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Ministry of Higher Education and Scientific Research

University Mohamed Boudiaf of M'Sila

Faculty of Technology — Department of Mechanical Engineering

LECTURE BOOKLET

Probability and Statistics

— An English Edition for Engineering Students —

Second Year Bachelor (L2) — Science & Technology

Semester S3

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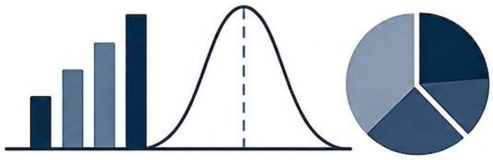


$$P(A) = \frac{Card(A)}{Card(\Omega)}$$

STATISTICS AND PROBABILITY

Preface

This booklet presents the material of the Probability and Statistics course delivered to second-year Bachelor students in Science and Technology at the Department of Mechanical Engineering, University of Mohamed Boudiaf of M'Sila. The content is organised in four parts, following the logical progression from purely descriptive statistics towards probabilistic modelling and, finally, inferential statistics.

The pedagogical approach adopted here is deliberately example-driven: each concept is first motivated by an engineering or everyday situation, then stated formally, and finally illustrated by a worked calculation. Every equation in the text is typeset using the Microsoft Word equation editor so that students and instructors can easily copy, modify, or extend the formulas in their own notes.

The material assumes only a first-year knowledge of calculus (derivatives, basic integration) and elementary set theory. Reference tables for the Poisson, standard normal, Student's t and chi-squared distributions are included in the appendix and are used throughout the worked examples.

Contents

Part I

Descriptive Statistics

Descriptive Statistics

Descriptive statistics organise, summarise and visualise raw data. Unlike inferential statistics, no probabilistic assumption is made here: we merely describe the sample we have in hand. Despite its apparent simplicity, this step is indispensable it is during the descriptive phase that outliers, trends and groupings first become visible.

1. Single-Variable Statistics

1.1. Vocabulary and Graphical Representation

Statistics is the study of populations, whose elementary units are called individuals. In most practical situations we cannot observe the entire population, so we work with a sample of it. The number of individuals in the sample is its size (or count).

For each individual we measure one or several characteristics (or variables). These may be qualitative (nationality, color, etc.) or quantitative. Quantitative variables are further split into discrete (shoe size, number of children) and continuous (height, area, duration).

When the data are continuous, or numerous, it is customary to group them into disjoint intervals called classes. The length of such an interval is the class width (or amplitude). For instance, adult heights may be grouped into the intervals [0, 100[, [100, 110[, ..., [190, 200[, [200, +∞[.

A statistical series is a set of pairs (x_i, n_i),

Where:

x_i denotes an observed value and

A statistical series is therefore a finite collection of pairs (x_i, n_i) where x_i is an observed value of the variable and n_i is the number of times this value occurs.

The total sample size is

$$n = \sum_{i=1}^{p} n_i$$

The relative frequency of x_i is defined as

$$f_i = \frac{n_i}{n}$$

Example 1 — Quality control.

From a daily production of one thousand parts, the number of defects found on each part is recorded:

Number of defects	0	1	2	3	4
Count n_i	570	215	140	60	15
Frequency f_i	0.570	0.215	0.140	0.060	0.015

Example 2 — Metal rods.

A technician measures the length of 500 metal rods and groups the values into the following classes:

Length (mm)	[330; 340[	[340; 343[	[343; 345[	[345; 350[	[350; 360[
Count	57	195	204	30	14
Frequency	0.114	0.390	0.408	0.060	0.028

For a histogram the classes are placed on the horizontal axis. The height of each rectangle equals the count divided by the class width, so the area of every rectangle is proportional to the count of that class. In this example the heights are 5.7, 65, 102, 6 and 1.4 respectively.

Warning it is the area, not the height, that carries the statistical meaning.

1.2. Measures of Central Tendency

Mode.

The mode is the most frequent value of the series. For grouped data one speaks of the modal class. A series may possess several modes. In Example 1 the mode is 0; in Example 2 the modal class is [343; 345[.

Median.

The median is the value M_e that splits the ordered sample into two equal halves. Equivalently, at least 50 % of the individuals have a value ≤ M_e and at least 50 % have a value ≥ M_e.

Arithmetic mean.

Definition — Sample Mean

$$\overline{x} = \frac{n_1 x_1 + n_2 x_2 + \cdots + n_p x_p}{n} = \sum_{i=1}^{p} f_i x_i$$

In Example 1 the mean number of defects per part is 0.215·1 + 0.14·2 + 0.06·3 + 0.015·4 = 0.735.

For grouped data an approximate mean is obtained by taking the midpoint of every class; in Example 2 this yields $\bar{x} \approx 342.52$ mm. The approximation is trustworthy only when the classes have comparable widths and comparable counts.

1.3. Measures of Dispersion

Two samples may share the same mean yet differ sharply in how tightly the individual values cluster around it. Dispersion measures quantify this spread.

Range.

The range is the simplest dispersion measure: it is the difference between the largest and the smallest values.

Variance and standard deviation.

Definition — Variance and Standard Deviation

$$\sigma'^2 = \frac{\sum_{i=1}^{p} n_i (x_i - \overline{x})^2}{n} = \sum_{i=1}^{p} f_i (x_i - \overline{x})^2$$

$$\sigma' = \sqrt{\sigma'^2}$$

In plain words the variance is the mean of the squared deviations from the mean. Expanding the square yields the computational form

$$\sigma'^2 = \overline{x^2} - \overline{x}^2$$

Note that the denominator used here is n (not n − 1). The distinction becomes important when the sample variance is used as an estimator of the population variance, a subject discussed in Chapter 10.

In Example 1 the variance is approximately 1.015 and therefore the standard deviation is $\sigma' \approx 1.01$.

2. Two-Variable Statistics and Linear Regression

When each individual carries two measurements, we speak of a bivariate series. The central questions become: are the two measurements related? If they are, how strong is the relation, and can it be described by a simple function?

2.1. The Scatter Plot and the Centroid

Example 1 — Ohm's law.

The current and the voltage across a resistor are measured simultaneously:

Current I (A)	**0.053**	**0.067**	**0.095**	**0.16**	**0.21**
Voltage U (V)	8.10	9.95	15.0	25.0	30.0

Each pair of measurements defines a point $M_i(x_i, y_i)$ in the plane; together they form the scatter plot. The centroid of the cloud is the point $G(\overline{x}, \overline{y})$. Here $G(0.117, 17.61)$.

2.2. Covariance and Correlation

Definition — Covariance

$$\sigma_{x,y} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})$$

$$\sigma_{x,y} = \frac{1}{n}\sum_{i=1}^{n} x_i y_i - \overline{x}\overline{y}$$

The two forms are algebraically equivalent; the second one is usually preferred for numerical computation. For Example 1 we find σ_{x,y} ≈ 0.503.

Definition — Coefficient of Linear Correlation

$$r = \frac{\sigma_{x,y}}{\sigma_x \sigma_y}$$

The correlation coefficient satisfies $-1 \le r \le 1$. When $|r|$ is close to 1 the points of the scatter plot lie almost on a straight line; when r is close to 0 there is no linear relation. In practice a value $|r| > 0.7$ is considered evidence of a linear link. For Example 1, $r \approx 0.994$.

Two important cautions.

(i) A small r does not rule out a relation it only rules out a linear one. Relations of the form $y = k \cdot e^{cx}$, $y = k \cdot x^c$, $y = 1/(ax + b)$, etc. are non-linear; a change of variables can sometimes reduce them to the linear case.

(ii) A large r does not imply causation. The classical Grenoble winter example is instructive: snowfall and air temperature are strongly correlated (on cold, clear days there is little snow), but the cold does not prevent the snow both phenomena are caused by the same anticyclone. Correlation is a statistical indicator, not a physical law.

2.3. Method of Least Squares — Regression Line of y on x

We seek the straight line D: y = ax + b that minimises the sum of the squared vertical deviations between the observed points and the line:

$$S(a,b) = \sum_{i=1}^{n} \left(y_i - (ax_i + b)\right)^2$$

Setting the partial derivatives of S to zero gives the normal equations, whose solution is

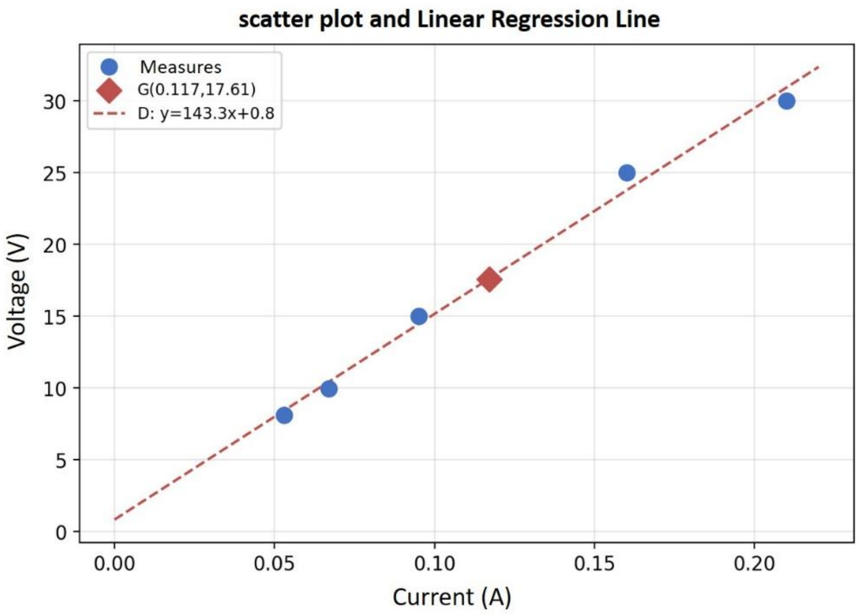
Least-Squares Regression Line of y on x

$$a = \frac{\sigma_{x,y}}{{\sigma_x}^2}$$

$$b = \overline{y} - a\overline{x}$$

(The line passes through the centroid $G(\bar{x}, \bar{y})$.)

For Example 1 the regression line of y on x is D: y = 143.27 x – 0.847. The symmetric regression of x on y, D′, gives y = 144.36 x + 0.719. Both lines are practically indistinguishable, confirming Ohm's law U = R·I with R ≈ 143–144 Ω.



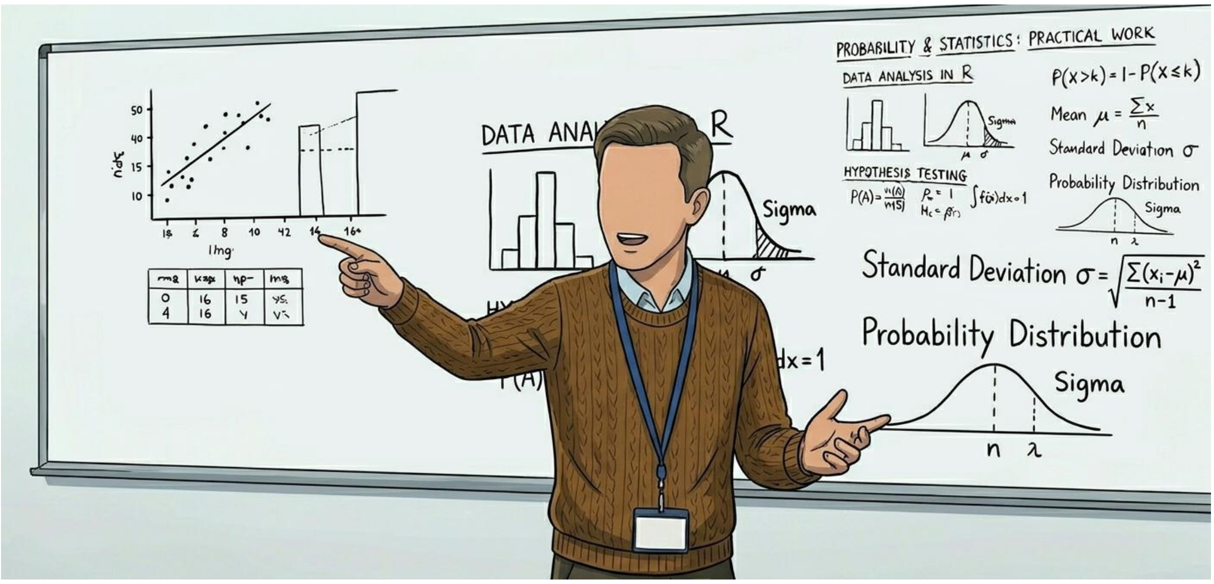
2.4. Regression Through the Origin

When a physical law is known to pass through the origin as Ohm's law U = R·I does one fits y = a·x without a constant term. The least-squares estimate becomes

$$a = \frac{\sum x_i y_i}{\sum {x_i}^2} = \frac{\overline{xy}}{\overline{x^2}}$$

For Example 1 this gives a ≈ 149.04 Ω. The small gap between this value and the one obtained with a free intercept reflects measurement noise and the fact that real resistors are never perfectly ohmic.

supervised practical work



TUTORIAL SERIES No. 01

Descriptive Statistics

Exercise No. 01:

Exercise No. 01:

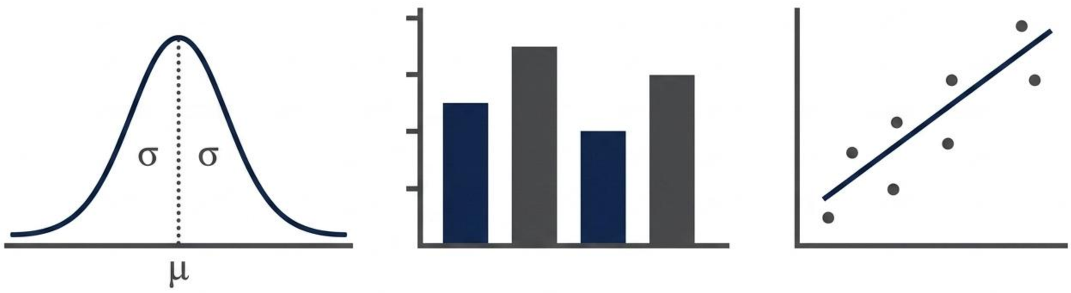
A medical study recorded the blood types of 400 donors at a blood bank. The results are shown in the following table:

Blood Type	A+	B+	O+	AB+	A-	B-	O-	AB-
Frequencies	148	72	116	28	16	8	10	2

1. What is the nature of this characteristic? Define the statistical variable associated with this characteristic.

2. Calculate the total frequency and the relative frequencies of this distribution.

3. Provide several graphical representations of this variable.



Solution of Exercise No. 01:

1. a) Nature of the characteristic:

The characteristic is qualitative nominal because it expresses a blood type (A+, B+, O+, AB+, A-, B-, O-, AB-), which is a biological category without any natural order.

b) Statistical variable:

The statistical variable X is the blood type of the blood bank donors. It can take the following modalities: A+, B+, O+, AB+, A-, B-, O-, AB-.

2. a) Total frequency:

$$N = \Sigma n_i = 148 + 72 + 116 + 28 + 16 + 8 + 10 + 2 = 400$$

Therefore, the total frequency is 400 donors.

b) Relative frequencies:

$$f_i = n_i / N \qquad \Sigma f_i = 1$$

$$fA+ = 148/400 = 0.370 \qquad fB+ = 72/400 = 0.180$$

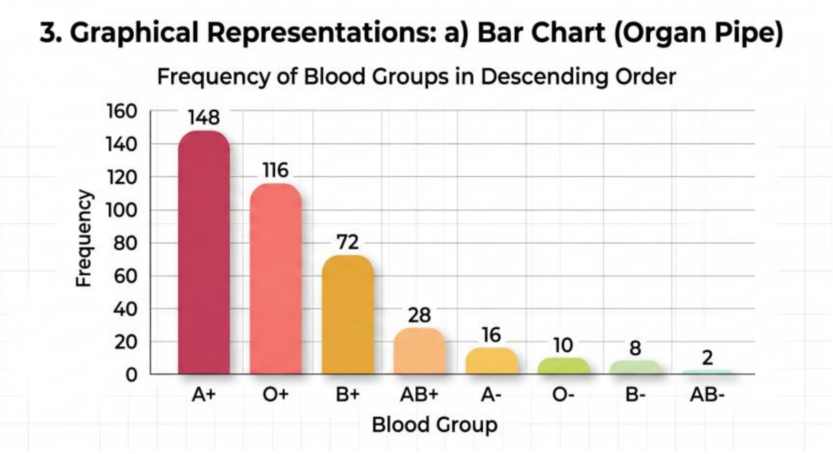
$$fO+ = 116/400 = 0.290 \qquad fAB+ = 28/400 = 0.070$$

$$fA- = 16/400 = 0.040 \qquad fB- = 8/400 = 0.020$$

$$fO- = 10/400 = 0.025 \qquad fAB- = 2/400 = 0.005$$

$$\Sigma f_i = 0.370 + 0.180 + 0.290 + 0.070 + 0.040 + 0.020 + 0.025 + 0.005 = 1.000$$

3. Graphical Representations:



b) Pie Chart:

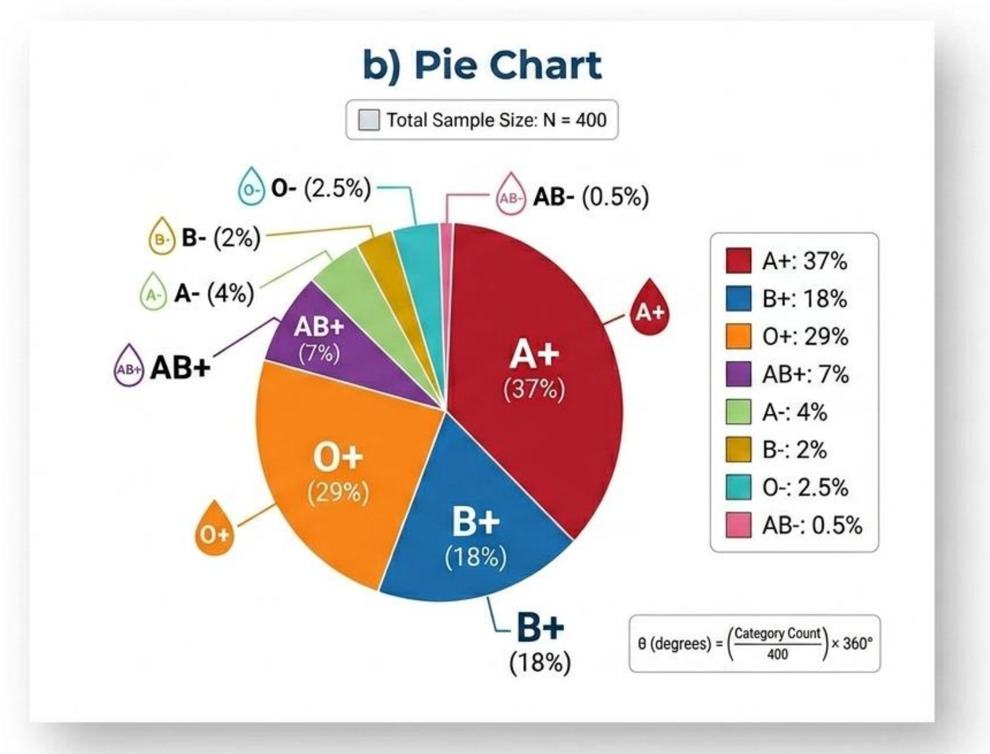
$$N = 400 \rightarrow 360°$$

$$\theta A+ = (148 \times 360°)/400 = 133.2° \quad \theta B+ = (72 \times 360°)/400 = 64.8°$$

$$\theta O+ = (116 \times 360°)/400 = 104.4° \quad \theta AB+ = (28 \times 360°)/400 = 25.2°$$

$$\theta A- = (16 \times 360°)/400 = 14.4° \quad \theta B- = (8 \times 360°)/400 = 7.2°$$

$$\theta O- = (10 \times 360°)/400 = 9.0° \quad \theta AB- = (2 \times 360°)/400 = 1.8°$$



Exercise No. 02:

In a survey of 50 students about the number of books they read during the summer vacation, the following results were obtained:

Number of books x_i	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**Σ**
Frequencies n_i	4	8	12	10	7	4	3	1	1	N=50

1. Determine the range and the mode of this series.

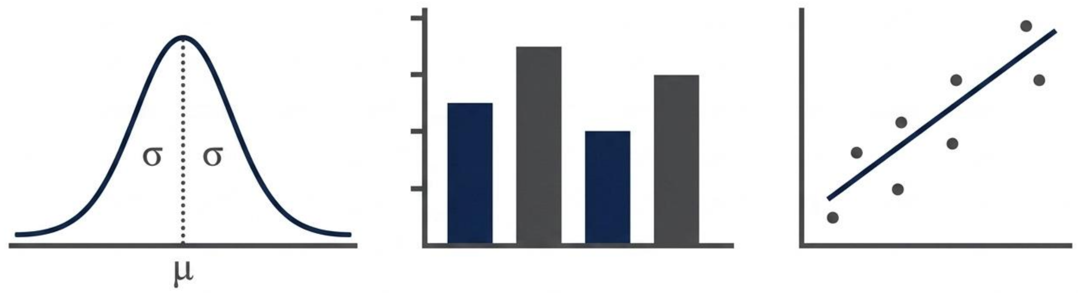
2. Calculate the mean of this series.

