overrightarrow{F_{c/a}} = k \dfrac{q_c q_a}{\|\overrightarrow{ca}\|^3} \overrightarrow{ca} \end{cases}$$

$$a(0\,;\,0) \quad b(0;\,2d) \quad c(d\,;\,0)$$

$$\overrightarrow{ba} = (x_a - x_b)\vec{\imath} + (y_a - y_b)\vec{\jmath} = -2d\ \vec{\jmath} \quad ; \|\overrightarrow{ba}\| = 2d$$

$$\overrightarrow{ca} = (x_a - x_c)\vec{\imath} + (y_a - y_c)\vec{\jmath} = -\,d\,\vec{\imath} \quad ; \|\overrightarrow{ca}\| = d$$

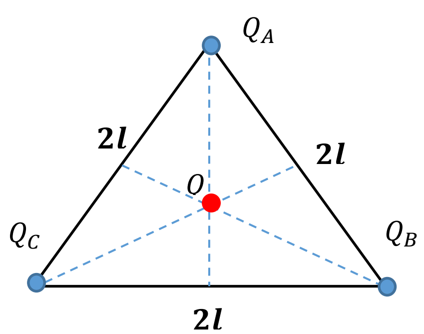
$$\begin{cases} \overrightarrow{F_{b/a}} = k \dfrac{2q(-q)}{(2d)^3} (-2d\ \vec{\jmath}) = k \dfrac{q^2}{2\ d^2} \vec{\jmath} \\ \overrightarrow{F_{c/a}} = k \dfrac{(-q)(-q)}{(d)^3} (-\,d\,\vec{\imath}) = -\,k \dfrac{q^2}{d^2} \vec{\imath} \end{cases}$$

The net force acting on the charge $\boldsymbol{q_a}$

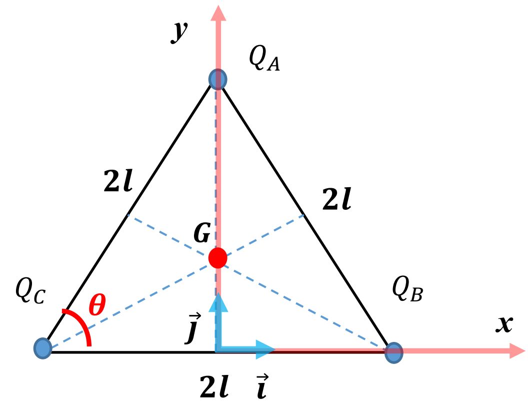
$$\overrightarrow{F_{net/d}} = \overrightarrow{F_{b/a}} + \overrightarrow{F_{c/a}} = -\,k \frac{q^2}{d^2}\vec{\imath} + k \frac{q^2}{2\ d^2}\vec{\jmath} = = k \frac{q^2}{d^2}\left(-\vec{\imath} + \frac{1}{2}\vec{\jmath}\right)$$

Exercise 07:

Consider three charges $\boldsymbol{q_A} = \boldsymbol{q_B} = \boldsymbol{q_C} = \boldsymbol{q}$ *at the vertices of an equilateral triangle of side* $2\boldsymbol{l}$. *What is the force on a charge* Q *placed at the centroid of the triangle, as shown in the Figure.*



Solution:



$$\overrightarrow{F_{net/G}} = \overrightarrow{F_{A/G}} + \overrightarrow{F_{B/G}} + \overrightarrow{F_{C/G}}$$

$$\begin{cases} \overrightarrow{F_{A/G}} = k\,\dfrac{q_A q_G}{\|\overrightarrow{AG}\|^3}\,\overrightarrow{AG} \\ \overrightarrow{F_{B/G}} = k\,\dfrac{q_B q_G}{\|\overrightarrow{BG}\|^3}\,\overrightarrow{BG} \\ \overrightarrow{F_{C/G}} = k\,\dfrac{q_C q_G}{\|\overrightarrow{CG}\|^3}\,\overrightarrow{CG} \end{cases}$$

$$A(0\,;\,y_A)\quad B(l\,;\,0)\quad C(-l\,;\,0)\quad G\,(0\,;\,y_G)$$

To find y_A : $\sin\theta = \frac{y_A}{2l} \Rightarrow y_A = 2l\sin\theta$ *where* $\theta = \frac{\pi}{3}$

$$y_A = 2l\sin\frac{\pi}{3} = 2l\,\frac{\sqrt{3}}{2} = l\sqrt{3} \quad \Rightarrow A\left(0\,;\,l\sqrt{3}\right)$$

To find y_G : $\tan\frac{\theta}{2} = \frac{y_G}{l} \Rightarrow y_G = l\tan\frac{\theta}{2}$ *where* $\frac{\theta}{2} = \frac{\pi}{6}$

$$y_G = l\tan\frac{\pi}{6} = l\,\frac{1}{\sqrt{3}} = \frac{l}{\sqrt{3}} \quad \Rightarrow G\left(0\,;\,\frac{l}{\sqrt{3}}\right)$$

$$A\left(0\,;\,l\sqrt{3}\right)\quad B(l\,;\,0)\quad C(-l\,;\,0)\quad G\left(0\,;\,\frac{l}{\sqrt{3}}\right)$$

$$\overrightarrow{AG} = (x_G - x_A)\vec{\imath} + (y_G - y_A)\vec{\jmath} = \left(\frac{l}{\sqrt{3}} - l\sqrt{3}\right)\vec{\jmath} = \frac{-2l\sqrt{3}}{3}\vec{\jmath} \quad ; \|\overrightarrow{AG}\| = \frac{2l\sqrt{3}}{3}$$

$$\overrightarrow{BG} = (x_G - x_B)\vec{\imath} + (y_G - y_B)\vec{\jmath} = -l\,\vec{\imath} + \frac{l}{\sqrt{3}}\vec{\jmath} \quad ; \|\overrightarrow{BG}\| = \sqrt{(-l)^2 + \left(\frac{l}{\sqrt{3}}\right)^2} = \frac{2l\sqrt{3}}{3}$$

$$\overrightarrow{CG} = (x_G - x_C)\vec{\imath} + (y_G - y_C)\vec{\jmath} = l\,\vec{\imath} + \frac{l}{\sqrt{3}}\vec{\jmath} \quad ; \|\overrightarrow{CG}\| = \sqrt{(l)^2 + \left(\frac{l}{\sqrt{3}}\right)^2} = \frac{2l\sqrt{3}}{3}$$

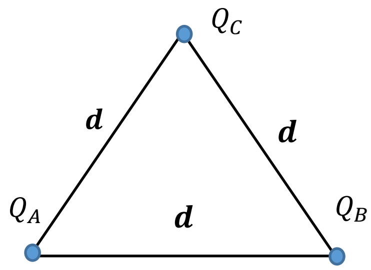
$$\begin{cases} \overrightarrow{F_{A/G}} = k\dfrac{qQ}{\left(\dfrac{2l\sqrt{3}}{3}\right)^3}\left(\dfrac{-2l\sqrt{3}}{3}\right)\vec{j} \\ \overrightarrow{F_{B/G}} = k\dfrac{qQ}{\left(\dfrac{2l\sqrt{3}}{3}\right)^3}\left(-l\,\vec{i} + \dfrac{l}{\sqrt{3}}\vec{j}\right) \\ \overrightarrow{F_{C/G}} = k\dfrac{qQ}{\left(\dfrac{2l\sqrt{3}}{3}\right)^3}\left(l\,\vec{i} + \dfrac{l}{\sqrt{3}}\vec{j}\right) \end{cases}$$

$$\overrightarrow{F_{net/G}} = \overrightarrow{F_{A/G}} + \overrightarrow{F_{B/G}} + \overrightarrow{F_{C/G}} = k\dfrac{qQ}{\left(\dfrac{2l\sqrt{3}}{3}\right)^3}\left((-l+l)\;\vec{i} + \left(\dfrac{-2l\sqrt{3}}{3} + \dfrac{2l\sqrt{3}}{3}\right)\vec{j}\right)$$

$$\overrightarrow{F_{net/G}} = \vec{0}$$

Exercise 08:

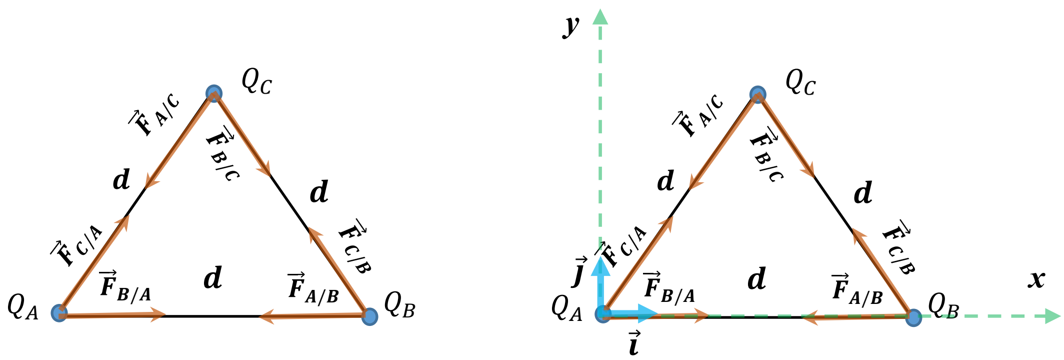
Consider the charges $\boldsymbol{q_A} = +\boldsymbol{q}$ *and* $\boldsymbol{q_B} = +2\boldsymbol{q}$*, and* $\boldsymbol{q_C} = -\boldsymbol{q}$ *placed at the vertices of an equilateral triangle, as shown in opposite figure.*



- ♣ *Represent on the diagram the forces acting on each charge.*
- ♣ *Write the expression of the force acting on each charge*

Solution:

- ♣ *Represent on the diagram the forces acting on each charge*



- ♣ *Write the expression of the force acting on each charge*

Forces acting on charge Q_C $\begin{cases} \overrightarrow{F_{A/C}} = k\dfrac{q_Aq_C}{\left\|\overrightarrow{AC}\right\|^3}\overrightarrow{AC} \\ \overrightarrow{F_{B/C}} = k\dfrac{q_Bq_C}{\left\|\overrightarrow{BC}\right\|^3}\overrightarrow{BC} \end{cases}$

♣ *Forces acting on charge* Q_B $\begin{cases} \overrightarrow{F_{A/B}} = k \dfrac{q_A q_B}{\|\overrightarrow{AB}\|^3} \overrightarrow{AB} \\ \overrightarrow{F_{C/B}} = k \dfrac{q_B q_C}{\|\overrightarrow{CB}\|^3} \overrightarrow{CB} \end{cases}$

♣ *Forces acting on charge* Q_A $\begin{cases} \overrightarrow{F_{C/A}} = k \dfrac{q_A q_C}{\|CA\|^3} \overrightarrow{CA} \\ \overrightarrow{F_{B/A}} = k \dfrac{q_B q_A}{\|\overrightarrow{BA}\|^3} \overrightarrow{BA} \end{cases}$

$$A(0\,;\,0)\quad B(d\,;\,0)\quad C(x_C\,;\,y_C)$$

To find x_C *qnd* y_C: $\begin{cases} \sin\theta = \dfrac{x_C}{d} & \Rightarrow x_C = d\sin\theta \\ \cos\theta = \dfrac{y_C}{d} & \Rightarrow y_C = d\cos\theta \end{cases}$ *where* $\theta = \dfrac{\pi}{3}$

$$\begin{cases} x_C = \dfrac{d}{2} \\ y_C = \dfrac{d\sqrt{3}}{2} \end{cases} \Rightarrow C\left(\frac{d}{2}\,;\,\frac{d\sqrt{3}}{2}\right)$$

$$\overrightarrow{AB} = -\overrightarrow{BA} = (x_B - x_A)\vec{\imath} + (y_B - y_A)\vec{\jmath} = d\,\vec{\imath} \quad ; \|\overrightarrow{AB}\| = \|\overrightarrow{BA}\| = d$$

$$\overrightarrow{AC} = -\overrightarrow{CA} = (x_C - x_A)\vec{\imath} + (y_C - y_A)\vec{\jmath} = \frac{d}{2}\,\vec{\imath} + \frac{d\sqrt{3}}{2}\,\vec{\jmath} \quad ; \|\overrightarrow{AC}\| = \|\overrightarrow{CA}\| = d$$

$$\overrightarrow{BC} = -\overrightarrow{CB} = (x_C - x_B)\vec{\imath} + (y_C - y_B)\vec{\jmath} = -\frac{d}{2}\,\vec{\imath} + \frac{d\sqrt{3}}{2}\,\vec{\jmath} \quad ; \|\overrightarrow{BC}\| = \|\overrightarrow{CB}\| = d$$

♣ *Forces acting on charge* Q_C

$$\begin{cases} \overrightarrow{F_{A/C}} = k \dfrac{q\,(-q)}{d^3}\left(\dfrac{d}{2}\,\vec{\imath} + \dfrac{d\sqrt{3}}{2}\,\vec{\jmath}\right) \\ \overrightarrow{F_{B/C}} = k \dfrac{2q\,(-q)}{d^3}\left(-\dfrac{d}{2}\,\vec{\imath} + \dfrac{d\sqrt{3}}{2}\,\vec{\jmath}\right) \end{cases}$$

$$\overrightarrow{F_{net/C}} = \frac{-k\,q^2}{d^2}\left(\left(\frac{1}{2}\,\vec{\imath} + \frac{\sqrt{3}}{2}\,\vec{\jmath}\right) + \left(-\,\vec{\imath} + \sqrt{3}\,\vec{\jmath}\right)\right) = \frac{-k\,q^2}{d^2}\left(-\frac{1}{2}\,\vec{\imath} + \frac{3\sqrt{3}}{2}\,\vec{\jmath}\right)$$

♣ *Forces acting on charge* Q_B

$$\begin{cases} \overrightarrow{F_{A/B}} = k \dfrac{q\,(2q)}{d^3}\,d\,\vec{\imath} \\ \overrightarrow{F_{C/B}} = k \dfrac{-q\,(2q)}{d^3}\left(\dfrac{d}{2}\,\vec{\imath} - \dfrac{d\sqrt{3}}{2}\,\vec{\jmath}\right) \end{cases}$$

$$\overrightarrow{F_{net/B}} = \frac{k\,q^2}{d^2}\left(2\,\vec{\imath} + \left(-\,\vec{\imath} + \sqrt{3}\,\vec{\jmath}\right)\right) = \frac{k\,q^2}{d^2}\left(\vec{\imath} + \sqrt{3}\,\vec{\jmath}\right)$$

- *Forces acting on charge Q_A*

$$\begin{cases} \overrightarrow{F_{B/A}} = k\,\dfrac{2q\,q}{d^3}\,(-d\,\vec{\imath}\,) \\ \overrightarrow{F_{C/A}} = k\,\dfrac{-q\,q}{d^3}\left(-\dfrac{d}{2}\,\vec{\imath} - \dfrac{d\sqrt{3}}{2}\,\vec{\jmath}\right) \end{cases}$$

$$\overrightarrow{F_{net/A}} = \frac{k\,q^2}{d^2}\left(-2\,\vec{\imath} + \left(\frac{1}{2}\,\vec{\imath} + \frac{\sqrt{3}}{2}\,\vec{\jmath}\right)\right) = \frac{k\,q^2}{d^2}\left(-\frac{3}{2}\,\vec{\imath} + \frac{\sqrt{3}}{2}\,\vec{\jmath}\right)$$

Exercise 09:

In the Millikan experiment, two horizontal metal plates are separated by a distance of 1.0 cm, with a potential difference of 3 kV. One plate is positively charged, the other negatively charged. Small oil droplets, negatively charged, are present between the plates in equilibrium.

- Calculate the charge of a spherical oil droplet and compare it to the charge of an electron.

Given values: oil density (ρ) = 900 kg/m³, droplet radius (R) = 2.0 µm, gravitational field strength (g) = 9.8 m/s²

Solution:

To calculate the charge of an oil droplet, we must first determine its mass and volume:

$V = \frac{4}{3}\pi R^3 = 3.35 \times 10^{-17}\,m^3$

$$\rho = \frac{m}{V} \Rightarrow m = \rho \times V = 900 \times 3.35 \times 10^{-17} = 3.015 \times 10^{-14}\,Kg$$

Then we need to calculate the value of the electric field, the electric force, as well as the gravitational force that this droplet will be subjected to.

The electric field between the plates:

$$E = \frac{V}{d} = \frac{3000}{0.01} = 3 \times 10^5\,V/m$$

$$F = q\,E = q \times 3 \times 10^5$$

The gravitational force

$$P = m\,g = 3.015 \times 10^{-14} \times 9.8 = 2.955 \times 10^{-13}\,N$$

In a state of equilibrium, the gravitational force is equal to and opposite in direction to the electric force

$$m\,g = q\,E \;\Rightarrow q = m\frac{g}{E} = \frac{2.955\times10^{-13}}{3\times10^{5}} = 9.85\times10^{-19}\,C$$

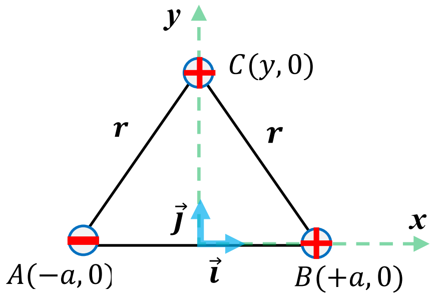
♣ *Compare the oil droplet charge to the charge of an electron.*

$$q_e = 1.6021\times10^{-19}\,C$$

$$\frac{q}{q_e} = \frac{9.85\times10^{-19}}{1.6021\times10^{-19}} \approx 6 \;\Rightarrow q = 6\,q_e$$

Exercise 10:

*A negative point charge of magnitude **-q** is located on the x-axis at point x = -a, and a positive point charge **+q** of the same magnitude is located at x = +a, see the opposite figure. A third positive point charge **q_0** is located on the y-axis with a coordinate (0, y).*



(a) *What is the expression of the net force on q_0 when its coordinate is (0, y)?*

(b) *What is the magnitude and direction of the net force exerted on q_0 when it placed at the origin (0,0)?*

Solution:

a) *The expression of the net force on q_0 when its coordinate is (0, y)*

$$\text{Forces acting on charge } Q_C \begin{cases} \overrightarrow{F_{A/C}} = k\,\dfrac{q_A q_C}{\left\|\overrightarrow{AC}\right\|^3}\,\overrightarrow{AC} \\[2ex] \overrightarrow{F_{B/C}} = k\,\dfrac{q_B q_C}{\left\|\overrightarrow{BC}\right\|^3}\,\overrightarrow{BC} \end{cases}$$

$$A(-a\;;\;0)\quad B(+a\;;\;0)\quad C(o\;;\;y)$$

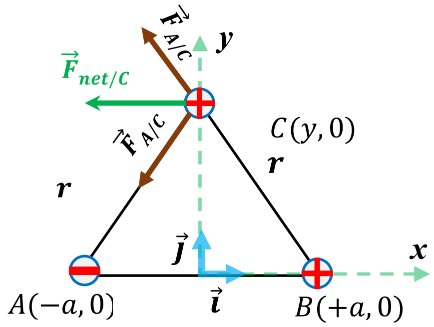
$$\overrightarrow{AC} = (x_C - x_A)\vec{\imath} + (y_C - y_A)\vec{\jmath} = a\,\vec{\imath} + y\,\vec{\jmath}\;\;;\left\|\overrightarrow{AC}\right\| = \sqrt{a^2+y^2}$$

$$\overrightarrow{BC} = (x_C - x_B)\vec{\imath} + (y_C - y_B)\vec{\jmath} = -a\,\vec{\imath} + y\,\vec{\jmath}\;\;;\left\|\overrightarrow{BC}\right\| == \sqrt{a^2+y^2}$$

$$\begin{cases} \overrightarrow{F_{A/C}} = k\,\dfrac{(-q)\,q_0}{(a^2+y^2)^{3/2}}\,(a\,\vec{\imath} + y\,\vec{\jmath}) \\[2ex] \overrightarrow{F_{B/C}} = k\,\dfrac{q\,q_0}{(a^2+y^2)^{3/2}}\,(-a\,\vec{\imath} + y\,\vec{\jmath}) \end{cases}$$

$$\overrightarrow{F_{net/C}} = \frac{k\,q\,q_0}{(a^2+y^2)^{3/2}}(-a\,\vec{\imath} - y\,\vec{\jmath} - a\,\vec{\imath} + y\,\vec{\jmath}) = \frac{-\,2\,a\,k\,q\,q_0}{(a^2+y^2)^{3/2}}\vec{\imath}$$

We can observe that the resultant of the forces acting on charge q_0 *has a single component parallel to the unit vector* $\vec{\imath}$*, and this result is consistent with what was obtained by representing the vectors graphically*



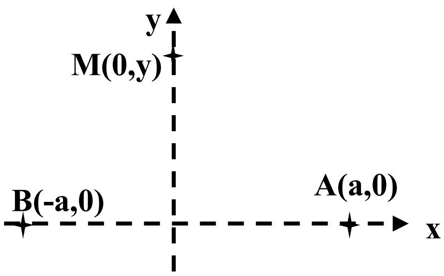
b) *The magnitude and direction of the net force exerted on* q_0 *when it placed at the origin*

$$y = 0 \Rightarrow \overrightarrow{F_{net/C}} = \frac{-2\,a\,k\,q\,q_0}{(a^2)^{3/2}}\vec{\imath} = \frac{-2\;k\,q\,q_0}{a^2}\vec{\imath}$$

Exercise 11:

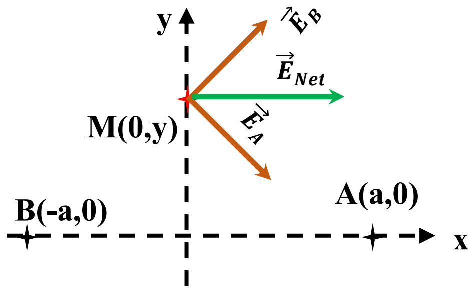
In a Cartesian coordinate system (xy), two charges -Q and Q are placed at points A(a, 0) and B(- a, 0), respectively, where a is a positive number.

1) *Represent the electric field generated* E_A *by electric charge A at point M*
2) *Represent the electric field generated* E_B *by electric charge B at point M*
3) *Determine the direction and magnitude of the electrostatic field at point M(y, 0), as a function of a, Q, and y.*
4) *Calculate the electric field for M(0,0)*
5) *Plot the curve E(y).*
6) *Deduce the net electric force applied by these two charges at a negative point charge placed at M (Q=-3µC), and a positive point charge place at M (Q=+4µC).*
7) *Deduce the electric potential V created by these charges at point M.*



Solution:

- *Represent the electric fields generated* $\vec{E}_A$ $\vec{E}_B$ *and by electric charges A and B at point M*



- *Determine the direction and magnitude of the electrostatic field at point M(y, 0), as a function of a, Q, and y. The net electric field at the point M*

$$\overrightarrow{E_{Net}}(M) = \sum_i \overrightarrow{E_\imath}(M) = \sum_i K \frac{q_i}{\left\|\overrightarrow{P_\imath M}\right\|^3} \overrightarrow{P_\imath M} \Rightarrow \overrightarrow{E_{Net}}(M) = \overrightarrow{E_A}(M) + \overrightarrow{E_B}(M)$$

$$\begin{cases} \overrightarrow{E_A}(M) = k \dfrac{q_A}{\left\|\overrightarrow{AM}\right\|^3} \overrightarrow{AM} \\ \overrightarrow{E_B}(M) = k \dfrac{q_B}{\left\|\overrightarrow{BM}\right\|^3} \overrightarrow{BM} \end{cases}$$

$$A(a\ ;\ 0) \quad B(-a\ ;\ 0) \quad M(o\ ;\ y)$$

$$\overrightarrow{AM} = (x_M - x_A)\vec{\imath} + (y_M - y_A)\vec{\jmath} = -a\,\vec{\imath} + y\,\vec{\jmath} \ \ ; \left\|\overrightarrow{AM}\right\| = \sqrt{a^2 + y^2}$$

$$\overrightarrow{BM} = (x_M - x_B)\vec{\imath} + (y_M - y_B)\vec{\jmath} = a\,\vec{\imath} + y\,\vec{\jmath} \ \ ; \left\|\overrightarrow{BM}\right\| = \sqrt{a^2 + y^2}$$

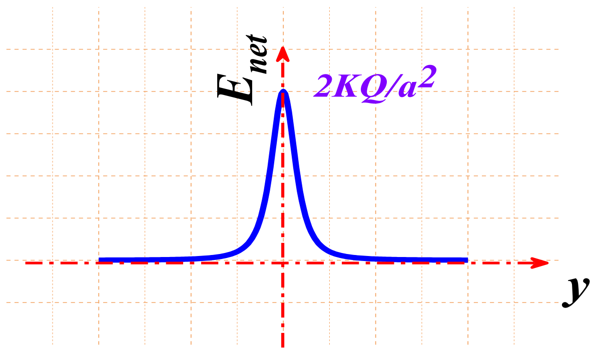
$$\begin{cases} \overrightarrow{E_A}(M) = k \dfrac{-Q}{(a^2 + y^2)^{3/2}} (-a\,\vec{\imath} + y\,\vec{\jmath}) \\ \overrightarrow{E_B}(M) = k \dfrac{Q}{(a^2 + y^2)^{3/2}} (a\,\vec{\imath} + y\,\vec{\jmath}) \end{cases}$$

$$\overrightarrow{E_{Net}}(M) = \overrightarrow{E_A}(M) + \overrightarrow{E_B}(M) = k \frac{2\,a\,Q}{(a^2 + y^2)^{3/2}} \vec{\imath}$$

- *Calculate the electric field for M(0,0) , in this case we replace the value of y by 0*

$$\overrightarrow{E_{Net}}(M) = k \frac{2\ Q}{a^2} \vec{\imath}$$

- *Plot the curve E(y):* $\left\|\overrightarrow{E_{Net}}(M)\right\| = E_{Net} = k \frac{2\,a\,Q}{(a^2+y^2)^{3/2}}$



♣ *Deduce the net electric force applied by these two charges at a negative point charge placed at M (Q=-3μC), and a positive point charge place at M (Q=+4μC).*

$$\overrightarrow{F_{Net/M}}(M) = Q_M \overrightarrow{E_{Net}}(M) = k\,\frac{2\,a\,Q\,Q_M}{(a^2+y^2)^{3/2}}\,\vec{\imath}$$

For $Q_M = -3\mu C \Rightarrow \overrightarrow{F_{Net/M}}(M) = 9\times10^9 \times \frac{2\,a\,Q(-3\times10^{-6})}{(a^2+y^2)^{3/2}}\,\vec{\imath} = -54\ \times 10^3 \times \frac{a\,Q}{(a^2+y^2)^{3/2}}\,\vec{\imath}$

For $Q_M = 4\mu C \Rightarrow \overrightarrow{F_{Net/M}}(M) = 9\times10^9 \times \frac{2\,a\,Q(4\times10^{-6})}{(a^2+y^2)^{3/2}}\,\vec{\imath} = 72\ \times 10^3 \times \frac{a\,Q}{(a^2+y^2)^{3/2}}\,\vec{\imath}$

♣ *Deduce the electric potential V created by these charges at point M.*

1st Method

$$V_{Net}(M) = \sum_i V_i(M) = \sum_i K\frac{q_i}{\|\overrightarrow{P_iM}\|} \Rightarrow V_{Net}(M) = V_A(M) + V_B(M)$$

$$\begin{cases} V_A(M) = k\dfrac{q_A}{\|\overrightarrow{AM}\|} + C_A = k\dfrac{q_A}{\sqrt{a^2+y^2}} + C_A \\ V_B(M) = k\dfrac{q_B}{\|\overrightarrow{BM}\|} + C_B = k\dfrac{q_A}{\sqrt{a^2+y^2}} + C_A \end{cases}$$

$$V_{Net}(M) = k\frac{q_A}{\sqrt{a^2+y^2}} + C_A + k\frac{q_B}{\sqrt{a^2+y^2}} + C_B = \frac{k}{\sqrt{a^2+y^2}}(q_A+q_B) + (C_A + C_B)$$

We assume $C_A + C_B = C$

$$V_{Net}(M) = \frac{k}{\sqrt{a^2+y^2}}(-Q+Q) + \mathrm{C} = C$$

To determine the value of C, we rely on the principle that the electric potential vanishes at infinity, and thus we assume:

$$V_{Net}(y\to\infty) = 0 \Rightarrow C = 0 \Rightarrow V_{Net}(M) = 0$$

***2nd Method :** We can derive the electric potential expression from the electric field vector expression as follows:*

$$dV = dV_x + dV_y + dV_z = -\vec{E}.d\vec{r} \ \ where \begin{cases} \vec{E} = E_x\,\vec{\imath} + E_y\,\vec{\jmath} + E_z\,\vec{k} \\ d\vec{r} = dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k} \end{cases}$$

$$dV = dV_x + dV_y + dV_z = -E_x\,dx - E_y\,dy - E_z\,dz \begin{cases} dV_x = -E_x\,dx \\ dV_y = -E_y\,dy \\ dV_z = -E_z\,dz \end{cases}$$

$$\vec{E} = k\,\frac{2\,a\,Q}{(a^2+y^2)^{3/2}}\,\vec{\imath} = \begin{cases} dV_x = -k\,\frac{2\,a\,Q}{(a^2+y^2)^{3/2}}\,dx \\ dV_y = 0\,dy \\ dV_z = 0\,dz \end{cases}$$

$$V = \int dV = \int dV_x + \int dV_y + \int dV_z \Rightarrow \begin{cases} \int dV_x = \int -k\,\frac{2\,a\,Q}{(a^2+y^2)^{3/2}}\,dx \\ \int dV_y = \int 0\,dy \\ \int dV_z = \int 0\,dz \end{cases}$$

$$\begin{cases} \int dV_x = -k\,\frac{2\,a\,Q}{(a^2+y^2)^{\frac{3}{2}}}\,x + c1 \\ \int dV_y = c2 \\ \int dV_z = c3 \end{cases} \Rightarrow V = -k\,\frac{2\,a\,Q}{(a^2+y^2)^{\frac{3}{2}}}\,x + c \ \ where \ \ c = c1 + c2 + c3$$

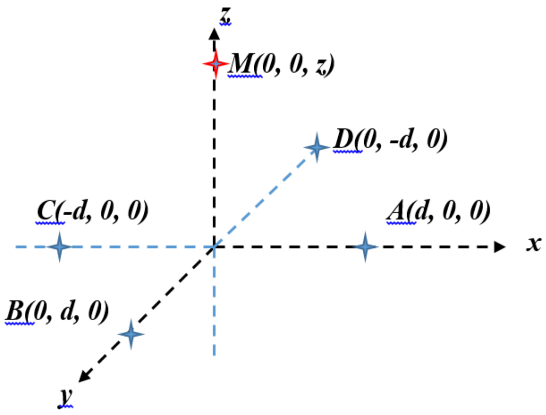
For a point M $M\,(0, y)$, *We replace x by 0 and we find the expression:*

$$V\,(M\,(0,y)) = -k\,\frac{2\,a\,Q}{(a^2+y^2)^{\frac{3}{2}}}\,0 + c = c$$

$$V_{Net}\,(y \to \infty) = 0 \Rightarrow C = 0 \Rightarrow V_{Net}\,(M) = 0$$

Exercise 12:

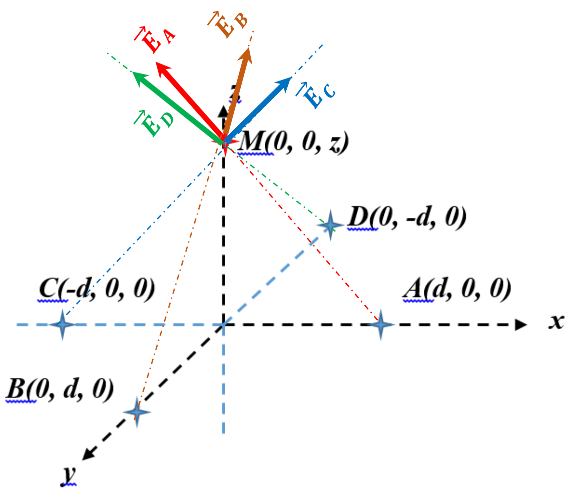
Four charges Q_A, Q_B, Q_C, and Q_D are arranged as shown in the opposite figure, where $Q_A = Q_B = Q_C = Q_D = Q = +3\mu C$



1) *Represent on the figure the electric field generated at the point M (0,0,z), and write its analytical expression*
2) *Deduce the net electric force applied by these four charges at negative point charge placed at M (Q=-2μC).*
3) *Deduce the electric potential V created by these four charges at point M.*
4) *Calculate the net electric field, the net electric force, and the net electric potential for M (0,0,0).*

Solution:

