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On spectral continuity of the essential spectrum

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Notations

$\mathcal{L}(X)$	The algebra of all bounded linear operators from X into itself
$\mathcal{N}(A)$	The null space of A
$R(A)$	The range space of A
$\mathcal{K}(X)$	The ideal of all compact operators on X
$\mathcal{C}(X)$	The set of all closed densly defined
$\mathcal{I}(X)$	An arbitrary non zero two sided ideal of $\mathcal{L}(X)$
σ_w	Essential spectrum of Weyl
σ_f	Essential spectrum of Wolf
$F(X)$	The set of Fredholm operators
$\alpha(A)$	The nullity of A is defined as the dimension of $N(A)$
$\beta(A)$	The deficiency of A is defined as the codimension of $R(A)$
$\mathcal{F}(X,Y)$	The set of fredholm perturbation
$\Phi_-(X,Y)$	Fredholm operators
$\mathcal{R}(X)$	Riesz operator
I	Operator of identity
$A_n \xrightarrow{n} A$	The norm convergence of A_n to A
$A_n \xrightarrow{cc} A$	The collectively compact convergence of A_n to A
$A_n \xrightarrow{\nu} A$	The ν-convergence of A_n to A

Introduction

This work is devoted to study of the spectral continuity of the essential spectrum by using a new mode of convergence which was provided by M.Ahus and A.Largillier in 1994, this work is section of the spectral analysis widely used in mathematical and physical. It is composed of three chaptre:

The first chapter, we recall the basic properities of the spectral theory of bounded linear operators which is an important part of function analysis. We begin by presenting generalies about linear operators and some classical quantities such as the closed and closable of an operators. In addition, special attention will be paid to the study of the compact operators and the basic notation necessary to study linear adjoint operators and self-adjiont for Norm and Hilbert space. We recall some fundamontal results and notation relating to the theory Fredholm and Fredholm perturbation. In the final section of this chapiter, we finish by the definition of the spectrum and after that we studies two classes of the spectrum essential and present also a characterization of essential spectra.

The second chapter, we study the convergence and approximation of operators, it consists of two section. In first section, introdusing the notion of the spectral approximation of the operators and the existence theorem, we defined some mode of convergence in order of as follows (the norm convergence $T_n \stackrel{n}{\to} T$, pointwise convergence $T_n \stackrel{p}{\to} T$, collectivly compact convergence $T_n \stackrel{cc}{\to} T$). In the last section, we will be paid to the study of a new mode of convergence it is called $\upsilon-$convergence, and we recall some properties and gives the links that may exist between $\upsilon-$convergence and anothre mode of convergence.

The third chapter, witch gives an extension to the previous chapter, is devotes to the study of the spectral continuity of some types of operators under $\upsilon-$convergence. First, we define the continuity of the spectrum and pertubation. Then, we expose the properties of the essentiel spectra using the $\upsilon-$convergence, we recall in it the convergence of the essential spectra of Wolf, Weyl and essential approximation point specrum under $\upsilon-$convergence and define the continuity of the essential spectrum $(\sigma_w, \sigma_f, \sigma_{eap})$. In the last, we recall the $\upsilon-$continuity of wolf and weyl essential spectra using the $\upsilon-$convergence.

Chaptre 1

Basic properties

In this chapter we recall the basic properties of a linear operators in a Banach spaces. Let us start by introducing the basic notions necessary to study linear operators.

1.1 Generalities about linear operators

Definition 1.1.1 *Let X and Y be two vectors space over a field K. We say that the operator A is called linear operator from X to Y if,*

$$A(x+y) = A(x) + A(y),\ A(\lambda x) = \lambda A(x) \text{ for all } x,\ y \in X \text{ and } \lambda \in K.$$

Definition 1.1.2 *Let A a linear operator from X to Y and we have $x_0 \in X$ the operator A is continuous at x_0 if and only if ,*

$$\lim_{x \to x_0} A(x) = A(x_0)$$

If for evry $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|x - x_0\| < \delta \Rightarrow \|Ax - Ax_0\| < \varepsilon$$

We say the operator A between normed spaces is continuous.

