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Numerical traitement of an integral equation of the second kind by the radial basis function (RBF).

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Dedication

I dedicate this work to:

From my lord and enlightenment my way and helped me pray, to the most precious person in this presence, my beloved mother, God rewarded me.

To those who worked hard for me and taught me the meaning of the struggle and brought me to what i am my holy father, God has given me.

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Introduction

The theory and application of integral equation is an important subject within applied mathematics. Integral equations are used as mathematical models for many and varied physical situations, For this result, we need to solve this kind of equations. There are a few numerical and analytical methods to estimate the solution of integral equations such as RBF.

The RBF method in solving integral equations was initially proposed in 2006. See[1].

In this work, we use RBFs collocation method to solve problems in integral equations. For this, in chapter 1, we talk about integral equations, in chapter 2, about the space which we will study this problem. In chapter 3, we explain the RBFs and types of them, RBF interpolation and properties of them, in chapter 4, a numerical example at the Gaussian radial basis functions (RBFs) is applied to solve the one-dimensional Fredholm integral equation of the second kind, is solved and compared with the results of the method.

Chapter 1

Overview of functional Analysis

Definition 1 *Bounded functions.*

Let S be a set and let Y be a Banach space over $\mathbb{K}$ with norm $y \rightarrow |y|$. We define the set of bounded functions (or bounded maps) on S with values in Y by

$$B(S; Y) = \{f : S \rightarrow Y; f(S) \text{ is a bounded subset of } Y\}.$$

Definition 2 *Continuous functions on compact sets.*

If $S \subset \mathbb{R}^n$ is closed and bounded and Y is a Banach space over $\mathbb{K}$ with norm $y \rightarrow |y|$, then

$$C^0(S; Y) \text{ or } (C(S; Y)) = \{f : S \rightarrow Y; f \text{ is continuous on } S\}$$

is the set of continuous functions on S with values in Y . Then $C^0(S; Y)$ is a closed subspace of $B(S; Y)$, and so if it is equipped with the supremum norm

$$\|f\|_{C^0(S)} := \|f\|_{B(S)} = \sup_{x \in S} |f(x)|$$

it becomes a Banach space.

Definition 3 *Continuous functions.*

let Y be a Banach space over $\mathbb{K}$ and let

$$C^0(S; Y) := \{f : S \rightarrow Y; f \text{ is continuous on } S\}.$$

be the set of continuous functions on S with values in Y .

Definition 4 *Support of a function.*

Let $S \subset \mathbb{R}^n$ and let Y be a Banach space. For a map $f : S \rightarrow Y$ we call

$$\text{supp}(f) := \text{clos}(\{x, f(x) = 0\}) \subset \text{clos}(S)$$

the support of f . For $S \subset \mathbb{R}^n$ we then define

$$C_0^0(S; Y) := \{f \in C^0(S; Y) ; \text{supp}(f) \text{ is a compact subset of } S\}.$$

1.1 Metric space

Definition 5 A metric space is a pair (X, d) , where X is a set and a function

$$d : X \times X \rightarrow \mathbb{R} \text{ for all } x, y, z \in X$$

has the following properties:

$$(M1) \ d(x, y) \geq 0 \text{ and } d(x, y) = 0 \Leftrightarrow x = y. \quad (\text{Positivity}),$$

$$(M2) \ d(x, y) = d(y, x). \quad (\text{Symmetry}),$$

$$(M3) \ d(x, y) \leq d(x, z) + d(z, y). \quad (\text{Triangle Inequality}).$$

We then call $d(x, y)$ **the distance** between the points x and y . The map $d : X \times X \rightarrow \mathbb{R}$ is called a metric on X .

Example 6 If we let $d(x, y) = |x - y|$, $(\mathbb{R}^n, d)$ is a metric space. the first two conditions are obviously satisfied, and the third follows from the triangle inequality for vectors the same way as above:

$$d(x, y) = |x - y| = |(x - z) + (z - y)| \leq |x - z| + |z - y| = d(x, z) + d(z, y).$$

1.2 Normed space

Definition 7 Let X be a $\mathbb{K}$ -vector space The pair $(X, \|\cdot\|)$ is called a **normed space** if $\|\cdot\| : X \rightarrow \mathbb{R}$ satisfies the following conditions for $x, y \in X$ and $\alpha \in \mathbb{K}$:

$$(N1) \ \|x\| \geq 0. \text{ and } \|x\| = 0 \Leftrightarrow x = 0 \quad (\text{Positivity}),$$

$$(N3) \ \|\alpha x\| = |\alpha| \|x\|. \quad (\text{homogeneity}),$$

$$(N4) \ \|x + y\| \leq \|x\| + \|y\|. \quad (\text{triangle inequality}).$$

We then say that the map $\|\cdot\| : X \rightarrow \mathbb{R}$ is a norm on X . If a norm $\|\cdot\|_x : X \rightarrow \mathbb{R}$ is fixed on the vector space X , then we also call X a normed space. See [2]

1.3 Sequence space

Definition 8 Let (X, d) be a metric space. A sequence $\{x_n\}$ in X converges to a point $a \in X$ if there for every $\varepsilon \succ 0$ exists an $N \in \mathbb{N}$ such that $d(x_n, a) \prec \varepsilon$ for all $n \geq N$, we write $\lim_{n \rightarrow \infty} x_n = a$ or $x_n \rightarrow a$.

In other word, let (X, d) be a normed space. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in X is said to converge to $a \in X$ if

$$\forall \varepsilon \succ 0 \exists N_0 \in \mathbb{N} : \forall n \in N_0 : \|x_n - a\| \prec \varepsilon.$$

Proposition 9 A sequence $\{x_n\}$ in a metric space is a Cauchy sequence if for every $\varepsilon \succ 0$ there is an $N \in \mathbb{N}$ such that $d(x_m, x_n) \prec \varepsilon$, for all $m, n \geq N$.