1) *Represent on the figure the electric field generated at the point M (0,0,z), and write its analytical expression*



$$\overrightarrow{E_{Net}}(M) = \sum_i \overrightarrow{E_i}(M) = \sum_i K \frac{q_i}{\left\|\overrightarrow{P_iM}\right\|^3} \overrightarrow{P_iM} = \overrightarrow{E_A}(M) + \overrightarrow{E_B}(M) + \overrightarrow{E_C}(M) + \overrightarrow{E_D}(M)$$

$$\begin{cases} \overrightarrow{E_A}(M) = k \dfrac{q_A}{\|\overrightarrow{AM}\|^3} \overrightarrow{AM} \\ \overrightarrow{E_B}(M) = k \dfrac{q_B}{\|\overrightarrow{BM}\|^3} \overrightarrow{BM} \\ \overrightarrow{E_C}(M) = k \dfrac{q_C}{\|\overrightarrow{CM}\|^3} \overrightarrow{CM} \\ \overrightarrow{E_D}(M) = k \dfrac{q_D}{\|\overrightarrow{DM}\|^3} \overrightarrow{DM} \end{cases}$$

$$A(d;0;0) \quad B(0;d;0) \quad C(-d\,;0;0) \quad D(0;-d;0) \quad M(0;0;z)$$

$$\overrightarrow{AM} = (x_M - x_A)\vec{\imath} + (y_M - y_A)\vec{\jmath} + (z_M - z_A)\vec{k} = -d\,\vec{\imath} + z\,\vec{k} \;\; ; \|\overrightarrow{AM}\| = \sqrt{d^2 + z^2}$$

$$\overrightarrow{BM} = (x_M - x_B)\vec{\imath} + (y_M - y_B)\vec{\jmath} + (z_M - z_B)\vec{k} = -d\,\vec{\jmath} + z\,\vec{k} \;\; ; \|\overrightarrow{BM}\| = \sqrt{d^2 + z^2}$$

$$\overrightarrow{CM} = (x_M - x_C)\vec{\imath} + (y_M - y_C)\vec{\jmath} + (z_M - z_C)\vec{k} = d\,\vec{\imath} + z\,\vec{k} \;\; ; \|\overrightarrow{CM}\| = \sqrt{d^2 + z^2}$$

$$\overrightarrow{DM} = (x_M - x_D)\vec{\imath} + (y_M - y_D)\vec{\jmath} + (z_M - z_D)\vec{k} = d\,\vec{\jmath} + z\,\vec{k} \;\; ; \|\overrightarrow{DM}\| = \sqrt{d^2 + z^2}$$

$$\begin{cases} \overrightarrow{E_A}(M) = k \dfrac{Q}{(d^2+z^2)^{3/2}} (-d\,\vec{\imath} + z\,\vec{k}) \\ \overrightarrow{E_B}(M) = k \dfrac{Q}{(d^2+z^2)^{3/2}} (-d\,\vec{\jmath} + z\,\vec{k}) \\ \overrightarrow{E_C}(M) = k \dfrac{Q}{(d^2+z^2)^{3/2}} (d\,\vec{\imath} + z\,\vec{k}) \\ \overrightarrow{E_D}(M) = k \dfrac{Q}{(d^2+z^2)^{3/2}} (d\,\vec{\jmath} + z\,\vec{k}) \end{cases}$$

$$\overrightarrow{E_{Net}}(M) = \overrightarrow{E_A}(M) + \overrightarrow{E_B}(M) + \overrightarrow{E_C}(M) + \overrightarrow{E_D}(M) = \frac{4K\,z\,Q}{(d^2+z^2)^{3/2}}\,\vec{k} = \frac{108\,z\,10^3}{(d^2+z^2)^{3/2}}\,\vec{k}$$

♣ *Deduce the net electric force* ***applied*** *by these two charges at a negative point charge placed at M (Q=-2µC).*

$$\overrightarrow{F_{Net/M}}(M) = Q_M \overrightarrow{E_{Net}}(M) = \frac{Q_M z\,Kd\,Q}{(d^2+z^2)^{3/2}}\,\vec{k} = \frac{-216\,z\,10^{-3}}{(d^2+z^2)^{3/2}}\,\vec{k}$$

♣ *Deduce the electric potential V created by these four charges at point M.*

1st method:

$$V_{Net}(M) = \sum_i V_i(M) = \sum_i K \frac{q_i}{\|\overrightarrow{P_i M}\|} = V_A(M) + V_B(M) + V_C(M) + V_D(M)$$

$$\begin{cases} V_A(M) = k\dfrac{q_A}{\|\overrightarrow{AM}\|} + C_A = k\dfrac{q_A}{\sqrt{d^2+z^2}} + C_A \\ V_B(M) = k\dfrac{q_B}{\|\overrightarrow{BM}\|} + C_B = k\dfrac{q_B}{\sqrt{d^2+z^2}} + C_B \\ V_C(M) = k\dfrac{q_C}{\|\overrightarrow{CM}\|} + C_C = k\dfrac{q_C}{\sqrt{d^2+z^2}} + C_C \\ V_D(M) = k\dfrac{q_D}{\|\overrightarrow{DM}\|} + C_D = k\dfrac{q_D}{\sqrt{d^2+z^2}} + C_D \end{cases}$$

$$V_{Net}(M) = \frac{k}{\sqrt{d^2+z^2}}(q_A+q_B+q_C+q_D) + (C_A+C_B+C_C+C_D)$$

We assume $C_A + C_B + C_C + C_D = C$

$$V_{Net}(M) = \frac{4kQ}{\sqrt{d^2+z^2}} + \mathrm{C}$$

To determine the value of C, we rely on the principle that the electric potential vanishes at infinity, and thus we assume:

$$V_{Net}(z\to\infty) = 0 \Rightarrow 0 + C = 0 \Rightarrow V_{Net}(M) = \frac{4kQ}{\sqrt{d^2+z^2}} = \frac{108\times10^3}{\sqrt{d^2+z^2}}$$

2nd Method : *We can derive the electric potential expression from the electric field vector expression as follows:*

$$dV = dV_x + dV_y + dV_z = -\vec{E}.d\vec{r} \ \ where \begin{cases} \vec{E} = E_x\,\vec{\imath} + E_y\,\vec{\jmath} + E_z\,\vec{k} \\ d\vec{r} = dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k} \end{cases}$$

$$dV = dV_x + dV_y + dV_z = -E_x\,dx - E_y\,dy - E_z\,dz \begin{cases} dV_x = -E_x\,dx \\ dV_y = -E_y\,dy \\ dV_z = -E_z\,dz \end{cases}$$

$$\vec{E} = \frac{4Kz\,Q}{(d^2+z^2)^{3/2}}\vec{k} = \begin{cases} dV_x = 0\,dx \\ dV_y = 0\,dy \\ dV_z = -\dfrac{4Kz\,Q}{(d^2+z^2)^{3/2}}\,dz \end{cases}$$

$$V \Rightarrow \begin{cases} V_x = \int dV_x = \int 0\,dx \\ V_y = \int dV_y = \int 0\,dy \\ V_z = \int dV_z = -4KQ \int \frac{z}{(d^2+z^2)^{3/2}}\,dz \end{cases} \Rightarrow \begin{cases} V_x = \int dV_x = C_x \\ V_y = \int dV_y = C_y \\ V_z = -4KQ \int \frac{z}{(d^2+z^2)^{3/2}}\,dz \end{cases}$$

To calculate the integral $I_z = \int \frac{z}{(d^2+z^2)^{3/2}}\,dz$ *, we perform the variable conversion process as follows:*

$$t^2 = d^2 + z^2 \Rightarrow 2t\,dt = 2z\,dz \Rightarrow dt = z\,dz$$

$$I_z = \int \frac{z}{(d^2+z^2)^{3/2}}\,dz \Rightarrow I_t = \int \frac{t}{(t^2)^{3/2}}\,dt = \int \frac{1}{t^2}\,dt = -\frac{1}{t}$$

We replace $t = \sqrt{d^2+z^2}$

$$I_z = -\frac{1}{\sqrt{d^2+z^2}}$$

Thus, we find:

$$V_z = -4KQ \int \frac{z}{(d^2+z^2)^{\frac{3}{2}}}\,dz = \frac{4KQ}{\sqrt{d^2+z^2}} + C_z$$

$$V(M) = V_x+V_y+V_z = \frac{4KQ}{\sqrt{d^2+z^2}} + C_x + C_y + C_z = \frac{4KQ}{\sqrt{d^2+z^2}} + C$$

To determine the constant c, we use the condition:

$$V_{Net}(z \to \infty) = 0 \Rightarrow C = 0 \Rightarrow V_{Net}(M) = 0$$

$$V(M) = \frac{4KQ}{\sqrt{d^2+z^2}}$$

- *Calculate the net electric field for M (0,0,0) ⇒z=0*

$$\overrightarrow{E_{Net}}(M) = \frac{4K\,(0)\,Q}{(d^2+(0)^2)^{3/2}}\,\vec{k} = \vec{o}$$

- *Calculate the net electric force for M (0,0,0) ⇒z=0*

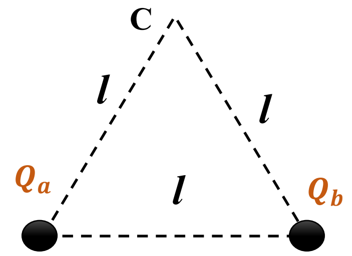
$$\overrightarrow{F_{Net/M}}(M) = Q_M \overrightarrow{E_{Net}}(M) = \vec{o}$$

- *Calculate the net electric potential for M (0,0,0) ⇒z=0*

$$V(M) = \frac{4KQ}{\sqrt{d^2+z^2}} = \frac{4KQ}{\sqrt{d^2+(0)^2}} = \frac{4KQ}{d}$$

Exercise 13:

*Two charges Q_a=+4μC and Q_b=+1μC are placed at the vertices of an equilateral triangle of side length **l** as shown the opposite figure.*



-Find the total electric potential at the point C.

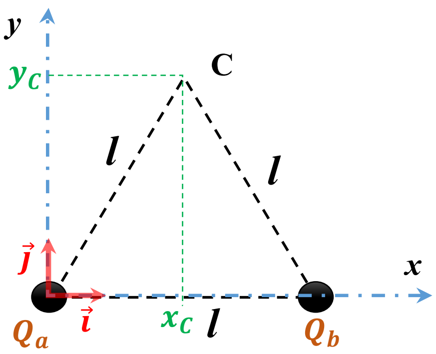
Calculate/Deduce the expression of the electric field generated by the two charges

Solution:

- *The total electric potential at the point C.*

$$V_{Net}(C) = \sum_i V_i(C) = \sum_i K\frac{q_i}{\|\overrightarrow{P_iC}\|} = V_A(C) + V_B(C)$$

First, we choose a coordinate system to determine the coordinates of the positions of charges A and B as well as the position of point C. as in the following figure.



Therefore, the positions are as follows:

$$A(0;0) \quad B(l;0) \quad C(x_C\,;y_C)$$

The coordinates of point C are calculated as follows:

A triangle is equilateral, so all of its angles are identical and equal to $\frac{\pi}{3}$

$$\begin{cases} x_C = l\cos\frac{\pi}{3} = \frac{l}{2} \\ y_C = l\sin\frac{\pi}{3} = \frac{l\sqrt{3}}{2} \end{cases} \Rightarrow C\left(\frac{l}{2}\,;\frac{l\sqrt{3}}{2}\right)$$

$$\overrightarrow{AC} = (x_C - x_A)\vec{\imath} + (y_C - y_A)\vec{\jmath} = \frac{l}{2}\,\vec{\imath} + \frac{l\sqrt{3}}{2}\,\vec{\jmath} \;\; ; \|\overrightarrow{AC}\| = \sqrt{\left(\frac{l}{2}\right)^2 + \left(\frac{l\sqrt{3}}{2}\right)^2} = l$$

$$\overrightarrow{BC} = (x_C - x_B)\vec{\imath} + (y_C - y_B)\vec{\jmath} = -\frac{l}{2}\,\vec{\imath} + \frac{l\sqrt{3}}{2}\,\vec{\jmath}\ \ ;\|\overrightarrow{BC}\| = \sqrt{\left(-\frac{l}{2}\right)^2 + \left(\frac{l\sqrt{3}}{2}\right)^2} = l$$

$$\begin{cases} V_A(C) = k\,\dfrac{Q_A}{\|\overrightarrow{AC}\|} + C_A = k\,\dfrac{Q_A}{l} + C_A \\ V_B(C) = k\,\dfrac{Q_B}{\|\overrightarrow{BC}\|} + C_B = k\,\dfrac{Q_B}{l} + C_B \end{cases}$$

$$V_{Net}(C) = \frac{k}{l}(Q_A + Q_B) + (C_A + C_B)$$

We assume $C_A + C_B = C$

$$V_{Net}(C) = \frac{k}{l}(Q_A + Q_B) + \mathrm{C}$$

To determine the constant c, we use the condition:

$$V_{Net}(l \to \infty) = 0 \Rightarrow C = 0 \Rightarrow V_{Net}(C) = 0$$

$$V(C) = \frac{k}{l}(Q_A + Q_B) = \frac{9 \times 10^9 \times 5 \times 10^{-6}}{l} = \frac{45 \times 10^3}{l}$$

♣ *Calculate/Deduce the expression of the electric field generated by the two charges*

$$\overrightarrow{E_{Net}}(C) = \sum_i \overrightarrow{E_i}(C) = \sum_i K\frac{q_i}{\|\overrightarrow{P_iC}\|^3}\,\overrightarrow{P_iC} = \overrightarrow{E_A}(C) + \overrightarrow{E_B}(C)$$

$$\begin{cases} \overrightarrow{E_A}(C) = k\,\dfrac{Q_A}{\|\overrightarrow{AC}\|^3}\,\overrightarrow{AC} = k\,\dfrac{Q_A}{l^3}\left(\dfrac{l}{2}\,\vec{\imath} + \dfrac{l\sqrt{3}}{2}\,\vec{\jmath}\right) = k\,\dfrac{Q_A}{l^2}\left(\dfrac{1}{2}\,\vec{\imath} + \dfrac{\sqrt{3}}{2}\,\vec{\jmath}\right) \\ \overrightarrow{E_B}(C) = k\,\dfrac{Q_B}{\|\overrightarrow{BC}\|^3}\,\overrightarrow{BC} = k\,\dfrac{Q_B}{l^3}\left(-\dfrac{l}{2}\,\vec{\imath} + \dfrac{l\sqrt{3}}{2}\,\vec{\jmath}\right) = k\,\dfrac{Q_B}{l^2}\left(-\dfrac{1}{2}\,\vec{\imath} + \dfrac{\sqrt{3}}{2}\,\vec{\jmath}\right) \end{cases}$$

$$\overrightarrow{E_{Net}}(C) = \overrightarrow{E_A}(C) + \overrightarrow{E_B}(C) = k\,\frac{Q_A}{l^2}\left(\frac{1}{2}\,\vec{\imath} + \frac{\sqrt{3}}{2}\,\vec{\jmath}\right) + k\,\frac{Q_B}{l^2}\left(-\frac{1}{2}\,\vec{\imath} + \frac{\sqrt{3}}{2}\,\vec{\jmath}\right)$$

$$\overrightarrow{E_{Net}}(C) = \frac{k}{l^2}\left[\left(\frac{Q_A}{2}\,\vec{\imath} + \frac{Q_A\sqrt{3}}{2}\,\vec{\jmath}\right) + \left(-\frac{Q_B}{2}\,\vec{\imath} + \frac{Q_B\sqrt{3}}{2}\,\vec{\jmath}\right)\right]$$

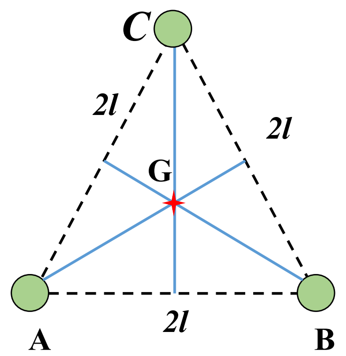
$$= \frac{k}{l^2}\left[\left(\frac{Q_A - Q_B}{2}\,\vec{\imath} + \frac{(Q_A + Q_B)\sqrt{3}}{2}\,\vec{\jmath}\right)\right]$$

Numerical application of the equation

$$\overrightarrow{E_{Net}}(C) = \frac{9 \times 10^9}{l^2}\left[\left(\frac{3}{2}\,\vec{\imath} + \frac{5\sqrt{3}}{2}\,\vec{\jmath}\right) \times 10^{-6}\right] = \frac{10^3}{l^2}\left[\left(\frac{27}{2}\,\vec{\imath} + \frac{45\sqrt{3}}{2}\,\vec{\jmath}\right)\right]$$

Exercise 14:

Consider three point charges positioned at the vertices of an equilateral triangle, the Length of the triangle's side is 2l as shown in the figure, where $Q_A = Q_B = Q_C = Q = +2\ \mu C$. Let G represent the centroid of the triangle.



1. *Determine the electric potential created by each charge at the centroid G.*
2. *Deduce the overall electric potential at the centroid G.*
3. *Compute the total electric field value at the centroid G.*

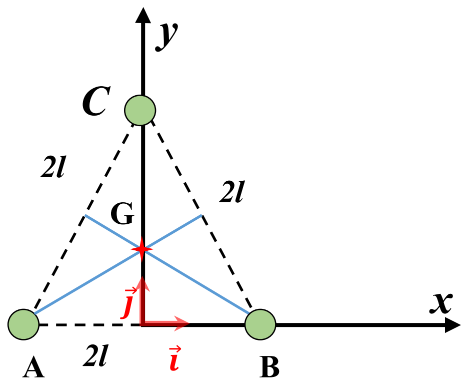
If the charge μC Q_G = -3 μC is positioned at the centroid G, calculate the resultant force acting upon it.

Solution:

Determine the electric potential created by each charge at the centroid G.

$$V_{Net}\,(G) = \sum_i V_i\,(G) = \sum_i K\frac{q_i}{\|\overrightarrow{P_iG}\|} = V_A\,(G) + V_B\,(G) + V_C\,(G)$$

To obtain the positions of charges A, B, and C as well as the position of the centroid of triangle G, an appropriate coordinate system should be chosen to facilitate the determination of these positions, as shown in the accompanying Figure.



Therefore, the positions are as follows:

$$A(-l;0)\quad B(+l;0)\quad C(0\,;y_C)\ and\ \ G(0\,;y_G)$$

The coordinates of $\boldsymbol{y_C}$ and $\boldsymbol{y_G}$ of the points C and G respectively are calculated as follows:

$$\begin{cases} y_C = 2\,l\sin\dfrac{\pi}{3} = l\sqrt{3} \\ y_G = l\tan\dfrac{\pi}{6} = \dfrac{l\sqrt{3}}{3} \end{cases} \Rightarrow C\left(0\,; l\sqrt{3}\right)\ and\ \ G\left(0\,;\dfrac{l\sqrt{3}}{3}\right)$$

$$\overrightarrow{AG} = (x_G - x_A)\vec{\imath} + (y_G - y_A)\vec{\jmath} = l\,\vec{\imath} + \frac{l\sqrt{3}}{3}\,\vec{\jmath}\ \ ; \|\overrightarrow{AG}\| = \sqrt{l^2 + \left(\frac{l\sqrt{3}}{3}\right)^2} = \frac{2l\sqrt{3}}{3}$$

$$\overrightarrow{BG} = (x_G - x_B)\vec{\imath} + (y_G - y_B)\vec{\jmath} = -l\,\vec{\imath} + \frac{l\sqrt{3}}{3}\,\vec{\jmath}\ \ ; \|\overrightarrow{BG}\| = \sqrt{l^2 + \left(\frac{l\sqrt{3}}{3}\right)^2} = \frac{2l\sqrt{3}}{3}$$

$$\overrightarrow{CG} = (x_G - x_C)\vec{\imath} + (y_G - y_C)\vec{\jmath} = 0\,\vec{\imath} - 2\frac{l\sqrt{3}}{3}\,\vec{\jmath}\ \ ; \|\overrightarrow{CG}\| = \sqrt{\left(-2\frac{l\sqrt{3}}{3}\right)^2} = \frac{2l\sqrt{3}}{3}$$

$$\begin{cases} V_A\,(C) = k\,\dfrac{Q_A}{\|\overrightarrow{AC}\|} + C_A = k\,\dfrac{3Q}{2l\sqrt{3}} + C_A \\ V_B\,(G) = k\,\dfrac{Q_B}{\|\overrightarrow{BG}\|} + C_B = k\,\dfrac{3Q}{2l\sqrt{3}} + C_B \\ V_C\,(G) = k\,\dfrac{Q_C}{\|\overrightarrow{CG}\|} + C_C = k\,\dfrac{3Q}{2l\sqrt{3}} + C_C \end{cases}$$

$$V_{Net}\,(G) = k\,\frac{9Q}{2l\sqrt{3}} + (C_A + C_B + C_C)$$

We assume $C_A + C_B + C_C = C$

$$V_{Net}\,(G) = k\,\frac{9Q}{2l\sqrt{3}} + \mathrm{C}$$

To determine the constant c, we use the condition:

$$V_{Net}\,(l \to \infty) = 0 \Rightarrow C = 0 \Rightarrow V_{Net}\,(G) = 0$$

$$V(G) = k\,\frac{3Q\sqrt{3}}{2l}$$

¤ *Compute the total electric field value at the centroid G.*

$$\overrightarrow{E_{Net}}(G) = \sum_i \overrightarrow{E_\imath}(G) = \sum_i K\,\frac{q_i}{\|\overrightarrow{P_\imath G}\|^3}\,\overrightarrow{P_\imath G} = \overrightarrow{E_A}(G) + \overrightarrow{E_B}(G) + +\overrightarrow{E_C}(G)$$

$$\begin{cases} \overrightarrow{E_A}(G) = k\dfrac{Q_A}{\|\overrightarrow{AG}\|^3}\overrightarrow{AG} = k\dfrac{Q}{\left(\dfrac{2l\sqrt{3}}{3}\right)^3}\left(l\,\vec{\imath} + \dfrac{l\sqrt{3}}{3}\,\vec{\jmath}\right) \\ \overrightarrow{E_B}(G) = k\dfrac{Q_B}{\|\overrightarrow{BG}\|^3}\overrightarrow{BG} = k\dfrac{Q}{\left(\dfrac{2l\sqrt{3}}{3}\right)^3}\left(-l\,\vec{\imath} + \dfrac{l\sqrt{3}}{3}\,\vec{\jmath}\right) \\ \overrightarrow{E_B}(G) = k\dfrac{Q_B}{\|\overrightarrow{BG}\|^3}\overrightarrow{CG} = k\dfrac{Q}{\left(\dfrac{2l\sqrt{3}}{3}\right)^3}\left(-2\dfrac{l\sqrt{3}}{3}\,\vec{\jmath}\right) \end{cases}$$

$$\overrightarrow{E_{Net}}(G) = \overrightarrow{E_A}(G) + \overrightarrow{E_B}(G) + \overrightarrow{E_C}(G)$$

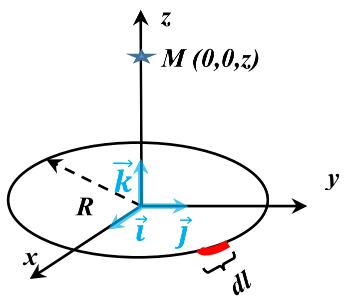
$$\overrightarrow{E_{Net}}(G) = k\dfrac{Q}{\left(\dfrac{2l\sqrt{3}}{3}\right)^3}\left[l\,\vec{\imath} + \dfrac{l\sqrt{3}}{3}\,\vec{\jmath} - l\,\vec{\imath} + \dfrac{l\sqrt{3}}{3}\,\vec{\jmath} - 2\dfrac{l\sqrt{3}}{3}\,\vec{\jmath}\right] = \vec{0}$$

¤ *If the charge μC $Q = -3\ \mu C$ is positioned at the centroid G, calculate the resultant force acting upon it.*

$$\overrightarrow{F_{Net/G}} = Q_G\overrightarrow{E_{Net}}(G) = \vec{0}$$

Exercise 15:

*Consider a charge **q** uniformly distributed, with a linear density **λ**, on a circumference centered at **O** and with a radius **R** as shown in the opposite figure*



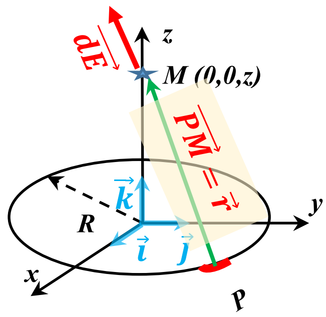
1- *Write the expression for the elemental charge **dq** carried along the elemental length **dl**.*
2- *Write the expression of the elemental electric field $\overrightarrow{\mathbf{dE}}$ generated by the elemental charge **dq** at the point **M***
3- *Calculate the net electric field $\vec{E}$ generated by the charge **q** at the point **M***
4- *Calculate/deduce the net electric potential at the point **M***
Calculate both electric field and electric potential at the point (0,0,0)

Solution:

♣ *Write the expression for the elemental charge **dq** carried along the elemental length **dl**.*

$$\lambda = \frac{q}{L} = \frac{dq}{dl} \Rightarrow dq = \ \ dl = \lambda\,R\,d\theta \qquad ;where\ dl = R\,d\theta$$

♣ *Write the expression of the elemental electric field $\overrightarrow{dE}$ generated by the elemental charge **dq** at the point **M***



$$d\vec{E}(M) = \frac{Kdq}{\|\overrightarrow{PM}\|^3}\,\overrightarrow{PM} = \frac{Kdq}{r^3}\,r\,\overrightarrow{u_r} = \frac{Kdq}{r^2}\,\overrightarrow{u_r}$$

$$where\ \overrightarrow{PM} = \vec{r} = r\,\overrightarrow{u_r}\ ;\ r = \|\overrightarrow{PM}\|$$

$$d\vec{E}(M) = \frac{K\lambda R\,d\theta}{\|\overrightarrow{PM}\|^3}\,\overrightarrow{PM} = \frac{K\lambda R\,d\theta}{r^2}\,\overrightarrow{u_r}$$

Coordinates of point P; The analytical expression for vector PM and its magnitude is written as follows.

$$P(x_P; y_P; 0) \qquad M(0; 0; z)$$

$$x_P = R\cos\theta \quad ; \quad y_P = R\sin\theta$$

$$\overrightarrow{PM} = (x_M - x_P)\vec{\imath} + (y_M - y_P)\vec{\jmath} + (z_M - z_P)\vec{k} = -R\cos\theta\,\vec{\imath} - R\sin\theta\,\vec{\jmath} + z\,\vec{k}$$

$$\|\overrightarrow{PM}\| = \sqrt{((-R\cos\theta)^2 + (-R\sin\theta)^2) + z^2} = \sqrt{R^2 + z^2}$$

$$d\vec{E}(M) = \frac{K\lambda R\,d\theta}{\|\overrightarrow{PM}\|^3}\,\overrightarrow{PM} \Rightarrow d\vec{E}(M) = \frac{K\lambda R\,d\theta}{(R^2+z^2)^{\frac{3}{2}}}\left(-R\cos\theta\,\vec{\imath} - R\sin\theta\,\vec{\jmath} + z\,\vec{k}\right)$$

♣ *Calculate the net electric field $\vec{E}$ generated by the charge **q** at the point **M***

$$\vec{E}(M) = \int d\vec{E}(M) = \int \frac{K\lambda R\,d\theta}{(R^2+z^2)^{\frac{3}{2}}}\left(-R\cos\theta\,\vec{\imath} - R\sin\theta\,\vec{\jmath} + z\,\vec{k}\right)$$

$$\vec{E}(M) = \int_0^{2\pi}\left(-\frac{R^2K\lambda\cos\theta\ d\theta}{(R^2+z^2)^{\frac{3}{2}}}\right)\vec{\imath} + \int_0^{2\pi}\left(\frac{-R^2K\lambda\sin\theta\,d\theta}{(R^2+z^2)^{\frac{3}{2}}}\right)\vec{\jmath} + \int_0^{2\pi}\left(\frac{K\lambda R\ d\theta z}{(R^2+z^2)^{\frac{3}{2}}}\right)\vec{k}$$

$$\int_0^{2\pi}\left(-\frac{R^2K\lambda\cos\theta\,d\theta}{(R^2+z^2)^{\frac{3}{2}}}\right)\vec{\imath} = \frac{-R^2K\lambda}{(R^2+z^2)^{\frac{3}{2}}}\int_0^{2\pi}\cos\theta\,d\theta\,\vec{\imath} = \frac{-R^2K\lambda}{(R^2+z^2)^{\frac{3}{2}}}[\sin\theta]_0^{2\pi}\vec{\imath} = 0\,\vec{\imath}$$

$$\int_0^{2\pi}\left(-\frac{R^2K\lambda\sin\theta\,d\theta}{(R^2+z^2)^{\frac{3}{2}}}\right)\vec{\jmath} = \frac{-R^2K\lambda}{(R^2+z^2)^{\frac{3}{2}}}\int_0^{2\pi}\sin\theta\,d\theta\,\vec{\jmath} = \frac{-R^2K\lambda}{(R^2+z^2)^{\frac{3}{2}}}[-\cos\theta]_0^{2\pi}\vec{\jmath} = 0\,\vec{\jmath}$$

$$\int_0^{2\pi}\left(\frac{K\lambda R\ d\theta z}{(R^2+z^2)^{\frac{3}{2}}}\right)\vec{k} = \frac{RK\lambda\,z}{(R^2+z^2)^{\frac{3}{2}}}\int_0^{2\pi}d\theta\,\vec{k} = \frac{RK\lambda\,z}{(R^2+z^2)^{\frac{3}{2}}}[\theta]_0^{2\pi}\vec{k} = \frac{2\,\pi RK\lambda\,z}{(R^2+z^2)^{\frac{3}{2}}}\vec{k}$$

$$\vec{E}(M) = \frac{2\,\pi RK\lambda\, z}{(R^2+z^2)^{\frac{3}{2}}}\vec{k}$$

♣ *Calculate/deduce the net electric potential at the point* ***M***

1st method

$$V(M) = \int dV(M) = \int \frac{Kdq}{\|\overrightarrow{PM}\|} = \int_0^{2\pi} \frac{K\lambda R\, d\theta}{\sqrt{R^2+z^2}} = \left[\frac{K\lambda R\,\theta}{\sqrt{R^2+z^2}}\right]_0^{2\pi} = \frac{2\pi K\lambda R}{\sqrt{R^2+z^2}}$$

2nd method

$$dV = -\vec{E}.d\vec{r} = dV_x + dV_y + dV_z \;\; where \begin{cases} \vec{E} = E_x\,\vec{\imath} + E_y\,\vec{\jmath} + E_z\,\vec{k} \\ d\vec{r} = dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k} \end{cases}$$

$$dV = dV_x + dV_y + dV_z = -E_x\,dx - E_y\,dy - E_z\,dz$$

$$\begin{cases} dV_x = -E_x\,dx \\ dV_y = -E_y\,dy \\ dV_z = -E_z\,dz \end{cases} \Rightarrow \begin{matrix} dV_x = 0\,dx \\ dV_y = 0\,dy \\ dV_z = -\dfrac{2\,\pi RK\lambda\, z}{(R^2+z^2)^{\frac{3}{2}}}\,dz \end{matrix} \Rightarrow \begin{matrix} V_x = \int dV_x = \int 0\,dx = C_x \\ V_y = \int dV_y = \int 0\,dy = C_y \\ \int dV_z = \int -\dfrac{2\,\pi RK\lambda\, z}{(R^2+z^2)^{\frac{3}{2}}}\,dz \end{matrix}$$

$$V_z = \int dV_z = -2\,\pi RK\lambda \int \frac{z}{(R^2+z^2)^{\frac{3}{2}}}\,dz$$

To calculate the integral $\int \frac{z}{(R^2+z^2)^{3/2}}\,dz$ *, we perform the variable conversion process as follows:*

$$t^2 += R^2 + z^2 \Rightarrow 2t\,dt = 2z\,dz \Rightarrow dt = z\,dz$$

$$\int \frac{z}{(R^2+z^2)^{3/2}}\,dz \Rightarrow \int \frac{t}{(t^2)^{3/2}}\,dt = \int \frac{1}{t^2}\,dt = -\frac{1}{t}$$

We replace $t = \sqrt{R^2+z^2}$

$$\int \frac{z}{(R^2+z^2)^{3/2}}\,dz = -\frac{1}{\sqrt{R^2+z^2}}$$

Thus, we find:

$$V_z = -2\,\pi RK\lambda \int \frac{z}{(R^2+z^2)^{\frac{3}{2}}}\,dz = \frac{2\,\pi RK\lambda}{\sqrt{R^2+z^2}} + C_z$$

$$V(M) = V_x + V_y + V_z = \frac{2\,\pi RK\lambda}{\sqrt{R^2+z^2}} + C_x + C_y + C_z = \frac{2\,\pi RK\lambda}{\sqrt{R^2+z^2}} + C$$