3. Construct a table giving the cumulative frequencies, the relative frequencies, and the cumulative relative frequencies.

4. Determine the median of this series.

5. What is the number of students who read strictly less than 3 books?

6. What is the percentage of students who read 4 books or more?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Solution of Exercise No. 02:

1. a) Range:

$$Range = Max(X_i) - Min(X_i) = 8 - 0 = 8$$

b) Mode:

It is the value with the highest frequency in the series. The number of books 2 has the highest frequency with 12 students.

The mode is therefore 2.

2. Mean:

$$\bar{X} = (1/N)\ \Sigma\ n_i \cdot x_i$$

With N = 50 (total frequency).

$$\bar{X} = (1/50)[(4\times0) + (8\times1) + (12\times2) + (10\times3) + (7\times4) + (4\times5) + (3\times6) + (1\times7) + (1\times8)]$$

$$\bar{X} = (0 + 8 + 24 + 30 + 28 + 20 + 18 + 7 + 8)/50 = 143/50 = 2.86$$

The mean number of books read is 2.86.

3. Table of cumulative frequencies, relative frequencies, and cumulative relative frequencies:

Books x_i	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**Σ**
Frequencies n_i	4	8	12	10	7	4	3	1	1	N=50
Cumul. Freq. n_ic	4	12	24	34	41	45	48	49	50	
Rel. Freq. f_i	0.080	0.160	0.240	0.200	0.140	0.080	0.060	0.020	0.020	$\Sigma f_i=1$
Cumul. Rel. Freq. f_ic	0.080	0.240	0.480	0.680	0.820	0.900	0.960	0.980	1.000	

4. Median:

The median corresponds to the central value of the distribution. For N = 50 (even), the median is the average of the 25th and 26th values.

According to the cumulative frequency table:

• The 25th value falls in the class $x_i = 2$ (cumulative frequency $24 < 25 \leq 34$)

• The 26th value also falls in the class $x_i = 2$

$$Median = (x(25) + x(26))/2 = (2 + 2)/2 = 2$$

The median is therefore 2 books.

5. Number of students who read strictly less than 3 books:

Numbers of books strictly less than 3 are: 0, 1, 2.

4 (for 0 books) + 8 (for 1 book) + 12 (for 2 books) = 24 students

Therefore, 24 students read strictly less than 3 books.

6. Percentage of students who read 4 books or more:

The total number of students who read 4 books or more is:

$$7 + 4 + 3 + 1 + 1 = 16 \text{ students}$$

The percentage is therefore:

$$(16/50) \times 100 = 32\%$$

Thus, 32% of students read 4 books or more during the summer vacation.

Exercise No. 03:

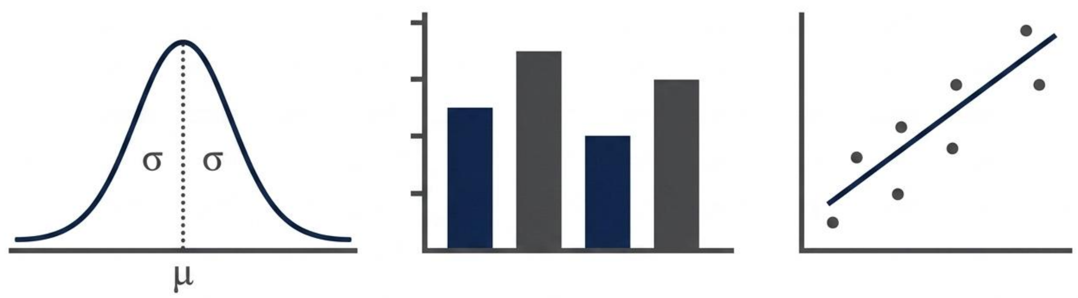
In a psychology experiment, the reaction time (in milliseconds) of 200 participants was measured. The times were grouped into 6 classes:

Reaction time (ms)	**[150; 200[**	**[200; 250[**	**[250; 300[**	**[300; 350[**	**[350; 400[**	**[400; 500[**
Number of participants	12	28	56	64	32	8

Present the data and perform calculations according to the directives.

1. Give: the characteristic / data table / frequency densities / histogram of frequencies.

2. Calculate: Arithmetic mean / Range / Variance / Standard deviation / Cumulative frequencies.



Solution of Exercise No. 03:

1. a) Studied characteristic:

The characteristic is quantitative continuous because it concerns the reaction times (in milliseconds) of participants, measured on a continuous scale.

b) Frequency density:

The frequency density is calculated by dividing the frequency of each class by the class width. The class width is A = Xmax - Xmin

Class	Width	Density = n_i/A
[150; 200[	50	12/50 = 0.24
[200; 250[	50	28/50 = 0.56
[250; 300[	50	56/50 = 1.12
[300; 350[	50	64/50 = 1.28
[350; 400[	50	32/50 = 0.64
[400; 500[	100	8/100 = 0.08

c) Histogram of frequencies:

The histogram represents the classes in ms on the x-axis and the frequency densities (n_i/A) on the y-axis.

Note: The last class [400; 500[has double the width (100 ms) compared to the others (50 ms). Therefore, frequency density must be used to avoid visual distortion.

2. a) Arithmetic mean:

The mean $\bar{X}$ is calculated from the class centers. The center C_i of each class is (Xmin + Xmax)/2

$$\bar{X} = (1/N)\ \Sigma\ n_i \cdot c_i = 59{,}200/200 = 296 \text{ ms}$$

b) Range:

$$Range = 500 - 150 = 350 \text{ ms}$$

c) Variance:

$$\sigma^2 = V(x) = (\Sigma\ n_i(C_i - \bar{X})^2)/N = \Sigma\ f_i(C_i - \bar{X})^2$$

d) Standard deviation (σx):

$$\sigma x = \sqrt{(\sigma^2)} = \sqrt{V(x)} = \sqrt{3724} = 61.02 \ ms$$

Detailed Calculation Table:

Time (ms) x_i	**[150;200[**	**[200;250[**	**[250;300[**	**[300;350[**	**[350;400[**	**[400;500[**	**Σ**
Participants n_i	12	28	56	64	32	8	N=200
Class center C_i	175	225	275	325	375	450	
$n_i \times C_i$	2,100	6,300	15,400	20,800	12,000	3,600	59,200
$C_i - \bar{X}$	-121	-71	-21	29	79	154	
$(C_i - \bar{X})^2$	14,641	5,041	441	841	6,241	23,716	
$f_i = n_i/N$	0.060	0.140	0.280	0.320	0.160	0.040	1
$f_i(C_i - \bar{X})^2$	878.46	705.74	123.48	269.12	998.56	948.64	$\sigma^2 \approx 3{,}924$
Cumul. Freq. f_ic	0.060	0.200	0.480	0.800	0.960	1.000	

Exercise No. 04:

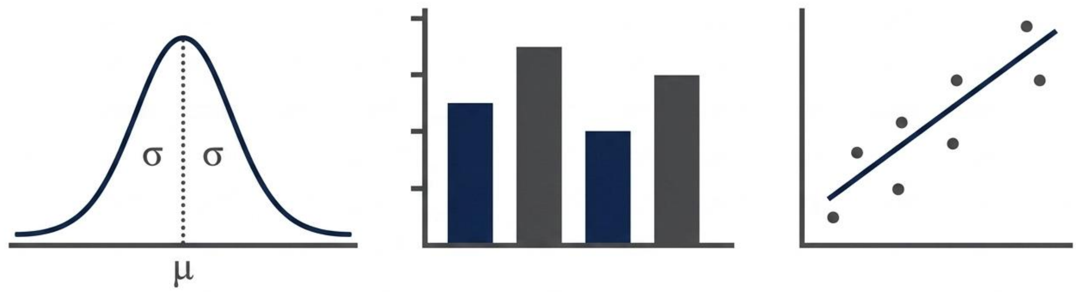
Distribution of monthly salaries (in thousands of DZD) of employees in a textile factory:

Salary (kDZD)	**[20; 30[**	**[30; 40[**	**[40; 60[**	**[60; 80[**	**[80; 120[**
Number of employees	45	68	92	38	17

1. Determine the mean monthly salary $\bar{x}$ from this grouping by class.

2. Knowing that the total monthly payroll is 10,240,000 DZD, calculate the exact mean monthly salary.

3. Compare with the value obtained in question 1 and explain the difference.



Solution of Exercise No. 04:

1. To calculate the mean monthly salary, we take the center of each class and use it to find the weighted mean by frequencies.

$$C_i = (Xmax + Xmin)/2$$

The mean monthly salary is:

$$\bar{X} = (\Sigma\, n_i \cdot C_i)/N = 9{,}180/260 = 35.31 \text{ kDZD} = 35{,}310 \text{ DZD}$$

Salary (kDZD) x_i	**[20;30[**	**[30;40[**	**[40;60[**	**[60;80[**	**[80;120[**	**Σ**
Employees n_i	45	68	92	38	17	N=260
C_i	25	35	50	70	100	
$n_i \times C_i$	1,125	2,380	4,600	2,660	1,700	9,180

2. The exact mean monthly salary is given by the total payroll divided by the total number of employees:

$$\text{Exact mean salary} = 10{,}240{,}000/260 \approx 39{,}384.62 \text{ DZD} \approx 39.38 \text{ kDZD}$$

3. Comparison:

$$\text{Difference} = 39.38 - 35.31 = 4.07 \text{ kDZD } (\approx 11.5\% \text{ underestimation})$$

The class-based mean (35.31 kDZD) underestimates the exact mean (39.38 kDZD). This difference is due to the right-skewed distribution: most employees are in lower salary classes, but the few high-salary employees in the [80; 120[class pull the exact mean upward. Using the class center (100) for this wide class underestimates the actual salaries of these high earners.

Exercise No. 05 (*):

A meteorologist recorded the maximum daily temperature (in °C) in the city of Constantine during the first ten days of August:

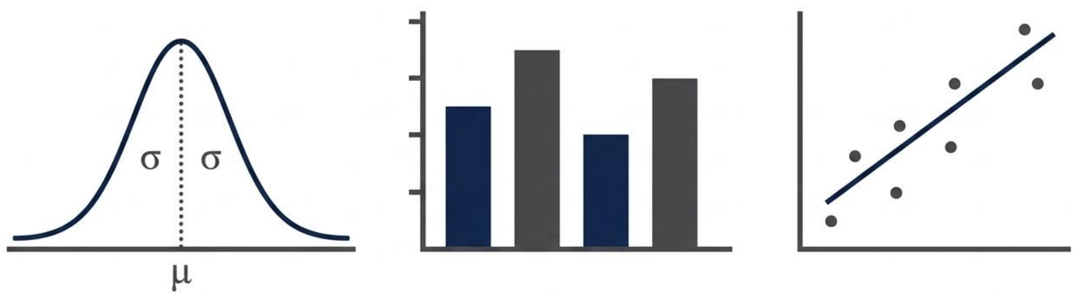
32.5 / 34.2 / 31.8 / 33.6 / 35.1 / 34.8 / 32.9 / 33.4 / 36.2 / 34.0

1. Give the median temperature of this series.

2. Calculate the mean temperature.

3. Calculate the range of this series.

4. Calculate the variance and standard deviation of this series.



Solution of Exercise No. 05:

1. Median temperature:

To find the median, we must first sort the temperatures in ascending order:

31.8 / 32.5 / 32.9 / 33.4 / 33.6 / 34.0 / 34.2 / 34.8 / 35.1 / 36.2

1 2 3 4 5 6 7 8 9 10

The median is the value in the middle of the ordered series. Since there are 10 values (even), the median is the average of the 5th and 6th values:

• 5th value: 33.6

• 6th value: 34.0

$$Median = [x(N/2) + x(N/2 + 1)]/2 = [x(5) + x(6)]/2 = (33.6 + 34.0)/2 = 33.8\ °C$$

The median temperature is therefore: 33.8 °C

2. Mean temperature:

$$\bar{X} = (\Sigma x_i)/N \quad \text{where } N = 10 \text{ (number of days)}$$

$$\bar{X} = (32.5 + 34.2 + 31.8 + 33.6 + 35.1 + 34.8 + 32.9 + 33.4 + 36.2 + 34.0)/10 = 338.5/10 = 33.85\ °C$$

3. Range:

The range is the difference between the largest and smallest value of the series:

• The smallest value is 31.8 °C

• The largest value is 36.2 °C

$$Range = 36.2 - 31.8 = 4.4\ °C$$

4. Variance and standard deviation:

First, calculate the squared deviations from the mean:

$$\Sigma(x_i - \bar{X})^2 = (32.5-33.85)^2 + (34.2-33.85)^2 + (31.8-33.85)^2 + (33.6-33.85)^2 + (35.1-33.85)^2$$

$$+ (34.8-33.85)^2 + (32.9-33.85)^2 + (33.4-33.85)^2 + (36.2-33.85)^2 + (34.0-33.85)^2$$

$$= 1.8225 + 0.1225 + 4.2025 + 0.0625 + 1.5625 + 0.9025 + 0.9025 + 0.2025 + 5.5225 + 0.0225$$

$$= 15.325$$

$$\sigma^2 = (\Sigma(x_i - \bar{X})^2)/N = 15.325/10 = 1.5325$$

$$\sigma x = \sqrt{(\sigma^2)} = \sqrt{1.5325} \approx 1.238\ °C$$

The variance is 1.5325 and the standard deviation is approximately 1.24 °C.

TUTORIAL SERIES No. 02

Contingency Tables, Correlation & Linear Regression

Exercise No. 01:

A survey of 60 students examined the relationship between weekly study hours (X) and exam grade ranges (Y). The contingency table below shows the joint frequencies:

/	**Y=1** **[0;10[**	**Y=2** **[10;14[**	**Y=4** **[14;20]**	**Marginal Freq.** $n_{i\cdot}$
X=2 **(0-5h)**	8	3	1	12
X=5 **(5-10h)**	4	10	6	20
X=7 **(10-15h)**	2	8	12	22
X=9 **(15h+)**	1	2	3	6
Marginal Freq. $n_{\cdot j}$	15	23	22	N=60

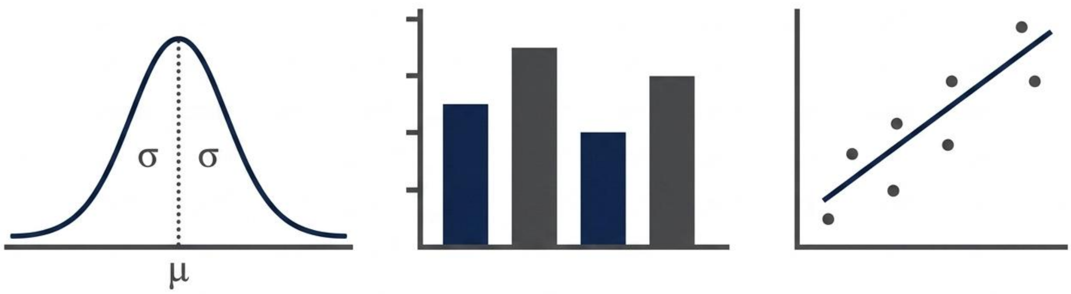
1. Calculate the marginal mean of X (study hours).

2. Calculate the marginal mean of Y (grade ranges).

3. Calculate the marginal variance of X.

4. Calculate the marginal variance of Y.

5. Calculate the marginal standard deviation of X and Y.



Solution of Exercise No. 01:

Marginal distributions:

Marginal distribution of X (Study Hours):

x_i	2	5	7	9	Total
$n_{i\cdot}$	12	20	22	6	N=60

Marginal distribution of Y (Grade Ranges):

y_j	1	2	4	Total
$n_{\cdot j}$	15	23	22	N=60

1. Marginal mean of X:

$$\bar{X} = (\Sigma\, n_{i\cdot} \cdot x_i) / n_{\cdot\cdot} = [(12\times2) + (20\times5) + (22\times7) + (6\times9)] / 60 = (24 + 100 + 154 + 54) / 60 = 332/60 = 5.533$$

The marginal mean of study hours is $\bar{X}$ = 5.53 hours.

2. Marginal mean of Y:

$$\bar{Y} = (\Sigma\, n_{\cdot j} \cdot y_j) / n_{\cdot\cdot} = [(15\times1) + (23\times2) + (22\times4)] / 60 = (15 + 46 + 88) / 60 = 149/60 = 2.483$$

The marginal mean of grade ranges is $\bar{Y}$ = 2.48.

3. Marginal variance of X:

$$\sigma x^2 = V(X) = (\Sigma\, n_{i\cdot} \cdot x_i^2)/n_{\cdot\cdot} - \bar{X}^2$$

$$\Sigma\, n_{i\cdot} \cdot x_i^2 = (12\times4) + (20\times25) + (22\times49) + (6\times81) = 48 + 500 + 1078 + 486 = 2112$$

$$\sigma x^2 = 2112/60 - (5.533)^2 = 35.20 - 30.62 = 4.58$$

4. Marginal variance of Y:

$$\sigma y^2 = V(Y) = (\Sigma\, n_{\cdot j} \cdot y_j^2)/n_{\cdot\cdot} - \bar{Y}^2$$

$$\Sigma\, n_{\cdot j} \cdot y_j^2 = (15\times1) + (23\times4) + (22\times16) = 15 + 92 + 352 = 459$$

$$\sigma y^2 = 459/60 - (2.483)^2 = 7.65 - 6.166 = 1.484$$

5. Marginal standard deviations:

$$\sigma x = \sqrt{(\sigma x^2)} = \sqrt{4.58} = 2.14$$

$$\sigma y = \sqrt{(\sigma y^2)} = \sqrt{1.484} = 1.22$$

Exercise No. 02:

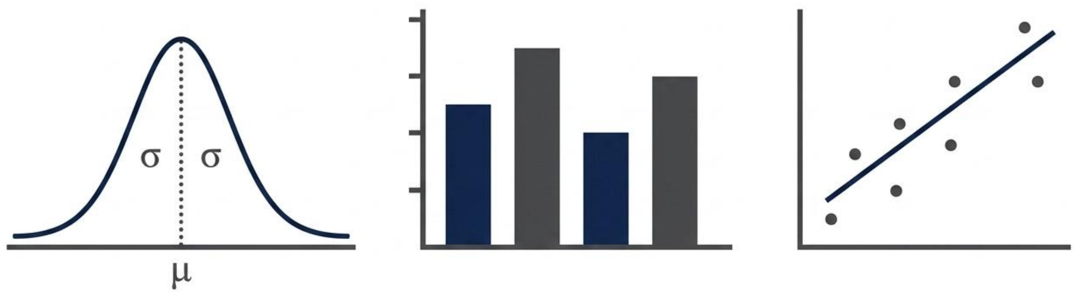
A marketing analyst studied the relationship between monthly advertising expenditure (in thousands of DZD) and monthly sales revenue (in thousands of DZD) for 6 different retail stores. The data obtained are as follows:

Store	**1**	**2**	**3**	**4**	**5**	**6**
Advertising X (kDZD)	10	15	20	25	30	35
Sales Y (kDZD)	45	58	72	85	98	112

1. Calculate the covariance between advertising expenditure and sales revenue.

2. Calculate the correlation coefficient to evaluate the strength of the relationship.

3. Find the equation of the linear regression line y = ax + b that models the relationship between advertising (X) and sales (Y) using the least squares method.



