PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA

MINISTRY OF HIGHER EDUCATION AND

SCIENTIFIC RESEARCH

Mohamed Boudiaf -M'sila University

Faculty of Mathematics and Computer Science

Department of Mathematics

N° = ...

T H E S I S

In order to obtain third-cycle Doctorate degree

Specialty : Mathematics

Option : Numerical analysis, PDEs and applications

Theme

Numerical solution of nonlinear integral equations

Presented by Abderrazak MEHELLOU

On 19/04/2026, in front of the jury composed of :

B. GAGUI	M.C.A.	President	Univ. of M'sila
A. KHIRANI	M.C.A.	Supervisor	Univ. of M'sila
M. NADIR	Prof.	Co-supervisor	Univ. of M'sila
A. RAHMOUNE	Prof.	Examiner	Univ. of Setif
R. ZEGHDANE	M.C.A.	Examiner	Univ. of Bordj Bou Arreridj
N. DJAIDJA	M.C.A.	Examiner	Univ. of M'sila

University year 2025 - 2026

Abstract

Many problems are related to physics, engineering, biology, and other applied sciences, leading to modeling phenomena in the form of integral equations. The aim of this thesis is to study nonlinear integral equations: classifications, existence and uniqueness of the solution using the Banach fixed point, and obtaining numerical solutions by using the successive approximations method, Nyström method, projection methods, and specifically highlight the wavelet method. All the methods used and examined for the numerical solution of nonlinear integral equations. Moreover, this method reduces the integral equations to systems of nonlinear algebraic equations that can be solved by Newton's method. We also estimate the error bounds and convergence of the presented methods. Using Mathematica 10.3 software, several numerical examples are mentioned to demonstrate its effectiveness and accuracy in solving nonlinear integral equations, specifically the type of nonlinear quadratic integral equations.

Key words: Nonlinear integral equations, Banach fixed-point, successive approximation method, projection method, Nyström method, wavelets method, convergence analysis.

Résumé

De nombreux problèmes sont liés à la physique, à l'ingénierie, à la biologie et à d'autres sciences appliquées. conduisent à la modélisation des phénomènes sous forme d'équations intégrales. L'objectif de cette thèse est d'étudier les équations intégrales non linéaires : classifications, existence et unicité de la solution en utilisant le point fixe de Banach, et obtenir des solutions numériques en utilisant la méthode des approximations successives, la méthode de Nyström, les méthodes de projection, et en particulier mettre en évidence la méthode des ondelettes. Toutes les méthodes utilisées et examinées pour la solution numérique des équations intégrales non linéaires. De plus, cette méthode réduit les équations intégrales à des systèmes d'équations algébriques non linéaires qui peuvent être résolus par la méthode de Newton. Nous estimons également les bornes d'erreur et la convergence des méthodes présentées. En utilisant le logiciel Mathematica 10.3, plusieurs exemples numériques sont mentionnés afin de démontrer son efficacité et sa précision dans la résolution des équations intégrales non linéaires, en particulier le type d'équations intégrales quadratiques non linéaires.

mots clés: Équations intégrales non linéaires, point fixe de Banach, méthode des approximations successives, méthode de projection, méthode de Nyström, méthode des wavelets, analyse de convergence.

ملخص

تؤدي الكثير من المشكلات الفيزيائية والهندسية والحيوية، وغيرها من العلوم التطبيقية، إلى نمذجة الظواهر في شكل معادلات تكاملية. تهدف هذه الأطروحة إلى دراسة المعادلات التكاملية غير الخطية من حيث تصنيفاتها، والبحث عن وجود ووحدانيته الحل باستخدام نظرية "نقطة الصامدة لبناخ ". كما تُعنى الأطروحة بإيجاد الحلول العددية باستخدام عدة طرق، منها: التقريبات المتعاقبة، وطريقة نايستروم، وطرق الإسقاط، مع التركيز بشكل خاص على "طريقة المويجات" (Wavelets). جرى فحص واختبار هذه الطرق للحل العددي للمعادلات التكاملية غير الخطية، حيث تعمل على تحويلها إلى أنظمة من المعادلات الجبرية غير الخطية التي يمكن حلها باستخدام "طريقة نيوتن". بالإضافة إلى ذلك، قمنا بتقدير حدود الخطأ ودراسة تقارب الطرق المقترحة. ولإثبات كفاءة ودقة هذه الطرق، تم عرض عدة أمثلة عددية باستخدام برنامج (Woflram Mathematica 10.3)، خاصة في معالجة المعادلات التكاملية غير الخطية من النوع التربيعي.

الكلمات المفتاحية: المعادلات التكاملية غير الخطية، نظرية النقطة الصامدة لبناخ، طريقة التقريبات المتتالية، طريقة الإسقاط، طريقة نستورم، طريقة الموجات، تحليل التقارب.

Dedication

I thank my parents, my wife, my children, my brothers and my sisters for their continued support and encouragement.

I dedicate this thesis to them.

Acknowledgments

We begin by expressing our deepest gratitude to Allah, the Most Merciful, for His blessings throughout this endeavor. We would also like to extend our sincere thanks to all who have contributed to the completion of this work.

We are profoundly indebted to our supervisors, Dr. Amina Khirani and Pr. Mostefa Nadir, for their invaluable guidance, unwavering encouragement, and immense patience during the preparation of this thesis. We are also thankful for their introduction to this field of study.

Furthermore, we wish to express our appreciation to the members of our thesis committee: Professors B. Gagui, A. Rahmoune, N. Dlaidja, and R. Zeghdane.

We also extend our thanks to all the respected professors in the Department of Mathematics at M'sila University for their knowledge and support.

Finally, our heartfelt gratitude goes to our families for their unconditional love, patience, and endless encouragement throughout this journey.

Contents

List of Tables

List of Figures

List of symbols and notation

X, Y, E	Metric spaces, Banach or Hilbert spaces.
$\overline{\Omega}$	Closure of Ω.
$B_R(\mathscr{Y}, X)$	Open ball $\{\mathscr{S} \in X : \lvert\mathscr{Y} - \mathscr{S}\rvert < R\}$.
$\overline{B}_R(\mathscr{Y}, X)$	Closed ball $\{\mathscr{S} \in X : \lvert\mathscr{Y} - \mathscr{S}\rvert \leq R\}$.
$\mathbb{R}^n$	Set of n-tuples $x = (x_1, x_2, \ldots, x_n)$.
$\langle x, y\rangle$	Euclidian inner product in $\mathbb{R}^n$, $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$.
$C^k\left(\overline{\Omega}\right)$	Stands for $C^k\left(\overline{\Omega}; \mathbb{R}\right)$.
$C^k\left(\Omega; \mathbb{R}^n\right)$	Space of k-times continuously differentiable functions $\mathscr{Y} : \Omega \longrightarrow \mathbb{R}^n$, ($\Omega \subset \mathbb{R}^n$ open).
$L^2\left(\Omega; \mathbb{R}^n\right)$	Set of measurable functions $\mathscr{Y} : \Omega \longrightarrow \mathbb{R}^n$, with $\int_\Omega \lvert\mathscr{Y}(x)\rvert^2 dx < \infty$, ($\Omega \subset \mathbb{R}^n$).
$\lVert\mathscr{Y}\rVert_\infty$	$\sup_{x \in \overline{\Omega}} \lvert\mathscr{Y}(x)\rvert$, ($\Omega \subset \mathbb{R}^n$ bounded open, $\mathscr{Y} \in C^k\left(\overline{\Omega}; \mathbb{R}^n\right)$).
$\{\xi_{h,r}\}$	Wavelets family with dilation parameter h and translation parameter r.
I_N	Lagrange interpolation operator.
$\mathfrak{K}$	Integral operator.
$\mathscr{O}(\cdot)$	Order of convergence.
NQVIE	Nonlinear quadratic Volterra integral equation.
NQFIE	Nonlinear quadratic Fredholm integral equation.
NVIE	Nonlinear Volterra integral equation.
NFIE	Nonlinear Fredholm integral equation.
IEs	Integral equations.

Introduction

One of the most practical topics in both pure and applied mathematics is the study of integral equations. As integral equations, many initial and boundary value issues related to ordinary differential equations (ODE) and partial differential equations (PDE) can be resolved analytically or numerically. The theory of Fourier integral introduced us to integral equations. Abel obtained another integral equation in 1826. In actuality, the Swedish mathematician Fredholm (1900) and the Italian mathematician Volterra (1896) are credited with founding the theory of integral equations. Different real-world occurrences are described by different types of integral equations. For example, they are used to model many scientific and technical problems, including gas kinetic theory, traffic theory, neutron transport theory, etc. [21, 7, 40]. The existence, uniqueness, positive solutions, monotonic solutions, and numerical techniques for approximating the solution of this kind of integral equation have been the focus of scholarly attention in recent years [15, 32, 39, 20]. These kinds of equations are typically challenging and occasionally impossible to solve. Therefore, we use numerical methods to concentrate on thesis.

Several approaches have been put forth recently to determine the approximate solutions of integral equations. For example, nonlinear multi-dimensional Volterra integral equations are solved using a mix of linear barycentric rational interpolation and the quasilinearization method [68]. A family of functional Volterra integral equations is solved iteratively using a methodology based on the quasilinearization method and the Jacobi-Galerkin

method [75]. These kinds of equations are frequently solved using spectral methods with orthogonal polynomials. As an illustration, consider the Legendre-collocation approach to the solution of second-kind Volterra integral equation systems [65]. the Muntz Legendre Wavelets Method (MLWM) [67], Homotopy Perturbation Methods (HPM) [4], Homotopy Analysis Method (HAM) [13], Fixed Point Method (FPM) [50], and a number of other techniques [12, 32]. High accuracy and computing efficiency are still difficult to achieve, though, particularly when working with intricate integral kernels and nonlinearities.

Our research in this thesis is concentrated on nonlinear integral equations. A nonlinear integral equation can be expressed in the following generic form:

$$\Phi(\mathscr{Y})(x) = \lambda \int_{\Omega} \Psi(x, t, \mathscr{Y}(t))\, dt \quad x, t \in \Omega,$$

where

Φ is a measurable function define on $\mathbb{R}^n$.

$\Psi(x, t, \mathscr{Y}(t))$ a measurable function define on $\Omega \times \Omega \times \mathbb{R}^n$ named by the kernel and the unknown function $\mathscr{Y}(t)$ is under the integral.

λ a non zero scalar from $\mathbb{R}$ or $\mathbb{C}$.

Ω the domain of the integration.

Therefore, studying integral equations from a numerical perspective is the goal of thesis. As a result, we start the first chapter with fundamental ideas and reminders. We discuss the existence, uniqueness, and generality of integral equations in the second chapter. The successive approximation approach, projection methods, Nyström methods, and wavelets method are covered in the third and fourth chapters. The convergence analysis is then conducted, and examples are provided using Mathematica 10.3 software in the last chapter. We wrap up by discussing the numerical results that were achieved.

Chapter 1

Recalls and basic concepts

In addition to presenting definitions and conclusions drawn from the literature, this chapter introduces the notational framework adopted for the remainder of the thesis.

1.1 Recall in functional analysis

1.1.1 Functional spaces

Definition 1.1.1 ***(inner product)***

Let X be a real linear space. A map $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{R}$ is an inner product if for all $x, y, z \in X$, and $\alpha \in \mathbb{R}$:

1. Linearity in the first argument: $\langle \alpha x + y, z\rangle = \alpha\langle x, z\rangle + \langle y, z\rangle$

2. Symmetry: $\langle y, x\rangle = \langle x, y\rangle$

3. Positive-definiteness: $\langle x, x\rangle > 0$ for all non-zero $x \in X$.

Example 1.1

The mapping that defined on continuously differentiable functions over $[0,1]$ with the form

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt + \int_0^1 f'(t)g'(t)dt$$

is an inner product.

Definition 1.1.2 ***(compact)***

A subset U of a normed space X is said to be compact if and only if

$$\forall V_j, j \in J(open); U \subset \bigcup_{j\in J} V_j, \exists V_{j(k)}, j(k) = 1,2,...,n : U \subset \bigcup_{k=1}^{n} V_j(k)$$

Definition 1.1.3 ***(relatively compact)*** *A subset S of a normed space$X$ is said to be relatively compact if its closure $\overline{S}$ is compact.*

Definition 1.1.4 ***(collectively compact)***

Let X and Y be normed spaces, B the closed unit ball in X and let Σ be a family of bounded operators from X into Y. Then, Σ is said to be collectively compact if and only if the set

$$\Sigma(B) = \{Ku,\ K \in \Sigma,\ u \in B\}$$

is relatively compact in Y.

Theorem 1.1.1 ***(Arzelà -Ascoli)***

Two conditions must be met for a set $S \subset C(\Omega)$ to be considered relatively compact in $C(\Omega)$:

(i) S is uniformly bounded subset.

(ii) S is equicontinuous, i.e., $\forall \varepsilon > 0,\ \exists \delta > 0$ such that, for all

$x_1, x_2 \in \Omega, |x_1 - x_2| < \delta$ implies $|\mathscr{Y}(x_1) - \mathscr{Y}(x_2)| < \varepsilon$ for all $\mathscr{Y} \in S$.

Proof 1.1.1 *see [11]*

1.1.2 Notions of operators

Linear operators

Definition 1.1.5 ***(Linear operators)***

An operator A is a function or a mapping that transforms elements from vector space X to elements in another vector space Y, which can be the same space. A is linear operator if

$$A(\mathscr{Y}_1 + \mathscr{Y}_2) = A\mathscr{Y}_1 + A\mathscr{Y}_2, \quad A(\alpha\mathscr{Y}_1) = \alpha A\mathscr{Y}_1, \quad \forall \mathscr{Y}_1, \mathscr{Y}_2 \in X, \text{and for all scalar } \alpha \in \mathbb{R}.$$

In the following, we assume that X and Y are normed spaces.

Proposition 1.1.1 ***(Continuous linear operators)***

An operator $A : X \to Y$ is said to be continuous if $\|\mathscr{Y}_n - \mathscr{Y}\|_X \to 0$ implies $\|A\mathscr{Y}_n - A\mathscr{Y}\|_Y \to 0$.

Proposition 1.1.2 ***(Bounded linear operators)***

A linear operator $A : X \to Y$ is continuous if and only if it is bounded, i.e., if there exists a non-negative real constant $c \geq 0$ such that for all $\mathscr{Y} \in X$, the following condition holds:

$$\|A\mathscr{Y}\|_Y \leq c\|\mathscr{Y}\|_X.$$

The norm of A is defined as the smallest such constant c, and it is given by

$$\|A\| = \sup\{\|A\mathscr{Y}\|_Y : \mathscr{Y} \in X, \|\mathscr{Y}\|_X \leq 1\}.$$

Definition 1.1.6 ***convergence of a sequence of operators***

A sequence of bounded linear operators $(A_n) : X \to Y$ *is said to converge (or converge strongly) if the sequence* $(A_n \mathscr{Y})$ *converge in* Y *for any* $\mathscr{Y} \in X$.

Theorem 1.1.2 ***(Banach-Steinhaus)***

Let $(A_n) : X \to Y$ *be a sequence of bounded linear operators.* (A_n) *converges if and only if it satisfies:*

(i) For every n*, there exists a positive constant* c *such that* $\|A_n\| \leq c$*;*

(ii) There exists a dense subset $S \subset X$ *such that, for every* $x \in S$*, the sequence* $(A_n x)$ *converges in* Y.

Also, under conditions (i) and (ii), the limit operator $A : X \to Y$*, defined by* $A\mathscr{Y} = \lim_{n\to\infty} A_n \mathscr{Y}$ *is bounded and linear.*

Adjoint operators

Definition 1.1.7 *[19] Let* H *be Hilbert space, and let* $A : H \to H$ *be a bounded linear operator. The adjoint of* A*, denoted by* A^**, is a unique bounded linear operator* $A^* : H \to H$ *that satisfies for all* $\mathscr{Y}_1, \mathscr{Y}_2 \in H$*:* $< A\mathscr{Y}_1, \mathscr{Y}_2 >= \ < \mathscr{Y}_1, A^* \mathscr{Y}_2 >$*. Moreover, we have*

$$\|A\| = \|A^*\|$$

Proposition 1.1.3

let $\mathscr{T}$ *be an integral operator related to the continuous kernel* Ψ *define on* $[a,b] \times [a,b]$ *as*

$$\mathfrak{K}(\mathscr{Y})x = \int_a^b \Psi(x,t)\mathscr{Y}(t)\,dt, \quad \text{for all } x \in [a,b].$$

Then, $\mathfrak{K}^*$ *is the unique adjoint operator related to* $\mathfrak{K}$ *for the usual inner product of* L^2

$$\left(\mathfrak{K}^* v\right)x = \int_a^b \Psi(t,x)v(t)\,dt, \quad \text{for all } x \in [a,b].$$

Corollary 1.1.1

If the integral operator of the kernel Ψ^* is $\Re^*$ and Ψ is the kernel of the integral operator $\Re$, thus

$$\Psi(x,t) = \Psi^*(t,x).$$

Definition 1.1.8 ***(Self-adjoint operator)*** *[19]*

An operator A on a Hilbert space H is said to be self-adjoint if it is equal to its own adjoint, i.e., $A^* = A$ or equivalently, for all

$$\forall \mathscr{Y}_1, \mathscr{Y}_2 \in H \quad \langle \mathscr{Y}_1, A\mathscr{Y}_2 \rangle = \langle A\mathscr{Y}_1, \mathscr{Y}_2 \rangle.$$

Corollary 1.1.2

The integral operator $\mathscr{T}$ is self-adjoint precisely when its kernel satisfies for all $x, t \in [a,b]$

$$\Psi(t,x) = \Psi(x,t).$$

Contraction operators

Definition 1.1.9

Let H be a Banach space and A a bounded operator not necessarily linear in H, the operator A is called contraction operator if there exists a constant k, satisfying that $0 \leq k < 1$ and

$$\| A\mathscr{Y}_1 - A\mathscr{Y}_2 \| \leq k \| \mathscr{Y}_1 - \mathscr{Y}_2 \|, \quad \forall \mathscr{Y}_1, \mathscr{Y}_2 \in H.$$

Theorem 1.1.3 *(Principle of Banach Contraction)*

Let A be a contraction operator on a Banach space H, then the equation

$$A\mathscr{Y} = \mathscr{Y},$$

admits a unique solution $\mathscr{Y}$ in H, this solution is called a fixed point of the operator A.

Compact operator

Definition 1.1.10 *(Compact operator)*

Let X, Y be vector spaces. An operator $A : X \to Y$ is called compact if it maps every bounded subset M of X into a relatively compact subset of Y.

Theorem 1.1.4 *Let X, Y be Banach spaces. We call that an operator $A : X \to Y$ is a compact if for every bounded sequence $\mathscr{Y}_n$ in X, the image sequence $A(\mathscr{Y}_n)$ in Y has a convergent subsequence.*

Definition 1.1.11 *(completely-continuous operator) If A is a continuous compact operator. Then, A is called completely continuous operator.*

Remark 1.1.1 *An important simplification occurs for linear operators between well-behaved spaces like Banach spaces: in this case, the concept of a "completely continuous operator" is entirely equivalent to that of a compact operator.*

Corollary 1.1.3 *Let X, Y, U, V be Banach spaces.*

(i) Every compact linear operator is bounded.

(ii) Every finite-dimensional bounded linear operator is compact.

(iii) Let $A_1, A_2 : X \to Y$ be compact linear operators and $\alpha_1, \alpha_2 \in \mathbb{R}$. Then the operator $\alpha_1 A_1 + \alpha_2 A_2$ is compact.

(iv) Let $T : X \to Y$ be a compact linear operator, and let $A : U \to X$ and $B : Y \to V$ be bounded linear operators, then the operators $TA : U \to Y$ and $BT : X \to V$ are compact.

(v) Let A be a compact operator and B be a continuous operator, then $A \circ B$ and $B \circ A$ are compact operators.

Theorem 1.1.5 *Let (A_n) the sequence of compact linear operators from X to Y, and let $A : X \to Y$ a bounded linear operator, such that $\|A_n - A\| \to 0$ as $n \to \infty$. Consequently $A : X \to Y$ is compact.*

Proof 1.1.2 *see [56]*

Theorem 1.1.6 ***(Fredholm alternative)*** *[19]*

Let $A : X \to X$ be a compact linear operator. Then

1. *The set of solutions to the equation* $\mathscr{Y} - A\mathscr{Y} = 0$ *forms a finite-dimensional subspace. This means there are only a finite number of linearly independent solutions,*

2. *The set of all vectors of the form* $f = \mathscr{Y} - A\mathscr{Y}$ *is a closed subspace of* X*,*

3. $N(I-A) = \{0\}$ *if and only if* $R(I-A) = E$*,*

4. *if* A^* *is the adjoint of* A *then* $\dim N(I-A) = \dim N(I-A^*)$*.*

Remark 1.1.2 *The Fredholm Alternative presents a strict dichotomy for equations of the form* $\mathscr{Y} - A\mathscr{Y} = f$*:*

- *Either the equation is always uniquely solvable for every* $f \in E$*.*

- *Or the homogeneous version* ($f = 0$) *has a finite number of non-trivial solutions, and in this case, the inhomogeneous equation is solvable if and only if* f *is orthogonal to all the solutions of the homogeneous adjoint equation.*

Compact integral operators

Definition 1.1.12 $\mathfrak{K}$ *is a linear integral operator from* $C(\Omega)$ *into* $C(\Omega)$ *if it has the form*

$$\mathfrak{K}(\mathscr{Y})(x) = \int_\Omega \Psi(x,t)\mathscr{Y}(t)\,dt, \quad x,t \in \Omega, \tag{1.1}$$

where Ω *is a compact subset on* $\mathbb{R}^n$ *and the continuous function* Ψ *is defined from* $\Omega \times \Omega$ *to* $\mathbb{R}^n$*. Moreover, we have* $\|\mathfrak{K}\| = \max\limits_{x\in\Omega} \int_\Omega |\Psi(x,t)|\,dt$*.*

Theorem 1.1.7 *Assume that* $\Psi(x,t)$ *is Riemann-integrable with respect to* t *for all* $x \in \Omega$*, and that the following conditions holds:*

i. $\lim\limits_{h\to 0} \max\limits_{x\in\Omega} \max\limits_{|x-\tau|\le h} \int_\Omega |\Psi(x,t) - \Psi(\tau,t)|\,dt = 0.$

ii. $\|\mathfrak{K}\| < \infty$*.*

Therefore, the integral operator given in (1.1) *is compact from* $C(\Omega)$ *into* $C(\Omega)$*.*

Remark 1.1.3 *Assumptions (i) and (ii) hold provided that* $\Psi(x,t)$ *is a continuous function on the domain* $\Omega \times \Omega$*.*

Weakly singular operator [69]

Let $\mathfrak{K}$ an integral operator defined by the kernel $\Psi(x,y)$ as

$$\mathfrak{K}(\mathscr{Y})(x) = \int_{\Omega} \Psi(x,t)\mathscr{Y}(t)\,dt, \quad x \in \Omega = [0,1], \quad \mathscr{Y} \in C(\Omega), \tag{1.2}$$

we call that $\mathfrak{K}$ weakly singular operator if the function Ψ is a weak singular kernel. A weakly singular kernel is a function $\Psi(x,t)$ used in an IEs that is unbounded (singular) at one or more points within the domain of integration (typically along the diagonal $x = t$), but the singularity is "weak" enough for the integral involving it to be absolutely integrable; It can be said the extremal understandings of weak singularity is to satisfies the two conditions:

(i) K is continuous on $D = \{(x,t) \in (\Omega \times \Omega) : x \neq t\}$.

(ii) $|\Psi(x,t)| \leq c_K(1 + |x-t|^{-\nu})$ for all $(x,t) \in D$ where $\nu < 1$.

The most common weak singularity kernel is in this form

$$\Psi(x,t) = \begin{cases} a(x,t)|x-t|^{-\nu}, & 0 < \nu < 1, \\ a(x,t)\log^{k}|x-t|, & 0 \leq k, \end{cases} \tag{1.3}$$

where $a \in C(\Omega \times \Omega)$ and many others are weak singular in this sense.

Nonlinear integral operator

The Fredholm (Urysohn) integral operator An integral operator $\mathfrak{K}$ is called nonlinear if its kernel Ψ is a nonlinear function as

$$\mathfrak{K}(\mathscr{Y})(x) = \int_{\Omega} \Psi(x,t,\mathscr{Y}(t))\,dt, \quad \forall x \in \Omega, \quad \mathscr{Y} \in C(\Omega), \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded set.

In the case that $\Omega = [a,b]$ and the kernel has the form

$$\Psi(x,t,\mathscr{Y}(t)) = \begin{cases} 0, & x < t, \\ \Psi(x,t,\mathscr{Y}(t)), & a \leq t < x \leq b, \end{cases} \tag{1.5}$$

then, the Fredholm integral operator (1.4) has the form:

$$\Re(\mathscr{Y})(x)=\int_a^x \Psi(x,t,\mathscr{Y}(t))\,dt, \quad x,t\in\Omega, \tag{1.6}$$

In this case $\Re$ is called nonlinear Volterra integral operator.