Where C_x, C_y *and* C_z *are constants*

To determine the constant C, we use the condition:

$$V_{Net}\,(z \to \infty) = 0 \;\Rightarrow C = 0 \;\Rightarrow\; V(M) = 0$$

$$V(M) \;= \frac{2\,\pi R K \lambda}{\sqrt{R^2 + z^2}}$$

♣ *Calculate both electric field and electric potential at the point (0,0,0)*

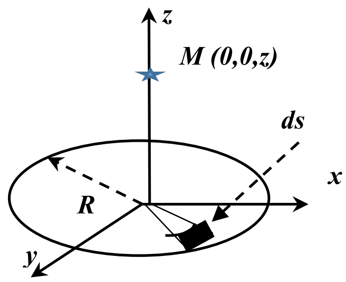
At the origin point (0,0,0) we set the value to zero.

$$z = 0 \;\Rightarrow \vec{E}(M) = \frac{2\,\pi R K \lambda\, z}{(R^2 + (0)^2)^{\frac{3}{2}}}\vec{k} = \vec{0}$$

$$z = 0 \;\Rightarrow V(M) = \frac{2\,\pi R K \lambda}{\sqrt{R^2 + (0)^2}} = 2\,\pi K \lambda$$

Exercise 16:

*We consider a positive charge **q** uniformly distributed over the surface of a disk (**σ**) with infinitesimal thickness as shown in the opposite figure*



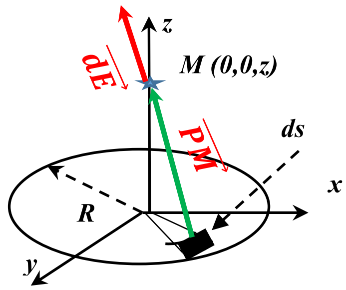
1- *Write the expression for the elemental charge **dq** carried along the elemental length **ds**.*
2- *Write the expression of the elemental electric field $\overrightarrow{\boldsymbol{dE}}$ generated by the elemental charge **dq** at the point **M***
3- *Calculate the net electric field $\overrightarrow{\boldsymbol{E}}$ generated by the charge **q** at the point **M***
4- *Calculate/deduce the net electric potential at the point **M***
5- *If the disk becomes an infinitely dimensional plane, find expressions for the electric field of this infinite plane.*

Solution:

♣ *Write the expression for the elemental charge **dq** carried along the elemental length **ds**.*

$$\sigma = \frac{q}{S} = \frac{dq}{dls} \Rightarrow dq = \sigma\, ds = \;\sigma\, r\, dr\, d\theta \qquad ; where\ ds = r\, dr\, d\theta$$

♣ *Write the expression of the elemental electric field* $\overrightarrow{dE}$ *generated by the elemental charge* ***dq*** *at the point* ***M***



$$d\vec{E}(M) = \frac{Kdq}{\|\overrightarrow{PM}\|^3}\,\overrightarrow{PM}$$

$$d\vec{E}(M) = \frac{K\sigma\, r\, dr\, d\theta}{\|\overrightarrow{PM}\|^3}\,\overrightarrow{PM}$$

Coordinates of point P; The analytical expression for vector PM and its magnitude is written as follows.

$$P(x_P; y_P; 0) \qquad M(0;0;z)$$

$$x_P = r\cos\theta \quad ; \quad y_P = r\sin\theta$$

$$\overrightarrow{PM} = (x_M - x_P)\vec{\imath} + (y_M - y_P)\vec{\jmath} + (z_M - z_P)\vec{k} = -r\cos\theta\,\vec{\imath} - r\sin\theta\,\vec{\jmath} + z\,\vec{k}$$

$$\|\overrightarrow{PM}\| = \sqrt{((-r\cos\theta)^2 + (-r\sin\theta)^2) + z^2} = \sqrt{r^2+z^2}$$

$$d\vec{E}(M) = \frac{K\sigma\, r\, dr\, d\theta}{\|\overrightarrow{PM}\|^3}\,\overrightarrow{PM} = \frac{K\sigma\, r\, dr\, d\theta}{(r^2+z^2)^{\frac{3}{2}}}\left(-r\cos\theta\,\vec{\imath} - r\sin\theta\,\vec{\jmath} + z\,\vec{k}\right)$$

♣ *Calculate the net electric field* $\vec{E}$ *generated by the charge* ***q*** *at the point* ***M***

$$\vec{E}(M) = \int d\vec{E}(M) = \int \frac{K\,\sigma\, r\, dr\, d\theta}{(r^2+z^2)^{\frac{3}{2}}}\left(-r\cos\theta\,\vec{\imath} - r\sin\theta\,\vec{\jmath} + z\,\vec{k}\right)$$

$$\vec{E}(M) = \int_0^R \frac{-r^2K\sigma}{(r^2+z^2)^{\frac{3}{2}}}dr \int_0^{2\pi} \cos\theta\; d\theta\,\vec{\imath} + \int_0^R \frac{-r^2K\sigma}{(r^2+z^2)^{\frac{3}{2}}}dr \int_0^{2\pi} \sin\theta\; d\theta\,\vec{\jmath} + \int_0^R \frac{zrK\sigma}{(r^2+z^2)^{\frac{3}{2}}}dr \int_0^{2\pi} d\theta\,\vec{k}$$

$$\vec{E}(M) = I_1\vec{\imath} + I_2\,\vec{\jmath} + I_3\vec{k}$$

$$I_1 = \int_0^R \frac{-r^2K\sigma}{(r^2+z^2)^{\frac{3}{2}}}dr \int_0^{2\pi} \cos\theta\; d\theta = \int_0^R \frac{-r^2K\sigma}{(r^2+z^2)^{\frac{3}{2}}}dr \times |\sin\theta|_0^{2\pi} = 0$$

$$I_2 = \int_0^R \frac{-r^2K\sigma}{(r^2+z^2)^{\frac{3}{2}}}dr \int_0^{2\pi} \sin\theta\; d\theta = \int_0^R \frac{-r^2K\sigma}{(r^2+z^2)^{\frac{3}{2}}}dr \times [\cos\theta]_0^{2\pi} = 0$$

$$I_3 = \int_0^R \frac{rK\sigma\, z}{(r^2+z^2)^{\frac{3}{2}}}dr \int_0^{2\pi} d\theta = \int_0^R \frac{rK\sigma\, z}{(r^2+z^2)^{\frac{3}{2}}}dr\; [\theta]_0^{2\pi} = 2\pi K\sigma\, z \int_0^R \frac{r}{(r^2+z^2)^{\frac{3}{2}}}dr$$

To achieve this integral $\int_0^R \frac{r}{(r^2+z^2)^{\frac{3}{2}}} dr$, *we perform a change of variable as follows:*

$$t^2 = r^2 + z^2 \Rightarrow 2t\,dt = 2r\,dr \Rightarrow \quad dt = r\,dr$$

$$\int \frac{r}{(r^2+z^2)^{3/2}}\,dr \Rightarrow \int \frac{t}{(t^2)^{3/2}}\,dt = \int \frac{1}{t^2}\,dt = -\frac{1}{t}$$

We replace $t = \sqrt{r^2+z^2}$

$$\int_0^R \frac{r}{(r^2+z^2)^{\frac{3}{2}}} dr = \left| -\frac{1}{\sqrt{r^2+z^2}} \right|_0^R = \frac{1}{\sqrt{z^2}} - \frac{1}{\sqrt{R^2+z^2}}$$

$$\vec{E}(M) = 2\pi K\sigma\, z\left(\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right)\vec{k} = 2\pi K\sigma \left(\frac{z}{|z|} - \frac{z}{\sqrt{R^2+z^2}}\right)\vec{k}$$

$$\begin{cases} For\ z > 0 \Rightarrow |z| = z \Rightarrow \vec{E}(M) = 2\pi K\sigma\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)\vec{k} \\ For\ z < 0 \Rightarrow |z| = -z \Rightarrow \vec{E}(M) = 2\pi K\sigma\left(-1 - \frac{z}{\sqrt{R^2+z^2}}\right)\vec{k} \end{cases}$$

♣ *Calculate/deduce the net electric potential at the point* ***M***

$$V(M) = \int dV(M) = \int \frac{Kdq}{\|\overrightarrow{PM}\|} = \int_0^R \frac{K\sigma\, r}{\sqrt{r^2+z^2}} dr \int_0^{2\pi} d\theta = \int_0^R \frac{K\sigma\, r}{\sqrt{r^2+z^2}} dr\, |\theta|_0^{2\pi}$$

$$V(M) = \pi K\sigma \int_0^R \frac{2\,r}{\sqrt{r^2+z^2}} dr = \pi K\sigma \int_0^R 2\,r\,(r^2+z^2)^{-\frac{1}{2}}\,dr$$

As a reminder:

$$\int (f(x))^n \frac{df(x)}{dx}\,dx = \frac{1}{n+1}(f(x))^{n+1}$$

By matching the integral we want to calculate with the integral given in the reminder, we find

$$\begin{cases} (r^2+z^2)^{-1/2}\ represents\ (f(r))^n\ where\ n = -\frac{1}{2} \\ 2r\,dr\ represents\ \frac{df(r)}{dr}\,dr \end{cases} \Rightarrow \begin{cases} (f(r))^{n+1} = (r^2+z^2)^{1/2} \\ \frac{1}{n+1} = \frac{1}{-\frac{1}{2}+1} = 2 \end{cases}$$

$$\int_0^R 2\,r\,(r^2+z^2)^{-\frac{1}{2}}\,dr = \left|2\ (r^2+z^2)^{1/2}\right|_0^R = 2\left(\sqrt{R^2+z^2} - \sqrt{z^2}\right)$$

$$V(M) = 2\pi K\sigma\left(\sqrt{R^2+z^2} - |z|\right)$$

$$\begin{cases} For\ z > 0 \Rightarrow |z| = z \ \Rightarrow \ V(M) = 2\pi K\sigma\left(\sqrt{R^2+z^2} - z\right) \\ For\ z < 0 \ \Rightarrow |z| = -z \ \Rightarrow \ V(M) = 2\pi K\sigma\left(\sqrt{R^2+z^2} + z\right) \end{cases}$$

♣ *If the disk becomes an infinitely dimensional plane, find expressions for the electric field of this infinite plane.*

$$\vec{E}(z \to \infty) = 2\pi K\sigma\left(\frac{z}{|z|}\right)\vec{k} = \begin{cases} For\ z > 0 \Rightarrow |z| = z \ \Rightarrow \ \vec{E}(M) = 2\pi K\sigma\,\vec{k} \\ For\ z < 0 \ \Rightarrow |z| = -z \ \Rightarrow \ \vec{E}(M) = -2\pi K\sigma\,\vec{k} \end{cases}$$

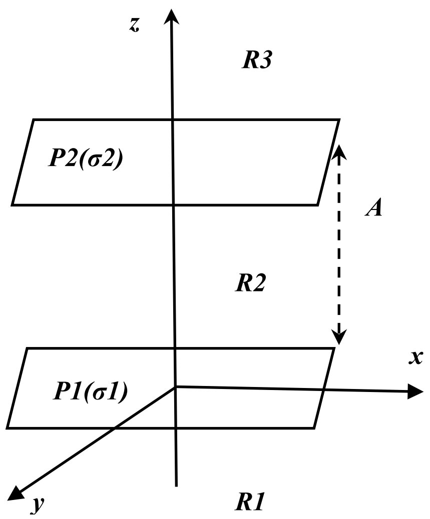
$$K = \frac{1}{4\pi\varepsilon_0}$$

$$\vec{E}(M) = \begin{cases} For\ z > 0 \ \Rightarrow \ \vec{E}(M) = \dfrac{\sigma}{2\varepsilon_0}\,\vec{k} \\ For\ z < 0 \ \Rightarrow \ \vec{E}(M) = -\dfrac{\sigma}{2\varepsilon_0}\,\vec{k} \end{cases}$$

Exercise 16:

As outlined in the second exercise, the electric field expression for an infinite plane with a positive surface charge density $\boldsymbol{\sigma}$ *is given as follows:* $\vec{E} = \begin{cases} \frac{\sigma}{2\varepsilon_0} & if\ z > 0 \\ \frac{-\sigma}{2\varepsilon_0} & if\ z < 0 \end{cases}$

Consider two infinite parallel planes, denoted as $\boldsymbol{P_1}$ *and* $\boldsymbol{P_2}$*, each carrying surface charge distributions* σ_1 *and* σ_2*, respectively. These planes are positioned at a separation distance A, as depicted in the accompanying figure.*

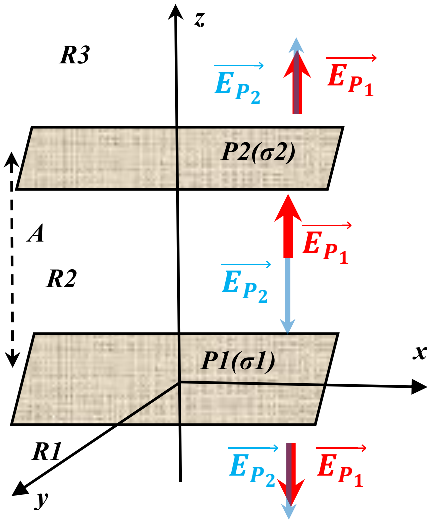


1- *Caculate the electric filed generated by each plane at the regions R1, R2 and R3*
2- *Calculate the electric potential V at each region*
3- *Discuss the following cases of charge distribution* $(\sigma 1 = \sigma 2 = \sigma)$, $(\sigma 1 = -\sigma 2 = \sigma)$

Solution:

♣ ***Caculate the electric filed generated by each plane at the regions R1, R2 and R3***

*The electric field vector at the points located **above** the plane is unform with constant magnitude and directed **upward** (in the same direction as the unit vector $\vec{k}$), while at the points located **below** the plane , the electric field vector is unform with constant magnitude and directed **downward** (opposite to the unit vector $\vec{k}$).*



Region R1: z < 0

According to the principle of superposition of electric fields

$$\vec{E} = \overrightarrow{E_{P_1}} + \overrightarrow{E_{P_2}} \text{ where } \begin{cases} \overrightarrow{E_{P_1}} = \dfrac{-\sigma_1}{2\varepsilon_0}\vec{k} \\ \overrightarrow{E_{P_2}} = \dfrac{-\sigma_2}{2\varepsilon_0}\vec{k} \end{cases}$$

$$\vec{E} = \frac{-\sigma_1 - \sigma_2}{2\varepsilon_0}\vec{k}$$

Region R2: 0 < z < A

$$\vec{E} = \overrightarrow{E_{P_1}} + \overrightarrow{E_{P_2}} \text{ where } \begin{cases} \overrightarrow{E_{P_1}} = \dfrac{+\sigma_1}{2\varepsilon_0}\vec{k} \\ \overrightarrow{E_{P_2}} = \dfrac{-\sigma_2}{2\varepsilon_0}\vec{k} \end{cases} \Rightarrow \vec{E} = \frac{\sigma_1 - \sigma_2}{2\varepsilon_0}\vec{k}$$

Region R3: A < z

$$\vec{E} = \overrightarrow{E_{P_1}} + \overrightarrow{E_{P_2}} \text{ where } \begin{cases} \overrightarrow{E_{P_1}} = \dfrac{+\sigma_1}{2\varepsilon_0}\vec{k} \\ \overrightarrow{E_{P_2}} = \dfrac{+\sigma_2}{2\varepsilon_0}\vec{k} \end{cases} \Rightarrow \vec{E} = \frac{\sigma_1 + \sigma_2}{2\varepsilon_0}\vec{k}$$

♣ ***Calculate the electric potential V at each region***

$$V = \int dV = \int -\vec{E}.d\vec{r} \quad where \begin{cases} \vec{E} = E_x\,\vec{\imath} + E_y\,\vec{\jmath} + E_z\,\vec{k} \\ d\vec{r} = dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k} \end{cases}$$

$$dV = dV_x + dV_y + dV_z = -E_x\,dx - E_y\,dy - E_z\,dz$$

$$\begin{cases} dV_x = -E_x\,dx & & V_x = \int dV_x = -\int E_x\,dx \\ dV_y = -E_y\,dy & \Rightarrow & V_y = \int dV_y = -\int E_y\,dy \\ dV_z = -E_z\,dz & & V_z = \int dV_z = -\int E_z\,dz \end{cases}$$

***Region R1:* $z < 0$**

$$\vec{E} = \frac{-\sigma_1 - \sigma_2}{2\varepsilon_0}\vec{k}$$

$$\begin{cases} V_x = -\int 0\, dx = C_x \\ V_y = -\int 0\, dy = C_y \\ V_z = -\int \frac{-\sigma_1 - \sigma_2}{2\varepsilon_0}\, dz = \frac{\sigma_1 + \sigma_2}{2\varepsilon_0}\, z + C_z \end{cases}$$

$$V = V_x + V_y + V_z = \frac{\sigma_1 + \sigma_2}{2\varepsilon_0}\, z + C_x + C_y + C_z = \frac{\sigma_1 + \sigma_2}{2\varepsilon_0}\, z + C_1 \; (where\; C_1 = C_x + C_y + C_z)$$

***Region R2:* $0 < z < A$**

$$\vec{E} = \frac{\sigma_1 - \sigma_2}{2\varepsilon_0}\vec{k}$$

$$\begin{cases} V_x = -\int 0\, dx = C_x \\ V_y = -\int 0\, dy = C_y \\ V_z = -\int \frac{\sigma_1 - \sigma_2}{2\varepsilon_0}\, dz = \frac{\sigma_2 - \sigma_1}{2\varepsilon_0}\, z + C_z \end{cases}$$

$$V = V_x + V_y + V_z = \frac{\sigma_2 - \sigma_1}{2\varepsilon_0}\, z + C_x + C_y + C_z = \frac{\sigma_2 - \sigma_1}{2\varepsilon_0}\, z + C_2 \; (where\; C_2 = C_x + C_y + C_z)$$

***Region R3:* $A < z$**

$$\vec{E} = \frac{\sigma_1 + \sigma_2}{2\varepsilon_0}\vec{k}$$

$$\begin{cases} V_x = -\int 0\, dx = C_x \\ V_y = -\int 0\, dy = C_y \\ V_z = -\int \frac{\sigma_1 + \sigma_2}{2\varepsilon_0}\, dz = \frac{\sigma_2 + \sigma_1}{2\varepsilon_0}\, z + C_z \end{cases}$$

$$V = V_x + V_y + V_z = \frac{\sigma_2 + \sigma_1}{2\varepsilon_0}\, z + C_x + C_y + C_z = -\frac{\sigma_2 + \sigma_1}{2\varepsilon_0}\, z + C_3 \; (where\; C_3 = C_x + C_y + C_z)$$

- ***Discuss the following cases of charge distribution*** $(\sigma 1 = \sigma 2 = \sigma)$, $(\sigma 1 = -\sigma 2 = \sigma)$

$$\begin{cases} \vec{E} = \frac{-\sigma_1 - \sigma_2}{2\varepsilon_0}\vec{k} & z < 0 \\ \vec{E} = \frac{\sigma_1 - \sigma_2}{2\varepsilon_0}\vec{k} & 0 < z < A \\ \vec{E} = \frac{\sigma_1 + \sigma_2}{2\varepsilon_0}\vec{k} & A < z \end{cases} \qquad \begin{cases} V = \frac{\sigma_1 + \sigma_2}{2\varepsilon_0}\, z + C_1 & z < 0 \\ V = \frac{\sigma_2 - \sigma_1}{2\varepsilon_0}\, z + C_2 & 0 < z < A \\ V = -\frac{\sigma_2 + \sigma_1}{2\varepsilon_0}\, z + C_3 & A < z \end{cases}$$

$\boldsymbol{Case\ of: \sigma_1 = \sigma_2 = \sigma}$

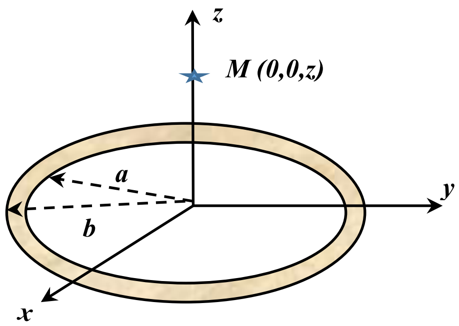
$$\begin{cases} \vec{E} = \frac{-\sigma}{\varepsilon_0}\vec{k} & z < 0 \\ \vec{E} = 0\,\vec{k} & 0 < z < A \\ \vec{E} = \frac{\sigma}{\varepsilon_0}\vec{k} & A < z \end{cases} \qquad \begin{cases} V = \frac{\sigma}{\varepsilon_0}\, z + C_1 & z < 0 \\ V = C_2 & 0 < z < A \\ V = -\frac{\sigma}{\varepsilon_0} + C_3 & A < z \end{cases}$$

$\boldsymbol{Case\ of: \sigma 1 = -\sigma 2 = \sigma}$

$$\begin{cases} \vec{E} = \frac{-\sigma_1 - \sigma_2}{2\varepsilon_0}\vec{k} & z < 0 \\ \vec{E} = \frac{\sigma_1 - \sigma_2}{2\varepsilon_0}\vec{k} & 0 < z < A \\ \vec{E} = \frac{\sigma_1 + \sigma_2}{2\varepsilon_0}\vec{k} & A < z \end{cases} \qquad \begin{cases} V = \frac{\sigma_1 + \sigma_2}{2\varepsilon_0}\, z + C_1 & z < 0 \\ V = \frac{\sigma_2 - \sigma_1}{2\varepsilon_0}\, z + C_2 & 0 < z < A \\ V = -\frac{\sigma_2 + \sigma_1}{2\varepsilon_0}\, z + C_3 & A < z \end{cases}$$

Exercise 17:

*A charge is uniformly distributed on the surface of a ring of inner radius "**a**" and outer radius "**b**".*

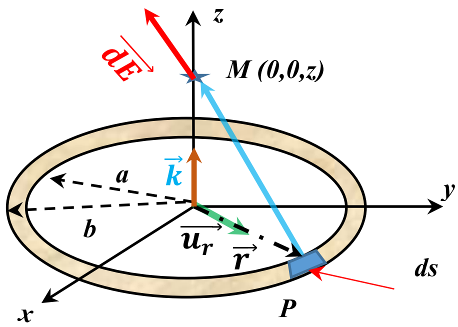


1- *Write the expression for the differential charge in the magnitude of a differential surface. Then write the expressions for the resulting electric field and electric potential at point M(0; 0; z)*

*2- Derive the expressions for the electric field at the point M(0; 0; z) when **a** → **0** and **b** → ∞.*

Solution:

- *Write the expression for the differential charge in the magnitude of a differential surface. Then write the expressions for the resulting electric field and electric potential at point M(0; 0; z)*



$$\sigma = \frac{q}{S} = \frac{dq}{dls} \Rightarrow dq = \sigma\, ds$$

$$= \sigma\, r\, dr\, d\theta \quad ; where\ ds = r\, dr\, d\theta$$

The differential electric field

$$d\vec{E}(M) = \frac{Kdq}{\left\|\overrightarrow{PM}\right\|^3}\,\overrightarrow{PM}$$

$$\overrightarrow{PM} = -r\,\overrightarrow{u_r} + z\,\vec{k} \Rightarrow \left\|\overrightarrow{PM}\right\| = \sqrt{r^2+z^2}$$

$$d\vec{E}(M) = \frac{K\sigma\, r\, dr\, d\theta}{(r^2+z^2)^{\frac{3}{2}}}\left(-r\,\overrightarrow{u_r} + z\,\vec{k}\right) = -\frac{K\sigma\, r^2\, dr\, d\theta}{(r^2+z^2)^{\frac{3}{2}}}\,\overrightarrow{u_r} + \frac{K\sigma\, r\, dr\, d\theta}{(r^2+z^2)^{\frac{3}{2}}}\,z\,\vec{k}$$

The expression for the resulting electric field

$$\vec{E}(M) = \int d\vec{E}(M) = -\iint \frac{K\sigma\, r^2\,\overrightarrow{u_r}}{(r^2+z^2)^{\frac{3}{2}}}\,dr\, d\theta + \iint \frac{K\sigma\, rz}{(r^2+z^2)^{\frac{3}{2}}}\,dr\, d\theta\,\vec{k} = -I_r + I_z\vec{k}$$

A highly important note:* *The unit vector* $\vec{k}$ *can be taken out of the integral since it is a constant vector, whereas the unit vector* $\overrightarrow{u_r}$ *cannot be taken out because it is not constant and varies with the variable .

$$I_r = \iint \frac{K\sigma\, r^2\,\overrightarrow{u_r}}{(r^2+z^2)^{\frac{3}{2}}}\,drd\theta = \int_a^b \frac{K\sigma\, r^2\, dr}{(r^2+z^2)^{\frac{3}{2}}}\int_0^{2\pi}\overrightarrow{u_r}\, d\theta$$

To calculate this integral, we write the unit vector analytical expression in Cartesian coordinates, as previously discussed in the Physics 1 lessons.

$$\vec{u}_r = \cos\theta\ \vec{\imath} + \sin\theta\ \vec{\jmath}$$

$$\int_0^{2\pi}\overrightarrow{u_r}\, d\theta = \int_0^{2\pi}\cos\theta\, d\theta\ \vec{\imath} + \int_0^{2\pi}\sin\theta\ d\theta\,\vec{\jmath} = [\sin\theta]_0^{2\pi}\,\vec{\imath} + [-\cos\theta]_0^{2\pi}\,\vec{\jmath} = \vec{0}$$

Thus

$$I_r = \iint \frac{K\sigma\, r^2\, \overrightarrow{u_r}}{(r^2+z^2)^{\frac{3}{2}}} dr d\theta = \int_a^b \frac{K\sigma\, r^2\, dr}{(r^2+z^2)^{\frac{3}{2}}} \int_0^{2\pi} \overrightarrow{u_r}\, d\theta = \vec{0}$$

$$I_z = \iint \frac{K\sigma\, rz}{(r^2+z^2)^{\frac{3}{2}}}\, dr\, d\theta = \int_a^b \frac{rK\sigma\, z}{(r^2+z^2)^{\frac{3}{2}}} dr \int_0^{2\pi} d\theta = \int_a^b \frac{rK\sigma\, z}{(r^2+z^2)^{\frac{3}{2}}} dr\ [\theta]_0^{2\pi}$$

$$I_z = 2\pi K\sigma\, z \int_a^b \frac{r}{(r^2+z^2)^{\frac{3}{2}}} dr$$

To complete this integral $\int_a^b \frac{r}{(r^2+z^2)^{\frac{3}{2}}} dr$, *we carry out a change of variable as follows:*

$$t^2 = r^2 + z^2 \Rightarrow 2t\, dt = 2r\, dr \Rightarrow \quad dt = r\, dr$$

$$\int \frac{r}{(r^2+z^2)^{3/2}}\, dr \Rightarrow \int \frac{t}{(t^2)^{3/2}}\, dt = \int \frac{1}{t^2}\, dt = -\frac{1}{t}$$

$We\ replace \quad t = \sqrt{r^2+z^2}$

$$\int_a^b \frac{r}{(r^2+z^2)^{\frac{3}{2}}} dr = \left| -\frac{1}{\sqrt{r^2+z^2}} \right|_a^b = \frac{1}{\sqrt{a^2+z^2}} - \frac{1}{\sqrt{b^2+z^2}}$$

$$\vec{E}(M) = 2\pi K\sigma\, z \left(\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{\sqrt{b^2+z^2}} \right) \vec{k}$$

The differential electric potential

Method 01

$$dV(M) = \frac{Kdq}{\|\overrightarrow{PM}\|} = \frac{K\sigma\, r\, dr\, d\theta}{\sqrt{r^2+z^2}}$$

$$V(M) = \int dV(M) = \iint \frac{K\sigma\, r\, dr\, d\theta}{\sqrt{r^2+z^2}} = \int_a^b \frac{K\sigma\, r}{\sqrt{r^2+z^2}} dr \int_0^{2\pi} d\theta$$

$$V(M) = \int_a^b \frac{K\sigma\, r}{\sqrt{r^2+z^2}} dr\, |\theta|_0^{2\pi}$$

$$V(M) = \pi K\sigma \int_a^b \frac{2\, r}{\sqrt{r^2+z^2}} dr$$

As a reminder:

$$\int (f(x))^n \frac{df(x)}{dx}\; dx = \frac{1}{n+1}(f(x))^{n+1}$$

By matching the integral we want to calculate with the integral given in the reminder, we find

$$\begin{cases} (r^2+z^2)^{-1/2} \textit{ represents } (f(r))^n \textit{ where } n = -\dfrac{1}{2} \\ 2r\, dr \textit{ represents } \dfrac{df(r)}{dr}\; dr \end{cases} \Rightarrow \begin{cases} (f(r))^{n+1} = \; (r^2+z^2)^{1/2} \\ \dfrac{1}{n+1} = \dfrac{1}{-\frac{1}{2}+1} = 2 \end{cases}$$

$$\int_a^b 2\, r\, (r^2+z^2)^{-\frac{1}{2}}\; dr = \left|2\; (r^2+z^2)^{1/2}\right|_a^b = 2\left(\sqrt{b^2+z^2}-\sqrt{a^2+z^2}\right)$$

$$V(M) = 2\pi K\sigma\left(\sqrt{b^2+z^2}-\sqrt{a^2+z^2}\right)$$

Method 02

$$dV = dV_r + dV_\theta + dV_z = -\vec{E}.d\vec{r} \;\; where \begin{cases} \vec{E} = E_r\, \overrightarrow{u_r} + E_\theta\; \overrightarrow{u_\theta} + E_z\; \vec{k} \\ d\vec{r} = dr\, \overrightarrow{u_r} + r\, d\theta\;\; \overrightarrow{u_\theta} + dz\; \vec{k} \end{cases}$$

$$dV = dV_r + dV_\theta + dV_z = -E_r\, dr - E_\theta\, r\, d\theta - E_z\, dz$$

$$\begin{cases} dV_r = -E_r\, dr \\ dV_\theta = -\, E_\theta\, r\, d\theta \\ dV_z = -E_z\, dz \end{cases}$$

$$\vec{E}(M) = 2\pi K\sigma\, z\left(\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{\sqrt{b^2+z^2}}\right)\vec{k}$$

$$\begin{cases} dV_r = -E_r\, dr = -0\, dr \;\Rightarrow V_r = C_r \\ dV_\theta = -\, E_\theta\, r\, d\theta = -0\, dr \;\Rightarrow V_\theta = C_\theta \\ dV_z = -E_z\, dz = -2\pi K\sigma\, z\left(\dfrac{1}{\sqrt{a^2+z^2}} - \dfrac{1}{\sqrt{b^2+z^2}}\right)\, dz \end{cases}$$

$$V_z = -\pi K\sigma \int\left(\frac{2z}{\sqrt{a^2+z^2}} - \frac{2z}{\sqrt{b^2+z^2}}\right)\, dz = -\pi K\sigma\left(\int \frac{2z}{\sqrt{a^2+z^2}}dz - \int \frac{2z}{\sqrt{b^2+z^2}}dz\right)$$

$$\int \frac{2\, z}{\sqrt{a^2+z^2}}dz = 2\sqrt{a^2+z^2}\;\; ; \;\int \frac{2\, z}{\sqrt{b^2+z^2}}dz = 2\sqrt{b^2+z^2}$$

$$V(M) = 2\pi K\sigma\;\left(\sqrt{b^2+z^2}-\sqrt{a^2+z^2}\right)$$

♣ *Derive the expressions for the electric field at the point M(0; 0; z) when **a** → 0 and **b** → ∞.*

$$\vec{E}(M) = 2\pi K\sigma\, z\left(\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{\sqrt{b^2+z^2}}\right)\vec{k}$$

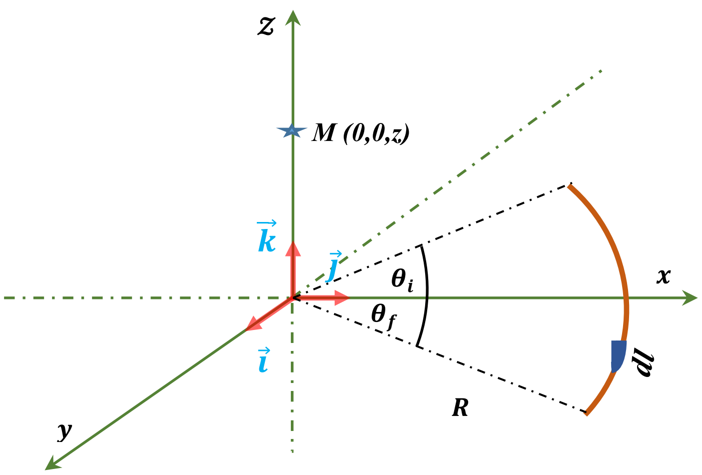
For $\boldsymbol{a} \rightarrow 0$ *and* $\boldsymbol{b} \rightarrow \infty$

$$\vec{E}(M) = 2\pi K\sigma\left(\frac{z}{\sqrt{z^2}}\right)\vec{k} = \frac{2\pi\sigma}{4\pi\varepsilon_0}\left(\frac{z}{|z|}\right)\vec{k}$$

$$\vec{E}(M) = \begin{cases} For\ z > 0 \ \Rightarrow\ \vec{E}(M) = \dfrac{\sigma}{2\varepsilon_0}\vec{k} \\ For\ z < 0 \ \Rightarrow\ \vec{E}(M) = -\dfrac{\sigma}{2\varepsilon_0}\vec{k} \end{cases}$$