Solution of Exercise No. 02:

1. Calculation of Covariance:

Step 1: Calculate the means

$$\bar{X} = (10 + 15 + 20 + 25 + 30 + 35) / 6 = 135/6 = 22.5$$

$$\bar{Y} = (45 + 58 + 72 + 85 + 98 + 112) / 6 = 470/6 = 78.33$$

Step 2: Calculate the covariance

$$Cov(X,Y) = (1/n)\ \Sigma(x_i - \bar{X})(y_i - \bar{Y})$$

$$= (1/6)[(10-22.5)(45-78.33) + (15-22.5)(58-78.33) + (20-22.5)(72-78.33)$$

$$+ (25-22.5)(85-78.33) + (30-22.5)(98-78.33) + (35-22.5)(112-78.33)]$$

$$= (1/6)[416.63 + 152.48 + 15.83 + 16.68 + 147.53 + 420.88]$$

$$= (1/6) \times 1170.03 = 195.00$$

The covariance between advertising and sales is Cov(X,Y) = 195.

2. Calculation of the Correlation Coefficient:

Step 1: Calculate the variances

$$\sigma x^2 = [(10-22.5)^2 + (15-22.5)^2 + (20-22.5)^2 + (25-22.5)^2 + (30-22.5)^2 + (35-22.5)^2] / 6$$

$$= [156.25 + 56.25 + 6.25 + 6.25 + 56.25 + 156.25] / 6 = 437.5/6 = 72.92$$

$$\sigma y^2 = [(45-78.33)^2 + (58-78.33)^2 + (72-78.33)^2 + (85-78.33)^2 + (98-78.33)^2 + (112-78.33)^2] / 6$$

$$= [1110.89 + 413.31 + 40.04 + 44.49 + 386.91 + 1133.67] / 6 = 3129.31/6 = 521.55$$

Step 2: Calculate standard deviations

$$\sigma x = \sqrt{72.92} = 8.54$$

$$\sigma y = \sqrt{521.55} = 22.84$$

Step 3: Calculate the correlation coefficient

$$r = Cov(X,Y) / (\sigma x \times \sigma y) = 195 / (8.54 \times 22.84) = 195 / 195.05 \approx 0.9997$$

The correlation coefficient is r ≈ 0.9997, indicating an almost perfect positive linear correlation between advertising expenditure and sales revenue.

3. Linear Regression Line (Least Squares Method):

The regression line has the form y = ax + b, where:

$$a = Cov(X,Y) / \sigma x^2 = 195 / 72.92 = 2.674$$

$$b = \bar{Y} - a \cdot \bar{X} = 78.33 - (2.674 \times 22.5) = 78.33 - 60.17 = 18.16$$

Equation of the regression line:

$$y = 2.674x + 18.16$$

Conclusion:

The regression line y = 2.674x + 18.16 allows us to predict sales revenue based on advertising expenditure. The correlation coefficient very close to 1 (0.9997) confirms a very strong positive linear relationship: when advertising increases by 1,000 DZD, sales increase by approximately 2,674 DZD.

Exercise No. 03:

In an agricultural research center, the relationship between fertilizer dosage (in kg/hectare) and wheat yield (in quintals/hectare) was studied on 10 different plots. The results are recorded in the following table:

Plot	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Fertilizer X (kg/ha)	50	75	100	125	150	175	200	225	250	275
Yield Y (q/ha)	18	24	30	35	40	46	51	55	60	64

We want to determine if knowing the fertilizer dosage allows us to estimate the wheat yield.

1. Is a linear adjustment justified? Justify your answer.

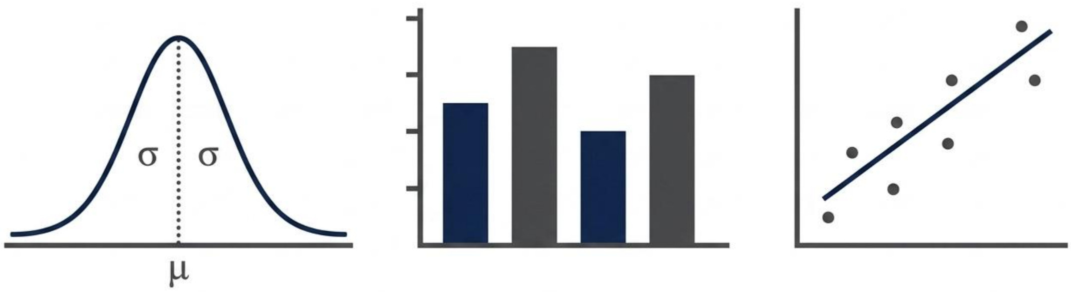
2. Perform a linear adjustment of yield as a function of fertilizer dosage using:

   a) Mayer's method (method of mean points)

   b) The extreme points method

3. Verify that the mean point lies on the Mayer adjustment line.

4. If a farmer uses 300 kg/ha of fertilizer, what yield can be expected using the Mayer method?



Solution of Exercise No. 03:

1. Justification of Linear Adjustment:

To determine if a linear adjustment is justified, we first calculate the correlation coefficient.

$$\bar{X} = (50+75+100+125+150+175+200+225+250+275)/10 = 162.5$$

$$\bar{Y} = (18+24+30+35+40+46+51+55+60+64)/10 = 42.3$$

Calculation of Cov(X,Y):

$$\Sigma(x_i - \bar{X})(y_i - \bar{Y}) = (-112.5)(-24.3) + (-87.5)(-18.3) + (-62.5)(-12.3) + (-37.5)(-7.3)$$

$$+ (-12.5)(-2.3) + (12.5)(3.7) + (37.5)(8.7) + (62.5)(12.7) + (87.5)(17.7) + (112.5)(21.7)$$

$$= 2733.75 + 1601.25 + 768.75 + 273.75 + 28.75 + 46.25 + 326.25 + 793.75 + 1548.75 + 2441.25$$

$$= 10{,}562.5$$

$$Cov(X,Y) = 10{,}562.5/10 = 1056.25$$

Calculation of variances:

$$\sigma x^2 = [(50-162.5)^2 + ... + (275-162.5)^2]/10 = 50{,}625/10 = 5062.5$$

$$\sigma y^2 = [(18-42.3)^2 + ... + (64-42.3)^2]/10 = 1961.1/10 = 196.11$$

$$r = 1056.25 / (\sqrt{5062.5} \times \sqrt{196.11}) = 1056.25 / (71.15 \times 14.00) = 1056.25/996.1 = 0.999$$

Since r ≈ 0.999 (very close to 1), a linear adjustment is fully justified.

2. a) Mayer's Method (Method of Mean Points):

We divide the 10 plots into two groups of 5 and calculate the mean point of each group:

Group 1 (Plots 1-5):

$$\bar{X}_1 = (50 + 75 + 100 + 125 + 150)/5 = 500/5 = 100$$

$$\bar{Y}_1 = (18 + 24 + 30 + 35 + 40)/5 = 147/5 = 29.4$$

Point A = (100, 29.4)

Group 2 (Plots 6-10):

$$\bar{X}_2 = (175 + 200 + 225 + 250 + 275)/5 = 1125/5 = 225$$

$$\bar{Y}_2 = (46 + 51 + 55 + 60 + 64)/5 = 276/5 = 55.2$$

Point B = (225, 55.2)

The Mayer line passes through A and B. We solve:

$$A \in D \rightarrow 29.4 = 100a + b \ \ ...(1)$$

$$B \in D \rightarrow 55.2 = 225a + b \ \ ...(2)$$

$$(2) - (1): 55.2 - 29.4 = (225 - 100)a \rightarrow 25.8 = 125a \rightarrow a = 0.2064$$

$$\text{From (2): } b = 55.2 - (0.2064 \times 225) = 55.2 - 46.44 = 8.76$$

Mayer's regression line:

$$y = 0.206x + 8.76$$

2. b) Extreme Points Method:

We use the minimum and maximum points:

$$A = (xmin, ymin) = (50, 18)$$

$$B = (xmax, ymax) = (275, 64)$$

The extreme points line passes through A and B:

$$A \in D \rightarrow 18 = 50a + b \ \ ...(1)$$

$$B \in D \rightarrow 64 = 275a + b \ \ ...(2)$$

$$(2) - (1): 64 - 18 = (275 - 50)a \rightarrow 46 = 225a \rightarrow a = 0.2044$$

$$\text{From (2): } b = 64 - (0.2044 \times 275) = 64 - 56.21 = 7.79$$

Extreme points regression line:

$$y = 0.204x + 7.79$$

3. Verification that the mean point lies on the Mayer line:

$$\text{Mean point } G = (\bar{X}, \bar{Y}) = (162.5, 42.3)$$

$$\text{Substitute into Mayer's equation: } y = 0.206 \times 162.5 + 8.76 = 33.48 + 8.76 = 42.24 \approx 42.3$$

The mean point (162.5, 42.3) lies approximately on the Mayer adjustment line. ✓

4. Prediction for 300 kg/ha using Mayer's method:

$$y = 0.206 \times 300 + 8.76 = 61.8 + 8.76 = 70.56$$

The expected yield for 300 kg/ha of fertilizer is approximately 70.6 quintals/hectare.

Comparison of Predictive Methods:

Method	Principle	Precision	Use Case
Extreme Points	Uses min and max values only	Less precise, sensitive to outliers	Quick estimation
Least Squares	Minimizes sum of squared errors	Most precise, optimal globally	Scientific analysis
Mayer's Method	Uses mean points of subgroups	Balanced estimate, less sensitive to extremes	When data has subgroups

Part II

Probability

Part II

Probability

Probability

Whereas descriptive statistics summarises data already collected, probability theory builds mathematical models of random phenomena before the data are observed. The two subjects are closely linked: a probabilistic model tells us what patterns to expect, and descriptive statistics lets us check whether those patterns are actually present.

3. Combinatorics

Probability computations frequently reduce to counting the elements of finite sets. The present chapter reviews the five counting techniques that recur throughout this course.

3.1. Cardinality

If E is a finite set, its cardinality $\text{card}(E)$ is the number of its elements. For instance card({1, 2, 3, 4, 5, 6}) = 6, while the set of natural numbers ℕ has infinite cardinality.

3.2. Permutations

The number of ways to order n distinct objects is the factorial:

Factorial

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

By convention 0! = 1. Reading: "n factorial."

3.3. Lists — Sampling with Replacement

An ordered list of length k whose entries are chosen from a set of n elements, with replacement, can be built in n^k ways.

3.4. Arrangements — Sampling without Replacement, Ordered

The number of ordered selections of p objects drawn from n distinct objects is

$$A_n^p$$

$$A_n^p = n \times (n-1) \times \cdots \times (n-p+1) = \frac{n!}{(n-p)!}$$

Example — the number of possible Olympic podiums (gold, silver, bronze) with ten athletes is $A_{10}{}^3 = 10 \times 9 \times 8 = 720$.

3.5. Combinations

When order is irrelevant the number of ways to choose p elements from n is the binomial coefficient:

Binomial Coefficient — "n choose p"

$$\binom{n}{p} = \frac{n \times (n-1) \times \cdots \times (n-p+1)}{p \times (p-1) \times \cdots \times 1} = \frac{n!}{p!\,(n-p)!}$$

Useful identities.

$$\binom{n}{p} = \binom{n}{n-p}$$

$$\binom{n}{p} = \binom{n-1}{p} + \binom{n-1}{p-1}$$

(Pascal's identity — basis of Pascal's triangle.)

Binomial theorem.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

4. Elementary Definitions of Probability

4.1. Random Experiments and Sample Space

A random experiment is an experiment whose outcome is not known in advance. The set Ω of all possible outcomes is called the sample space. Examples:

- Rolling a fair six-sided die: $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- Inspecting a manufactured item: $\Omega = \{\text{conforming, non-conforming}\}$.
- Choosing a non-negative integer: $\Omega = \mathbb{N}$ (infinite, discrete).
- Choosing a point in the plane: $\Omega = \mathbb{R}^2$ (infinite, continuous).

4.2. Events

An event is a subset $A \subseteq \Omega$. The empty set $\emptyset$ is the impossible event; Ω itself is the certain event. A singleton event is called elementary. The set of all events is the power set $\mathcal{P}(\Omega)$.

4.3. Operations on Events

For two events A and B:

- Complement: $\bar{A} = \Omega \setminus A$ ("A does not occur").
- Union: $A \cup B$ ("A or B, possibly both, occurs").
- Intersection: $A \cap B$ ("A and B occur simultaneously").

Two events are said to be mutually exclusive (or incompatible) when $A \cap B = \emptyset$. An event and its complement are always mutually exclusive.

4.4. Axioms of Probability

A probability is a mapping $p: \mathcal{P}(\Omega) \to [0,1]$ such that $p(\Omega) = 1$ and, for any two mutually exclusive events A and B,

$$p(A \cup B) = p(A) + p(B)$$

Elementary consequences.

$$0 \le p(A) \le 1$$

$$p(\emptyset) = 0$$

$$p(\overline{A}) = 1 - p(A)$$

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

4.5. Finite Sample Spaces and Equiprobability

When Ω is finite it suffices to know the probability of each elementary event; any other probability is obtained by summation. A particularly important case is equiprobability, in which every elementary event has the same probability 1/card(Ω). Then for any event A,

Equiprobable Case

$$p(A) = \frac{\text{card}(A)}{\text{card}(\Omega)} = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}$$

4.6. Continuous Probability — Probability Density

When Ω is an interval of the real line (for instance [0, 1] or ℝ), the probability of any single outcome is zero, so we need a different description. We associate to the probability a non-negative, integrable probability density f such that

$$\int_{\Omega} f(x)\mathrm{d}x = 1$$

For an event $A \subseteq \Omega$,

$$p(A) = \int_{A} f(x)\mathrm{d}x$$

Example — the uniform density on [0, 1] is f(x) = 1 on the interval and 0 outside. The probability that the outcome lies in [a, b] with $0 \le a \le b \le 1$ is simply p([a, b]) = b − a.

4.7. Conditional Probability

Definition — Conditional Probability

$$p(B \mid A) = \frac{p(A \cap B)}{p(A)}, \text{whenever } p(A) > 0$$

Read: "the probability of B given A." The conditional probability turns out to satisfy the same axioms as an ordinary probability, with A playing the role of the sample space.

Multiplicative rule.

$$p(A \cap B) = p(A)p(B \mid A) = p(B)p(A \mid B)$$

Law of total probability.

If $A_1, A_2, \dots, A_n$ are pairwise mutually exclusive events that together cover Ω, then for every event B,

$$p(B) = \sum_{i=1}^{n} p(A_i)p(B \mid A_i)$$

Bayes' theorem.

Bayes' Formula

$$p(A_k \mid B) = \frac{p(A_k)p(B \mid A_k)}{\sum_{i=1}^{n} p(A_i)p(B \mid A_i)}$$

Worked example — diagnostic testing.

A screening test for a disease that affects 1 person in 10 000 detects 99 % of true cases but gives a 0.5 % rate of false positives. What is the probability that a randomly selected individual who tested positive is actually ill? Writing M for "ill" and P for "positive", Bayes' formula gives

$$p(M \mid P) = \frac{p(P \mid M)p(M)}{p(P \mid M)p(M) + p(P \mid \overline{M})p(\overline{M})} \approx 1.94 \,\%$$

Despite an apparently excellent test, only about 2 % of positive results correspond to genuinely ill patients the counter-intuitive consequence of a very low base rate.

4.8. Independent Events

Definition — Independence of Two Events

Two events A and B are independent when the occurrence of one does not alter the probability of the other, that is,

$$p(A \cap B) = p(A)p(B)$$

Do not confuse independence and incompatibility. Two events of non-zero probability that are mutually exclusive can never be independent knowing that one has occurred tells us with certainty that the other has not.

5. Random Variables

5.1. Definition and Distribution

A random variable on a probability space (Ω, p) is a function $X: \Omega \to \mathbb{R}$. Typical examples are the outcome of a die roll, the measured length of a metal rod, or the number of breakdowns of a machine per day.

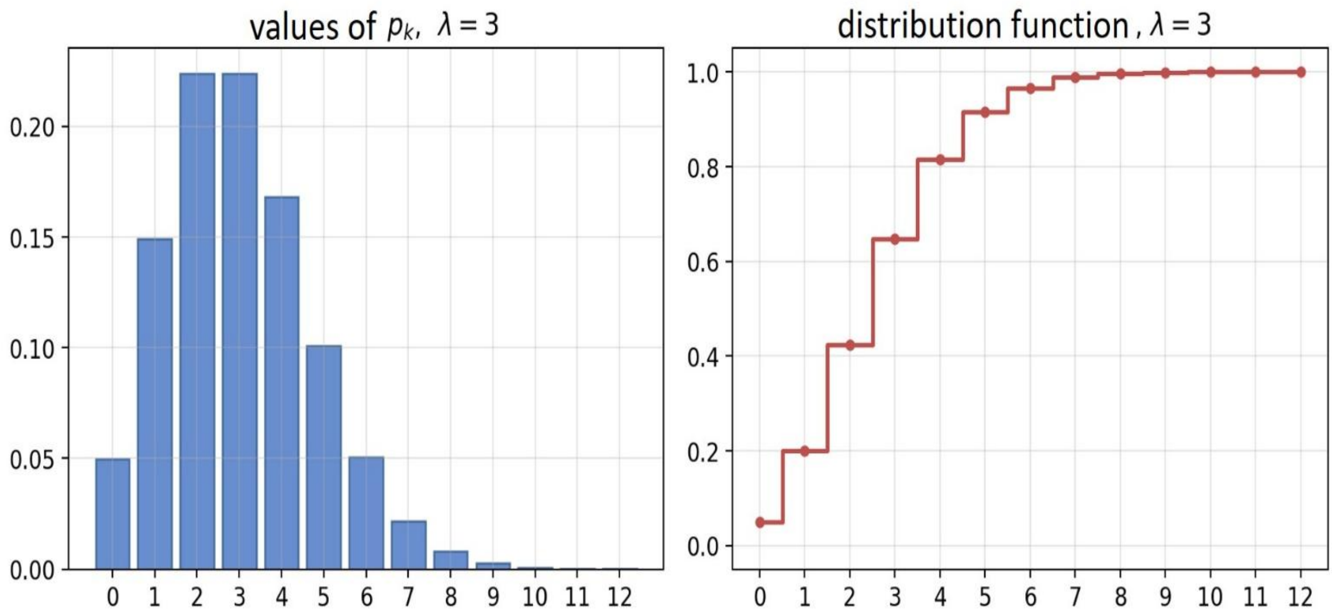
Each random variable induces a probability on the real line through

$$p_X(A) = p(X \in A) = p(\{\omega \in \Omega \mid X(\omega) \in A\})$$

The cumulative distribution function (c.d.f.) of X is

$$F_X(x) = p(X < x)$$

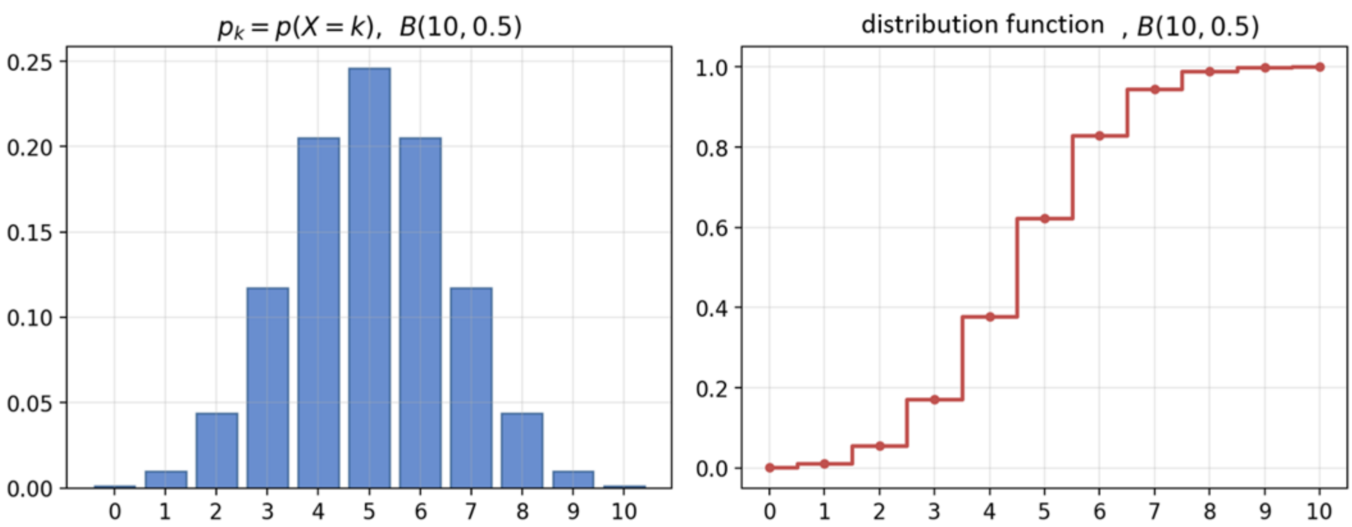
It is non-decreasing, tends to 0 at −∞ and to 1 at +∞. It is, in general, enough to know F_X on all intervals of the form]−∞, x[to reconstruct every probability associated with X.



