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## Theme

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# *ON THE SPECTRAL CONTINUITY USING THE $\nu$-CONVERGENCE*

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# ON THE SPECTRAL CONTINUITY USING THE $\nu$-CONVERGENCE

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In The Name of **ALLAH**, The Most Beneficent, The Most Merciful.
All praise belongs to **ALLAH** alone, blessings and peace be upon the final Prophet.

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# NOTATIONS

| | |
|---|---|
| $\mathcal{L}(X)$ | $The\ algebra\ of\ all\ bounded\ linear\ operators\ from\ X\ into\ itelf$ |
| $N(A)$ | $The\ null\ space\ of\ A$ |
| $R(A)$ | $The\ range\ space\ of\ A$ |
| $K(X)$ | $The\ ideal\ of\ all\ compact\ operators\ on\ X$ |
| $\sigma(A)$ | $The\ spectrum\ of\ A$ |
| $\rho(A)$ | $The\ resolvent\ set\ of\ A$ |
| $F(X)$ | $The\ set\ of\ Fredholm\ operators$ |
| $\alpha(A)$ | $The\ nullity\ of\ A\ is\ defined\ as\ the\ dimension\ of\ N(A)$ |
| $\beta(A)$ | $The\ deficiency\ of\ A\ is\ defined\ as\ the\ codimension\ of\ R(A)$ |
| $I$ | $Operator\ of\ identity$ |
| $\Phi_{\pm}(X)$ | $The\ set\ of\ semi-Fredholm\ operators$ |
| $\Phi_{+}(X)$ | $The\ set\ of\ upper\ semi-Fredholmo\ perators$ |
| $\Phi_{-}(X)$ | $The\ set\ of\ lower\ semi-Fredholm\ operators$ |
| $\mathcal{F}(X,Y)$ | $The\ set\ of\ fredholm\ perturbation$ |
| $T_n \overset{n}{\to} T$ | $The\ norm\ convergence\ of\ T_n\ to\ T$ |
| $T_n \overset{p}{\to} T$ | $The\ pointwise\ convergence\ of\ T_n\ to\ T$ |
| $T_n \overset{c}{\to} T$ | $The\ compact\ convergence\ of\ T_n\ to\ T$ |
| $T_n \overset{cc}{\to} T$ | $The\ collectively\ compact\ convergence\ of\ T_n\ to\ T$ |
| $T_n \overset{\nu}{\to} T$ | $The\ \nu-convergence\ of\ T_n\ to\ T$ |

# GENERAL INTRODUCTION

The spectrum of a linear operators is one of the most useful objects in functional analysis. However, since exact computations of the spectrum are almost always impossible, it is relevant to know the spectrum or any of its parts in approximate way. Several authors have studied this problem and other related topic using various types of convergence in $\mathcal{L}(X)$.

There are several notions of convergence of a sequence of operators which yield spectral results: the norm convergence, pointwise convergence, compact convergence, collectively compact convergence. Moreover, in 1994 M. Ahues, and A. Lagillier introuduced the $\nu$-convergence.

The aim of this works is study the spectral continuity using the $\nu$-convergence. This works is composed of three chapters:

The first chapter: We will present the basic properties of the spectral theory of bounded linear operator which is an important part of functional analysis we begin by presenting generalies about linear opeators. In addition, special attension will be paid to the study of the linear adjoint operator and self adjoint for norm and hilbert space, and the basic notation necessary to study of the spectrum and compact operators, we recall some fundamontal results and notion relating to the theory fredholm and semi fredholm pertubation. In the final section of this chapter, we study two classes of the spectrum essential and present also a characterisation of essential spectrum.

The second chapter: we present some types convergence of a sequence of bounded linear operators (the norm convergence, pointwise convergence, compact convergence, collectively compact convergence). Subsequently, we show the v-convergence (definition and properties).

The third chapter: Is devoted to the study of the spectral continuity of some classes of essential spectrum using the $\nu$-convergence, first we define the spectrum continuity. In the last, we recall the $\nu$-continuty of the spectrum and the approximate point spectrum.

# 1

# BASIC PROPERTIES

In this chapter, we will talk about some classes of bounded linear operators, and we will be interested in presented the basic properties of a linear operators in a Banach spaces

## 1.1 GENERALITIES ABOUT LINEAR OPERATORS

**Definition 1.1.1** *Let $X$ and $Y$ be two vector space a field $K$, we say $A : X \to Y$ is a lineare operator if:*

*$\forall x, y \in X$ and $\lambda \in K$*

- $A(x + y) = A(x) + A(y)$
- $A(\lambda x) = \lambda A(x)$

**Definition 1.1.2** *[12] Let $X$ and $Y$ be a normed vector space and that the lineare operator $A : X \to Y$ is bounded if there existe a constant $C$ such that*

$$\|A(x)\| \leq C\|x\| \quad \forall x \in X \tag{1.1}$$

*the norme of such an operator is defined by*

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

*Again , it is the smallest $C$ wich works in (1.1). An operator $A$ is called continuous at a point $x_0 \in X$ if $x_n \to x$ in implie $Ax_n \to Ax$ in $Y$.A bounded linear operator is continuous at each point . for if $x_n \to x$ in $X$, then $\|Ax_n - Ax\| \leq \|A\|\|x_n - x\| \to 0$*

**Notation** *$\mathcal{L}(X, Y)$ denote the space of bounded linear operators from $X$ in $Y$, and if $X = Y$, we denote by: $\mathcal{L}(X, Y) = \mathcal{L}(X)$*

**Definition 1.1.3** *for* $A \in \mathcal{L}(X)$*, the image of* $A$ *is the set defined by* $R(A) = \{y = A(x); y \in Y\}$ *and the kernel of* $A$ *is the set defined by* $Ker(A) = \{Ax = 0; x \in X\}$ *a linear operator on a normed space* $X$ *to a normed space* $Y$ *is continuous at every point* $x$*; if is continous at a single point in* $x$*.*

**Proposition 1.1.1** *Let* $X$ *and* $Y$ *be normed lineare space and let* $A$ *be a lineare operator* $A : X \to Y$*. the following statements are equivalent:*

- $A$ *is bounded*
- $A$ *is continous at a point* $x_0$
- $A$ *is uniformly continous on* $X$

### 1.1.1 *Properties of the space of bounded linear operators*

- The space of all linear operators from $X$ to $Y$ is denoted by $\mathcal{L}(X, Y)$ and is normed space.
- If $Y$ is Banach, then so is $\mathcal{L}(X, Y)$, from which it follows that dual spaces are Banach.
- For any $A$ in $\mathcal{L}(X, Y)$, the kernel of $A$ is a closed linear subspaces of $X$.
- If $\mathcal{L}(X, Y)$ is Banach and $X$ is non trivial, then $Y$ is Banach.

## 1.2 ADJOINT OPERATORS

**Definition 1.2.1** *Let* $A$ *be an operator defined from a Hilbert space* $H_1$ *into a Hilbert space* $H_2$ *then, there exists an adjoint operator of* $A$ *Denoted by* $A^*$ *defined from* $H_2$ *into* $H_1$ *such that*

$$\langle Ax, y\rangle_{H_2} = \langle x, A^*y\rangle_{H_1}, \text{for all } x \in H_1 \text{ and } y \in H_2.$$

*Besides, we have*

$$\|A\| = \|A^*\|.$$

### 1.2.1 *Properties*

Let $H_1, H_2$ be Hilbert space $A_1 : H_1 \to H_2$ and $A_2 : H_2 \to H_1$ bounded linear operators and a any scalar. then we have

1. $(A_1 + A_2)^* = A_1^* + A_2^*$
2. $(\alpha A)^* = \overline{\alpha} A^*$
3. $(A_1 A_2)^* = A_2^* A_1^*$
4. $A^{**} = A$
5. $\|A^* A\| = \|A\|^2$

**Proposition 1.2.1** *Let $A$ be an operator defined from a Hilbert space $H_1$ into a Hilbert space $H_2$ then, we have*

$N(A) = \{\varphi \in H_1, A\varphi = 0\} = R^{\perp}(A^*).$

$N(A^*) = \{\psi \in H_2, A^*\psi = 0\} = R^{\perp}(A).$

### 1.2.2 *Self-adjoint operators*

**Definition 1.2.2** *[7] A bounded linear operator $A : H \to H$ on a complex Hilbert space $H$ to be self-adjoint if*

$$A = A^*$$

*Equivalently, a bounded linear operator $A$ to be self-adjoint if*

$$\langle Ax, y \rangle = \langle x, Ay \rangle \ \forall x, y \in H$$

**Theorem 1.2.1** *[7] Let $A : H \to H$ be a bounded linear operator on a Hilbert space $H$, then:*

1. *if $A$ is self-a djoint, then $\langle Ax, x \rangle$ is real for all $x \in H$*