Exercise 18:

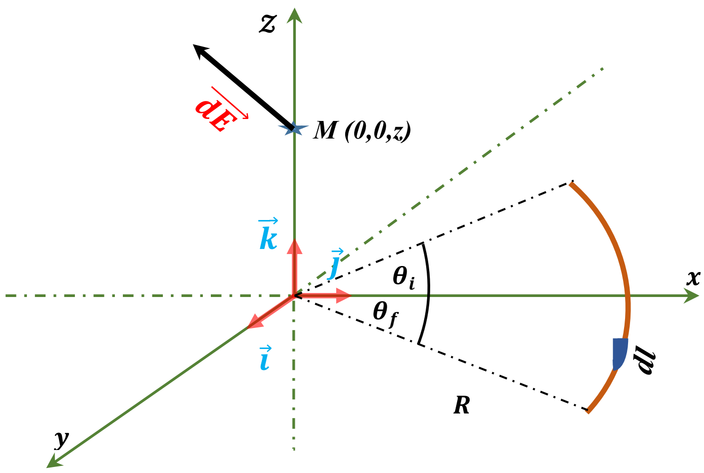
The given figure depicts a circular arc with radius $\boldsymbol{R}$ *extending in the plane (oxy) between two boundary angles* $(\boldsymbol{\theta_i})$*and*$(\boldsymbol{\theta_f})$ *with respect to (oy) axis. Along this arc, a positive electric charge is distributed evenly with a linear charge density* $\boldsymbol{\lambda}$*.*



- ♣ *Represent at point* ***M (0; 0; z)*** *on the figure above the vector of the differential electric field* $\overrightarrow{\mathbf{dE}}$ *resulting from the differential charge* $\mathbf{dq}$ *dustributed along the differential length* $\mathbf{dl}$*, then express its analytical expression in the Cartesian coordinates.*
- ♣ *Determine the expression for the electric field* $\overrightarrow{\boldsymbol{E}}$ *at the point M, then derive the expression for the electric potential* ***V*** *at this point .*
- ♣ *Discuss the expressions of the electric field and the electric potential at the point M (0; 0; 0) For the following cases of angles:* $\left(\boldsymbol{\theta_i} = -\boldsymbol{\theta_f} = \frac{\pi}{4}\right)$ *;and* $\left(\boldsymbol{\theta_i} = -\boldsymbol{\theta_f} = \boldsymbol{\pi}\right)$

Solution

- ♣ *Representing the differential electric field* $\overrightarrow{\mathbf{dE}}$ *at point* ***M (0; 0; z)***



♣ *Determining the expression for the differential electric field* $\overrightarrow{\boldsymbol{dE}}$ *at the point M*

$$d\vec{E}(M) = \frac{Kdq}{\|\overrightarrow{PM}\|^3}\,\overrightarrow{PM}$$

$$\lambda = \frac{q}{L} = \frac{dq}{dl} \Rightarrow dq = \quad dl = \lambda\, R\, d\theta \qquad ;where\ dl = R\, d\theta$$

Coordinates of point P; The analytical expression for vector $\overrightarrow{PM}$ *and its magnitude is written as follows.*

$$P(x_P; y_P; 0) \qquad M(0;0;z)$$

$$x_P = R\cos\theta \quad ; \quad y_P = R\sin\theta$$

$$\overrightarrow{PM} = (x_M - x_P)\vec{\imath} + (y_M - y_P)\vec{\jmath} + (z_M - z_P)\vec{k} = -R\cos\theta\,\vec{\imath} - R\sin\theta\,\vec{\jmath} + z\,\vec{k}$$

$$\|\overrightarrow{PM}\| = \sqrt{((-R\cos\theta)^2 + (-R\sin\theta)^2) + z^2} = \sqrt{R^2 + z^2}$$

$$d\vec{E}(M) = \frac{K\lambda R\, d\theta}{\|\overrightarrow{PM}\|^3}\,\overrightarrow{PM} \Rightarrow d\vec{E}(M) = \frac{K\lambda R\, d\theta}{(R^2+z^2)^{\frac{3}{2}}}\left(-R\cos\theta\,\vec{\imath} - R\sin\theta\,\vec{\jmath} + z\,\vec{k}\right)$$

♣ *Calculate the net electric field* $\vec{\boldsymbol{E}}$ *generated at the point* ***M***

$$\vec{E}(M) = \int d\vec{E}(M) = \int_{\theta_i}^{\theta_f} \frac{K\lambda R\, d\theta}{(R^2+z^2)^{\frac{3}{2}}}\left(-R\cos\theta\,\vec{\imath} - R\sin\theta\,\vec{\jmath} + z\,\vec{k}\right)$$

$$\vec{E}(M) = \int_{\theta_i}^{\theta_f}\left(-\frac{R^2K\lambda\cos\theta\ d\theta}{(R^2+z^2)^{\frac{3}{2}}}\right)\vec{\imath} + \int_{\theta_i}^{\theta_f}\left(\frac{-R^2K\lambda\sin\theta\, d\theta}{(R^2+z^2)^{\frac{3}{2}}}\right)\vec{\jmath} + \int_{\theta_i}^{\theta_f}\left(\frac{K\lambda R\, d\theta z}{(R^2+z^2)^{\frac{3}{2}}}\right)\vec{k}$$

$$\int_{\theta_i}^{\theta_f}\left(-\frac{R^2K\lambda\cos\theta\, d\theta}{(R^2+z^2)^{\frac{3}{2}}}\right)\vec{\imath} = \frac{-R^2K\lambda}{(R^2+z^2)^{\frac{3}{2}}}\int_{\theta_i}^{\theta_f}\cos\theta\, d\theta\,\vec{\imath} = \frac{-R^2K\lambda}{(R^2+z^2)^{\frac{3}{2}}}[\sin\theta]_{\theta_i}^{\theta_f}\vec{\imath}$$

$$= \frac{-R^2K\lambda}{(R^2+z^2)^{\frac{3}{2}}}\left[\sin\theta_f - \sin\theta_i\right]\vec{\imath} = \frac{R^2K\lambda}{(R^2+z^2)^{\frac{3}{2}}}\left[\sin\theta_i - \sin\theta_f\right]\vec{\imath}$$

$$\int_{\theta_i}^{\theta_f}\left(-\frac{R^2K\lambda\sin\theta\, d\theta}{(R^2+z^2)^{\frac{3}{2}}}\right)\vec{\jmath} = \frac{-R^2K\lambda}{(R^2+z^2)^{\frac{3}{2}}}\int_{\theta_i}^{\theta_f}\sin\theta\, d\theta\,\vec{\jmath} = \frac{-R^2K\lambda}{(R^2+z^2)^{\frac{3}{2}}}\left[-\cos\theta\right]_{\theta_i}^{\theta_f}\vec{\jmath}$$

$$= \frac{-R^2K\lambda}{(R^2+z^2)^{\frac{3}{2}}}\left[-\cos\theta_f + \cos\theta_i\right]\vec{\jmath} = \frac{R^2K\lambda}{(R^2+z^2)^{\frac{3}{2}}}\left[\cos\theta_f - \cos\theta_i\right]\vec{\jmath}$$

$$\int_{\theta_i}^{\theta_f}\left(\frac{K\lambda R\, d\theta z}{(R^2+z^2)^{\frac{3}{2}}}\right)\vec{k} = \frac{RK\lambda\, z}{(R^2+z^2)^{\frac{3}{2}}}\int_{\theta_i}^{\theta_f} d\theta\vec{k} = \frac{RK\lambda\, z}{(R^2+z^2)^{\frac{3}{2}}}\left[\theta\right]_{\theta_i}^{\theta_f}\vec{k} = \frac{RK\lambda\, z}{(R^2+z^2)^{\frac{3}{2}}}\left(\theta_f - \theta_i\right)\vec{k}$$

$$\vec{E}(M) = \frac{R^2K\lambda}{(R^2+z^2)^{\frac{3}{2}}}\left[\sin\theta_i - \sin\theta_f\right]\vec{\imath} + \frac{R^2K\lambda}{(R^2+z^2)^{\frac{3}{2}}}\left[\cos\theta_f - \cos\theta_i\right]\vec{\jmath} + \frac{RK\lambda\, z}{(R^2+z^2)^{\frac{3}{2}}}\left(\theta_f - \theta_i\right)\vec{k}$$

$$\vec{E}(M) = \frac{RK\lambda}{(R^2+z^2)^{\frac{3}{2}}}\left(R\left[\sin\theta_i - \sin\theta_f\right]\vec{\imath} + R\left[\cos\theta_f - \cos\theta_i\right]\vec{\jmath} + z\left(\theta_f - \theta_i\right)\vec{k}\right)$$

♣ *Derive the expression for the electric potential V at the point M*

$$V(M) = \int dV(M) = \int \frac{Kdq}{\left\|\overrightarrow{PM}\right\|} = \int_{\theta_i}^{\theta_f}\frac{K\lambda R\, d\theta}{\sqrt{R^2+z^2}} = \left[\frac{K\lambda R\,\theta}{\sqrt{R^2+z^2}}\right]_{\theta_i}^{\theta_f} = \frac{K\lambda R\left(\theta_f - \theta_i\right)}{\sqrt{R^2+z^2}}$$

♣ *Discuss the expressions of the electric field and the electric potential at the point M (0; 0; 0) For the following cases of angles:* $\left(\boldsymbol{\theta_i = -\theta_f = \frac{\pi}{4}}\right)$ *;and* $\left(\boldsymbol{\theta_i = -\theta_f = \pi}\right)$

For $\boldsymbol{\theta_i = -\theta_f}$

$$\vec{E}(M) = \frac{2RK\lambda}{(R^2+z^2)^{\frac{3}{2}}}\left(-R\sin\theta_f\ \vec{\imath} + z\ \theta_f\,\vec{k}\right)$$

$$V(M) = \frac{2K\lambda R\,\theta_f}{\sqrt{R^2+z^2}}$$

At thee point at the point M (0; 0; 0) $\Rightarrow z = 0$ *and* $\boldsymbol{\theta_f = -\frac{\pi}{4}}$

$$\vec{E}(M) = \frac{\sqrt{2}K\lambda}{R}\vec{\imath}$$

$$V(M) = -K\lambda\frac{\pi}{2}$$

At thee point at the point M (0; 0; 0) $\Rightarrow z = 0$ *and* $\boldsymbol{\theta_f} = -\boldsymbol{\pi}$

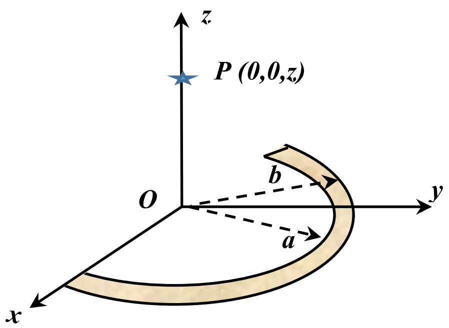
$$\vec{E}(M) = \vec{0}$$

$$V(M) = -2K\lambda\,\boldsymbol{\pi}$$

Exercise 19:

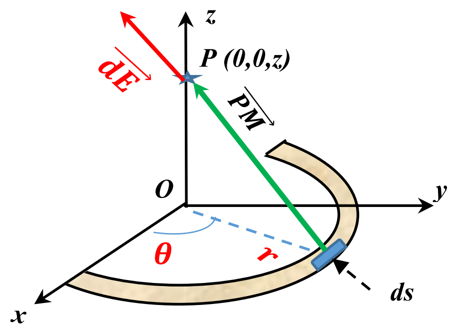
A semicircle with an inner radius of $\boldsymbol{a}$ *and an outer radius of* $\boldsymbol{b}$ *carries a uniform surface charge with a surface density of* $\boldsymbol{\sigma}$. *Calculate:*

- ♣ *Electric field and electric potential at point* $\boldsymbol{P}$, *which lies on the central axis of the ring at a distance of* σ *from its center.*
- ♣ *The electric field and the electric potential at point* $\boldsymbol{O}$ *(center of the ring).*



Solution:

- ♣ *The electric potential at point* $\boldsymbol{P}$, *which lies on the central axis of the ring at a distance of* σ *from its center.*



$$dV = \frac{Kdq}{\left\|\overrightarrow{PM}\right\|}$$

Coordinates of point P; The analytical expression for vector $\overrightarrow{PM}$ *and its magnitude in the Cartesian coordinates system is written as follows.*

$$P(x_P; y_P; 0) \qquad M(0;0;z)$$

$$x_P = r\cos\theta \quad ; \quad y_P = r\sin\theta$$

$$\overrightarrow{PM} = (x_M - x_P)\vec{\imath} + (y_M - y_P)\vec{\jmath} + (z_M - z_P)\vec{k} = -r\cos\theta\,\vec{\imath} - r\sin\theta\,\vec{\jmath} + z\,\vec{k}$$

$$\left\|\overrightarrow{PM}\right\| = (r^2 + z^2)^{\frac{1}{2}}$$

$$\sigma = \frac{q}{S} = \frac{dq}{dls} \Rightarrow dq = \sigma\, ds = \sigma\, r\, dr\, d\theta \qquad ; where\ ds = r\, dr\, d\theta$$

$$dV = \frac{Kdq}{\left\|\overrightarrow{PM}\right\|} = \frac{K\sigma\, r\, dr\, d\theta}{(r^2+z^2)^{\frac{1}{2}}} \Rightarrow V = \int dV = \int_a^b \int_0^\pi \frac{K\sigma\, r\, dr\, d\theta}{(r^2+z^2)^{\frac{1}{2}}} = \int_a^b \frac{K\sigma\, r\, dr}{(r^2+z^2)^{\frac{1}{2}}} \int_0^\pi d\theta$$

$$V = \int_a^b \frac{K\sigma\, r\, dr}{(r^2+z^2)^{\frac{1}{2}}} \int_0^\pi d\theta = \int_a^b \frac{K\sigma\, r\, dr}{(r^2+z^2)^{\frac{1}{2}}} \left|\theta\right|_0^\pi = \pi K\sigma \int_a^b \frac{r\, dr}{(r^2+z^2)^{\frac{1}{2}}}$$

This integral $\int_a^b \frac{r\,dr}{(r^2+z^2)^{\frac{1}{2}}}$ *can be calculated by following the rule:*

$$\int (f(x))^n \frac{df(x)}{dx}\, dx = \frac{1}{n+1}(f(x))^{n+1}$$

$$\int_a^b r\,(r^2+z^2)^{-\frac{1}{2}}\, dr = \left|\,(r^2+z^2)^{1/2}\right|_a^b = \left(\sqrt{b^2+z^2} - \sqrt{a^2+z^2}\right)$$

$$V = \pi K\sigma\left(\sqrt{b^2+z^2} - \sqrt{a^2+z^2}\right)$$

The electric field at point ***P***

$$\vec{E} = E_x\,\vec{\imath} + E_y\,\vec{\jmath} + E_z\,\vec{k}\ where \begin{cases} E_x = -\dfrac{\partial V(x,y,z)}{\partial x} \\ E_y = -\dfrac{\partial V(x,y,z)}{\partial y} \\ E_z = -\dfrac{\partial V(x,y,z)}{\partial z} \end{cases}$$

$$\begin{cases} E_x = -\dfrac{\partial\left(\pi K\sigma\left(\sqrt{b^2+z^2} - \sqrt{a^2+z^2}\right)\right)}{\partial x} = 0 \\ E_y = -\dfrac{\partial\left(\pi K\sigma\left(\sqrt{b^2+z^2} - \sqrt{a^2+z^2}\right)\right)}{\partial y} = 0 \\ E_z = -\dfrac{\partial\left(\pi K\sigma\left(\sqrt{b^2+z^2} - \sqrt{a^2+z^2}\right)\right)}{\partial z} = -\pi K\sigma\left(\dfrac{z}{\sqrt{b^2+z^2}} - \dfrac{z}{\sqrt{a^2+z^2}}\right) \end{cases}$$

$$\vec{E} = -\pi K\sigma\left(\frac{z}{\sqrt{b^2+z^2}} - \frac{z}{\sqrt{a^2+z^2}}\right)\vec{k}$$

♣ *The electric field and the electric potential at point **O** (center of the ring) : In this case, we replace the value of z with zero, and thus we will obtain:*

$$V = \pi K\sigma\left(\sqrt{b^2+z^2}-\sqrt{a^2+z^2}\right) = \pi K\sigma(b-a)$$

$$\vec{E} = \vec{0}$$

Exercise 20:

*Using Gauss's theorem, find the electric field and electric potential inside and outside a sphere of radius **R** in the following two cases:*

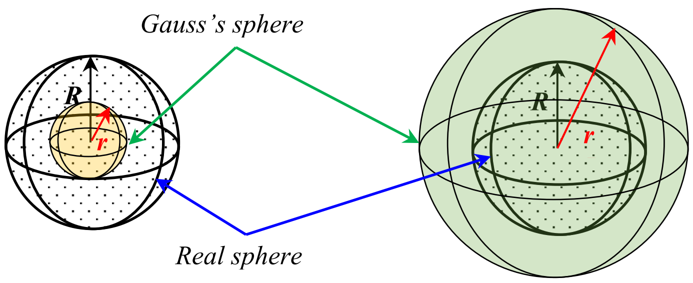
*1- The charge is distributed on its surface with a surface charge distribution density of **σ.** Plot the curves E=f(r) and V=g(r)*

*2 - The charge is distributed on the volume with a charge distribution density of **ρ.** Plot the curves E=f(r) and V=g(r)*

Solution:

♣ *The charge is distributed on its surface with a surface charge distribution density of **σ***

a) Calculate the electric field



1st region: inside the sphere $(\boldsymbol{0 < r < R})$

*To calculate the electric field by applying Gauss's theorem inside the sphere, we impose an imaginary spherical surface of variable radius **r** inside the real sphere called a Gaussian surface* $(\boldsymbol{0 < r < R})$

$$\Phi_{\vec{E}} = \oint \vec{E}.\vec{dS} = \frac{\sum Q_{in}}{\varepsilon_0}$$

$$\oint \vec{E}.\vec{dS} = E.S = E\ 4\pi r^2$$

Since charge is distributed over the real surface of the sphere, the total charge sums to zero inside the spherical Gaussian surface ($\sum Q_{in} = 0$).

$$E\ 4\pi r^2 = 0\ \Rightarrow E = 0$$

2nd region: outside the sphere $\quad(\boldsymbol{R < r})$

$$\Phi_{\vec{E}} = \oint \vec{E}.\overrightarrow{dS} = \frac{\sum Q_{in}}{\varepsilon_0}$$

$$\oint \vec{E}.\overrightarrow{dS} = E.S = E\ 4\pi r^2$$

Since charge is distributed over the surface of the real sphere, the sum of the total charges inside the spherical Gaussian surface is equal to the total electric charge on the surface of the real sphere and is calculated as:

$$\sigma = \frac{Q}{S} \Rightarrow Q = \sigma.S = \sigma\, 4\,\pi\, R^2$$

Therefore, the electric field expression is written as

$$E\ 4\pi r^2 = \frac{\sigma\, 4\,\pi\, R^2}{\varepsilon_0}\ \Rightarrow E = \frac{\sigma\ R^2}{\varepsilon_0\, r^2}$$

b) Calculate the electric potential

1st region: inside the sphere $\quad(\boldsymbol{0 < r < R})$

$$\vec{E} = -\overrightarrow{grad}\,V\ \ \Rightarrow V = -\int E.dr$$

$$V_1(r) = V(r) = -\int E.dr\ = -\int 0\,dr = C_1$$

2nd region: outside the sphere $\quad(\boldsymbol{R < r})$

$$\vec{E} = -\overrightarrow{grad}\,V\ \ \Rightarrow V = -\int E.dr$$

$$V_2(r) = V(r) = -\int E.dr\ = -\int \frac{\sigma\ R^2}{\varepsilon_0\, r^2}\,dr = \frac{\sigma\ R^2}{\varepsilon_0\, r} + C_2$$

The values of $\boldsymbol{C_1}$ *and* $\boldsymbol{C_2}$ *constants are determined by the initial or continuity condition between regions*

$$V(r\to\infty) = 0 \Rightarrow 0\ =\ 0 + C_2\ \Rightarrow C_2 = 0$$

$$V_2(r) = V(r) = \frac{\sigma\ R^2}{\varepsilon_0\, r}$$

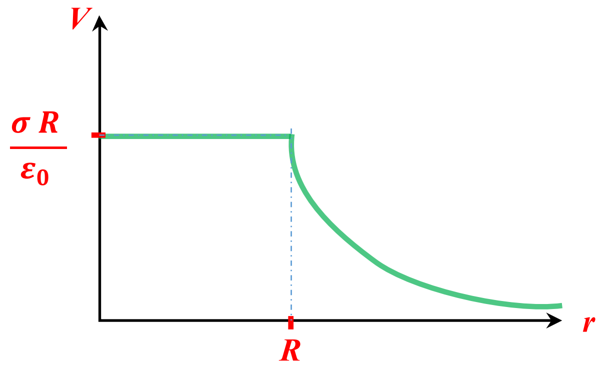
Applying the condition of continuity of the electrical potential function, we find:

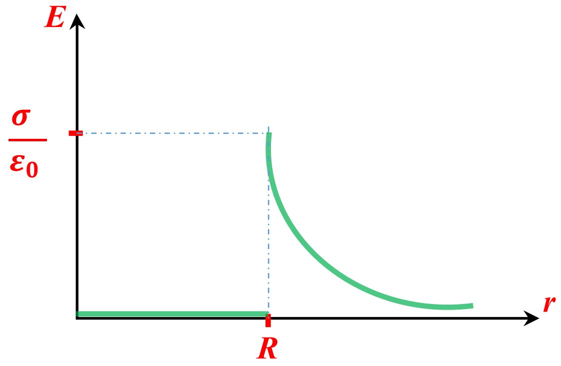
$$V_1(R) = V_2(R) \Rightarrow C_1 = \frac{\sigma \; R^2}{\varepsilon_0 \, R} = \frac{\sigma \; R}{\varepsilon_0}$$

$$V_1(r) = \frac{\sigma \; R}{\varepsilon_0}$$

❖ *Plot the curves E=f(r) and V=g(r)*

$$for\; r < R \begin{cases} E(r) = 0 \\ V(r) = \dfrac{\sigma \; R}{\varepsilon_0} \end{cases} \qquad for\; r > R \begin{cases} E(r) = \dfrac{\sigma \; R^2}{\varepsilon_0 \, r^2} \\ V(r) = \dfrac{\sigma \; R^2}{\varepsilon_0 \, r} \end{cases}$$





❖ *The charge is distributed on the volume with a charge distribution density of* $\boldsymbol{\rho}$

1st region: inside the sphere $(\mathbf{0} < \boldsymbol{r} < \boldsymbol{R})$

The electric field can be calculated by applying Gauss's theorem inside the sphere, therefore we assume an imaginary spherical surface of variable radius ***r*** *inside the real sphere called a Gaussian surface* $(\mathbf{0} < \boldsymbol{r} < \boldsymbol{R})$

$$\Phi_{\vec{E}} = \oint \vec{E}.\overrightarrow{dS} = \frac{\sum Q_{in}}{\varepsilon_0}$$

$$\oint \vec{E}.\overrightarrow{dS} = E.S = E\ 4\pi r^2$$

The electric charge is distributed over the real surface of the sphere, and the total charge inside the spherical Gaussian surface is calculated as follows:

$$\rho = \frac{Q}{V} \Rightarrow Q = \rho.V = \rho\frac{4}{3}\pi\, r^3$$

$$E\ 4\pi r^2 = \rho\frac{4}{3\ \varepsilon_0}\pi\, r^3 \ \Rightarrow E = \frac{\rho}{3\ \varepsilon_0}\ r$$

2nd region: outside the sphere $\quad (\boldsymbol{R < r})$

$$\Phi_{\vec{E}} = \oint \vec{E}.\overrightarrow{dS} = \frac{\sum Q_{in}}{\varepsilon_0}$$

$$\oint \vec{E}.\overrightarrow{dS} = E.S = E\ 4\pi r^2$$

Since charge is distributed over the surface of the real sphere, the sum of the total charges inside the spherical Gaussian surface is equal to the total electric charge on the surface of the real sphere and is calculated as:

$$\rho = \frac{Q}{V} \Rightarrow Q = \rho.V = \rho\frac{4}{3}\pi\, R^3$$

Therefore, the electric field expression is written as

$$E\ 4\pi r^2 = \rho\frac{4}{3\ \varepsilon_0}\pi\, R^3 \ \Rightarrow E = \frac{\rho}{3\ \varepsilon_0}\frac{R^3}{r^2}$$

c) Calculate the electric potential

1st region: inside the sphere $\quad (\boldsymbol{0 < r < R})$

$$\vec{E} = -\overrightarrow{grad}\, V \ \ \Rightarrow V = -\int E.dr$$

$$V_1(r) = V(r) = -\int \frac{\rho}{3\ \varepsilon_0}\ r.dr\ = -\frac{\rho}{6\ \varepsilon_0}\ r^2 + \ C_1$$

2nd region: outside the sphere $\quad (\boldsymbol{R < r})$

$$\vec{E} = -\overrightarrow{grad}\, V \quad \Rightarrow V = -\int E.dr$$

$$V_2(r) = V(r) = -\int E.dr = -\int \frac{\rho}{3\,\varepsilon_0}\frac{R^3}{r^2}\,dr = \frac{\rho}{3\,\varepsilon_0}\frac{R^3}{r} + C_2$$

*The values of **C_1** and **C_2** constants are determined by the initial or continuity condition between regions*

$$V(r \to \infty) = 0 \Rightarrow 0 = 0 + C_2 \Rightarrow C_2 = 0$$

$$V_2(r) = V(r) = \frac{\rho}{3\,\varepsilon_0}\frac{R^3}{r}$$

Applying the condition of continuity of the electrical potential function, we find:

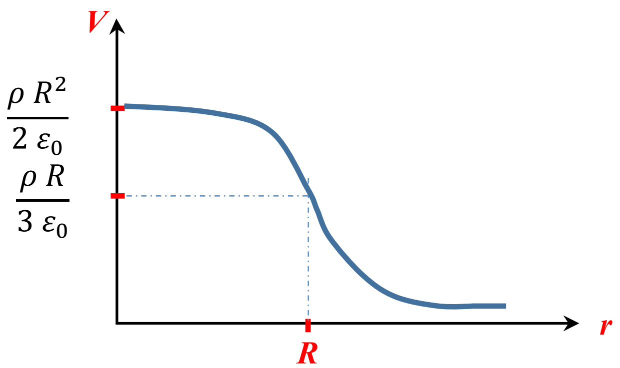
$$V_1(R) = V_2(R) \Rightarrow -\frac{\rho}{6\,\varepsilon_0}\,R^2 + C_1 = \frac{\rho}{3\,\varepsilon_0}\frac{R^3}{R}$$

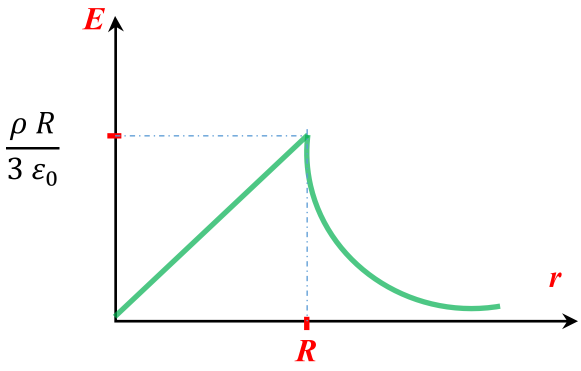
$$C_1 = \left(\frac{\rho}{3\,\varepsilon_0} + \frac{\rho}{6\,\varepsilon_0}\right) R^2 \Rightarrow C_1 = \frac{\rho}{2\,\varepsilon_0}\,R^2$$

$$V_1(r) = -\frac{\rho}{6\,\varepsilon_0}\,r^2 + \frac{\rho}{2\,\varepsilon_0}\,R^2$$

❖ *Plot the curves E=f(r) and V=g(r)*

$$for\ r < R \begin{cases} E(r) = \dfrac{\rho}{3\,\varepsilon_0}\,r \\ V(r) = -\dfrac{\rho}{6\,\varepsilon_0}\,r^2 + \dfrac{\rho}{2\,\varepsilon_0}\,R^2 \end{cases} \qquad for\ r > R \begin{cases} E(r) = \dfrac{\rho}{3\,\varepsilon_0}\dfrac{R^3}{r^2} \\ V(r) = \dfrac{\rho}{3\,\varepsilon_0}\dfrac{R^3}{r} \end{cases}$$





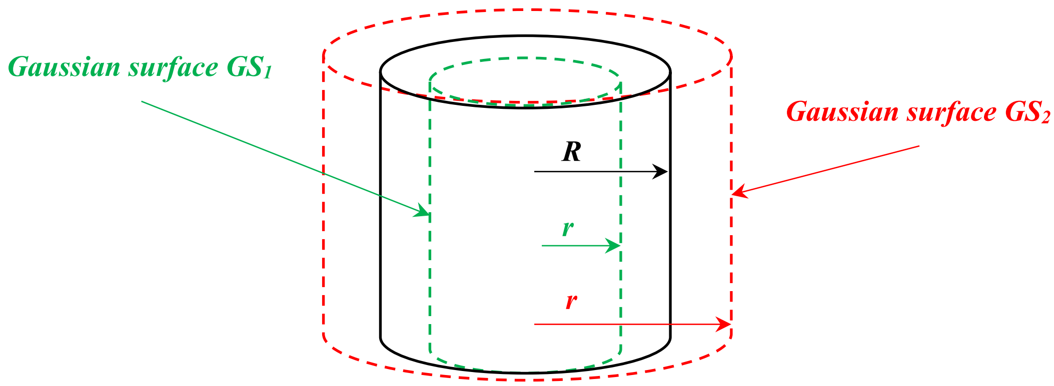
Exercise 21:

Two cylinders of equal height (H) are considered: the first (C_1) *is a solid cylinder of radius* $\mathbf{R}$ *with uniformly volume charge distribution density* $\boldsymbol{\rho}$*; the second* (C_2) *is hollow cylinder of radius* $\mathbf{2R}$ *, carrying a uniform surface charge density* $\boldsymbol{\sigma}$.