Theorem 1.1.8 *[62] The Fredholm operator $\Re: C(\Omega)\to C(\Omega)$, defined in (1.4) with a continuous kernel $\Psi:\Omega^2\times\mathbb{R}^n\to\mathbb{R}^n$, is completely continuous.*

Proof 1.1.3 *Firstly we determine that the operator $\Re$ is continuous . Assume that $\mathscr{Y}\in C\left(\Omega;\mathbb{R}^n\right)$ and we take a number $R>|u_0|_\infty$. and let Assume that $\varepsilon>0$, while Ψ is uniformly continuous on the compact $\Omega^2\times\overline{B}_R\left(0;\mathbb{R}^n\right)$, there exists a constant $\delta_\varepsilon>0$ such that for each $\mathscr{Y}\in C\left(\Omega;\mathbb{R}^n\right)$ fulfilling $|\mathscr{Y}-\mathscr{Y}_0|_\infty\le\delta_\varepsilon$ one has $\mathscr{Y}(t)\in\overline{B}_R\left(0;\mathbb{R}^n\right)$ and $|\Psi(x,t,\mathscr{Y}(t))-\Psi(x,t,\mathscr{Y}_0(t))|\le\varepsilon$,then*

$$|\Re(\mathscr{Y})(x)-\Re(\mathscr{Y}_0)(x)|\le\int_\Omega[\Psi(x,t,\mathscr{Y}(t))-\Psi(x,t,\mathscr{Y}_0(t))]\,dy\le\varepsilon\mu(\Omega)\quad\forall x\in\Omega.$$

Therefore, we get

$$|\mathscr{Y}-\mathscr{Y}_0|_\infty\le\delta_\varepsilon\Rightarrow|\Re(\mathscr{Y})-\Re(\mathscr{Y}_0)|_\infty\le\varepsilon\mu(\Omega).$$

it mean that $\Re$ is continuous at $\mathscr{Y}_0$.

secondly, let E a given bounded subset of $C\left(\Omega;\mathbb{R}^n\right)$, and we prove that $\Re(E)$ is relatively compact in $C\left(\Omega;\mathbb{R}^n\right)$.

because E is bounded there exists a constant $c>0$ such that

$$|\mathscr{Y}|_\infty\le c \text{ for all } \mathscr{Y}\in E.$$

This implies that for any $\mathscr{Y}\in E$ we get

$$|\Re(\mathscr{Y})|_\infty\le M\mu(\Omega), \text{where} \quad M=\max_{\Omega^2\times\overline{B}_c(0;\mathbb{R}^n)}|\Psi(x,t,\mathscr{Y})|.$$

Then the set $\Re(E)$ is bounded in $C\left(\Omega;\mathbb{R}^n\right)$.

The uniform continuity of Ψ on the compact set $\Omega^2\times\overline{B}_c\left(0;\mathbb{R}^n\right)$, implies that for each $\varepsilon>0$, there exists $\delta_\varepsilon>0$ such that

$$\left|\Psi(x,t,\mathscr{Y}(t))-\Psi\left(x',t,\mathscr{Y}(t)\right)\right|\le\varepsilon$$

for all $x, x', t \in \Omega$ *with* $\left|x - x'\right| \leq \delta_\varepsilon$ *and* $\mathscr{Y} \in E$. *It follows that*

$$\left|\Re(\mathscr{Y})(x) - \Re(\mathscr{Y})(x')\right| \leq \varepsilon \mu(\Omega),$$

so that $\Re(E)$ *is equicontinuous. Therefore, the set* $\Re(E)$ *is compact According to the Ascoli-Arzela theorem. Finally,* $\Re$ *is a completely continuous.*

Corollary 1.1.4 *The operator* $\Re : \overline{B_R}(0; C(\overline{\Omega})) \to C(\overline{\Omega})$, *defined in* (1.4) *with a continuous kernel* $\Psi : \overline{\Omega}^2 \times \overline{B_R}(0; \mathbb{R}^n) \to \mathbb{R}^n$ *where* $R > 0$, *is completely continuous operator.*

The following result follows by an argument parallel to that of the preceding theorem (1.1.8).

Theorem 1.1.9 *[62] The Volterra operator* $\Re : C[a,b] \to C[a,b]$, *defined in* (1.6) *with a continuous kernel* $\Psi : [a,b]^2 \times \mathbb{R}^n \to \mathbb{R}^n$ *where* $R > 0$, *is completely continuous.*

Corollary 1.1.5 *The operator* $\Re : \overline{B_R}(0; C[a,b]) \to C[a,b]$, *defined in* (1.6) *with a continuous kernel* $\Psi : [a,b]^2 \times \overline{B_R}(0; \mathbb{R}^n) \to \mathbb{R}^n$ *where* $R > 0$, *is completely continuous.*

1.1.3 The Fréchet derivative

There are several ways to generalize the concept of differentiability to Banach space. For example, Fréchet differentiability, which is called after Maurice Fréchet (1878-1973), is a differential calculus technique that enables additional research on nonlinear operators by connecting them to linear operators. More specifically, it uses the method of local approximation to a nonlinear operator by a linear one. We pick a fixed point $x_0 \in \Omega$ and assume that there exists a continuous linear operator F from X to Y such that, for each $x \in X$, there exists an operator A mapping an open set Ω of a Banach space X into a set D of a Banach space Y.

$$\lim_{r \to 0} \frac{A(x_0 + rx) - A(x_0)}{r} = F(x), \tag{1.7}$$

if the limit in equation (1.7) is uniform for all $x \in X$ with $\|x\| = 1$, then A is said to be Fréchet differentiable at x_0 and in this case $F = A'(x_0)$ is called the Fréchet derivative (which is unique) of A at x_0.

The Fréchet derivative of an operator A at x_0, denoted $A'(x_0)$, is a bounded linear operator whose action on a direction $h \in X$ provides formally as

$$\lim_{h\to 0} \frac{\|A(x_0+h) - A(x_0) - F(h)\|_Y}{\|h\|_X} = 0.$$

Example 1.2

Given a polynomial $P(x) \in \mathbb{R}[x]$ and $X = C[0,1], Y = \mathbb{R}$, the nonlinear operator $A : X \to Y$ is given as

$$A(\mathscr{Y}) = \int_0^1 P(t)[\mathscr{Y}(t)]^3 dt$$

is Fréchet differentiable on X, Indeed, for any fixed $u, h \in X$ and $r > 0$, we get

$$\begin{aligned} A'(\mathscr{Y})(x) &= \lim_{r\to 0} \frac{A(\mathscr{Y}+rx) - A(\mathscr{Y})}{r} \\ &= \lim_{r\to 0} \frac{1}{r} \left\{ \int_0^1 P(t)[\mathscr{Y}(t) + rx(t)]^3 dt - \int_0^1 P(t)[\mathscr{Y}(t)]^3 dt \right\} \\ &= \lim_{r\to 0} \int_0^1 P(t) \left[3\mathscr{Y}^2(t)x(t) + 3r\mathscr{Y}(t)x^2(t) + r^2x^3(t) \right] dt \\ &= \int_0^1 3P(t)\mathscr{Y}^2(t)x(t)dt \end{aligned}$$

Therefore, the Fréchet derivative of A at $\mathscr{Y}$ is the linear operator given by

$$F(x) = \int_0^1 3P(t)[\mathscr{Y}(t)]^2 x(t)dt$$

since

$$\|F\| = \sup_{x\neq 0} \frac{|F(x)|}{\|x\|_\infty} \leq 3\|P\|_\infty \|\mathscr{Y}\|_\infty^2$$

F is a bounded linear operator as a result. We now demonstrate that the F is Fréchet derivative of A at $\mathscr{Y}$:

$$\lim_{h\to 0}\frac{|A(\mathscr{Y}+h)-A(\mathscr{Y})-F(h)|}{\|h\|_\infty}=\lim_{h\to 0}\frac{1}{\|h\|_\infty}\Big|\int_0^1 P(t)[\mathscr{Y}(t)+h(t)]^3dt-\int_0^1 P(t)[\mathscr{Y}(t)]^3dt$$
$$-\int_0^1 3P(t)[\mathscr{Y}(t)]^3h(t)dt\,|$$
$$=\lim_{h\to 0}\frac{1}{\|h\|_\infty}\left|\int_0^1 P(t)\left[3\mathscr{Y}(t)h^2(t)+h^3(t)\right]dt\right|$$
$$\leq\lim_{h\to 0}\frac{\|h\|_\infty^2}{\|h\|_\infty}\int_0^1\left|P(t)\left([\mathscr{Y}(t)]^2+h(t)\right)\right|dt=0$$

for small h the integral is bounded . Therefore, for each $\mathscr{Y}\in X$, A is Fréchet differentiable on X, and we obtain

$$A'(\mathscr{Y})(x)=\int_0^1 P(t)[\mathscr{Y}(t)]^2x(t)dt,\quad x\in X$$

Differentiation of nonlinear integral operators

Let $\mathfrak{K}$ be the integral operator formed as

$$\mathfrak{K}(\mathscr{Y})(z)=\int_0^1\Psi(z,t,\mathscr{Y}(t))dt \tag{1.8}$$

Theorem 1.1.10 *[44] Let $\Psi(z,t,\mathscr{Y})$ a continuous function are given on Ω, with continuous derivatives $\Psi'_{\mathscr{Y}}(z,t,\mathscr{Y})$ and $\Psi''_{\mathscr{Y}^2}(z,t,\mathscr{Y})$, Then, the operator $\mathfrak{K}$ in (1.8) maps Ω into $C[0,1]$ and is twice differentiable at every interior point $\mathscr{Y}_0\in\Omega$, also we have*

$$\begin{aligned}\mathfrak{K}'(\mathscr{Y}_0)(x(z))&=\int_0^1\Psi'_{\mathscr{Y}}(z,t,\mathscr{Y}_0(t))\,x(t)dt\\ \mathfrak{K}''(\mathscr{Y}_0)((x,y)(z))&=\int_0^1\Psi''_{\mathscr{Y}^2}(z,t,\mathscr{Y}_0(t))\,x(t)y(t)dt\end{aligned} \tag{1.9}$$

Proof 1.1.4 *see [44].*

1.2 Recall from numerical analysis

1.2.1 Interpolation

Theorem 1.2.1 *[46] Let $U_n\subset C([a,b])$ be an n-dimensional subspace that is unisolvent with respect to the nodes $x_1,\ldots,x_n\in[a,b]$. Then, for any $g_1,\ldots,g_n\in\mathbb{R}$, there exists a unique $\mathscr{Y}\in U_n$ with the interpolation property*

$$\mathscr{Y}(x_j) = g_j, \quad j = 1, \ldots, n. \tag{1.10}$$

Consequently, given a function $g \in C([a,b])$, the mapping $g \mapsto \mathscr{Y}$, where $\mathscr{Y}$ is the unique interpolate to the data $g_j = g(x_j)$, defines a bounded linear operator $P_n : C([a,b]) \to U_n$, called the interpolation operator.

Theorem 1.2.2 *[46] Let $g \in C^2[a,b]$. Then, for the error in piecewise linear interpolation we have the estimate*

$$\|P_n g - g\|_\infty \leq \frac{1}{8} h^2 \|g''\|_\infty. \tag{1.11}$$

(Lagrange interpolation) Let $X = C[a,b]$ denote the space of continuous functions on $[a,b]$, and let P_N be the space of polynomials of degree $\leq N$. The Lagrange interpolation operator $I_N : X \to P_N$ is defined for any $\mathscr{Y} \in X$ by

$$I_N(x) = \sum_{j=0}^{N} \mathscr{Y}\left(x_j\right) L_j(x), \quad L_j(x) = \prod_{i=1, i\neq j}^{N} \frac{x - x_i}{x_j - x_i}, \quad j = 1, ..., N,$$

where $\{L_j\}_{j=0}^{N}$ is the Lagrange interpolation basic functions form $\mathscr{P}_N$, and satisfy the conditions $L_j(x_i) = \delta_{ji}$ and $I_N(x_j) = y_j, \quad i, j = 1, ..., N$.

1.2.2 Orthogonal polynomials

This part provides a quick introduction and highlights the significance of orthogonal polynomials utilized in the thesis.

A sequence of polynomials that are pairwise "orthogonal" to one another with regard to a specified inner product or weight function are known as orthogonal polynomials. This means that the integral of the product of any two different polynomials is zero. Numerous disciplines, including physics, approximation theory, and numerical analysis, use these polynomials, including the classical polynomials like Legendre, Chebyshev, Jacobi, etc.

Jacobi Polynomials

The Jacobi polynomials is the family $\{J_k^{\alpha\beta}, k = 0,1,\ldots,n\}$, for $\alpha,\beta > -1$. $J_k^{\alpha\beta}$ can **defined** as express of series as follows

$$J_n^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)} \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(\alpha+m+1)} \left(\frac{z-1}{2}\right)^m.$$

The **Rodrigues' formula** of the Jacobi polynomials is given by

$$J_n^{(\alpha,\beta)}(z) = \frac{(-1)^n}{2^n n!}(1-z)^{-\alpha}(1+z)^{-\beta}\frac{d^n}{dz^n}\left\{(1-z)^{\alpha}(1+z)^{\beta}(1-z^2)^n\right\}.$$

Orthogonality: the Jacobi polynomials satisfy the orthogonality condition

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}J_m^{(\alpha,\beta)}(x)J_n^{(\alpha,\beta)}(x)\,dx = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!}\delta_{nm},$$

Differential equation: the Jacobi polynomials $J_n^{(\alpha,\beta)}$ is, up to scaling, the unique polynomial solution of the Sturm–Liouville problem

$$(1-x^2)y'' + (\beta-\alpha-(\alpha+\beta+2)x)y' = \lambda y$$

where $\lambda = -n(n+\alpha+\beta+1)$. The other solution involves the logarithm function. Bochner's theorem states that the Jacobi polynomials are uniquely characterized as polynomial solutions to Sturm–Liouville problems with polynomial coefficients.

The first three Jacobi polynomials is

$$\begin{aligned}
J_0^{(\alpha,\beta)}(z) &= 1,\\
J_1^{(\alpha,\beta)}(z) &= (\alpha+1)+(\alpha+\beta+2)\frac{z-1}{2},\\
J_2^{(\alpha,\beta)}(z) &= \frac{(\alpha+1)(\alpha+2)}{2}+(\alpha+2)(\alpha+\beta+3)\frac{z-1}{2}+\frac{(\alpha+\beta+3)(\alpha+\beta+4)}{2}\left(\frac{z-1}{2}\right)^2.
\end{aligned}$$

If we set $\alpha = \beta = 0$ we have the Legendre polynomials, and if we choose $\alpha = \beta = -1/2$ we get the Chebyshev polynomials. etc.

Legendre polynomials

Legendre polynomials are a set of orthogonal polynomials that appear in many fields of mathematics, particularly in solving problems involving spherical coordinates and in approximating functions. The Legendre polynomials are **defined** on the domain $[-1,1]$ with a weight factor 1 and can be expressed using the **Rodrigues formula** as:

$$P_n(x) = \left(\frac{1}{2^n n!}\right) \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(x^2-1)^n\right]. \tag{1.12}$$

The Legendre polynomials satisfy the homogeneous second order ordinary differential equation given by

$$(x^2-1)y'' + 2xy' = \lambda y \tag{1.13}$$

Orthogonality: Legendre polynomials satisfy the following orthogonality

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0, & n \neq m, \\ \dfrac{2}{2n+1}, & n = m. \end{cases} \tag{1.14}$$

Recurrence Relation: the Legendre polynomials they can be generated using the relation

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \tag{1.15}$$

The first few Legendre polynomials are:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \tfrac{1}{2}(3x^2-1) \\ P_3(x) &= \tfrac{1}{2}(5x^3-3x) \\ P_4(x) &= \tfrac{1}{8}(35x^4-30x^2+3) \end{aligned}$$

The next figures represent graphs of the four Legendre polynomials P_1, P_2, P_3, P_4.

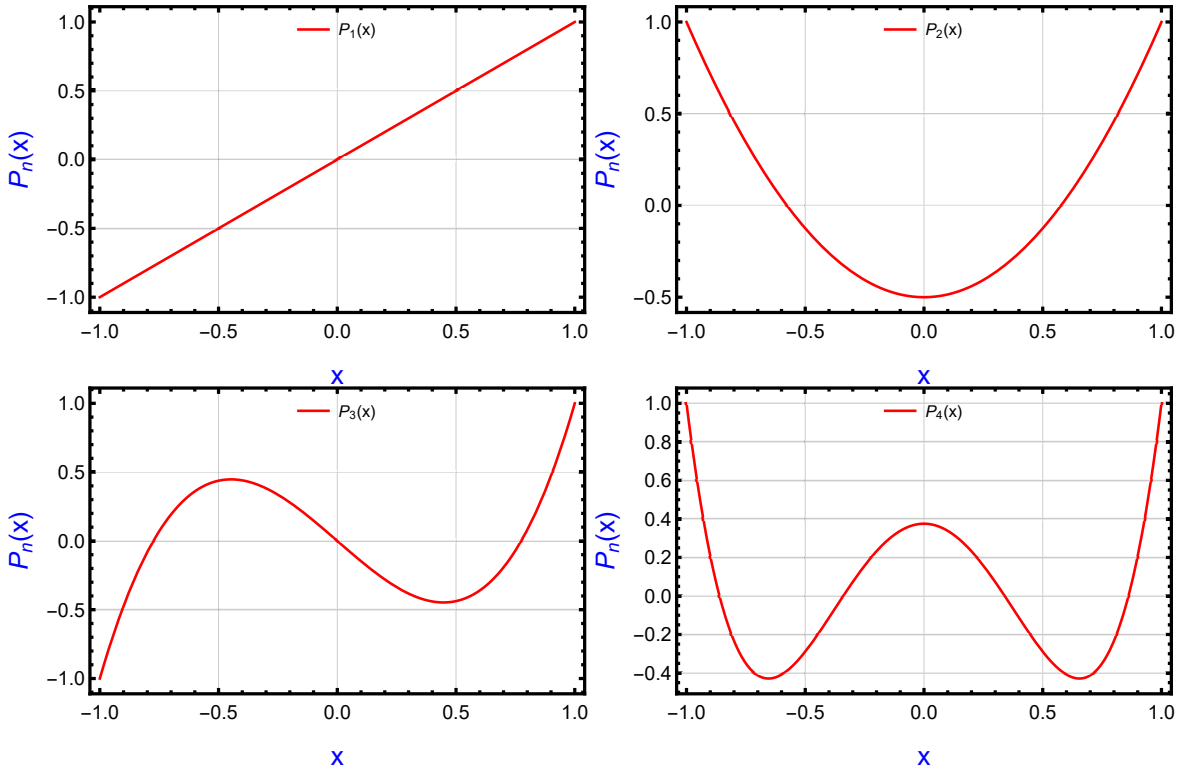

Figure 1.1: Graph of some Legendre polynomials.

Vieta-Lucas polynomials

The polynomials of the Vieta-Lucas of degree m ($m \in \mathbb{N}$) is $V_m(x)$ such that [45, 41]:

$$V_m(x) = 2\cos\left(m \arccos\left(\frac{x}{2}\right)\right), \quad x \in [-2,2]. \tag{1.16}$$

Recurrence relation: the Vieta-Lucas polynomials they can be generated using the relation

$$\begin{cases} V_0(x) = 2, \\ V_1(x) = x, \\ V_m(x) = xV_{m-1}(x) - V_{m-2}(x), \quad m = 2,3,4,\ldots. \end{cases} \tag{1.17}$$

The first five Vieta-Lucas polynomials is

$$\begin{aligned} V_0(x) &= 2, \\ V_1(x) &= x, \\ V_2(x) &= x^2 - 2, \\ V_3(x) &= x^3 - 3x, \\ V_4(x) &= x^4 - 4x^2 + 2. \end{aligned} \tag{1.18}$$

The next figures represent graphs of the four Vieta-Lucas polynomials V_1, V_2, V_3, V_4.

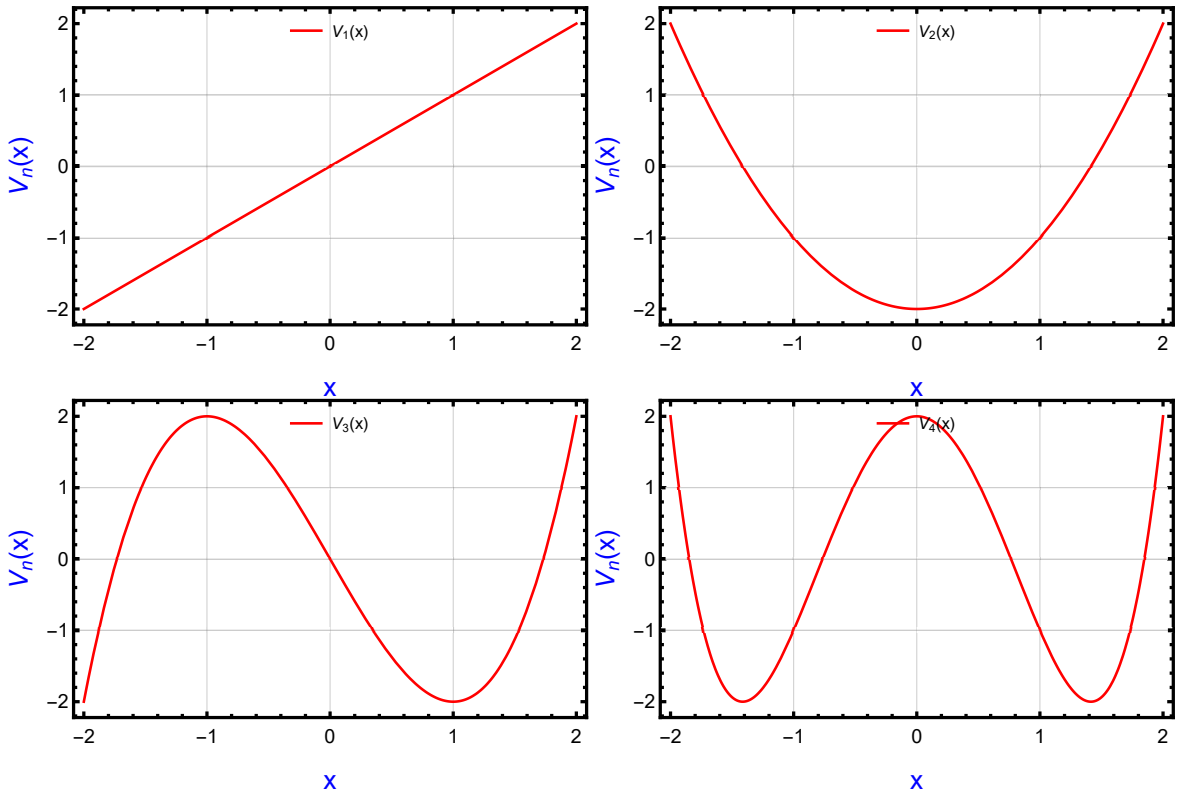

Figure 1.2: Graph of some Vieta-Lucas polynomials.

Express series: the Vieta-Lucas polynomials can be expressed as a power of series as

$$V_m(x) = \begin{cases} 2, & m = 0, \\ \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \dfrac{m(m-i-1)!}{i!(m-2i)!} x^{m-2i}, & m = 1,2,3,4,\ldots, \end{cases} \tag{1.19}$$

here, $\lfloor \frac{m}{2} \rfloor$ denotes the floor function of $\frac{m}{2}$.

Orthogonality: the Vieta-Lucas polynomials $V_n(x)$ and $V_m(x)$ are orthogonal over $[-2,2]$ With respect to the weight function $w(x) = \dfrac{1}{\sqrt{4-x^2}}$. Therefore, we have the orthogonality condition:

$$\langle V_n(x), V_m(x) \rangle_{w(x)} = \int_{-2}^{2} V_n(x) V_m(x) w(x)\, dx = \begin{cases} 4\pi, & n = m = 0, \\ 2\pi, & n = m \neq 0, \\ 0, & n \neq m \neq 0. \end{cases} \tag{1.20}$$

The generating function for Vieta-Lucas polynomials is defined as:

$$\sum_{m=0}^{\infty} V_m(x)\, t^m = \frac{2 - xt}{1 - xt + t^2}. \tag{1.21}$$

Chapter 2

Generality in integral equations

This chapter addresses the concept and classification of the IEs, followed by the existence and uniqueness of their solutions.

2.1 Concept of integral equations

Definition 2.1.1 ***(nonlinear integral equation)***

A nonlinear IE is one where the unknown function $\mathscr{Y}$ appears under the integral sign, and the equation is nonlinear with respect to $\mathscr{Y}$. This nonlinearity can occur inside the integrand, outside of it, or in both places at the same time.

Equations with the following form are one of the common categories

$$\Phi(\mathscr{Y})(x) = \lambda \int_{\Omega} \Psi(x,t,\mathscr{Y}(t))\,dt, \quad x,t \in \Omega, \tag{2.1}$$

where

Φ is a measurable function define on $\mathbb{R}^n$.

$\Psi(x,t,\mathscr{Y}(t))$ a measurable function given on $\Omega \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n$ named by the kernel and the unknown function $\mathscr{Y}$ is under the integral.

λ a non zero scalar from $\mathbb{R}$ or $\mathbb{C}$.

Ω the domain of the integration.

The most standard type of IE and the main concern is of the form

$$\Phi(x)\mathscr{Y}(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} \Psi(x,t,\mathscr{Y}(t))\,dt, \tag{2.2}$$

where $f(x)$ continuous function.

Remark 2.1.1

- In integral equation (2.2), $f(x) = 0$ results in a homogeneous, otherwise it is nonhomogeneous IE.

- If the kernel $\Psi(x,t,\mathscr{Y}(t)) = \Psi(x,t)\mathscr{Y}(t)$, the integral equation (2.2) is categorized as linear, otherwise it is nonlinear IE.
- $\alpha(x), \beta(x)$ will determine type of the IE.
- The type of the integral equation (2.2) is classified based on the function Φ:
  - First kind: when $\Phi = 0$.
  - Second kind: when Φ is a non-zero constant.
  - Third kind: when Φ is a non-constant function.

Example 2.1

$0 = \lambda \int_5^{x^2} \Psi(x,t)\mathscr{Y}^2(t)\,dt$ *homogeneous integral equation of first type*

$\mathscr{Y}(x) = x^2 + \lambda \int_a^b \Psi(x,t)\sin(\mathscr{Y}(x))\,dt$ *nonhomogeneous integral equation of second type*

$\sqrt{x}\mathscr{Y}(x) = \lambda \int_a^b \Psi(x,t)\mathscr{Y}(t)\,dt$ *linear integral equation of third type*

$\mathscr{Y}(x) = x + \lambda \int_a^{x+1} \Psi(x,t)\ln(\mathscr{Y}(t))\,dt$ *nonlinear integral equation of second type*

Solutions to integral (or differential) equations generally fall into two categories:

Exact Solutions: These are solutions that can be written in a closed form, typically composed of elementary functions like polynomials, exponentials, ..., or finite combinations thereof.