•The null spaces of a linear operator A is $N(A) = \{x \in X : Ax = 0\}$, it is also called the kernel of A and denoted by $\ker(A)$.

•The image of a linear operator A is $R(A) = \{y \in Y : y = A(x)\}$, it is also called the range of A and denoted by $\operatorname{Ran}(A)$.

A linear operator on a normed space X to a normed space Y is continuous at every point x, if it is continuous at a single point in x.

Definition 1.1.3 *Let X and Y be a normed vector spaces and that the linear operator $A : X \to Y$ is bounded, then*

$$\sup \|Ax\| < \infty$$

This number is called the norm of the operator A and it is denoted by $\|A\|$ then

$$\|Ax\| \leq \|A\| . \|x\|$$

And $\|A\|$ is the smallest canstant C, such that

$$\|Ax\| \leq C \|x\| \text{ for any } x \in X.$$

Proposition 1.1.1 *Let X and Y be normed linear spaces and let A be a linear operator with domain X and range in Y. The following statements are equivalent:*

(a) A is continuous at a point.

(b) A is uniformly continuous on X.

(c) A is bounded. i.e, there exists a canstant C, such that for all $x \in X$

$$\|Ax\| \leq C \|x\|$$

1.1.1 Properties of the space of bounded linear operators

The space of all bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$ and is a normed space.

- If Y is Banach, then so is $\mathcal{L}(X, Y)$, from which it follows that dual spaces are Banach.
- For any A in $\mathcal{L}(X, Y)$, the kernel of A is a closed linear subspace of X.

Example 1.1.1 *Many integral transforms are bounded linear operators, for instance if $K : [a, b] \times [c, d] \rightarrow \mathbb{R}$ is a continuous function then the operaor L defined on the space of continuous functions on $[a, b]$ endowed with the uniform norm and with values in the space with given by the formula*

$$(LF) = \int_a^b k(x, y) f(x) dx \text{ is bounded.}$$

Proposition 1.1.2 *A linear operators between normed spaces X and Y is bounded if and only if, it is a continuous linear operators.*

1.2 Closed linear operators

Definition 1.2.1 *An unbounded linear operator from X into Y is a linear map $A : D(A) \subset X \rightarrow Y$, defined on a linear subspace $D(A)$, the set $D(A)$ is called the domain of A.*

Definition 1.2.2 *Let $X \times Y$ is defined as the normed linear space of all ordered pairs (x, y), $x \in X$, $y \in Y$ with the usual definitions of addition and scalar multiplicaton and with norm given by $\|(x, y)\| = \max\{\|x\|, \|y\|\}$.*

· The graph $G(A)$ of a linear operator A is the set $\{(x, Ax) \mid x \in D(T)\}$. The graph $G(A)$ is a subspace of $X \times Y$.

If the graph of A is closed in $X \times Y$, then A is said to be closed in X. When there is no ambiguity concerning the space X, we say that A is closed.

Lemma 1.2.1 *A subspace G of $X \times Y$ is a graph if and only if, $\{0, y\} \in G$ implies $y = 0$.*

Definition 1.2.3 *The linear operator A with domain $D(A) \subset X$ and range $R(A) \subset Y$, it said to be closed whenever $G(A)$ it a closed to subspace of $X \times Y$.*

Equivalently, A is closed operator if and only if, for any sequence $(x_n)_n \in D(A)$ for $n = 1, 2, ...,$ such that $x_n \to x \in X$ and $Ax_n \to y \in Y$ it follows that $x \in D(A)$ and $Ax = y$.

Remark 1.2.1 *Let $A : X \to Y$ be a linear operator with the domain $D(A)$.*

1· If A is closed then the null space $N(A)$ is closed, however $R(A)$ need not be closed.

2·The linear operator A_1 with domain $D(A_1) \subset X$ and range $R(A_1) \subset Y$ is called an extention of A if $D(A) \subset D(A_1)$ and $A_1 x = Ax$ for all $x \in D(A)$. If in addition A_1 is a closed linear operator, then A_1 is called a closed linear extension of A.