- *Using Gauss's theorem, write the expression for the electric field and the electric potential inside and outside the cylinder.*
- *If we insert coaxially cylinder* (C_1) *into* (C_2)*, do the electric field and electric potential change inside and outside each cylinder?*

Solution:

- *Using Gauss's theorem, write the expression for the electric field and the electric potential inside and outside the cylinder.*



- ***Inside the Cylinder*** (C_1)

For a point M located inside cylinder$(\boldsymbol{C_1})$*, we suppose there is a Gaussian surface (**GS₁**) in the form of a cylinder with height h and variable radius* $\boldsymbol{r}$ *where* $\mathbf{0} < \boldsymbol{r} < \boldsymbol{R}$*. the electric field flux is as follows:*

$$\Phi_{\vec{E}} = \oint \vec{E}.\overrightarrow{dS} = \frac{\sum Q_{in}}{\varepsilon_0}$$

$$\oint \vec{E}.\overrightarrow{dS} = E.S = E\ 2\pi\ r\ H$$

The electric charge is distributed over the entire cylinder $(\boldsymbol{C_1})$*, and the total charge inside the spherical Gaussian surface **GS₁** is calculated as follows:*

$$\rho = \frac{Q}{V} \Rightarrow Q = \rho.V = \rho\ H\ \pi\ r^2$$

$$E\ 2\pi\ r\ H = \frac{\rho\ H\ \pi\ r^2}{\varepsilon_0} \Rightarrow E = \frac{\rho}{2\ \varepsilon_0}\ r$$

The electric potential can be calculate as:

$$\vec{E} = -\overrightarrow{grad}\ V \quad \Rightarrow V = -\int E.dr$$

$$V_1(r) = V(r) = -\int E.dr = -\int \frac{\rho}{2\ \varepsilon_0}\ r\ dr = \frac{\rho}{4\ \varepsilon_0}\ r^2 + C_1$$

- ***Outside the Cylinder*** $(\boldsymbol{C_1})$

*In this case, we assume the presence of a Gaussian surface (**GS₂**) in the form of a cylinder with height h and variable radius* $\boldsymbol{r}$ *where* $\boldsymbol{R} < \boldsymbol{r}$*. Therefore, the electric field flux is written follows:*

$$\Phi_{\vec{E}} = \oint \vec{E}.\overrightarrow{dS} = \frac{\sum Q_{in}}{\varepsilon_0} \Rightarrow = E.S = E\ 2\pi\ r\ H$$

*The total charge inside the Gaussian surface **GS₂** is calculated as follows:*

$$\rho = \frac{Q}{V} \Rightarrow Q = \rho.V = \rho\ H\ \pi\ R^2$$

$$E\ 2\pi\ r\ H = \frac{\rho\ H\ \pi\ R^2}{\varepsilon_0} \Rightarrow E = \frac{\rho\ R^2}{2\ \varepsilon_0\ r}$$

The electric potential can be calculate as:

$$\vec{E} = -\overrightarrow{grad}\ V \quad \Rightarrow V = -\int E.dr$$

$$V_2(r) = -\int E.dr = -\int \frac{\rho\ R^2}{2\ \varepsilon_0\ r}\ dr = \frac{\rho\ R^2}{2\ \varepsilon_0} \ln\frac{1}{r} + C_2$$

*The constants **C_1** and **C_2** are determined according to initial conditions and also by adopting the condition of continuity of the potential function.*

❖ *The 2nd Cylinder*

- ***Inside the Cylinder** (C_2)*

We perform the same previous steps, where for point M inside the cylinder we have:

The electric field flux is as follows:

$$\Phi_{\vec{E}} = \oint \vec{E}.\overrightarrow{dS} = \frac{\sum Q_{in}}{\varepsilon_0}$$

$$\oint \vec{E}.\overrightarrow{dS} = E.S = E\ 2\pi\ r\ H$$

*For the hollow cylinder (C_2) there are no charges inside the Gaussian surface **GS_1**, so:*

$$\sum Q_{in} = 0 \ \Rightarrow\ E\ 2\pi\ r\ H = 0 \Rightarrow E = 0$$

The electric potential can be calculate as:

$$\vec{E} = -\overrightarrow{grad}\ V \quad \Rightarrow V = -\int E.dr$$

$$V_1(r) = V(r) = -\int E.dr = -\int 0\ dr = C_1$$

- ***Outside the Cylinder** (C_2)*

For calculating the field outside the cylinder (C_2)*, we assume the presence of a Gaussian surface (GS₂) in the form of a cylinder with height h and variable radius* $\boldsymbol{r}$ *where* $\boldsymbol{R} < \boldsymbol{r}$*. Therefore, the electric field flux is written follows:*

$$\Phi_{\vec{E}} = \oint \vec{E}.\overrightarrow{dS} = \frac{\sum Q_{in}}{\varepsilon_0} \Rightarrow = E.S = E\ 2\pi\ r\ H$$

The total charge inside the Gaussian surface GS_2 *is calculated as follows:*

$$\sigma = \frac{Q}{S} \Rightarrow Q = \sigma.S = \sigma\ H\ 2\ \pi\ (2R)$$

$$E\ 2\pi\ r\ H = \frac{\sigma\ H\ 2\ \pi\ R}{\varepsilon_0} \Rightarrow E = \frac{2\ \sigma\ \ R}{\varepsilon_0\ r}$$

The electric potential can be calculate as:

$$\vec{E} = -\overrightarrow{grad}\ V \ \ \Rightarrow V = -\int E.dr$$

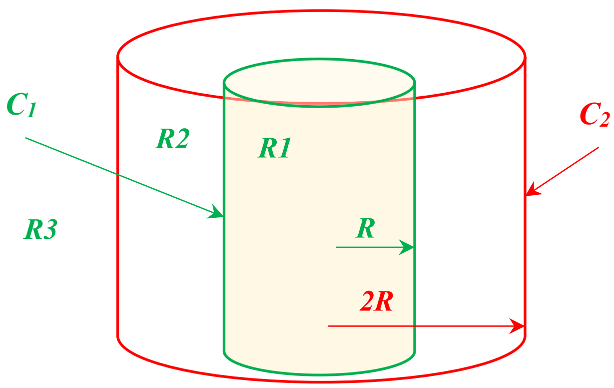
$$V_2(r) = -\int E.dr\ = -\int \frac{2\sigma\ \ R}{\varepsilon_0\ r}\ dr = \frac{2\sigma\ \ R}{\varepsilon_0}\ \ln\frac{1}{r} + C_2$$

The constants C_1 *and* C_2 *are determined according to initial conditions and also by adopting the condition of continuity of the potential function.*

- ***Case of inserting coaxially cylinder*** $(\boldsymbol{C_1})$ ***into*** $(\boldsymbol{C_2})$***, do the electric field and electric potential change inside and outside each cylinder***

  In this case, we divide the space into three regions:

  - ***Region (R1):*** *Inside the first cylinder* $r < R$
  - ***Region (R2):*** *Between the two cylinders* $R < r < 2R$
  - ***Region (R3):*** *Outside the second cylinder* $r > 2R$



- Region (R1): Inside the first cylinder $r < R$

The electric field expression inside this region is written as follows:

$$\Phi_{\vec{E}} = \oint \vec{E}.\overrightarrow{dS} = \frac{\sum Q_{in}}{\varepsilon_0} \quad \Rightarrow \oint \vec{E}.\overrightarrow{dS} = E.S = E\ 2\pi\, r\, H$$

Where

$$\sum Q_{in} = Q = \rho.V = \rho\, H\, \pi\, r^2$$

$$E\ 2\pi\, r\, H = \frac{\rho\, H\, \pi\, r^2}{\varepsilon_0} \Rightarrow E = \frac{\rho}{2\,\varepsilon_0}\ r$$

The electric potential can be calculate as:

$$\vec{E} = -\overrightarrow{grad}\, V \quad \Rightarrow V = -\int E.dr$$

$$V_1(r) = V(r) = -\int E.dr = -\int \frac{\rho}{2\,\varepsilon_0}\ r\, dr = \frac{\rho}{4\,\varepsilon_0}\ r^2 + C_1$$

- Region (R2): Between the two cylinders $R < r < 2R$

The electric field expression inside this region is written as follows:

$$\Phi_{\vec{E}} = \oint \vec{E}.\overrightarrow{dS} = \frac{\sum Q_{in}}{\varepsilon_0} \Rightarrow = E.S = E\ 2\pi\, r\, H$$

The total charge inside the Gaussian surface is calculated as follows:

$$\sum Q_{in} = Q = \rho.V = \rho\, H\, \pi\, R^2$$

$$E\ 2\pi\, r\, H = \frac{\rho\, H\, \pi\, R^2}{\varepsilon_0} \Rightarrow E = \frac{\rho\, R^2}{2\,\varepsilon_0\, r}$$

The electric potential can be calculate as:

$$\vec{E} = -\overrightarrow{grad}\, V \quad \Rightarrow V = -\int E.dr$$

$$V_2(r) = -\int E.dr = -\int \frac{\rho\, R^2}{2\,\varepsilon_0\, r}\, dr = \frac{\rho\, R^2}{2\,\varepsilon_0} \ln\frac{1}{r} + C_2$$

- Region (R3): Outside the second cylinder $r > 2R$

$$\Phi_{\vec{E}} = \oint \vec{E}.\overrightarrow{dS} = \frac{\sum Q_{in}}{\varepsilon_0} \Rightarrow= E.S = E\ 2\pi\ r\ H$$

The total charge enclosed within the Gaussian surface is the sum of the charges contained in both cylinders- the first and the second.

$$\sum Q_{in} = Q_1 + Q_2 = \rho.V + \sigma.S = \rho\ H\ \pi\ R^2 + \sigma\ H\ 2\ \pi\ (2R) = H\ R\ \pi\ (\rho R + 4\sigma)$$

$$E\ 2\pi\ r\ H = \frac{H\ R\ \pi\ (\rho R + 4\sigma)}{\varepsilon_0} \Rightarrow E = \frac{R\ (\rho R + 4\sigma)}{2\ \varepsilon_0\ r}$$

The electric potential can be calculate as:

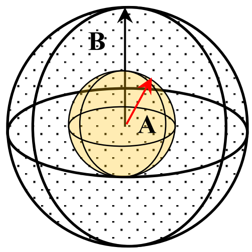
$$\vec{E} = -\overrightarrow{grad}\ V \quad \Rightarrow V = -\int E.dr$$

$$V_3(r) = -\int E.dr = -\int \frac{R\ (\rho R + 4\sigma)}{2\ \varepsilon_0\ r}\ dr = \frac{R\ (\rho R + 4\ \sigma)}{2\ \varepsilon_0} \ln\frac{1}{r} + C_3$$

The constants $\boldsymbol{C_1}$, $\boldsymbol{C_2}$ *and* $\boldsymbol{C_3}$ *are determined according to initial conditions and also by adopting the condition of continuity of the potential function.*

Exercise 22:

*The following figure represents two uniformly charged spheres, the first with a radius of **A** and a volume density distribution* $\boldsymbol{\rho}$ *, and the second with a radius of **B** and a surface density distribution* $\boldsymbol{\sigma}$.



- *Calculate the electric field and potential inside and outside the spheres (given* $V(r \to \infty) = 0$*)*
- *Plot the curves V(r) and E(r)*

Exercise 22:

The space inside and outside the spheres is divided into three regions:

Region (R1): *Inside the first sphere* $r < A$

- ***Region (R2):*** *Between the two spheres* $A < r < B$
- ***Region (R3):*** *Outside the second sphere* $r > B$

Region (R1): Inside the first cylinder $\boldsymbol{r < A}$

The electric field expression inside this region is written as follows:

$$\Phi_{\vec{E}} = \oint \vec{E}.\overrightarrow{dS} = \frac{\sum Q_{in}}{\varepsilon_0} \Rightarrow \oint \vec{E}.\overrightarrow{dS} = E.S = E\ 4\pi\ r^2$$

Where

$$\sum Q_{in} = Q = \rho.V = \rho\frac{4}{3}\,\pi\, r^3$$

$$E\ 4\pi\, r^2 = \frac{\rho\frac{4}{3}\,\pi\, r^3}{\varepsilon_0} \Rightarrow E = \frac{\rho}{3\,\varepsilon_0}\ r$$

The electric potential can be calculate as:

$$\vec{E} = -\overrightarrow{grad}\,V \ \Rightarrow V = -\int E.dr$$

$$V_1(r) = V(r) = -\int E.dr = -\int \frac{\rho}{3\,\varepsilon_0}\ r\,dr = \frac{\rho}{6\,\varepsilon_0}\ r^2 + C_1$$

- Region (R2): Between the two cylinders $A < r < B$

The electric field expression inside this region is written as follows:

$$\Phi_{\vec{E}} = \oint \vec{E}.\overrightarrow{dS} = \frac{\sum Q_{in}}{\varepsilon_0} \Rightarrow= E.S = E\ 4\pi\, r^2$$

The total charge inside the Gaussian surface is calculated as follows:

$$\sum Q_{in} = Q = \rho.V = \rho\frac{4}{3}\,\pi\, A^3$$

$$E\ \ 4\pi\, r^2 = \frac{\rho\frac{4}{3}\,\pi\, A^3}{\varepsilon_0} \Rightarrow E = \frac{\rho\, A^3}{3\,\varepsilon_0\, r^2}$$

The electric potential can be calculate as:

$$\vec{E} = -\overrightarrow{grad}\,V \ \Rightarrow V = -\int E.dr$$

$$V_2(r) = -\int E.dr = -\int \frac{\rho\, A^3}{3\,\varepsilon_0\, r^2}\,dr = \frac{\rho\, A^3}{3\,\varepsilon_0}\frac{1}{r} + C_2$$

- Region (R3): Outside the second cylinder $r > B$

$$\Phi_{\vec{E}} = \oint \vec{E}.\overrightarrow{dS} = \frac{\sum Q_{in}}{\varepsilon_0} \Rightarrow= E.S = \ E\ 4\pi\, r^2$$

The total charge enclosed within the Gaussian surface is the sum of the charges contained in both the first and the second spheres.

$$\sum Q_{in} = Q_1 + Q_2 = \rho . V + \sigma . S = \rho \frac{4}{3} \pi A^3 + \sigma 4 \pi B^2$$

$$E\ 4\pi r^2 = \frac{\rho \frac{4}{3} \pi A^3 + \sigma 4 \pi B^2}{\varepsilon_0} \Rightarrow E = \frac{\frac{\rho}{3} A^3 + \sigma B^2}{\varepsilon_0 r^2}$$

The electric potential can be calculate as:

$$\vec{E} = -\overrightarrow{grad}\, V \quad \Rightarrow V = -\int E.dr$$

$$V_3(r) = -\int E.dr = -\int \frac{\frac{\rho}{3} A^3 + \sigma B^2}{\varepsilon_0 r^2} dr = \frac{\frac{\rho}{3} A^3 + \sigma B^2}{\varepsilon_0\ r} + C_3$$

The constants $\boldsymbol{C_1}$*,* $\boldsymbol{C_2}$ *and* $\boldsymbol{C_3}$ *are determined according to initial conditions and also by adopting the condition of continuity of the potential function.*

First condition:

$$V_3(r \to \infty) = 0 \Rightarrow 0 + C_3 = 0 \Rightarrow C_3 = 0$$

$$V_3(r) = \frac{\frac{\rho}{3} A^3 + \sigma B^2}{\varepsilon_0 r}$$

Second condition:

$$V_3(B) = V_2(B) \Rightarrow \frac{\rho\ A^3 + 3\sigma B^2}{3\ \varepsilon_0\ B} = \frac{\rho\ A^3}{3\ \varepsilon_0 B} + C_2$$

$$\frac{\rho\ A^3 + 3\sigma B^2}{3\ \varepsilon_0\ B} - \frac{\rho\ A^3}{3\ \varepsilon_0 B} = C_2 \Rightarrow C_2 = \frac{\sigma B}{\varepsilon_0}$$

$$V_2(r) = \frac{\rho\ A^3}{3\ \varepsilon_0} \frac{1}{r} + \frac{\sigma B}{\varepsilon_0}$$

Third condition

$$V_1(A) = V_2(A) \Rightarrow \frac{\rho A^2}{6\ \varepsilon_0} + C_1 = \frac{\rho\ A^2}{3\ \varepsilon_0} + \frac{\sigma B}{\varepsilon_0}$$

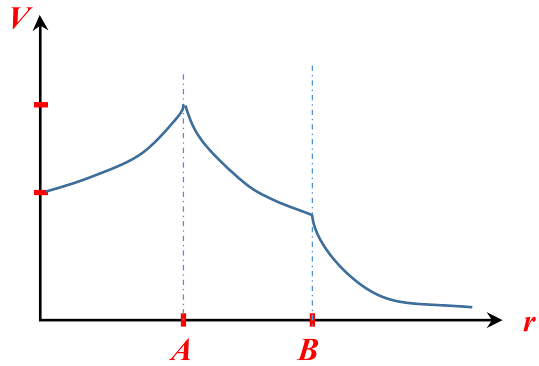
$$C_1 = \frac{\rho\ A^2}{3\ \varepsilon_0} - \frac{\rho A^2}{6\ \varepsilon_0} + \frac{\sigma B}{\varepsilon_0} \Rightarrow C_1 = \frac{\rho\ A^2}{6\ \varepsilon_0} + \frac{\sigma B}{\varepsilon_0}$$

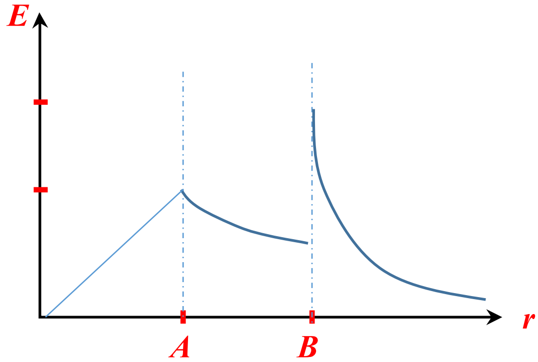
$$V_1(r) = \frac{\rho}{6\,\varepsilon_0}\ r^2 + \frac{\rho\,A^2}{6\,\varepsilon_0} + \frac{\sigma\,B}{\varepsilon_0}$$

- *Plot the curves V(r) and E(r)*

$$\begin{cases} V_1(r) = \dfrac{\rho}{6\,\varepsilon_0}\ r^2 + \dfrac{\rho\,A^2}{6\,\varepsilon_0} + \dfrac{\sigma\,B}{\varepsilon_0} \\[2ex] V_2(r) = \dfrac{\rho\,A^3}{3\,\varepsilon_0}\dfrac{1}{r} + \dfrac{\sigma\,B}{\varepsilon_0} \\[2ex] V_3(r) = \dfrac{\frac{\rho}{3}\ A^3 + \sigma\,B^2}{\varepsilon_0\,r} \end{cases}$$

$$\begin{cases} E_1(r) = \dfrac{\rho}{3\,\varepsilon_0}\ r & \text{for } r < A \\[2ex] E_2(r) = \dfrac{\rho\,A^3}{3\,\varepsilon_0\,r^2} & \text{for } A < r < B \\[2ex] E_3(r) = \dfrac{\frac{\rho}{3}\ A^3 + \sigma\,B^2}{\varepsilon_0\,r^2} & \text{for } B < r \end{cases}$$





Chapter 2: Conductors

1. Introduction

In this section, we will rely on the concepts reached from the study of the electric field, the electric potential, as well as the applications of Gauss's theory in understanding and simplifying electrical conductors, as well as the gradual understanding of electrical capacitors in their various forms and their role in storing electricity.

2. Conductor definition

A conductor is defined as any material that possesses relatively free-moving charge carriers (weakly attached to the nuclei of the atoms forming the material). Charge carriers move in a conductive material when subjected to an electric force resulting from an external electric field. This field generates a potential difference across the material, causing the charge carriers (such as free valence electrons in metals) to accelerate and begin moving within the material's crystal structure, causing an electric current to flow.

*We say that a **conductor is electrically balanced** if the charge carriers are stationary, meaning they are not subjected to any electric force. An electrically balanced conductor is characterized by the following:*

- *The electric field inside the balanced conductor is zero ($\overrightarrow{E_{inside}} = \vec{0}$), which can be explained by the fact that the charge carriers (electrons) are stationary and therefore not subjected to any forces (or the net forces are zero).*

$$\vec{F} = q\vec{E} \Rightarrow \vec{F} = \vec{0} \text{ then } \vec{E} = \vec{0}$$

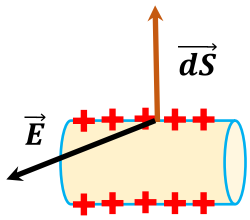
- *The electric charge inside a balanced conductor is zero. This is explained by applying Gauss's theorem.*

$$\oiint \overrightarrow{E_{inside}} \, . \overrightarrow{dS} = \frac{\sum Q_{inside}}{\varepsilon_0} \Rightarrow \overrightarrow{E_{inside}} = \vec{0} \text{, then} \sum Q_{inside} = 0$$

- *The hollow balanced conductor has the same properties as the non-hollow balanced conductor.*
- *The electric potential is constant across all points of the conductor, and thus the balanced conductor forms an equipotential volume.*

$$V = -\int \vec{E} \, \overrightarrow{dr} = -\int \vec{0} \, \overrightarrow{dr} = Constant$$

- *Free charges (electrons) are located on the surface of a balanced conductor, (due to the repulsive forces between them, the charges move away to the maximum distance, which is the surface of the conductive material), and the direction of the electric field is perpendicular to the surface vector.*



- *When an electric conductor is charged, the added charge rapidly distributes throughout the material due to the mobility of free charge carriers. Subsequently, the charges rearrange themselves and settle on the outer surface of the conductor, bringing the system back to electrostatic equilibrium—characterized by a zero electric field inside the conductor and a uniform electric potential throughout its volume.*

3. The electric pressure on the outer surface of the conductor

As a result of the repulsive forces between the charges located on the surface of the balanced conductor, an outward electric pressure is generated on its surface.

The electric pressure P_e generated by the elemental charge dq on the elemental surface dS can be expressed in terms of the surface charge density σ and the surface electric field E_S as follow:

$$P_e = \frac{dF}{dS}$$

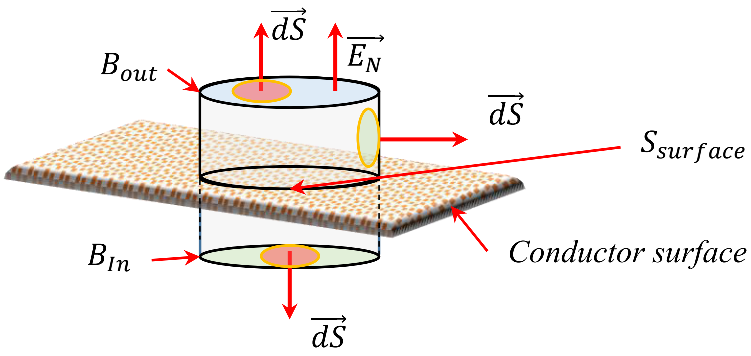
$$\text{Where } dF = dq\, E_S \text{ and } q = \sigma\, dS \Rightarrow dF = \sigma\, dS\, E_S$$

$$P_e = \frac{\sigma\, dS\, E_S}{dS} \Rightarrow P_e = \sigma\ E_S$$

E_S : can be calculated by Gauss's theorem

4. Electric field next to a conductor

To calculate the electric field at points very close to the conductor's outer surface but not belonging to it, we use Gauss' theorem, which involves selecting an imaginary Gaussian surface in the shape of a cylinder whose axis perpendicular to the conductor's surface, its inner base inside the conductor B_{In} , and its outer base outside the conductor B_{out}, as illustrated in the figure.



The electric field flux through the imaginary Gaussian surface represented by the cylinder (as shown in the picture) is equal to the sum of the field fluxes through the lateral surface, the inner base and the outer base and is written as follows:

$$\Phi_{cylinder} = \Phi_{lateral} + \Phi_{inner\ base} + \Phi_{outer\ base}$$

¤ ***Through the lateral surface:*** *The electric field flux through the lateral surface is zero because the electric field vector is perpendicular to the surface vector.* ($\overrightarrow{E_N} \perp \overrightarrow{dS_{lateral}}$), *then:*

$$\Phi_{lateral} = \oiint \overrightarrow{E_N} \,.\, \overrightarrow{dS_{lateral}} = 0$$

¤ ***Through the inner base***: *The electric field flux through the inner base is zero because the electric field vector is zero inside the balanced conductor* ($\overrightarrow{E_{inside}} = \vec{0}$), *then:*

$$\Phi_{inner} = \oiint \overrightarrow{E_{inside}} \,.\, \overrightarrow{dS_{inner}} = 0$$

¤ ***Through the outer base***: *The electric field flow through the outer base is non-zero because the electric field vector is parallel to the surface vector and is written as follows*

($\overrightarrow{E_N} \parallel \overrightarrow{dS_{outer}}$), *then:*

$$\Phi_{outer} = \oiint \overrightarrow{E_N} \,.\, \overrightarrow{dS_{outer}} = E_N\, S_{outer}$$

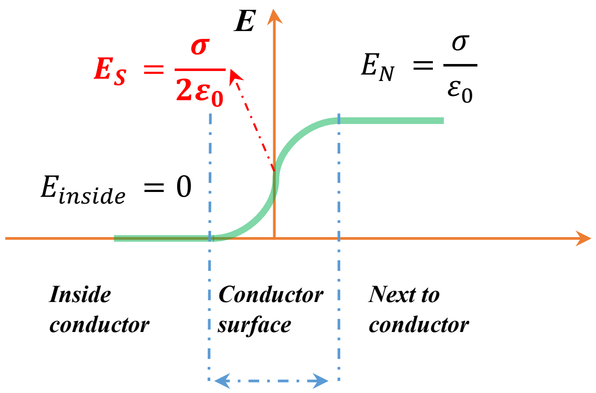
$$\Phi_{cylinder} = E_N\, S_{outer} = \frac{\sum Q_{inside}}{\varepsilon_0}$$

$$Q_{inside} = S_{surface}\ \sigma = S_{outer}\ \sigma$$

$$E_N\, S_{outer} = \frac{S_{outer}\ \sigma}{\varepsilon_0} \Rightarrow E_N = \frac{\sigma}{\varepsilon_0}$$

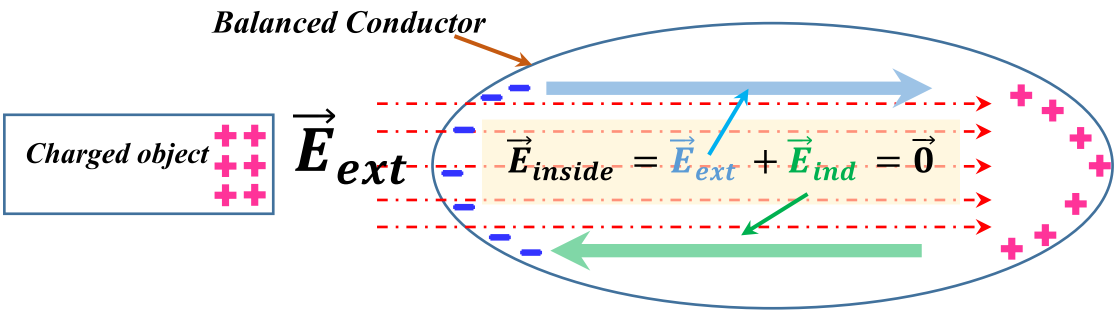
$E_N = \frac{\sigma}{\varepsilon_0}$ *is the expression for the electric field near the surface. The expression for the electric field directly at the surface is given as:*

$$E_S = \frac{E_N + E_{inside}}{2} = \frac{\frac{\sigma}{\varepsilon_0} + 0}{2} = \frac{\sigma}{2\,\varepsilon_0}$$



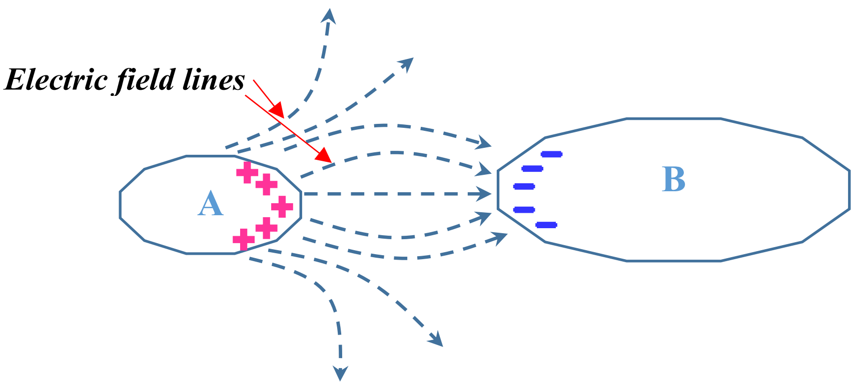
5. The influence of an external electric field on a balanced conductor

In this section, we explain how a charged object affects a balanced conductor, as shown in the below figure. When a positively charged object is brought close to a balanced conductor, the free negative charges within the conductor begin to move in the opposite direction of the external electric field $\vec{E}_{ext}$ generated by the charged object. The negative charges accumulate at the end nearest to the object, while positive charges appear at the far end of the conductor as a result of the migration of negative charges from that end. This causes the conductor to become electrically polarized. As a result of this inhomogeneous charge distribution, an induced electric field $\vec{E}_{ind}$ is generated within the conductor whose direction is opposite to the direction of the external field which leads to canceling out the total field within the conductor, making the electric field inside it zero ($\vec{E}_{inside} = \vec{E}_{ext} + \vec{E}_{ind} = \vec{0}$). In this case, the effect of the charged object on the conductor is only partial, because the electric field lines emanating from it do not all reach the conductor; rather, they are partially propagated into the surrounding space. This means that the external field is not completely canceled outside the conductor, although it is canceled inside it. Thus the conductor becomes electrically balanced again.

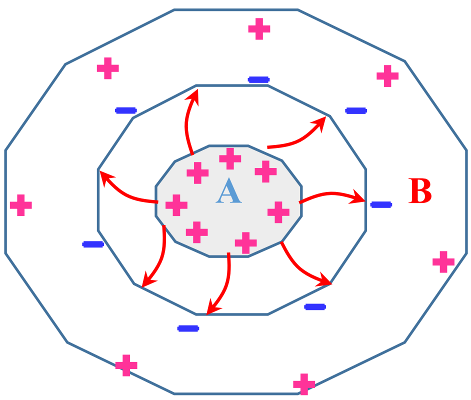


6. Total and partial effect between conductors

The influence between conductors can be either complete (total) or partial. The influence is said to be complete (see figure below) when all electric field lines originating from one conductor reach the other conductor. In contrast, the influence is considered partial when only a portion of the electric field lines reaches the other conductor.



Partial effect

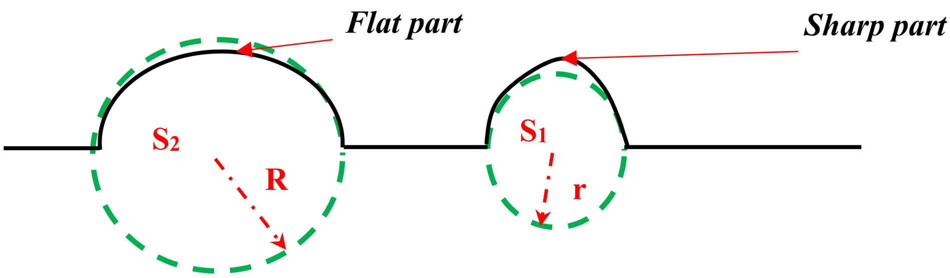


Total effect

*If conductor **A** carries a positive charge **+Q**, then the inner surface of conductor **B** will carry a negative charge **-Q**, while the outer surface of the conductor will carry a positive charge **+Q**. The total charge of conductor **B** remains conserved and equals **+Q - Q = 0**.*

7. Geometric Shape of a Conductor influence on its Surface Charge Density

The distribution of electric charge depends strongly on the degree of curvature of the conductor's surface. Charges tend to accumulate more densely at the sharper, and the highly curved parts of the conductor's surface, meanwhile, charges are distributed with lower density on flatter parts of the surface. To justify this phenomena, a sharply curved part of a conductor's surface can be modeled by the surface of a sphere conductor with a very small radius $\boldsymbol{r}$, *while we the nearly flat part of a conductor's surface can be modeled by a sphere conductor with a large radius* $\boldsymbol{R}$ *as shown in the following figure:*



Applying Gauss's theorem, we can find the expression for the potential at the surface of each sphere:

$$V_1 = k\frac{q_1}{r} \quad , \qquad V_2 = k\frac{q_2}{R}$$

When the two conductors are connected with a thin conductor wire, the charges move from one conductor to the other until equilibrium occurs, i.e. the potential difference between the conductors becomes zero.

$$V_1 = V_2 \Rightarrow \; k\frac{q_1}{r} = k\frac{q_2}{R} \;\Rightarrow q_1 = \frac{r}{R}\, q_2$$

Since $r < R$, *then* $\frac{r}{R} < 1 \;\;\Rightarrow\; q_1 < \; q_2$

$$q_1 = \sigma_1\, 4\,\pi\, r^2 \; and \; q_2 = \sigma_2\, 4\,\pi\, R^2$$

$$q_1 = \frac{r}{R}\, q_2 \quad \Rightarrow \quad \sigma_1\, 4\,\pi\, r^2 = \frac{r}{R}\sigma_2 \;\; 4\,\pi\, R^2 \;\Rightarrow \sigma_1\, r = R\, \sigma_2 \quad \Rightarrow \sigma_1 \; = \frac{R}{r}\; \sigma_2$$

Since $r < R$, *then* $\frac{R}{r} > 1 \;\;\Rightarrow\; \boldsymbol{\sigma_1} > \; \boldsymbol{\sigma_2}$

Consequently, the surface charge density on the surface of sphere $\boldsymbol{S_1}$ *is greater than that on the surface of the larger sphere* $\boldsymbol{S_2}$.

8. Conductor capacitance and its internal energy

*The capacitance of a balanced conductor is an intrinsic property that measures the conductor's ability to store electric charge, it defined as the ratio of the magnitude charge it carries (**Q**) to the potential (**V**) on its surface. Capacitance is denoted by **C**, and its unit is the farad (**F**).*

$$C = \frac{Q}{V}$$

The capacity of a conductor and its ability to store charges depends mainly on its geometric shape and size, in addition to the nature of the surrounding medium, such as the presence of other conductors or its presence within an external electric potential or field.

***The internal energy U** of a balanced conductor represents the work required to charge it with a charge **Q** until it reaches a potential **V**. The internal energy expression can be formulated in terms of potential **(V)**, capacitance **(C)** and charge **(Q)** as follows:*

$$U = \frac{1}{2} Q\,V = \frac{1}{2}\,C\,V^2 = \frac{1}{2}\,\frac{Q^2}{C}$$

9. Parallel Plate, spherical, and cylindrical capacitors

A capacitor is a very important electrical component in many electronic systems. Its role is to store electrical charges. A capacitor consists of two conductors in a state of total influence between them, separated by a dielectric medium with a permittivity e (air or any other insulator). In an electrical circuit, this is represented by the symbol ┤├.