Approximate Solutions: When an exact solution is unattainable for a concrete problem, an approximate solution is sought. Such solutions, which may be expressed in a spectral form or other basis, aim to converge toward the exact solution. The accuracy of these approximations is vital for numerical purposes and generally improves as more terms are included in the solution representation. Formally, an approximate solution is a function that satisfies the IE with an acceptably low error.

2.2 Important categories of integral equations

IEs appear in many types, classes, and categorizations; see [70, 63, 43, 30]. The types of these equations are primarily determined by the nature of the integration limits and the properties of the kernel. In this section, we mention some of this categorization.

Nonlinear integral equations

Definition 2.2.1 ***(Nonlinear quadratic integral equations (NQIEs))***

The predominant form of quadratic IEs is formulated as

$$\Phi(x)\mathscr{Y}(x) = f(x) + g(x,\mathscr{Y}(x))\int_{\alpha(x)}^{\beta(x)} \Psi(x,t,\mathscr{Y}(t))\,dt, \tag{2.3}$$

where f, g, Ψ three functions are given.

Definition 2.2.2 ***(Nonlinear quadratic Fredholm integral equations (NQFIEs))***

In Eq. (2.3), if the two limits of the integration be a (constants). Then the name of Eq. (2.3) becomes a nonlinear quadratic Fredholm IE, where its form

$$\phi(x)\mathscr{Y}(x) = f(x) + g(x,\mathscr{Y}(x))\int_{a}^{b} \Psi(x,t,\mathscr{Y}(t))\,dt, \quad a \le x,t \le b. \tag{2.4}$$

Eq. (2.4), named as nonlinear Fredholm IE for $g(x,\mathscr{Y}(x)) = \lambda$ (constant).

Definition 2.2.3 ***(Nonlinear quadratic Volterra integral equations (NQVIEs))***

In Eq. (2.4), if $\Psi(x,t,\mathscr{Y}(t)) = 0$ for all $t > x$. Then the name of Eq. (2.4) become a nonlinear quadratic Volterra IE where her form as

$$\phi(x)\mathscr{Y}(x) = f(x) + \lambda\int_{a}^{x} \Psi(x,t,\mathscr{Y}(t))\,dt, \quad a \le x,t \le b. \tag{2.5}$$

Eq. (2.5), named as nonlinear Volterra IE for $g(x,\mathscr{Y}(x)) = \lambda$ (constant).

Example 2.2

$$\mathscr{Y}(x) = x^2 - 2 + \frac{4}{\mathscr{Y}(x)+1}\int_{a}^{x} t\mathscr{Y}^2(t)\,dt, \tag{2.6}$$

this equation is named as the NQVIE of the second kind.

Definition 2.2.4 ***(Nonlinear Volterra-Fredholm integral equations)***

The forms of nonlinear Volterra-Fredholm IE (mixed IEs) of the second kind is

$$\mathscr{Y}(x) = f(x) + \lambda_1 \int_a^b \Psi_1(x,t,\mathscr{Y}(t))\,dt + \lambda_2 \int_0^x \Psi_2(x,t,\mathscr{Y}(t))\,dt, \tag{2.7}$$

and second form is

$$\mathscr{Y}(x) = f(x) + \lambda \int_0^x \int_a^b \Psi(r,t,\mathscr{Y}(t))\,dt\,dr, \tag{2.8}$$

where f, Ψ, Ψ_1, Ψ_2 are given functions and $\lambda, \lambda_1, \lambda_2$ are constant parameter.

Definition 2.2.5 ***(Nonlinear integro-differential equations)***

The form of nonlinear integro-differential equation like as

$$\mathscr{Y}^{(n)}(t) = f(t) + \lambda \int_G \Psi(t,x,\mathscr{Y}(x))\,dx\ , \qquad x,y \in G \tag{2.9}$$

and the more standard form is like

$$\int_G \Psi_1(t,x,\mathscr{Y}(x))\,dx + \int_G \Psi_2\left(t,x,\mathscr{Y}^{(n)}(x)\right)dx = f(t)\ , \quad \Psi_2\left(t,x,\mathscr{Y}^{(n)}(x)\right) \neq 0, \tag{2.10}$$

Let $\mathscr{Y}^{(n)}$ denote the n-th derivative of $\mathscr{Y}(x)$. The Volterra-Fredholm integro-differential equation arises similarly to the Volterra-Fredholm IE, but incorporates one or more ordinary derivatives into the IE. Here, the kernels Ψ, Ψ_1, Ψ_2 and the function $f(x)$ are known functions.

For example

$$\mathscr{Y}'(x) = e^x + \int_0^2 x^2 \cos(\mathscr{Y}(t))\,dt$$

Definition 2.2.6 ***(Hammerstein integral equations)***

The second kind of Hammerstein IE has the form

$$\phi(x)\mathscr{Y}(x) + \lambda \int_a^b \Psi(x,t)F(t,\mathscr{Y}(t))\,dt = f(x) \tag{2.11}$$

For example

$$\mathscr{Y}(x) + \int_0^3 \sqrt{(x-t)}\sin(\mathscr{Y}(t)+t)\,dt = e^{-2x}$$

Definition 2.2.7 ***(Hammerstein-Volterra integral equations)***

We named the Hammerstein-Volterra IE; every equation has the form

$$\phi(x)\mathscr{Y}(x)+\lambda\int_a^x \Psi(x,t)F(t,\mathscr{Y}(t))\,dt=f(x) \tag{2.12}$$

For example

$$x\mathscr{Y}(x)+2\int_{-2}^{x}\frac{x-t}{2}\sin(\mathscr{Y}(t)+t)\,dt=2x^3$$

Definition 2.2.8 ***(Lalesco integral equations)***

The second kind of Lalesco IE has the form

$$\mathscr{Y}(x)=f(x)+\int_0^x[\Psi_1(x,t)\mathscr{Y}(t)+\Psi_2(x,t)\mathscr{Y}^2(t)+\Psi_3(x,t)\mathscr{Y}^3(t)+\ldots+\Psi_n(x,t)\mathscr{Y}^n(t)]\,dt \tag{2.13}$$

Definition 2.2.9 ***(Bratu integral equations)***

$$\mathscr{Y}(x)=\lambda\int_a^b G(x,t)e^{\mathscr{Y}(t)}\,dt \tag{2.14}$$

This form of equations called by nonlinear Bratu IE. where $b>0$ and $G(x,t)$ is Green function:

$$G(x;t)=\begin{cases}\dfrac{(b-x)(t-a)}{b-a}, t\le x\\ \dfrac{(b-t)(x-a)}{b-a}, t\ge x\end{cases}$$

For example

$$\mathscr{Y}(x)=\int_1^3 G(x,t)e^{\mathscr{Y}(t)}\,dt$$

with

$$G(x,t)=\begin{cases}\dfrac{(3-x)(t-1)}{2};\ t\le x\\ \dfrac{(3-t)(x-1)}{2};\ x\le t\end{cases}$$

Singular integral equations

Definition 2.2.10 ***(Singular integral equations)***

A singular IE is an IE involving infinite limits of integration or whose kernel becomes unbounded at a point within the interval.

Definition 2.2.11 ***(Abel integral equations)***

One of the forms of Abel IEs of second kind is

$$\mathscr{Y}(x)+\frac{\lambda}{\Gamma(1-\alpha)}\int_a^x (x-t)^{-\alpha}\mathscr{Y}(t)\,dt = f(x) \tag{2.15}$$

When $0<\alpha<1$ *and* λ *is a constant.*

Definition 2.2.12 ***(Weakly singular Fredholm integral equations)***

The second kind of weakly singular Fredholm IE define as

$$\mathscr{Y}(x) = f(x)+\lambda\int_a^b M(x,t)\Psi(x,t)\mathscr{Y}(t)\,dt \quad , x\in[0,T] \tag{2.16}$$

where $\Psi(x,t)$ *must be a weak singular kernel*

$$\Psi(x;t)=\begin{cases}(x-t)^{-\alpha}, & 0<\alpha<1,\\ \log^k(x-t), & 0<k.\end{cases}$$

For example

$$\mathscr{Y}(x) = e^{-x}+\int_0^1 \sqrt{x-t}\log(x-t)\mathscr{Y}(t)\,dt$$

Remark 2.2.1

If the upper limit of integration is $b=x$*, then equation* (2.16) *is termed as weakly singular Volterra equation.*

Definition 2.2.13 ***(Carleman integral equations)***

The form of Carleman IE is

$$p(t)\frac{1}{\pi i}\int_{-1}^1 \frac{\mathscr{Y}(t)}{(t-x)}\,dt+\frac{1}{\pi i}\int_{-1}^1 \frac{q(t)}{(t-x)}\mathscr{Y}(t)\,dt = f(x) \tag{2.17}$$

where p,q *are two given functions.*

Definition 2.2.14 ***(Cauchy singularity)***

For a bounded and connected domain D *in the complex plane, the Cauchy integral formula is*

$$\frac{1}{\pi i}\int_{\partial D}\frac{\mathscr{Y}(t)}{(t-x)}\,dt = f(x) \quad t\in\mathbb{C} \tag{2.18}$$

For example

$$\frac{1}{i\pi}\int_{\partial(D(0;1))}\frac{\mathscr{Y}(t)}{t-x}dt = x^2$$

2.3 Differential equations and integral equations

2.3.1 Linear differential and integral equations

Volterra integral equations are fundamentally linked to ordinary differential equations. For instance, consider a differential equation of the form

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_n(x)y = F(x) \tag{2.19}$$

with continuous $\{a_i\}$, combined with its associated initial conditions

$$y(0) = c_0, y'(0) = c_1, y''(0) = c_2, ..., y^{(n-1)}(0) = c_{n-1} \tag{2.20}$$

can be reduced to the solution of a corresponding Volterra IEs

$$\mathscr{Y}(x) + \int_0^x \Psi(x,t)\mathscr{Y}(t)dt = f(x) \tag{2.21}$$

To do this, we apply the following transformation:

$$\frac{d^n y}{dx^n} = \mathscr{Y}(x) \tag{2.22}$$

We integrate the two said with respect to x from 0 to x

$$\frac{d^{n-1}y}{dx^{n-1}} = \int_0^x \mathscr{Y}(t)dt + c_{n-1}$$

Then, by successive integrals we have

$$\begin{aligned}
\frac{d^{n-2}y}{dx^{n-2}} &= \int_0^x\int_0^x \mathscr{Y}(t)dtdt + c_{n-1}x + c_{n-2} \\
\frac{d^{n-3}y}{dx^{n-3}} &= \int_0^x\int_0^x\int_0^x \mathscr{Y}(t)dtdtdt + c_{n-1}\frac{x^2}{2!} + c_{n-2}x + c_{n-3} \\
..... &= .....
\end{aligned}$$

This leads to

$$\begin{aligned} y(x) = & \underbrace{\int_0^x \int_0^x \int_0^x \ldots \int_0^x}_{n-fold integration} \mathscr{Y}(t)dtdtdt...dt \\ & + c_{n-1}\frac{x^{n-1}}{(n-1)!} + c_{n-2}\frac{x^{n-2}}{(n-2)!} + \ldots + c_1 x + c_0 \\ = & \int_0^x \frac{(x-t)^{n-1}}{(n-1)!}\mathscr{Y}(t)dt + c_{n-1}\frac{x^{n-1}}{(n-1)!} + c_{n-2}\frac{x^{n-2}}{(n-2)!} + \ldots + c_1 x + c_0 \end{aligned} \tag{2.23}$$

From the differential equation (2.19), we can written (2.19) as

$$\begin{aligned} \mathscr{Y}(x) + & \int_0^x \left\{ a_1(x) + a_2(x)(x-t) + a_3(x)\frac{(x-t)^2}{2!} + \ldots + a_n(x)\frac{(x-t)^{n-1}}{(n-1)!} \right\} \mathscr{Y}(t)dt \\ = & F(x) - c_{n-1}a_1(x) - (c_{n-1}+c_{n-2})a_2(x) - \ldots \\ & -(c_{n-1}\frac{(x)^{n-1}}{(n-1)!} + \ldots + c_1 x + c_0)a_n(x) \end{aligned}$$

and this become the equation

$$\mathscr{Y}(x) + \int_0^x \Psi(x,t)\mathscr{Y}(t)dt = f(x)$$

where

$$\Psi(x,t) = \sum_{\nu=1}^{n} a(x)\frac{(x-t)^{\nu-1}}{(\nu-1)!} \tag{2.24}$$

and

$$f(x) = F(x) - c_{n-1}a_1(x) - (c_{n-1}+c_{n-2})a_2(x) - \ldots - \left(c_{n-1}\frac{(x)^{n-1}}{(n-1)!} + \ldots + c_1 x + c_0\right)a_n(x). \tag{2.25}$$

Conversely, we get the (unique) solution of Eq (2.19) that fulfill the initial conditions(2.20) by using Eq (2.21) with Ψ and f given by (2.24) and (2.25) and replacing the values for $\mathscr{Y}(x)$ from the last Eq of (2.23). If the leading coefficient in equation (2.19) is not unity but $a_0(x)$, Eq (2.21) becomes

$$a_0(x)\mathscr{Y}(x) + \int_0^x \Psi(x,t)\mathscr{Y}(t)dt = f(x) \tag{2.26}$$

Remark 2.3.1

If $a_0(x) \neq 0$ in the interval considered, nothing is altered; nevertheless, if $a_0(x)$ vanishes at some point, we can tell that the Eq of the type (2.26) is equivalent to a singular differential equation, at least, when $\Psi(x,t)$ is a polynomial of t.

Example 2.3

Let the following differential equation

$$y''(x) + 4y(x) = sin(x), \quad and \quad y(0) = 0, y'(0) = 0.$$

It can be reduce to Volterra IE in the following step

Solution

Assuming

$$\begin{aligned} y''(x) &= \mathscr{Y}(x) \\ y'(x) &= \int_0^x \mathscr{Y}(t)dt \\ y(x) &= \int_0^x \int_0^x \mathscr{Y}(t)dt^2 \\ &= \int_0^x (x-t)\mathscr{Y}(t)dt \end{aligned}$$

Then, the corresponding integral equation is giving by

$$\mathscr{Y}(x) + 4\int_0^x (x-t)\mathscr{Y}(t)dt = sin(x)$$

2.3.2 Nonlinear differential and integral equations

IEs are just as significant as differential equations and can be found in a lot of applicable fields. Since a wide class of initial and boundary-value problems may be transformed into VIEs or FIEs, many problems can really be expressed (equivalently) as either a differential or an IE, see [77, 63, 3].

Let $f(x,u)$ be a continuous real-valued function on $[a,b] \times [c,d]$. The Cauchy initial value problem is to find a continuous differentiable function u on $[a,b]$ satisfying the differential equation

$$\frac{d\mathscr{Y}}{dx} = f(x,\mathscr{Y}), \quad \mathscr{Y}(x_0) = u_0. \tag{2.27}$$

We take the Banach space $C[a,b]$ of continuous real-valued functions, where the supremum norm is determined by $\|\mathscr{Y}\| = \sup\{|\mathscr{Y}(x)| : x \in [a,b]\}$. Integrating (2.27), we obtain an IE

$$\mathscr{Y}(x) = u_0 + \int_{x_0}^{x} f(t, \mathscr{Y}(t))\, dt. \tag{2.28}$$

The integral equation (2.28) is equivalent to the problem (2.27).

Example 2.4 *We consider the following boundary-value problem*

$$\mathscr{Y}'(x) + \lambda e^{\mathscr{Y}(x)} = 0 \quad \mathscr{Y}(0) = 0, 0 \leq x \leq 1,$$

by integrate the two said from 0 *to* x *we get*

$$\mathscr{Y}(x) = -\lambda \int_0^x e^{\mathscr{Y}(t)} dt, 0 \leq x \leq 1,$$

this is the nonlinear IE associated to the considered a boundary-value problems.

2.4 Existence and uniqueness of the solution of integral equations

2.4.1 Nonlinear Fredholm integral equations

The following theorem is a consequence of Banach's fixed point theorem.

Theorem 2.4.1 *[76]*

Let $\Psi(t, x, \mathscr{Y})$ be a continuous function on the set $\Omega \times \Omega \times \mathbb{R}$ where $\Omega = [a, b]$ and satisfies a Lipschitz condition with respect to the third variable, e. i.,

$$|\Psi(t, x, \mathscr{Y}_1) - \Psi(t, x, \mathscr{Y}_2)| < c\,|\mathscr{Y}_1 - \mathscr{Y}_2|,$$

suppose further that $f \in C(\Omega)$. Then, the NFIEs

$$\mathscr{Y}(t) = \lambda \int_a^b \Psi(t, x, \mathscr{Y}(x)) dx + f(t)$$

has a unique solution on Ω whenever $|\lambda| < 1/(c(b-a))$.

Proof. See, [76].

2.4.2 Nonlinear quadratic Volterra integral equations

The existence and uniqueness of solutions NQVIEs was examined before, such as [15, 20, 77]. Our contribution is to identify a novel case in which these equations admit a unique solution under specific conditions.

Let $\Omega = [0,1]$ and we put the space $D = C(\Omega)$ with sup norm $\|\mathscr{Y}\| = \sup_{x\in I} |\mathscr{Y}(x)|$. We consider the NQVEIs of the form

$$\mathscr{Y}(x) = f(x) + g(x,\mathscr{Y}(x)) \int_0^x \Psi(x,t,\mathscr{Y}(t))\,dt, \quad x,t \in \Omega. \tag{2.29}$$

here we study three cases of $g(x,\mathscr{Y}(x))$;

case1 $g(x,\mathscr{Y}(x)) = 1$,

case2 $g(x,\mathscr{Y}(x))$ *explicitly given,*

case3 $g(x,\mathscr{Y}(x)) = \int_0^x v(x,t,\mathscr{Y}(t))\,dt.$

For **case 1** suppose that

(i) $f : \Omega \to \mathbb{R}$ is a continuous function.

(ii) $\Psi : \Omega \times \Omega \times D \subset \mathbb{R} \to \mathbb{R}$ is a continuous function.

(iii) Ψ is Lipschitzian function with respect to the third variable; there exist positive constant L_1 so that $|\Psi(x,t,\mathscr{Y}_1) - \Psi(x,t,\mathscr{Y}_2)| \le L_1 |\mathscr{Y}_1 - \mathscr{Y}_2|$.

We can prove the next theorem easily

Theorem 2.4.2 *Eq.* (2.29) *has a unique solution* $\mathscr{Y} \in D$ *for* ***case*** 1 *under the conditions* (i) – (iii) *and* $L_1 < 1$.

For **case** 2 and **case** 3 With some specific assumptions:

(i) $f : \Omega \to \mathbb{R}$ is a continuous function.

(ii) $g : \Omega \times D \subset \mathbb{R} \to \mathbb{R}$ is a continuous function.

(iii) $\Psi, v : \Omega \times \Omega \times D \subset \mathbb{R} \to \mathbb{R}$ are continuous functions.

(iv) g, v, Ψ satisfy the Lipschitz condition; there exist two positive constants L_1 and L_2 so that

$|\Psi(x,t,\mathscr{Y}_1)-\Psi(x,t,\mathscr{Y}_2)| \leq L_1|\mathscr{Y}_1-\mathscr{Y}_2|$ and $\left|g(x,\mathscr{Y}_1)-g(x,\mathscr{Y}_2)\right| \leq L_2|\mathscr{Y}_1-\mathscr{Y}_2|$ (respectively, $|v(x,t,\mathscr{Y}_1)-v(x,t,\mathscr{Y}_2)| \leq L_2|\mathscr{Y}_1-\mathscr{Y}_2|$) for **case** 2 (respectively, for **case** 3) for all $x,t \in \Omega$ and $\mathscr{Y}_1,\mathscr{Y}_2 \in D$.

(v) $\lambda_1 = \sup\{|\Psi(x,t,\mathscr{Y})| : x,t \in \Omega, \mathscr{Y} \in D\}$.

(vi) $\lambda_2 = \sup\{|g(x,\mathscr{Y})| : x \in \Omega, \mathscr{Y} \in D\}$ (respectively, $\lambda_2 = \sup\{|v(x,t,\mathscr{Y})| : x,t \in \Omega, \mathscr{Y} \in D\}$) for **case** 2 (respectively, for **case** 3).

Theorem 2.4.3 *Suppose that the conditions (i)-(vi) and* $(L_2\lambda_1 + L_1\lambda_2) < 1$ *are satisfied. Then, the NQVIEs* (2.29) *has a unique solution* $\mathscr{Y} \in D$ *for* ***case*** 2 *and* ***case*** 3.

Proof 2.4.1 *Suppose* F *is a mapping that has been defined as* $F : D \to D$, *where* $F(\mathscr{Y})(x) = f(x) + g(x,\mathscr{Y}(x))\int_0^x \Psi(x,t,\mathscr{Y}(t))\,dt$, *for the* ***case*** 2*: Let* $\mathscr{Y}_1,\mathscr{Y}_2 \in D$, *then*

$$\begin{aligned}
F(\mathscr{Y}_1)(x) - F(\mathscr{Y}_2)(x) &= g(x,\mathscr{Y}_1(x))\int_0^x \Psi(x,t,\mathscr{Y}_1(t))\,dt - g(x,\mathscr{Y}_2(x))\int_0^x \Psi(x,t,\mathscr{Y}_2(t))\,dt,\\
&= g(x,\mathscr{Y}_1(x))\int_0^x \Psi(x,t,\mathscr{Y}_1(t))\,dt - g(x,\mathscr{Y}_2(x))\int_0^x \Psi(x,t,\mathscr{Y}_1(t))\,dt\\
&+ g(x,\mathscr{Y}_2(x))\int_0^x \Psi(x,t,\mathscr{Y}_1(t))\,ds - g(x,\mathscr{Y}_2(x))\int_0^x \Psi(x,t,\mathscr{Y}_2(t))\,dt,\\
&= \big(g(x,\mathscr{Y}_1(x)) - g(x,\mathscr{Y}_2(x))\big)\int_0^x \Psi(x,t,\mathscr{Y}_1(t))\,dt\\
&+ g(x,\mathscr{Y}_2(x))\left(\int_0^x \Psi(x,t,\mathscr{Y}_1(t)) - \Psi(x,t,\mathscr{Y}_2(t))\,dt\right),
\end{aligned} \tag{2.30}$$

so, we have

$$\begin{aligned}
\|F(\mathscr{Y}_1)(x) - F(\mathscr{Y}_2)(x)\| &\leq \sup_{x\in I}\left|g(x,\mathscr{Y}_1(x)) - g(x,\mathscr{Y}_2(x))\right|\int_0^x |\Psi(x,t,\mathscr{Y}_1(t))|\,dt\\
&+ \sup_{x\in I}\left|g(x,\mathscr{Y}_2(x))\right|\int_0^x |\Psi(x,t,\mathscr{Y}_1(t)) - \Psi(x,t,\mathscr{Y}_2(t))|\,dt,\\
&\leq L_2\lambda_1 \sup_{x\in I}|\mathscr{Y}_1(x) - \mathscr{Y}_2(x)| + L_1\lambda_2 \sup_{x\in I}|\mathscr{Y}_1(x) - \mathscr{Y}_2(x)|,\\
&\leq (L_2\lambda_1 + L_1\lambda_2)\,\|\mathscr{Y}_1(x) - \mathscr{Y}_2(x)\|,
\end{aligned} \tag{2.31}$$

*in this case (****case2****)* F *is a contraction mapping on the Banach space* D *under the condition* $L_2\lambda_1 + L_1\lambda_2 < 1$. *Then, there exists a unique solution on* D *of the Eq.* (2.29).

For ***case*** 3, *Employing the same technique and a similar steps, we establish that* F *is a contraction mapping on the Banach space* D *under the condition* $L_2\lambda_1 + L_1\lambda_2 < 1$ *and the assumptions (i)-(vi).*

Chapter 3

Certain numerical methods for solving NIEs

This chapter presents and analyzes different kinds of numerical methods for solving nonlinear IEs. The core principle of these methods is to discretize the equation $\mathscr{Y} = \mathfrak{K}\mathscr{Y}$ by replacing it with a sequence of finite-dimensional approximation problems $\mathscr{Y}_n = \mathfrak{K}_n\mathscr{Y}_n$, where $n \to +\infty$. The types of discretization discussed include:

- Successive approximation methods,
- Nyström method,
- Projection methods, primarily collocation and Galerkin,
- Wavelets method.

A detailed examination of wavelets-based method will be provided in the subsequent chapter.

3.1 Successive approximation method

The description of a straightforward yet effective strategy for dealing with nonlinear equations takes up this part. This technique is sometimes referred to as the Picard iteration method or the method of successive approximation. By beginning with an initial guess, known as the zeroth approximation, this method finds in sequence approximations to the solution, so solving any problem. Any chosen real-valued function that will be employed in a recurrence relation to ascertain the other approximations is the zeroth approximation, as will be observed later. Numerous kinds of nonlinear IEs can frequently be effectively solved using the successive approximation method. (see, [38, 51, 64, 70]). For NVIE of the second kind in the Urysohn form

$$\mathscr{Y}(x) = f(x) + \int_a^x \Psi(x,t,\mathscr{Y}(t))\,dt, \quad a \le x \le b, \tag{3.1}$$

The form of the corresponding recursive expression is

$$\mathscr{Y}_{i+1}(x) = f(x) + \int_a^x \Psi(x,t,\mathscr{Y}_i(t))\,dt, \quad i = 0,1,2,.., \tag{3.2}$$

Typically, the first approximation is made using either the form $\mathscr{Y}_0(x) = 0$ or in the form $\mathscr{Y}_0(x) = f(x)$:

Theorem 3.1.1 *[38] Assume $f \in C[a,b]$ and let $\Psi(x,t,\mathscr{Y})$ be continuous for $a \le x \le t \le b$, . Additionally, assume*

$$|\Psi(x,t,\mathscr{Y}_1) - \Psi(x,t,\mathscr{Y}_2)| \le M|\mathscr{Y}_1 - \mathscr{Y}_2| \quad a \le x \le t \le b, \quad \mathscr{Y}_1, \mathscr{Y}_2 \in \mathbb{R}$$

for a constant M. Then, $\mathscr{Y} \in C[a,b]$ is the unique solution to the IE (3.1). Additionally, for every initial function $\mathscr{Y}_0 \in C[a,b]$, the iterative method (3.2) converges.