3· We denote by $\mathcal{C}(X, Y)$ the set of all closed densely defined linear operators from X into Y. If $X = Y$ then $\mathcal{C}(X, X) = \mathcal{C}(X)$.

4· The continuity of A does not necessarily imply that A is closed. Conversely, A closed does not necessarily imply that A is continuous.

Proposition 1.2.1 *we have:*

(a) A is closed if and only if, the sequence $\{x_n\}$ in $D(A)$ such that $x_n \to x$, $Ax_n \to y$, imply $x \in D(A)$ and $Ax = y$.

(c) If $D(A)$ is closed and A is continuous, then A is closed.

Example 1.2.1 *Let $X = Y = C([0, 1])$ and let $C\prime[(0, 1)]$ be the subspace of X consisting of the functions with continuous first derivatives.*

Define the linear differential operator A mapping $C\prime([0, 1])$ into Y by $(Ax)(t) = x\prime(t)$, $t \in [0, 1]$. A is closed, for if $x_n \to x$ and $Ax_n \to y$, then

$\{x_n\}$ converges uniformly to x and $\{x\prime_n\}$ converges uniformly to y on $[0, 1]$. It follows from taking antiderivatives of $x\prime_n$ and y that x is in

$C\prime([O, 1])$ and that $Ax = x\prime = y$ on $[0, 1]$. Thus A is closed.

However, A is unbounded, since the sequence $\{x_n(t)\} = \{t^n\}$ has the properties $\|Ax_n\| = n$ and $\|x_n\| = 1$.

1.2.1 Closable Operators

Definition 1.2.4 *Let A and B are operators from X to Y and $D(B) \subset D(A)$ with $Bx = Ax$ for $x \in D(B)$, we say that A is an extension of B and B is a restriction of A and we write $B \subset A$.Equivley $B \subset A$ if and only if,*

1st, case. If A is bounded

We can simply take the operator $\overline{A}$ with graph $\overline{G(A)}$, here $\overline{G(A)}$ is a graph since $x_n \to 0$ in X imlies $Ax_n \to 0$.

2nd, case. If A is unbounded

We can not be certain that it has a closed extainsion. But if A has a closed extension A_1, then $G(A_1)$ is a closed subspaces of $X \times Y$ containing $G(A)$, in that case $\overline{G(A)}$ is a graph, it is in fact the graph of the smallest closed extention of A, we call it the closure of A and denote it $\overline{A}$.

Definition 1.2.5 *An operator A is called closable if it has a closed extension, the smallest closed extenion of A whose graph equals $\overline{G(A)}$ is denoted by $\overline{A}$ and called the closure of A, every closable operator has a closure.*

Proposition 1.2.2 *Let $A : X \to Y$ be an operator, the following conditions are equivalent:*

(a) A is closable.

(b) The graph $\overline{G(A)}$ is a graph of an operator.

(b) If $(0, y) \in \overline{G(A)}$ then $y = 0$.

(d) If for any sequence $(x_n)_n \subset D(A)$ such that $x_n \to 0 \in X$ and $Ax_n \to y \in Y$ implies $y = 0$.

1.3 Compact operators

Definition 1.3.1 *Let X and Y be Banach spaces then $A : X \to Y$ is called a compact linear operator or completely continuous linear operator if for every bounded subset of X the range $A(G)$ is relatively compact in Y. In other words, the closure $\overline{A(G)}$ is compact.*

By other hand, then A is compact if and only if, for all bounded sequence $(x_n)_n$ in X, the sequence $(Ax_n)_n$ has a subsequence converges in Y.

Theorem 1.3.1 *Let be $A, A_n \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, Z)$ with X, Y and Z is Banach spaces:*

(a) The sequence A_n of compact operators defined from a normed space X into a Banach space Y converges uniformly to an operator A, say

$$\lim \|A_n - A\| = 0$$

Then the limit operator A is compact.

(b) the product AB is compact if the operator A or B is compact.