Lemma 10 Every convergent sequence is a Cauchy sequence.

Definition 11 A metric space is called complete if all Cauchy sequences converge.

1.4 Uniform convergence

Definition 12 (*Pointwise convergence*)

suppose $\{f_n : n \geq 0\}$ is a sequence of functions defined on an interval I . We say that $f_n(x)$ converges pointwise to the function $f(x)$ on interval I if

$$f_n(x) \rightarrow f(x), \text{ as } n \rightarrow \infty \text{ for each } x \in I$$

Now we come to the definition of **uniform convergence**.

Definition 13 A sequence $\{f_n : n \geq 1\}$ of functions converges uniformly to f on D whenever

$$\forall \varepsilon \succ 0, \exists N(\varepsilon) \in \mathbb{N} \text{ such that } n \geq N(\varepsilon) \Rightarrow |f_n(x) - f(x)| \prec \varepsilon \text{ for every } x \in D.$$

the force of this definition is that the same $N = N(\varepsilon)$ must work for every $x \in D$ simultaneously.

we write this as

$$\|f_n(x) - f(x)\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

then

$$\lim_{n \rightarrow +\infty} f_n = f \quad \text{uniformly on } D.$$

Example 14 $f_n(x) = e^{-nx}$ on $[1, 3]$. We have seen above that $f_n(x)$ converges pointwise to $f(x) = 0$ for each $x \in [1, 3]$. Then we have $|f_n(x) - f(x)| = |f_n(x)|$ and we then have

$$\begin{aligned} \|f_n(x) - f(x)\| &= \sup_{x \in [1, 3]} |f_n(x)| \\ &= \sup_{x \in [1, 3]} |e^{-nx}| \\ &= \sup_{x \in [1, 3]} e^{-nx} \\ &= e^{-n} \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

Thus we have uniform in this case. Note that the last step follow from the observation that e^{-nx} is strictly decreasing for $x \geq 0$ with $n \geq 0$, so that $e^{-n} \geq e^{-nx}$ for all $x \geq 1$.

1.5 Completion

Theorem 15 *Every normed space X can be embedded isometrically in a unique Banach space which contains X as a dense subset, only simpler. The unique Banach space containing X is called the completion of X . See [3]*

1.6 Banach space

Definition 16 *A complete normed space is said to be a Banach space, B .*

Example 17 *The space $C[a, b]$ of continuous functions defined in a closed interval $a \leq x \leq b$ with the (uniform) norm*

$$\|f\|_c = \max_{a \leq x \leq b} |f(x)|$$

and the metric

$$\rho(\phi_1, \phi_2) = \|\phi_1 - \phi_2\| = \max_{a \leq x \leq b} |\phi_1(x) - \phi_2(x)|$$

is a normed and complete space. Indeed, every fundamental (convergent) functional sequence $\{\phi_n(x)\}$ of continuous functions converges to a continuous function in the $C[a, b]$ -metric.

My thesis focus in the space of all continuous space $C([a, b], \|\cdot\|_\infty)$ is a popular a Banach space.

Chapter 2

Integral equations

2.1 Definition

In mathematics, an integral equation is an equation in which an unknown function of one or more variables appears under an integral sign. The general formula of an integral equation is given by:

$$\varphi(x) - \lambda \int_a^x k(x, t) \varphi(t) dt = f(x).$$

where $f(x)$, $k(x, t)$ are given functions, the function $\varphi(x)$ which appears inside and outside the integral sign is the unknown to be determined, λ is a real or complex parameter different from zero ($\lambda \neq 0$), the function $k(x, t)$ is called the kernel of the integral equation.

2.2 Classification of integral equations

An integral equation can be classified as a linear or nonlinear integral equation. The most frequently used integral equations fall under two major classes, namely Volterra and Fredholm integral equations. Of course, we have to classify them as homogeneous or nonhomogeneous.

2.2.1 Volterra integral equations

The general form that is studied is

$$x(t) + \int_a^t K(t, s, x(s))ds = y(t), \quad t \geq a. \quad (2.1)$$

The functions $K(t, s, u)$ and $y(t)$ are given, and $x(t)$ is sought. This is a nonlinear integral equation, and it is in this form that the equation is most commonly applied and solved. Such equations can be thought of as generalizations of the initial value problem for ordinary differential equations. This equation is equivalent to the integral equation

$$x(t) = x_0 + \int_a^t f(s, x(s))ds, \quad t \geq a \quad (2.2)$$

which is a special case of (2.1).

The numerical methods for solving (2.1) are closely related to those for solving the initial value problem (2.2). Volterra integral equations are most commonly studied for functions

x of one variable, as above, but there are examples of Volterra integral equations for functions of more than one variable.

2.2.2 Fredholm integral equations

The general form of such an integral equation is

$$\lambda x(t) - \int_D K(t, s)x(s)ds = y(t), \quad t \in D, \lambda \neq 0 \quad (2.3)$$

with D a closed bounded set in $\mathbb{R}^m$, some $m \geq 1$. The kernel function $K(t, s)$ is assumed to be absolutely integrable.

For $y \neq 0$, we have λ and y given, and we seek x , this is the nonhomogeneous problem. For $y = 0$, equation (2.3) becomes an eigenvalue problem, and we seek both the eigenvalue λ and the eigenfunction x . See [4]

2.3 Existence and Uniqueness of the solution of the integral equation

Kernel operator the equation that interest in the sequence is Freholm's equation

$$\lambda \varphi(x) - \int_{\Omega} K(x, t) \varphi(t) dt = f(t) \quad \Omega \subset \mathbb{R}^n$$

Ω denotes a compact set included in $\mathbb{R}^n$ and let $C(\Omega)$ be the space of continuous functions in Ω , this space is provided with the uniform norm.

$$\|f\|_{\infty} = \max_{x \in \Omega} |f(x)|$$

we will consider equations involving integrals, in the form of an integral operator.