Next, for the second-kind NFIE of the form

$$\mathscr{Y}(t) = \lambda \int_a^b \Psi(t,x,\mathscr{Y}(x))\,dx + f(t), \quad a \le t \le b \tag{3.3}$$

we assume that

$$\Psi \in C([a,b] \times [a,b] \times \mathbb{R}) \text{ and } f \in C[a,b]. \tag{3.4}$$

Additionally, we assume that Ψ satisfies a uniform Lipschitz condition for the third argument:

$$|\Psi(t,x,\mathscr{Y}_1) - \Psi(t,x,\mathscr{Y}_2)| \le M|\mathscr{Y}_1 - \mathscr{Y}_2|, \quad a \le x,t \le b, \quad \mathscr{Y}_1, \mathscr{Y}_2 \in \mathbb{R}. \tag{3.5}$$

We can implement the fixed point iteration since (3.3) has the form $\mathscr{Y} = \Re\mathscr{Y}$,

$$\mathscr{Y}_n(t) = \lambda \int_a^b \Psi(t,x,\mathscr{Y}_{n-1}(x))\,dx + f(t), \quad a \le t \le b, \quad n \ge 1. \tag{3.6}$$

Theorem 3.1.2 *[38] Let us assume that f and Ψ satisfy the conditions specified (3.4), (3.5). Additionally, suppose that*

$$|\lambda| M(b-a) < 1.$$

The iteration approach of (3.6) can be used to approximate the unique solution $\mathscr{Y} \in C[a,b]$ of the IE (3.3).

Example 3.1 *[70] The NFIE can be solved using the successive approximations approach.*

$$\mathscr{Y}(t) = \sin t + 1 - \frac{\pi}{12} - \frac{5\pi^2}{144} + \frac{1}{36}\int_0^{\pi} x\left(\mathscr{Y}(x) + \mathscr{Y}^2(x)\right) dx. \tag{3.7}$$

We can choose the zeroth approximation $\mathscr{Y}_0(t)$.

$$\mathscr{Y}_0(t) = 1. \tag{3.8}$$

The iteration formula can be used with the successive approximation method

$$\mathscr{Y}_{n+1}(t) = \sin t + 1 - \frac{\pi}{12} - \frac{5\pi^2}{144} + \frac{1}{36}\int_0^{\pi} x\left(\mathscr{Y}_n(x) + \mathscr{Y}_n^2(x)\right) dx, \quad n \geq 0 \tag{3.9}$$

We find the approximations by substituting (3.8) *into* (3.9)

$$\mathscr{Y}_1(t) \simeq \sin t + 0.669661693,$$

$$\mathscr{Y}_2(t) \simeq \sin t + 0.821457305,$$

$$\mathscr{Y}_3(t) \simeq \sin t + 0.899785379,$$

$$\mathscr{Y}_4(t) \simeq \sin t + 0.942674306,$$

$$\mathscr{Y}_5(t) \simeq \sin t + 0.966871003,$$

and so forth. As a result, the solution $\mathscr{Y}(t)$ *of* (3.7) *is provided by*

$$\mathscr{Y}(t) = \lim_{n\to\infty} \mathscr{Y}_{n+1}(t) = 1 + \sin t.$$

An alternate solution to this equation is provided using the direct computing method and is given by

$$\mathscr{Y}(t) = \sin t - 2 - \frac{4}{\pi}\left(1 - \frac{18}{\pi}\right).$$

3.2 Nyström methods

This section will discuss the so-called quadrature or Nyström approach, which uses an ordinary quadrature rule to approximate the integral terms in a class of NIEs of the second kind. Let $\mathscr{Q} : X = C[a,b] \to \mathbb{R}$ be an integral operator described as

$$\mathcal{Q}(g) = \int_a^b g(t)\,dt$$

and let $\mathcal{Q}_n : X \to \mathbb{R}$ be a discrete operator defined by the quadrature rule

$$\mathcal{Q}_n(g) = \sum_{j=1}^{n} \omega_j^{(n)} g\left(x_j^{(n)}\right) \tag{3.10}$$

The quadrature nodes are denoted by the values $\left\{x_j^{(n)}\right\}_{j=1}^{n}$, while the weights are denoted by $\left\{\omega_j^{(n)}\right\}_{j=1}^{n}$.

Definition 3.2.1 *A sequence of quadrature rules $\mathcal{Q}_n(g)$ is convergent if*

$$\mathcal{Q}_n(g) \to \mathcal{Q}(g) \text{ as } n \to \infty, \text{ for all } g \in X$$

The Newton-Cotes methods are a set of numerical techniques with a step-size of h and nodes that are evenly distributed across the integration interval. Newton-Cotes methods can be divided into two categories. The trapezoidal rule and Simpson's rule are two instances of closed Newton-cotes, which use the end points of the integration interval. Open Newton-Cotes is the other kind. Since the nodes are inside the interval, this type does not employ the interval's end points. The midway rule is an illustration of an open Newton-Cotes. We shall solely look at open Newton-Cotes methods in this thesis.

Understanding that the error terms for the composite trapezium rule (sometimes called the trapezoidal rule) and composite Simpson rule are of the order $O(h^2)$ and $O(h^4)$, respectively, is the importance of the following two results. This demonstrates that as the step size h approaches zero, the error for Simpson's rule converges to zero more quickly than the error for the trapezoidal method.

Corollary 3.2.1 *(Trapezoidal rule)*

Assume that the G subintervals $[x_i, x_{i+1}]$ divided from $[a,b]$ with the width $h = (b-a)/G$. The composite trapezoidal rule

$$T\left(g,h\right)=\frac{h}{2}\left(g\left(a\right)+g\left(b\right)\right)+h\sum_{i=1}^{G-1}g\left(x_i\right),$$

is provide an approximation to the following integral

$$\int_a^b g\left(x\right)\,dx=T\left(g,h\right)+E_T\left(g,h\right).$$

Moreover, if $g\in C^2\left[a,b\right]$*, there exists a constant* $c\in\left]a,b\right[$ *so that the error term* $E_T\left(g,h\right)$ *is*

$$E_T\left(g,h\right)=\frac{\left(a-b\right)g^{(2)}\left(c\right)h^2}{12}=\mathcal{O}\left(h^2\right).$$

Corollary 3.2.2 *(Simpson's rule)*

Assume that the $2G$ *subintervals* $[x_i,x_{i+1}]$ *divided from* $[a,b]$ *of the width* $h=(b-a)/(2G)$*. The composite Simpson rule*

$$S\left(g,h\right)=\frac{h}{3}\left(g\left(a\right)+g\left(b\right)\right)+\frac{2h}{3}\sum_{i=1}^{G-1}g\left(x_{2i}\right)+\frac{4h}{3}\sum_{i=1}^{G}g\left(x_{2i-1}\right)$$

is provide an approximation to the integral

$$\int_a^b g\left(x\right)\,dx=S\left(g,h\right)+E_S\left(g,h\right).$$

Moreover, if $g\in C^4\left[a,b\right]$*, there exists a constant* $c\in\left]a,b\right[$ *so that the error term* $E_S\left(g,h\right)$ *is*

$$E_S\left(g,h\right)=\frac{\left(a-b\right)g^{(4)}\left(c\right)h^4}{180}=\mathcal{O}\left(h^4\right).$$

Corollary 3.2.3 *(Legendre-Gauss Quadrature Rule)*

Let σ_i *be the roots of the Legendre polynomial* $L_G(x)$ *of degree* G*, where* $i=1,\ldots,G$ *and* G *is an integer number. integral over the domain* $[a,b]$ *can be approximated using the* G*-point Legendre-Gauss quadrature rule as follows [42]:*

$$\int_a^b g(x)\,dx=L\left(g\right)+E_L\left(g\right),\tag{3.11}$$

where the Legendre-Gauss quadrature approximation is

$$L(g) = \sum_{i=1}^{G} w_i g(\eta_i),$$

and the weights w_i and nodes η_i are given by:

$$w_i = \frac{b-a}{(1-\sigma_i^2)(L'_G(\sigma_i))^2}, \quad \eta_i = \frac{b-a}{2}\sigma_i + \frac{b+a}{2}, \quad i = 1,\dots,G. \tag{3.12}$$

If $g \in C^{2G+2}[a,b]$. Then, there exists a constant $c \in]a,b[$ such that the error of the Legendre-Gauss quadrature estimation is given by [42]

$$E_L(g) = \frac{2^{2G+3}((G+1)!)^4 g^{(2G+2)}(c)}{(2G+3)((2G+2)!)^3}. \tag{3.13}$$

So that

$$\left|E_L(g)\right| \leq \Gamma_G, \tag{3.14}$$

where

$$\Gamma_G = \frac{2^{2G+3}((G+1)!)^4 \max_{c\in]a,b[}\left|g^{(2G+2)}(c)\right|}{(2G+3)((2G+2)!)^3}.$$

Example 3.2 *Let approximate the following integral*

$$\int_0^{\pi} \exp(t)\,dt = e^{\pi} - 1 \simeq 22.140692632779269006,$$

using the Legendre-Gauss Quadrature Rule, where $g(t) = \exp(t)$. Then, we have this table

G	2	4	6	8
$L(g)$	21.76515773314	22.14060693956	22.14069263003	22.1406926327792451
$E_L(g)$	0.3755	$8.57e^{-05}$	$2.74e^{-09}$	$2.39e^{-14}$

3.2.1 Principle of Nyström methods

Consider the NFIE of the form

$$\mathscr{Y}(x) = f(x) + \int_a^b \Psi(x,t,\mathscr{Y}(t))\,dt, \quad x \in \Omega = [a,b] \tag{3.15}$$

where $f \in X = C(\Omega)$ and $\Psi(x,t,\mathscr{Y}(t))$ are two given functions with an appropriate smoothness assumption, The right-hand side of (3.15) defines a completely continuous operator over an open domain $D \subset X$ into X, specifically

$$\mathfrak{K}(\mathscr{Y})(x) = \int_a^b \Psi(x,t,\mathscr{Y}(t))\,dt, \quad x \in \Omega$$

Therefore, the problem of solving equation (3.15) is equivalent to that of solving the operator equation

$$\mathscr{Y} = f + \mathfrak{K}(\mathscr{Y}), \tag{3.16}$$

using the quadrature formula (Nyström method) (3.10) to approximate the integral in (3.15). Now, we find $\mathscr{Y}_n(x)$ such that

$$\mathscr{Y}_n(x) = f(x) + \sum_{j=1}^{n} \omega_j^{(n)} \Psi\left(x, t_j^{(n)}, \mathscr{Y}_n\left(t_j^{(n)}\right)\right), \quad x \in \Omega \tag{3.17}$$

where $\mathscr{Y}_n(x)$ is an approximation to $\mathscr{Y}(x)$, the operator equation equivalent of the equation (3.17) in notation as

$$\mathscr{Y}_n = f + \mathfrak{K}_n(\mathscr{Y}_n) \tag{3.18}$$

where the integral operators $\mathfrak{K}_n, n \geq 1$ has the form

$$\mathfrak{K}_n(\mathscr{Y})(x) = \sum_{j=1}^{n} \omega_j^{(n)} \Psi\left(x, t_j^{(n)}, \mathscr{Y}\left(t_j^{(n)}\right)\right), \quad x \in \Omega \tag{3.19}$$

To solve the equation (3.17) we determine $\left\{\mathscr{Y}_n\left(x_i^{(n)}\right)\right\}$, thus the equation (3.17) is reduced to the finite of nonlinear algebraic system

$$z_i = f\left(x_i^{(n)}\right) + \sum_{j=1}^{n} \omega_j^{(n)} \Psi\left(x_i^{(n)}, t_j^{(n)}, z_j^{(n)}\right), \quad i = 1, \cdots, n, \tag{3.20}$$

where the interpolatory function

$$z(x) = f(x) + \sum_{j=1}^{n} \omega_j^{(n)} \Psi\left(x, t_j^{(n)}, z_j^{(n)}\right), \quad x \in \Omega$$

satisfies (3.17), (for more details see[[48],[11]]). For the theoretical convergence analysis we use (3.17), however we solve (3.20) practically.

3.2.2 Convergence analysis

This section examines the behavior (existence and convergence) of the approximate solution defined by equation (3.18) in a neighborhood of an isolated solution to equation (3.16). Our analysis rests on two key premises:

- Equation (3.18) admits an isolated solution $\mathscr{Y}_0 \in X$.
- The theory of collectively compact operators (see [5, 24]) provides the framework for determining the order of convergence.

Under these conditions, we demonstrate the existence of the approximate solution and quantify its rate of convergence.

An isolated solution $\mathscr{Y}_0 \in X$,e.i , there exists a ball

$$B(\mathscr{Y}_0, \delta) = \{\mathscr{Y} \in X : \|\mathscr{Y} - \mathscr{Y}_0\| \leq \delta, \delta > 0\}$$

The compact operator $\mathfrak{K}$ has a continuous first and bounded second derivative on $B(\mathscr{Y}_0, \delta)$, and there are no other than $\mathscr{Y}_0$ solutions to (3.16). For reference, the required assumptions are listed in [8] and [71].

I. $\{\mathfrak{K}_n : n \geq 1\}$ is a family of collectively compact on X.

II. The operators $\mathfrak{K}_n$ is convergent to $\mathfrak{K}$ on X.

III. For $n \geq 1, \mathfrak{K}_n$ possesses a first and second Fréchet derivatives are continuous and bound respectively on $B(\mathscr{Y}_0, \delta)$. Additionally, $\left\|\mathfrak{K}_n''\right\| \leq \lambda < \infty$.

Lemma 3.2.1 *(Weiss [57]) Under hypotheses* I – III, *if* $\left[I - \mathfrak{K}'(\mathscr{Y}_0)\right]$ *is nonsingular, then for all sufficiently large n (i.e., n* $\geq n_1$*), the linear operators* $\left[I - \mathfrak{K}'(\mathscr{Y}_0)\right]$ *are also nonsingular, and*

$$\left\|\left[I - \mathfrak{K}_n'(\mathscr{Y}_0)\right]^{-1}\right\| \leq \beta < \infty. \tag{3.21}$$

Theorem 3.2.1 *Under the assumptions of Lemma 3.2.1, there exists a constant* $n_1 \in \mathbb{N}$ *such that* $\forall n \geq n_1$, *equation (3.18) possesses a unique solution* $\mathscr{Y}_n \in B(\mathscr{Y}_0, \delta)$. *Moreover, there exists a positive constant C, separate of n, for which the following bound holds:*

$$\|\mathscr{Y}_0 - \mathscr{Y}_n\| \leq C \|\mathfrak{K}(\mathscr{Y}_0) - \mathfrak{K}_n(\mathscr{Y}_0)\|$$

Proof 3.2.1 *Subtract* (3.16) *from* (3.18) *to have*

$$\mathscr{Y}_0 - \mathscr{Y}_n = \mathfrak{K}(\mathscr{Y}_0) - \mathfrak{K}_n(\mathscr{Y}_n),$$

now, adding the expression $\mathfrak{K}'_n(\mathscr{Y}_0)(\mathscr{Y}_0 - \mathscr{Y}_n)$ *on two sides, we get*

$$\left[I - \mathfrak{K}'_n(\mathscr{Y}_0)\right](\mathscr{Y}_0 - \mathscr{Y}_n) = \mathfrak{K}(\mathscr{Y}_0) - \mathfrak{K}_n(\mathscr{Y}_0) - \left[\mathfrak{K}_n(\mathscr{Y}_n) - \mathfrak{K}_n(\mathscr{Y}_0) - \mathfrak{K}'_n(\mathscr{Y}_0)(\mathscr{Y}_n - \mathscr{Y}_0)\right].$$

The expression $\mathfrak{K}_n(\mathscr{Y}_n) - \mathfrak{K}_n(\mathscr{Y}_0) - \mathfrak{K}'_n(\mathscr{Y}_0)(\mathscr{Y}_n - \mathscr{Y}_0)$ *is limited by the expression* $\frac{1}{2}\lambda\|\mathscr{Y}_0 - \mathscr{Y}_n\|^2$ *(see [6]), using lemma 3.2.1, we get*

$$\|\mathscr{Y}_0 - \mathscr{Y}_n\| \leq \beta\left[\|\mathfrak{K}(\mathscr{Y}_0) - \mathfrak{K}_n(\mathscr{Y}_0)\| + \frac{1}{2}\lambda\|\mathscr{Y}_0 - \mathscr{Y}_n\|^2\right],$$

moreover

$$\begin{aligned}\|\mathscr{Y}_0 - \mathscr{Y}_n\| &\leq \frac{\beta\|\mathfrak{K}(\mathscr{Y}_0) - \mathfrak{K}_n(\mathscr{Y}_0)\|}{1 - \frac{\beta\lambda\delta}{2}} \\ &\leq \frac{\beta}{1 - \frac{\beta\lambda\delta}{2}}\|\mathfrak{K}(\mathscr{Y}_0) - \mathfrak{K}_n(\mathscr{Y}_0)\|.\end{aligned}$$

This completes the proof.

Based on this theorem, we may conclude that the rate of convergence of $\mathscr{Y}_n$ to $\mathscr{Y}_0$ is the numerical integration approach used to $\mathfrak{K}(\mathscr{Y}_0)$, which is typically simple to obtain.

3.3 Projection methods

Projection methods approximate the solution of the nonlinear IE

$$\mathscr{Y}(x) = \int_\Omega \Psi(x, t, \mathscr{Y}(t))\,dt, \quad x \in \Omega = [a, b] \tag{3.22}$$

by seeking a function $\mathscr{Y}_n(x)$ within a finite-dimensional subspace approach to the exact solution $\mathscr{Y}(x)$. This approximation is chosen to satisfy Eq. (3.22) in an approximate sense. Different interpretations of this approximation yield distinct methods; the most prominent

are collocation and Galerkin methods, defined subsequently. For a comprehensive theoretical background on projection methods, see [11, 10, 9, 36, 35, 47, 66, 72].

3.3.1 Projection operators

Definition 3.3.1 *[46] Let Y be a nontrivial subspace on normed space X. A projection operator $P : X \to Y$ is a bounded linear operator with the property*

$$P(\mathscr{Y}) = \mathscr{Y} \quad \text{for all } \mathscr{Y} \in Y. \tag{3.23}$$

From (3.23), it follows that for any $\mathscr{Y} \in Y$,

$$P^2(\mathscr{Y}) = P(P(\mathscr{Y})) = P(\mathscr{Y}),$$

since $P(\mathscr{Y}) \in Y$. This implies the idempotency property:

$$P^2 = P,$$

Furthermore, using the submultiplicative property of the operator norm,

$$\|P\| = \left\|P^2\right\| \leq \|P\|^2,$$

moreover, we get

$$|P| \geq 1. \tag{3.24}$$

Orthogonal projection operators

Definition 3.3.2 *A projection operator P on a Hilbert space X be orthogonal if and only if*

$$\langle P\mathscr{Y}, (I-P)\mathscr{S}\rangle = 0, \quad \forall \mathscr{Y}, \mathscr{S} \in X \tag{3.25}$$

Example 3.3 *Let $\{\varphi_n\}_{n\geq 1}$ be an orthonormal basis of X. For each n, define the finite-dimensional subspace $X_n = \operatorname{span}\{\varphi_1, \cdots, \varphi_n\}$. The orthogonal projection $P_n : X \to X_n$ is given by*

$$P_n(\mathscr{Y}) = \sum_{i=1}^{n} \langle \mathscr{Y}, \varphi_i \rangle \varphi_i, \quad \forall \mathscr{Y} \in X.$$

Interpolatory projection operators

Let us define the interpolation framework:

X: a normed vector space.

X_n: an n-dimensional subspace of X with basis $\{\varphi_1, \cdots, \varphi_n\}$.

$x_1, \cdots, x_n \in \Omega$: distinct interpolation nodes.

The objective is to approximate a function $\mathscr{Y} \in X$ by an element $\mathscr{Y}_n \in X_n$. The corresponding interpolation problem is:

For given data $\{y_i\}_{i=1}^n$, find coefficients $\{\alpha_j\}_{j=1}^n$ such that the function

$$\mathscr{Y}_n(x) = \sum_{j=1}^{n} \alpha_j \varphi_j(x), \tag{3.26}$$

satisfies $\mathscr{Y}_n(x_i) = y_i$ for all $\quad i = 1, \cdots, n$.

The coefficients $\alpha_1, \cdots, \alpha_n$ are therefore found by solving the linear system

$$\sum_{j=1}^{n} \alpha_j \varphi_j(x_i) = y_i, \quad i = 1, \cdots, n. \tag{3.27}$$

Solution of (3.27) to be unique, it is necessary and sufficient that

$$\det\left[\varphi_j(x_i)\right] \neq 0 \quad i, j = 1...n. \tag{3.28}$$

Then, the solution of the interpolation problem has the form

$$P_n(\mathscr{Y})(x) = \sum_{i=1}^{n} \mathscr{Y}(x_i)\, \varphi_i(x) \tag{3.29}$$

where the interpolatory projection operator from X onto X_n is P_n.

3.3.2 Principle of projection methods

Let X be $C(\Omega)$ or $L^2(\Omega)$ or another Banach space. And we consider the operator of the NFIEs of the form

$$\mathfrak{K}(\mathscr{Y})(x) = \int_{\Omega} \Psi(x, t, \mathscr{Y}(t))\, dt, \quad x \in \Omega = [a, b] \tag{3.30}$$

Our objective is to find a solution to the operator IE

$$\mathscr{Y} = \mathfrak{K}\mathscr{Y}, \tag{3.31}$$

Consider a sequence $X_n, n \geq 1$ of finite-dimensional subspaces of X, each of a fixed dimension N, and let P_n denote a projection operator from X onto X_n. The analysis typically relies on the following assumption:

$$P_n\mathscr{Y} \to \mathscr{Y}, \text{ as } n \to \infty, \quad \forall \mathscr{Y} \in X, \tag{3.32}$$

The projection method approximates the solution of (3.31) by finding $\mathscr{Y}_n \in X_n$ such that

$$\mathscr{Y}_n = P_n\mathfrak{K}(\mathscr{Y}_n), \tag{3.33}$$

Consequently, we arrive at an approximate fixed-point problem, expressible in the equivalent form:

$$P_n(I - \mathfrak{K})(\mathscr{Y}_n) = 0, \quad \mathscr{Y}_n \in X_n \tag{3.34}$$

Lemma 3.3.1 *[11] Suppose X is a Banach space and P_n is a family of bounded projections on X converging to the identity. Then for any compact operator $\mathfrak{K} : X \to X$,*

$$\|\mathfrak{K} - P_n\mathfrak{K}\| \to 0 \text{ as } n \to \infty \tag{3.35}$$

Theorem 3.3.1 *[11] Let X be a Banach space and $\mathfrak{K} : X \to X$ is a bounded. And assume that $\lambda - \mathfrak{K} : X \xrightarrow[onto]{1-1} X$. Suppose P_n is a sequence of bounded projections on X for which*

$$\|\mathfrak{K} - P_n\mathfrak{K}\| \to 0 \text{ as } n \to \infty \tag{3.36}$$

Then there exists a bounded operator $(\lambda - P_n\mathfrak{K})^{-1} : X \to X$ for all large n; $n \geq N$.

Furthermore

$$\sup_{n \geq N} \left\| (\lambda - P_n\mathfrak{K})^{-1} \right\| < \infty \tag{3.37}$$

3.3.3 Convergence of projection methods

To demonstrate that the approximating system (3.33) admits a unique solution for all sufficiently large n and to derive the bounds of convergence rates for $\mathscr{Y}_n \to \mathscr{Y}$, we apply a linearization technique. The core of this method is to replace the original nonlinear function with its first-order Taylor expansion around a point $\mathscr{Y}_0$, thereby linearizing the problem and enabling the application of the Banach Fixed-Point Theorem.

$$\mathfrak{K}(\mathscr{Y}) = \mathfrak{K}(\mathscr{Y}_0) + \mathfrak{K}'(\mathscr{Y}_0)(\mathscr{Y} - \mathscr{Y}_0), \tag{3.38}$$

Let $\mathfrak{K}'(\mathscr{Y}_0)$ denote the Fréchet derivative of $\mathfrak{K}$ at $\mathscr{Y}_0$. We now examine the error incurred by linearizing $\mathfrak{K}(\mathscr{Y})$ around $\mathscr{Y}_0$, given by:

$$R(\mathscr{Y};\mathscr{Y}_0) = \mathfrak{K}(\mathscr{Y}) - \left[\mathfrak{K}(\mathscr{Y}_0) + \mathfrak{K}'(\mathscr{Y}_0)(\mathscr{Y} - \mathscr{Y}_0)\right], \tag{3.39}$$

The linearization error $R(\mathscr{Y};\mathscr{Y}_0)$ possesses key properties that are instrumental in analyzing the convergence.

Lemma 3.3.2 *[11] Let X be a Banach space, $E \subset X$ an open subset, and $\mathfrak{K} : E \to X$ a twice continuously differentiable map whose second derivative$\mathfrak{K}(\mathscr{Y})$ is bounded on bounded subsets of E. Consider a closed, bounded, convex set $B \subset E$ with non-empty interior and a point $\mathscr{Y}_0$ in the interior of B. Then, for the linearization error $R(\mathscr{Y};\mathscr{Y}_0)$ defined previously and for all $\mathscr{Y}_1, \mathscr{Y}_2 \in B$, the following holds:*

$$\|R(\mathscr{Y}_1;\mathscr{Y}_2)\| \leq \frac{1}{2} M \|\mathscr{Y}_1 - \mathscr{Y}_2\|^2, \tag{3.40}$$

where $M = \sup_{\mathscr{Y} \in B} \left\|\mathfrak{K}''(u)\right\|$. In addition we have

$$\left\|\mathfrak{K}'(\mathscr{Y}_2) - \mathfrak{K}'(\mathscr{Y}_1)\right\| \leq M \|\mathscr{Y}_2 - \mathscr{Y}_1\| \tag{3.41}$$

which implies that $\mathfrak{K}'(\mathscr{Y})$ is Lipschitz continuous, and

$$\|R(\mathscr{Y}_1;\mathscr{Y}_0) - R(\mathscr{Y}_2;\mathscr{Y}_0)\| \leq M \left[\|\mathscr{Y}_1 - \mathscr{Y}_0\| - \frac{1}{2}\|\mathscr{Y}_1 - \mathscr{Y}_2\|\right] \|\mathscr{Y}_1 - \mathscr{Y}_2\|. \tag{3.42}$$

We now state the following assumptions on the operators $\Re$ and $\{P_n\}$ the following assumptions.