Proof. (a) Let $(x_m)_m$ in $\mathcal{B}(0,1)$. for all n there existe a subsequence$\left(x_{\varphi_{n(m)}}\right)_m$ such that $\left(Ax_{\varphi n(m)}\right)_m$ is convergente because A_n is compact $\left\|Ax_{\varphi n(n)} - Ax_{\varphi m(m)}\right\| \leq \|A - A_n\| + \|A_m - A\| + \left\|A_n x_{\varphi n(n)} - A_m x_{\varphi m(m)}\right\|_{n,m\to\infty} \to 0$

(b) Let x_n be a bounded sequence in X, then if we consider B as a bounded operator, the sequence $Bx_n(x)$ is bounded and from the compactness of the operator A gives a convergent subsequence $A\left(Bx_{nk}\left(x\right)\right)$ of $A(Bx_n(x))$. Hence, the operator AB is compact. On the other hand, if we consider B as a compact, one can extract from $Bx_n(x)$ a convergent subsequence $Bx_{nk}(x)$ and from the boundedness of the operator A gives the convergence of the sequence $A(Bx_{nk}(x))$. Hence, the operator AB is compact. ■

Theorem 1.3.2 *A linear operator A defiened from a normed space X into a normed space Y is called a linear compact operator or completely continuous linear operator if and only if, for every bounded sequence x_n in X the sequence Ax_n in Y has a convergent subsequence x_{n_k}.*

Proof. Let x_n be a bounded sequence in X, since the operator A is compact, then the set $\{Ax_n\}$ is relatively compact in Y. Where this property shows that Ax_n contains a convergent subsequence. Conversely, let us consider any bounded subset Ω in X and let y_n be any sequence in $A(\Omega)$, then there exists a bounded sequence x_n in X, such that $y_n = Ax_n$. By assumption, $Ax_n = y_n$ contains a convergent subsequence y_{nk} in F. Thus

$A(\Omega)$ is relatively compact because for any bounded sequence y in $A(\Omega)$, there exits a convergent subsequence y_{nk} in Y. In other words, for all bounded set X the set $A(\Omega)$ is relatively compact in F. Hence A is compact. ■

Theorem 1.3.3 *The linear combination $A = \alpha A_1 + \beta A_2$ of compact operators A_1 and A_2 is a compact operator, for every scalars α and β.*

1.3.1 Relatively Boundedness and Relatively Compactness

Definition 1.3.2 *Let X, Y and Z be Banach spaces and let A and S be two linear operators from X into Y and from X into Z, respectively.*

(a) S is called relatively bounded with respect to A, or A-bounded, if $D(A) \subset D(S)$, there exist two constants $a_s \geq 0$ and $b_s \geq 0$, such that

$$\|Sx\| \leq a_s \|x\| + b_s \|Ax\|, x \in D(A) \tag{(1.3)}$$

The infimum δ of all b_s that holds for some $a_s \geq 0$ is called relative bound of S with respect to A (or A-bounded of S).

(b) S is called relatively compact with respect to A (or A-compact), if $D(A) \subset D(S)$ and for every bounded sequence $(x_n)_n \in D(A)$ such that $(Ax_n)_n \subset Y$ is bounded, the sequence $(Sx_n)_n \subset Z$ contains a convergent subsequence.

Remark 1.3.1 *The inequality (1.3) is equivalent to*

$\|SX\|^2 \leq a^2 \|x\|^2 + b^2 \|Ax\|^2$ for all $x \in D(A)$, where $a = \sqrt{a_s^2 + a_s b_s}$ and $b = \sqrt{b_s^2 + a_s b_s}$

Lemma 1.3.1 *If S is $A-$bounded with an $A-$bound $\delta < 1$, then S is $(A+S)$-bounded with an A-bound $\leq \frac{\delta}{1-\delta}$.*

Proof. First of all, it should be mentioned that $A + S$ is well defined as $D(A + S) = D(S) \cap D(A) = D(A) \subset D(S)$.