*When a potential difference is applied between the conductors, a charge of **+Q** appears on the surface of one of them, and a charge of **-Q** appears on the surface of the other conductor.*

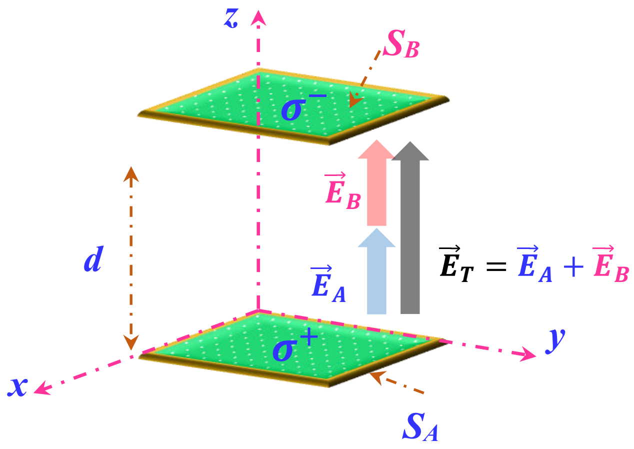
*The capacitance C of a capacitor is equal to the ratio of the magnitude of the electric charge Q stored on either conductor to the potential difference **ΔV** between the conductors.*

$$C = \frac{Q}{\Delta V}$$

The capacitance of a capacitor depends on its geometric shape and size. Based on this principle, there are three types:

- ***Parallel Plate Capacitors:***

A parallel-plate capacitor, as shown in the figure, consists of two parallel conducting plates, each with an area of $\boldsymbol{S}$*, separated by an insulating medium where the distance between them being* $\boldsymbol{d}$.



To calculate the capacitance $\boldsymbol{C}$*, we calculate the electric field* $\vec{\boldsymbol{E}}$ *between the two plates and then calculate the potential difference* $\Delta \boldsymbol{V}$ *generated between the plates as follows:*
The total electric field between the plates is equal to the sum of the electric fields generated by each plate $\vec{\boldsymbol{E}}_T = \vec{\boldsymbol{E}}_A + \vec{\boldsymbol{E}}_B$
As, we demonstrated previously, the electric field of an infinite plane is given by the following relation:

$$\overrightarrow{E_{S_A}} = \begin{cases} +\dfrac{\sigma}{2\varepsilon_0}\vec{k} & for\ z > 0 \\ -\dfrac{\sigma}{2\varepsilon_0}\vec{k} & for\ z < 0 \end{cases}$$

$$\overrightarrow{E_{S_B}} = \begin{cases} -\dfrac{\sigma}{2\varepsilon_0}\vec{k} & for\ z > d \\ +\dfrac{\sigma}{2\varepsilon_0}\vec{k} & for\ z < d \end{cases}$$

The region between $(\boldsymbol{0} < \boldsymbol{z} < \boldsymbol{d})$ *the two plates corresponds to* $\boldsymbol{z} > \boldsymbol{0}$ *for the plate* $\boldsymbol{S_A}$ *and* $\boldsymbol{z} < \boldsymbol{d}$ *for the plate* $\boldsymbol{S_B}$*, thus* **:**

$$\begin{cases} \overrightarrow{E_{S_A}} = +\dfrac{\sigma}{2\varepsilon_0}\vec{k} \\ \overrightarrow{E_{S_B}} = +\dfrac{\sigma}{2\varepsilon_0}\vec{k} \end{cases} \Rightarrow \vec{E} = \frac{\sigma}{\varepsilon_0}\vec{k}$$

Calculate the electric potential V

$$V = \int -\vec{E}.d\vec{r}$$

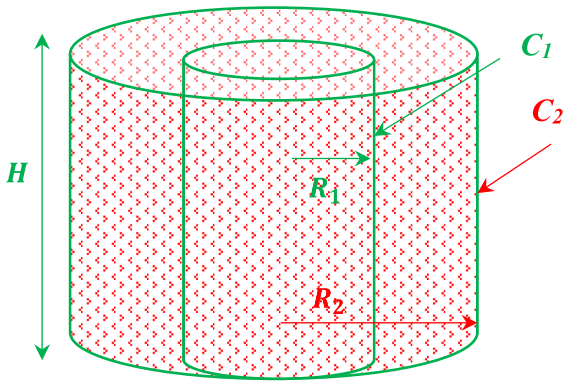
$$\Delta V = V^{+} - V^{-} = V_A - V_B = \int_{S_B}^{S_A} dV = -\int_{d}^{0} \frac{\sigma}{\varepsilon_0} dz = -\frac{\sigma}{\varepsilon_0}(0-d) = \frac{\sigma\, d}{\varepsilon_0}$$

$$C = \frac{Q}{\Delta V} = \frac{Q}{\frac{\sigma\, d}{\varepsilon_0}} = \frac{\varepsilon_0\, Q}{\sigma\, d} = \frac{\varepsilon_0\, \sigma\, S}{\sigma\, d} = \frac{\varepsilon_0\; S}{d}$$

A larger plate area (S) and a smaller separation distance (d) between the plates result in a greater capacitance for a parallel-plate capacitor.

Cylindrical capacitors:

*A cylindrical capacitor, as shown in the figure below, consists of two hollow cylindrical conductors, one with a radius of **R_1** carrying a positive charge on its surface (inner cylinder) and the other with a radius of **R_2** carrying a negative charge on its surface (outer cylinder). The two cylinders are separated by an insulating medium (air). Calculating the flow of the electric field in the space between the two cylinders through the side surface of the cylinder using Gauss's theorem paves the way for presenting the difference in electric potential between the two cylinders.*



The electric field expression the two cylinders is written as follows:

$$\Phi_{\vec{E}} = \oint \vec{E}.\overrightarrow{dS} = \frac{\sum Q_{in}}{\varepsilon_0} \Rightarrow = E.S = E\; 2\pi\, r\, H$$

The total charge inside the Gaussian surface is calculated as follows:

$$E\; 2\pi\, r\, H = \frac{Q}{\varepsilon_0} \Rightarrow E = \frac{Q}{\varepsilon_0\;\, 2\pi\, r\, H}$$

The electric potential can be calculate as:

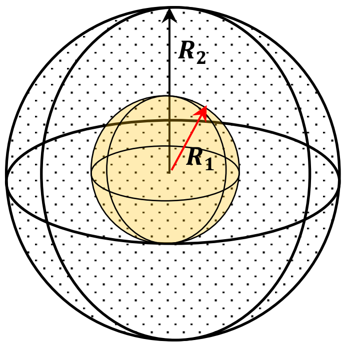
$$\vec{E} = -\overrightarrow{grad}\, V \;\Rightarrow V = -\int E.dr = -\int \frac{Q}{\varepsilon_0 \;\, 2\pi\, r\, H}\, dr$$

$$\Delta V = V^{+} - V^{-} = -\int_{R_2}^{R_1} \frac{Q}{\varepsilon_0 \;\, 2\pi\, r\, H}\, dr = -\frac{Q}{\varepsilon_0 \;\, 2\pi \;\, H}\, |\ln r|_{R_2}^{R_1}$$

$$\Delta V = \frac{Q}{\varepsilon_0 \;\, 2\pi \;\, H}\left(\ln\frac{R_2}{R_1}\right)$$

$$C = \frac{Q}{\Delta V} = \frac{Q}{\frac{Q}{\varepsilon_0 \;\, 2\pi \;\, H}\left(\ln\frac{R_2}{R_1}\right)} = \frac{2\, H\, \pi\, \varepsilon_0}{\left(\ln\frac{R_2}{R_1}\right)}$$

Spherical capacitors:



The spherical capacitor consists of two spherical conductors with the same center, separated by an insulating medium. The surface of the inner spherical conductor (with radius $\boldsymbol{R_1}$*) carries a positive charge* $\boldsymbol{+Q}$*, while the surface of the outer spherical conductor (with radius* $\boldsymbol{R_2}$*) carries a negative charge* $\boldsymbol{-Q}$.

We calculate the electric field and then the electric potential difference between the two ends of the spherical conductors using Gauss's theorem as follows:

The electric field expression between the two conducting spheres are written as follows:

$$\Phi_{\vec{E}} = \oint \vec{E}.\overrightarrow{dS} = \frac{\sum Q_{in}}{\varepsilon_0} \Rightarrow = E.S = E\, 4\,\pi\, r^2$$

The total charge inside the Gaussian surface is calculated as follows:

$$E \;\, 4\pi\, r^2 = \frac{Q}{\varepsilon_0} \Rightarrow E = \frac{Q}{\varepsilon_0\, 4\,\pi\, r^2}$$

The electric potential can be calculate as:

$$\vec{E} = -\overrightarrow{grad}\, V \;\Rightarrow V = -\int E.dr$$

$$\Delta V = V^{+} - V^{-} = -\int_{R_2}^{R_1} \frac{Q}{\varepsilon_0\, 4\,\pi\, r^2}\, dr = \left|\frac{Q}{\varepsilon_0\, 4\,\pi\, r}\right|_{R_2}^{R_1}$$

$$\Delta V = \frac{Q}{\varepsilon_0\, 4\,\pi}\left(\frac{1}{\boldsymbol{R_1}} - \frac{\boldsymbol{1}}{\boldsymbol{R_2}}\right) = \frac{Q}{\varepsilon_0\, 4\,\pi}\left(\frac{\boldsymbol{R_2 - R_1}}{\boldsymbol{R_2\, R_1}}\right)$$

$$C = \frac{Q}{\Delta V} = \frac{Q}{\frac{Q}{\varepsilon_0\, 4\,\pi}\left(\frac{\boldsymbol{R_2 - R_1}}{\boldsymbol{R_2\, R_1}}\right)} = \boldsymbol{\varepsilon_0\, 4\,\pi}\left(\frac{\boldsymbol{R_2\, R_1}}{\boldsymbol{R_2 - R_1}}\right)$$

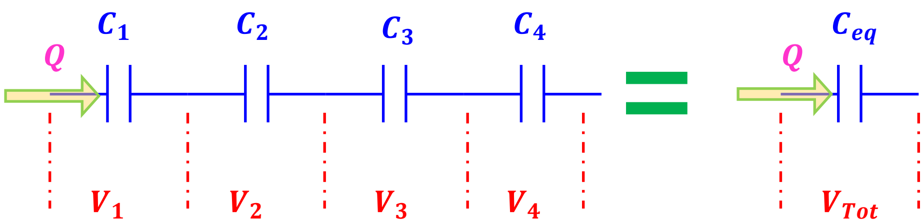
$$\boldsymbol{C = \varepsilon_0\, 4\,\pi\left(\frac{R_2\, R_1}{R_2 - R_1}\right)}$$

10. Capacitors combination

Capacitors in electrical circuits can be arranged in series, in parallel, or in mixed combination.

1- Series combination of Capacitors

Capacitors are installed in series combination when each pair of successive capacitors shares a single connection point (a common terminal), and the same amount of charge flows through all capacitors, while the potential difference across each one may vary.



The equivalent capacitor $\boldsymbol{C_{eq}}$ *for the capacitors* $\boldsymbol{C_i}$ *connected in series shown in the figure above carries the same amount of electric charge* $\boldsymbol{Q_{Tot}}$

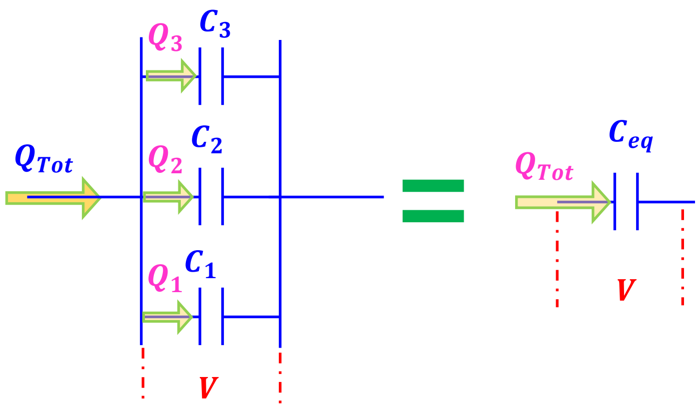
, however the total potential difference $\boldsymbol{V_{Tot}}$ *across its terminals is equal to the sum of the individual potential difference* $\boldsymbol{V_i}$ *across each capacitor. The capacitance value of the equivalent capacitor can be calculated as follows:*

$$C_{eq} = \frac{\boldsymbol{Q_{Tot}}}{V_{Tot}} \qquad \begin{cases} \boldsymbol{Q_{Tot}} = Q_i \\ C_i = \dfrac{Q_i}{V_i} \\ V_{Tot} = \displaystyle\sum_i V_i \end{cases}$$

$$\boldsymbol{\frac{1}{C_{eq}} = \frac{V_{Tot}}{Q} = \frac{\sum_i V_i}{Q} = \sum_i \frac{V_i}{Q_i} = \sum_i \frac{1}{C_i}}$$

2- Parallel Combination of Capacitors

Capacitors are installed a parallel combination, all capacitors are connected to the same two terminals, so the potential difference across each capacitor is the same, while the total charge is distributed among them according to their individual capacitances.



In the parallel capacitors combination method, the potential difference $\boldsymbol{V_{Tot}}$ *across the equivalent capacitor* $\boldsymbol{C_{eq}}$ *is equal to the potential difference* $\boldsymbol{V_i}$ *across each individual capacitor* $\boldsymbol{C_i}$*, while the total charge* $\boldsymbol{Q_{Tot}}$ *stored in the equivalent capacitor is equal to the sum of the of the charge* $\boldsymbol{Q_i}$ *stored in the individual capacitors. The capacitance of the equivalent capacitor is calculated as follows:*

$$C_{eq} = \frac{\boldsymbol{Q_{Tot}}}{V_{Tot}} \quad \begin{cases} \boldsymbol{Q_{Tot}} = \sum_i Q_i \\ C_i = \dfrac{Q_i}{V_i} \\ V_{Tot} = V_i \end{cases}$$

$$C_{eq} = \frac{\boldsymbol{Q_{Tot}}}{V_{Tot}} = \frac{\sum_i Q_i}{V_{Tot}} = \sum_i \frac{Q_i}{V_{Tot}} = \sum_i \frac{Q_i}{V_i} = \sum_i C_i$$

3- Mixed Combination of Capacitors

A mixed combination involves connecting some capacitors in series and others in parallel within the same circuit. In the mixed arrangement of capacitors, the electrical circuit is simplified step by step, and each time the equivalent capacitor for the capacitors connected in series or parallel is calculated until the final equivalent capacitor is obtained.

11. Solved exercises

Exercise 01:

Two conductive spheres, with radii R_1 and R_2, carry an electric charge of Q_1 and a charge of Q_2 respectively. Each sphere is initially placed far from the other to neglect any mutual influence.

1- *Find the expression for the electric potential at any point on the surface of the conducting sphere.*

For the following two cases:

a) $R_1 = R_2$ and $Q_2 = 2\,Q_1$

b) $R_2 = 2\,R_1$ and $Q_1 = Q_2$

2- *Calculate the electric potential at the surface of each sphere*

2- Determine the direction of charge flow if the two conducting spheres are connected by a thin conducting wire. Then, calculate the final electric charge on each sphere once electrostatic equilibrium is reached.

Solution:

The electric potential of each sphere in the following two cases:

We can calculate the electric potential at the surface of a spherical conductor using Gauss's theorem.

$$\Phi_{\vec{E}} = \oint \vec{E}.\overrightarrow{dS} = \frac{\sum Q_{in}}{\varepsilon_0} \Rightarrow E.S = E\,4\,\pi\,r^2$$

The total charge inside the Gaussian surface is calculated as follows:

$$E\ \ 4\pi\,r^2 = \frac{Q}{\varepsilon_0} \Rightarrow E = \frac{Q}{\varepsilon_0\,4\,\pi\,r^2}$$

Inside the conducting sphere

$$\sum Q_{in} = 0 \Rightarrow E_{in} = 0$$

The electric potential can be calculate as:

$$\vec{E} = -\overrightarrow{grad}\,V \Rightarrow V = -\int E.dr$$

$$V_{in} = -\int 0 \; dr = C_{in}$$

Outside the conducting sphere

$$\sum Q_{in} = Q = \sigma\, S = \sigma\, 4\,\pi\, R^2$$

$$E_{out} = \frac{\sigma\, 4\,\pi\, R^2}{\varepsilon_0\, 4\,\pi\, r^2} = \frac{\sigma\;\, R^2}{\varepsilon_0\;\, r^2}$$

$$\vec{E} = -\overrightarrow{grad}\, V \;\; \Rightarrow V = -\int \frac{\sigma\;\, R^2}{\varepsilon_0\;\, r^2} . dr$$

$$V_{out} = \frac{\sigma\;\, R^2}{\varepsilon_0\;\, r} + C_{out}$$

$$V_{out}(r \to \infty) = 0 \;\Rightarrow\; C_{out} = 0 \;\Rightarrow V_{out} = \frac{\sigma\;\, R^2}{\varepsilon_0\;\, r}$$

$$V_{out}(r = R) = V_{in}(r = R) \;\Rightarrow\; C_{in} = \frac{\sigma\;\, R}{\varepsilon_0} \Rightarrow V_{in} = \frac{\sigma\;\, R}{\varepsilon_0}$$

$$\frac{Q}{4\pi\, R^2} = \sigma$$

$$V_{in} = \frac{\frac{Q}{4\pi\, R^2}\;\; R}{\varepsilon_0} = \frac{Q}{4\pi\;\varepsilon_0\; R} = K\frac{Q}{R}$$

$$V_{Surface} = K\frac{Q}{R}$$

- *Calculating the electric potential at the surface of each sphere*

a) ***R_1 = R_2 and Q_2 = 2 Q_1***

$$V_{S1} = K\frac{\boldsymbol{Q_1}}{R_1}, V_{S2} = K\frac{\boldsymbol{Q_2}}{R_2} = \; K\frac{\boldsymbol{2Q_1}}{R_1} = 2V_{S1}$$

b) ***R_2 = 2 R_1 and Q_1 = Q_2***

$$V_{S1} = K\frac{\boldsymbol{Q_1}}{R_1}, V_{S2} = K\frac{\boldsymbol{Q_2}}{R_2} = \; K\frac{\boldsymbol{Q_1}}{2R_1} = \frac{1}{2}V_{S1}$$

- *Determine the direction of charge flow if the two conducting spheres are connected by a thin conducting wire.*

  a) ***R_1 = R_2 and Q_2 = 2 Q_1***

$$, V_{S2} = 2\, V_{S1}$$

*In this case, the electric charges (positive) move from the high potential to the low potential, and thus from the conducting sphere **S_2** to the conducting sphere **S_1***

b) **$R_2 = 2\, R_1$** *and* **$Q_1 = Q_2$**

$$V_{S2} = \frac{1}{2} V_{S1}$$

*According to the previous explanation, the electric charges (positive) are moved from spherical conductor **S_1** to spherical conductor **S_2**.*

- *Calculating the final electric charge on each sphere once electrostatic equilibrium is reached.*

*The charge continues to transfer until equilibrium is reached, where the potential of the first conducting sphere **S_1** will equal the potential of the second conducting sphere **S_2**. At equilibrium, the conducting sphere **S_1** will carry a new electric charge of **Q'_1** and a new electric potential of **V'_1**, also, the conducting sphere **S_2** will carry a new electric charge of **Q'_2** and a potential of **V'_2**. During this process, the total charge remains conserved.*

$$Q_1 + Q_2 = Q'_1 + Q'_2$$

$$V'_1 = K\frac{Q'_1}{R_1}, \qquad V'_2 = K\frac{Q'_2}{R_2}$$

At the equilibrium state $V'_1 = V'_2 \Rightarrow K\frac{Q'_1}{R_1} = K\frac{Q'_2}{R_2} \Rightarrow Q'_1 = R_1\frac{Q'_2}{R_2}$

$$\begin{cases} Q'_1 = R_1\dfrac{Q'_2}{R_2} \\ Q_1 + Q_2 = Q'_1 + Q'_2 \end{cases} \Rightarrow Q_1 + Q_2 = R_1\frac{Q'_2}{R_2} + Q'_2$$

$$Q_1 + Q_2 = \left(\frac{R_1}{R_2} + 1\right) Q'_2 = \left(\frac{R_1 + R_2}{R_2}\right) Q'_2$$

$$Q'_2 = \left(\frac{R_2}{R_1 + R_2}\right)(Q_1 + Q_2)$$

$$Q'_1 = R_1\frac{Q'_2}{R_2} = \left(\frac{R_1}{R_1 + R_2}\right)(Q_1 + Q_2)$$

$$\begin{cases} Q'_2 = \left(\dfrac{R_2}{R_1 + R_2}\right)(Q_1 + Q_2) \\ Q'_1 = \left(\dfrac{R_1}{R_1 + R_2}\right)(Q_1 + Q_2) \end{cases}$$

- *For* $R_1 = R_2$ *and* $Q_2 = 2\,Q_1$

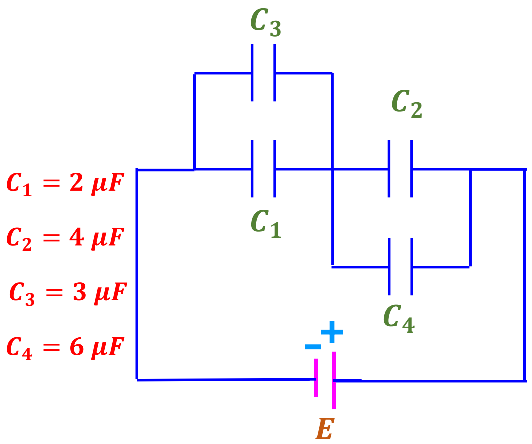
$$\begin{cases} Q'_2 = \left(\dfrac{R_1}{R_1 + R_1}\right)(Q_1 + 2Q_1) = \dfrac{3}{2}Q_1 \\ Q'_1 = \left(\dfrac{R_1}{R_1 + R_1}\right)(Q_1 + 2Q_1) = \dfrac{3}{2}Q_1 \end{cases}$$

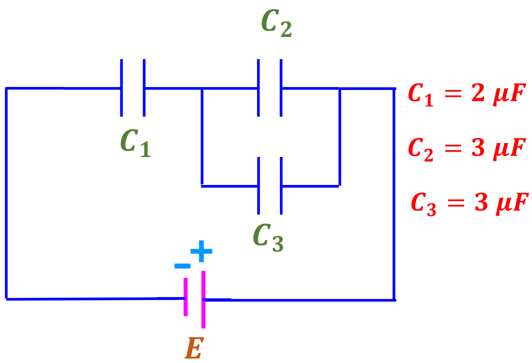
- *For* $R_2 = 2\,R_1$ *and* $Q_1 = Q_2$

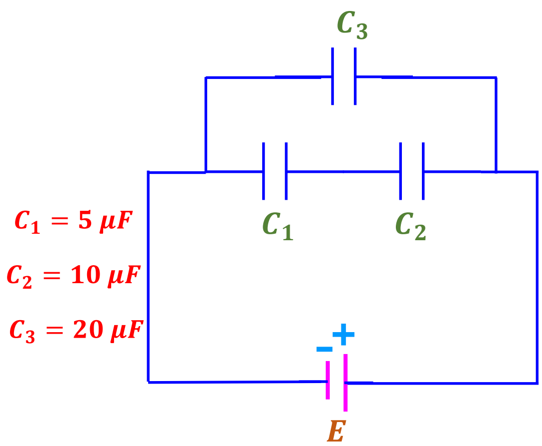
$$\begin{cases} Q'_2 = \left(\dfrac{2\,R_1}{R_1 + 2\,R_1}\right)(Q_1 + Q_1) = \dfrac{4}{3}Q_1 \\ Q'_1 = \left(\dfrac{R_1}{R_1 + 2\,R_1}\right)(Q_1 + Q_1) = \dfrac{2}{3}Q_1 \end{cases}$$

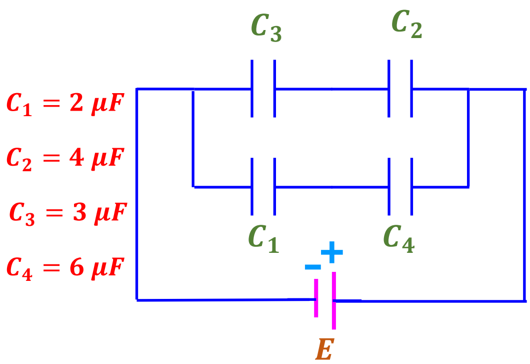
Exercise 02:

Find the equivalent capacitance of the following combinations:





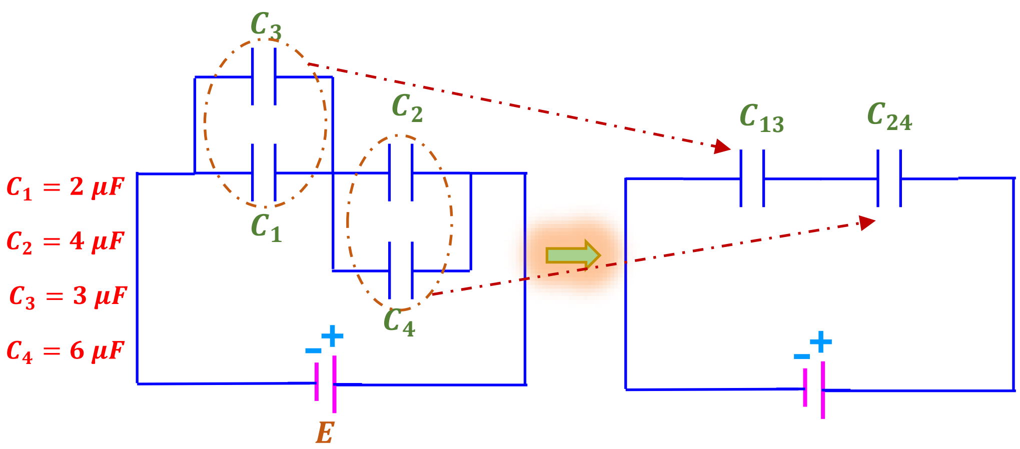




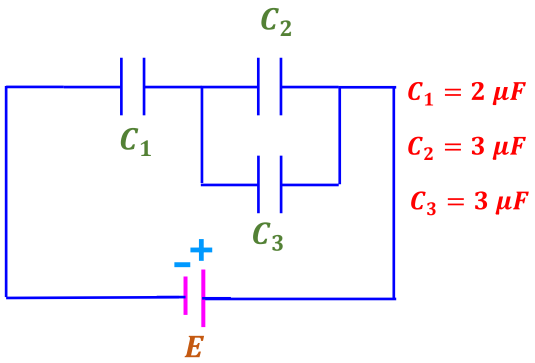
Solution:

The equivalent capacitance

$$\begin{cases} C_1 \text{ and } C_3 \text{ in parallel} \Rightarrow C_{13} = C_1 + C_3 = 2 + 3 = 5\ \mu F \\ C_2 \text{ and } C_4 \text{ in parallel} \Rightarrow C_{24} = C_2 + C_4 = 4 + 6 = 10\ \mu F \end{cases}$$

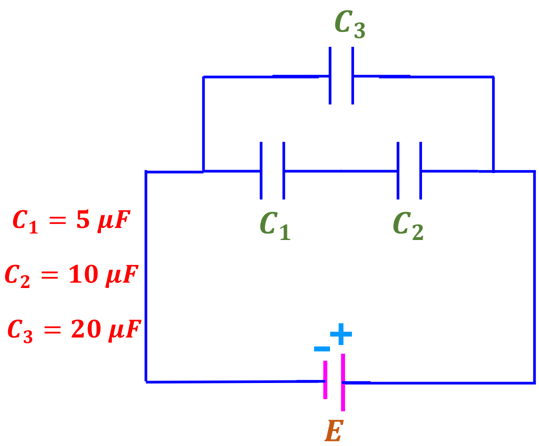


$$C_{13} \text{ and } C_{24} \text{ in series} \quad \Rightarrow \frac{1}{C_{eq}} = \frac{1}{C_{13}} + \frac{1}{C_{24}} \Rightarrow C_{eq} = \frac{C_{13}C_{24}}{C_{13}+C_{24}} = \frac{5\times 10}{10+5} = \frac{10}{3}\ \mu F$$



$$C_2 \text{ and } C_3 \text{ in parallel} \quad \Rightarrow C_{23} = C_2 + C_3 = 3 + 3 = 6\ \mu F$$

$$C_1 \text{ and } C_{23} \text{ in series} \quad \Rightarrow \frac{1}{C_{eq}} = \frac{1}{C_{23}} + \frac{1}{C_1} \Rightarrow C_{eq} = \frac{C_1C_{23}}{C_1+C_{23}} = \frac{2\times 6}{2+6} = \frac{3}{2}\ \mu F$$



$$C_1 \text{ and } C_2 \text{ in series} \quad \Rightarrow \frac{1}{C_{12}} = \frac{1}{C_2} + \frac{1}{C_1} \Rightarrow C_{12} = \frac{C_1C_2}{C_1+C_2} = \frac{5\times 10}{5+10} = \frac{10}{3}\ \mu F$$

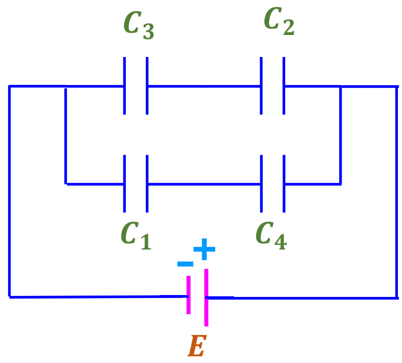
$$C_{12} \text{ and } C_3 \text{ in parallel} \quad \Rightarrow C_{eq} = C_{12} + C_3 = \frac{10}{3} + 20 = \frac{70}{3}\ \mu F$$

$C_1 = 2\ \mu F$

$C_2 = 4\ \mu F$

$C_3 = 3\ \mu F$

$C_4 = 6\ \mu F$



$$\begin{cases} C_1 \text{ and } C_4 \text{ in series } \Rightarrow C_{14} = \dfrac{C_1 C_4}{C_1 + C_4} = \dfrac{2 \times 6}{2+6} = \dfrac{3}{2}\ \mu F \\ C_2 \text{ and } C_3 \text{ in series } \Rightarrow C_{23} = \dfrac{C_2 C_3}{C_2 + C_3} = \dfrac{4 \times 3}{4+3} = \dfrac{12}{7}\ \mu F \end{cases}$$

$$C_{14} \text{ and } C_{23} \text{ in parallel } \Rightarrow C_{eq} = C_{14} + C_{23} = \frac{3}{2} + \frac{12}{7} = \frac{45}{14}\ \mu F$$

Exercise 03:

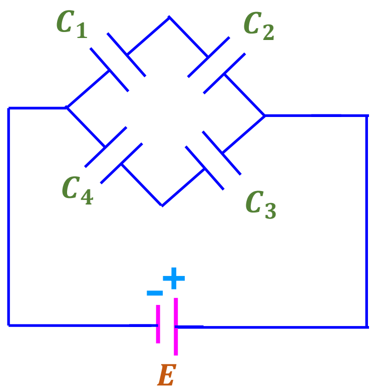
Calculate the equivalent capacitance of the following combinations:

$C_1 = 4\ \mu F$

$C_2 = 8\ \mu F$

$C_3 = 2\ \mu F$

$C_4 = 2\ \mu F$



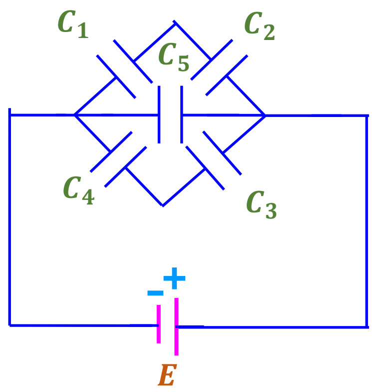
$C_1 = 4\ \mu F$

$C_2 = 8\ \mu F$

$C_3 = 2\ \mu F$

$C_4 = 2\ \mu F$

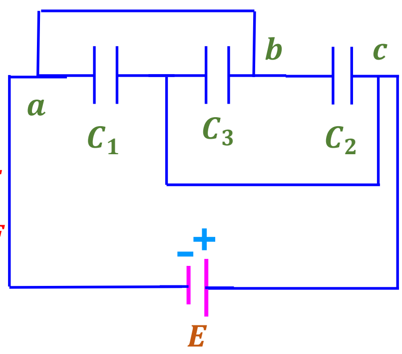
$C_5 = 10\ \mu F$



$C_1 = 5\ \mu F$

$C_2 = 10\ \mu F$

$C_3 = 20\ \mu F$



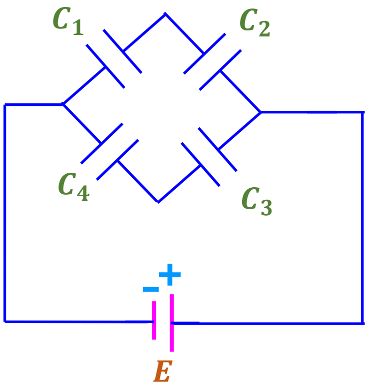
Solution:

$C_1 = 4\ \mu F$

$C_2 = 8\ \mu F$

$C_3 = 2\ \mu F$

$C_4 = 2\ \mu F$



$$\begin{cases} C_1 \text{ and } C_2 \text{ in series} & \Rightarrow C_{12} = \frac{C_1C_2}{C_1+C_2} = \frac{4\times8}{4+8} = \frac{8}{3}\ \mu F \\ C_4 \text{ and } C_3 \text{ in series} & \Rightarrow C_{34} = \frac{C_4C_3}{C_4+C_3} = \frac{2\times2}{2+2} = 1\ \mu F \end{cases}$$

$$C_{12} \text{ and } C_{34} \text{ in parallel} \quad \Rightarrow C_{eq} = C_{12} + C_{34} = \frac{8}{3} + 1 = \frac{11}{3}\ \mu F$$

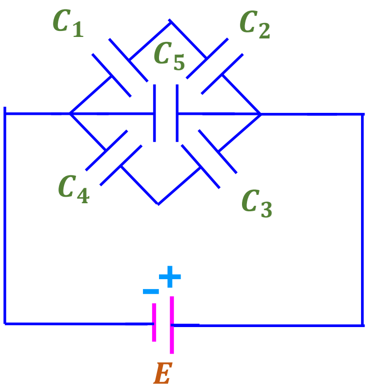
$C_1 = 4\ \mu F$

$C_2 = 8\ \mu F$

$C_3 = 2\ \mu F$

$C_4 = 2\ \mu F$

$C_5 = 10\ \mu F$



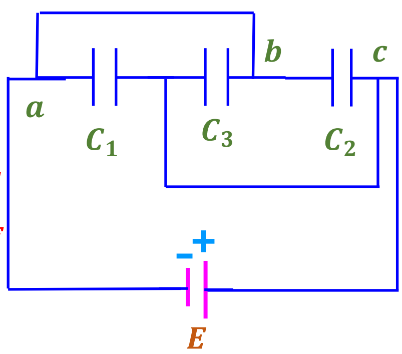
$$\begin{cases} C_1 \text{ and } C_2 \text{ in series} & \Rightarrow C_{12} = \frac{C_1C_2}{C_1+C_2} = \frac{4\times8}{4+8} = \frac{8}{3}\ \mu F \\ C_4 \text{ and } C_3 \text{ in series} & \Rightarrow C_{34} = \frac{C_4C_3}{C_4+C_3} = \frac{2\times2}{2+2} = 1\ \mu F \end{cases}$$

$$C_5\,, C_{12} \text{ and } C_{34} \text{ in parallel} \quad \Rightarrow C_{eq} = C_5 + C_{12} + C_{34} = 10 + \frac{8}{3} + 1 = \frac{41}{3}\ \mu F$$

$C_1 = 5\ \mu F$

$C_2 = 10\ \mu F$

$C_3 = 20\ \mu F$

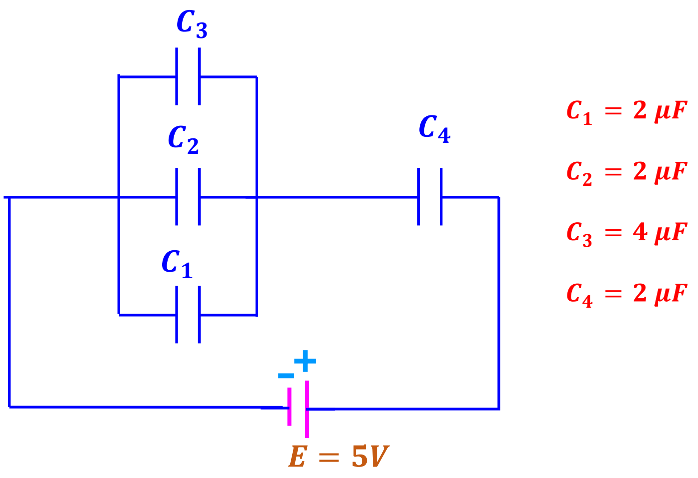


$$C_5\,, C_2 \text{ and } C_3 \text{ in parallel} \quad \Rightarrow C_{eq} = C_1 + C_2 + C_3 = 5 + 10 + 20 = 35\ \mu F$$

Exercise 04:

1- *Calculate the capacitance of the equivalent capacitor*

2- *Calculate the electrical charge that passes through each capacitor and the difference in electrical potential between the two ends of each capacitor*



Solution:

- *The capacitance of the equivalent capacitor*

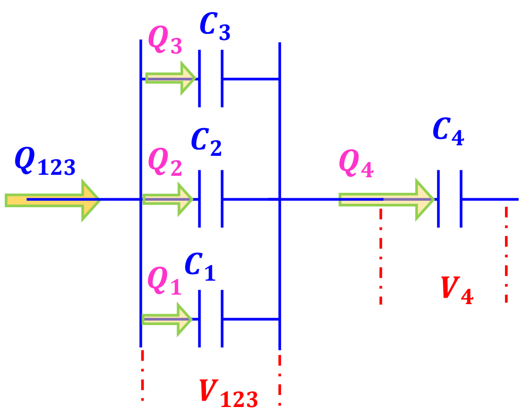
$$C_1, C_2 \text{ and } C_3 \text{ in parallel} \Rightarrow C_{123} = C_1 + C_2 + C_3 = 2 + 2 + 4 = 8\ \mu F$$

$$C_{123} \text{ and } C_4 \text{ in series} \Rightarrow \frac{1}{C_{eq}} = \frac{1}{C_{123}} + \frac{1}{C_4} \Rightarrow C_{eq} = \frac{C_{123}C_4}{C_{123} + C_4} = \frac{8 \times 2}{8 + 2} = \frac{8}{5}\ \mu F$$

- *The electrical charge that passes through each capacitor and the difference in electrical potential between the two ends of each capacitor*

$$E = V_4 + V_{123}$$

$C_4 = \frac{Q_4}{V_4}$, $C_{123} = \frac{Q_{123}}{V_{123}}$ *where* $Q_{123} = Q_4$ *Because the two capacitors are connected in series*



$$E = \frac{Q_4}{C_4} + \frac{Q_{123}}{C_{123}} = Q_4\left(\frac{1}{C_4} + \frac{1}{C_{123}}\right) = Q_4 \frac{1}{C_{eq}} \Rightarrow Q_4 = C_{eq}\, E = \frac{8}{5} \times 5 \times 10^{-6} = 8 \times 10^{-6} C = 8\ \mu C$$

$$Q_4 = Q_{123} = 8\,\mu C \Rightarrow \begin{cases} V_4 = \dfrac{Q_4}{C_4} = \dfrac{8\times 10^{-6}}{2\times 10^{-6}} = 4V \\ V_{123} = \dfrac{Q_{123}}{C_{123}} = \dfrac{8\times 10^{-6}}{8\times 10^{-6}} = 1V \end{cases}$$

$V_{123} = V_1 = V_2 = V_3 = 1V$ *since the three capacitors* C_1 , C_2 and C_3 *are connected in parallel.*

$$C_1 = \frac{Q_1}{V_1} \Rightarrow Q_1 = C_1 \times V_1 = 2 \times 1 \times 10^{-6} C = 2\,\mu C$$

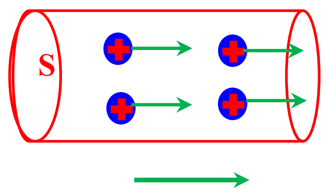
$$C_2 = \frac{Q_2}{V_2} \Rightarrow Q_2 = C_2 \times V_2 = 2 \times 1 \times 10^{-6} C = 2\,\mu C$$

$$C_3 = \frac{Q_3}{V_3} \Rightarrow Q_3 = C_3 \times V_3 = 4 \times 1 \times 10^{-6} C = 4\,\mu C$$

Chapter 3: Electric Circuits (Electrokinetics)

1) Electric current

Electric current is defined as the instantaneously collective and orderly movement of electric charges within a conductor, occurring when the conductor is subjected to an electric field created by a potential difference between its ends. This flow of charges constitutes an electric current, whose intensity is measured as the amount of charge passing through a given cross-sectional area per unit time. The direction of the conventional current is taken to be the direction positive charges would move, even though in most conductors the actual charge carriers are electrons moving in the opposite direction.



Current I direction

*Electric current intensity (I) is defined as the amount of electric charge (Q) that passes through the surface of a conductor during a unit of time (t). Its unit in the **SI** unit is **ampere (A)** and its expression is given as :*

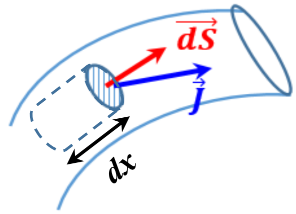
$$I = \frac{Q}{t} = \frac{dQ}{dt}$$

2) Electric Current Density

***Electric Current Density** is a vector quantity that expresses the magnitude and direction of the electric current passing through a unit area perpendicular to the direction of the current at a given point within the conductor. It shows how the current is distributed within the conductor.*

The relationship between electric current and current density is written as follows:

$$i(t) = \int \vec{J}\,.\,\overrightarrow{dS}$$



The current density can be written in terms of the average velocity $\boldsymbol{v_a}$ *of the electric charges. Suppose that n charges are inside a cylinder of elemental volume (length* **dx** *and area* **dS***) and an electric current di flows inside it, the volume density of the charge inside it* $\boldsymbol{\rho}$*. On this basis, the number* $\boldsymbol{n_c}$ *of charges inside the cylinder of elemental volume* **dV** *that will pass through the surface element* **dS***, covering a distance* **dx***, is expressed as follows:*

$$dV = dS \times dx \quad where \begin{cases} dx = v_a\, dt \\ \\ dq = \rho\, dV \end{cases} \Rightarrow dq = \rho\, v_a\, dt\, dS$$

$$dq = \rho\, \vec{v}_a\, dt\, \overrightarrow{dS} = i\, dt$$

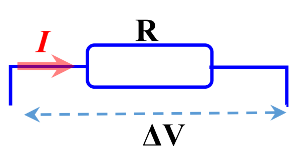
$$i = \vec{J}.\overrightarrow{dS} = \rho\, \vec{v}_a\, \overrightarrow{dS} \Rightarrow \vec{J} = \rho\, \overrightarrow{v_a}$$

If the number of free charges in a unit volume is n, then it is possible to write:

$$\rho = n\, q \Rightarrow \begin{cases} \vec{J} = n\, q\, \overrightarrow{v_a} \\ \\ i = n\, q\, \overrightarrow{v_a}\,\vec{S} \end{cases}$$

3) Ohm's Law

Ohm's law states that the ratio of the potential difference between the two ends of a conductor to the electric current passing through it is a constant ratio. It is symbolized by ***R*** *and its unit is the* Ω.



*This ratio is called electrical resistance **R** and represents the electrical obstruction that a conductor shows to the movement of charges when an electric current passes through it.*

$$R = \frac{\Delta V}{I} = \frac{\int_A^B \vec{E}.\overrightarrow{dr}}{\int_S \vec{J}\,.\overrightarrow{dS}}$$

The simplified formula of Ohm's law is reached by finding the relationship that links the electric field applied to these moving charges and their speed. This is done by applying the basic principle of motion, where the charges during their movement are subject to two forces, the first of which is an electric driving force, given by expression (1), and the second is a frictional force, where its direction is opposite to the direction of the charge movement, and its intensity is proportional to the speed of the charges, given by expression (2).

$$\begin{cases} \overrightarrow{F_e} = q\,\vec{E} & (1) \\ \overrightarrow{F_f} = -k\ \vec{v} & (2) \end{cases}$$

$$\sum \overrightarrow{F_{ext}} = m\,\vec{a} = m\frac{d\vec{v}}{dt} \Rightarrow \quad \overrightarrow{F_e} + \overrightarrow{F_f} = q\,\vec{E} - k\ \vec{v} = m\frac{d\vec{v}}{dt}$$

$$q\,\vec{E} - k\ \vec{v} = m\frac{d\vec{v}}{dt}$$

If the current intensity is constant, the speed of charge movement will also reach a constant value, which is the entrained velocity, and therefore:

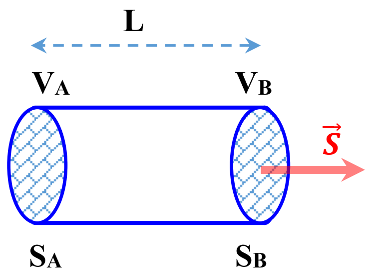
$$\frac{d\vec{v}}{dt} = 0 \Rightarrow q\,\vec{E} = k\ \vec{v} \Rightarrow \vec{v} = \frac{q}{k}\,\vec{E} = \mu\ \vec{E}$$

*Multiplying both sides of this equation by **(n q)** we get*

$$n\,q\,\vec{v} = n\,q\,\mu\ \vec{E} = \vec{J} = \sigma\,\vec{E}\ where\ \sigma = n\,q\,\mu$$

σ *is called conductivity of matter.*

*In the case of a cylindrical conveyor with a cross-sectional area of **S** and a length of **L***



The electric field is uniform, so the electric potential difference between terminals A and B becomes:

$$\Delta V = V_B - V_A = \int_A^B \vec{E}.\overrightarrow{dr} = E\,L$$

The electric current intensity is written as follows:

$$I = \int_S \sigma\,\vec{E}\,.\overrightarrow{dS} = \sigma\,E\,S$$

$$R = \frac{V}{I} = \frac{E\,L}{\sigma\,E\,S} = \frac{L}{\sigma\,S} = \rho\frac{L}{S} \quad where\ \rho = \frac{1}{\sigma}$$

ρ *Represents the specific resistivity of the material.*

4) Electric circuits

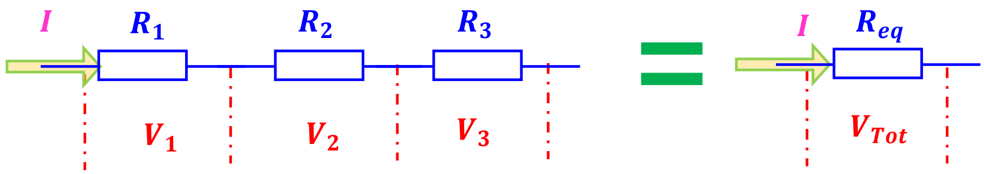
An electric circuit is an arrangement of interconnected electrical components that allows electric current to flow through them. These components may include sources of electromotive force (such as batteries or voltage supplies), resistors, inductors, capacitors, and semiconductor devices such as diodes and transistors.

5) Resistors combination

Resistors in electrical circuits can be arranged in series, in parallel, or in mixed combination.

6) 5.1) Series combination of Resistors

Resistors are installed in series combination when each pair of successive capacitors shares one connection point (one common terminal), and the same electric intensity passes through all resistors, while the potential difference across each one may vary.



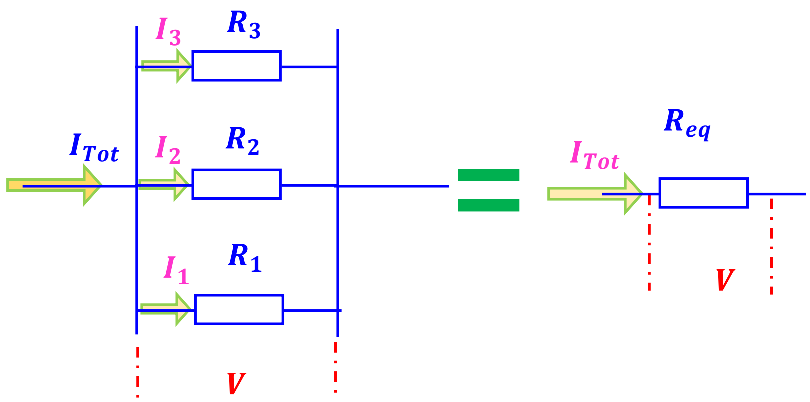
The electric current intensity passing through the equivalent resistor $\boldsymbol{R_{eq}}$ *is equal to the electric current intensity passing through the individual resistors* $\boldsymbol{R_i}$ *, meanwhile, the total potential difference* $\boldsymbol{V_{Tot}}$ *between the equivalent resistor ends is equal to the sum of the individual potential difference* $\boldsymbol{V_i}$ *between each resistor ends. The resistance value of the equivalent resistor can be calculated as follows:*

$$R_{eq} = \frac{\boldsymbol{V_{Tot}}}{I_{Tot}} \quad \begin{cases} \boldsymbol{I_{Tot}} = I_i \\ R_i = \dfrac{V_i}{I_i} \\ V_{Tot} = \displaystyle\sum_i V_i \end{cases}$$

$$R_{eq} = \frac{\boldsymbol{V_{Tot}}}{I_{Tot}} = \frac{\sum_i V_i}{I_i} = \sum_i \frac{V_i}{I_i} = \sum_i R_i$$

7) 5. 2) Parallel Combination of resistors

Resistors are connected in a parallel combination, all resistors are connected to the same two terminals, so the potential difference across each resistor is the same, while the total electric current is distributed among them according to their individual resistance value.



In the parallel capacitors combination method, the potential difference $\boldsymbol{V_{Tot}}$ *across the equivalent resistor* $\boldsymbol{R_{eq}}$ *is equal to the potential difference* $\boldsymbol{V_i}$ *across each individual resistor* $\boldsymbol{R_i}$*, while the total electric current* $\boldsymbol{I_{Tot}}$ *passes in the equivalent resistor is equal to the sum of the of the electric currents* $\boldsymbol{I_i}$ *passing through the individual resistors. The resistance of the equivalent resistor is calculated as follows:*

$$R_{eq} = \frac{\boldsymbol{V_{Tot}}}{I_{Tot}} \quad \begin{cases} \boldsymbol{I_{Tot}} = \sum_i I_i \\ R_i = \frac{V_i}{I_i} \\ V_{Tot} = V_i \end{cases}$$

$$R_{eq} = \frac{\boldsymbol{V_{Tot}}}{I_{Tot}} = \frac{V_i}{\sum_i I_i} \Rightarrow \frac{1}{R_{eq}} = \frac{\sum_i I_i}{V_i} = \sum_i \frac{I_i}{V_i} = \sum_i \frac{1}{R_i}$$

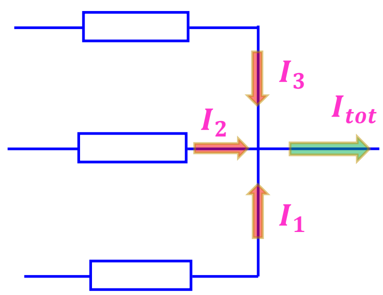
8) Kirchhoff's laws

To facilitate the study of electrical circuits and to know the current intensity passing through each branch of the electrical circuit, or the electrical voltage between the two ends of each element, we rely on Kirchhoff's laws.

9) 6.1) Junction rule:

At any junction in a circuit, the sum of the ingoing currents must equal the sum of the outgoing currents. That is:

$$\sum I_{in} = \sum I_{out}$$

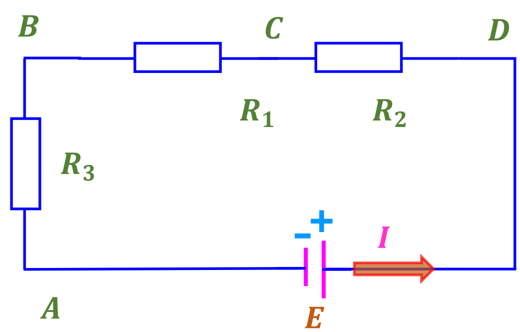


$$\boldsymbol{I_{tot} = I_1 + I_2 + I_3}$$

10) 6.2) Loop rule:

For any closed loop in a circuit, the sum of the potential differences across all elements must be zero. That is:

$$\sum_{closed\ loop} V_I = 0$$



To study this circuit, we note the following:

- *The conventional direction of current is from the positive pole (**+**) to the negative pole (**-**) .*
- *When an electric current **I** passes through a resistor **R**, a potential difference **V** is generated between its two ends, the direction of which is opposite to the direction of the current.*
- *According to Ohm's law, this potential difference is equal to* $\boldsymbol{V = R \times I}$
- *The potential difference between the same point is zero* $\boldsymbol{V_{AA} = O}$
- *We mark the two ends of each electrical component with letters, as shown in the figure.*
- *We apply Kirchhoff's loop law to the circuit as follows:*

$$\sum_{closed\ loop} V_I = 0 \Rightarrow V_{AA} = 0$$

$$V_{AA} = V_{A\to D} + V_{D\to C} + V_{C\to B} + V_{B\to A}$$

We choose the following conventional direction $\boldsymbol{A \to D \to C \to B \to A}$ *as the positive directio:*

$$V_{A\to D} = E\ ;\ V_{D\to C} = -\boldsymbol{R_2} \times I\ ;\ V_{C\to B} = -\boldsymbol{R_1} \times I\ ;\ V_{B\to A} = -\boldsymbol{R_3} \times I$$

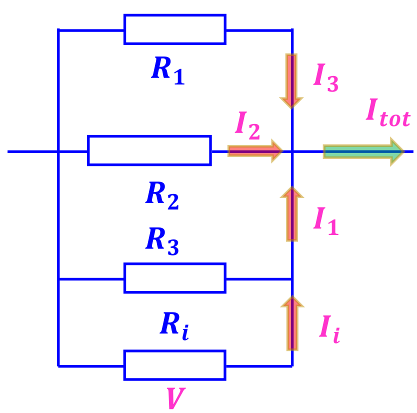
$$V_{AA} = 0 \Rightarrow E - \boldsymbol{R_2} \times I - \boldsymbol{R_1} \times I - \boldsymbol{R_3} \times I = 0$$

$$E = \boldsymbol{R_2} \times I + \boldsymbol{R_1} \times I + \boldsymbol{R_3} \times I$$

11) Current and Voltage Division Law

12) 7.1) Current Division Law

In a circuit that includes a group of resistors in parallel, the total electric current is divided according to the values of the resistors, where the resistor with the larger value has a smaller electric current passing through it.



$$I_{tot} = I_1 + I_2 + I_3 + I_i$$

$$I_i = \frac{V_i}{R_i} \quad and \quad V_i = V \quad \Rightarrow I_i = \frac{V}{R_i}$$

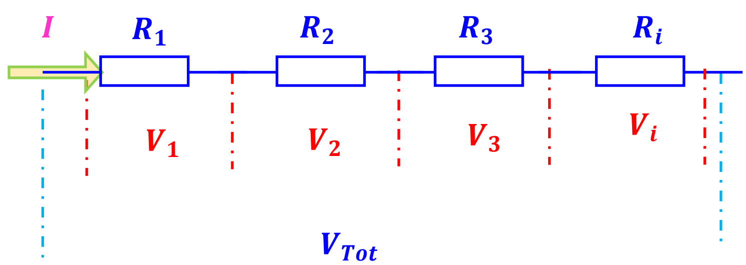
$$I_{tot} = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3} + \frac{V}{R_i} = V\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_i}\right) \Rightarrow V = \frac{I_{tot}}{\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_i}\right)}$$

$$I_i = \frac{V}{R_i} = \frac{1}{R_i} \times \frac{I_{tot}}{\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_i}\right)}$$

$$\boxed{I_i = \frac{1}{R_i} \times \frac{I_{tot}}{\sum_i \frac{1}{R_i}}}$$

13) 7.2) Voltage Division Law

In a circuit that includes a group of resistors in series, the total electrical voltage is divided according to the values of the resistors, where the resistor with the higher value has the greater voltage between its two ends.



$$V_i = R_i \times I_i \quad where\ I_i = I_{tot} = I = \frac{V_{tot}}{R_1 + R_1 + R_3 + R_i}$$

$$V_i = \frac{R_i}{R_1 + R_1 + R_3 + +R_i} \times V_{tot}$$

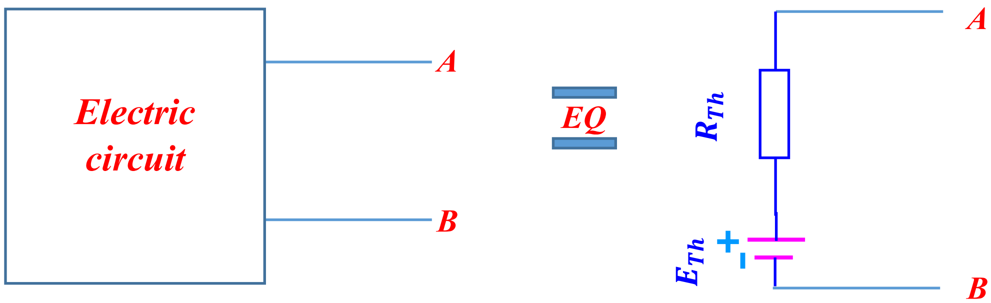
$$\boxed{V_i = \frac{R_i}{\sum_i R_i} \times V_{tot}}$$

14) Thevenin's and Norton's theorems

Thevenin and Norton methods are two of the most important methods for analyzing linear electrical circuits.

15) 8.1) Thévenin's theorem

Thévenin's theorem states that every linear circuit between two terminals A and B can be simplified into a simple circuit called a Thévenin circuit, which includes a Thévenin's generator $\boldsymbol{E_{Th}}$ *and a Thévenin's resistor* $\boldsymbol{R_{Th}}$ *connected in series.*

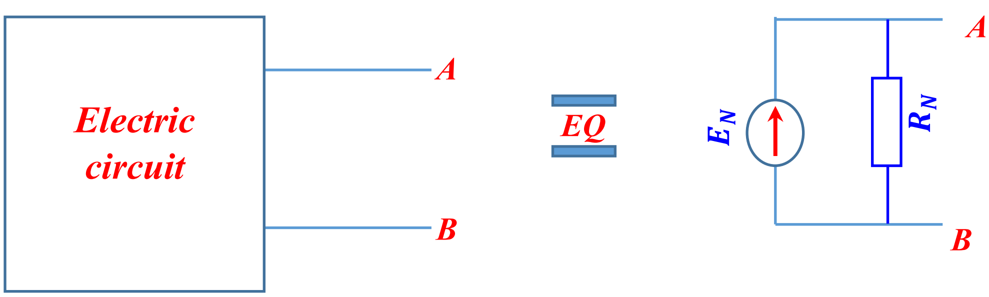


To convert any electrical network to its equivalent Thévenin circuit and to calculate the value of the Thévenin resistance and the electromotive force of the Thévenin generator, we perform the following steps:

- *Isolate the resistor to be measured.*
- *Short-circuit all voltage generators.*
- *Calculate the equivalent resistance.*
- *Calculate the potential difference between the two terminals of the branch.*

16) 8.2) Norton's theorems

The basis of Norton's theory is to convert an electrical circuit into a circuit consisting of a current generator (Norton generator) connected in parallel with a resistance (Norton resistor).

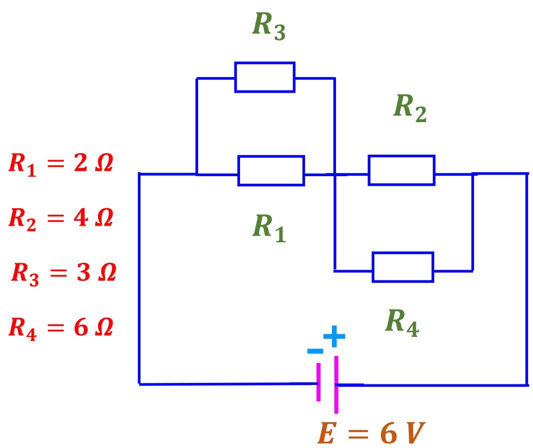


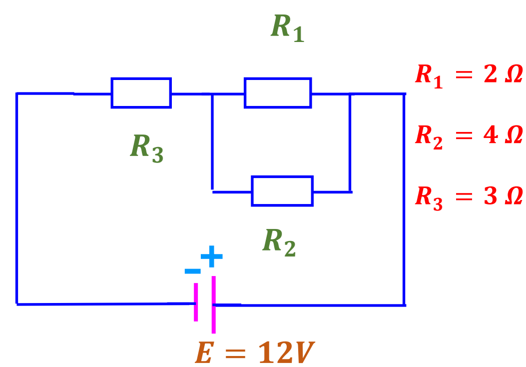
The steps followed in the Norton method are almost similar to the Thévenin method, where the Norton resistor is calculated in the same way as the Thévenin method, while the Norton current is calculated by shorting the target resistor in the calculation.