A_1. *The nonlinear operator $K : E \subseteq X \to X$ is a completely continuous.*

A_2. *Eq (3.31) has a unique isolated solution $\mathscr{Y}^* \in X$. Furthermore, there exists an $\varepsilon > 0$ such that this solution is unique within the closed ball*

$$B\left(\mathscr{Y}^*, \varepsilon\right) = \left\{\mathscr{Y} \in E / \left\|\mathscr{Y} - \mathscr{Y}^*\right\| \leq \varepsilon\right\}. \tag{3.43}$$

A_3. *The operator $\Re$ is twice continuously Fréchet differentiable on E, and its second derivative is uniformly bounded on $B\left(\mathscr{Y}^*, \varepsilon\right)$:*

$$M = \sup_{\mathscr{Y} \in B(\mathscr{Y}^*, \varepsilon)} \left\|\Re''(\mathscr{Y})\right\| < \infty. \tag{3.44}$$

A_4. *The sequence of projection operators $\{P_n\}$ converges to the identity operator on X:*

$$P_n(\mathscr{Y}) \to \mathscr{Y} \text{ as } n \to \infty, \quad \forall u \in X. \tag{3.45}$$

It follows from Assumption A_1 that the linear operator $L = \Re'\left(\mathscr{Y}^*\right)$ is compact. Simultaneously, Assumption A_4 satisfies the conditions of Lemma 3.3.1, from which we obtain

$$\|(I - P_n) L\| \to 0 \text{ as } n \to \infty, \tag{3.46}$$

which implies that for all $n, n \geq N$ (with N sufficiently large), the inverse operator $(I - P_n L)^{-1}$ exists and

$$\sup_{n \geq N} \left\|(I - P_n L)^{-1}\right\| < \infty. \tag{3.47}$$

By Theorem 3.3.1, it suffices to show that for all sufficiently large n, (3.33) has a unique solution in some ball $B\left(\mathscr{Y}^*, \varepsilon_1\right)$ with $0 \leq \varepsilon_1 \leq \varepsilon$. Expanding $\Re(\mathscr{Y}_n)$ about $\mathscr{Y}^*$ gives

$$\Re(\mathscr{Y}_n) = \Re\left(\mathscr{Y}^*\right) + L\left(\mathscr{Y}_n - \mathscr{Y}^*\right) + R\left(\mathscr{Y}_n; \mathscr{Y}^*\right), \tag{3.48}$$

we therefore obtain the following equivalent formulation of (3.33)

$$(I - P_n L)\left(\mathscr{Y}_n - \mathscr{Y}^*\right) = P_n\mathscr{Y}^* - \mathscr{Y}^* + P_n R\left(\mathscr{Y}_n; \mathscr{Y}^*\right), \tag{3.49}$$

we put $\gamma_n = \mathscr{Y}_n - \mathscr{Y}^*$, *then we get*

$$\begin{aligned}\gamma_n &= (I - P_n L)^{-1}\left(P_n\mathscr{Y}^* - \mathscr{Y}^*\right) + (I - P_n L)^{-1} R\left(\gamma_n + \mathscr{Y}^*; \mathscr{Y}^*\right)\\ &\equiv F_n\left(\gamma_n\right)\end{aligned}$$

By combining the established results with the Banach fixed-point theorem, we can show that the fixed-point equation (3.49) has a unique solution γ_n. This, in turn, proves that the approximating equation (3.33) has a unique solution $\mathscr{Y}_n$ in a ball of fixed radius around $\mathscr{Y}^*$.

The rate of convergence

Theorem 3.3.2 *[9] Suppose 1 is not eigenvalue of* $L = \mathfrak{K}'\left(\mathscr{Y}^*\right)$, *Then we can find a nonnegative sequence* $\{w_n\}$ *convergent to zero and satisfies*

$$\left\|\mathscr{Y}_n - \mathscr{Y}^*\right\| \leq C\,(1 + w_n)\left\|P_n\mathscr{Y}^* - \mathscr{Y}^*\right\|. \tag{3.50}$$

Proof 3.3.1 *Eq.* (3.48) *can be rewritten as the equivalent identity*

$$(I - L)\left(\mathscr{Y}_n - \mathscr{Y}^*\right) = (P_n - I)\,L\left(\mathscr{Y}_n - \mathscr{Y}^*\right) + (P_n - I)\,\mathscr{Y}^* + P_n R\left(\mathscr{Y}_n; \mathscr{Y}^*\right) \tag{3.51}$$

A bound on the right-hand side of this identity then leads to (3.50), *where the constants are given by*

$$w_n = \frac{C\,(a_n + Pr_n)}{1 - C\,(a_n + Pr_n)}, \quad a_n = \|(I - P_n)\,L\|, \quad r_n = \frac{\|R\left(\mathscr{Y}_n; \mathscr{Y}^*\right)\|}{\|\mathscr{Y}_n - \mathscr{Y}^*\|},$$
$$C = \left\|(I - L)^{-1}\right\|, \quad \sup_n \|P_n\| \leq P < \infty.$$

Collocation method

Consider the NFIEs of the form

$$\mathscr{Y}(x) = f(x) + \int_{\Omega} \Psi(x,t,\mathscr{Y}(t))\,dt, \quad x \in \Omega. \tag{3.52}$$

Let $\mathscr{Y}_n(x) = \sum_{i=1}^{N} \alpha_j \varphi_j(x), \quad x \in \Omega$. We consider X the space of continuous functions $C(\Omega)$, and X_n be a finite dimensional subsequence of X, let P_n be the interpolatory projection operator of X into X_n.

$\{\varphi_j\}_{j=1}^{N}$ be a basis of X_n, and let $\{x_i\}_{i=1}^{N} \in \Omega$ be a set of distinct node points. The collocation method requires that the approximate solution $\mathscr{Y}_n(x)$ satisfy the original equation exactly at these nodes. This leads to the system:

$$R_n(x_i) \equiv \sum_{i=1}^{N} \alpha_j \varphi_j(x_i) - f(x_i) - \int_{\Omega} \Psi\left(x_i, t, \sum_{i=1}^{N} \alpha_j \varphi_j(t)\right) dt = 0 \tag{3.53}$$

whereR_n is the residual, which measures the error in the equation when $\mathscr{Y}$ is replaced by the approximation $\mathscr{Y}_n$.

By forcing the residual to be zero at the collocation points $\{x_i\}_{i=1}^{N}$, we obtain a system of equations to determine the coefficients$\{\alpha_j\}_{i=1}^{N}$. The goal is for the resulting function $\mathscr{Y}_n(x)$ to be a close approximation to the exact solution $\mathscr{Y}(x)$.

Galerkin method

Consider a Hilbert space X, such as $L^2(\Omega)$, and a finite-dimensional subspace $X_n \subset X$. The orthogonal projection operator $P_n : X \to X_n$ is defined by the condition

$$\langle P_n\mathscr{Y}, \mathscr{S}\rangle = \langle \mathscr{Y}, \mathscr{S}\rangle, \quad \text{for all } \mathscr{S} \in X_n \tag{3.54}$$

where $\langle .,. \rangle$ denotes the inner product on $L^2(\Omega)$.

The Galerkin method for the NFIEs (3.52) is defined as follows. Let $\{\varphi_j\}_{j=1}^{N}$ be a basis of the finite-dimensional subspace X_n. The method seeks an approximate solution $\mathscr{Y}_n \in X_n$ of the form

$$\mathscr{Y}_n(x) = \sum_{j=1}^{N} \alpha_j \varphi_j(x)$$

such that

$$\langle R_n, \varphi_j \rangle = 0, \quad j = 1, \cdots, N$$

The result is a non-trivial system

$$\sum_{j=1}^{N} \alpha_j \langle \varphi_j, \varphi_i \rangle = \langle f, \varphi_i \rangle + \left\langle \int_\Omega K\left(\cdot, t, \sum_{j=1}^{N} \alpha_j \varphi_j(t)\right) dt, \varphi_i \right\rangle, \quad i = 1, \cdots, N \tag{3.55}$$

The coefficients $\{\alpha_j\}_{j=1}^{N}$, obtained by solving this system, to define the approximate solution $\mathscr{Y}_n(x)$. The accuracy of the method is gauged by the proximity of $\mathscr{Y}_n(x)$ to the exact solution $\mathscr{Y}(x)$.

3.4 Wavelets method

Emerging in the late 20th century, one of the projection methods known as the wavelet method has developed into a strong tool in numerical analysis, signal processing, and applied mathematics because of its capacity to produce local time-frequency representations. Wavelets also serve as a tool for analyzing transient, non-stationary, and time-varying phenomena. A wavelet is a wave oscillation whose amplitude, initially zero, grows monotonically and then decays to zero. The wavelet series is often represented by a square-integrable function or by a set of orthonormal basis functions. Three main classes of wavelets are generally distinguished: discrete, continuous, and multiresolution wavelets. Continuous wavelets are projected onto a continuous function space, while discrete wavelets are generally considered on a discrete subset of the upper half-plane. The analysis of discrete wavelets only considers a finite number of coefficients, which can sometimes lead to numerical complexity. In this case, multiresolution wavelets are preferred.

The application of wavelets in analyzing dynamic systems and differential equations arising in other sciences has greatly improved numerical solutions by giving sparse representations of operators and effective computational methods [74, 28, 29, 18, 26, 25, 37]. The largest development development for wavelets was in the mid 1980s. Orthonormal wavelets were initially studied by Meyer [74] over the real line $\mathbb{R}$. Daubechie's (1992) work on compactly supported wavelets and Mallat's (1989) multiresolution analysis (MRA) framework led to the first application of Wavelets in IEs early in the 1990s.

A mother wavelet is a type of function generate a whole family of wavelets that is derived through the dilation and translation. The next figure is a graphs of some famous mother wavelets,

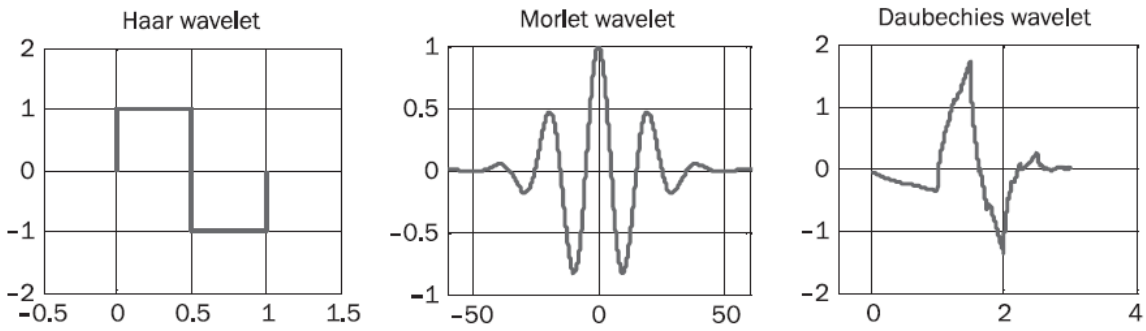

Figure 3.1: Graphs of some famous mother wavelets

The continuous wavelets family with dilation parameter h and translation parameter r is defined as [74]:

$$\xi_{h,r}(x) = |h|^{-1/2}\xi\left(\frac{x-r}{h}\right), \quad h,r \in \mathbb{R}, h \neq 0. \tag{3.56}$$

For practical applications, one prefers continuously differentiable functions with compact support as mother wavelet. However, to satisfy analytical requirements (in the continuous WT) and in general for theoretical reasons, one chooses the wavelet functions from a subspace of the space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Being in this space ensures that mother wavelet can satisfy $\int_{-\infty}^{\infty} \xi(x)\,dx = 0$ and $\int_{-\infty}^{\infty} |\xi(x)|^2\,dx = 1$.

If we put $h = h_0^{-k}, r = sr_0h_0^{-k}, h_0 > 1, r_0 > 0$ in (3.56), then the discrete wavelet family consists

of the following members:

$$\xi_{k,s}(x) = |h_0|^{k/2}\,\xi\left(h_0^k x - s r_0\right), \quad k,s \in \mathbb{Z}^+, \tag{3.57}$$

For a specific choice of $h_0 = 2$ and $r_0 = 1, \xi_{k,s}(x) = 2^{k/2}\xi\left(2^k x - s\right)$ constitutes the wavelet basis for $L_2(\mathrm{R})$.

Orthogonal polynomial wavelets

we can define a orthogonal polynomial wavelets $\xi_{n,m}(x)$ over the interval $[0;1]$ as follows:

$$\xi_{n,m}(x) = \begin{cases} \dfrac{2^{\frac{k}{2}}}{||P_m(x)||_{L^2}} P_m\left(2^k x - n\right), & \dfrac{n}{2^k} \le x \le \dfrac{n+1}{2^k} \\ 0, & otherwise, \end{cases} \tag{3.58}$$

where k is assumed to be any positive integer, $n = 0,1,2,...,2^k - 1$ and the coefficient $||P_m(x)||_{L^2}$ is for normality and $m = 0,1,...,M+1$. Here $P_m(x)$ is a orthogonal polynomials of degree m we can choice any polynomials like Legendre, Chebyshev, Hermit... for define a orthogonal polynomial wavelets.

Remark 3.4.1 *We can shift the polynomials if he not define on* $[0,1]$.

Function approximation

we can approximate $\mathscr{Y}$ function from $L^2([0;1))$ using polynomial wavelets as follows

$$\mathscr{Y}(x) \simeq \mathscr{Y}^*(x) = \sum_{n=0}^{2^k-1} \sum_{m=0}^{M+1} y_{n,m}\xi_{n,m}(x) = U^T \xi(x), \tag{3.59}$$

with

$$y_{n,m} = \langle \mathscr{Y}(x), \xi_{n,m}(x) \rangle_{w(x)},$$

where $w(x)$ is a proper weight function of orthogonal polynomials P_m, U and $\xi(x)$ are $\mathfrak{M} \times 1$ matrices, with $\mathfrak{M} = 2^k(M+2)$. Their structures are given by:

$$\begin{aligned} U &= \left[y_{0,0},\ldots,y_{0,M+1} \mid y_{1,0},\ldots,y_{1,M+1} \mid \ldots \mid y_{2^{k-1},0},\ldots,y_{2^{k-1},M+1}\right]^T, \\ \xi(t) &= \left[\xi_{0,0}(t),\ldots,\xi_{0,M+1}(t) \mid \xi_{1,0}(t),\ldots,\xi_{1,M+1}(t) \mid \ldots \mid \xi_{2^{k-1},0}(t),\ldots,\xi_{2^{k-1},M+1}(t)\right]^T. \end{aligned} \tag{3.60}$$

3.4.1 Principle of wavelets method

Consider the NFIE of the form

$$\mathscr{Y}(x) = f(x) + \int_0^1 \Psi(x,t,\mathscr{Y}(t))dt, \quad x \in \Omega = [0,1]. \tag{3.61}$$

For $f \in X = C(\Omega)$ and $\Psi(x,t,\mathscr{Y}(t))$ satisfying appropriate smoothness assumptions, the right-hand side of (3.61) defines a completely continuous operator $\Psi : D \subset X \to X$, explicitly defined as:

$$\Re(\mathscr{Y})(x) = \int_0^1 \Psi(x,t,\mathscr{Y}(t))dt, \quad x \in \Omega$$

Therefore, the problem of solving equation (3.61) is equivalent to that of solving the operator equatio

$$\mathscr{Y} = f + \Re(\mathscr{Y}), \tag{3.62}$$

using the orthogonal polynomial wavelets (3.58) to approximate $\mathscr{Y}$ in (3.61) to $\mathscr{Y}^*$ where $\mathscr{Y}^*(x) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M+1} y_{n,m}\xi_{n,m}(x)$ such that

$$\mathscr{Y}^* = f + \Re(\mathscr{Y}^*), \tag{3.63}$$

The functional Eq. (3.63) can be solved by finding the set of coefficients $\{y_{n,m}\}$, which transforms the problem into a finite nonlinear system. Using Collocation method with the Newton-Cotes nodes $x_j = \dfrac{2j-1}{2\mathfrak{M}}$, for $j = 1,\ldots,\mathfrak{M}$ where $\mathfrak{M} = 2^k(M+2)$. Resulting

$$\mathscr{Y}^*(x_j) = f(x_j) + \Re(\mathscr{Y}^*)(x_j), \quad j = 1,\ldots,\mathfrak{M}. \tag{3.64}$$

Which is subsequently solved using Newton's method.

Chapter 4

Numerical solution of nonlinear quadratic IEs using Vieta-Lucas wavelets method

In this chapter, we will solve the quadratic IE in three cases using the Vieta-Lucas wavelet method, studying the convergence and estimating the error in each case, along with testing this method with examples accompanied by tables and graphs programmed by Mathematica 10.3 software.

4.1 Vieta-Lucas wavelets

The functions $\xi_{s,m}$ defined on area $[0,1]$ by $\xi_{s,m}(x) = \xi(k,s,m,x)$ are called Vieta-Lucas wavelets if they have the following form [41]:

$$\xi_{s,m}(x) = \begin{cases} \frac{2^{\frac{k+2}{2}}}{\sqrt{\alpha_m}} V_m\left(2^{k+2}x - 4s + 2\right), & \frac{s-1}{2^k} \le x \le \frac{s}{2^k}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

The parameters are defined as follows: $k \in \mathbb{Z}^+$, $s = 1,2,\ldots,2^k$, and $m = 0,1,2,\ldots,M+1$, where $M+1$ denoting the highest polynomial degree. α_m is the normalization value. Where

$$\alpha_m = \begin{cases} 4\pi, & m = 0, \\ 2\pi, & m \ge 1. \end{cases}$$

The next figures represent some of graphs Vieta-Lucas wavelet for $k = 0$ where define on the domain $[0,1]$.

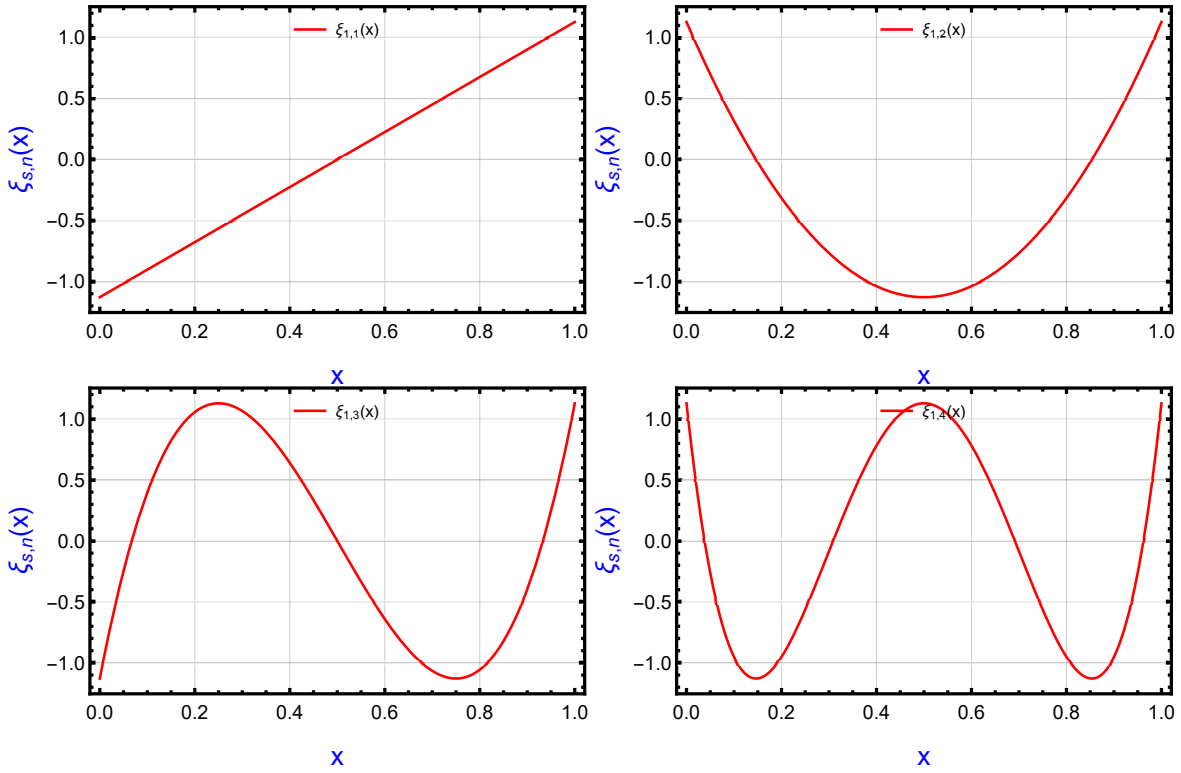

Figure 4.1: Graph of some Vieta-Lucas wavelet for $k = 0$.

The next figures represent some of graphs Vieta-Lucas wavelet for $k = 1$ and $s = 1, 2$, where define on the domain $[0, 0.5]$ and $[0.5, 1]$ respectively.

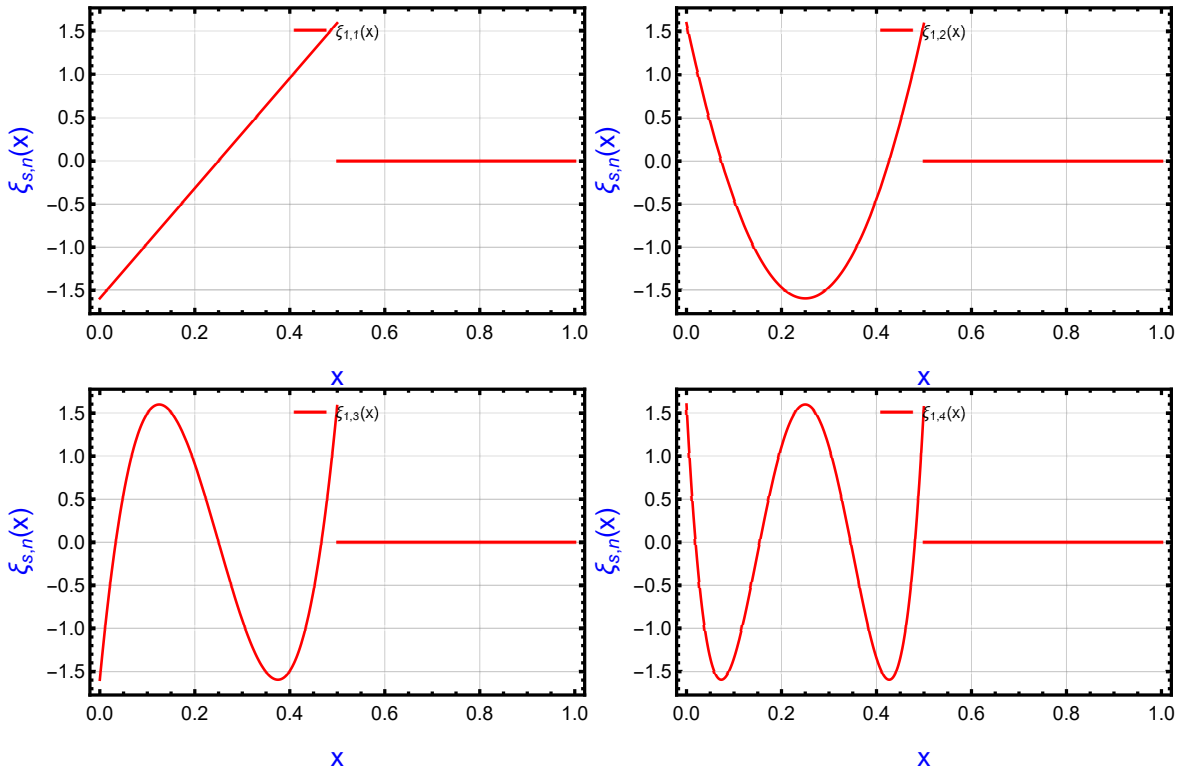

Figure 4.2: Graph of some Vieta-Lucas wavelet for $k = 1$ and $s = 1$.