2. *if* $H$ *is complex* $\langle Ax, x\rangle$ *is real for all* $x \in H$. *then the operator* $A$ *self-adjoint.*

**Proof.**

1. if $A$ is self-adjoint, then for all $x \in H$

   $$\langle \overline{Ax, x} \rangle = \langle x, Ax \rangle \qquad (1)$$

   by definition $\langle Ax, y\rangle = \langle x, A^* y\rangle$ and since $A$ is self-adjoint, we have

   $$\langle Ax, x\rangle = \langle x, Ax\rangle \qquad (2)$$

   combining equation (1) and (2) gives

   $$\langle \overline{Ax, x} \rangle = \langle Ax, x\rangle$$

   Hense $\langle Ax, x\rangle$ is equal to its complex conjugate which implie that is real

2. if $\langle Ax, x\rangle$ is real for all $x \in H$. then

   $$\langle Ax, x\rangle = \langle \overline{Ax, x}\rangle = \langle \overline{x, A^* x}\rangle = \langle A^* x, x\rangle$$

   Hense

   $$0 = \langle Ax, x\rangle - \langle A^* x, x\rangle = \langle (A - A^*)x, x\rangle$$

   thus, $A - A^* = 0$

   there for $A = A^*$

∎

## 1.3 SPECTRUM

**Definition 1.3.1** *[6] Let $A \in \mathcal{L}(X)$, the resolvent set of $A$ is the set*

$$\rho(A) = \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \text{ is invertible}\}$$

**Definition 1.3.2** *[6] Let $A \in \mathcal{L}(X)$, the spectrum of $A$ is:*

$$\sigma(A) = \mathbb{C} \setminus \rho(A) = \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \text{ is not invertible}\}$$

*There may be several reasons why the operator $\lambda - A$ be not invertible:*

1. *$N(\lambda - A) \neq \{0\}$, i.e. $\exists v \in X$ such that $v \neq 0$ and $Av = \lambda v$, i.e. $\lambda$ is an eigenvalue. The subset of $\delta(A)$ defined by*

   $$\sigma_p(A) = \{\lambda \in \mathbb{C}; N(\lambda - A) \neq \{0\}\}$$

   *is composed of eigenvalues and is called the* ***punctual spectrum*** *of $A$.*

2. *$N(\lambda - A) = \{0\}$ but $R(\lambda - A) \neq X$. Then we may again consider two subcases:*
   - *a) $N(\lambda - A) = \{0\}$ and $R(\lambda - A)$ is not dense in $X$, then we say that $\lambda$ is a* ***residual spectral value*** *of $A$. The subset*

     $$\sigma_r(A) = \{\lambda \in \mathbb{C}; N(\lambda - A) = \{0\}; \overline{R(\lambda - A)} \neq X\}$$

     *is called the* ***residual spectrum*** *of $A$.*
   - *b) $N(\lambda - A) = \{0\}$, $R(\lambda - A) \neq X$ but $R(\lambda - A)$ is dense in $X$, then we say that $\lambda$ is a* ***continuous spectral value*** *of $A$. The subset*

     $$\sigma_c(A) = \{\lambda \in \mathbb{C}; N(\lambda - A) = \{0\}; \overline{R(\lambda - A)} = X\}$$

     *is called the* ***continuous spectrum*** *of $A$.*

## 1.4 COMPACT LINEAR OPERATORS

**Definition 1.4.1** *A linear operator $A$ defined from a normed space $X$ into a normed space $Y$ is called a compact linear operator if for every bounded subset $G$ of $X$, the image $A(G)$ is relatively compact in $Y$. In other words, the closure $\overline{A(G)}$ is compact.*

**Theorem 1.4.1** *(Compactness criterion) A linear operator $A$ defined from a normed space $X$ into a normed space $Y$ is called a linear compact operator or completely continuous linear operator if and only if for every bounded sequence $\varphi_n$ in $X$, the sequence $A\varphi_n$ in $Y$ has a convergent subsequence $A\varphi_{nk}$*

**Theorem 1.4.2** *The product $AB$ of two bounded operators $A$ and $B$ is compact if either of operators $A$ or $B$ is compact.*

**Proof.** Let $\varphi_n$ be a boudned sequence in $X$, then if we consider $B$ as a bounded operator the sequence $B\varphi_n(x)$ is bounded, and from the compactness of the operator $A$ gives a convergent subsequence $A(B\varphi_{nk}(x))$ of $A(B\varphi_n(x))$. Hence the operator $AB$ is compact.

On the other hand, if we consider $B$ as a compact, one can extract from $B\varphi_n(x)$ a convergent subsequence $A(B\varphi_{nk}(x))$, and from the boundedness of the operator $A$ gives the convergence of the sequence $A(B\varphi_{nk}(x))$. Hence the operator $AB$ is compact. ■

**Theorem 1.4.3** *The sequence $A_n$ of compact operators defined from a normed space $X$ into a Banach space $Y$ converges uniformly to an operator $A$, say*

$$\lim_{n\to\infty} \|A_n - A\| = 0.$$

*Then the limit operator $A$ is compact.*

## 1.5 FREDHOLM OPERATORS

**Definition 1.5.1** *[12] Let $X$ and $Y$ be two Banach spaces and let $A : X \to Y$ be a bounded linear operator is called a fredholm operator if the three conditions following are satisfied:*

- $\alpha(A) = \dim(N(A)) < \infty$.
- $R(A)$ *is closed.*
- $\beta(A) = co\dim(R(A)) < \infty$.

**Remark 1.5.1** *[4] If $A \in \Phi_{\pm}(X,Y)$, The number $ind(A) = \alpha(A) - \beta(A)$ is called the index of $A$. Cearly, $ind(A)$ is an integer or $\pm\infty$. The subset of all compact operators of $\mathcal{L}(X,Y)$ is denoted by $K(X,Y)$. If $X = Y$ then $\Phi_+(X,Y)$, $\Phi_-(X,Y)$, $\Phi_{\pm}(X,Y)$, $\Phi(X,Y)$ and $\Phi^b(X,Y)$ are replaced respectively by $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_{\pm}(X)$, $\Phi(X)$ and $\Phi^b(X)$.*

*Observe that in the case $X = Y$ the class $\Phi(X)$ is non-empty since the identity trivially is a Fredholm operators. but for certain different Banach spaces X, Y no bounded Fredholm operators from X to Y may exist. Then we have*

- *Let $\Phi(X)$ denoted the set of **fredholm operators**, then we have*

  $\Phi(X) = \Phi_+(X) \cap \Phi_-(X).$

- *Let $\Phi_{\pm}(X)$ denoted the set of **semi-Fredholm operators**, then we have*

  $\Phi_{\pm}(X) = \Phi_+(X) \cup \Phi_-(X).$

- *Let $\Phi_+(X)$ denoted the set of **upper semi-Fredholm operators**, then we have*

  $\Phi_+(X) = \{A \in \mathcal{L}(X) : \alpha(A) < \infty \text{ and } R(A) \text{ is closed in } X\}.$

- *Let $\Phi_-(X)$ denoted the set of **lower semi-Fredholm operators**, then we have*

  $\Phi_-(X) = \{A \in \mathcal{L}(X) : \beta(A) < \infty \text{ and } R(A) \text{ is closed in } X\}.$

- *Let $\Phi^b(X)$ denoted the set of **bounded Fredholm operators**, then we have*

  $\Phi^b(X) = \Phi_+(X) \cap \mathcal{L}(X).$

**Theorem 1.5.1** *[9] Let $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,Z)$, where $X, Y$ and $Z$ are Banach spaces.*

1. *If $A \in \Phi_+(X,Y)$ and $B \in \Phi_+(Y,Z)$, then $BA \in \Phi_+(X,Z)$ and*
   $ind(BA) = ind(A) + ind(B).$

2. *If* $A \in \Phi_-(X,Y)$ *and* $B \in \Phi_-(Y,Z)$, *then* $BA \in \Phi_-(X,Z)$ *and* $ind(BA) = ind(A) + ind(B)$.
3. *If* $A \in \Phi(X,Y)$ *and* $B \in \Phi(Y,Z)$, *then* $BA \in \Phi(X,Z)$ *and* $ind(BA) = ind(A) + ind(B)$.
4. *If* $BA \in \Phi_+(X,Z)$, *then* $A \in \Phi_+(Y,Z)$.
5. *If* $BA \in \Phi_-(X,Z)$, *then* $B \in \Phi_-(Y,Z)$.
6. *If* $BA \in \Phi(X,Z)$, *then* $B \in \Phi_-(Y,Z)$ *and* $A \in \Phi_+(X,Y)$.