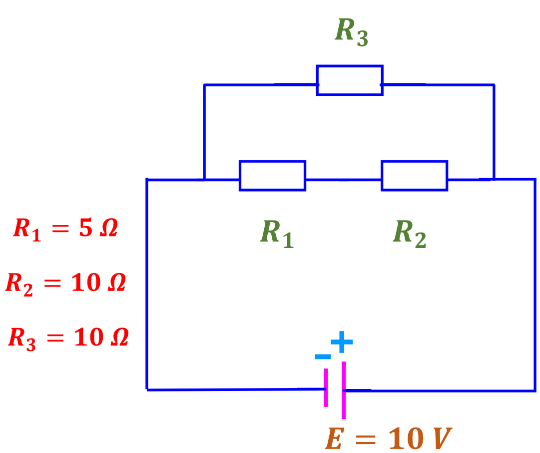
17) Solved exercise

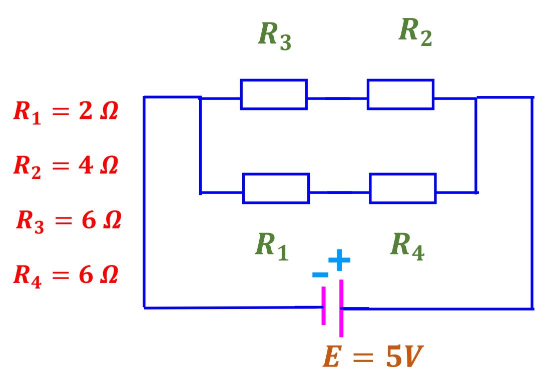
Exercise 01

Find the equivalent resistor of the following combinations:

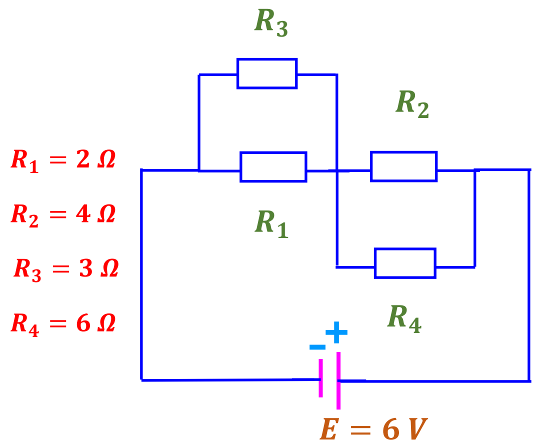








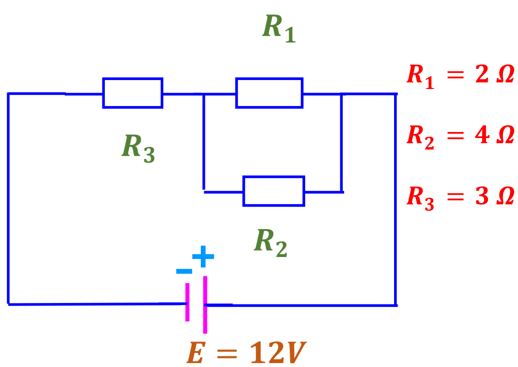
Solution



$$R_{13}\ eq\ of\ R_1\ and\ R_3 \Rightarrow R_{13} = \frac{R_1 \times R_3}{R_1 + R_3} = \frac{6}{5} = 1.2\ \Omega$$

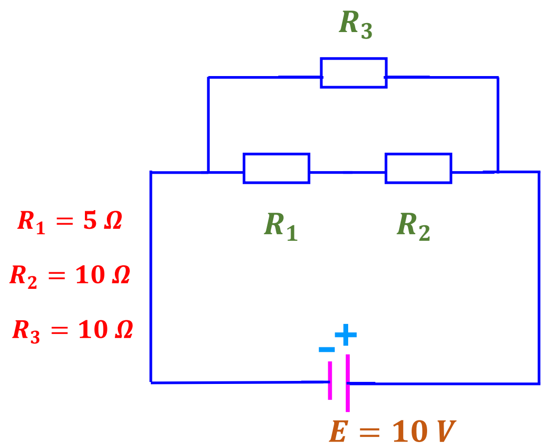
$$R_{24}\ eq\ of\ R_2\ and\ R_4 \Rightarrow R_{24} = \frac{R_2 \times R_4}{R_2 + R_4} = \frac{24}{10} = 2.4\ \Omega$$

$$R_{eq}\ eq\ of\ R_{13}\ and\ R_{24} \Rightarrow R_{eq} = R_{13} + R_{24} = 1.2 + 2.4 = 3.6\ \Omega$$



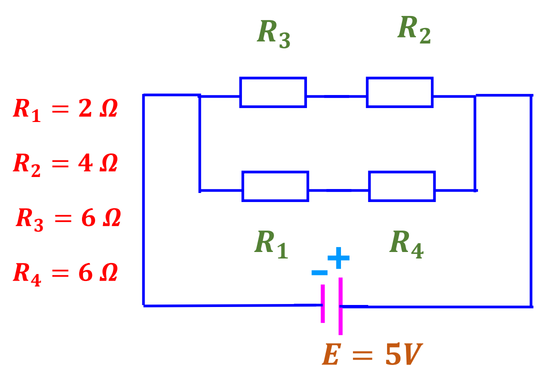
$$R_{12}\ eq\ of\ R_1\ and\ R_2 \Rightarrow R_{12} = \frac{R_1 \times R_2}{R_1 + R_2} = \frac{8}{6} = \frac{4}{3}\ \Omega$$

$$R_{eq}\ eq\ of\ R_3\ and\ R_{12} \Rightarrow R_{eq} = R_{12} + R_3 = 1.333 + 3 = 4.333\ \Omega$$



$$R_{12}\ eq\ of\ R_1\ and\ \ R_2\ \ \Rightarrow R_{12}\ =\ R_1 +\ \ R_2 = 10 + 5 = 15\ \Omega$$

$$R_{eq}\ \ eq\ of\ R_{12}\ and\ \ R_3\ \Rightarrow\ R_{eq}\ = \frac{R_{12} \times R_3}{R_{12} + R_3} = 6\ \Omega$$



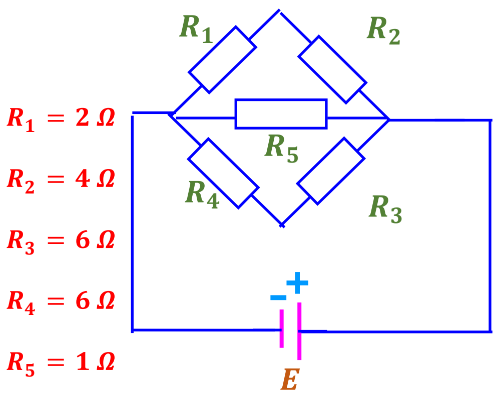
$$R_{14}\ eq\ of\ R_1\ and\ \ R_4\ \ \Rightarrow R_{14}\ =\ R_1 +\ \ R_4 = 8\ \Omega$$

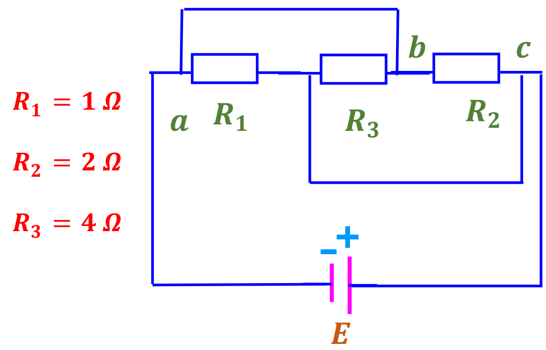
$$R_{23}\ eq\ of\ R_2\ and\ \ R_3\ \ \Rightarrow R_{23}\ =\ R_2 +\ \ R_3 = 10\ \Omega$$

$$R_{eq}\ \ eq\ of\ R_{14}\ and\ \ R_{23}\ \Rightarrow\ R_{eq}\ = \frac{R_{14} \times R_{23}}{R_{14} + R_{23}} = 4.44\ \Omega$$

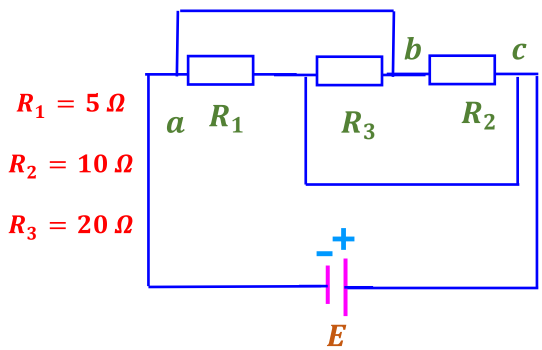
Exercise 02

Calculate the equivalent resistor of the following combinations:



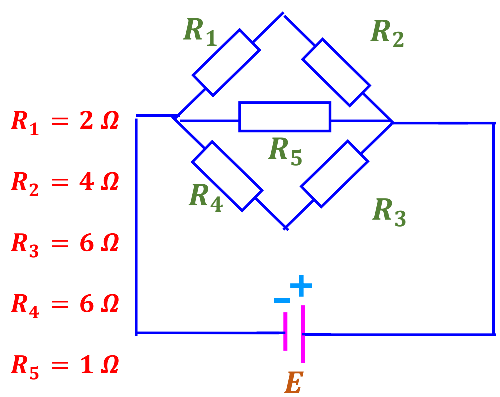


Solution



$$R_{eq}\ eq\ of\ R_1, R_2\ \ and\ \ R_3 \Rightarrow \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} = 1 + \frac{1}{2} + \frac{1}{4} = 1.75\ \Omega$$

$$R_{eq} = \frac{1}{1.75} = 0.57\ \Omega$$



$$R_{12}\ eq\ of\ R_1\ and\ \ R_2 \ \ \Rightarrow R_{12} = R_1 + \ R_2 = 6\ \Omega$$

$$R_{34}\ eq\ of\ R_4\ and\ \ R_3 \ \ \Rightarrow R_{34} = R_4 + \ R_3 = 12\ \Omega$$

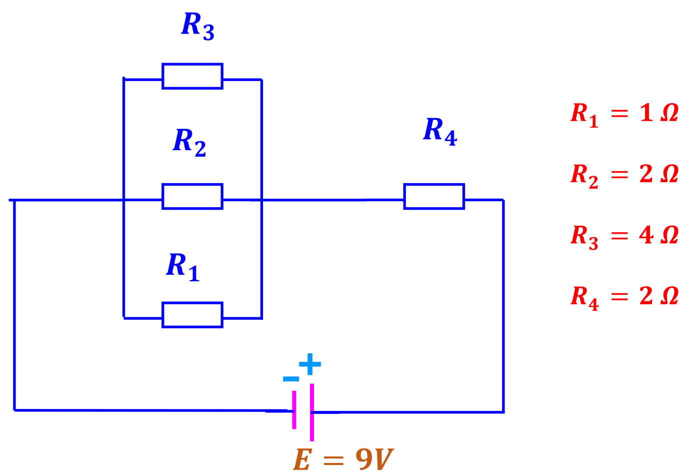
$$R_{eq}\ eq\ of\ R_{12}, R_{34}\ \ and\ \ R_5 \Rightarrow \frac{1}{R_{eq}} = \frac{1}{R_{12}} + \frac{1}{R_{34}} + \frac{1}{R_5} = \frac{1}{6} + \frac{1}{12} + 1 = 1.25\ \Omega$$

$$R_{eq} = \frac{1}{1.25} = 0.8\ \Omega$$

Exercise 03

1- *Calculate the equivalent resistor*

2- *Calculate the electric current that passes through each resistor and the difference in electrical potential between the two ends of each resistor*



Solution:

The resistance of the equivalent resistor

$$R_1, R_2 \text{ and } R_3 \text{ in parallel} \Rightarrow \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} = 1 + \frac{1}{2} + \frac{1}{4} = 1.75\ \Omega$$

$$R_{eq} = \frac{4}{7}\Omega$$

$$R_{123} \text{ and } R_4 \text{ in series} \Rightarrow R_{eq} = R_{123} + R_4 = 2 + \frac{4}{7}\ \Omega = \frac{18}{7}\ \Omega$$

The electric current that passes through each resistor and the difference in electrical potential between the two ends of each capacitor

$$E = V_4 + V_{123} = (R_{123} + R_4) \times I \Rightarrow I = \frac{E}{(R_{123} + R_4)} = \frac{9}{\frac{18}{7}} = 3.5\ A$$

$$V_4 = R_4 \times I = 2 \times 3.5 = 7\ V$$

$$V_{123} = R_{123} \times I = \frac{4}{7} \times 3.5 = 2\ V$$

$$V_{123} = V_1 = V_2 = V_3 = 2V$$

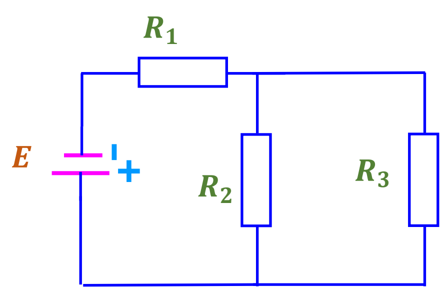
$$I_1 = \frac{V_1}{R_1} = \frac{2}{1} = 2\ A$$

$$I_2 = \frac{V_2}{R_2} = \frac{2}{2} = 1\ A$$

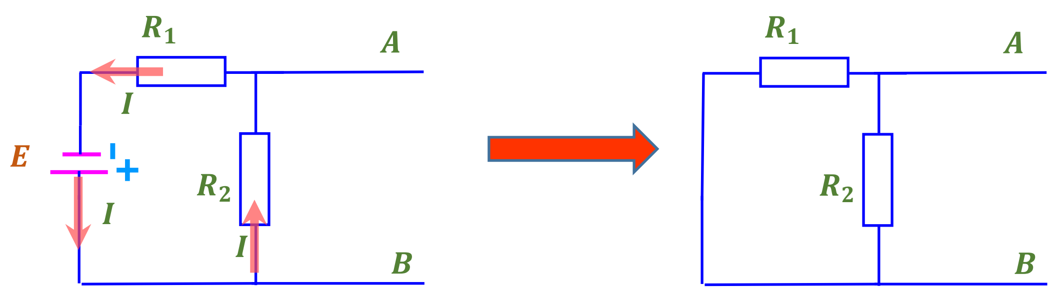
$$I_3 = \frac{V_3}{R_3} = \frac{2}{4} = 0.5\ A$$

Exercise 04

*Using **Thevenin's** method, calculate the current and voltage across the two ends of resistor **R_3**.*



Solution



$$\frac{1}{R_{eq}} = \frac{1}{R_2} + \frac{1}{R_1} \Rightarrow R_{eq} = \frac{R_1 \times R_2}{R_1 + R_2}$$

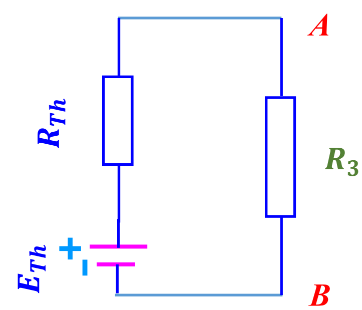
$$R_{Th} = R_{eq} = \frac{R_1 \times R_2}{R_1 + R_2}$$

$$V_{AB} = V_2 = R_2 \times I$$

$$E = (R_1 + R_2) \times I \quad \Rightarrow I = \frac{E}{R_1 + R_2}$$

$$V_{AB} = V_2 = R_2 \times \frac{E}{R_1 + R_2}$$

$$E_{Th} = R_2 \times \frac{E}{R_1 + R_2}$$



Applying Kirchhoff's Loop law

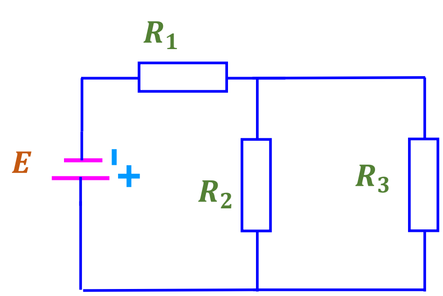
$$E_{Th} = (R_{Th} + R_3) \times I \quad \Rightarrow I = \frac{E_{Th}}{R_{Th} + R_3}$$

$$V_3 = R_3 \times I = R_3 \frac{E_{Th}}{R_{Th} + R_3} = R_3 \frac{R_2 \times \frac{E}{R_1 + R_2}}{\frac{R_1 \times R_2}{R_1 + R_2} + R_3} = \frac{\frac{R_3 \times R_2 \times E}{R_1 + R_2}}{\frac{R_1 \times R_2 + R_3(R_1 + R_2)}{R_1 + R_2}}$$

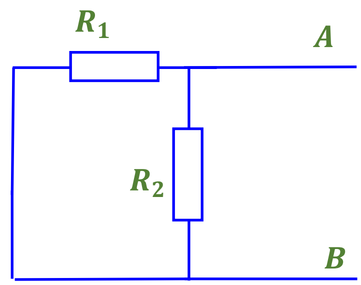
$$V_3 = \frac{R_3 \times R_2 \times E}{R_1 \times R_2 + R_3(R_1 + R_2)}$$

Exercise 05

*Using **Norton** method, calculate the current and voltage across the two ends of resistor **R_3**.*

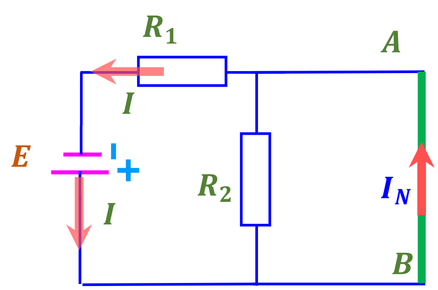


Solution

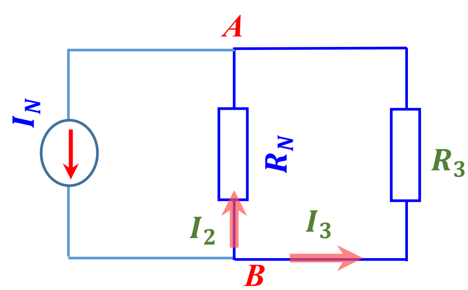


$$\frac{1}{R_{eq}} = \frac{1}{R_2} + \frac{1}{R_1} \Rightarrow R_{eq} = \frac{R_1 \times R_2}{R_1 + R_2}$$

$$R_N = R_{eq} = \frac{R_1 \times R_2}{R_1 + R_2}$$



$$E = (R_1) \times I_N \quad \Rightarrow I_N = \frac{E}{R_1}$$



$$V_3 = R_3 \times I_3 = V_N = R_N \times I_2 \Rightarrow I_2 = R_3 \times \frac{I_3}{R_N}$$

$$I_N = I_2 + I_3 = R_3 \times \frac{I_3}{R_N} + I_3 = I_3\left(\frac{R_N + R_3}{R_N}\right) \Rightarrow I_3 = \frac{I_N \times R_N}{R_N + R_3}$$

$$I_3 = \frac{\frac{E}{R_1} \times \frac{R_1 \times R_2}{R_1 + R_2}}{\frac{R_1 \times R_2}{R_1 + R_2} + R_3} = \frac{\frac{E}{R_1} \times \frac{R_1 \times R_2}{R_1 + R_2}}{\frac{R_1 \times R_2 + R_3(R_1 + R_2)}{R_1 + R_2}} = \frac{E\, R_2}{R_1 \times R_2 + R_3(R_1 + R_2)}$$

$$V_3 = \frac{E\, R_2\, R_3}{R_1 \times R_2 + R_3(R_1 + R_2)}$$

Chapter 4: Magnetostatics

1. Introduction

Magneto-statics is a branch of electromagnetism that studies time-invariant magnetic fields—fields produced by constant electric currents (that do not change with time $\Rightarrow \frac{dI}{dt} = 0$ *). It is considered a magnetic counterpart to electrostatics, which studies electric fields produced by stationary charges.*

A static electric charge produces an electric field, while a moving electric charge produces, in addition to the electric field, a magnetic field related to the direction and speed of the charge's movement.

2. Electromagnetic force (Lorentz's law)

This law states that every charge (q) moving at a velocity $\vec{v}$ *within an electromagnetic field will be subject to the influence of the magnetic field as well as the electric field. The expression for this force is written as follows:*

$$\vec{F} = \overrightarrow{F_{elec}} + \overrightarrow{F_{mag}} \quad where \begin{cases} \overrightarrow{F_{elec}} = q\,\vec{E} \\ \overrightarrow{F_{mag}} = q\,\vec{v} \times \vec{B} \end{cases}$$

3. Properties of magnetic force:

- *Magnetic force is nonexistent if the charge is stationary.*
- *Magnetic force is directed perpendicular to both the velocity and magnetic field.*
- *The magnetic force is zero if the charge's velocity vector is parallel to the magnetic field vector.*
- *The magnetic force reaches its maximum value when the charge's velocity vector is perpendicular to the magnetic field vector.*
- *The magnetic field does not change the velocity of the moving charge, but rather changes its direction.*

- ***Kinetic energy is conserved :*** *the change in kinetic energy of a charged particle moving within a magnetic field is equal to zero*

$$\Delta E_K(A \to B) = W(A \to B) = \int_A^B \overrightarrow{F_{mag}} \,.\, \overrightarrow{dr} = \int_A^B (q\,\vec{v} \times \vec{B})\,.\,\overrightarrow{dr}$$

$$= \int_A^B (q\,\vec{v} \times \vec{B})\,.\,\overrightarrow{dr}\frac{dt}{dt} = \int_A^B (q\,\vec{v} \times \vec{B}).\frac{\overrightarrow{dr}}{dt}dt$$

$$\int_A^B (q\,\vec{v} \times \vec{B}).\vec{v}\,dt = 0$$

$$\Delta E_K(A \to B) = 0 \Rightarrow E_K(B) = E_K(A)$$

4. The effect of a magnetic field on an electric current

Considering that the electric current flows through a group of charges.

*Each element length **dl** of wire passes through a current **I** and **N** electrons pass through it.*

$$\boldsymbol{N = n\,dV = n\,S\,dl}$$

*Where **S** is Cross-sectional area of wire, and **n** is the concentration of charges in the volume of the element dv. The expression for the magnetic force acting on the longitudinal element dl through which a current I passes is given by:*

$$\overrightarrow{F_{mag}} = q\,\vec{v} \times \vec{B}$$

$$d\overrightarrow{F_{mag}} = (-n\,e\,\vec{v})S\,d\vec{l} \times \vec{B}$$

$$d\overrightarrow{F_{mag}} = I\,d\vec{l} \times \vec{B}$$

$$\overrightarrow{F_{mag}} = L\,\vec{I} \times \vec{B}$$

As we learned in the previous chapter

$$I = \vec{J}.\overrightarrow{dS}$$

So,

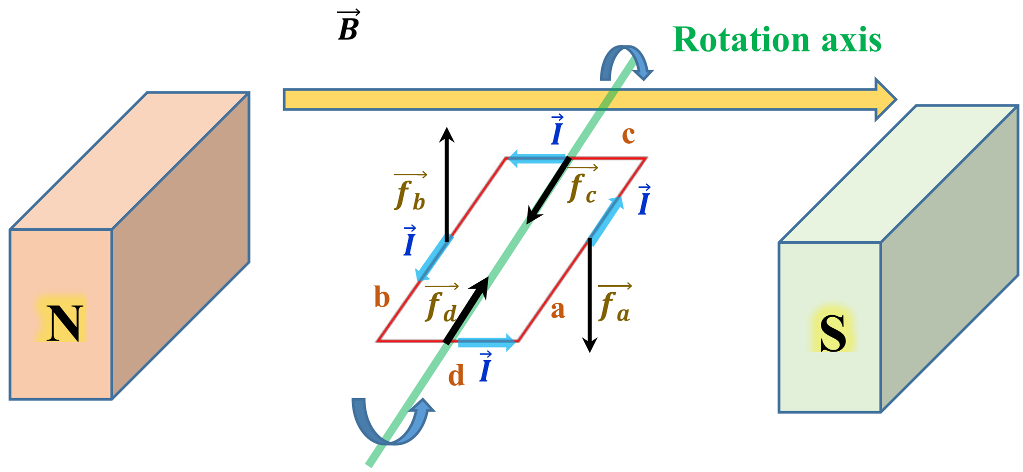
$$d\overrightarrow{F_{mag}} = (\vec{J}.\overrightarrow{dS})d\vec{l} \times \vec{B}$$

$$d\overrightarrow{F_{mag}} = \vec{J}.\overrightarrow{dS}\,d\vec{l} \times \vec{B}$$

$$d\overrightarrow{F_{mag}} = (\vec{J} \times \vec{B})\,dV$$

5. Torque on a Current Loop

The following figure represents a conducting wire structured as a rectangular loop with sides ABCD (the side's length of each A and B are labeled as L, and the sides C and D are labeled as K). A constant electric current flows through this loop. The loop is free to move and can rotate about an axis passing through the middle of sides D and C. The loop is placed under the influence of a constant magnetic field in the x-direction, parallel to sides C and D.

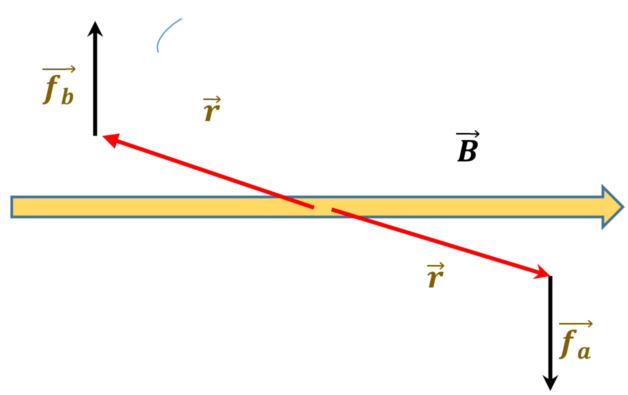


The magnetic field exerts a magnetic force on the electric current passing through each side.

By representing the forces, it is clear that the two forces $\vec{f}_c$ and $\vec{f}_d$ *have zero torque of force.*

The two magnetic forces applied to sides A and B form a double torque. The value of each force is given as follows:

$$\|\overrightarrow{f_b}\| = \|\overrightarrow{f_a}\| = L \times I \times B$$



$$\overrightarrow{\mathcal{M}}_{/0}(\overrightarrow{f_a}) = \vec{r} \wedge \overrightarrow{f_a} \quad \text{and } \overrightarrow{\mathcal{M}}_{/0}(\overrightarrow{f_b}) = \vec{r} \wedge \overrightarrow{f_b}$$

$$\|\vec{r}\| = \frac{K}{2}$$

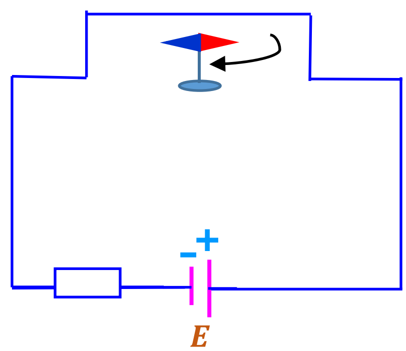
$$\overrightarrow{\mathcal{M}}_{/0} = \overrightarrow{\mathcal{M}}_{/0}(\overrightarrow{f_a}) + \overrightarrow{\mathcal{M}}_{/0}(\overrightarrow{f_b})$$

$$\|\overrightarrow{\mathcal{M}}_{/0}\| = \frac{K}{2} L \times I \times B \sin\theta + \frac{K}{2} L \times I \times B \sin\theta = K\, L \times I \times B \sin\theta$$

This idea is the working principle of electromagnetic motors.

6. Oersted's experiment:

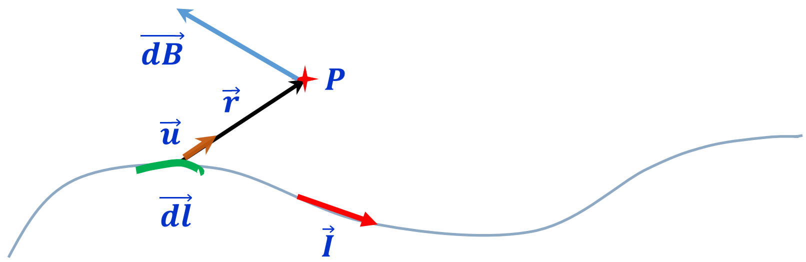
In 1820, Oersted observed a magnetic needle deflect when placed next to a wire carrying an electric current. It then returned to its normal position when the electric current was cut off. This experiment was the first step in generating a magnetic field through induction.



It was also shown through the same experiment that the direction of deflection of the magnetized needle is related to the direction of the electric current.

7. Magnetic field created by a current (The Biot–Savart Law)

*This law states that a magnetic field is generated at a point **P** in the space surrounding a conducting wire (Any electric charge moving at a speed of **v**) as a result of an electric current passing through it. This phenomenon is called "magnetic induction".*



The expression for the elemental magnetic field formed by the passage of an electric current ***I*** *through an element of length* ***dl*** *at any point* ***P*** *in space is as follows:*

$$d\vec{B} = \frac{\mu_0\, I}{4\,\pi\, r^2}\,.\overrightarrow{dl}\ \times \vec{u}$$

For an electric charge moving at a velocity of $\vec{v}$

$$\vec{B} = \frac{\mu_0\, q}{4\,\pi\, r^2}\,.\vec{v}\ \times \vec{u} = \frac{\mu_0\, q}{4\,\pi\, r^3}\,.\vec{v}\ \times \vec{r}$$

μ_0 *denotes a constant called the permeability of free space', its value equal to* $\boldsymbol{4\,\pi\,10^{-7}\,T.m/A}$

8. Solved exercises

Exercise 01

A point charge of $q = 1\,\mu\,C$ *moves in a straight path parallel to the x-axis at a constant speed of 200 m/s.*

Find the value of the field generated by this moving charge at the following points in space when it is at position (1 m, 0, 0)

Point N1 (0, 0, 0)

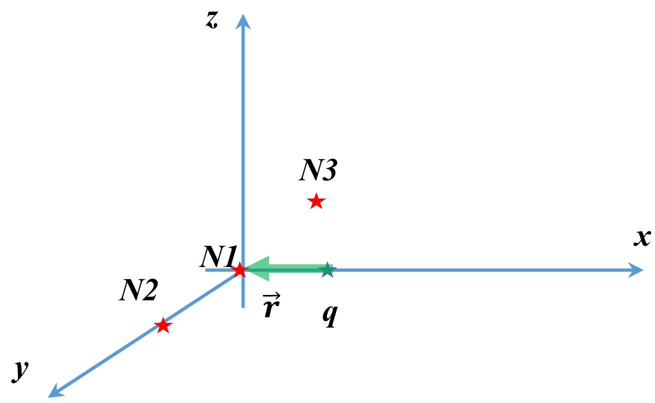
Point N2 (0, 1, 0)

Point N3 (,1 1,1)

Solution

The magnetic field generated by a moving charge is calculated using the following relationship:

$$\vec{B} == \frac{\mu_0\, q}{4\,\pi\, r^3}\,.\vec{v}\ \times \vec{r}$$



$$\vec{v} = 200\,\vec{\imath} \quad \mu_0 = \mathbf{4\,\pi\,10^{-7}}\ \boldsymbol{T.m/A}$$

At the Point N1 (0, 0, 0)

$$\vec{r} = -\vec{\imath} \quad \Rightarrow r = \|\vec{r}\| = 1$$

$$\vec{B} = \frac{\mu_0\, q}{4\,\pi\, r^3}\,.\vec{v}\times\vec{r} \;=\frac{(\mathbf{4\,\pi\,10^{-7}})\,(\mathbf{1\,10^{-6}})}{4\,\pi}\,.(200\,\vec{\imath}\times-\vec{\imath}\,) = \vec{0}$$

At the Point N2 (0, 1, 0)

$$\vec{r} = -\vec{\imath}+\vec{\jmath} \quad \Rightarrow r = \|\vec{r}\| = \sqrt{2}$$

$$\vec{B} = \frac{\mu_0\, q}{4\,\pi\, r^3}\,.\vec{v}\times\vec{r} \;=\frac{(\mathbf{4\,\pi\,10^{-7}})\,(\mathbf{1}\times\mathbf{10^{-6}})}{4\,\pi}\,.(200\,\vec{\imath}\times(-\vec{\imath}+\vec{\jmath}\,)\,)$$

$$= \mathbf{2}\times\ \mathbf{10^{-11}}\,\vec{k}$$

At the Point N1 (0, 0, 0)

$$\vec{r} = \vec{\jmath}+\vec{k} \quad \Rightarrow r = \|\vec{r}\| = \sqrt{2}$$

$$\vec{B} = \frac{\mu_0\, q}{4\,\pi\, r^3}\,.\vec{v}\times\vec{r} \;=\frac{(\mathbf{4\,\pi\,10^{-7}})\,(\mathbf{1\,10^{-6}})}{4\,\pi}\,.\left(200\,\vec{\imath}\times\left(\vec{\jmath}+\vec{k}\right)\right)$$

$$= \mathbf{2}\times\ \mathbf{10^{-11}}(\;\vec{k}-\vec{\jmath}\,)$$

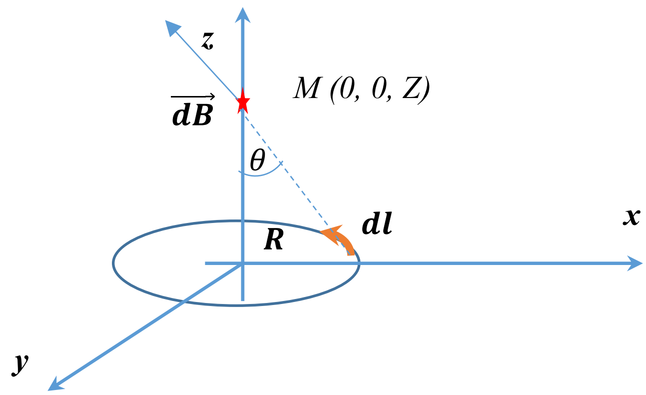
Exercise 02

A constant electric current passes through a conducting circle of radius R.

Find the expression for the magnetic field resulting from this current at point M (0, 0, Z).

Deduce the value of the magnetic field at point 0.

Solution



$$d\vec{B} = \frac{\mu_0 I}{4\,\pi\, r^3} \,.\vec{dl} \times \vec{r}$$

$$dB = \frac{\mu_0 I}{4\,\pi\, r^2}\, dl$$

Due to symmetry, the components on the x and y-axes cancel.

$$dB_z = \frac{\mu_0 I}{4\,\pi\, r^2}\, dl \sin\theta$$

$$B_z = \int \frac{\mu_0 I}{4\,\pi\, r^2}\, dl \sin\theta$$

$$B_z = \frac{\mu_0 I}{4\,\pi\, r^2} \sin\theta \int dl = \frac{\mu_0 I}{4\,\pi\, r^2} \sin\theta\, 2\,\pi\, R$$

$$\vec{B} = B_z\,\vec{k} = \frac{\mu_0 I}{2\, r^2} \sin\theta\;\, R\,\vec{k}$$

$$\sin\theta = \frac{R}{r} = \frac{R}{\sqrt{R^2 + z^2}}$$

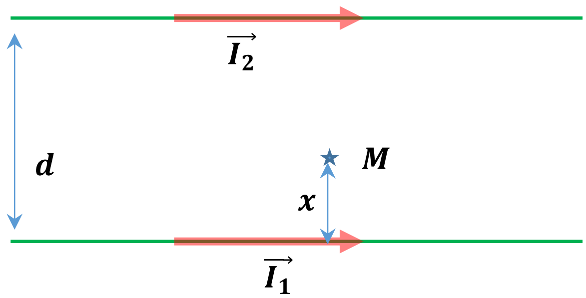
$$\vec{B} = \frac{\mu_0\, I\, R^2}{2\,(R^2 + z^2)^{\frac{3}{2}}}\,\vec{k}$$

At the point O ⇒ z = 0

$$\vec{B} = \frac{\mu_0\, I\, R^2}{2\,(R^2 + 0)^{\frac{3}{2}}}\,\vec{k} = \frac{\mu_0 I}{2R}\,\vec{k}$$

Exercise 03

Two straight parallel wires, each with an electric current passing through it.



- *Calculate the value of the magnetic field at point M located between the two wires.*
- *Discuss the case $x = d/2$ and $I_1 = I_2$*

Solution

The magnetic field at point M, which is the median point between the two straight wires carrying an electric current, is the vector summation of the field resulting from the first and second wires.

$$\vec{B}_M = \vec{B}_1 + \vec{B}_2$$

The magnetic fields 1 and 2 resulting from the electric current passing through each wire are opposite in direction at point M according to the right-hand rule.

$$B_M = B_1 - B_2$$

Where in the case of an infinitely long wire, the expression for the magnetic field at a point x away from it is given as follows:

$$B = \frac{\mu_0\, I}{2\,\pi\, b}$$

$$B_1 = \frac{\mu_0\, I_1}{2\,\pi\, x}$$

$$B_2 = \frac{\mu_0\, I_2}{2\,\pi\,(d-x)}$$

$$B_M = \frac{\mu_0\, I_1}{2\,\pi\, x} - \frac{\mu_0\, I_2}{2\,\pi\,(d-x)}$$

- *Discuss the case $x = d/2$ and $I_1 = I_2$*

$$B_M = \frac{\mu_0\, I_1}{2\,\pi\, x} - \frac{\mu_0\, I_2}{2\,\pi\,(d-x)}\;0$$

In this case, the magnetic field will be zero for points located at the same distance from the two wires.

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