5.2. Discrete Random Variables

If the set of values taken by X is countable, $X(\Omega) = \{x_1, x_2, \dots\}$, the distribution is completely specified by the elementary probabilities $p_i = p(X = x_i)$ with

$$\sum_i p_i = 1$$



Expectation, variance, standard deviation.

Definitions — Discrete Case

$$E(X) = \sum_i p_i x_i$$

$$\text{Var}(X) = \sum_i p_i\left(x_i - E(X)\right)^2 = E(X^2) - \left(E(X)\right)^2$$

$$\sigma(X) = \sqrt{\text{Var}(X)}$$

Basic properties.

$$E(X + Y) = E(X) + E(Y)$$

$$E(aX + b) = aE(X) + b$$

$$\text{Var}(aX + b) = a^2\text{Var}(X)$$

$$\sigma(aX + b) = |a|\,\sigma(X)$$

A random variable is said to be centred when E(X) = 0 and reduced (standardised) when Var(X) = 1. For any random variable with non-zero standard deviation, the variable

$$Z = \frac{X - E(X)}{\sigma(X)}$$

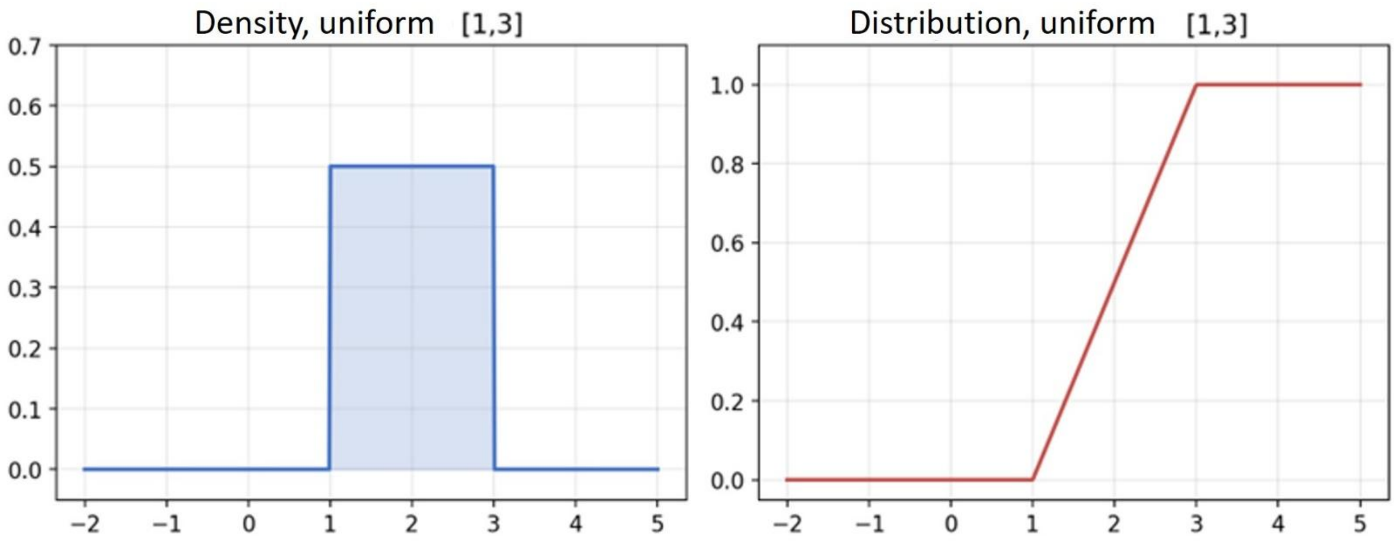
is always centred and reduced a transformation we shall use repeatedly in the chapters that follow.

5.3. Continuous Random Variables

A random variable X admits a density f_X when, for every real x,

$$F_X(x) = p(X < x) = \int_{-\infty}^{x} f_X(t)\,\mathrm{d}t$$

The density is therefore the derivative of the c.d.f. It follows that p(a ≤ X < b) = F_X(b) − F_X(a), while the probability of any single value is zero.



Definitions — Continuous Case

$$E(X) = \int_{-\infty}^{+\infty} x f_X(x)\, \mathrm{d}x$$

$$\mathrm{Var}(X) = \int_{-\infty}^{+\infty} \left(x - E(X)\right)^2 f_X(x)\, \mathrm{d}x$$

5.4. Chebyshev's Inequality

A remarkable feature of the mean and standard deviation is that they bound, independently of the specific distribution, the probability of large deviations. This is Chebyshev's inequality.

Chebyshev's Inequality

For any random variable X with finite variance and any λ > 0:

$$p\big(|X - E(X)| \geq \lambda\sigma(X)\big) \leq \frac{1}{\lambda^2}$$

$$p\big(E(X) - \lambda\sigma(X) < X < E(X) + \lambda\sigma(X)\big) \geq 1 - \frac{1}{\lambda^2}$$

Choosing λ = 2 gives a lower bound of 3/4 for the probability that X lies within two standard deviations of its mean; with λ = 3, the bound is 8/9. These bounds are loose in most practical situations but have the great merit of being universal.

6. Pairs of Random Variables

Many engineering problems involve two variables measured on the same population the height and weight of an individual, the current and voltage across a component, the stress and strain of a sample. The pair (X, Y) is itself a random variable taking values in $\mathbb{R}^2$.

6.1. Joint Distribution — Discrete Case

When Ω is discrete, the joint distribution of (X, Y) is given by the joint elementary probabilities

$$p_{xy} = p(X = x, Y = y)$$

The marginal distributions are obtained by summation:

$$p(X = x) = \sum_y p_{xy} \;, \qquad p(Y = y) = \sum_x p_{xy}$$

Independence (discrete case).

Two discrete variables X and Y are independent if, for every pair (x, y),

$$p(X = x, Y = y) = p(X = x)p(Y = y)$$

The practical payoff of independence is that the two marginal distributions are then sufficient to recover the joint distribution.

6.2. Joint Distribution — Continuous Case

A continuous pair (X, Y) admits a joint density $f(x, y) \geq 0$ satisfying

$$\iint f(x,y)\,\mathrm{d}x\,\mathrm{d}y = 1$$

The marginal densities are obtained by integration:

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,\mathrm{d}y$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,\mathrm{d}x$$

X and Y are independent if and only if the joint density factorises into the product of the marginal densities:

$$f(x,y) = f_X(x) f_Y(y)$$

6.3. Covariance and Correlation Coefficient

Definitions

$$\operatorname{cov}(X,Y) = E\Big(\big(X - E(X)\big)\big(Y - E(Y)\big)\Big)$$

$$\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$\rho(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sigma(X)\sigma(Y)}$$

Key properties.

$$|\operatorname{cov}(X,Y)| \leq \sigma(X)\sigma(Y)$$

$$-1 \leq \rho(X,Y) \leq 1$$

$$\operatorname{Var}(aX + bY) = a^2\operatorname{Var}(X) + 2ab\operatorname{cov}(X,Y) + b^2\operatorname{Var}(Y)$$

If X and Y are independent then $\operatorname{cov}(X, Y) = \rho(X, Y) = 0$ and $E(X\,Y) = E(X) \cdot E(Y)$; the variance is additive: $\operatorname{Var}(X + Y) = \operatorname{Var}(X) + \operatorname{Var}(Y)$.

Caution the converse is false: $\rho(X, Y) = 0$ does not imply independence. It merely means there is no linear dependence. When $|\rho| = 1$, however, X and Y are exactly linearly related: $Y = aX + b$ for some constants a and b.

7. Classical Probability Distributions

7.1. Discrete Distributions

Bernoulli distribution B(p).

A Bernoulli trial has two outcomes: success (probability p) and failure (probability q = 1 − p). The associated Bernoulli random variable X takes the value 1 on success and 0 on failure.

Bernoulli Distribution

$$p(X = 1) = p, p(X = 0) = 1 - p$$

$$E(X) = p, \mathrm{Var}(X) = p(1 - p), \sigma(X) = \sqrt{p(1 - p)}$$

Binomial distribution B(n, p).

The number of successes X in n independent Bernoulli trials with common success probability p follows a binomial distribution of parameters (n, p).

Binomial Distribution B(n, p)

$$p(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$E(X) = np, \mathrm{Var}(X) = np(1 - p), \sigma(X) = \sqrt{np(1 - p)}$$

Worked example. A candidate has a 5 % success rate in a competitive examination held in a room of 40 people. The number of successful candidates follows B(40, 0.05), so the expected number of admissions per room is E(X) = n p = 2; the variance is 1.90 and the standard deviation 1.38. On average one expects two successes per room, with a typical dispersion of about ± 1.4.

Poisson distribution P(λ).

The Poisson distribution is the canonical model for rare events that occur independently in time (or in space) at a constant average rate. Typical examples are radioactive decays, phone calls reaching a switchboard or cosmic-ray events observed by a detector.

Poisson Distribution P(λ)

$$p(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \ k \in \mathbb{N}$$

$$E(X) = \lambda, \mathrm{Var}(X) = \lambda, \sigma(X) = \sqrt{\lambda}$$

A remarkable property is that the expectation and variance are equal a diagnostic often used to decide whether real data is Poisson-like.

If $X \sim P(\lambda_1)$ and $Y \sim P(\lambda_2)$ are independent, then $X + Y \sim P(\lambda_1 + \lambda_2)$.

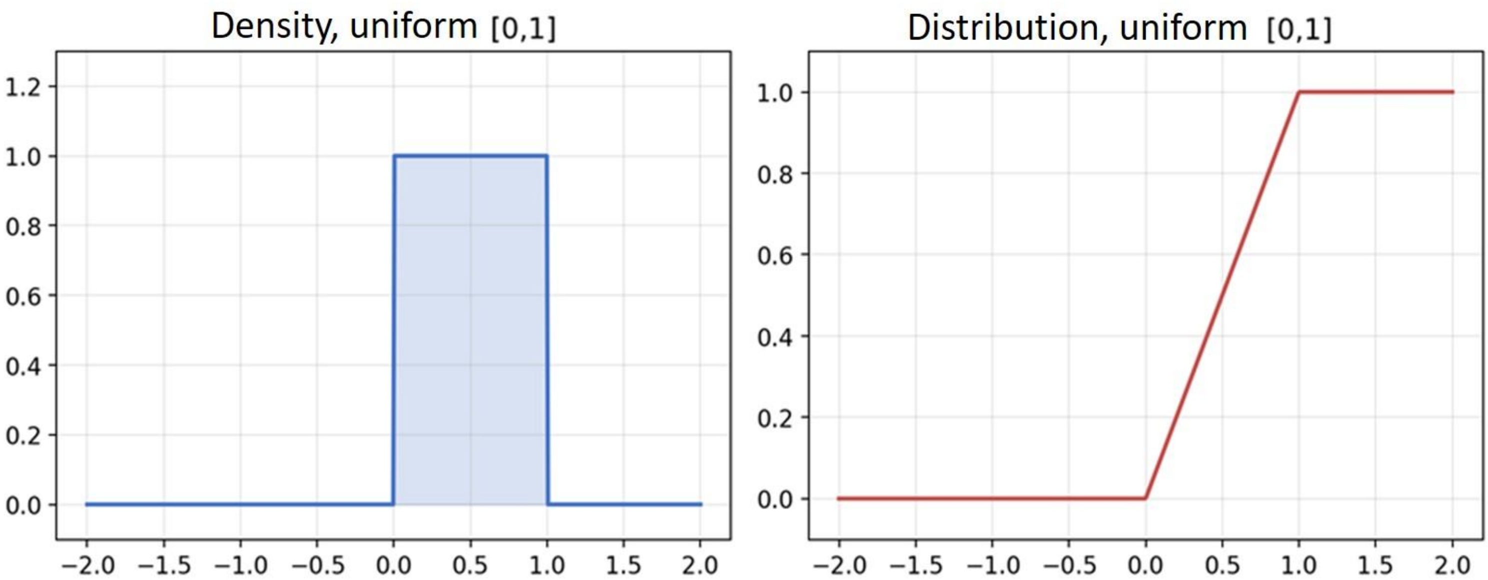
Worked example — telephone switchboard.

A switchboard receives an average of 10 calls per hour. The number of calls in a period of length Δt (in hours) follows P(10 Δt). Standard questions are answered by reading the Poisson table:

- Probability of exactly 2 calls in one hour: $p(X = 2) = e^{-10} \cdot 10^2 / 2! \approx 0.23$ %.
- Probability of exactly 3 calls in 5 minutes: $X \sim P(5/6)$, so $p(X = 3) \approx 4.2$ %.
- Probability of at least 2 calls in 15 minutes: $X \sim P(2.5)$, so $p(X \geq 2) \approx 71.3$ %.
- Probability of at most 8 calls in one hour: $X \sim P(10)$, so $p(X \leq 8) \approx 33.3$ %.

7.2. Continuous Distributions

Uniform distribution U(a, b).



Uniform Distribution on [a, b]

$$f_X(x) = \frac{1}{b - a} \quad \text{for } a \leq x \leq b, \ \text{0 otherwise}$$

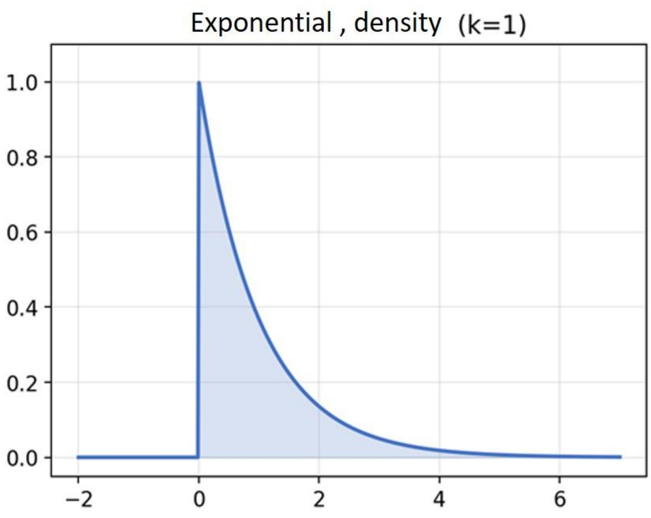
$$E(X) = \frac{a + b}{2}, \quad \text{Var}(X) = \frac{(b - a)^2}{12}$$

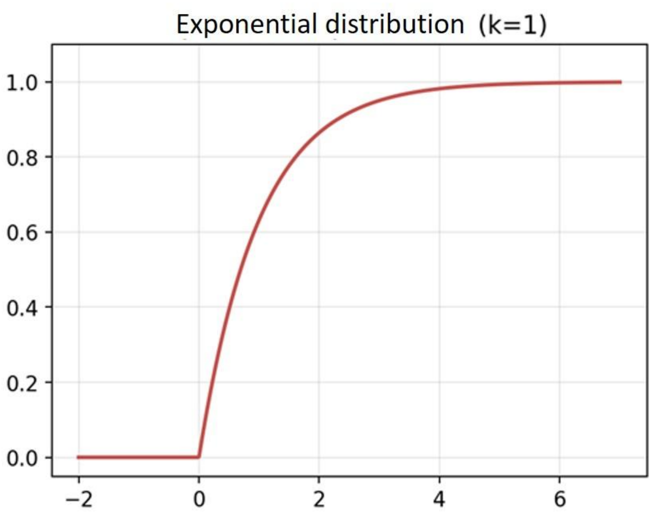
Remark no uniform distribution exists on the whole real line: the integral of a non-zero constant over ℝ is infinite, and the integral of zero is zero.

Exponential distribution E(k).

The exponential distribution models the lifetime of a memoryless component that is, a component whose probability of surviving an additional interval t is independent of its age:

$$p(X \geq t + T \mid X \geq T) = p(X \geq t)$$





Only exponential distributions satisfy this property. The density, c.d.f. and moments are:

Exponential Distribution

$$f_X(t) = ke^{-kt} \quad \text{for } t \geq 0$$

$$F_X(t) = 1 - e^{-kt} \quad \text{for } t \geq 0$$

$$E(X) = \frac{1}{k}, \quad \text{Var}(X) = \frac{1}{k^2}, \quad \sigma(X) = \frac{1}{k}$$

The mean 1/k is called the MTBF (Mean Time Between Failures), and k is the failure rate. For example, a sensor with failure rate 0.2 per year has probability $e^{-0.2t}$ of functioning at time t, and an expected lifetime of 5 years.

Normal (Gaussian) distribution N(μ, σ²).

By far the most widely used distribution in engineering and physics. Its density is the familiar bell curve:

Normal Distribution N(μ, σ²)

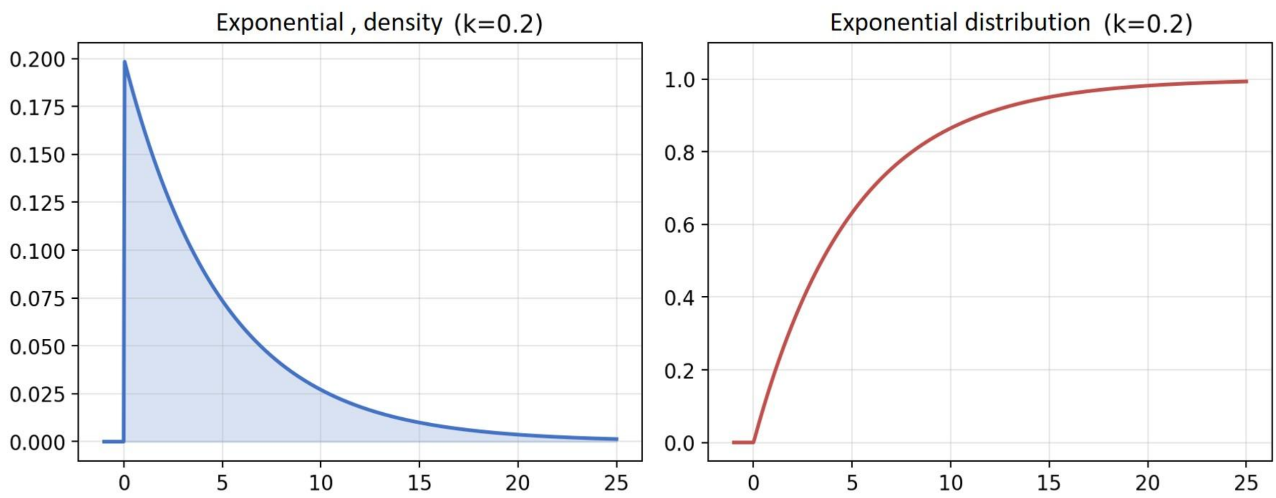
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

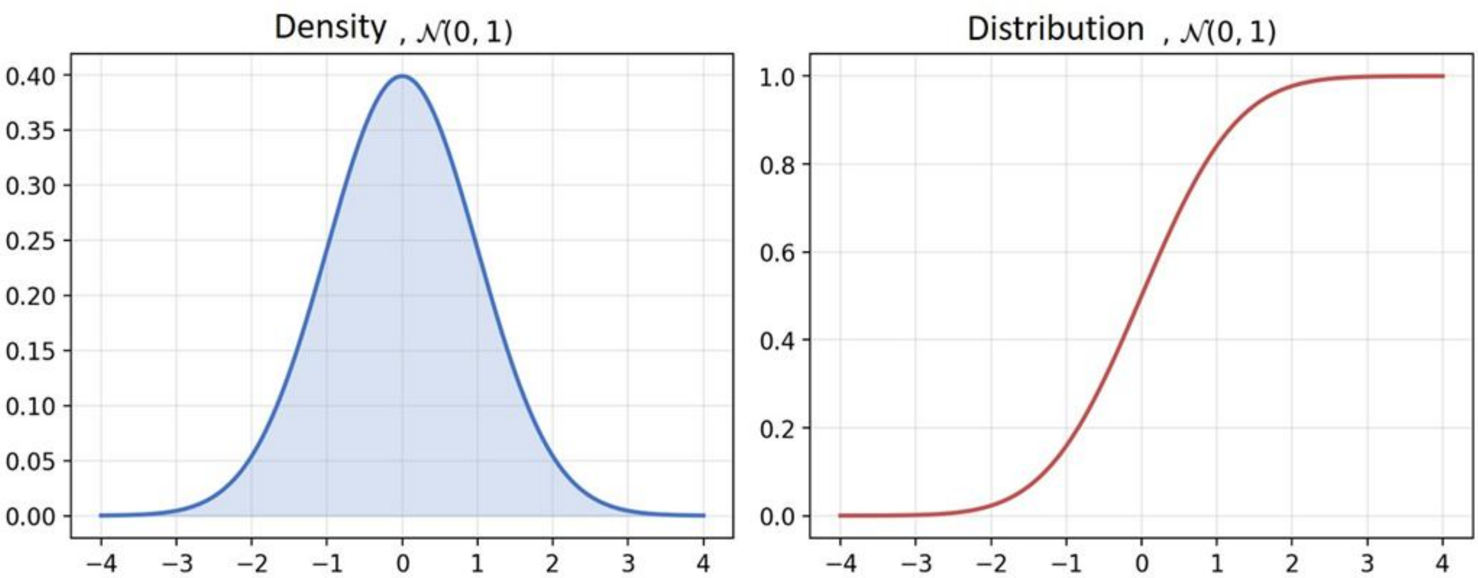
$$E(X) = \mu, \quad \text{Var}(X) = \sigma^2, \quad \sigma(X) = \sigma$$

The parameter μ controls the centre of the bell; σ controls its width. The c.d.f. cannot be expressed with elementary functions, so numerical tables of the standardised distribution N(0, 1) are used. Given $X \sim N(\mu, \sigma^2)$, the linear change of variable

$$Z = \frac{X - \mu}{\sigma}$$

produces a standard normal $Z \sim N(0, 1)$. All probability computations for a normal variable can therefore be reduced to look-ups in a single table.