**Theorem 1.5.2** *[8] Let* $S, T \in \mathcal{L}(X,Y)$, *then:*

i) *If* $\beta(T) = \infty$, *then* $\beta(TS) = \infty$.

ii) *If* $\beta(S) < \infty$ *and* $\alpha(T) = \infty$, *then* $\alpha(TS) = \infty$.

**Theorem 1.5.3** *[4] Let* $A \in \mathcal{L}(X,Y)$ *so:*

- $A \in \Phi_+(X,Y) \Leftrightarrow A^* \in \Phi_-(Y^*,X^*)$.
- $A \in \Phi_-(X,Y) \Leftrightarrow A^* \in \Phi_+(Y^*,X^*)$.
- $A \in \Phi(X,Y) \Leftrightarrow A^* \in \Phi(Y^*,X^*)$.
- *If* $A \in \Phi_\pm(X)$, *then* $ind(A) = -ind(A^*)$.

**Theorem 1.5.4** *(Alternative de Fredholm) Let* $A \in \mathcal{L}(X)$ *so:*

1. $N(\lambda I - A) < \infty$.
2. $R(\lambda I - A)$ *is closed.*
3. $N(\lambda I - A) = \{0\} \Leftrightarrow R(\lambda I - A) = X$.
4. $\dim N(\lambda I - A) = \dim N(\lambda I - A^*)$.

### 1.5.1 *Fredholm and semi-fredholm perturbation*

**Definition 1.5.2** *[9] Let X and Y be two Banach spaces and let $A : X \to Y$ be a bounded linear operator, $A \in \mathcal{L}(X, Y)$.*

- *A is called a* ***Fredholm perturbation*** *if*

  $U + A \in \Phi(X, Y)$ *when ever* $U \in \Phi(X, Y)$.

- *A is called an* ***upper semi-Fredholm perturbation*** *(resp. a* ***lower semiFredhlom perturbation****) if*

  $U + A \in \Phi_+(X, Y)$ *(resp.* $U + A \in \Phi_-(X, Y)$*) for all* $U \in \Phi_+(X, Y)$ *(resp.* $U \in \Phi_-(X, Y)$*).*

*The set of* ***Fredholm perturbations, upper semi-Fredholm perturbation*** *(resp. a* ***lower semi-Fredholm perturbation****), are respectively denoted by*

$$\mathcal{F}(X, Y), \mathcal{F}_+(X, Y), \mathcal{F}_-(X, Y).$$

**Proposition 1.5.1** *Let X, Y and Z be three Banach spaces. If at least one of the sets $\phi^b(X, Y)$ and $\phi^b(Y, Z)$ is not empty, then*

- $F \in \mathcal{F}^b(X, Y)$, $A \in \mathcal{L}(Y, Z)$ *imply* $AF \in \mathcal{F}^b(X, Z)$.
- $F \in \mathcal{F}^b(Y, Z)$, $A \in \mathcal{L}(X)$ *imply* $FA \in \mathcal{F}^b(X, Z)$.

## 1.6 ESSENTIAL SPECTRUM

**Definition 1.6.1** *[9] Let X be a Banach space and let $A : X \to Y$ be a linear operator, the essential spectre denoted by $\sigma_{ess}$:*

$$\sigma_e(A) = \{\lambda \in \mathbb{C}; \lambda - A \textit{ is not Fredholm}(A) \textit{ indice } 0\}$$

**Remark 1.6.1** *Let $\mathbb{C}$ be the set of complex numbers. If $A \in \mathcal{L}(x)$ we define the sets*

$\Phi_A = \{ \lambda \in \mathbb{C} \textit{ such that } \lambda - A \in \Phi(x) \}$

$$\Phi_{+A} = \{\ \lambda \in \mathbb{C} \text{ such that } \lambda - A \in \Phi_{+}(x)\ \}$$

$$\Phi_{-A} = \{\ \lambda \in \mathbb{C} \text{ such that } \lambda - A \in \Phi_{-}(x)\ \}$$

*$\phi_A, \phi_{+A}$ and $\phi_{-A}$ are open subsets of the complex plane.*

**Proposition 1.6.1** *[9] There are sevral definitions of the essential spectrum, let us define the following sets:*

$$\sigma_{e1}(A) = \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \phi_{+}(X)\} = \mathbb{C}\backslash\phi_{+A}$$

$$\sigma_{e2}(A) = \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \phi_{-}(X)\} = \mathbb{C}\backslash\phi_{-A}$$

$$\sigma_{e3}(A) = \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \phi_{\pm}(X)\} = \mathbb{C}\backslash\phi_{\pm A}$$

$$\sigma_{e4}(A) = \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \phi(X)\} = \mathbb{C}\backslash\phi_{A}$$

$$\sigma_{e5}(A) = \bigcap_{k \in K(x)} \sigma(A+K)$$

$$\sigma_{e6}(A) = \mathbb{C}\backslash\rho_6(A)$$

$$\sigma_{e7}(A) = \bigcap_{k \in K(x)} \sigma_{ap}(A+K)$$

$$\sigma_{e8}(A) = \bigcap_{k \in K(x)} \sigma_{\delta}(A+K)$$

$$\sigma_{pa}(A) = \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \text{ is not one-to-one with closed range}\}$$

*with:*

$$\sigma_{ap}(A) = \left\{\lambda \in \mathbb{C}\backslash \inf_{\|x\| \leq 1} \|\lambda - A\| = 0\right\}$$

$$\sigma_{\delta}(A) = \{\lambda \in \mathbb{C}\backslash\lambda - A \text{ not surjective}\}$$

$$\rho_6(A) = \{\lambda \in \rho_5(A)\backslash \text{ all } \lambda \text{ in } \rho(A)\}$$

$$\rho_5(A) = \{\lambda \in \phi_A \backslash i(\lambda - A) = 0\}$$

**Proposition 1.6.2** *[9] We have:*

- $\sigma_{e1}(A)$ *and* $\sigma_{e2}(A)$*, its is essential spectrum of Gustafson and Weidmann.*
- $\sigma_{e3}(A)$ *is essential spectrum of Kato.*
- $\sigma_{e4}(A)$ *is essential spectrum of Wolf.*
- $\sigma_{e5}(A)$ *is essential spectrum of Schechter or essential spectrum of Weyl.*
- $\sigma_{e6}(A)$ *is essential spectrum of Browder.*
- $\sigma_{e7}(A)$ *is essential spectrum of Rakovcèvic.*
- $\sigma_{e8}(A)$ *is essential spectrum introduced by Schmoeger.*

**Proposition 1.6.3** *we have:*

- $\sigma_{ess1}(A) \cap \sigma_{ess2}(A) = \sigma_{ess3}(A) \subseteq \sigma_{ess4}(A) \subseteq \sigma_{ess5}(A) \subseteq \sigma_{ess6}(A)$.
- $\sigma_{ess5}(A) = \sigma_{ess7}(A) \cup \sigma_{ess8}(A)$.
- $\sigma_{ess1}(A) \subset \sigma_{ess7}(A)$.
- $\sigma_{ess2}(A) \subset \sigma_{ess8}(A)$.

### 1.6.1 *Essential spectrum of wolf and weyl*

**Definition 1.6.2** *[2] In a Banach space X, the Wolf essential spectrum of the operator* $A \in \mathcal{L}(X)$ *is denoted by*

$$\sigma_f(A) = \mathbb{C} \backslash \{\lambda \in \mathbb{C} : \lambda - A \in \Phi(X)\}.$$

*The Weyl essential spectrum of the operator* $A \in \mathcal{L}(X)$ *is denoted by*

$$\sigma_w(A) = \bigcap_{K \in K(x)} \sigma(A + K),$$

*where* $K(X)$ *stands for the ideal of all compact operators on X.*

**Proposition 1.6.4** *[2] Let* $U \in \mathcal{L}(X)$*. Then* $\lambda \notin \sigma_w(U)$ *if, and only if,* $\lambda - U \in \Phi(X)$ *and* $i(\lambda - U) = 0$*.*

*the relationship between the Weyl essential spectrum and the Wolf essential spectrum of* $U$ *by:*
$\sigma_w(U) = \sigma_f \cup \{\lambda \in \mathbb{C} : (i\lambda - U) \neq 0\}$.