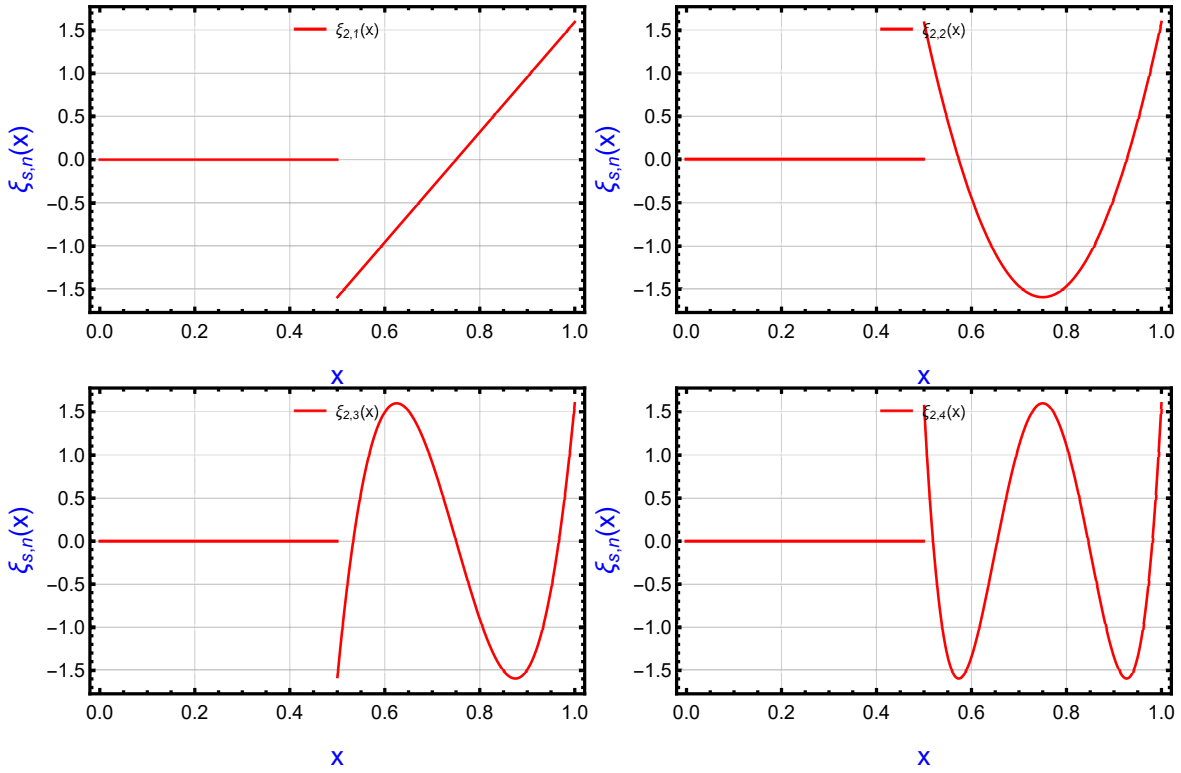

Figure 4.3: Graph of some Vieta-Lucas wavelet for $k = 1$ and $s = 2$.

The Vieta-Lucas Wavelets are orthogonal in $L^2_{w_s}[0,1]$ where the weight function is $w_s(x) = w\left(2^{k+2}x - 4s + 2\right)$. That is, they satisfy the condition:

$$\left\langle \xi_{s,m}(x), \xi_{s',m'}(x) \right\rangle_{w_s(x)} = \int_0^1 \xi_{s,m}(x)\xi_{s',m'}(x)\, w_s(x)\, dt = \begin{cases} 1, & (s,m) = (s',m'), \\ 0, & (s,m) \neq (s',m'). \end{cases} \tag{4.2}$$

Let $\mathscr{Y}(x)$ a function from $L^2_{w_s}[0,1]$. Then, $\mathscr{Y}(x)$ can be approximate using the Vieta-Lucas Wavelets series

$$\mathscr{Y}(x) = \sum_{s=1}^{\infty} \sum_{m=0}^{\infty} y_{s,m} \xi_{s,m}(x), \tag{4.3}$$

The coefficients $y_{s,m}$ in this expansion are calculated as follows

$$y_{s,m} = \int_0^1 \mathscr{Y}(x)\xi_{s,m}(x)\, w_s(x)\, dx. \tag{4.4}$$

The truncated series expansion in terms of Vieta-Lucas wavelets is given by

$$\mathscr{Y}(x) \simeq \mathscr{Y}^*(x) = \sum_{s=1}^{2^k} \sum_{m=0}^{M+1} y_{s,m} \xi_{s,m}(x) = U^T \xi(x), \tag{4.5}$$

where U and $\xi(x)$ are $\mathfrak{M}\times 1$ matrices, with $\mathfrak{M}=2^k(M+2)$. Their structures are given by:

$$\begin{aligned} U &= \left[y_{1,0}, y_{1,1}, \ldots, y_{1,M+1} \mid y_{2,0}, \ldots, y_{2,M+1} \mid \ldots \mid y_{2^k,0}, \ldots, y_{2^k,M+1}\right]^T, \\ \xi(t) &= \left[\xi_{1,0}(t), \ldots, \xi_{1,M+1}(t) \mid \xi_{2,0}(t), \ldots, \xi_{2,M+1}(t) \mid \ldots \mid \xi_{2^k,0}(t), \ldots, \xi_{2^k,M+1}(t)\right]^T. \end{aligned} \tag{4.6}$$

4.2 Solution of nonlinear IEs using Vieta-Lucas wavelets

4.2.1 Nonlinear quadratic Volterra IEs

In this section, we solve three cases of the quadratic Volterra IEs numerically by applying the Vieta-Lucas Wavelets method, supported by Legendre-Gauss Quadrature Rule. Let consider the NQVIEs of the form:

$$\mathscr{Y}(x) = f(x) + g(x, \mathscr{Y}(x)) \int_0^x \Psi(x, s, \mathscr{Y}(s))\, ds, \quad 0 \le x < 1. \tag{4.7}$$

First, we approximate the function $\mathscr{Y}(x)$ in (4.7) using the Vieta-Lucas Wavelets method in the following way:

$$\mathscr{Y}(x) \simeq \mathscr{Y}^*(x) = U^T \xi(x), \tag{4.8}$$

U denotes an $\mathfrak{M}$-dimensional column vector of unknown coefficients, and $\xi(x)$ as in (4.5) and (4.6). Substituting (4.8) into (4.7), we have

$$U^T \xi(x) = f(x) + g\left(x, U^T \xi(x)\right) \int_0^x \Psi\left(x, s, U^T \xi(s)\right) ds. \tag{4.9}$$

Next, we set $s = x\epsilon$, where $\epsilon \in [0,1]$, and apply Legendre-Gauss Quadrature Rule to approximate the integral:

$$\begin{aligned} \int_0^x \Psi\left(x, s, U^T \xi(s)\right) ds &= \int_0^1 x\Psi\left(x, x\epsilon, U^T \xi(x\epsilon)\right) d\epsilon \\ &= \sum_{i=1}^{G} x w_i \Psi\left(x, x\eta_i, U^T \xi(x\eta_i)\right), \end{aligned} \tag{4.10}$$

where w_i and η_i are the weights and nodes of the Legendre-Gauss Quadrature Rule, respectively.

Substituting (4.10) into (4.9), we derive

$$U^T \xi(x) = f(x) + g\left(x, U^T \xi(x)\right) \sum_{i=1}^{G} x w_i \Psi\left(x, x\eta_i, U^T \xi(x\eta_i)\right). \tag{4.11}$$

We now consider three cases for the function $g(x,\mathscr{Y}(x))$:

Case 1: If $g(x,\mathscr{Y}(x))=\lambda$ then the Quadratic Volterra IEs become the stander Volterra IEs

$$U^T\xi(x)=f(x)+\lambda\sum_{i=1}^{G}xw_i\Psi\left(x,x\eta_i,U^T\xi(x\eta_i)\right), \tag{4.12}$$

we applying collocation method with the nodes of Newton-Cotes $x_j=\dfrac{2j-1}{2\mathfrak{M}}$, for $j=1,\ldots,\mathfrak{M}$ at the Eq (4.12). Yields, a system of nonlinear algebraic equations for the unknown U, which is subsequently solved using Newton's method.

Case 2: If $g(x,\mathscr{Y}(x))$ is explicitly given, we applying collocation method with the nodes of Newton-Cotes $x_j=\dfrac{2j-1}{2\mathfrak{M}}$, for $j=1,\ldots,\mathfrak{M}$ at the Eq (4.11), Yields, a system of nonlinear algebraic equations for the unknown U, which is subsequently solved using Newton's method.

Case 3: If $g(x,\mathscr{Y}(x))=\displaystyle\int_0^x v\left(x,s,\mathscr{Y}(s)\right)ds$, in eq. (4.7), we approximate the integral using the same Legendre-Gauss Quadrature Rule method from (4.10), yielding

$$\int_0^x v\left(x,s,U^T\xi(s)\right)ds=\sum_{l=1}^{G'}xw_l v\left(x,x\eta_l,U^T\xi(x\eta_l)\right). \tag{4.13}$$

Substituting Eqs. (4.13) and (4.10) into (4.9), we obtain

$$U^T\xi(x)=g(x)+\sum_{l=1}^{G'}xw_l v\left(x,x\eta_l,U^T\xi(x\eta_l)\right)\sum_{i=1}^{G}xw_i\Psi\left(x,x\eta_i,U^T\xi(x\eta_i)\right). \tag{4.14}$$

With G consistently chosen as G, applying collocation method with the nodes of Newton-Cotes $x_j=\dfrac{2j-1}{2\mathfrak{M}}$, for $j=1,\ldots,\mathfrak{M}$ at the Eq (4.14). Yields, a system of nonlinear algebraic equations for the unknown U, which is subsequently solved using Newton's method.

4.2.2 Nonlinear quadratic Fredholm IEs

In this section, we solve three cases of the Quadratic Fredholm IEs numerically by applying the Vieta-Lucas Wavelets method, supported by Legendre-Gauss Quadrature Rule. Let consider the NQFIEs of the form:

$$\mathscr{Y}(x)=f(x)+g\left(x,\mathscr{Y}(x)\right)\int_0^1\Psi\left(x,s,\mathscr{Y}(s)\right)ds,\quad 0\le x<1. \tag{4.15}$$

First, we approximate the function $\mathscr{Y}(x)$ in (4.15) using the Vieta-Lucas Wavelets method in the following way:

$$\mathscr{Y}(x) \simeq \mathscr{Y}^*(x) = U^T \xi(x), \tag{4.16}$$

U denotes an $\mathfrak{M}$-dimensional column vector of unknown coefficients, and $\xi(x)$ as in (4.5) and (4.6). Substituting (4.16) into (4.15), we have

$$U^T \xi(x) = f(x) + g\left(x, U^T \xi(x)\right) \int_0^1 \Psi\left(x, s, U^T \xi(s)\right) ds. \tag{4.17}$$

Next, the integral is approximated using the Legendre-Gauss quadrature rule:

$$\int_0^1 \Psi\left(x, s, U^T \xi(s)\right) ds = \sum_{i=1}^{G} w_i \Psi\left(x, \eta_i, U^T \xi(\eta_i)\right), \tag{4.18}$$

where w_i and η_i are the weights and nodes of the Legendre-Gauss Quadrature Rule, respectively. Substituting (4.18) into (4.17), we derive

$$U^T \xi(x) = f(x) + g\left(x, U^T \xi(x)\right) \sum_{i=1}^{G} w_i \Psi\left(x, \eta_i, U^T \xi(\eta_i)\right). \tag{4.19}$$

We now consider three cases for the function $g(x, \mathscr{Y}(x))$:

Case 1: If $f(x, \mathscr{Y}(x)) = \lambda$ then the Quadratic Fredholm IEs become the stander FIEs

$$U^T \xi(x) = f(x) + \lambda \sum_{i=1}^{G} w_i \Psi\left(x, \eta_i, U^T \xi(\eta_i)\right), \tag{4.20}$$

we applying collocation method with the nodes of Newton-Cotes $x_j = \dfrac{2j-1}{2\mathfrak{M}}$, for $j = 1, \ldots, \mathfrak{M}$ at the Eq (4.20), Yields, a system of nonlinear algebraic equations for the unknown U, which is subsequently solved using Newton's method.

Case 2: If $g(x, \mathscr{Y}(x))$ is explicitly given, we applying collocation method with the nodes of Newton-Cotes $x_j = \dfrac{2j-1}{2\mathfrak{M}}$, for $j = 1, \ldots, \mathfrak{M}$ at the Eq (4.19), Yields, a system of nonlinear algebraic equations for the unknown U, which is subsequently solved using Newton's method.

Case 3: If $g(x, \mathscr{Y}(x)) = \displaystyle\int_0^1 v(x, s, \mathscr{Y}(s))\, ds$, in Eq.(4.15), we approximate the integral using the same Legendre-Gauss Quadrature Rule method from (4.18), yielding

$$\int_0^1 v\left(x, s, U^T \xi(s)\right) ds = \sum_{l=1}^{G'} w_l v\left(x, \eta_l, U^T \xi(\eta_l)\right). \tag{4.21}$$

Substituting Eqs. (4.21) and (4.18) into (4.17), we obtain

$$U^T\xi(x) = f(x) + \sum_{l=1}^{G'} w_l v\left(x,\eta_l,U^T\xi(\eta_l)\right)\sum_{i=1}^{G} w_i \Psi\left(x,\eta_i,U^T\xi(\eta_i)\right). \tag{4.22}$$

With G consistently chosen as G, applying collocation method with the nodes of Newton-Cotes $x_j = \dfrac{2j-1}{2\mathfrak{M}}$, for $j = 1,\ldots,\mathfrak{M}$ at the Eq (4.22). Yields, a system of nonlinear algebraic equations for the unknown U, which is subsequently solved using Newton's method.

4.3 Convergence analysis

Theorem 4.3.1 *Suppose that $\mathscr{Y}(x) \in L^2_{w_s}[0,1]$ has a bounded second derivative $H > 0$ $\left(i.e., \left|\mathscr{Y}''(x)\right| \le H\right)$. Then $\mathscr{Y}(x)$ admits an infinite expansion in terms of Vieta-Lucas wavelets, where the series is uniformly convergent as $k, M \to \infty$. Moreover:*

$$\left|y_{s,m}\right| \le \frac{H\sqrt{\pi}}{2^{\frac{5}{2}} s^{\frac{5}{2}}\left(m^2-1\right)}, m > 1, s \ge 1 \text{ and } \left|y_{s,1}\right| \le \frac{\sqrt{\pi}}{2^{\frac{5}{2}} s^{\frac{3}{2}}} \max_{0\le t\le 1}\left|\mathscr{Y}'(x)\right|, m = 1, s \ge 1.$$

where the coefficients $y_{s,m}$ given in (4.4).

Proof 4.3.1 *see [41].*

Lemma 4.3.1 *Suppose $f(t)$ is a positive, continuous, decreasing function for all $t \ge s$ and let $f(k) = \Lambda_k$, If the series $\sum \Lambda_s$ converges, then the following inequality holds $\sum_{k=s+1}^{\infty} \Lambda_k \le \int_s^{\infty} f(t)dt$.*

Theorem 4.3.2 *Let $\mathscr{Y}(x)$ be a function that satisfies Theorem 4.3.1. And let $\mathscr{Y}_{M,k}(x)$ be the approximation of $\mathscr{Y}(x)$ using Vieta-Lucas wavelets. Then, the error bound estimate is*

$$\left|\mathscr{Y}(x) - \mathscr{Y}_{M,k}(x)\right| \le \frac{H(2M+3)}{3\times 2^{k+1}(M+1)(M+2)} = \Theta_M^k. \tag{4.23}$$

Proof 4.3.2 *Considering the Vieta-Lucas Wavelets expansion as*

$$\mathscr{Y}_{M,k}(x) = \sum_{s=1}^{2^k}\sum_{m=0}^{M+1} y_{s,m}\xi_{s,m}(x),$$

from (4.3), *we obtain*

$$|\mathscr{Y}(x)-\mathscr{Y}_{M,k}(x)| \leq \sum_{s=2^k+1}^{\infty}\sum_{m=M+2}^{\infty}\left|y_{s,m}\right|\left|\xi_{s,m}(x)\right|.$$

Using the definition of $\xi_{s,m}(x)$ *in* (4.1) *then Theorem* 4.3.1 *for* $m>1$, *we can be express*

$$\begin{aligned}
|\mathscr{Y}(x)-\mathscr{Y}_{M,k}(x)| &\leq \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}\sum_{s=2^k+1}^{\infty}\sum_{m=M+2}^{\infty}\left|y_{s,m}\right|,\\
&\leq \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}\sum_{s=2^k+1}^{\infty}\sum_{m=M+2}^{\infty}\frac{H\sqrt{\pi}}{2^{\frac{5}{2}}s^{\frac{5}{2}}\left|m^2-1\right|},\\
&= H2^{\frac{k-2}{2}}\left(\sum_{s=2^k+1}^{\infty}\frac{1}{s^{\frac{5}{2}}}\right)\left(\sum_{m=M+2}^{\infty}\frac{1}{\left|m^2-1\right|}\right),\\
&= H2^{\frac{k-2}{2}}\left(\sum_{s=2^k+1}^{\infty}\frac{1}{s^{\frac{5}{2}}}\right)\left(\frac{2M+3}{2(M+1)(M+2)}\right),
\end{aligned} \tag{4.24}$$

from Lemma 4.3.1, *we obtain*

$$\begin{aligned}
|\mathscr{Y}(x)-\mathscr{Y}_{M,k}(x)| &\leq H2^{\frac{k-2}{2}}\left(\int_{\xi=2^k}^{\infty}\frac{d\xi}{\xi^{\frac{5}{2}}}\right)\left(\frac{2M+3}{2(M+1)(M+2)}\right),\\
&= \Theta_M^k \xrightarrow[k,m\to\infty]{} 0.
\end{aligned} \tag{4.25}$$

Inequality (4.25) *shows that the Vieta-Lucas wavelet approximation error* $|\mathscr{Y}(x)-\mathscr{Y}_{M,k}(x)|$ *is* $\mathscr{O}(1/(kM))$, *and thus decays with increasing* M *or* k.

Theorem 4.3.3 *Suppose* $\mathscr{Y}(x)$ *and* $\mathscr{Y}^*_{M,k}(x)$ *denote the exact and approximate solutions to equation* (2.29), *respectively. Provided that conditions (i)–(vi) hold, the inequality* $(L_1\lambda_2+L_2\lambda_1)<1$ *is satisfied, and the requirements of Theorem* 4.3.1 *are met, then for both* ***case 1*** *and* ***case 2*** *we obtain the following:*

Let $\mathscr{Y}(x)$ *and* $\mathscr{Y}^*_{M,k}(x)$ *be the exact and approximate solutions of equation* (2.29). *The assumption for* ***case 1*** *that the conditions* (i) – (iii) *and* $L_1<1$ *hold, and the requirements of Theorem* 4.3.1 *are met.*

The assumption for ***case 2*** *and* ***case 3*** *that the conditions (i)–(vi)hold, the inequality* $(L_1\lambda_2+L_2\lambda_1)<1$ *is satisfied, and the requirements of Theorem* 4.3.1 *are met.*

Then, respectively in ***case 1***, ***case 2*** *and* ***case 3***, *we get*

$$\|\mathscr{Y}(x)-\mathscr{Y}^*_{M,k}(x)\| \le \max\left\{\Theta^k_M, \lambda L_1\Theta^k_M+\Gamma_G\right\},$$

$$\|\mathscr{Y}(x)-\mathscr{Y}^*_{M,k}(x)\| \le \max\left\{\Theta^k_M, (L_2\lambda_1+L_1\lambda_2)\Theta^k_M+\lambda_2\Gamma_G\right\} \textit{ and}$$

$$\|\mathscr{Y}(x)-\mathscr{Y}^*_{M,k}(x)\| \le \max\left\{\Theta^k_M, L_1(\lambda_2+\Gamma'_G)\Theta^k_M+\Gamma_G(\lambda_2+\Gamma'_G)+\Gamma'_G\lambda_1\right\}.$$

Proof 4.3.3 *Consider Eq.* (4.12) *in* ***case*** 1*:*

$$\mathscr{Y}^*_{M,k}(x) = f(x)+\lambda\sum_{i=1}^{G} x w_i \Psi\left(x, x\eta_i, \mathscr{Y}^*_{M,k}(x\eta_i)\right), \tag{4.26}$$

where $\mathscr{Y}^*_{M,k}(x) = U^T\xi(x)$*, by subtracting* (2.29) *and* (4.30)*, we get*

$$\begin{aligned}\mathscr{Y}(x)-\mathscr{Y}^*_{M,k}(x) &= \lambda\int_0^x \Psi(x,s,\mathscr{Y}(s))ds-\lambda\sum_{i=1}^{G} x w_i\Psi\left(x,x\eta_i,\mathscr{Y}^*_{M,k}(x\eta_i)\right),\\ &= \lambda\left(\int_0^x\left(\Psi(x,s,\mathscr{Y}(s))-\Psi(x,s,\mathscr{Y}^*_{M,k}(s))\right)ds+\int_0^x\Psi(x,s,\mathscr{Y}^*_{M,k}(s))ds\right.\\ &\quad\left.-\sum_{i=1}^{G} x w_i\Psi\left(x,x\eta_i,\mathscr{Y}^*_{M,k}(x\eta_i)\right)\right).\end{aligned} \tag{4.27}$$

Assume $\mathscr{Y}(x)$ *satisfies conditions* (i) $--$(iii)*, and from Eq.* (3.14)*, we get*

$$\begin{aligned}\|\mathscr{Y}(x)-\mathscr{Y}^*_{M,k}(x)\| &\le \lambda\left(\int_0^x\left|\Psi(x,s,\mathscr{Y}(s))-\Psi(x,s,\mathscr{Y}^*_{M,k}(s))\right|ds+\left|\int_0^x\Psi(x,s,\mathscr{Y}^*_{M,k}(s))ds\right.\right.\\ &\quad\left.\left.-\sum_{i=1}^{G} x w_i\Psi\left(x,x\eta_i,\mathscr{Y}^*_{M,k}(x\eta_i)\right)\right|\right),\\ &\le \lambda\left(L_1\|\mathscr{Y}(x)-\mathscr{Y}^*_{M,k}(x)\|+\Gamma_G\right).\end{aligned} \tag{4.28}$$

Assume $\mathscr{Y}(x)$ *satisfies Theorem* 4.3.1*, and using Theorem* 4.3.2*. Then, we have*

$$\|\mathscr{Y}(x)-\mathscr{Y}^*_{M,k}(x)\| \le \max\left\{\Theta^k_M, \lambda\left(L_1\Theta^k_M+\Gamma_G\right)\right\}. \tag{4.29}$$

In ***case*** 2*:*

$$\mathscr{Y}^*_{M,k}(x) = f(x)+g(x,\mathscr{Y}^*_{M,k}(x))\sum_{i=1}^{G} x w_i\Psi\left(x,x\eta_i,\mathscr{Y}^*_{M,k}(x\eta_i)\right), \tag{4.30}$$

where $\mathscr{Y}^*_{M,k}(x) = U^T\xi(x)$*, by subtracting* (2.29) *and* (4.30)*, we get*

$$\begin{aligned}
\mathscr{Y}(x)-\mathscr{Y}^*_{M,k}(x) &= g(x,\mathscr{Y}(x))\int_0^x \Psi(x,s,\mathscr{Y}(s))ds - g(x,\mathscr{Y}^*_{M,k}(t))\sum_{i=1}^{G} xw_i\Psi\left(x,x\eta_i,\mathscr{Y}^*_{M,k}(x\eta_i)\right),\\
&= g(x,\mathscr{Y}(x))\int_0^x \Psi(x,s,\mathscr{Y}(s))ds - g(x,\mathscr{Y}^*_{M,k}(x))\int_0^x \Psi(x,s,\mathscr{Y}(s))ds\\
&+ g(x,\mathscr{Y}^*_{M,k}(x))\int_0^x \Psi(x,s,\mathscr{Y}(s))ds - g(x,\mathscr{Y}^*_{M,k}(x))\sum_{i=1}^{G} xw_i\Psi\left(x,x\eta_i,\mathscr{Y}^*_{M,k}(x\eta_i)\right),\\
&= \left(g(x,\mathscr{Y}(x))-g(x,\mathscr{Y}^*_{M,k}(x))\right)\int_0^x \Psi(x,s,\mathscr{Y}(s))ds\\
&+ g(x,\mathscr{Y}^*_{M,k}(x))\left(\int_0^x \Psi(x,s,\mathscr{Y}(s))ds - \sum_{i=1}^{G} xw_i\Psi\left(x,x\eta_i,\mathscr{Y}^*_{M,k}(x\eta_i)\right)\right),\\
&= \left(g(x,\mathscr{Y}(x))-g(x,\mathscr{Y}^*_{M,k}(x))\right)\int_0^x \Psi(x,s,\mathscr{Y}(s))ds + g(x,\mathscr{Y}^*_{M,k}(x))\left(\int_0^x\Big(\Psi(x,s,\mathscr{Y}(s))\right.\\
&\left.-\Psi(x,s,\mathscr{Y}^*_{M,k}(s))\Big)ds + \int_0^x \Psi(x,s,\mathscr{Y}^*_{M,k}(s))ds - \sum_{i=1}^{G} xw_i\Psi\left(x,x\eta_i,\mathscr{Y}^*_{M,k}(x\eta_i)\right)\right),
\end{aligned} \tag{4.31}$$

Assume $\mathscr{Y}(x)$ satisfies conditions (i)–(vi), and from Eq. (3.14), we obtain

$$\begin{aligned}
\|\mathscr{Y}(x)-\mathscr{Y}^*_{M,k}(x)\| &\le \left|g(x,\mathscr{Y}(x))-g(x,\mathscr{Y}^*_{M,k}(x))\right|\int_0^x |\Psi(x,s,\mathscr{Y}(s))|ds + |g(x,\mathscr{Y}^*_{M,k}(x))|\times\\
&\left(\int_0^x |\Psi(x,s,\mathscr{Y}(s))-\Psi(x,s,\mathscr{Y}^*_{M,k}(s))|ds + \left|\int_0^x \Psi(x,s,\mathscr{Y}^*_{M,k}(s))ds\right.\right.\\
&\left.\left.-\sum_{i=1}^{G} xw_i\Psi\left(x,x\eta_i,\mathscr{Y}^*_{M,k}(x\eta_i)\right)\right|\right),\\
&\le L_2\|\mathscr{Y}(x)-\mathscr{Y}^*_{M,k}(x)\|\lambda_1 + \lambda_2\left(L_1\|\mathscr{Y}(x)-\mathscr{Y}^*_{M,k}(x)\| + \Gamma_G\right).
\end{aligned} \tag{4.32}$$

Suppose $\mathscr{Y}(x)$ satisfies the assumptions of Theorem 4.3.1, and using Theorem 4.3.2. Then, we get

$$\|\mathscr{Y}(x)-\mathscr{Y}^*_{M,k}(x)\| \le \max\left\{\Theta^k_M, (L_2\lambda_1+L_1\lambda_2)\Theta^k_M + \lambda_2\Gamma_G\right\}. \tag{4.33}$$

*For **case** 3, from (4.12) we have*

$$\mathscr{Y}^*_{M,k}(x) = f(x) + \sum_{l=1}^{G'} xw_l\upsilon\left(x,x\eta_l,\mathscr{Y}^*_{M,k}(x\eta_l)\right)\sum_{i=1}^{G} xw_i\Psi\left(x,x\eta_i,\mathscr{Y}^*_{M,k}(x\eta_i)\right). \tag{4.34}$$

Employing the same technique, we able to establish the next inequalities

$$\begin{aligned}\|\mathscr{Y}(x)-\mathscr{Y}^*_{M,k}(x)\| &\leq \left|\int_0^x v(x,s,\mathscr{Y}(s))ds-\sum_{l=1}^{G'} xw_l v\left(x,x\eta_l,\mathscr{Y}^*_{M,k}(x\eta_l)\right)\right|\int_0^x |\Psi(x,s,\mathscr{Y}(s))|ds\\ &+\left|\sum_{l=1}^{G'} xw_l v\left(x,x\eta_l,\mathscr{Y}^*_{M,k}(x\eta_l)\right)\right|\left|\int_0^x \Psi(x,s,\mathscr{Y}(s))ds-\sum_{i=1}^{G} xw_i\Psi\left(x,x\eta_i,\mathscr{Y}^*_{M,k}(x\eta_i)\right)\right|\\ &\leq \Gamma'_G\lambda_x+(\lambda_2+\Gamma'_G)\left(L_1\|\mathscr{Y}(x)-\mathscr{Y}^*_{M,k}(x)\|+\Gamma_G\right),\\ &= L_1(\lambda_2+\Gamma'_G)\|\mathscr{Y}(x)-\mathscr{Y}^*_{M,k}(x)\|+\Gamma_G(\lambda_2+\Gamma'_G)+\Gamma'_G\lambda_1.\end{aligned} \tag{4.35}$$

Suppose $\mathscr{Y}(x)$ satisfies the assumptions of Theorem 4.3.1, and using Theorem 4.3.2. Then, we obtain

$$\|\mathscr{Y}(x)-\mathscr{Y}^*_{M,k}(x)\|\leq \max\left\{\Theta^k_M, L_1(\lambda_2+\Gamma'_G)\Theta^k_M+\Gamma_G(\lambda_2+\Gamma'_G)+\Gamma'_G\lambda_1\right\}. \tag{4.36}$$

Observe that $\|\mathscr{Y}(x)-\mathscr{Y}^*_{M,k}(x)\|_\infty\to 0$, as $G,k,M\to\infty$ since $\Theta^k_M\xrightarrow[k,M\to\infty]{}0$, $\Gamma_G\xrightarrow[J\to\infty]{}0$, and $\Gamma'_G\xrightarrow[J\to\infty]{}0$.