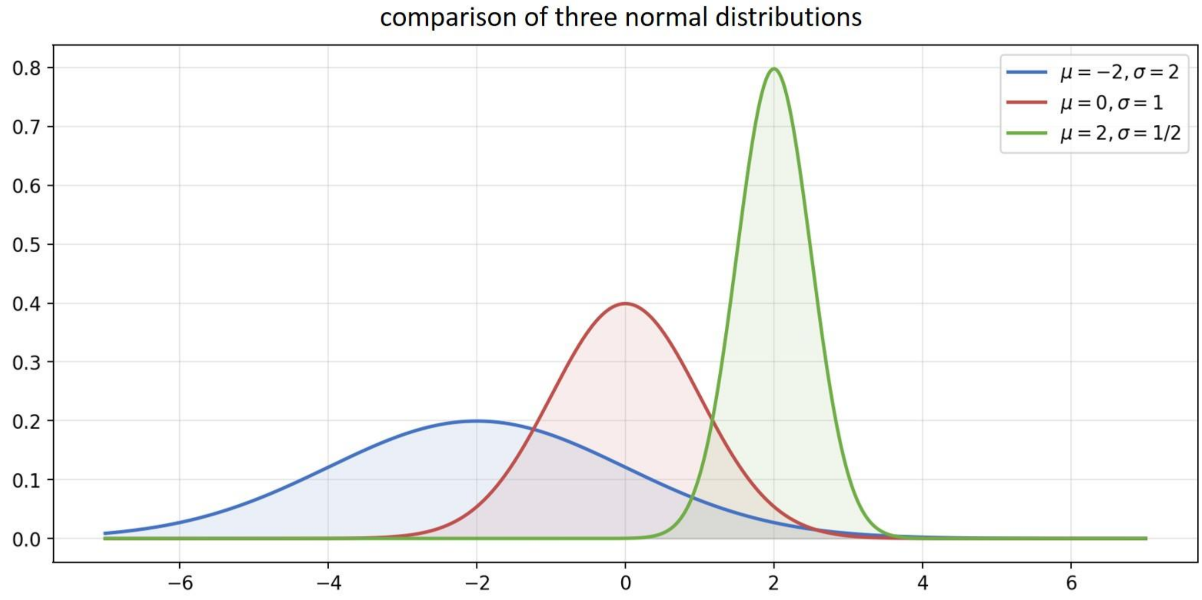




Worked example.

If $X \sim N(1, 4)$, that is $\mu = 1$ and $\sigma = 2$, we compute:

- $p(X \leq 3) = p(Z \leq 1) = 0.8413$.
- $p(X > 2) = 1 - p(Z \leq 0.5) = 1 - 0.6915 = 0.3085$.
- $p(-2.5 \leq X \leq 3) = p(-1.75 \leq Z \leq 1) = 0.8413 - (1 - 0.9599) = 0.8012$.



Three-sigma rule (sharpening of Chebyshev for a normal law).

Empirical (3σ) Rule for the Normal Law

$$p(\mu - \sigma < X < \mu + \sigma) \approx 0.683 = 68.3\ \%$$

$$p(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.954 = 95.4\ \%$$

$$p(\mu - 3\sigma < X < \mu + 3\sigma) \approx 0.997 = 99.7\ \%$$

Stability under sum. If $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ are independent, then $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$. In particular the mean $\bar{X}$ of n i.i.d. variables $N(\mu, \sigma^2)$ follows $N(\mu, \sigma^2/n)$ a fact that is central to the theory of sampling.

Part III

Statistics

Statistics

We now combine the language of descriptive statistics (Part I) with the models of probability (Part II) to perform inference: from a sample we extract information about the population it comes from, and we quantify the uncertainty attached to that information.

8. Approximation of One Distribution by Another

8.1. Motivation

Statistical tables cannot be exhaustive it is impractical to tabulate the binomial distribution for every pair (n, p). When n is large, however, the binomial B(n, p) is close to a Poisson or a normal distribution with the same mean and variance, and the tables of those two distributions suffice.

8.2. Binomial → Poisson

When n is large and p is small, the binomial B(n, p) is well approximated by the Poisson $P(np)$:

$$p(X = k) \approx \frac{e^{-np}(np)^k}{k!}$$

Practical rule of thumb: $n \geq 30$, $p \leq 0.1$ and $n\,p \leq 5$.

8.3. Binomial → Normal

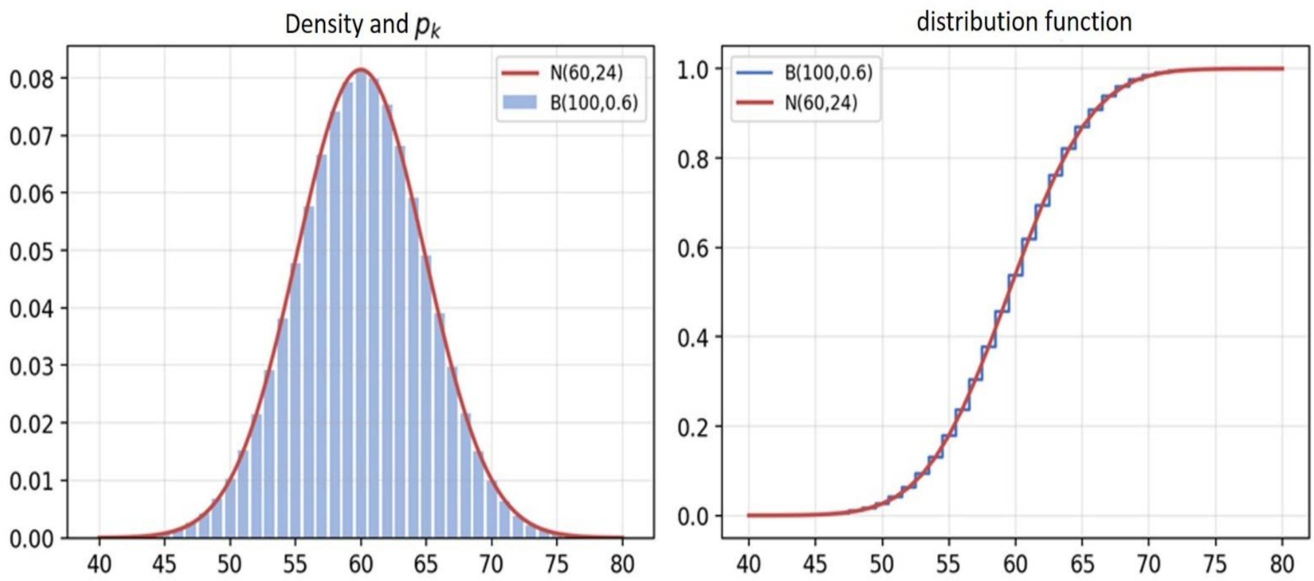
When n is large but p is moderate, the Poisson approximation no longer applies. One then uses the normal distribution with the same mean and variance:

$$\mathrm{B}(n,p) \approx \mathrm{N}\big(np, np(1-p)\big)$$

Since the binomial is discrete while the normal is continuous, each discrete value k is replaced by the interval $]k - 0.5, k + 0.5[$. This continuity correction gives

$$p(X = k) \approx p(k - 0.5 < X' < k + 0.5)$$

Practical rule of thumb: n ≥ 50 and n p (1 − p) ≥ 18.



8.4. Poisson → Normal

For λ > 10 the Poisson P(λ) is well approximated by the normal N(λ, λ) again with the continuity correction p(X = k) ≈ p(k − 0.5 < X′ < k + 0.5). These two approximations are particular instances of the Central Limit Theorem, to which we now turn.

9. Sampling

9.1. The Weak Law of Large Numbers

Consider an event of probability p that we replicate in n independent trials. Let S_n be the number of successes. Then S_n / n is the observed frequency of the event.

Weak Law of Large Numbers

For every ε > 0,

$$p\left(\left|\frac{S_n}{n} - p\right| \geq \varepsilon\right) \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

Equivalently, the empirical frequency S_n/n converges in probability to the theoretical probability p.

Remark randomness has no memory. Getting one hundred "heads" in a row does not make the next toss more likely to come up "tails"; the law of large numbers is a statement about long-run averages, not individual outcomes.

9.2. The Sampling Problem

A population is characterised by a random variable X with (possibly unknown) mean μ and variance σ^2. A sample of size n is modelled by n independent copies $X_1, X_2, \ldots, X_n$ of X. The sample mean

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

is itself a random variable. Using linearity of expectation and independence of the variance,

$$E(\overline{X}) = \mu, \quad \mathrm{Var}(\overline{X}) = \frac{\sigma^2}{n}$$

The modelling assumption of independent copies is legitimate only when the sample is drawn from a much larger population, so that removing an individual does not appreciably *change the population. In practice this holds whenever the sample size does not exceed 5 – 10 % of the population.*

9.3. Sampling from a Normal Population

When the population itself is normal, the sample mean is normal too:

Sampling Distribution of the Mean (Normal Population)

If $X \sim N(\mu, \sigma^2)$ and $(X_1, \ldots, X_n)$ is an i.i.d. sample, then

$$\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

Worked example.

A machine produces discs whose diameters follow $N(15\text{ cm}, 0.25^2\text{ cm}^2)$. For a random sample of 10 discs, the sample mean $\bar{X}$ follows $N(15, 0.00625)$. Consequently, in 95.4 % of samples, the sample mean lies in

$$\left[15 - 2 \times \frac{0.25}{\sqrt{10}}, 15 + 2 \times \frac{0.25}{\sqrt{10}}\right] \approx [14.84, 15.16]$$

9.4. The Central Limit Theorem

Central Limit Theorem

If $X_1, X_2, \ldots, X_n$ are i.i.d. with common mean μ and variance σ^2, then for n sufficiently large

$$\overline{X} \approx \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

no matter what the distribution of X is. A rule of thumb is $n \geq 30$.

Worked example — voltages.

A factory manufactures batteries whose voltages have mean 1.6 V and standard deviation 0.1 V, with no assumption of normality. For a sample of 50 batteries, the CLT gives

$$\overline{X} \approx \mathrm{N}\left(1.6, \frac{(0.1)^2}{50}\right)$$

with standard deviation ≈ 0.0141 V. Hence, with 95.4 % probability, the mean voltage of the sample lies in [1.572, 1.628].

9.5. Application to Proportions

Consider a population in which a proportion p of individuals possess some property (defective, left-handed, in favour of candidate A, ...). For a sample of size n, the number of individuals with the property follows B(n, p). When n is large enough, the observed frequency F = X / n follows approximately a normal distribution:

Sampling Distribution of a Proportion

$$F \approx \mathrm{N}\left(p, \frac{p(1-p)}{n}\right)$$

Worked example.

The proportion of left-handed individuals in the population is p = 0.10. In a group of 30 persons, the probability of observing more than 15 % of left-handers is p(F > 0.15), where F ~ N(0.1, 0.09/30), giving ≈ 19 %.

10. Estimation

10.1. The Inverse Problem

Sampling theory (Chapter 9) gives the distribution of sample statistics when the population parameters are known. Estimation is the inverse problem: from a sample, deduce plausible values for the unknown parameters μ and σ^2 of the population.

Example 1 — metal rods.

A sample of 300 metal rods yields the diameter distribution:

Diameter d (mm)	**42**	**43**	**44**	**45**	**46**	**47**	**48**
Count	13	26	63	149	30	15	4

Example 2 — voltages.

Five successive measurements of a voltage give (V): 11.83, 11.91, 12.01, 12.03, 12.10.

10.2. Point Estimates

Point Estimates

$$\overline{x} = \frac{x_1 + \cdots + x_n}{n} \qquad \text{(estimate of } \mu\text{)}$$

$$\sigma'^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n}$$

$$s^2 = \frac{n}{n-1}\sigma'^2 \qquad \text{(unbiased estimate of } \sigma^2\text{)}$$

The sample variance σ'^2 is a biased estimator of the population variance σ^2 it tends to under-estimate σ^2 when n is small. The corrected estimator s^2 with divisor $n - 1$ is unbiased, that is $E(s^2) = \sigma^2$. This is why statistical software reports the "sample variance" with the factor $1/(n - 1)$.

Numerical results.

Example 1: $\bar{x} = 44.73$, $\sigma'^2 = 1.292$, hence $s^2 = 1.296$, $\sigma' = 1.136$, $s = 1.138$.

Example 2: $\bar{x} = 11.97$, $\sigma'^2 = 0.009$, hence $s^2 = 0.01128$, $\sigma' = 0.094$, $s = 0.106$.

10.3. Confidence-Interval Estimates

A single point estimate gives no sense of how close it is to the unknown truth. A confidence interval solves the problem: it is an interval, centred on the point estimate, that contains the true parameter with a pre-specified probability (typically 95 %, 99 %).

10.3.1. Mean with known σ.

When σ is known, $\sqrt{n}\,(\bar{X} - \mu)/\sigma$ follows $N(0, 1)$. Writing F for the standard normal c.d.f. and choosing t so that $2F(t) - 1 = 1 - \alpha$, we obtain

Confidence Interval for μ (σ known)

$$\mu \in \left[\overline{x} - t\frac{\sigma}{\sqrt{n}},\ \overline{x} + t\frac{\sigma}{\sqrt{n}}\right] \text{ with probability } 1 - \alpha$$

Standard values (widths expressed in units of $\sigma/\sqrt{n}$):

- 68.3 % confidence: $\bar{x} \pm 1.00\ \sigma/\sqrt{n}$
- 95.0 % confidence: $\bar{x} \pm 1.96\ \sigma/\sqrt{n}$
- 95.4 % confidence: $\bar{x} \pm 2.00\ \sigma/\sqrt{n}$
- 99.7 % confidence: $\bar{x} \pm 3.00\ \sigma/\sqrt{n}$

10.3.2. Mean with unknown σ — Student's t-distribution.

When σ is unknown and is replaced by the unbiased estimate s, the ratio

$$T = \sqrt{n}\frac{\overline{X} - \mu}{s}$$

follows a Student's t-distribution with $n - 1$ degrees of freedom. For $n \geq 30$ this distribution is indistinguishable from N(0, 1); for smaller samples the Student table is used.

Confidence Interval for μ (σ unknown)

$$\mu \in \left[\overline{x} - t(\alpha)\frac{s}{\sqrt{n}},\ \overline{x} + t(\alpha)\frac{s}{\sqrt{n}}\right] \text{ with probability } 1 - \alpha$$

where t(α) is read from the Student table at (n – 1) degrees of freedom and risk α.

Results for our two examples.

Example 1 ($n = 300 \gg 30$, Student ≈ normal):

- 95 % confidence interval: [44.60, 44.86]
- 99.7 % confidence interval: [44.53, 44.92]

Example 2 ($n = 5$, use Student's t with 4 d.o.f.):

- 95 % confidence interval: [11.83, 12.10]
- 99 % confidence interval: [11.75, 12.19]

Note how the smaller sample yields a wider interval for the same confidence level the correct statistical price paid for less information.

10.3.3. Confidence interval for a proportion.

The observed frequency f in a sample of size n estimates the unknown population proportion p. Since $F \approx N(p, p(1-p)/n)$ for large n, and since p is unknown, one substitutes the observed frequency into the variance, obtaining

Confidence Interval for a Proportion

$$p \in \left[f - t_\alpha \sqrt{\frac{f(1-f)}{n}},\ f + t_\alpha \sqrt{\frac{f(1-f)}{n}} \right]$$

Worked example.

Out of 50 parts, 12 are defective, so f = 0.24. The 95 % confidence interval for the defective rate is

$$\left[0.24 \pm 1.96 \times \sqrt{\frac{0.24 \times 0.76}{50}} \right] \approx [0.122, 0.358]$$

Had the same proportion been observed on 1 000 parts, the 95 % interval would have shrunk to [0.226, 0.254] four times narrower, as expected from the $1/\sqrt{n}$ dependence.

11. Hypothesis Testing

A hypothesis test compares an observed statistic with a hypothesis H_0 about the population. The general procedure is:

1. State the null hypothesis H_0 (the "default" hypothesis) and the alternative hypothesis H_1.
2. Fix the risk of type I error $\alpha = p(\text{reject } H_0 \mid H_0 \text{ true})$. Typical values: 1 % or 5 %.

3. Assuming H_0 is true, construct the acceptance region the confidence interval at level $1 - \alpha$ for the test statistic.
4. If the observed value falls in the acceptance region, H_0 is retained; otherwise it is rejected.

Retaining H_0 does not prove it true it only means the data are compatible with it. Rejecting H_0 is a stronger statement, because it is made against an explicit null.

11.1. Two-Sided Test of the Mean

Example.

A manufacturer claims his discs are 45 mm in diameter. A sample of 300 discs gives $\bar{x}$ = 44.73 and s = 1.139. Test H_0: $\mu = 45$ against H_1: $\mu \neq 45$ at risk α = 5 %.

Under H_0, $\sqrt{n}\,(\bar{X} - \mu)/s \approx N(0, 1)$ because n is large. The 95 % acceptance region is

$$\left[45 - 1.96 \times \frac{1.139}{\sqrt{300}}, \quad 45 + 1.96 \times \frac{1.139}{\sqrt{300}}\right] \approx [44.87, 45.13]$$

Since $\bar{x} = 44.73 \notin [44.87, 45.13]$, H_0 is rejected at the 5 % level.

11.2. One-Sided Test of the Mean

Example.

A manufacturer claims his components have a lifetime of at least 1000 h. A sample of 50 components gives $\bar{x}$ = 998 h and s = 15 h. Test H_0: $\mu \geq 1000$ against H_1: $\mu < 1000$ at risk α = 1 %.