2

# CONVERGENCE OF A SEQUENCE OF BOUNDED LINEAR OPERATOR

$T$ and $T_n$ denote bounded linear operators on a complex Banach space $X$, that is, $T; T_n \in \mathcal{L}(X)$. Unless otherwise mentioned, the convergence is as $n \to \infty$.
$B$ denotes the unit sphere of $X$ , $B = \{x \in X; \|x\| \leq 1\}$. the sequence $\{x_n\}_N$, $x_n \in X$, is relatively compact in $X$ if the set of its values $\bigcup\limits_{n=1}^{\infty} x_n$ is relatively compact in $X$, that is, if each subsequence $\{x_n\}_{N_1 \subset N}$ has a converging subsequence $\{x_n\}_{N_2 \subset N_1}$. For exemple, let $T$ be compact .the sequence $\{Tx_n\}_N$ is relatively compact if $\{x_n\}_N$ is a abitrary bounded sequence in $X$.

## 2.1 MODES OF CONVERGENCE

Now we discuss convergence of bounded sequences of linear operators. There are three different types of convergence that arise naturally.

### 2.1.1 *The norm convergence*

**Definition 2.1.1** *Let $\{T_n\}$ be a sequence of operators in $\mathcal{L}(X)$, and $\lim\limits_{n\to\infty} \|T_n - T\| = 0$ for some $T \in \mathcal{L}(X)$. then we say that :*

*The **norm convergence**, denoted by $T_n \overset{n}{\to} T$ :*

$$\|T_n - T\| \to 0.$$

### 2.1.2 *The pointwise convergence*

**Definition 2.1.2** *Let $\{T_n\}$ be a sequence of operators in $\mathcal{L}(X)$, and $T : X \to Y$ are bounded linear operators. We say that:*

*The* ***pointwise convergence****, denoted by $T_n \xrightarrow{p} T$ :*

$$\|T_n x - Tx\| \to 0 \, \textit{forevery} \; x \in X.$$

**Example 2.1.1** *[5] $X$ is taken to be $\ell^2$ : $\{e_i\}_i^x$ is the canonical basic $X_n = \{e_1, e_2 ... e_n\}$, $n = 1, 2...$*

$T_x = x$. $T_n x = \sum_{i=1}^{n} (x.e_i)e_i$. $T_n x \to x : T_n \xrightarrow{p} T$.

**Theorem 2.1.1** *[5] Let X and Y be Complex Banach spaces. The necessary and sufficient conditions for the sequence $\{T_n\}$ of bounded operators of $\mathcal{L}(X : Y)$ to be pointwise convergent on X to a linear operator are*

- $\sup_n \|T_n\| < M$.
- *$T_n x$ is convergent for all $x$ in some set $F$ dense in $X$.*

**Theorem 2.1.2** *[5] Suppose that $T_n x \to Tx, x \in X$. then for any relatively compact set $U$,*

$\sup_{x \in u} \|(T_n - T)x\| \to 0$.

***Proof.*** *Let U be a relatively compact set. For any $\varepsilon > 0$, U hase a finite cover with setes of diameter less than $\varepsilon$.*
*For each $x \in U$ there exists $x_\varepsilon$ such that $\|x - x_\varepsilon\| > \varepsilon$; there are a finite number of such $\varepsilon$.*

*then*
$\|(T_n - T)x\| \leq \|(T_n - T)(x - x_\varepsilon\| \|(T_n - T)x_\varepsilon\| \leq (\|T_n\| + \|T\|)\varepsilon + \max_{x_\varepsilon} \|(T_n - T)x_\varepsilon\|$.

■

### 2.1.3 *Compact convergence*

**Definition 2.1.3** *Let $\{T_n\}$ be a sequence of operators in normed linear space converging to $T$ by* ***compact convergence*** $T_n \overset{c}{\to} T$ *if and only if,*

- $T_n \overset{p}{\to} T$.
- *The following condition is satisfed:*
  *for any sequence $\{x_n\}_N$ in $B$, the sequence $\{(T - T_n)x_n\}_N$ is a relatively compact in $X$.*

### 2.1.4 *Collectivelly compact convergence*

**Definition 2.1.4** *Let $\{T_n\}$ be a sequence of operators in normed linear space converging to $T$ by* ***collectively compact convergence*** $T_n \overset{cc}{\to} T$ *if and only if,*

- $T_n \overset{p}{\to} T$.
- *The following condition is satisfed:*
  *the set $K = \bigcup_{n=1}^{\infty} \{(T - T_n)x : x \in X; \|x\| \leq 1\}$ is a relatively compact in $X$.*

***Example 2.1.2*** *Consider $X = \ell^2$, $X_n = \{e_1, e_2, ...e_n\}, n = 1, 2, ...$*

$Tx = 0.\ T_n x = \left(\dfrac{1}{n}\right) x.\ \|T_n - T\| = \dfrac{1}{n} \to 0.$
*and $T_n \overset{c}{\to} T$ but $T_n \overset{cc}{\nrightarrow} T$*

#### 2.1.4.1 *Relation with compact convergence*

It is clear that $T_n \overset{cc}{\to} T$ implies $T_n \overset{c}{\to} T$ but the reciprocal is not true in general, as show by Exemple(2.1.2). From a practical point of view, cases when the collectively compact convergence lets itself be characterized by the compact convergence are interesting. we first not that $T_n \overset{cc}{\to} T$ implies that each $T - T_n$ is compact.
since for $n \in \mathbb{N}$, $(T - T_n)B \subset \bigcup_{n=1}^{x} (T - T_n)B$ which is relatively compact.

**Proposition 2.1.1** *[5] the following are equivallent:*

- $T - T_n$ *is compact for any integer* $n$ *and* $T_n \overset{c}{\to} T$.
- $T_n \overset{cc}{\to} T$.

**Proposition 2.1.2** *[5] Suppose that* $V_n \overset{cc}{\to} V, V$ *compact. and for* $U_n \in \mathcal{L}(X)$, $U_n \overset{p}{\to} U$, *then* $V_n U_n \overset{cc}{\to} VU$ *and* $U_n V_n \overset{cc}{\to} UV$

***Proof.*** $V$ *and* $V - V_n$ *are compact ;hence* $V_n, VU$ *and* $UV$ *are compact.*
$V_n U_n x \to VUx$ *and* $U_n V_n x \to UV$ *for* $x \in X$.

*Let* $x_n \in B$ *: it is then clear that for any infinite subset* $N_1$ *of* $N$, *each sequence* $\{V_n U_n x\}_{N_1}$ *and* $\{U_n V_n x\}_{N_1}$ *has a converging subsequence.*

∎

**Proposition 2.1.3** *[5] If* $T_n$ *is compact,* $T_n \overset{cc}{\to} T$ *implies* $T$ *compact.*

***Proof.*** $T_n \overset{cc}{\to} T$ *implies* $T_n - T$ *compact for* $n \in \mathbb{N}$, *which in turn implies that* $T = T_n - (T_n - T)$ *is compact.* ∎

### 2.1.4.2 *Relation with norm convergence*

In general, $\|T_n - T\| \to 0$ does not imply $T_n \overset{cc}{\to} T$

**Proposition 2.1.4** *[5] Let* $S \in \mathcal{L}(X)$ *be compact and suppose that* $S_n \overset{cc}{\to} S$. *if* $T_n \overset{p}{\to} T$, *then* $\|(T_n - T)S_n\| \to 0$.

***Proof.*** $\bigcup_{n=1}^{\infty} S_n B$ *is relatively compact since* $S$ *is compact. the result follows by Theorem 2.1.2.*
∎

## 2.2 $\nu$-CONVERGENCE

This mode of convergence encompasses a wide variety of approximation methods, also this convergence allows to approximate non-compact operators through finite rank operators.

**Definition 2.2.1** *[2] A sequence $(T_n)$ of bounded linear operators mapping from X into X is said to be ν-convergent to T, denoted by* $T_n \overset{\nu}{\to} T$, *if*

- $\|T_n\|$ *is bounded.*
- $\|(T_n - T)T\| \to 0.$
- $\|(T_n - T)T_n\| \to 0.$

*These conditions on the norms of the operators* $T_n$, $(T_n - T)T$ *and* $(T_n - T)T_n$ *have evolved.*

*Simple examples show that none of these conditions is implied by the remaining two:*

- *Let X be a Banach space,* $T_n$ *be a sequence of operators denoted on* $X \bigotimes X$ *by* $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *and let* $T_n = \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}$ *fro* $n = 1, 2, ...$
  *Then* $\|(T_n - T)T\| = 0 = \|(T_n - T)T_n\|$, *but* $(\|T_n\|)$ *is unbounded.*
- *Let X be a Banach,* $C \in \mathcal{L}(X)$, $A_n$, $B_n \in \mathcal{L}(X)$ *and* $T_n$ *be a sequence of operators denoted on* $X \bigotimes X$ *by* $T = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$ *and let* $T_n = \begin{pmatrix} A_n & 0 \\ 0 & B_n \end{pmatrix}$
  *If* $A_n \to 0$ *and* $B_n \to 0$ *then* $\|T_n\| = \max\{\|A_n\|, \|V_n\|\} \to 0.$
  *So,* $T_n \to T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
  *This implies that* $T_n \overset{\nu}{\to} T.$

- *If* $T_n \xrightarrow{\nu} T$*, this does not genaerally imply that* $T_n \to T$.