The fact that S is A-bounded, there exist $a_s \geq 0$ and $\delta \leq b_s < 1$ such that, for all $x \in D(A)$, we have

$$\begin{aligned}\|Sx\| &\leq a_s\|x\| + b_s\|Ax\| \\ &= a_s\|x\| + b_s\|Ax + Sx - Sx\| \\ &= a_s\|x\| + b_s\|Ax + Sx\| + b_s\|Sx\|\end{aligned}$$

Since $b_s < 1$, it follows that

$$\|Sx\| \leq \frac{a_s}{1-b_s}\|x\| + \frac{b_s}{1-b_s}\|(A+S)x\|,\ x \in D \ \blacksquare$$

Proposition 1.3.1 *If A is closed and B is closable, then $D(A) \subset D(B)$ implies that B is A-bounded.*

Definition 1.3.3 *We say that an operator J is A-closed, if $x_n \to x$, $Ax_n \to y$, $Jx_n \to z$ for $(x_n)_n \subseteq D(A)$ implies that $x \in D(J)$ and $Jx = z$. An operator J will be called A-closable, if $x_n \to 0$, $Ax_n \to 0$, $Jx_n \to z$ implies $z = 0$.*

1· If J is bounded, then J is A-bounded.

2· If J is closed, then J is A-closed.

3· If J is closable, then J is A-closable.

4· If J is closed, then J is A-closed if and only if, J is A-closable if and only if, J is A-bounded.

1.4 Adjoint operators

Recall that when X a Banach space, the dual space $X^* :\ = \mathcal{L}(X; \mathbb{C})$ consists of the bounded linear functionals x^* on X, it is a Banach space with the norm

$$\|x\|_{x^*} = \inf\{|x^*(x)| : x \in X, \|x\| = 1\}$$

When $A : X \to Y$ is densely defned, we can define the adjoint operator $A^* : Y^* \to X^*$ as follows:

The domain $D(A^*)$ consists of the $y \in Y^*$, for which the linear functional $x \to y^*(Ax)$, $x \in D(A)$

is continuous, this means that there is a canstant C such that $|y^*(Ax)| \leq C\|x\|_X$ for all $x \in D(A)$.

Since, $D(A)$ is dense in X the mapping extends by continuity to X, so there is a uniquely determined $x^* \in X$ so that

$$y^*(Ax) = x^*(x), \text{ for all } x \in D(A)$$

When x^* is determined from y^*, we can define the operator A^* from Y^* to $X^*$by

$$A^* y^* = x^*, \text{ for all } y^* \in D(A^*)$$

Theorem 1.4.1 *Let A be a bounded operator defined from a Hilbert space H_1 in a Hilbert space H_2. then, there exists an adjoint operator of A denoted by A^*defined from H_2 into H_1 such that*

$$\langle Ax, y\rangle_{H_2} = \langle x, A^*y\rangle_{H_1}, \text{ for all } x \in H_1 \text{ and } y \in H_2.$$

Besides, we have

$$\|A\| = \|A^*\|$$

Theorem 1.4.2 *Let A be a compact operator defined from a Hilbert space H_1 into a Hilbert space H_2. then, the adjoint operator A^* defined from H_2 into H_1 is also a compact operator.*

Theorem 1.4.3 *Let $A \in \mathcal{C}(X, Y)$, then there is an adjoint operator $A^* : Y^* \to X^*$. Moreover, $A^*$is closed, if A is a bounded operator then A^* is also a bounded operator from Y^* into X^* and moreover,$\|A\| = \|A^*\|$, for nonempty sets $M \subseteq X$ and $N \subseteq X^*$. We define the annihilators:*

$$M^\perp = \{f \in X^* : f(x) = 0 \text{ for all } x \in M\}.$$
$$N^\perp = \{x \in X : f(x) = 0 \text{ for all } x \in N\}.$$

Even if M and N are not subspaces and $M^\perp$ and $N^\perp$ are closed subspaces of X^* and X respectively, we have $M^\perp = X^*(resp \cdot N^\perp = X)$ if and only if, $M = \{0\}\,(resp \cdot N = \{0\})$.

Theorem 1.4.4 *Let A be a closed linear operator with domain $D(A)$ dense in X and range $R(A) \subset Y$. Then $D(A^*)$ it dense in Y^*.*

Hence, A^{**}exists and we have $A^{**} = A$.