Definition 18 Let $K : \Omega \times \Omega \rightarrow \mathbb{R}$ a continuous function is called integral operator with kernel. $K(.,.)$ the operator defined by

$$k : \varphi \in C(\Omega) \rightarrow k(\varphi) \in C(\Omega)$$

$$k(\varphi)(x) = \int_{\Omega} K(x, t) \varphi(t) dt$$

This operator is continuous of the norm

$$\|k\|_{l(C(\Omega), C(\Omega))} = \max_{x \in \Omega} \int_{\Omega} |K(x, t)| dt$$

Theorem 19 of the geometric series of Neumann

Let V be a banach space, L a bounded linear operator, $L \in l(V)$ and I identical operator. suppose that :

$$\|L\| < 1$$

So, the operator is continuous. $(I - L)$ is invertible in V and $(I - L)^{-1}$ bounded such that

$$\|(I - L)^{-1}\| \leq \frac{1}{1 - \|L\|} \quad (2.4)$$

Proof. Let (M_n) the sequence define by

$$M_n = \sum_{i=0}^n L^i \quad n \geq 0$$

■

We have

$$\| M_{n+p} - M_n \| = \left\| \sum_{i=n+1}^{n+p} L^i \right\| \leq \sum_{i=n+1}^{n+p} \| L^i \| \leq \sum_{i=n+1}^{n+p} \| L \|^i$$

from where

$$\| M_{n+p} - M_n \| \leq \frac{\| L \|^{n+1}}{1 - \| L \|}$$

there for we have

$$\sup_{p \geq 1} \| M_{n+p} - M_n \| \rightarrow 0 \quad \text{quand } n \rightarrow \infty$$

So, the sequence (M_n) is cauchy sequence in the complete space $l(V)$ so there exists $M \in l(V)$ such that.

$$\| M_n - M \| \rightarrow 0$$

We also notice

$$(I - L) M_n = M_n (I - L) = I - L^{n+1}$$

if $n \rightarrow \infty$ we obtain

$$(I - L) M = M (I - L) = I$$

Which allows the operator to say $(I - L)$ is invertible and we have

$$M = (I - L)^{-1} = \lim_{n \rightarrow \infty} \sum_{i=0}^n L^i = \sum_{i=0}^{+\infty} L^i$$

here $M = (I - L)^{-1}$ is the sum of the Neumann series $\sum L^i$. it remains to show (2.4), since

$$\| M_n \| = \left\| \sum_{i=0}^n L^i \right\| \leq \sum_{i=0}^n \| L \|^i \leq \frac{1}{1 - \| L \|}$$

2.3. EXISTENCE AND UNIQUENESS OF THE SOLUTION OF THE INTEGRAL EQUATION 13

So

$$\lim_{n \rightarrow \infty} \|M_n\| = \|M\| \leq \frac{1}{1 - \|L\|}$$

$$\|(I - L)^{-1}\| \leq \frac{1}{1 - \|L\|}$$

Result Let the hypothesis of the theorem (19) for any $f \in V$ the equation

$$(I - L)\varphi = f$$

admits a unique solution in V such that

$$\varphi = (I - L)^{-1} f \quad f \in V$$

Example 20 consider Fredholm's integral equation of the second type.

$$\lambda \varphi(x) - \int_{\Omega} K(x, t) \varphi(t) dt = f(x) \quad \Omega \subset \mathbb{R}^n \quad (2.5)$$

We assume that $\lambda \neq 0$ the function $K : \Omega \times \Omega \rightarrow \mathbb{R}$ is continuous on $\Omega \times \Omega$ of plus $V = C(\Omega)$ equipped with the norm $\|\cdot\|_{\infty}$ symbolically the equation (2.3) is written in the form

$$(\lambda I - k)\varphi = f \quad (2.6)$$

Noticing that.

$$(I - L)\varphi = \frac{1}{\lambda} f$$

with $L = \frac{1}{\lambda} k$

and applying the result of the theorem (19)

$$\|L\| = \frac{1}{|\lambda|} \|k\| < 1$$

or

$$\|k\| = \max_{x \in \Omega} \int_{\Omega} |K(x, t)| dt < |\lambda| \quad (2.7)$$

In this case the operator $(\lambda I - k)^{-1}$ exists and we have

$$\|(\lambda I - k)^{-1}\| \leq \frac{1}{|\lambda| - \|k\|}$$

Thus under the hypothesis (2.7) and for all $f \in C(\Omega)$ the integral Fredholm equation of the second type (2.5) admits a unique solution $\varphi \in C(\Omega)$ and we have

$$\|\varphi\|_{\infty} \leq \|(\lambda I - K)^{-1}\| \cdot \|f\|_{\infty} \leq \frac{\|f\|_{\infty}}{|\lambda| - \|K\|}$$

2.4 numerical resolution of the integral equations

2.4.1 Projection Methods

$X_n; n \geq 1$: finite dimensional approximation spaces

$P_n : X \rightarrow X_n$, bounded projection operators

Approximate $(\lambda - K)x = y$ by

$$(\lambda - P_n K)x_n = P_n y$$

Motivation: Assume

$$x(t) \approx x_n(t) = \sum_{j=1}^n c_{j;n} \varphi_{j;n}(s)$$

with $\{\varphi_{j;n}\}$ a basis for X_n . Calculate

$$r(c_{1;n}, \dots, c_{n;n}) \equiv (\lambda - K)x_n - y$$

Choose $[c_{1;n}, \dots, c_{n;n}]^T$ so that r is small in some sense.

Collocation method:

Set to 0 the values of $r(t)$ at a set of points $\{t_1, \dots, t_n\} \subseteq [a, b]$. P_n is an interpolatory projection.