Under H_0, the one-sided 99 % acceptance region is

$$\left[\mu - 2.33 \times \frac{s}{\sqrt{50}}, +\infty\right[\subseteq \left[1000 - 2.33 \times \frac{15}{\sqrt{50}}, +\infty\right[\approx [995, +\infty[$$

Since $\bar{x} = 998 \in [995, +\infty[$, we retain H_0 at the 1 % risk: the data are compatible with the manufacturer's claim.

11.3. Test for Comparing Two Means

Two populations are characterised by (μ_1, σ_1^2) and (μ_2, σ_2^2). Samples of sizes n_1 and n_2 give observed means $\bar{x}_1$ and $\bar{x}_2$. Under H_0: $\mu_1 = \mu_2$, and assuming the samples are large (or both populations normal),

$$\frac{\overline{X_1} - \overline{X_2}}{\sqrt{\frac{\sigma_1{}^2}{n_1} + \frac{\sigma_2{}^2}{n_2}}} \sim \mathrm{N}(0,1)$$

A risk α is fixed; one computes the acceptance interval for the difference of means and checks whether the observed difference $\bar{x}_1 - \bar{x}_2$ falls inside.

Example.

For 40 lamps of type 1: $\bar{x}_1 = 1025$ h, $s_1 = 120$ h. For 40 lamps of type 2: $\bar{x}_2 = 1070$ h, $s_2 = 140$ h. Can we state, at $\alpha = 5$ %, that type-1 lamps have a greater mean lifetime than type-2 lamps?

Under H_0: $\mu_1 = \mu_2$, the 95 % acceptance interval for $\bar{x}_1 - \bar{x}_2$ is approximately [−57.1, +57.1]. The observed difference is −45, which lies inside the interval; therefore H_0 is retained the difference is not statistically significant at the 5 % level.

11.4. The Chi-Squared (χ^2) Goodness-of-Fit Test

Whereas the preceding tests focus on a single parameter, the χ^2 test judges whether an entire distribution is compatible with a hypothesised model.

Chi-Squared Test Procedure

Given a statistical series (x_i, n_i) of total size n and a hypothesised distribution {p_i}:

$$\chi_o{}^2 = \sum_{i=1}^{k} \frac{(n_i - np_i)^2}{np_i}$$

Compute the critical value c^2 from the χ^2 table at ν degrees of freedom and risk α: $\nu = k - 1$ in general, or $\nu = k - 2$ when X is hypothesised to follow a binomial, Poisson or normal distribution (one extra degree of freedom is lost to the estimation of the parameter).

If $\chi\_o^2 \le c^2$, H_0 is accepted; otherwise it is rejected at risk α.

Worked example — fairness of a die.

A die is thrown 100 times, with observed counts:

Face d	1	2	3	4	5	6
Count	10	17	11	15	19	28

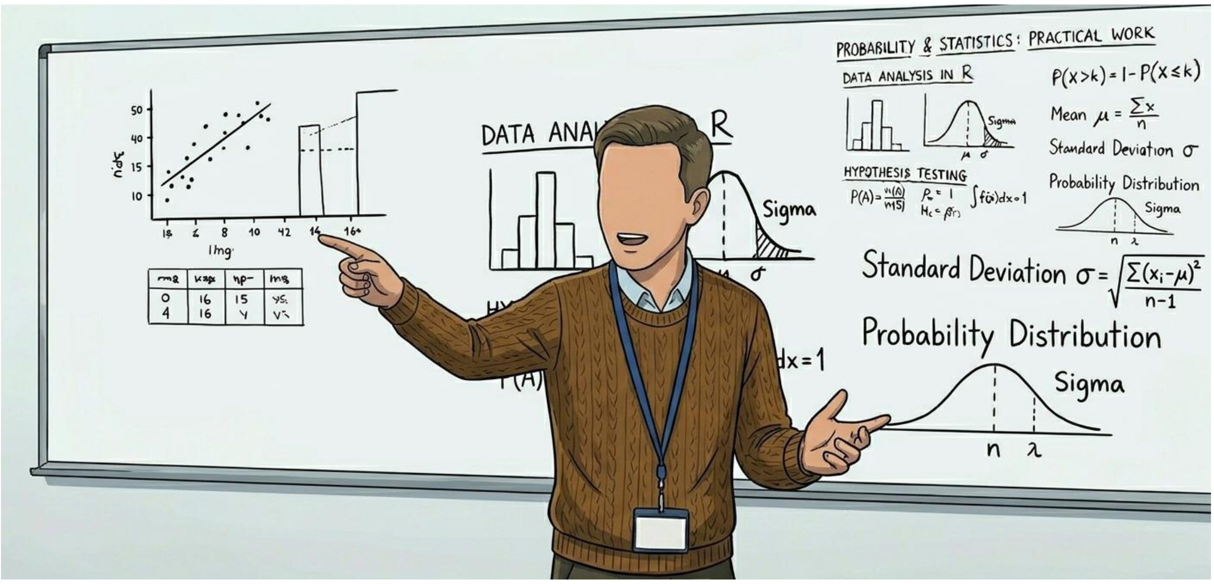
If the die is fair, p_i = 1/6 and n p_i = 100/6. Computing,

$$\chi_o{}^2 = \frac{6}{100}\sum_{i=1}^{6}\left(n_i - \frac{100}{6}\right)^2 = \frac{6}{100} \times \frac{640}{3} = 12.8$$

The hypothesised distribution is uniform, so $\nu = k - 1 = 5$ degrees of freedom. From the χ^2 table, $c^2(5, 5\ \%) = 11.07$ and $c^2(5, 1\ \%) = 15.09$. The observed $\chi^2 = 12.8$ exceeds 11.07 but is smaller than 15.09: the fairness of the die is rejected at the 5 % level but retained at the 1 % level. In practice, one would collect more data to disambiguate.

Two practical points. (i) The χ^2 test is valid only when every expected count n p_i is at least 5 otherwise one pools neighbouring classes. (ii) Accepting H_0 does not prove the postulated law is correct; it only means the data does not contradict it.

supervised practical work



TUTORIAL SERIES No. 03

Combinatorial Analysis

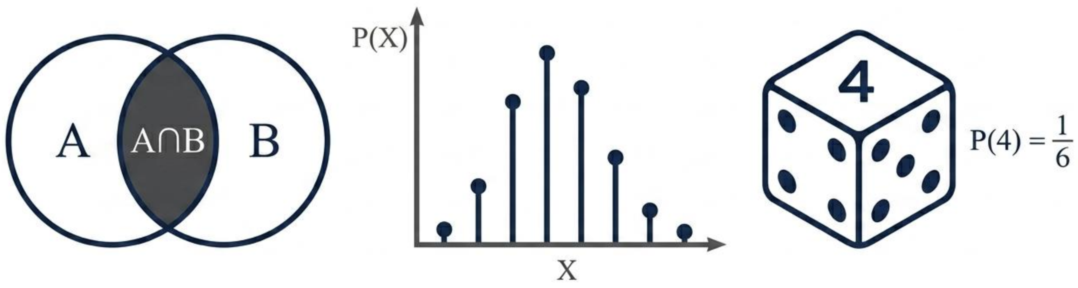
Exercise No. 01:

A specialty coffee shop offers 6 different coffee bean origins: Ethiopian, Colombian, Brazilian, Kenyan, Vietnamese, and Indonesian. A customer wants to create a custom blend using 4 different origins.

1. How many different ways can the customer choose the origins if the order of selection matters (e.g., Ethiopian first, Colombian second, etc.)?

2. How many different ways can the customer choose the origins if the order does not matter (the blend is just a mixture)?

3. If the customer must include Ethiopian beans in every blend, how many different 4-origin blends are possible (without considering order)?



Solution of Exercise No. 01:

1. Arrangements (order matters):

When the order of selection matters, we use the formula for arrangements (permutations of p elements from n):

$$\boldsymbol{A_n^P = \frac{n!}{(n-P)!}}$$

Where n = 6 (total origins) and p = 4 (origins chosen).

$$A_6^4 = \frac{6!}{(6-4)!} = \frac{6 \times 5 \times 4 \times 3 \times 2!}{2!} = 360$$

There are 360 different ways to choose 4 origins when order matters.

2. Combinations (order does not matter):

When the order does not matter, we use the formula for combinations:

$$\boldsymbol{C_n^P = \frac{n!}{(n-P)! \cdot P!}}$$

$$C_6^4 = \frac{6!}{(6-4)! \cdot 4!} = \frac{56!}{2! \cdot 4!} = \frac{6 \times 5 \times 4!}{2 \cdot 4!} = \frac{30}{2} = 15$$

There are 15 different ways to choose 4 origins when order does not matter.

Alternative calculation using the property $\boldsymbol{C_n^P = C_{n-p}^P}$

$$C_6^4 = C_6^2 = \frac{6 \times 5}{2} = 15$$

3. Combinations including Ethiopian beans:

If Ethiopian must be included, we fix Ethiopian as one origin and choose the remaining 3 origins from the other 5 origins:

C_5^3 = *5! / [(5 - 3)! × 3!] = 5! / (2! × 3!) = (5 x 4 × 3!) / (2 ×3!) = 20 / 2 = 10*

There are 10 different 4-origin blends that must include Ethiopian beans.

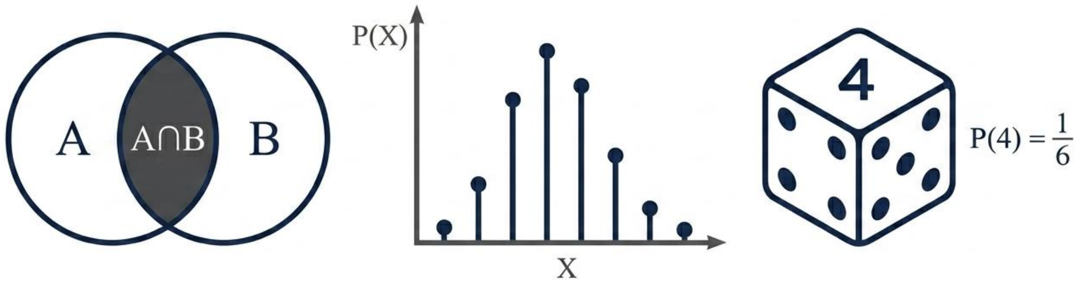
Exercise No. 02:

A university chess tournament has 8 registered players labeled P1, P2, P3, P4, P5, P6, P7, and P8.

1. How many different ways can the 8 players be arranged in a single-elimination tournament bracket?

2. If two of the players are twins (P3 and P4) and must always be placed in opposite halves of the bracket (to avoid meeting before the final), how many valid arrangements exist?

3. The tournament organizer wants to award 1st, 2nd, and 3rd place trophies. How many different podium outcomes are possible?



Solution of Exercise No. 02:

1. Simple permutations of 8 players:

When all players are distinct and order matters (position in the bracket), we use simple permutations:

$$\boldsymbol{P_n = n!}$$

$$P_8 = 8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 24$$

There are 40,320 different ways to arrange the 8 players in the tournament bracket.

2. Permutations with constraint (twins in opposite halves):

Step 1: Place the twins P3 and P4 in opposite halves.

- There are 2 choices for which twin goes to the top half (P3 or P4)

- Once one twin is placed, the other twin's position is fixed (opposite half)

Step 2: Arrange the remaining 6 players in the remaining 6 positions.

$$\textit{Number of arrangements} = 2 \times 6! = 2 \times 720 = 1{,}440$$

There are 1,440 valid arrangements where the twins are in opposite halves.

3. Podium outcomes (permutations of 3 from 8):

For 1st, 2nd, and 3rd place, order matters and we select 3 players from 8:

$$A_8^3 = 8! / (8 - 3)! = 8! / 5! = 8 \times 7 \times 6 = 336$$

There are 336 different possible podium outcomes.

Exercise No. 03:

A cybersecurity expert is analyzing password combinations for a system that uses:

- 26 lowercase letters (a-z)

- 10 digits (0-9)

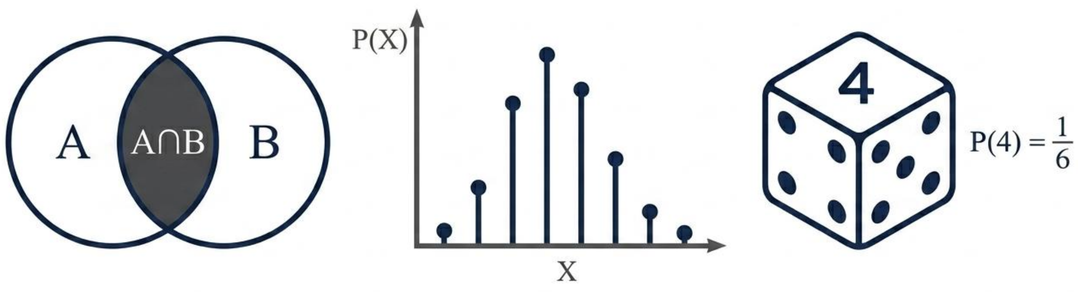
- 8 special characters

1. How many different 8-character passwords can be formed if characters can be repeated and any character type can be used?

2. How many different 8-character passwords can be formed if characters can be repeated but the password must start with a letter and end with a digit?

3. How many different 8-character passwords can be formed if no character can be repeated (all characters must be distinct)?

4. How many different 8-character passwords can be formed if exactly 2 digits, 3 letters, and 3 special characters must be used (no repetition within each category)?



Solution of Exercise No. 03:

Total available characters: 26 + 10 + 8 = 44 characters

1. Passwords with repetition allowed:

Each of the 8 positions can be filled with any of the 44 characters:

$$\text{Total passwords} = 44^8 = 44 \times 44 \times 44 \times 44 \times 44 \times 44 \times 44 \times 44$$

$$44^8 = 14{,}048{,}223{,}625{,}216 = 1.40 \times 10^{13}$$

There are approximately 14 trillion possible 8-character passwords with repetition.

2. Passwords starting with a letter and ending with a digit (with repetition):

- Position 1 (letter): 26 choices

- Positions 2-7 (any character): 44 choices each

- Position 8 (digit): 10 choices

$$\text{Total} = 26 \times 44^6 \times 10 = 26 \times 7{,}256{,}313{,}856 \times 10$$

$$= 26 \times 72{,}563{,}138{,}560 = 1{,}886{,}641{,}602{,}560 = 1.89 \times 10^{12}$$

There are approximately 1.89 trillion such passwords.

3. Passwords with no repetition (all distinct characters):

We select and arrange 8 distinct characters from 44 available characters:

$$A_{44}^{8} = 44! / (44 - 8)! = 44! / 36!$$

$$= 44 \times 43 \times 42 \times 41 \times 40 \times 39 \times 38 \times 37$$

$$= 16{,}158{,}162{,}538{,}560 = 1.62 \times 10^{13}$$

There are approximately 16.2 trillion passwords with all distinct characters.

4. Passwords with exactly 2 digits, 3 letters, and 3 special characters (no repetition within categories):

Step 1: Choose the characters from each category:

- *Choose 2 digits from 10:* C_{10}^{2} = *10! / (8! × 2!) = 45*

- *Choose 3 letters from 26:* C_{26}^{3} = *26! / (23! × 3!) = 2,600*

- *Choose 3 special characters from 8:* C_{8}^{3} = *8! / (5! × 3!) = 56*

Step 2: Arrange the 8 selected characters:

Number of arrangements = 8! = 40,320

Step 3: Total passwords:

$$Total = C_{10}^{2} \times C_{26}^{3} \times C_{8}^{3} \times 8!$$

$$= 45 \text{ x } 2{,}600 \times 56 \times 40{,}320$$

$$= 45 \text{ x } 2{,}600 \times 2{,}257{,}920$$

$$= 117{,}000 \times 2{,}257{,}920$$

$$= 264{,}176{,}640{,}000 = 2.64 \times 10^{11}$$

There are approximately 264 billion such passwords.

Exercise No. 04:

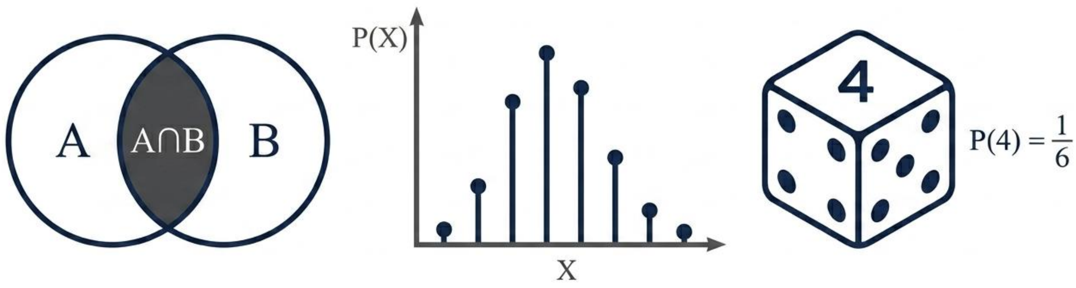
A faculty committee must be formed from 12 professors (7 men and 5 women) and 8 doctoral students (4 men and 4 women). The committee must have exactly 6 members.

1. How many different committees can be formed with no restrictions?

2. How many committees can be formed with exactly 3 professors and 3 doctoral students?

3. How many committees can be formed with at least 2 women?

4. How many committees can be formed with at least 1 professor and at least 1 doctoral student?



Solution of Exercise No. 04:

Total people available: 12 professors + 8 doctoral students = 20 people

1. Committees with no restrictions:

We simply choose 6 people from 20:

$$C_{20}^{6} = 20! / (14! \times 6!) = (20 \times 19 \times 18 \times 17 \times 16 \times 15) / (6 \times 5 \times 4 \times 3 \times 2 \times 1)$$

$$= 27{,}907{,}200 / 720 = 38{,}760$$

There are 38,760 different possible committees.

2. Committees with exactly 3 professors and 3 doctoral students:

We choose 3 professors from 12 AND 3 doctoral students from 8:

$$C_{12}^{3} = 12! / (9! \times 3!) = (12 \times 11 \times 10) / 6 = 220$$

$$C_{8}^{3} = 8! / (5! \times 3!) = (8 \times 7 \times 6) / 6 = 56$$

$$Total\ committees = C(12,3) \times C(8,3) = 220 \times 56 = 12{,}320$$

There are 12,320 committees with exactly 3 professors and 3 doctoral students.

3. Committees with at least 2 women:

Total women available: 5 (professors) + 4 (doctoral students) = 9 women

Total men available: 7 (professors) + 4 (doctoral students) = 11 men

Method: Calculate total committees minus committees with 0 women and 1 woman.

Committees with 0 women (all 6 are men):

$$C_{11}^{6} = = 11! / (5! \times 6!) = 462$$

Committees with 1 woman and 5 men:

$$C_9^1 \times C_{11}^5 = 9 \times 462 = 4{,}158$$

Committees with at least 2 women:

Total - (0 women + 1 woman) = 38,760 - (462 + 4,158)

= 38,760 - 4,620 = 34,140

There are 34,140 committees with at least 2 women.

Alternative verification (direct calculation):

2 women + 4 men: $C_9^2 \times C_{11}^4$ = 36 x 330 = 11,880

3 women + 3 men: $C_9^3 \times C_{11}^3$ = 84 x 165 = 13,860

4 women + 2 men: $C_9^4 \times C_{11}^2$ = 126 x 55 = 6,930

5 women + 1 man: $C_9^4) \times C_{11}^1$ = 126 x 11 = 1,386

6 women + 0 men: $C_9^6 \times C_{11}^0$ = 84 x 1 = 84

Sum = 11,880 + 13,860 + 6,930 + 1,386 + 84 = 34,140 [check mark]

4. Committees with at least 1 professor and at least 1 doctoral student:

Method: Calculate total committees minus all-professor committees and all-doctoral-student committees.

All 6 are professors:

$$C_{12}^6 = 12! / (6! \times 6!) = 924$$

All 6 are doctoral students:

$$C_8^6 = 8! / (6! \times 2!) = 28$$

Committees with at least 1 of each:

Total - (all professors + all doctoral students)

= 38,760 - (924 + 28) = 38,760 - 952 = 37,808

There are 37,808 committees with at least 1 professor and at least 1 doctoral student.