About the three cases of the quadratic Fredholm IEs by the similar steps and techniques again we can have the same three result in quadratic Volterra IEs respectively.

Chapter 5

Numerical testing of some of the proposed methods

The accuracy of the proposed method (using Vieta-Lucas wavelets) in chapter 4 and successive approximation method is assessed using the maximum absolute error, defined in this chapter as:

$$E = \|\mathscr{Y}(t) - \mathscr{Y}^*(t)\| = \sup_{0\le t\le 1} |\mathscr{Y}(t) - \mathscr{Y}^*(t)|, \tag{5.1}$$

where $\mathscr{Y}(t)$ is the exact solutions and $\mathscr{Y}^*(t)$ represent the approximate solutions. To demonstrate that the method's efficacy, several numerical examples are presented. All computations were performed in Wolfram Mathematica 10.3 over the interval $0 \le t \le 1$ with the parameter $k = 1$ for Vieta-Lucas wavelets method.

Example 1. An examination of the following NQVIE:

$$\mathscr{Y}(t) = \sin(t) - \frac{t^3 \sin(t)}{30(1+\sin(t))} + g(t,\mathscr{Y}(t)) \int_0^t \Psi(t,s,\mathscr{Y}(s))\,ds, \tag{5.2}$$

where

$$g(t,\mathscr{Y}(t)) = \frac{|\mathscr{Y}(t)|}{3(1+|\mathscr{Y}(t)|)}, \qquad |g(t,\mathscr{Y}(t))| \le \frac{1}{3} = \lambda_2, \qquad \left|\frac{\partial g(t,\mathscr{Y})}{\partial \mathscr{Y}}\right| \le \frac{1}{3} = L_2,$$
$$\Psi(t,s,\mathscr{Y}(s)) = \frac{t}{5}\arcsin(|\mathscr{Y}(s)|), \quad |\Psi(t,s,\mathscr{Y}(s))| \le \frac{\pi}{10} = \lambda_1, \quad \left|\frac{\partial \Psi(t,s,\mathscr{Y})}{\partial \mathscr{Y}}\right| \le \frac{1}{5} = L_1,$$

The uniqueness of the solution for Eq. (5.2), $\mathscr{Y}(t) = \sin(t)$, is established by Theorem 2.4.3. The performance of the proposed method in chapter 4 is quantified by the maximum absolute error E, computed at $M = 2,6$ and $G = 2,4,6$; these results are compiled in Table 5.1. Figures 5.1 and 5.2 represents the absolute errors at $M = 2,6$ and $G = 2,6$ respectively. Figure 5.3 is the graph of the exact versus approximate solution at $M = 8$ and $G = 6$.

	$G=2$		$G=4$		$G=6$	
M	2	6	2	6	2	6
E	$4.8155e^{-5}$	$3.14880e^{-11}$	$4.7377e^{-5}$	$3.1650e^{-11}$	$4.7239e^{-5}$	$3.1217e^{-11}$
CPU Time	0.031192	0.137650	0.047399	0.190661	0.051935	0.24920

Table 5.1: Maximum absolute error E for Example 1.

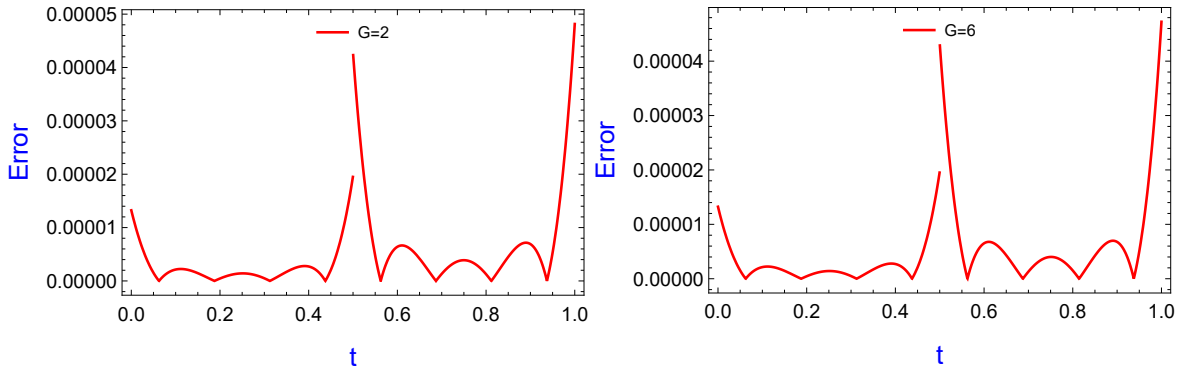

Figure 5.1: Absolute errors at $M = 2$ for Example 1.

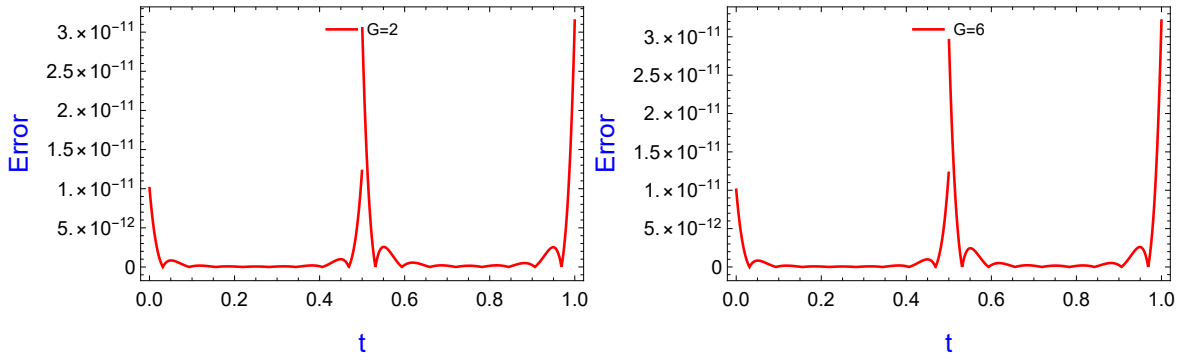

Figure 5.2: Absolute errors at $M = 6$ for Example 1.

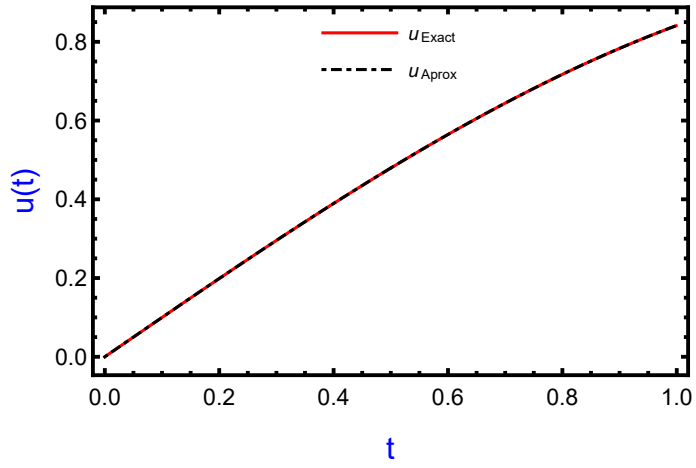

Figure 5.3: Exact versus approximate solution at $M = 8$ and $G = 6$ for Example 1.

Now, the next Table 5.2, represents the maximum absolute error of successive approximation method (Picard method) at $n = 2,4,6$, and Figure 5.4 shows the graphs of the absolute error for the same numbers.

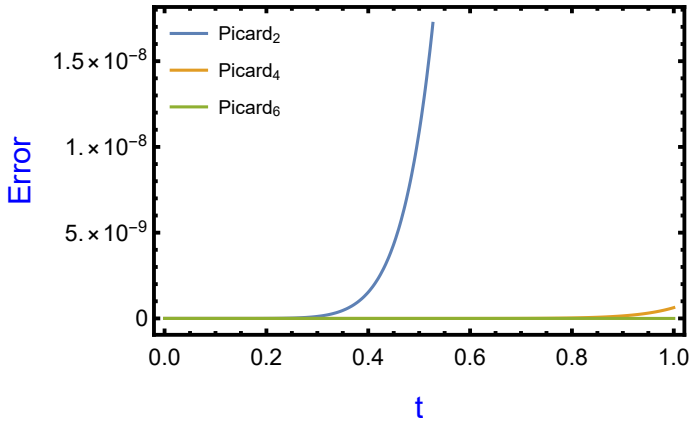

Figure 5.4: Absolute errors at $n = 2,4,6$ for Example 1.

n	2	4	6
E	$4.41251e^{-6}$	$6.29814e^{-10}$	$6.93751e^{-14}$

Table 5.2: Maximum absolute error E for Example 1.

Example 2. An examination of the following NQVIE [50]:

$$\mathscr{Y}(t) = e^t + \frac{t\sqrt{e^t}}{10}\left(\ln(e^{-t}+1) - \ln 2\right) + \sqrt{\mathscr{Y}(t)}\int_0^t \frac{0.1t}{1+\mathscr{Y}(s)}\,ds. \tag{5.3}$$

The unique solution of Eq. (5.3) is established as $\mathscr{Y}(t) = e^t$ [50]. To evaluate the accuracy of the proposed method in chapter 4, the maximum absolute error E was computed for parameters $G = 4,6$ and $M = 4,6,8$; these results are summarized in Table 5.3. The efficacy of the method is further demonstrated through a comparative analysis with the state-of-the-art FPM [50] and Picard method in Table 5.4. Finally, the convergence of the method is visualized in Figure 5.5, which shows the absolute errors across different values of M for $G = 6$. Figure 5.6 represents the graph of Exact versus approximate solution at $M = 8$ and $G = 6$.

	$G=4$			$G=6$		
M	4	6	8	4	6	8
E	$1.6552e^{-7}$	$1.5465e^{-10}$	$2.1207e^{-10}$	$1.6477e^{-7}$	$9.9786e^{-11}$	$7.2868e^{-14}$
CPU Time	0.129105	0.25691	0.46221	0.169522	0.342608	0.594012

Table 5.3: Maximum absolute error E for Example 2.

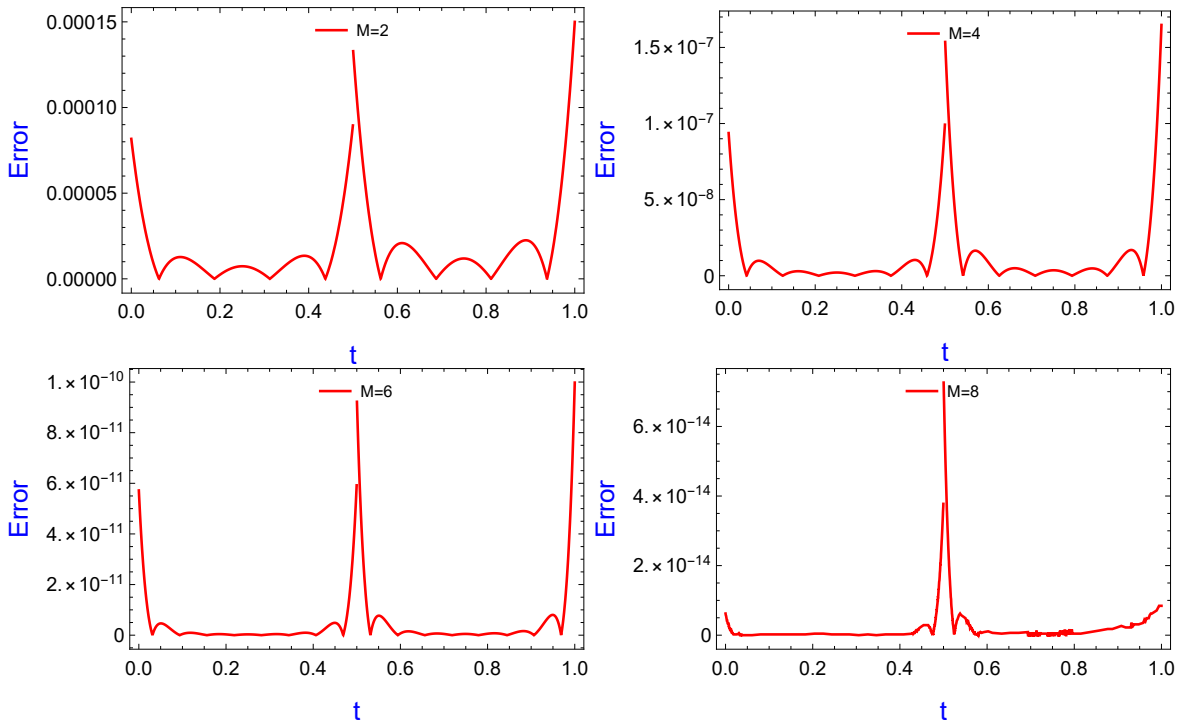

Figure 5.5: Absolute errors for different values of M at $G=6$ for Example 2.

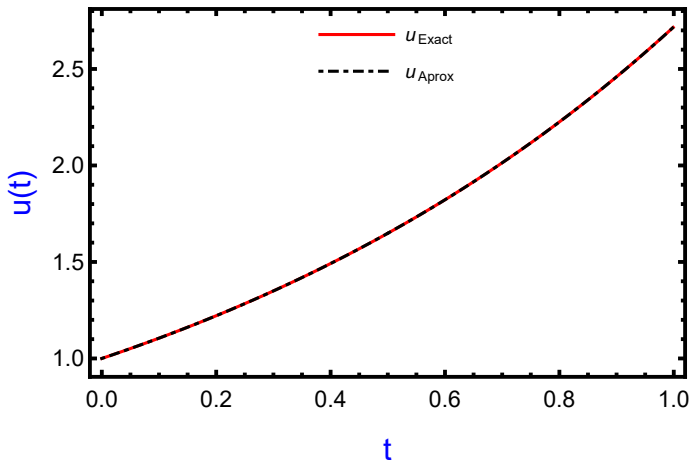

Figure 5.6: Exact versus approximate solution at $M=8$ and $G=6$ for Example 2.

FPM [50] $N=100$	Picard method $n=6$	method in Ch 4 $M=8, G=6$
$1.166e^{-07}$	$1.6875e^{-14}$	$7.2868e^{-14}$

Table 5.4: Comparing the maximum absolute error for Example 2.

Now, the next Table 5.5, represents the maximum absolute error of successive approximation method (Picard method) at $n=2,4,6$, and Figure 5.7 shows the graphs of the absolute error for the same numbers.

n	2	4	6
E	$3.04044e^{-6}$	$2.50171e^{-10}$	$1.68754e^{-14}$

Table 5.5: Maximum absolute error E for Example 2.

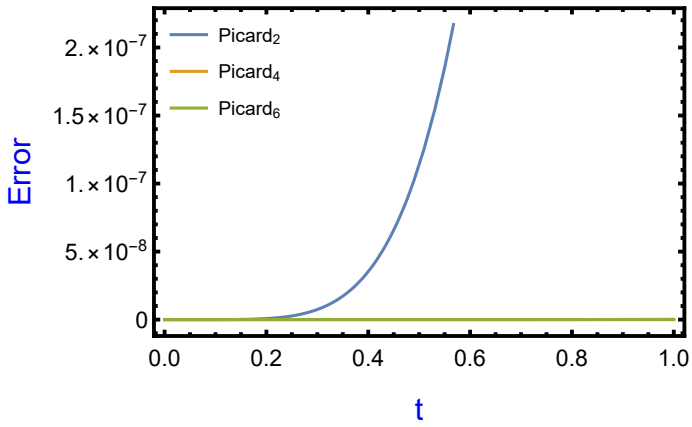

Figure 5.7: Absolute errors at $n=2,4,6$ for Example 2.

Example 3. An examination of the following NQVIE in form **case 3**:

$$\mathscr{Y}(t)=e^{\frac{3t}{2}}+\frac{4t}{90}\left(-\frac{4}{3}e^{\frac{3t}{4}}+t+\frac{4}{3}\right)\left(te^{3t}-\frac{1}{3}e^{3t}+\frac{1}{3}\right)+\left(\int_0^t\frac{(t-s)}{10}\sqrt{\mathscr{Y}(s)}\,ds\right)\left(\int_0^t st\mathscr{Y}^2(s)\,ds\right). \quad (5.4)$$

$\mathscr{Y}(t)=e^{\frac{3t}{2}}$ is the exact solution of Eq. (5.4). Applying the proposed method in chapter 4 for parameters $G=6,8$ and $M=8,10$, yields the maximum absolute errors E compiled in Table

5.6. Figure 5.8 demonstrates the absolute errors for $M = 8, 10$ and $G = 6, 8$.

	$G = 6$		$G = 8$	
M	8	10	8	10
E	$1.436135e^{-10}$	$1.415686e^{-10}$	$3.185916e^{-12}$	$4.018390e^{-14}$

Table 5.6: Maximum absolute error E for Example 3.

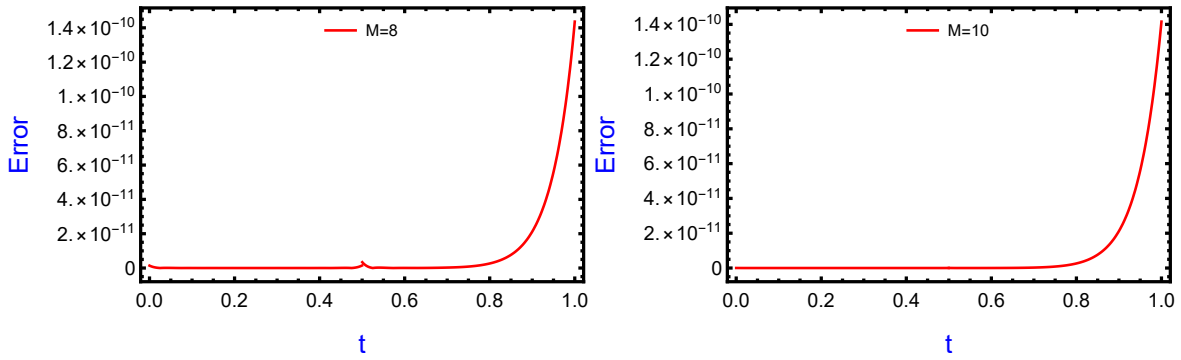

Figure 5.8: The absolute errors at $G = 6, M = 8, 10$ for Example 3.

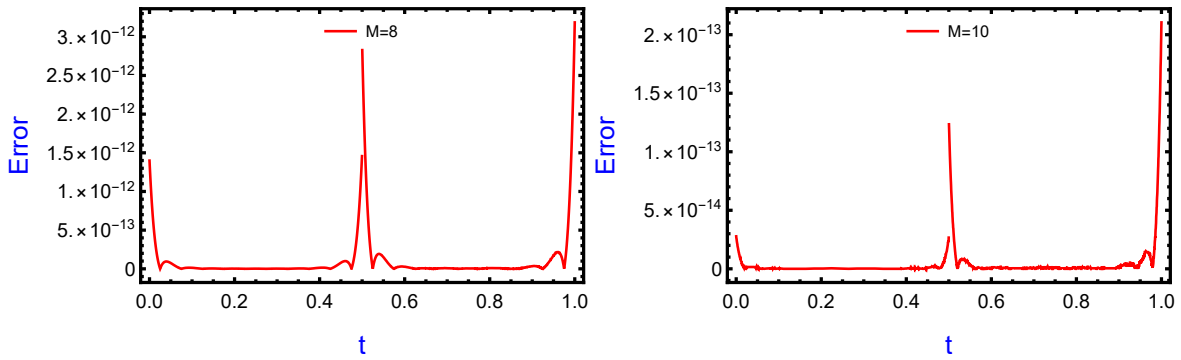

Figure 5.9: The absolute errors at $G = 8, M = 8, 10$ for Example 3.

Example 4. An examination of the following NQVIE in form **case 3** [67]:

$$\mathscr{Y}(t) = t^2 - \frac{t^{15}}{1350} + \left(\int_0^t s\mathscr{Y}^2(s)\,ds\right)\left(\int_0^t \frac{s^2}{25}\mathscr{Y}^3(s)\,ds\right). \tag{5.5}$$

The exact solution of Eq.(5.5) is $\mathscr{Y}(t) = t^2$. Applying the proposed method in chapter 4 for parameters $G = 4, 5$ and $M = 2, 4$ yields the maximum absolute errors E compiled in Table

5.7. Table 5.8 provides a comparison between the best errors obtained by the MLWM [67] and method in chapter 4. Also, Figures 5.10 and 5.11 show the absolute errors for different values of M and G.

	$G=4$		$G=5$	
M	2	4	2	4
E	$1.117076e^{-07}$	$1.457188e^{-07}$	$2.688612e^{-16}$	$8.950217e^{-16}$

Table 5.7: Maximum absolute error E for Example 4.

MLWM [67] $\hat{m}=16$	Method in Ch 4 at $M=2, G=5$	Our CPU time
$7.2494e^{-06}$	$2.68861e^{-16}$	0.04940

Table 5.8: Comparing the maximum absolute error for Example 4.

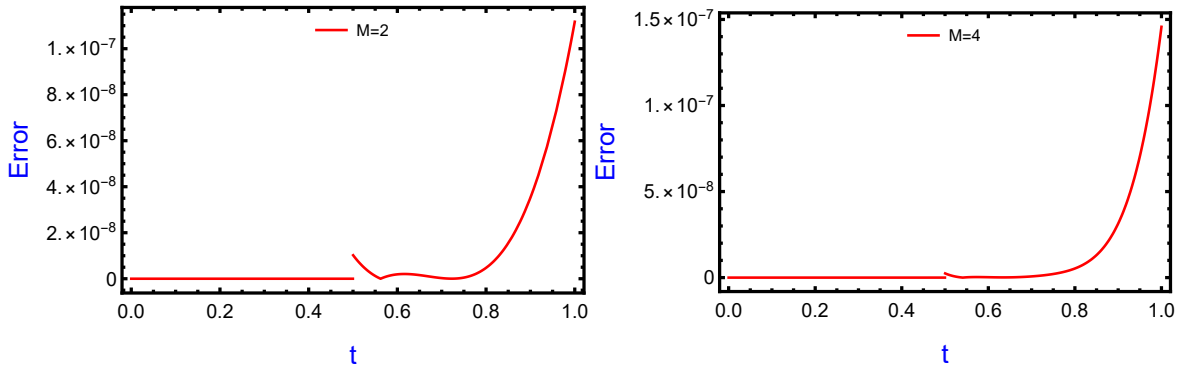

Figure 5.10: The absolute errors at $G=4$ for Example 4.

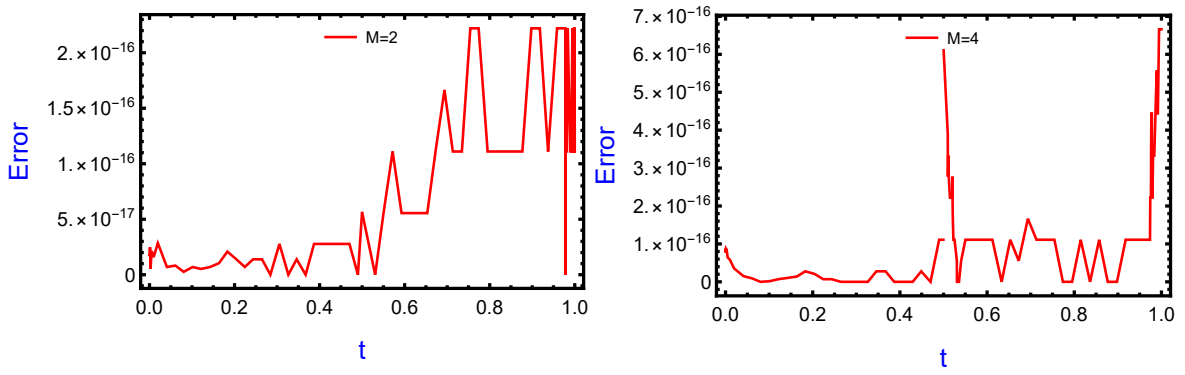

Figure 5.11: The absolute errors at $G = 5$ for Example 4.