  *in fact, take* $X = l_2(\mathbb{N}) = \left\{ (x_j)_{j\geq 1} \text{ such that } x_j \in \mathbb{C} \text{ and } \sum_{j=1}^{\infty} |x_j|^2 < \infty \right\}$

  *with cononical Hilbert basis* $e_1, e_2, \ldots$ *for* $n = 1, 2, \ldots$

  *Let* $T_n$ *be defined by* $T_n x := x(n+1)e_n$ *for* $n = 1, 2, \ldots$ *and* $x = \sum_{j=1}^{\infty} x(k)_{e_k} \in X$

  *then* $T_n \xrightarrow{\nu} 0$ *but* $T_n \nrightarrow 0$.

### 2.2.1 *Properties of the* $\nu$*-convergence*

**Lemma 2.2.1** *[1]*

a) *If* $T_n \xrightarrow{n} T$*, then* $T_n \xrightarrow{\nu} T$*. Conversely, if* $0 \notin \sigma(T)$ *and* $T_n \xrightarrow{\nu} T$*, then* $T_n \xrightarrow{n} T$

b) *Let* $T_n \xrightarrow{\nu} T$ *and* $U_n \xrightarrow{n} U$*. then* $T_n + U_n \xrightarrow{n} T + U$ *if and only if* $(T_n - T)U \xrightarrow{n} 0$.

   *In particular,*

   1) *if* $T_n \xrightarrow{\nu} T$ *and* $U_n \xrightarrow{n} 0$*, then* $T_n + U_n \xrightarrow{\nu} T$.
   2) *if* $T_n \xrightarrow{\nu} 0$*,* $U_n \xrightarrow{n} U$ *and* $T_n U \xrightarrow{n} 0$*; then* $T_n + U_n \xrightarrow{\nu} U$.

c) *If* $T_n \xrightarrow{cc} T$ *and* $T$ *is a compact operator, then* $T_n \xrightarrow{\nu} T$.

***Proof.***

a) *Let* $T_n \xrightarrow{n} T$.

   *Since*

$$\|T_n\| \leq \|T_n - T\| + \|T\|; \; \|(T_n - T)T\| \leq \|T_n - T\|\|T\|$$

   *and*

$$\|(T_n - T)T_n\| \leq \|T_n - T\|\|T_n\|,$$

   *we see that* $T_n \xrightarrow{\nu} T$.

   *Conversely, let* $0 \notin \sigma(T)$ *and* $T_n \xrightarrow{\nu} T$.

*Then T is is invertible and*

$$\|T_n - T\| = \|(T_n - T)TT^{-1}\| \leq \|(T_n - T)T\|\|T^{-1}\|$$

*so that* $T_n \xrightarrow{n} T$

b) *Since*

$$\|T_n + U_n\| \leq \|T_n\| + \|U_n\|,$$

*we see that the sequence* $\|(T_n + U_n)\|$ *is bounded. Assume that* $(T_n - T)U \xrightarrow{n} 0$. *As*

$$\|(T_n + U_n - T - U)(T + U)\| \leq$$
$$\|(T_n - T)T\| + \|(T_n - T)U\| + \|(U_n - U)\|\|T + U\|,$$
$$\|(T_n + U_n - T - U)(T_n + U_n)\| \leq$$
$$\|(T_n - T)T_n\| + \|(T_n - T)U_n\| + \|(U_n - U)\|(\|T_n\| + \|U_n\|)$$

*where*

$$\|(T_n - T)U_n\| \leq \|(T_n - T)\|\|U_n - U\| + \|(T_n - T)U\|$$

*we see that* $T_n + U_n \xrightarrow{\nu} T + U$

*Conversely, assume that* $T_n + U_n \xrightarrow{\nu} T + U$.

*Since*

$$(T_n - T)U = (T_n + U_n - T - U)(T + U) - (T_n - T)T - (U_n - U)(T + U)$$

*we obtain* $(T_n - T)U \xrightarrow{n} 0$

*The particular cases (1) and (2) follow easily.*

c) *Let* $T_n \xrightarrow{cc} T$. *By the Uniform Boundedness Principle ([10] Theorem 9.1), the sequence* $(\|T_n\|)$ *is bounded and the pointwise convergence of* $(T_n)$ *to T s uniform on the relatively compact sets* $\{Tx : x \in X; \|x\| \leq 1\}$ *and* $\bigcup_{n=1}^{\infty} \{T_n x : x \in X; \|x\| \leq 1\}$

*Hence* $\|(T_n - T)T\| \to 0$ *and* $\|(T_n - T)T_n\| \to 0$.

*Thus* $T_n \xrightarrow{\nu} T$.

■

**Lemma 2.2.2** *[11] If $\|(T_n - T)T\| \to 0$ and $\|(T_n - U)U\| \to 0$, then for every $x \in R(T) \cap R(U)$, $Tx = Ux$.*

***Proof.*** *Let $x \in R(T) \cap R(U)$. then there exist $y, z \in X$ such that $x = Ty = Uz$ and* $0 \leq \|Ux - Tx\| = \|U^2z - T^2y\| = \|U^2z - T_nUz + T_nTy - T^2y\| \leq \|(T_n - U)Uz\| + \|(T_n - T)Ty\| \to 0$ *as* $n \to \infty$. ■

**Lemma 2.2.3** *[11] If $T_n \overset{\nu}{\to} T, T_n \overset{\nu}{\to} U, T_nT = TT_n$ and $T_nU = UT_n$, then*

$$T^2 = U^2.$$

**Proof.** Let $x \in X$, then

$$0 \leq \| T^2x - U^2x \| =$$
$$\| T^2x - T_nTx + T_nTx - T_n^2x + T_n^2x - T_nUx + T_nUx - U^2x \| \leq$$
$$\| (T_n - T)\, Tx \| + \| (T_n - T)\, T_nx \| + \| (T_n - U)\, T_nx \| + \| (T_n - U)\, Ux \| \to 0$$

as $n \to \infty$.

■

**Example 2.2.1** *[11]*

*Let $T_n, n \in \mathbb{N}$, be operators defined on $\mathbb{C} \otimes \mathbb{C}$ as*

$T_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $T_n = \begin{pmatrix} 0 & \frac{1}{n} \\ 0 & 0 \end{pmatrix}$ *,$n \in \mathbb{N}^*$.*

*Then*

$$\| T_n \| \leq \frac{1}{n}$$

*for all $n \geq 1$. Thus,*

$$T_n \overset{n}{\to} T_0$$

*and, hence,*

$$T_n \overset{\nu}{\to} T_0.$$

*On the other hand,* $\| (T_n - T_1) T_1 \| = o = \| (T_n - T_1) T_n \|$,
*therefor*

$$T_n \overset{\nu}{\rightarrow} T_1.$$

*Observe that* $T_1 \neq T_{0.}$. *These operators clearly satisfy*
$T_n T_1 = T_1 T_n$ *and* $T_n T_0 = T_0 T_n$.

**Lemma 2.2.4** *[2] Let* $U, V \in \mathcal{L}(X)$. *If there exists* $n_0 \in \mathbb{N} \setminus \{0\}$ *such that* $U^{n_0} = V^{n_0}$, $E = R\left(U^{n_0-1}\right) \cap R\left(V^{n_0-1}\right)$ *is closed and* $U|_E = V|_E$, *then* $\sigma_i(U) = \sigma_i(V)$ *for* $i = f, w$.