1.4.1 self-adjoint operators

Definition 1.4.1 *Let A be a linear operator from a Hilbert space H to itself.*

- If A is an extenson of A, that is $A \subset A^*$, then A is called a symmetric operator.
- If $A = A^*$, then a bounded operator A is called a self-adjoint operator.
- If $\overline{A} = A^*$, then A is called essentially self-adjoint.
- If A is a symmetric operator, then for any $x, y \in D(A)$,$\langle Ax, y\rangle = \langle x, Ay\rangle$
- If A is a symmetric bounded operator, then A is self-adjoint.
- If A is a symmetric operator, then $D(A) \subset D(A^*)$, so A^* is densely defined and A is closable and $\overline{A} = A^{**} \subset A^*$. Thus, if A is essentially self-adjoint, then $\overline{A} = A^{**} \subset A^*$.
- Let A be a self-adjoint operator and B be a symmetric operator such that $A \subset B$, then $A = B$, this is because $A \subset B \subset B^* \subset A^* = A$, we see that a symmetric operator can have diferent self-adjoint extension.

Proposition 1.4.1 *Let A be a densely defined operator on a Hilbert space H. Then*

$$A \text{ is essentially self–adjoint} \iff A \text{ has a unique self–adjoint extension.}$$

1.5 Fredholm operators

Definition 1.5.1 *Let X and Y be Banach spaces and let $A : X \to Y$ be a bounded linear operator, $A \in \mathcal{L}(X, Y)$ we say that:*

1. Fredholm operators denoted by $\Phi(X, Y)$ or $\Phi(X)$ when $X = Y$ if the following three condition satisfied:

$\cdot\alpha(A) = \dim(N(A)) < \infty$.

$\cdot R(A)$ is closed.

$\cdot\beta(A) = co\dim(R(A)) < \infty$

2.Let $\Phi_\pm(X)$ denoted the set of **semi-Fredholm operators**, then we have

$$\Phi_\pm(X) = \Phi_+(X) \cup \Phi_-(X), \text{ when } X = Y.$$

3.Let $\Phi_+(X)$ denoted the set of **upper semi-Fredholm operators**,then we have

$\Phi_+(A) = \{A \in \mathcal{C}(X) \text{ such that } \alpha(A) < \infty \text{ and } R(A) \text{ is closed in } X\}$, when $X = Y$.

4.Let $\Phi_-(X)$ denoted the set of **lower semi-Fredholm operators**, then we have

$\Phi_-(X) = \{A \in \mathcal{C}(X) \text{ such that } \beta(A) < \infty \text{ and } R(A) \text{ is closed in } X\}$, when $X = Y$.

5.Let $\Phi^b(X)$ denoted the set of **bounded Fredholm operators**, then we have

$$\Phi^b(X) = \Phi_+(X) \cap \mathcal{L}(X) \text{ when } X = Y.$$

6.$\Phi^b_{\mathcal{U}}(X,Y) = \left\{\lambda \in \mathbb{C} \text{ such that } \lambda - \mathcal{U} \in \Phi^b_{\mathcal{U}}(X,Y) \text{ denoted by } \Phi^b_{\mathcal{U}}(X), \text{ when } X = Y\right\}$.

Remark 1.5.1 *Like the image of an operator is closed if it is of finite codimension then*

(*i*) $\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y)$.

(*ii*) $\Phi(X)$ is a non-empty set as it contains the identity. On the other hand, $\Phi(X,Y)$ may be empty when $X \neq Y$.

Definition 1.5.2 *For all operator* $A \in \mathcal{L}(X,Y)$, *we denoted* $\alpha(A) = \dim(N(A))$ *and* $\beta(A) = co\dim(R(A))$, *the number* $i(A) = \alpha(A) - \beta(A)$ *is called the index of* A.

Remark 1.5.2 *We have* $i(A) \in \mathbb{Z} \cup \{\pm\infty\}$, *such that*

- If $A \in \Phi(X)$ then $i(A) < \infty$.
- If $A \in \Phi_+(X) \setminus \Phi(X)$ then $i(A) = -\infty$.
- If $A \in \Phi_-(X) \setminus \Phi(X)$ then $i(A) = +\infty$.