Error Analysis

Assuming

$$\|K - P_n K\| < \frac{1}{\|(\lambda - K)^{-1}\|}$$

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we have existence of $(\lambda - P_n K)^{-1}$. This follows from

$$\lambda I - P_n K = (\lambda I - K) [I - (\lambda I - K)^{-1} (K - P_n K)]$$

Moreover, for

$$(\lambda - K)x = y, \quad (\lambda - P_n K)x_n = P_n y,$$

we have

$$\begin{aligned} \frac{|\lambda| \|x - P_n x\|}{\|\lambda - P_n K\|} &\leq \|x - x_n\| \\ &\leq |\lambda| \|x - P_n x\| \|(\lambda - P_n K)^{-1}\| \end{aligned}$$

If K is compact and if $P_n z \rightarrow z$ for all $z \in X$, then

$$\|K - P_n K\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

This may still be true even when $\{P_n\}$ is not pointwise convergent to the identity operator on X .

Iterated Projection Solution

Let x_n be the solution of $(\lambda - P_n K)x_n = P_n y$. Define

$$\tilde{x}_n = \frac{1}{\lambda} [y + Kx_n]$$

Then $P_n \tilde{x}_n = x_n$ and

$$(\lambda - K P_n) \tilde{x}_n = y$$

We can show $(\lambda - K P_n)^{-1}$ exists if and only if $(\lambda - P_n K)^{-1}$ exists. Moreover,

$$x - \tilde{x}_n = (\lambda - K P_n)^{-1} K (I - P_n) x$$

The quantity $\tilde{x}_n$ is often called the Sloan iterate. it is always converges more rapidly to x than does x_n . This assumes $P_n z \rightarrow z$ for all $z \in X$.

$$\begin{aligned} K(I - P_n)x &= K(I - P_n)(I - P_n)x \\ \|K(I - P_n)x\| &\leq \|K(I - P_n)(I - P_n)x\| \\ &= \|(I - P_n)K^*\| \|(I - P_n)x\| \end{aligned}$$

Iterated Collocation Solution

With the collocation method, the projection P_n is interpolatory. For $z \in C[a, b]$, $P_n z$ is the unique element in X_n satisfying

$$P_n z(t_i) = z(t_i), \quad i = 1, \dots, n$$

Since $P_n \tilde{x}_n = x_n$, $\tilde{x}_n$ and x_n agree at the collocation node points.

We can sometimes show that $\tilde{x}_n$ converges more rapidly to x than does x_n . Applying $P_n \tilde{x}_n = x_n$ then shows superconvergence of x_n at the node points $\{t_i\}$. See [5]

Chapter 3

Radial basis functions

Radial basis function methods are modern ways to approximate multivariate functions. They have been known, tested and analysed for several years now and many positive properties have been identified. See [6]

3.1 Radial function

Definition 21 A function

$$\phi : \mathbb{R}^d \rightarrow \mathbb{R}$$

is defined as a radial if there is a univariate function $\varphi : [0; \infty) \rightarrow \mathbb{R}$, such that $\phi(x) = \varphi(r)$, where $r = \|x\|$; and $\|\cdot\|$ is some norm on $\mathbb{R}^d$ which is typically the Euclidean norm.

3.2 Radial basis function

Definition 22 Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ be the non-negative half-line and let $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a continuous function with $\psi(0) > 0$. Radial basis functions on $\mathbb{R}^d$ are functions of the form $\psi(\|x - x_j\|)$, in which $x, x_j \in \mathbb{R}^d$, and $\|\cdot\|$ denotes the Euclidean distance between x and x_j . If one chooses N points $\{x_j\}_{j=1}^N$ in $\mathbb{R}^d$ then by custom

$$f(x) = \sum_{j=1}^N \lambda_j \psi(\|x - x_j\|), \lambda_j \in \mathbb{R} \quad (3.1)$$

is called the radial basis functions as well. See [8]

3.3 Types of Radial basis functions

There are many types of RBF, we recall some in the table below

Name of RBF(Abbreviation)	$\phi(r), \quad r \geq 0$	Smoothness
Gaussian (GA)	e^{-cr^2}	Infinite
Multiquadric (MQ)	$\sqrt{c^2 + r^2}$	Infinite
Inverse Multiquadric (IMQ)	$\frac{1}{\sqrt{c^2 + r^2}}$	Infinite
Cubic (CU)	r^3	Piecewise
Thin Plate Spline (TPS)	$r^2 \log(r)$	Piecewise

The radial $r = \|x - x_i\|$ denotes the distance between x and the i -th nodal point x_i ($x \in \mathbb{R}^d$, x_i are centers) and c is the shape parameter which can play a substantial role in the accuracy of approximations. The standard radial basis functions are categorized into two major classes. See[9]

3.3.1 Class1.Infinitely smooth RBFs

These basis functions are infinitely differentiable and heavily depend on the shape parameter c e.g. Hardy multiquadric(MQ), Gaussian(GA), inverse multiquadric(IMQ), and inverse quadric(IQ).