**Proof.** Let $\lambda \notin \sigma_w(U)$. We shall divide the proof into two cases.

1

$1^{st}$case. If $\lambda \neq 0$ we have $\lambda - U \in \Phi^b(X)$ and $i(\lambda - U) = 0$. Since $U^{n_0} = V^{n_0}$ it follows that $U(E) \subseteq E$, $V(E) \subseteq E$, $U^{n_0}(X) \subseteq E$ and $V^{n_0}(X) \subseteq E$. By using ([3] Section 3) we have $\lambda - U$ is a Fredholm operator if, and only if, $\lambda - U|_E$ is a Fredholm operator and $i(\lambda - U) = i(\lambda - U|_E) = 0$. Thus $\lambda - V|_E$ is a Fredholm operator and $i(\lambda - V|_E) = 0$. So, $\lambda - V$ is a Fredholm operator and $i(\lambda - V) = i(\lambda - V|_E) = 0$. Consequently $\lambda \notin \sigma_w(V)$.

$2^{nd}$case. If $\lambda = 0$ then $U$ is a Fredholm operator and $i(U) = 0$. This implies that $U^{n_0}$ is a Fredholm operator and $i(U^{n_0}) = n_0 i(U) = 0$. Hence $V^{n_0}$ is a Fredholm operator and $i(V^{n_0}) = 0$ which implies that $V$ is a Fredholm operator and $i(V) = \frac{i(V^{n_0})}{n_0} = 0$. So, $0 \notin \sigma_w(V)$. Therefore $\sigma_w(V) \subseteq \sigma_w(U)$.
The opposite inclusion is analogous. By following the same reasoning, we obtain the following equality $\sigma_f(U) = \sigma_f(V)$.

■

**Theorem 2.2.1** *[2] Let* $U, V \in \mathcal{L}(X)$ *be two operators such that* $R(U) \cap R(V)$ *is closed and let* $(U_n)$ *be a sequence of bounded operators*

*commuting with both* $U$ *and* $V$. *If* $U_n \overset{\nu}{\rightarrow} U$ *and* $U_n \overset{\nu}{\rightarrow} V$
*then*

$$\sigma_w(U) = \sigma_w(V)$$

*and*

$$\sigma_f(U) = \sigma_f(V)$$

**Proof.** It is immediate consequence from, Lemma 2.2.3 and Lemma 2.2.4. ■

# 3

# SPECTRAL CONTINUITY

## 3.1 THE SPECTRUM CONTINUITY

Let $S$ be the collection of all non-empty compact subsets of $\mathbb{C}$. It is well know that the convergence of a sequence in $S$ with respect to the Hausdorff metric can be characterized through the concepts of limit inferior and superior.

### 3.1.1 *Limit inferior and Limit superior*

Let $(E_n)$ be a sequence of compact subsets of $\mathbb{C}$. Define the limit inferior and the limit superior of $(E_n)$, denoted respectively lim inf $E_n$,and lim sup $E_n$ can be denoted as follow:

$\liminf E_n = \{\lambda \in \mathbb{C} : \forall \varepsilon > 0, \exists\ n_0 \in \mathbb{N}$ such that $B\left(\lambda, \varepsilon\right) \cap E_n \neq emptyset\ \forall n \geq n_0\}$,

and

$\limsup E_n = \{\lambda \in \mathbb{C} : \forall \varepsilon > 0, \exists\ II \subseteq \mathbb{N}$ infinite such that $B\left(\lambda, \varepsilon\right) \cap E_n \neq \emptyset\ \forall n \in II\}$.

If lim inf $E_n$ = lim sup $E_n$, then lim $E_n$ is said to exist and is equal to this common limit.

**Remark 3.1.1** *[11] Let $\{E_n\}$ be a sequence of non-empty subsets of $\mathbb{C}$. The following properties of limit inferior and superior are known:*

*a)* lim inf $E_n$ *and* lim sup $E_n$ *are closed subsets of* $\mathbb{C}$.

*b)* $\lambda \in$ lim sup $E_n$ *if and only if, there exists an increasing sequence of natural numbers* $n_1 < n_2 < n_3 ...$ *and points* $\lambda_{nk} \in E_{nk}$*, for all* $k \in \mathbb{N}$*, such that* $\lim \lambda_{nk} = \lambda$.

c) *$\lambda \in \liminf E_n$ if and only if, there exists a sequence $\{\lambda_n\}$such that $\lambda_n \in E_n$ for all $n \in \mathbb{N}$, and $\lim \lambda_n = \lambda$.*

d) *Suppose $E, E_n \in S$ for all $n \in \mathbb{N}$ and there exists $k \in S$ such that $E_n \subseteq K$, for all $n \in \mathbb{N}$. Then $E_n \to E$ in the Hausdorff metric if and only if, $\limsup E_n \subseteq E$ and $E \subseteq \liminf E_n$.*

## 3.2 $\nu$-CONVERGENCE AND SPECTRAL PROPERTIES

### 3.2.1 *Semi-Fredholm sequence*

For $T \in \mathcal{L}(X)$ and $k \in \mathbb{N} \cup [-\infty, \infty]$, let $\rho^k_{s-F}(T)$ denote the set of $\lambda \in \mathbb{C}$ for which $\lambda - T \in \Phi_\pm(X)$ and $i(\lambda - T) = k$. put

$$\rho^+_{s-F}(T) = \bigcup_{1 \leq k \leq \infty} \rho^k_{s-F}(T)$$

and

$$\rho^-_{s-F}(T) = \bigcup_{-\infty \leq k \leq -1} \rho^k_{s-F}(T)$$

**Theorem 3.2.1** *[8]*

*Let $T \in \Phi^b(X)$ and $\{T_n\}$ be a sequnce in $\mathcal{L}(X)$ such that $T_n \xrightarrow{\nu} T$. then for any integre -$\infty \leq k \leq \infty$ and any $\lambda \in \rho^k_{s-F}(T)$ there exists $n_0 \in \mathbb{N}$ such that $\lambda \in \rho^k_{s-F}(T_n)$, for all $n \geq n_0$.*

**Proof.** Since $T_n \xrightarrow{\nu} T$ it is easy to see that $(T_n - T)T \xrightarrow{n} 0$ implies $(\lambda - T_n)T \xrightarrow{n} (\lambda - T)T$, for any $\lambda \in \mathbb{C}$.

Suppose that $\lambda \in \rho^k_{s-F}(T), -\infty \leq k \leq \infty$, then $(\lambda - T)T$ is a semi-Fredholm operator.

<u>Case 1:</u> Let $k$ be integer. Then

$$(\lambda - T)T \in \Phi^b(X).$$

by ([8] Theoerem 2.3.4), there exists $n_0 \in \mathbb{N}$ such that $(\lambda - T_n) T \in \Phi^b(X)$ and $i((\lambda - T_n)T) = i((\lambda - T)T)$ for all $n \geq n_0$.

Moreover, from Theoreom 1.5.2 $(i)$ and $(ii)$, $\lambda - T_n \in \Phi^b(X)$.

Thus by Theoreom 1.5.1, It follows that

$$i(\lambda - T_n) + i(T) = i(\lambda - T) + i(T) = k + i(T),$$

and hence

$$i(\lambda - T_n) = k \quad \text{for all } n \geq n_0.$$

Case 2: Let $k = \infty$. then $\alpha(\lambda - T) = \infty, \beta(\lambda - T) < \infty$, and

$$\lambda - T \in \Phi_-^b(X).$$

since $\beta(T) < \infty$, by Theoreom 1.5.2 (ii), it follows that

$$\alpha((\lambda - T)T) = \infty$$

Clearly, in view of Theoreom 1.5.1, we have

$$(\lambda - T)T \in \Phi_-^b(X)$$

and, consequently, by ([8] Theorem 2.3.5), there is a positive integer $n_0$ such that, for each $n \geq n_0$.

$$(\lambda - T_n)T \in \Phi_-^b(X)$$

and $\alpha((\lambda - T_n)T) = \infty$. Again, from Theoreom 1.5.2 (ii),

$$\beta(\lambda - T_n) < \infty.$$

If $\alpha(\lambda - T_n) < \infty$,then $(\lambda - T_n)T \in \Phi^b(X)$, which is a contradiction. Hence

$$\lambda \in \rho_{s-F}^{\infty}(T_n)$$

Case 3: Let $k = -\infty$. then $\lambda - T \in \Phi_+^b(X), \alpha(\lambda - T) < \infty$, and $\beta(\lambda - T) = \infty$.