Summary of Combinatorial Formulas:

Concept	Formula	When to Use
Permutation (simple)	P(n) = n!	All elements, order matters, no repetition
Arrangement	A_n^p= n! / (n-p)!	p elements from n, order matters, no repetition
Combination	C_n^p= n! / [(n-p)! x p!]	p elements from n, order does not matter
Permutation with repetition	n! / (n1! x n2! x ...)	All elements, some identical
Arrangement with repetition	n^p	p elements from n, order matters, repetition allowed
Combination with repetition	C_{n+p-1}^p	p elements from n, order does not matter, repetition allowed

TUTORIAL SERIES No. 04

Probability Calculations

Exercise No. 01:

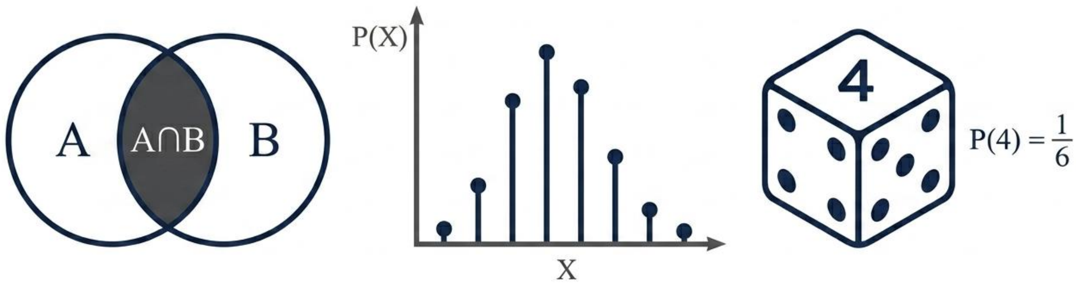
A fair six-sided die is rolled.

- Let A be the event "obtaining an even number".

- Let B be the event "obtaining a number greater than or equal to 4".

1. Calculate P(A U B), the probability of obtaining an even number OR a number greater than or equal to 4.

2. Calculate P(A n B), the probability of obtaining a number that is BOTH even AND greater than or equal to 4.



Solution of Exercise No. 01:

1. Union of Events:

We use the formula for the union of two events:

Omega = {1, 2, 3, 4, 5, 6}

A = {2, 4, 6}

B = {4, 5, 6}

P(A U B) = P(A) + P(B) - P(A ∩ B)

$$P(A) = \frac{Card(A)}{Card(\Omega)} = \frac{number\ of\ favorable\ cases}{number\ of\ possible\ cases}$$

P(A) = 3/6 = 0.5

$$P(B) = \frac{Card(B)}{Card(\Omega)} = \frac{number\ of\ favorable\ cases}{number\ of\ possible\ cases}$$

P(B) = 3/6 = 0.5

Now, we calculate P(A ∩ B), the intersection of A and B.

The intersection is the set of outcomes that are in both A and B:

A ∩ B = {4, 6}

$$P(A \cap B) = \frac{Card\ (A \cap B)}{Card(\Omega)} = \frac{number\ of\ favorable\ cases}{number\ of\ possible\ cases}$$

P(A ∩ B) = 2/6 = 0.3333

Finally, we can calculate P(A U B):

P(A U B) = 3/6 + 3/6 - 2/6 = 4/6 = 0.6667

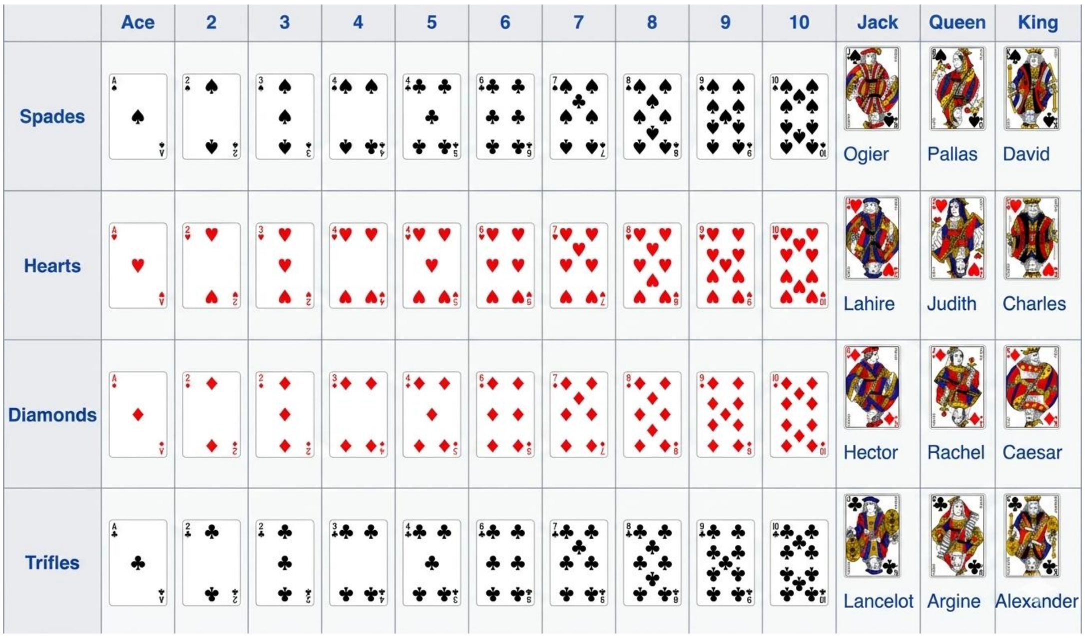
2. Intersection of Events:

As we have already seen:

P(A ∩ B) = 2/6 = 0.3333

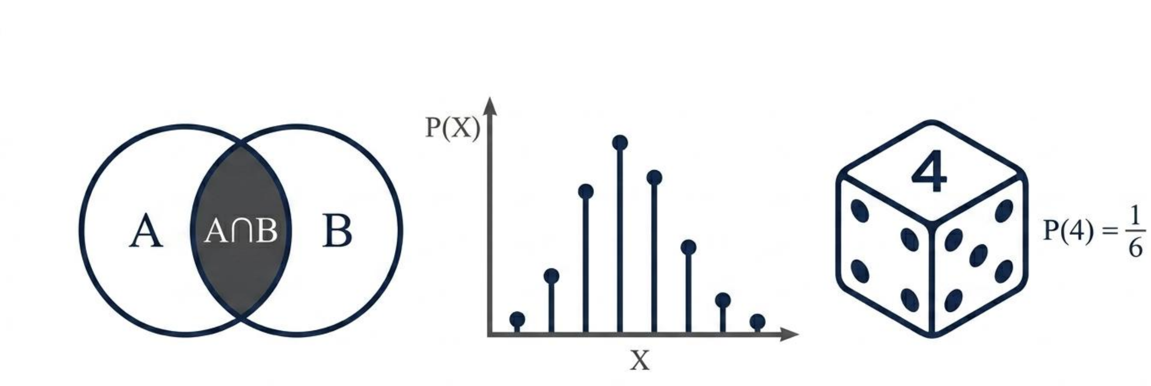
Exercise No. 02:

A card is drawn from a standard 52-card deck.



- Let A be the event "drawing a heart".

1. Calculate the probability of NOT drawing a heart P(A^c^).



Solution of Exercise No. 02:

1. Probability of Event A:

The sample space Omega consists of 52 cards, and there are 13 heart cards in a 52-card deck:

$$P(A) = \frac{Card(A)}{Card(\Omega)} = \frac{number\ of\ favorable\ cases}{number\ of\ possible\ cases}$$

$$P(A) = 13/52 = 0.25$$

2. Probability of the Complement A^c:

The probability of the complement of A (i.e., not drawing a heart) is given by:

$$P(A^c) = P(\bar{A}) = P(\Omega) - P(A)$$

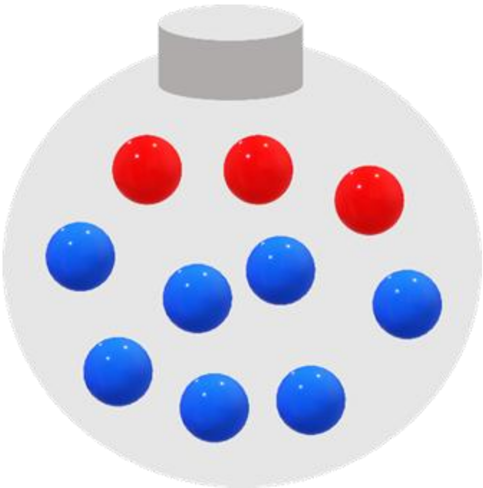
$$P(A^c) = P(\bar{A}) = 1 - 0.25 = 0.75$$

Conclusion:

The probability of not drawing a heart is $P(A^c) = 0.75$

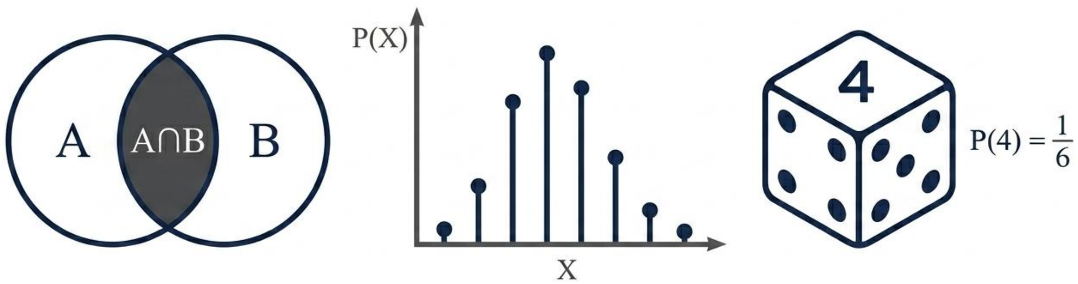
Exercise No. 03:

An urn contains 7 blue balls and 3 red balls (10 balls total).



1. Three balls are drawn simultaneously from the urn. What is the probability of drawing 2 blue balls and 1 red ball?

2. Three balls are drawn successively from the urn without replacement. What is the probability of drawing first a blue ball, then two red balls?



Solution of Exercise No. 03:

1. Let A be the event of drawing 2 blue balls and 1 red ball simultaneously.

$$P(A) = \frac{Card(A)}{Card(\Omega)} = \frac{number\ of\ favorable\ cases}{number\ of\ possible\ cases}$$

$$P(A) = \frac{C_7^2 \cdot C_3^1}{C_{10}^3} = \frac{\frac{7!}{(7-2)! \cdot 2!} \cdot \frac{3!}{(3-1)! \cdot 1!}}{\frac{10!}{(10-3)! \cdot 3!}} = \frac{\frac{7 \cdot 6 \cdot 5!}{5! \cdot 2 \cdot 1} \cdot \frac{3 \cdot 2!}{2!}}{\frac{10!}{7! \cdot 3!}} = \frac{7 \cdot 3 \cdot 3}{10 \cdot 3 \cdot 4} = \frac{63}{120}$$

$$= 0.525$$

The probability of drawing 2 blue balls and 1 red ball simultaneously is P(A) = 0.525

2. Let B be the event of drawing first a blue ball, then two red balls successively.

$$P(B) = \frac{Card(B)}{Card(\Omega)} = \frac{number\ of\ favorable\ cases}{number\ of\ possible\ cases}$$

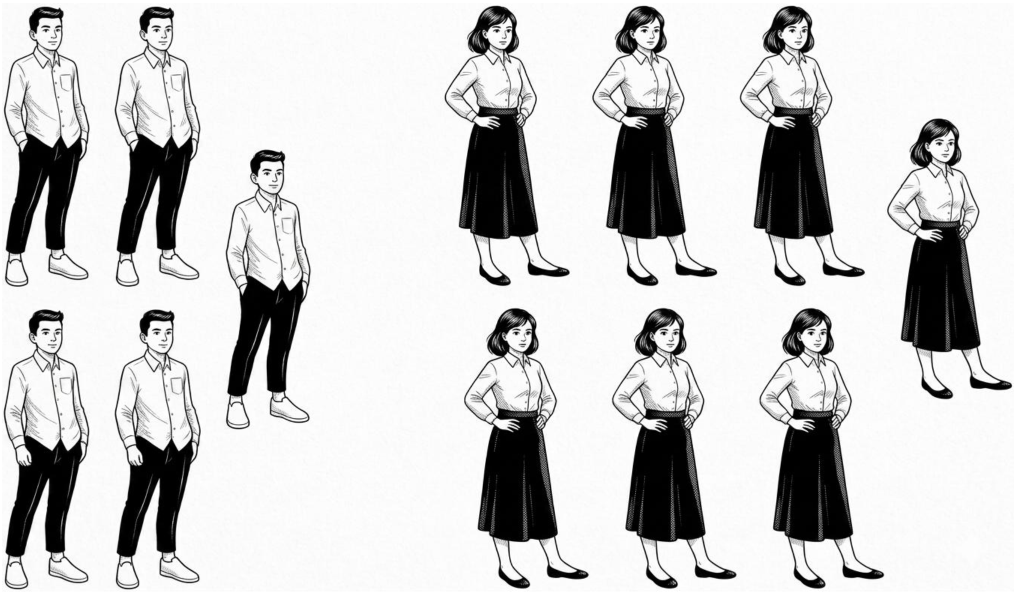
$$P(B) = \frac{A_7^1 \cdot A_3^2}{A_{10}^3} = \frac{\frac{7!}{(7-1)!} \cdot \frac{3!}{(3-2)!}}{\frac{10!}{(10-3)!}} = \frac{\frac{7!}{6!} \cdot \frac{3!}{1!}}{\frac{10!}{7!}} = \frac{7 \cdot 3 \cdot 2}{10 \cdot 9 \cdot 8} = \frac{42}{720} = 0.0583$$

The probability of drawing first a blue ball then two red balls successively is:

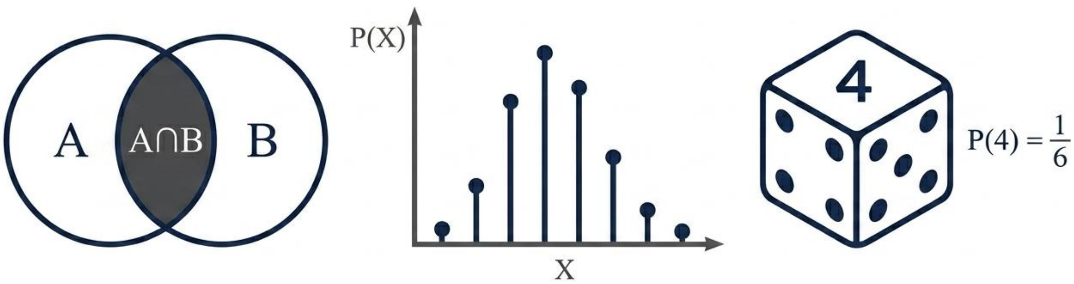
P(B) = 0.0583

Exercise No. 04:

A committee of 3 people is chosen from 5 men and 7 women.



1. What is the probability that the 3 chosen people are 2 men and 1 woman?



Solution of Exercise No. 04:

1.

$$P(A) = \frac{Card(A)}{Card(\Omega)} = \frac{number\ of\ favorable\ cases}{number\ of\ possible\ cases}$$

$$0 \leq P(A) \leq 1$$

$$P(A) = \frac{C_5^2 \cdot C_7^1}{C_{12}^3} = \frac{\frac{5!}{(5-2)! \cdot 2!} \cdot \frac{7!}{(7-1)! \cdot 1!}}{\frac{12!}{(12-3)! \cdot 3!}} = \frac{\frac{5 \cdot 4 \cdot 3!}{3! \cdot 2 \cdot 1} \cdot \frac{7 \cdot 6!}{6! \cdot 1!}}{\frac{12 \cdot 11 \cdot 10 \cdot 9!}{9! \cdot 3!}} = \frac{5 \cdot 2 \cdot 7}{\frac{12 \cdot 11 \cdot 10}{3 \cdot 2}} = \frac{70 \cdot 6}{1320}$$

$$= 0.3181 = 31.82\%$$

There is a 31.82% chance of having a committee composed of 2 men and 1 woman.

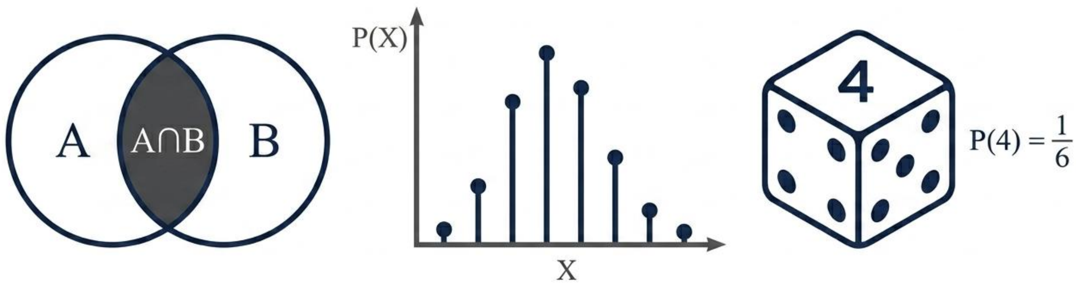
TUTORIAL SERIES No. 05

Conditional Probability - Random Variables

Exercise No. 01: Weighted Tree

A fair coin is tossed 3 times.

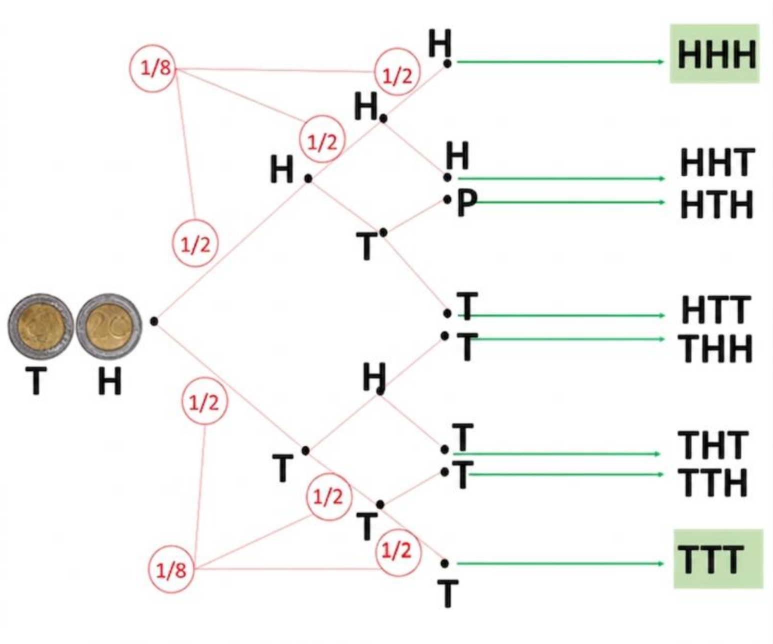
1. **What is the probability of obtaining the same face 3 times?**



Solution of Exercise No. 01:

The possible outcomes when tossing a coin 3 times:

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$



Total number of outcomes: Card(Ω) = 2^3 = 8

Favorable outcomes (same face 3 times):

$$A = \{HHH, TTT\}$$

$$Card(A) = 2$$

The probability of obtaining the same face 3 times is:

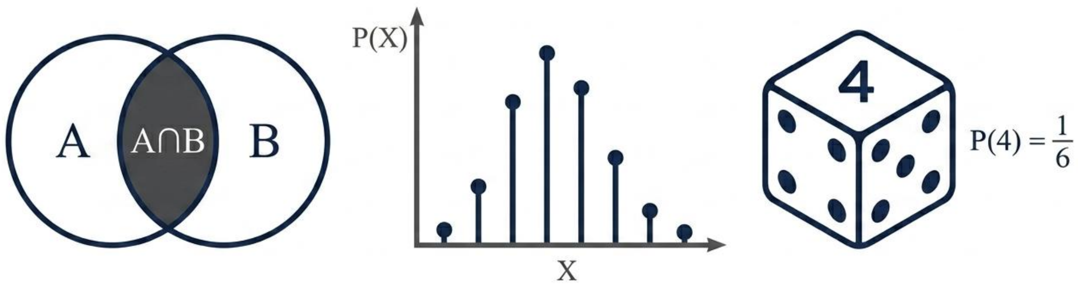
$$P(A) = \frac{Card(A)}{Card\ (\Omega)} = 2/8 = 1/4 = 0.25 = 25\%$$

Exercise No. 02: Conditional Probability

We wish to study the effectiveness of tutoring courses on the success rate of students in the BAC exam.