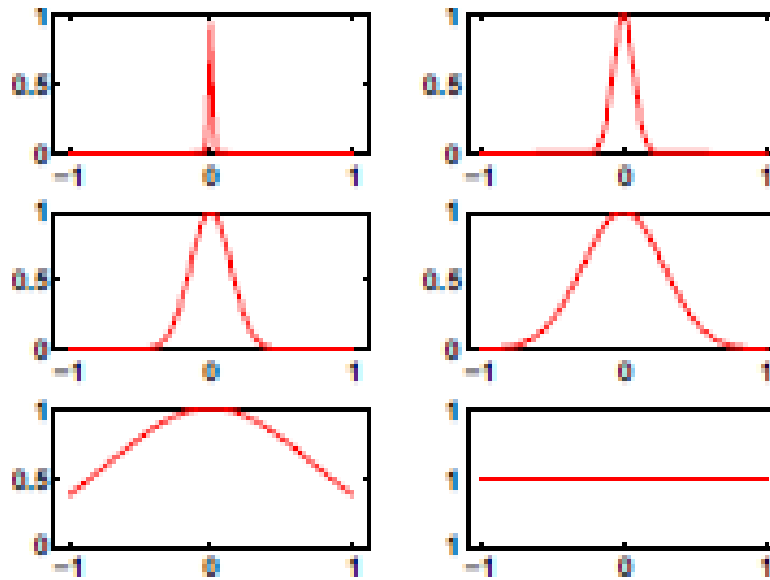
3.3.2 Class2 piecewise smooth RBFs

The basis functions of this category are not infinitely differentiable. These basis functions are shape parameter free and have comparatively less accuracy than the basis functions cited in the Class 1. For example, thin plate spline, etc.

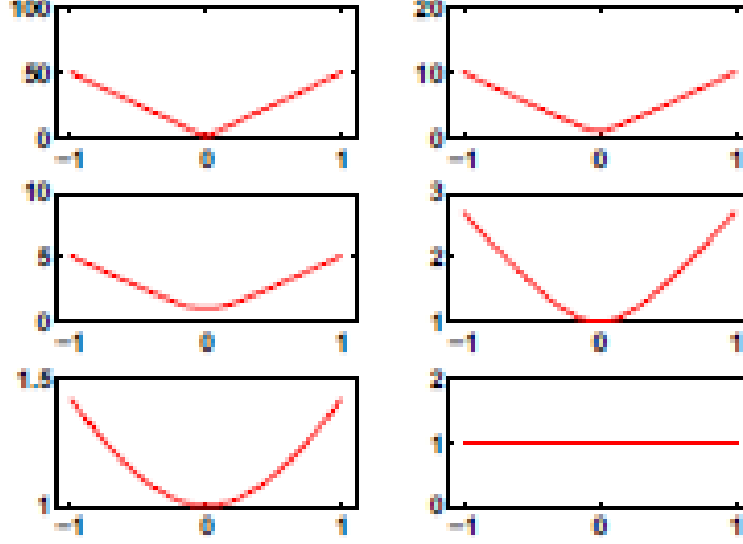
And there exist other class which is Compactly supported piecewise polynomials with free parameter for adjusting the support: Wendland functions.

The shape parameter can play a substantial role in the accuracy of ap-

proximations.



Gaussian RBF used a different shape parameters $\{50, 10, 5, 2.5, 1, 10^{-8}\}$.



Multiquadric RBF used a different shape parameters $\{50, 10, 5, 2.5, 1, 10^{-8}\}$.

3.4 RBF Interpolation

Definition 23 (*Scattered data interpolation problem*).

Let $X \in \mathbb{R}^d$, given the data $(x_i, f(x_i)); i = 1; \dots; N$, where $f : \mathbb{R}^d \rightarrow \mathbb{R}$ the multivariate scattered data interpolation problem is to find a function $S : \mathbb{R} \rightarrow \mathbb{R}$ such that $S(x_i) = f(x_i); i = 1; \dots; N$ where S is called the interpolant to the data

Interpretation of Mairhuber-Curtis

The Mairhuber-Curtis theorem tells us that if we want to have a well-posed multivariate scattered data interpolation problem we can no longer fix in advance the set of basis functions we plan to use for interpolation of arbitrary scattered data. Instead, the basis should depend on the data locations. See [14]

The definition of radial function can be explained as that for a finite set of distinct points $X \subset \mathbb{R}^d$ called the centers, that the function value $\phi(x), x \in X$, solely depends on $\|x\|$ but not on the coordinates of x . The function is radially symmetric about the origin. The reason that makes

radial functions most useful for applications is the fact that the interpolation problem becomes insensitive to the dimension d of the space in which the data set lies. Instead of having to deal with a multivariate function, whose complexity will increase with increasing space dimension d , one can work with the same univariate function for all choices of d .

Given the centers set $X = \{x_i; i = 1, \dots, N\}$, which is a set of distinct points in $\mathbb{R}^d$, and the function values $f(x_i), i = 1, \dots, N$. The RBF approximation to the function f is of the form

$$S(x) = \sum_{i=1}^N \alpha_i \phi(\|x - x_i\|_2). \quad (3.2)$$

The coefficients α_i are determined by enforcing the interpolation condition

$$S(x_i) = f(x_i), \quad (3.3)$$

at the center set X . Enforcing the interpolation condition at N centers results in a $N \times N$ linear system

$$A\alpha = f, \quad (3.4)$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)^t$, $A = (\phi(\|x_i - x_j\|_2)), 1 \leq i, j \leq N$ and $f = s_i$.

The above linear system has to be solved for the coefficients α . The matrix A is called the interpolation matrix or the system matrix and consist of the functions serving as the basis of the approximation space.