Moreover,

$$(\lambda - T)\,T \in \Phi_{+}^{b}\,(X)$$

and, by Theoreom 1.5.2 (i),

$$\beta\left((\lambda - T)\,T\right) = \infty$$

Again, by ([8] Theorem 2.3.5), there is a positive integer $n_0$ such that, for each $n \geq n_0$.

$$(\lambda - T_n)\,T \in \Phi_{+}^{b}\,(X)$$

and

$$i\left((\lambda - T_n)\,T\right) = i\left((\lambda - T)\,T\right) = -\infty$$

Additionally, $(\lambda - T_n)\,T \in \Phi_{+}^{b}\,(X)$ and $T \in \Phi^{b}\,(X)$ implies $\lambda - T_n \in \Phi_{+}^{b}\,(X)$ which

$$i\,(\lambda - T_n) = -\infty.$$

Hence,

$$\lambda \in \rho_{s-F}^{-\infty}\,(T_n)\,.$$

■

**Theorem 3.2.2** *[8] Let $T \in \Phi^{b}\,(X)$ and $\{T_n\}$ be a sequnce in $\mathcal{L}\,(X)$ such that $T_n \xrightarrow{\nu} T$ Then*

$$\bigcup_{k\in(\mathbb{Z}\setminus\{0\})\cup(\pm\infty)} \overline{\rho_{s-F}^{k}\,(T)} \subseteq \liminf \sigma\,(T_n)\,.$$

**Proof.** Let $\lambda \in \rho_{s-F}^{k}\,(T)$ for some $k \in (\mathbb{Z}\setminus\{0\}) \cup (\pm\infty)$ suppose that $\lambda \notin$ lim inf $\sigma\,(T_n)\,.$

then there exists an increasing sequence of natural numbers $n_1 < n_2 < n_3 ...$ such that

$$\lambda \notin \sigma\,(T_i) \quad \text{for all } i \in \mathbb{N}.$$

this implies that

$$\lambda \in \rho_{s-F}^{0}\,(T_{n_i}) \quad \text{for all } i \in \mathbb{N}.$$

on the other hand, since $T_{n_i} \xrightarrow{\nu} T.$ it follows by Theorem 3.2.1 that there exists $i_0 \in \mathbb{N}$ such that

$$\lambda \in \rho_{s-F}^{k}\,(T_{n_i}) \quad \text{for all } i \geq i_0.$$

therefore

$$\lambda \in \rho^{0}_{s-F}\left(T_{n_i}\right) \cap \, \rho^{k}_{s-F}\left(T_{n_i}\right) \quad \text{for all } i \geq i_0$$

which a contradiction. consequently,

$$\rho^{k}_{s-F}\left(T\right) \subseteq \liminf \sigma\left(T_n\right).$$

and by Remark 3.1.1

$$\overline{\rho^{k}_{s-F}\left(T\right)} \subseteq \liminf \sigma\left(T_n\right).$$

■

**Theorem 3.2.3** *[8] Let $T \in \Phi\left(X\right)$ and $\left\{T_n\right\}$ be a sequnce in $\mathcal{L}\left(X\right)$ such that $T_n \xrightarrow{\nu} T$, Then*

$$\overline{\rho^{+}_{s-F}\left(T\right)} \subseteq \liminf \sigma_{ap}\left(T_n\right)$$

**Remark 3.2.1** *[8] For $T \in \Phi\left(X\right)$, we have*

$$\sigma\left(T\right) \diagdown \sigma_{ap}\left(T\right) \subseteq \liminf \sigma\left(T_n\right)$$

*for all sequence $\left\{T_n\right\}$ in $B\left\{X\right\}$ such that*

$$T_n \xrightarrow{\nu} T.$$

*This follows from Theorem 3.2.2, since the set*

$$\bigcup_{-\infty \leq k \leq 0} \rho^{k}_{s-F}\left(T\right)$$

*contains*

$$\sigma\left(T\right) \backslash \sigma_{ap}\left(T\right).$$

### 3.2.2 *The wolf and weyl essential spectra of sequence of linear operators*

The goal of this section is to discuss the Wolf and Weyl essential spectra of a sequence of linear operators that are $\nu$-convergent in a Banach space $X$.

We should note that if $U \in \mathcal{L}(X)$, $\{U_n\} \subseteq \mathcal{L}(X)$, $0 \in \Phi_u^b(X)$ and $U_n \overset{\nu}{\to} U$, then not necessarily

$$\sigma_f(U_n) \subseteq \sigma_f(U).$$

In fact, it suffices to consider the following examples: Let

$$R : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$$

be the right shift operator defined by

$$R(x_1, x_2, x_3 ...) = (0, x_1, x_2, x_3 ...)$$

For each $n \in \mathbb{N}^*$, let $U_n = \left(1 - \frac{1}{n}\right)$ and $U = R$. Then $U, U_n \in \mathcal{L}(X)$, $0 \in \Phi_b^u$ and

$$U_n \overset{\nu}{\to} U,$$

but

$$\sigma_f(U_n) = \left\{\lambda \in \mathbb{C} \text{ such that } |\lambda| = \frac{n-1}{n}\right\}$$

and

$$\sigma_f(U) = \{\lambda \in \mathbb{C} \text{ such that } |\lambda| = 1\}.$$

Therefore

$$\sigma_f(U_n) \nsubseteq \sigma_f(U) \quad \text{for all } n \in \mathbb{N}^*.$$

However, we have the following result.

**Theorem 3.2.4** *[2]*

*Let $U \in \mathcal{L}(X)$ and $(U_n)$ be a sequence in $\mathcal{L}(X)$ such that $0 \in \Phi_u^b(X)$.*
*If $U_n \overset{\nu}{\to} U$ and $\mathcal{O}$ is an open set of $\mathbb{C}$ with $0 \in \mathcal{O}$, then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$*

$$\sigma_w(U_n) \subseteq \sigma_w(U) + \mathcal{O}$$

*and*

$$\sigma_f(U_n) \subseteq \sigma_f(U) + \mathcal{O}$$

**Proof.**

For $i = w$, assume that the assertion fails. Then by passing to a subsequence, it may be assumed that, for each $n \in \mathbb{N}$, there exists $\lambda_n \in \sigma_w(U_n)$such that

$$\lambda_n \notin \sigma_w(U_n) + \mathcal{O}.$$

Since $(\lambda_n)$ is bounded, we may assume that

$$\lim_{n\to\infty} \lambda_n = \lambda.$$

This implies that

$$\lambda \notin \sigma_w(U) + \mathcal{O}.$$

Using the fact that $0 \in \mathcal{O}$, we have

$$\lambda \notin \sigma_w(U)$$

and therefore,

$$\lambda - U \in \Phi^b(X)$$

and

$$i(\lambda - U) = 0.$$

Let

$$V_n = (\lambda_n - \lambda) U + (U - U_n) U. \qquad (3.2.4)$$

It follows from Equation (3.2.4), that the operator $(\lambda_n - U_n) U$ can be expressed in the form

$$(\lambda_n - U_n) U = (\lambda - U) U + V_n.$$

Using the fact that $V_n \to 0$ and $(\lambda - U) U \in \Phi^b(X)$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$(\lambda_n - U_n) U \in \Phi^b(X)$$

and

$$i[(\lambda_n - U_n) U] = i[(\lambda - U) U]$$

Moreover $\beta(U) < \infty$, then by Theoreom 1.5.2 (ii), we conclude that

$$(\lambda_n - U_n) \in \Phi^b(X)$$

with

$$i\left(\lambda_n - U_n\right) = i\left(\lambda - U\right).$$

We get

$$\left(\lambda_n - U_n\right) \in \Phi^b\left(X\right)$$

and

$$i\left(\lambda_n - U_n\right) = i\left(\lambda - U\right) = 0.$$

So

$$\lambda_n \notin \sigma_w\left(U_n\right),$$

which is a contradiction. This enables us to conclude that,

$$\sigma_w\left(U_n\right) \subseteq \sigma_w\left(U\right) + \mathcal{O}, \forall n \geq n_0.$$

The proof of the inclusion $\sigma_f\left(U_n\right) \subseteq \sigma_f\left(U\right) + \mathcal{O}$, follows the same reasoning.