Example 5. An examination of the following NQVIE [34, 13, 4]:

$$\mathscr{Y}(t) = t^2 - \frac{t^{10}}{35} + \frac{t}{5}\mathscr{Y}(t)\int_0^t s^2\mathscr{Y}^2(s)\,ds. \tag{5.6}$$

The exact solution of Eq.(5.6) is $\mathscr{Y}(t) = t^2$. By applying the proposed method in chapter 4 for $M = 2, G = 5$ we obtain the best absolute error. Table 5.9 provides a comparison of our result with the methods in [34, 13, 4].

ADM [34]	HAM [13]	HPM [4]	Method in Ch 4 at $M = 2, G = 5$	Our CPU time
$1.74487e^{-10}$	$6.436888e^{-06}$	$5.745e^{-09}$	$3.234720e^{-16}$	0.0721814

Table 5.9: Comparing the maximum absolute error for Example 5.

Example 6. An examination of the following NQVIE [34, 13, 4]:

$$\mathscr{Y}(t) = \left(t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110}\right) + \frac{t^3}{10}\mathscr{Y}^2(t)\int_0^t (s+1)\mathscr{Y}^3(s)\,ds. \tag{5.7}$$

The exact solution of Eq.(5.7) is $\mathscr{Y}(t) = t^3$. By applying the proposed method in chapter 4 for $M = 4, G = 8$ we obtain the best absolute error. Table 5.10 shows a comparison between our result and the methods in [34, 13, 4].

ADM [34]	HAM [13]	HPM [4]	Method in Ch 4 at $M = 4, G = 8$	Our CPU time
$4.72338e^{-10}$	$2.607894e^{-05}$	$5.210e^{-09}$	$7.863825e^{-16}$	0.154982

Table 5.10: Comparing the maximum absolute error for Example 6.

Example 7. An examination of the following NQFIE in form **case 3**:

$$\mathscr{Y}(x) = x^2 - \frac{e-1}{450} + \left(\int_0^1 s e^{\mathscr{Y}(s)}\, ds\right)\left(\int_0^1 \frac{s^2}{25}\mathscr{Y}^3(s)\, ds\right). \tag{5.8}$$

The exact solution of Eq.(5.8) is $\mathscr{Y}(x) = x^2$. Applying the Picard iteration method. And Vieta-Lucas wavelets method for parameters $G = 6, 8, 10$ and $M = 2, 4$ yields the maximum absolute errors E compiled in the next Tables.

	$M=2$			$M=4$		
G	6	8	10	6	8	10
E	$1.2505e^{-11}$	$1.8802e^{-15}$	$4.232e^{-16}$	$1.2505e^{-11}$	$1.9082e^{-15}$	$4.1430e^{-16}$
CPU Ti	0.0074718	0.0095706	0.0117185	0.0202526	0.0202526	0.0258019

Table 5.11: Vieta-Lucas wavelets method maximum absolute error E for Example 7.

n	2	4	6	8
E	$2.7302e^{-03}$	$9.3964e^{-08}$	$.2321e^{-11}$	$1.1117e^{-14}$
CPU Time	0.0026911	0.0052776	0.0076317	0.0100275

Table 5.12: Picard method maximum absolute error at $G = 10$ for Example 7.

Example 8. An examination of the following problem [12]:

$$\mathscr{Y}(t) = e^{-t} + \mathscr{Y}(t)\int_0^t \frac{t^2 \ln(1 + s|\mathscr{Y}(s)|)}{2e^{t+s}}\, ds. \tag{5.9}$$

Since Eq. (5.9) possesses at least one undetermined solution [15], we derive an approximate solution $\mathscr{Y}^*(t)$ over $[0, 1]$. The performance is evaluated using the maximum absolute residual error:

$$RE_M = \max_{0\le t<1}\left|\mathscr{Y}^*(t) - g(t) - f(t, \mathscr{Y}^*(t))\int_0^t \Psi(t, s, \mathscr{Y}^*(s))\, ds\right|.$$

Using method in chapter 4 at $G = 6$, Table 5.13 compares the RE_M values for $M = 6, 10, 14$ with the results from the Avazzadeh method [12]. Also, the behavior of the solution and its corresponding errors at a higher resolution ($M = 16$) is illustrated in Figure 5.12.

	Method of Avazzadeh [12]			method in Ch 4 at $G=6$		
M	5	10	15	6	10	14
RE_M	$5.426e^{-06}$	$1.161e^{-08}$	$5.172e^{-12}$	$7.92857e^{-07}$	$6.68971e^{-10}$	$1.94429e^{-12}$
CPU Time				0.039121	0.048757	0.095853

Table 5.13: Maximum absolute residual error RE_M for Example 8.

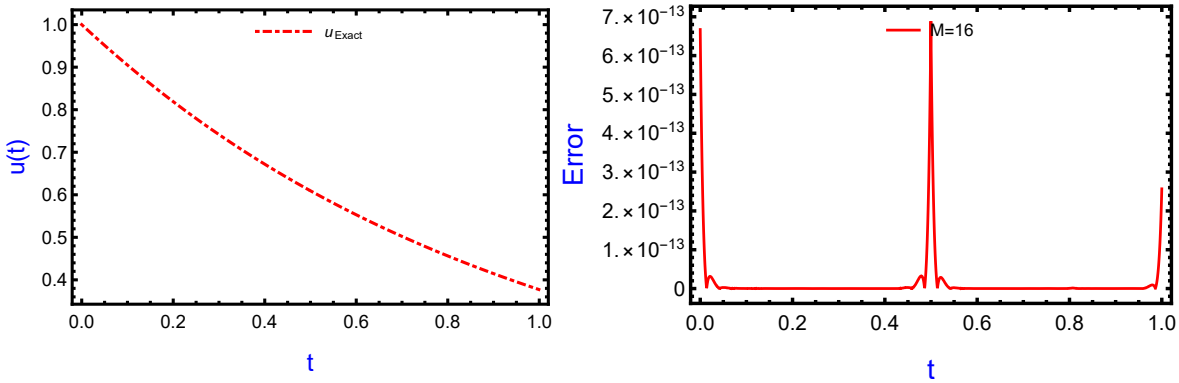

Figure 5.12: (a) Graph of approximate solution, (b) absolute residual error for Example 8

The Table 5.14 is represents the maximum absolute error of successive approximation method (Picard method) at $n = 2,4,6$ and Figure 5.13 of the graphs of the absolute error in the same number.

n	2	4	6
E	$4.23208e^{-6}$	$4.23455e^{-9}$	$3.96151e^{-12}$

Table 5.14: Maximum absolute error E for Example 8.

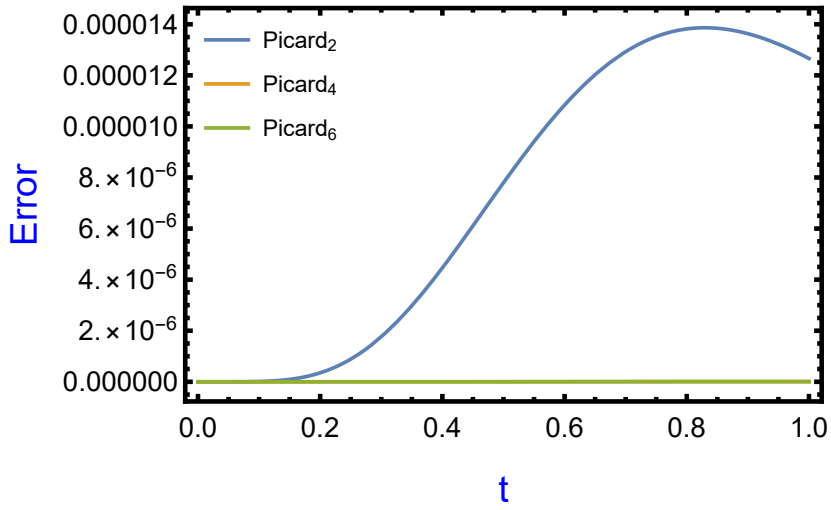

Figure 5.13: Absolute errors at $n = 2,4,6$ for Example 8.

Discussion:

In most of the error plots obtained using the method presented in chapter 4, we observe that the absolute error exhibits a jump at $t = 0.5$ (discontinuity). The reason for this is the way the wavelets method works; the Vieta-Lucas wavelets divide the domain $[0,1]$ into 2^k subfamilies. For $k = 1$, we obtain the two subfamilies $\{\xi 1, m(t)\}\, m$ and $\{\xi 2, m(t)\}\, m$, which are defined on the subintervals $[0,0.5[$ and $[0.5,1[$, respectively. The duration to move between these two subfamilies of polynomials is $t = 0.5$. Each interval contains independent polynomials. This phenomenon can be considered one of the wavelet methods' issues, even though all of the functions are continuous. Nevertheless, these jumps (discontinuity) in absolute error are negligible compared to the high convergence speed of this method, which offers a precise accuracy and rapid approximation to the exact solution. Furthermore, if we increase the value of M, we observe from the figures that these jumps decrease.

Conclusion, prospects and future work

Conclusion and prospects

In this thesis we classified some of IEs and studied the nonlinear IEs, specifically the type of quadratic IEs for three cases; the last one is a new case (**case 3**). Solving this class of equations is difficult and sometimes unsolvable.

So there are several numerical methods to solve nonlinear IEs. We addressed some of these numerical methods to obtain an approximate solution of nonlinear IEs by converting them into systems of nonlinear algebraic equations: We have presented each numerical method for solving nonlinear IE. Each method has converged, and the absolute error has approached zero. This demonstrates the accuracy of these numerical methods, such as the successive approximation method, Nyström methods, projection methods, and wavelet methods.

The successive approximation method is a numerical technique for finding solutions to equations by starting with an initial guess and iteratively improving it until the result converges to a stable value of the solution.

In order to get an approximate solution in a finite number of points, the Nyström or quadrature methods rely on approximating the definite integral that is present in our IEs throughout the interval $\Omega = [a, b]$ by a finite sum.

Our equation is projected in a finite-dimensional subspace using the projection method, which involves choosing a finite-dimensional set of functions that should include a function $\mathscr{Y}_n(x)$ that is close to the exact solution $\mathscr{Y}(x)$. The necessary numerical solution $\mathscr{Y}_n(x)$ is

selected by having it roughly satisfy our IE. The standard contractive mapping theorem and the linearization procedure were employed in the convergence analysis to demonstrate that the speed of convergence of $\mathscr{Y}_n$ to $\mathscr{Y}$ is precisely the same as that of $P_n\mathscr{Y}$ to $\mathscr{Y}$; therefore, it depends only on the approximation properties of $P_n\mathscr{Y}$ and not explicitly on the operator $\mathfrak{K}$. The wavelet method is shown to be more accurate, exact, and efficient for a limited number of Vieta-Lucas wavelets in numerical testing. Furthermore, the findings obtained indicate that the maximum absolute error reduces as the degree or number of polynomials M and the G-point Legendre-Gauss quadrature rule increase. Also, we observe from the absolute error tables and graphs in most experimental examples that the wavelet method is faster and more accurate in convergence.

We observe that all methods converge and that the absolute error tends towards zero, which demonstrates that the numerical results are acceptable. Then, the most accurate scheme is the projection method with the wavelets method, and the least accurate one is the successive approximation method. Although it is generally not possible to compare all the methods. For example, the proposed Vieta-Lucas wavelet method uses the collocation method with wavelet methods and Nyström methods. This combination of these methods is necessary for solving nonlinear IEs.

In addition, we have attempted to use the current numerical method to solve the class of NQVIEs utilizing the Legendre-Gauss Quadrature Rule and the collocation method with Vieta-Lucas wavelets. First, we establish the primary problem's existence and uniqueness under specific conditions. Next, we also estimate the provided method's convergence and error bound. Its precision and efficacy in solving NQVIEs are illustrated by a number of numerical examples.

But the wavelet method has a flaw: small jumps in the error plots, despite the equations' smooth functions.

Future work

The application of the system matrix for solving nonlinear Integral Equations (IEs) presents significant challenges regarding existence and convergence. Specifically, we observed a marked decrease in convergence rates toward the end of the domain $[0,1]$. This issue is well-documented in the literature, such as in [67]. As noted by Professor Ziada [77], fixed-point theory guarantees a solution on the domain $[0,T]$ only under the condition $T < \frac{1}{M}$, which frequently limits the practical domain to $T < 0.6$ in numerical examples. Consequently, the standard system matrix approach was excluded from the current study. However, to overcome these limitations, our future work will implement the Wavelet Galerkin system matrix for nonlinear integro-differential equations.

Bibliography

[1] Abd-Elhameed, W. M., & Youssri, Y. H. (2020).Connection formulae between generalized Lucas polynomials and some Jacobi polynomials: Application to certain types of fourth-order BVPs. *International Journal of Applied and Computational Mathematics, 6*(2), 45.

[2] Aeri, S., Kumar, R., Baleanu, D., & Sooppy Nisar, K. (2023). Vieta-Lucas wavelet based schemes for the numerical solution of the singular models. arXiv:2306.16876v2.

[3] Agarwal, R. P., O'Regan, D., & Sahu, D. R. (2009). *Fixed point theory for Lipschitzian-type mappings with applications* (Vol. 6, p. 368). New York: Springer.

[4] Al-badrani, H., Hendi, F. A., & Shammakh, W. (2017). Numerical solutions of a quadratic integral equations by using variational iteration and homotopy perturbation methods. *Journal of Mathematics Research, 9*(2), 134-145.

[5] Anselone, P., & Palmer, T. (1968). Collectively compact sets of linear operators. *Pacific Journal of Mathematics, 25*(3), 417-422.

[6] Argyros, I.K. (1985). Quadratic equations and applications to Chandrasekhars and related equations. *Bulletin of the Australian Mathematical Society, 32*, 275-292.

[7] Argyros, I. K. (1988). On a class of nonlinear integral equations arising in neutron transport. *Aequationes Mathematicae, 36*, 99-111.

[8] Atkinson, K. E. (1973). The numerical evaluation of fixed points for completely continuous operators. *SIAM Journal on Numerical Analysis, 10*(5), 799-807.

[9] Atkinson, K. E., & Potra, F. A. (1987). Projection and iterated projection methods for nonlinear integral equations. *SIAM journal on numerical analysis, 24*(6), 1352-1373.

[10] Atkinson, K. E. (1992). A survey of numerical methods for solving nonlinear integral equations. *The Journal of Integral Equations and Applications*, 15-46.

[11] Atkinson, K., & Han, W. (2005). *Theoretical numerical analysis: a functional analysis framework*. New York, NY: Springer New York.

[12] Avazzadeh, Z. (2012). A numerical approach for solving quadratic integral equations of Urysohns type using radial basis function. *Journal of Applied & Computational Mathematics, 1*(4), 1000116.

[13] Bairwa, R. K., & Kumar, A. (2022). Solution of the quadratic integral equation by homotopy analysis method. *Annals of Pure and Applied Mathematics, 25*(1), 17-40.

[14] Banas, J., Lecko, M., & El-Sayed, W. G. (1998). Existence theorems for some quadratic integral equations. *Journal of Mathematical Analysis and Applications, 222* , 276-285.

[15] Banas, J., & Martinon, A. (2004). Monotonic solutions of a quadratic integral equation of Volterra type. *Computers and Mathematics with Applications, 47*, 271-279.

[16] Banerjea, S., & Sarkar, C. (2006). On a singular integral equation with log kernel and its application. *International journal of mathematics and mathematical sciences, 2006*(1), 035630.

[17] Banerjea, S., Dutta, B., & Chakrabarti, A. (2011). Solution of Singular Integral Equations Involving Logarithmically Singular Kernel with an Application in a Water Wave Problem.*International Scholarly Research Notices, 2011*(1), 341564.

[18] Benedetto, J.J. (1993). *Wavelets: Mathematics and Applications*, vol. 13.Boca Raton, FL: CRC Press.

[19] Brezis, H. (2011). Functional Analysis, Sobolev Spaces and Partial Differential Equations. *Springer*.

[20] Cardinali, T., & Rubbioni, P. (2020). Existence theorems for generalized nonlinear quadratic integral equations via a new fixed point result. *Discrete & Continuous Dynamical Systems-Series S, 13*(7), 1947-1955.

[21] Case, K. M., & Zweifel, P. F. (1967). *Linear Transport Theory*, Addison-Wesley.

[22] Chakrabarti, A., & Manam, S. R. (2003). Solution of a logarithmic singular integral equation. *Applied mathematics letters, 16*(3), 369-373.

[23] Chakrabarti, A. (2006). Solution of certain weakly singular integral equations.*IMA journal of applied mathematics, 71*(4), 534-543.

[24] Chandler-Wilde, S. N., & Zhang, B. (2002). A generalized collectively compact operator theory with an application to integral equations on unbounded domains.*The Journal of Integral Equations and Applications*, 11-52.

[25] Chen, M.Q., Hwang, C., & Shih, Y.P. (1996). The computation of wavelet-Galerkin approximation on a bounded interval. *International Journal for Numerical Methods in Engineering 39* (17): 2921-2944.

[26] Chui, C.K. (2016).*An Introduction to Wavelets*. San Diego: Academic Press.

[27] Cicho, M., & Metwali, M.M. (2016). On the existence of solutions for quadratic integral equations in Orlicz spaces. *Mathematica Slovaca, 66*, 1413 - 1426.

[28] Daubechies, I. (1988). Orthonormal bases of compactly supported wavelets. *Communications on Pure and Applied Mathematics 41* (7): 909-996.

[29] Daubechies, I. (1992). *Ten Lectures on Wavelets*, vol. 61. Philadelphia, PA: SIAM.

[30] Davis, H. T. (1962). Introduction to Nonlinear Differential and Integral Equations, United States Atomic Energy Commission, Washington, DC, 1960. *Reissued by Dover, New York.*

[31] El-Sayed, W. G., & Rzepka, B. (2006). Nondecreasing solutions of a quadratic integral equation of Urysohn type. *Computers & Mathematics with Applications, 51*, 1065-1074.

[32] El-Sayed, A. M. A., Hashem, H. H. G., & Ziada, E. A. A. (2010). Picard and adomian methods for quadratic integral equation. *Computational & Applied Mathematics , 29*(3), 447-463.

[33] Gordji, M. E., Baghani, H., & Baghani, O. (2011). On existence and uniqueness of solutions of a nonlinear integral equation. *Journal of Applied Mathematics*, 2011(1), 743923.

[34] Fu, S. Z., Wang, Z., & Duan, J. S. (2013). Solution of quadratic integral equations by the adomian decomposition method. *Computer Modeling in Engineering & Sciences, 92*(4), 369-385.

[35] Golberg, M. A. (Ed.). (1979). *Solution methods for integral equations*, Springer, New York.

[36] Golberg, M. A. (1990). Introduction to the numerical solution of Cauchy singular integral equations. *In Numerical solution of integral equations* (pp. 183-308). Boston, MA: Springer US.

[37] Haar, A. (1910). Zur theorie der orthogonalen funktionensysteme. *Mathematische Annalen 69* (3): 331-371.

[38] Han, W., & Atkinson, K. E. (2009). *Theoretical numerical analysis: A functional analysis framework*. Springer New York.

[39] Hashem, H.H., & Alhejelan, A. (2017). Solvability of Chandrasekhars quadratic integral equations in Banach algebra. *Applied Mathematics-a Journal of Chinese Universities Series B, 08*, 846-856.

[40] Hu, S., Khavanin, M., & Zhuang, W. (1989). Integral equations arising in the kinetic theory of gases. *Applicable Analysis, 34*, 261-266.

[41] Idrees, S., & Saeed, U. (2022). Vieta-Lucas Wavelets method for fractional linear and nonlinear delay differential equations. *Engineering Computations, 39*(9), 0264-4401.

[42] Jedrzejewski, F. (2005). *Introduction aux methodes numériques*. 2nd ed. Springer Science & Business Media.

[43] Jerri, A. J. (1999). *Introduction to integral equations with applications*. John Wiley & Sons, New York.

[44] Kantorovich, L., & Akilov, G. (1964). Functional analysis in normed spaces.*New York: Mamillan*.

[45] Khirallah, M. Q. (2024). Vieta-Lucas spectral collocation method for solving fractional order Volterra integro-differential equations. *Results in Nonlinear Analysis, 7*(1), 14-23.

[46] Kress, R. (2014). *Linear integral equations* . 3rd ed, Springer Science & Business Media, New York.

[47] Kress, R. (1999). Singular integral equations. *In Linear integral equations* (pp. 94-124). New York, NY: Springer New York.

[48] Linz, P. (1985). *Analytical and numerical methods for Volterra equations*. Society for Industrial and Applied Mathematics.

[49] Liu, Z., & Kang, S. M. (2007). Existence of monotone solutions for a nonlinear quadratic integral equation of Volterra type. *The Rocky Mountain Journal of Mathematics,37*(6), 1971-1980.

[50] Maleknejad, K., Torabi, P., & Mollapourasl R. (2011). Fixed point method for solving nonlinear quadratic Volterra integral equations. *Computers and Mathematics with Applications, 62*, 2555-2566.

[51] Manzhirov, A.V & Polyanin, A.D. (2008).*Handbook of integral equation*, Taylor & Francis Group, second edition, London and New York, 2008.

[52] Mastroianni, G., & Occorsio, D. (2001). Optimal systems of nodes for Lagrange interpolation on bounded intervals. A survey. *Journal of computational and applied mathematics*, 134(1-2), 325-341.

[53] Meyer, Y. (1989). Orthonormal wavelets. In: *Wavelets* (ed. J.-M. Combes, A. Grossmann & P. Tchamitchian), 21-37. Berlin, Heidelberg: Springer.

[54] Morris, K. A. (2020). *Controller design for distributed parameter systems* (pp. 59-67). Cham, Switzerland: Springer.

[55] Moshtaghi, N., & Saadatmandi, A. (2021). Polynomial-Sinc collocation method combined with the Legendre-Gauss quadrature rule for numerical solution of distributed order fractional differential equations. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 47*, 115.

[56] Nadir, M. (2004). Cours d'analyse fonctionnelle. *http://www.mostefanadir.com/Functional Analysis.htm*, University of Msila.

[57] Nadir, M., & Gagui, B. (2014). Numerical Solution of Hammerstein Integral Equations In Lp Spaces . *Fasciculi Mathematici*, (52), 83-91.

[58] Nadir, M., & Khirani, A. (2016). Adapted Newton-Kantorovich method for nonlinear integral equations. *J. Math. Stat*, 12(3), 176-181.

[59] Nadir, M., & Rahmoune, A. (2007). Modifed method for solving linear Volterra integral equations of the second kind, *International Journal Mathematical Manuscripts* (IJMM), 2, 133-140.

[60] Nahid, N., Das, P., & Nelakanti, G. (2019). Projection and multi projection methods for nonlinear integral equations on the half-line. *Journal of Computational and Applied Mathematics*, 359, 119-144.

[61] Nevai, P. (1984). Mean convergence of Lagrange interpolation. III. *Transactions of the American Mathematical Society*, 669-698.

[62] Precup, R. (2002). *Methods in nonlinear integral equations*. Springer Science & Business Media.

[63] Rahman, M. (2007). *Integral equations and their applications*. WIT press.

[64] Rall, L. (1969). *Computational solution of nonlinear operator equations*. John Wiley & Sons, New York.

[65] Samadi, O. R. N., & Tohidi, E. (2012). The spectral method for solving systems of Volterra integral equations. *Journal of Applied Mathematics and Computing, 40*, 477-497.

[66] Shen, J., Tang, T., & Wang, L. L. (2011). *Spectral methods: algorithms, analysis and applications* (Vol. 41). Springer Science & Business Media.

[67] Shiralashetti, S. C., & Lamani, L. (2021). Nonlinear quadratic integral equations using Muntz Legendre Wavelets. *Poincare Journal of Analysis and Applications, 8*(1)(II), 69-89.

[68] Torkaman, S., Heydari, M., & Loghmani, G. B. (2023). A combination of the quasi-linearization method and linear barycentric rational interpolation to solve nonlinear multi-dimensional Volterra integral equations . *Mathematics and Computers in Simulation, 208*, 366-397.

[69] Vainikko, G. (2006). *Weakly singular integral equations*. Lecture Notes.

[70] Wazwaz, A. (2011). *Linear and nonlinear integral equations methods and applications*, Higher Education Press Springer-Verlag Berlin Heidelberg, London, New York.

[71] Weiss, R. (1974). On the approximation of fixed points of nonlinear compact operators. *SIAM Journal on Numerical Analysis*, 11(3), 550-553.

[72] Xiu, D. (2010). *Numerical methods for stochastic computations: a spectral method approach*. Princeton university press.

[73] Xu, Y., Zhang, Y., & Zhao, J. (2019). Error analysis of the Legendre-Gauss collocation methods for the nonlinear distributed-order fractional differential equation. *Applied Numerical Mathematics, 142*, 122-138.

[74] Yuan, G. X. Z. (1998). *An introduction to wavelet theory and its applications in statistics*. Dalhousie University.

[75] Zare, F., Heydari, M., & Loghmani, G. B. (2024). Convergence analysis of an iterative scheme to solve a family of functional Volterra integral equations. *Applied Mathematics and Computation, 477*, 128799.

[76] Zemyan, S. M. (2012). *The classical theory of integral equations: a concise treatment*. Springer Science & Business Media.

[77] Ziada, E. A. A. (2013). Adomian solution of a nonlinear quadratic integral equation. *Journal of the Egyptian Mathematical Society, 21*(1), 52-56.