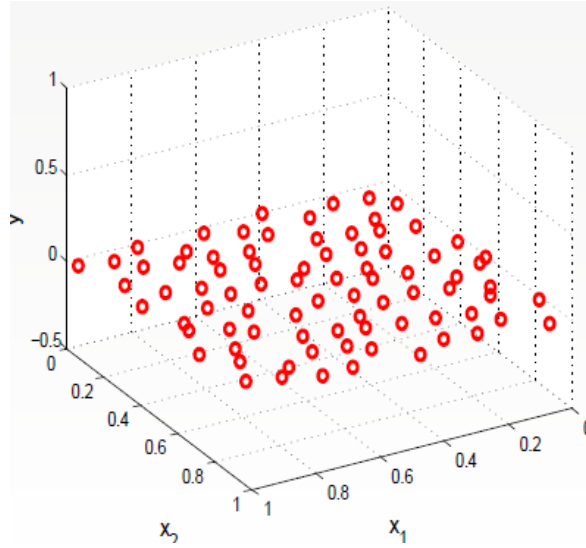


Figure 3.1: Example of 2-D scattered data

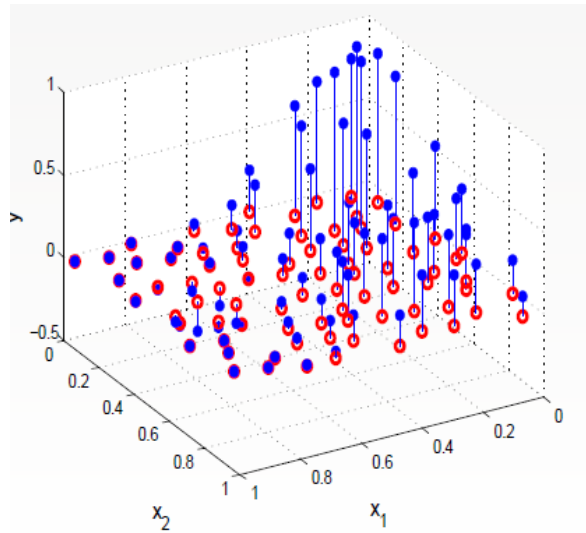


Figure 3.2: Basis function set - one rotated Gaussian is located at each data point.

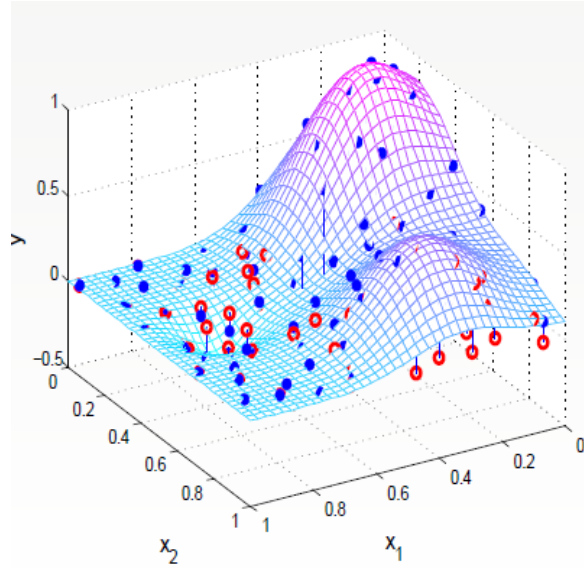


Figure 3.3: The unique linear combination of the Gaussian which agrees with all the provided data.

3.5 Properties of radial function interpolants

- **Invariance under all Euclidean transformations** (i.e, translations, rotations, and reflections).
- **Insensitive to the dimension s** of the space in which the data sites lie. Instead of having to deal with a multivariate function ϕ we can work with the same univariate function φ for all choices of s .
- Spectral accuracy for smooth solutions.

3.6 Positive definite matrices and functions

Definition 24 (*Positive definite matrix*).

A real symmetric $N \times N$ matrix A is called *positive semi-definite* if its associated quadratic form is non-negative, i.e.,

$$\sum_{j=1}^N \sum_{k=1}^N \lambda_j \lambda_k A_{jk} \geq 0,$$

for $\lambda = (\lambda_1, \dots, \lambda_N)^t$. If the only λ vector that turns the above quadratic form into an equality is the zero vector, then A is called positive definite.

The positive definite matrix A has an inverse matrix A^{-1} , because its determinant is not zero.

Definition 25 (*Positive definite function*).

A real valued continuous function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is positive definite on $\mathbb{R}^d$ if and only if it is even and

$$\sum_{j=1}^N \sum_{k=1}^N \lambda_j \lambda_k \phi(x_j - x_k) \geq 0,$$

for any N pairwise different points $\{x_1, \dots, x_N\} \subset \mathbb{R}^d$, and $\lambda = (\lambda_1, \dots, \lambda_N)^t$. The function ϕ is strictly positive definite on $\mathbb{R}^d$ if the only vector that turns the above into the equality is the zero vector. See[11]

3.7 Numerical Example

RBF approximation methods may be either global or local. The global approach uses information from every center in the domain to approximate a function value or derivative at a single point. In contrast, the local method only uses a small subset of the available centers. In this section, the mechanics of the global interpolation method is illustrated in the following example. For this example, the MQ RBF, as defined in the last Table, is used to interpolate a function.

Let

$$f(x) = e^{\sin(\pi x)}$$

be restricted to the interval $[0, 1]$. This function is interpolated using the following three centers that are not evenly spaced: $x_1^c = 0, x_2^c = 0.6$, and $x_3^c = 1$.

The interpolant is evaluated at the five evenly spaced evaluation points $x_1 = 0, x_2 = 0.25, x_3 = 0.5, x_4 = 0.75$, and $x_5 = 1$. For the MQ, a shape parameter is also required. We will

let

$$\varepsilon = 1.25$$

Let $\lambda = (\lambda_1 \ \lambda_2 \ \lambda_3)^t$ be the unknown vector of the expansion coefficients,
let $f = (f(x_1^c) \ f(x_2^c) \ f(x_3^c))^t = (1 \ e^{\sin(0.6\pi)} \ 1)^t$

$$A = \begin{pmatrix} \phi(\|x_1^c - x_1^c\|_2) & \phi(\|x_1^c - x_2^c\|_2) & \phi(\|x_1^c - x_3^c\|_2) \\ \phi(\|x_2^c - x_1^c\|_2) & \phi(\|x_2^c - x_2^c\|_2) & \phi(\|x_2^c - x_3^c\|_2) \\ \phi(\|x_3^c - x_1^c\|_2) & \phi(\|x_3^c - x_2^c\|_2) & \phi(\|x_3^c - x_3^c\|_2) \end{pmatrix}$$