- 20% of students take tutoring courses
- 95% of students who take tutoring courses pass their BAC
- 6% of students did not pass their BAC

1. Determine the probability that a student who did NOT take tutoring courses also did NOT pass their BAC?



Solution of Exercise No. 02:

Let A be the event "taking tutoring courses".

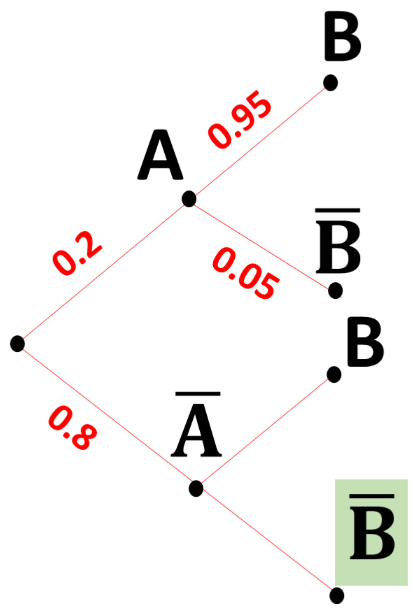
Let B be the event "passing the BAC".

Given data:

$$P(A) = 0.20 \quad \text{(20\% take tutoring)}$$

$$P_{\bar{A}}(\bar{B}) = 0.95 \quad \text{(95\% of tutored students pass)}$$

$$P(\bar{B}) = 0.06 \quad \text{(6\% fail overall)}$$



We need to find $P_{\bar{A}}(\bar{B}) = = P(\bar{B}| \bar{A})$

Using the law of total probability:

$$P(\bar{B}) = P(A) \times P_{\bar{A}}(\bar{B}) + P(\bar{A}) \times P_{\bar{A}}(\bar{B})$$

$$P(\bar{B}) = P(A) \times [1\ P_{A}(\bar{B})] \times \ P(\bar{A}) \times P_{\bar{A}}(\bar{B})$$

Substituting the known values:

$$0.06 = 0.20 \times 0.05 + 0.80 \times P_{\overline{A}}(\overline{B})$$

$$0.06 = 0.01 + 0.80 \times P_{\overline{A}}(\overline{B})$$

$$0.05 = 0.80 \times P_{\overline{A}}(\overline{B})$$

$$P_{\overline{A}}(\overline{B}) = 0.05 / 0.80 = 0.0625 = 6.25\%$$

Therefore:

$$P_{\overline{A}}(\overline{B}) = 6.25\%$$

$$P_{\overline{A}}(B) = 1 - 0.0625 = 93.75\%$$

$$B_A(B) = 95\%$$

Conclusion:

Since $B_A(B) = P_A(B) = 95\%$ and $P_{\overline{A}}(B) == 93.75\%$, the difference is only 1.25%. Therefore, tutoring courses are not very effective in improving BAC success rates.

Exercise No. 03: Independence and Incompatibility

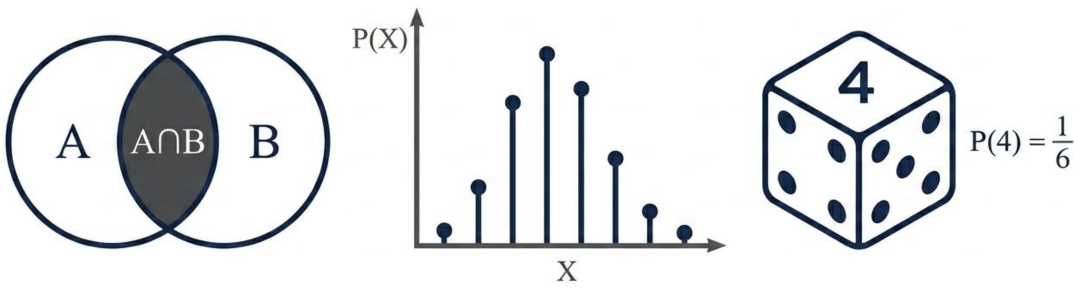
Let A and B be two events such that:

$$P(A) = 1/5$$

$$P(A \cup B) = 1/2$$

1. Suppose that A and B are incompatible. Calculate P(B)?

2. Suppose that A and B are independent. Calculate P(B)?



Solution of Exercise No. 03:

1. A and B are incompatible:

If A and B are incompatible, then A n B = empty set

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$1/2 = 1/5 + P(B) - 0$$

$$P(B) = 1/2 - 1/5 = 5/10 - 2/10 = 3/10 = 0.3 = 30\%$$

2. A and B are independent:

If A and B are independent, then P(A ∩ B) = P(A) x P(B)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{1}{2} = \frac{1}{5} + P(B) - [P(A) \cdot P(B)]$$

$$1/2 = 1/5 + P(B) - [1/5 \,.\, P(B)]$$

$$1/2 = 1/5 + P(B).\,(1 - 1/5)$$

$$1/2 = 1/5 + P(B) \times (4/5)$$

$$1/2 - 1/5 = P(B)\,.\,(4/5)$$

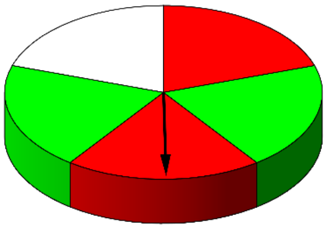
$$3/10 = P(B)\,.\,(4/5)$$

$$P(B) = (3/10) \,/\, (4/5) = (3/10)\,.\,(5/4) = 15/40 = 0.375 = 37.5\%$$

Exercise No. 04: Random Variables

A balanced wheel has 3 colors: green, red, and white. The wheel has 2 green faces, 2 red faces, and 1 white face (5 faces total).

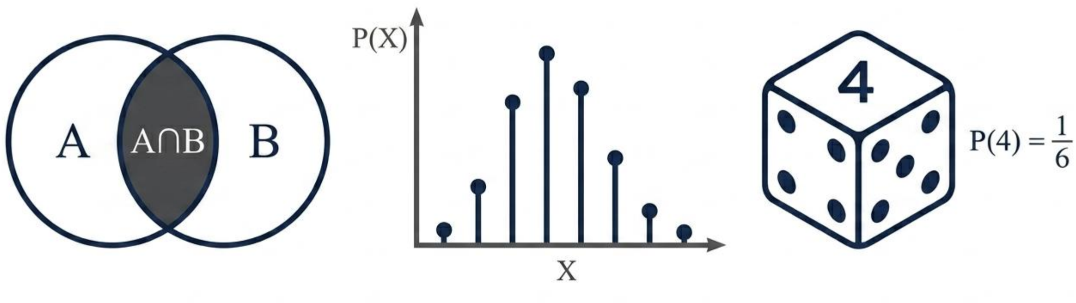
We note the color on which the arrow stops.



- If the face is green -> +600 DA
- If the face is red -> -200 DA
- If the face is white -> -300 DA

1. Give the probability distribution of X?

2. Calculate the mathematical expectation, the variance, and the standard deviation?



Solution of Exercise No. 04:

1. Probability Distribution of X:

The wheel has 5 faces: 2 green, 2 red, 1 white

Gain X	-300 DA	-200 DA	+600 DA	Sum
P(X = Xi)	1/5	2/5	2/5	Sum P(X=Xi) = 1

2. Mathematical Expectation:

$$E(X) = \sum P_i \cdot X_i = P_1 \cdot X_1 + P_2 \cdot X_2 + \cdots + P_n \cdot X_n$$

$$E(X) = \frac{1}{5}(-300) + \frac{2}{5}(-200) + \frac{2}{5}(+600) = 100DA$$

$$E(X) = -60 - 80 + 240 = 100 \text{ DA}$$

This means that on average, one wins 100 DA per game in the long run.

Interpretation of Expectation:

- **Negative** expectation: The average of results will be negative, indicating losses in the long run.

- **Positive** expectation: The average of results will be positive, indicating gains in the long run.

- Zero expectation: Gains and losses compensate each other, there is no net trend, like in a coin toss game where the probability of winning and losing is the same, and the amounts are symmetric.

3. Variance and Standard Deviation:

$$(X) = P_i\left(X_i - E(X)\right)^2 = P_1\left(X_1 - E(X)\right)^2 + P_2\left(X_2 - E(X)\right)^2 \cdots + P_n\left(X_n - E(X)\right)^2$$

$$V(X) = \frac{1}{5}[(-300) - (100)]^2 + \frac{2}{5}[(-200) - (100)]^2 + \frac{2}{5}[(+600) - (100)]^2$$

$$= 168000$$

Standard Deviation:

$$\sigma_X = \sqrt{V(X)}$$

$$\sigma_X = \sqrt{168000} = 409.8780$$

References

[1] M. Morin, Polycopié de Statistiques et Probabilités, Department of Physical Measurements, IUT 1, Grenoble, 1981.

[2] S. Olympieff, Handwritten Lecture Notes, Department of Physical Measurements, IUT 1, Grenoble.

[3] B. Verlant & G. Saint-Pierre, Statistiques et Probabilités, Éditions Foucher, ISBN 2-216-08885-4.

[4] J.-M. Bernabotto, Cours de Statistiques, http://infiniment.free.fr/stats.html.

[5] K. Protassov, Analyse Statistique des Données Expérimentales, EDP Sciences, ISBN 2-86883-590-2.

[6] G. Laget, Probabilités et Statistiques, IUT 1, Grenoble, 2009; available at http://maths.tetras.org.

[7] R. V. Hogg, J. W. McKean & A. T. Craig, Introduction to Mathematical Statistics, 8th ed., Pearson, 2019.

[8] S. M. Ross, Introduction to Probability and Statistics for Engineers and Scientists, 6th ed., Academic Press, 2020.

Part IV

Appendix: Tables of Common Distributions

The four tables that follow are reproduced, with minor typographical adjustments, from the original French lecture notes of G. Laget (IUT 1, Grenoble, 2009). They are reproduced here in English with row and column labels translated. All numerical values are language-independent.

A.1. Poisson Distribution

A.2. Standard Normal Distribution N(0, 1)

For Z ~ N(0, 1), the table gives the cumulative distribution function to four decimal places:

$$F(x) = p(Z \le x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} \, dt$$

The first two digits of x are read on the row and the second decimal on the column; their intersection is F(x). For negative x, use F(−x) = 1 − F(x).

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545

1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Example. F(2.14) is read at row 2.1 column 0.04, giving 0.9838.

A.3. Student's t-Distribution

For T a Student random variable with n degrees of freedom, the table entry at row n and column α is the value t(α) such that

$$p\big(|T| \leq t(\alpha)\big) = 1 - \alpha$$

As $n \to \infty$, the Student distribution converges to N(0, 1); the last row gives the normal values.

n\α	0.9	0.7	0.5	0.2	0.1	0.05	0.02	0.01	0.002	0.001
1	0.1584	0.5095	1.0000	3.0777	6.3138	12.7062	31.8205	63.6567	318.3088	636.6192
2	0.1421	0.4447	0.8165	1.8856	2.9200	4.3027	6.9646	9.9248	22.3271	31.5991
3	0.1366	0.4242	0.7649	1.6377	2.3534	3.1824	4.5407	5.8409	10.2145	12.9240
4	0.1338	0.4142	0.7407	1.5332	2.1318	2.7764	3.7469	4.6041	7.1732	8.6103
5	0.1322	0.4082	0.7267	1.4759	2.0150	2.5706	3.3649	4.0321	5.8934	6.8688
6	0.1311	0.4043	0.7176	1.4398	1.9432	2.4469	3.1427	3.7074	5.2076	5.9588
7	0.1303	0.4015	0.7111	1.4149	1.8946	2.3646	2.9980	3.4995	4.7853	5.4079
8	0.1297	0.3995	0.7064	1.3968	1.8595	2.3060	2.8965	3.3554	4.5008	5.0413
9	0.1293	0.3979	0.7027	1.3830	1.8331	2.2622	2.8214	3.2498	4.2968	4.7809
10	0.1289	0.3966	0.6998	1.3722	1.8125	2.2281	2.7638	3.1693	4.1437	4.5869
11	0.1286	0.3956	0.6974	1.3634	1.7959	2.2010	2.7181	3.1058	4.0247	4.4370
12	0.1283	0.3947	0.6955	1.3562	1.7823	2.1788	2.6810	3.0545	3.9296	4.3178
13	0.1281	0.3940	0.6938	1.3502	1.7709	2.1604	2.6503	3.0123	3.8520	4.2208
14	0.1280	0.3933	0.6924	1.3450	1.7613	2.1448	2.6245	2.9768	3.7874	4.1405
15	0.1278	0.3928	0.6912	1.3406	1.7531	2.1314	2.6025	2.9467	3.7328	4.0728
16	0.1277	0.3923	0.6901	1.3368	1.7459	2.1199	2.5835	2.9208	3.6862	4.0150
17	0.1276	0.3919	0.6892	1.3334	1.7396	2.1098	2.5669	2.8982	3.6458	3.9651
18	0.1274	0.3915	0.6884	1.3304	1.7341	2.1009	2.5524	2.8784	3.6105	3.9216
19	0.1274	0.3912	0.6876	1.3277	1.7291	2.0930	2.5395	2.8609	3.5794	3.8834
20	0.1273	0.3909	0.6870	1.3253	1.7247	2.0860	2.5280	2.8453	3.5518	3.8495
21	0.1272	0.3906	0.6864	1.3232	1.7207	2.0796	2.5176	2.8314	3.5272	3.8193
22	0.1271	0.3904	0.6858	1.3212	1.7171	2.0739	2.5083	2.8188	3.5050	3.7921
23	0.1271	0.3902	0.6853	1.3195	1.7139	2.0687	2.4999	2.8073	3.4850	3.7676
24	0.1270	0.3900	0.6848	1.3178	1.7109	2.0639	2.4922	2.7969	3.4668	3.7454
25	0.1269	0.3898	0.6844	1.3163	1.7081	2.0595	2.4851	2.7874	3.4502	3.7251

26	0.1269	0.3896	0.6840	1.3150	1.7056	2.0555	2.4786	2.7787	3.4350	3.7066
27	0.1268	0.3894	0.6837	1.3137	1.7033	2.0518	2.4727	2.7707	3.4210	3.6896
28	0.1268	0.3893	0.6834	1.3125	1.7011	2.0484	2.4671	2.7633	3.4082	3.6739
29	0.1268	0.3892	0.6830	1.3114	1.6991	2.0452	2.4620	2.7564	3.3962	3.6594
30	0.1267	0.3890	0.6828	1.3104	1.6973	2.0423	2.4573	2.7500	3.3852	3.6460
40	0.1265	0.3881	0.6807	1.3031	1.6839	2.0211	2.4233	2.7045	3.3069	3.5510
50	0.1263	0.3875	0.6794	1.2987	1.6759	2.0086	2.4033	2.6778	3.2614	3.4960
60	0.1262	0.3872	0.6786	1.2958	1.6706	2.0003	2.3901	2.6603	3.2317	3.4602
70	0.1261	0.3869	0.6780	1.2938	1.6669	1.9944	2.3808	2.6479	3.2108	3.4350
80	0.1261	0.3867	0.6776	1.2922	1.6641	1.9901	2.3739	2.6387	3.1953	3.4163
90	0.1260	0.3866	0.6772	1.2910	1.6620	1.9867	2.3685	2.6316	3.1833	3.4019
100	0.1260	0.3864	0.6770	1.2901	1.6602	1.9840	2.3642	2.6259	3.1737	3.3905
200	0.1258	0.3859	0.6757	1.2858	1.6525	1.9719	2.3451	2.6006	3.1315	3.3398
500	0.1257	0.3855	0.6750	1.2832	1.6479	1.9647	2.3338	2.5857	3.1066	3.3101
1000	0.1257	0.3854	0.6747	1.2824	1.6464	1.9623	2.3301	2.5808	3.0984	3.3003
∞	0.1257	0.3853	0.6745	1.2816	1.6449	1.9600	2.3263	2.5758	3.0902	3.2905

A.4. Chi-Squared Distribution (χ^2)

For X^2 following a chi-squared distribution with n degrees of freedom, the table gives the critical value c^2 such that

$$p(X^2 \geq c^2) = \alpha$$

n\α	0.999	0.99	0.975	0.95	0.9	0.5	0.1	0.05	0.025	0.01	0.001
1	0.000	0.000	0.001	0.004	0.016	0.455	2.706	3.841	5.024	6.635	10.828
2	0.002	0.020	0.051	0.103	0.211	1.386	4.605	5.991	7.378	9.210	13.816
3	0.024	0.115	0.216	0.352	0.584	2.366	6.251	7.815	9.348	11.345	16.266
4	0.091	0.297	0.484	0.711	1.064	3.357	7.779	9.488	11.143	13.277	18.467
5	0.210	0.554	0.831	1.145	1.610	4.351	9.236	11.070	12.833	15.086	20.515
6	0.381	0.872	1.237	1.635	2.204	5.348	10.645	12.592	14.449	16.812	22.458
7	0.598	1.239	1.690	2.167	2.833	6.346	12.017	14.067	16.013	18.475	24.322
8	0.857	1.646	2.180	2.733	3.490	7.344	13.362	15.507	17.535	20.090	26.124
9	1.152	2.088	2.700	3.325	4.168	8.343	14.684	16.919	19.023	21.666	27.877
10	1.479	2.558	3.247	3.940	4.865	9.342	15.987	18.307	20.483	23.209	29.588
11	1.834	3.053	3.816	4.575	5.578	10.341	17.275	19.675	21.920	24.725	31.264
12	2.214	3.571	4.404	5.226	6.304	11.340	18.549	21.026	23.337	26.217	32.909
13	2.617	4.107	5.009	5.892	7.042	12.340	19.812	22.362	24.736	27.688	34.528
14	3.041	4.660	5.629	6.571	7.790	13.339	21.064	23.685	26.119	29.141	36.123
15	3.483	5.229	6.262	7.261	8.547	14.339	22.307	24.996	27.488	30.578	37.697
16	3.942	5.812	6.908	7.962	9.312	15.338	23.542	26.296	28.845	32.000	39.252
17	4.416	6.408	7.564	8.672	10.085	16.338	24.769	27.587	30.191	33.409	40.790
18	4.905	7.015	8.231	9.390	10.865	17.338	25.989	28.869	31.526	34.805	42.312
19	5.407	7.633	8.907	10.117	11.651	18.338	27.204	30.144	32.852	36.191	43.820
20	5.921	8.260	9.591	10.851	12.443	19.337	28.412	31.410	34.170	37.566	45.315
21	6.447	8.897	10.283	11.591	13.240	20.337	29.615	32.671	35.479	38.932	46.797
22	6.983	9.542	10.982	12.338	14.041	21.337	30.813	33.924	36.781	40.289	48.268
23	7.529	10.196	11.689	13.091	14.848	22.337	32.007	35.172	38.076	41.638	49.728
24	8.085	10.856	12.401	13.848	15.659	23.337	33.196	36.415	39.364	42.980	51.179
25	8.649	11.524	13.120	14.611	16.473	24.337	34.382	37.652	40.646	44.314	52.620
26	9.222	12.198	13.844	15.379	17.292	25.336	35.563	38.885	41.923	45.642	54.052
27	9.803	12.879	14.573	16.151	18.114	26.336	36.741	40.113	43.195	46.963	55.476
28	10.391	13.565	15.308	16.928	18.939	27.336	37.916	41.337	44.461	48.278	56.892

29	10.986	14.256	16.047	17.708	19.768	28.336	39.087	42.557	45.722	49.588	58.301
30	11.588	14.953	16.791	18.493	20.599	29.336	40.256	43.773	46.979	50.892	59.703
40	17.916	22.164	24.433	26.509	29.051	39.335	51.805	55.758	59.342	63.691	73.402
50	24.674	29.707	32.357	34.764	37.689	49.335	63.167	67.505	71.420	76.154	86.661
60	31.738	37.485	40.482	43.188	46.459	59.335	74.397	79.082	83.298	88.379	99.607
70	39.036	45.442	48.758	51.739	55.329	69.334	85.527	90.531	95.023	100.425	112.317
80	46.520	53.540	57.153	60.391	64.278	79.334	96.578	101.879	106.629	112.329	124.839
90	54.155	61.754	65.647	69.126	73.291	89.334	107.565	113.145	118.136	124.116	137.208
100	61.918	70.065	74.222	77.929	82.358	99.334	118.498	124.342	129.561	135.807	149.449

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