■

**Corollary 3.2.1** *[2]*

*Let $U \in \mathcal{L}\left(X\right)$ and $\left(U_n\right)$ be a sequence of linear operators in $\mathcal{L}\left(X\right)$ such that $0 \in \Phi_u^b$ and $U_n \xrightarrow{v} U$. If $\mathcal{O} \subseteq \mathbb{C}$ is an open set with $0 \in \mathcal{O}$, $V \in \mathcal{L}\left(X\right)$ and $VU \in \mathcal{F}^b\left(X\right)$ $\left(\text{or } UV \in \mathcal{F}^b\left(X\right)\right)$, then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,*

$$\sigma_i\left(U_n + V\right) \subseteq \sigma_i\left(U + V\right) + \mathcal{O} \text{ for } i = f, w.$$

**Proof.** By applying Theorem 3.2.4, there exists $n_0 \in$ such that

$$\sigma_i\left(U_n\right) \subseteq \sigma_i\left(U\right) + \mathcal{O}, \text{ for } i = f, w.$$

Now, we prove that

$$\sigma_i\left(U_n + V\right) = \sigma_i\left(U_n\right) \text{ for } i = f, w.$$

Since $U$ is a Fredholm linear operator, then there exists $U_0 \in \mathcal{L}\left(X\right)$ such that $UU_0 = I - K$ where $K \in \mathbb{k}\left(X\right)$. The fact that, $VUU_0 = V\left(I - K\right) = V - VK \in \mathcal{F}^b\left(X\right)$ and $VK \in \mathcal{F}^b\left(X\right)$, we see that $V \in \mathcal{F}^b\left(X\right)$. Hence $\sigma_i\left(U_n + V\right) = \sigma_i\left(U_n\right)$,

for $i = f, w$.
For $UV \in \mathcal{F}^b(X)$, the proof can be checked in the same way.
The aim of the following theorem is to extend Theorem 3.2.4 to a sequence of closed linear operators.

■

## 3.3 $\nu$-CONTINUITY OF THE SPECTRUM AND THE APPROXIMATE POINT SPECTRUM

**Definition 3.3.1** *A mapping $\tau$ on $\mathcal{L}(X)$ whose values are compact subsets of $\mathbb{C}$ is said to be $\nu$-upper semi continuous at $T$ when*

$$T_n \overset{\nu}{\to} T \Rightarrow \limsup \tau(T_n) \subset \tau(T)$$

*and to be $\nu$-lower semi continuous at $T$ when*

$$T_n \overset{\nu}{\to} T \Rightarrow \tau(T) \subset \liminf \tau(T_n)$$

*If $\tau$ is both $\nu$-upper and $\nu$-lower semi continuous, then it is said to be $\nu$-continuous.*

**Remark 3.3.1** *[2]*

1) *If $\tau$ is $\nu$-lower semi continuous at $T$, then $\tau$ is lower semi continuous at $T$.*

2) *If $\tau$ is $\nu$-upper semi continuous at $T$, then $\tau$ is upper semi continuous at $T$.*

3) *If $\tau$ is bounded on $\nu$-convergent sequences and $\tau$ is $\nu$-continuous at $T$, then $\tau$ is continuous in the Hausdor metric at $T$.*

4) *Generally, the converse of (1), (2) and (3) is not true. But if $0 \in \rho(T)$ we have:*
   i) *$\tau$ is $\nu$-lower semi continuous at $T$ if, and only if, $\tau$ is lower semi continuous at $T$.*
   ii) *$\tau$ is $\nu$-upper semi continuous at $T$ if, and only if, $\tau$ is upper semi continuous at $T$.*
   iii) *If $\tau$ is bounded on $\nu$-convergent sequences then, $\tau$ is $\nu$-continuous at $T$ if, and only if, $\tau$ is continuous in the Hausdor metric at $T$.*

**Theorem 3.3.1** *Let $T \in \Phi(X)$. If $(T_n)$ is a sequence in $B(X)$ such that $T_n \xrightarrow{\nu} T$ then*

$$\left[\limsup \sigma_{ap}(T_n)\right] \setminus \{0\} \subseteq \sigma_{ap}(T).$$

**Proof.** Let $D, E$ be subspaces of $X$ with dim $E \langle \infty$ such that

$$X = N(T) \oplus D \text{ and } X = R(T) \oplus E. \quad (3.3.1)$$

Consider $(T_n)$ a sequence in $B(X)$ which is $\nu$-convergent to $T$.

Let $\lambda \in \limsup \sigma_{ap}(T_n)$ with $\lambda \neq 0$. By Remark 3.1.1, there exist an increasing sequence of natural numbers $(n_k)$ and points $\lambda_{nk} \in \sigma_{ap}(T_{nk})$ such that $\lambda_{nk} \to \lambda$.

Suppose that $\lambda \notin \sigma_{ap}(T)$. Then $T - \lambda \in \Phi_+(X)$ and $\alpha(T - \lambda) = 0$.
By (3.3.1), $R(T|_D) = R(T)$ and so $T|_D$ is bounded below, therefore by ([12], Theorem 5.26),

$$(T - \lambda) T|_D \in \Phi_+(D, X) \text{ and } \alpha((T - \lambda) T|_D) = 0.$$

On the other hand, observe that
$(T_{nk} - \lambda_{nk}) T|_D = (T - \lambda) T|_D + (T_{nk} - T) T|_D + (\lambda - \lambda_{nk}) T|_D$.
From $\| (T_n - T) T \| \to 0$ we have that $(T_{nk} - \lambda_{nk}) T|_D$ converges in norm to $(T - \lambda) T|_D$.
Consequently by ([12], Theorem 5.23), there exists $k_0 \in \mathbb{N}$ such that every $k \geq k_0$,

$$(T_{nk} - \lambda_{nk}) T|_D \in \Phi_+(D, X) \textit{ and } \alpha((T_{nk} - \lambda_{nk}) T|_D) = 0.$$

Suppose that for each $k \geq k_0$,

$$N(T_{nk} - \lambda_{nk}) \cap E \neq \{0\}.$$

Take $v_k \in N(T_{nk} - \lambda_{nk}) \cap E$ with $\| v_k \| = 1$ for all $k \geq k_0$. Since dim $E < \infty$ it follows that $F = \{e \in E : \| e \| = 1\}$ is a compact set. Therefore we may assume without loss of generality that there exists $v \in F$ such that $v_k \to v$.
Observe that
$\| (T_{nk} - T) T_{nk} \| \geq \| (T_{nk} - T) T_{nk} v_k \| = \| (T_{nk} - T) \lambda_{nk} v_k \| = | \lambda_{nk} | \| \lambda_{nk} v_k - T v_k \|$

for all $k \geq k_0$, and $| \lambda_{nk} ||| \lambda_{nk} v_k - T v_k \| \rightarrow | \lambda ||| \lambda v - Tv \|$.

This implies that

$| \lambda ||| \lambda v - Tv \| = \lim | \lambda_{nk} ||| \lambda_{nk} v_k - T v_k \| \leq \lim \| (T_{nk} - T) T_{nk} \| = 0$,

and so $\| \lambda v - Tv \| = 0, i.e. Tv = \lambda v$. Consequently, $\lambda \in \sigma_p (T) \left(\subseteq \sigma_{ap} (T)\right)$,

which is a contradiction. Therefore there exists $k^{'} \geq k_0$ such that

$N \left(T_{nk'} - \lambda_{nk'}\right) \cap E = \{0\}$.

Thus

$$X = R(T) \oplus N(T_{nk'} - \lambda_{nk'}) \oplus E. \quad (3.3.2)$$

Then by (3.3.1) and (3.3.2),

$$\dim E = \dim X / R(T) = \dim \left[N(T_{nk'} - \lambda_{nk'}) \oplus E\right] = \dim N(T_{nk'} - \lambda_{nk'}) + \dim E.$$

Hence $\dim N(T_{nk'} - \lambda_{nk'}) = 0$ and so $N(T_{nk'} - \lambda_{nk'}) = \{0\}$.

From (3.3.2),

$$R(T_{nk'} - \lambda_{nk'}) = (T_{nk'} - \lambda_{nk'}) T(D) + (T_{nk'} - \lambda_{nk'})(E),$$

which implies that $R(T_{nk'} - \lambda_{nk'})$ is closed.

Therefore $\lambda_{nk'} \neq \sigma_{ap} \left(T_{nk'}\right)$, a contradiction. Consequently, $\lambda \in \sigma_{ap} (T)$.

■

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# ملخص

في هذه المذكرة قمنا بدراسة استمرارية الطيف باستخدام التقارب الخامس. حيث قدمنا تعريف للطيف وبعد ذلك قمنا بدراسة فئتين من الطيف الأساسي وكذلك التوصيف الحالي للاطياف الأساسية.

**الكلمات المفتاحية :** الطيف الأساسي. التقارب الخامس. استمرارية الطيف.

# Abstract

In this work we study the spectral continuity using the v-convergence. Where we have given a definition of the spectrum and after that we stady two classes of the essential spectrum and present Also characterization of essential spectra.

**Key words:** Essential spectrum, The v-convergence, Spectral continuity.

# Résumé

Dans ce mémoire. Nous étudions la continuité spectrale en utilisant la v-convergence . ou nous avons donné une definition du spectre et après cela nous étudions deux classes du spectre essential et presentons également la caractérisation des spectres essentiels.

**Mots clés:** Spectre essentiel, La v-convergence, Continuité spectrale.