be the 3×3 system matrix. The linear system $A\lambda = f$ can now be solved resulting in the following expansion coefficients:

$$\lambda = (30.8318 \quad -82.8848 \quad 48.3411)^t.$$

The 5×3 evaluation matrix, H , is used to evaluate the interpolant. and let

$$H = \begin{pmatrix} \phi(\|x_1 - x_1^c\|_2) & \phi(\|x_1 - x_2^c\|_2) & \phi(\|x_1 - x_3^c\|_2) \\ \phi(\|x_2 - x_1^c\|_2) & \phi(\|x_2 - x_2^c\|_2) & \phi(\|x_2 - x_3^c\|_2) \\ \phi(\|x_3 - x_1^c\|_2) & \phi(\|x_3 - x_2^c\|_2) & \phi(\|x_3 - x_3^c\|_2) \\ \phi(\|x_4 - x_1^c\|_2) & \phi(\|x_4 - x_2^c\|_2) & \phi(\|x_4 - x_3^c\|_2) \\ \phi(\|x_5 - x_1^c\|_2) & \phi(\|x_5 - x_2^c\|_2) & \phi(\|x_5 - x_3^c\|_2) \end{pmatrix}$$

The next step is to calculate $H\lambda = f_a$. The interpolated values are:

$$f_a = (1 \ 2.1809 \ 2.6528 \ 2.2185 \ 1)^t.$$

The exact solutions, the approximate solutions and the point-wise errors are which we calculate them by Matlab as a follow:

Exact solutions	Aproximate solutions	error
1	1	0.0000000000000000
2.0281	2.18097	0.088631801660779
2.7183	2.6528	0.088631801660779
2.0281	2.2185	0.171280064870869
1	1	0.0000000000000000

Chapter 4

Application

In this chapter we offer steps for solving Fredholm integral equation by Gaussian-RBF method and we applied it for an example.

4.1 Gaussian-RBF for Fredholm integral equation

We consider the following integral equation of Fredholm type

$$y(x) = f(x) + \int_a^b k(x, t) y(t) dt. \quad (4.1)$$

In RBF approximation, the unknown function $y(x)$ is expressed as a combination of RBF i.e: $y(x) \approx \sum_{j=1}^N \lambda_j \phi_j(x)$, then the equation (4.1) become:

$$\sum_{j=1}^N \lambda_j \phi_j(x) = f(x) + \int_a^b k(x, t) \left(\sum_{j=1}^N \lambda_j \phi_j(t) \right) dt \quad (4.2)$$

if $\phi_j(x) = \phi(|x - x_j|) = e^{-c^2|x-x_j|^2}$ then the equation (4.2) become:

$$\sum_{j=1}^N \lambda_j e^{-c^2|x-x_j|^2} = f(x) + \int_a^b k(x, t) \left(\sum_{j=1}^N \lambda_j e^{-c^2|t-x_j|^2} \right) dt \quad (4.3)$$

Furthermore, for any collocation point x_i in $[a, b]$ by using the linear property of integral, we obtain the specific expression of the above equation as

$$\sum_{j=1}^N \lambda_j e^{-c^2|x-x_j|^2} = f(x_i) + \int_a^b k(x_i, t) \left(\sum_{j=1}^N \lambda_j e^{-c^2|t-x_j|^2} \right) dt \quad x_i \text{ collocation points.} \quad (4.4)$$

when choosing M collocation point, hence forming the collocation equations in matrix form as

$$[\Phi - K] [\lambda] = [F], \quad (4.5)$$

where

$$F = (f(x_1), f(x_2), \dots, f(x_M)),$$

and

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$$

The element in matrix Φ and K are defined:

$$\Phi_{ij} = e^{-c^2|x_i-x_j|^2} \quad (4.6)$$

$$K_{ij} = \int_a^b k(x_i, t) e^{-c^2|t-x_j|^2} dt \quad (4.7)$$

The notation x_i and x_j represent correspondingly the collocation point and RBF center, while the element K_{ij} is the definite integral of function $h(t)$ which can be calculated through the following Gauss quadratic formula:

$$\int_{-1}^1 h(\xi) d\xi \approx \sum_{q=1}^Q w_q h(\xi_q) \quad (4.8)$$

The integral range can be transformed with formula $t = (a+b)/2 + (b -$

$a)\xi/2 = p(\xi)$, so the final calculation formula is

$$\begin{aligned} K_{ij} &= \int_a^b k(x_i, t) e^{-c^2|t-x_j|^2} dt \\ &= \frac{b-a}{2} \int_{-1}^1 k(x_i, p(\xi)) e^{-c^2|p(\xi)-x_j|^2} d\xi \\ &\approx \frac{b-a}{2} \sum_{q=1}^Q w_q k(x_i, p(\xi_q)) e^{-c^2|p(\xi_q)-x_j|^2} d\xi \end{aligned}$$

4.1.1 Numerical illustration

Here, we give the result of numerical experiments and compared it with the exact solution to confirm the accuracy and efficiency of the RBF method

Example 26 *see(4.1) In special case, We consider the linear Fredholm equation with exact solution $y(x) = x^2$.*

$$\begin{aligned} \int_0^1 \sqrt{x+ty} (t) dt &= f(x), \\ f(x) &= -\frac{16}{105}x^{7/2} + \frac{16}{105}x^2(x+1)^{3/2} - \frac{8}{35}x(x+1)^{3/2} + \frac{2}{7}(x+1)^{3/2}, \text{ where } \\ 0 \leq x &\leq 1. \end{aligned}$$

We apply RBF-GA $\left(\phi(r) = e^{-(cr)^2}\right)$ with shape parameter $c = 0.2$, the collocation points are the same as the center points $N = 11$.

For Gauss quadratique formula we choose $Q = 20$. In this case the equation (4.5) write as:

$$K\lambda = F$$

$$(k_{ij})(\lambda_i) = (f_i)$$

where

$$k_{ij} = \int_a^b k(x_i, t) e^{-c^2|t-x_j|^2} dt.$$

$$\lambda = (\lambda_1, \dots, \lambda_n)^t.$$

$$F = (f_1, \dots, f_N)^t.$$

we obtain:

$$\begin{pmatrix} 0.7322 & 0.7345 & 0.7364 & 0.7378 & 0.7387 & 0.7391 & 0.7390 & 0.7385 & 0.7375 & 0.7359 & 0.7339 \\ 0.7952 & 0.7977 & 0.7996 & 0.8010 & 0.8020 & 0.8023 & 0.8022 & 0.8015 & 0.8003 & 0.7986 & 0.7964 \\ 0.8522 & 0.8548 & 0.8568 & 0.8583 & 0.8592 & 0.8596 & 0.8593 & 0.8586 & 0.8572 & 0.8553 & 0.8529 \\ 0.9049 & 0.9076 & 0.9097 & 0.9112 & 0.9122 & 0.9125 & 0.9122 & 0.9114 & 0.9099 & 0.9078 & 0.9052 \\ 0.9542 & 0.9571 & 0.9593 & 0.9608 & 0.9618 & 0.9621 & 0.9618 & 0.9608 & 0.9593 & 0.9571 & 0.9542 \\ 1.0009 & 1.0038 & 1.0061 & 1.0077 & 1.0087 & 1.0090 & 1.0086 & 1.0076 & 1.0059 & 1.0036 & 1.0006 \\ 1.0453 & 1.0483 & 1.0507 & 1.0524 & 1.0533 & 1.0536 & 1.0532 & 1.0521 & 1.0503 & 1.0479 & 1.0447 \\ 1.0878 & 1.0909 & 1.0933 & 1.0951 & 1.0960 & 1.0963 & 1.0959 & 1.0947 & 1.0928 & 1.0902 & 1.0869 \\ 1.1285 & 1.1318 & 1.1343 & 1.1360 & 1.1370 & 1.1373 & 1.1368 & 1.1356 & 1.1336 & 1.1309 & 1.1275 \\ 1.1678 & 1.1711 & 1.1737 & 1.1755 & 1.1765 & 1.1768 & 1.1763 & 1.1750 & 1.1729 & 1.1701 & 1.1275 \\ 1.2058 & 1.2092 & 1.2118 & 1.2136 & 1.2147 & 1.2149 & 1.2144 & 1.2130 & 1.2109 & 1.2080 & 1.2043 \end{pmatrix}$$

$$\times \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \\ \lambda_9 \\ \lambda_{10} \\ \lambda_{11} \end{pmatrix} = \begin{pmatrix} 0.3033 \\ 0.3198 \\ 0.3354 \\ 0.3503 \\ 0.3645 \\ 0.3782 \\ 0.3914 \\ 0.4042 \\ 0.4166 \\ 0.4286 \\ 0.4402 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \\ \lambda_9 \\ \lambda_{10} \\ \lambda_{11} \end{pmatrix} = \begin{pmatrix} 0.4895 \\ 0.4919 \\ -3.5796 \\ 3.4090 \\ -1.2215 \\ -0.1488 \\ 2.3039 \\ -2.0041 \\ 0.9861 \\ -1.6515 \\ 0.9250 \end{pmatrix} \times 1.0e + 06.$$

The absolute errors are reported in table below:

x_i	y	y_i	$ y - y_i $
x_1	0.0083	0.0082	$1.75 \times 1.0e - 05$
x_2	0.0331	0.0330	$1.41 \times 1.0e - 05$
x_3	0.0744	0.0744	$9.0 \times 1.0e - 06$
x_4	0.1322	0.1323	$2.85 \times 1.0e - 05$
x_5	0.2066	0.2066	$3.07 \times 1.0e - 05$
x_6	0.2975	0.2975	$1.22 \times 1.0e - 05$
x_7	0.4050	0.4049	$2.02 \times 1.0e - 05$
x_8	0.5289	0.5289	$5.02 \times 1.0e - 05$
x_9	0.6694	0.6694	$5.29 \times 1.0e - 05$
x_{10}	0.8264	0.8265	$4.2 \times 1.0e - 03$
x_{11}	1.0000	1.0002	$1.592 \times 1.0e - 04$

The maximum error is $1.592 \times 1.0e - 04$.

Comments This table illustrates, numerical results show simplicity and very good accuracy of the method this errors decreases by increasing the number of collocation point N for GA- RBFs.

Conclusion

This work describes using RBF to solve integral equation; the proposed method is a collocation method.

According to the numerical results that we have obtain, we infer that RBFs collocation method is a powerful tool for solving the integral equations, and from the example which is presented to show the efficiency of the method.

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Abstract

In this thesis, we apply RBF collocation method to solve the linear one dimensional integral equations of Fredholm.

To illustrate the method some examples are provided.

The results show the efficiency and accuracy of the method.

Key words

Integral equations, radial basis functions, Fredholm.

Résumé

Dans ce mémoire, on applique la méthode de collocation en RBF pour résoudre les équations intégrales linéaires de Fredholm.

Pour illustrer la méthode quelques exemples sont présentés. Les résultats montrent l'efficacité et la précision de la méthode.

Mots clés

Equations intégrales, fonctions de base radiale, Fredholm.

الخلاصة

في هذه المذكرة، حاولنا تطبيق طريقة التجميع للدوال القطرية الأساسية في المعادلات التكاملية الخطية نوع فريدهولم. وقد قدمنا أمثلة لعرض هذه الطريقة.

النتائج اثبتت مدى جدية و دقة الطريقة.

كلمات مفتاحية

معادلات تكاملية، الدوال القطرية الأساسية، فريدهولم.