Theorem 1.5.1 *Let* $A \in \mathcal{L}(X,Y)$ *a closed range operator*

(a) A is **upper semi-Fredholm** if and only if, A^* is **lower semi-Fredholm**.

(b) A is **lower semi-Fredholm** if and only if, A^* is **upper semi-Fredholm**.

(c) A is **semi-Fredholm** if and only if, A^* is **semi-Fredholm**.

(d) A is **Fredholm** if and only if, A^* is **Fredholm**.

And we have with all cas $i(A^*) = -i(A)$.

Proof. For $R(A)$ so $R(A^*)$ is closed, such that $R(A) = N(A^*)^\perp$ and $R(A^*) = N(A)^\perp$ then, $\alpha(A^*) = \dim N(A^*) = co\dim R(A) = \beta(A)$ and $\beta(A^*) = co\dim R(A^*) = \dim N(A) = \alpha(A)$. For $\dim N(A^*)$ and $co\dim R(A^*) < \infty$, then A^* is Fredholm and $i(A^*) = \alpha(A^*) - \beta(A^*) = \beta(A) - \alpha(A) = -i(A)$. ■

Theorem 1.5.2 *Let $A \in \mathcal{L}(X;Y)$ and $S \in \mathcal{L}(X,Y)$ such that*

(a) If $A \in \Phi_+(X,Y)$ and $S \in \Phi_+(Y,Z)$, so $SA \in \Phi_+(X,Z)$.

(b) If $A \in \Phi_-(X,Y)$ and $S \in \Phi_-(Y,Z)$, so $SA \in \Phi_-(X,Z)$.

(c) If $A \in \Phi(X,Y)$ and $S \in \Phi(Y,Z)$, so $SA \in \Phi(X,Z)$.

Lemma 1.5.1 *If $A, B \in \Phi(X)$ then we have $BA \in \Phi(X)$ suth that*

$$i(BA) = i(A) + i(B)$$

Theorem 1.5.3 *(Alternative de Fredholm) Let A be a compact operator $A \in \mathcal{L}(X)$ for all* ■ $\in C^*$*:*

(a) $N(\lambda I - A) < \infty$.

(b) $R(\lambda I - A)$ is closed.

(c) $N(\lambda I - A) = \{0\} \Leftrightarrow R(\lambda I - A) = X$.

1.5.1 Fredholm perturbations

Definition 1.5.3 *Let X and Y be two Banach spaces and let $A \in \mathcal{L}(X)$.*

- A is called a **Fredholm perturbation** if $A + U \in \Phi$, whenever $U \in \Phi(X,Y)$.
- A is called a **upper semi-Fredholm perturbation**(resp. a lower semi-Fredhlom perturbation)if $A + U \in \Phi_+(X,Y)$ (resp. Φ_-) for all $U \in \Phi_+(X,Y)$(resp. $\Phi_-(X)$).

·The set of Fredholm perturbations, upper semi-Fredholm perturbation (resp. a lower semi-Fredholm pe denoted by $\mathcal{F}(X,Y)$,$\mathcal{F}+(X,Y)$, $\mathcal{F}-(X,Y)$.

Proposition 1.5.1 *Let X, Y and Z be three Banach spaces. If at least of the set $\Phi^b(X,Y)$ and $\Phi^b(Y,Z)$ is not empty, then*

(i) $F \in \mathcal{F}(X,Y)$, $A \in \mathcal{L}(Y,Z)$ imply $AF \in \mathcal{F}(X,Z)$.

(ii) $F \in \mathcal{F}^b(Y,Z)$, $A \in \mathcal{L}(X)$ imply $FA \in \mathcal{F}(X,Z)$.

Proposition 1.5.2 *Let X be a complex Banach space of infinite dimension and let $F \in \mathcal{F}(X)$*

$$i(A+F) = i(A) \text{ for all } A \in \mathcal{L}(X).$$

Lemma 1.5.2 *Let $A \in \mathcal{C}(X)$ and $U \in \mathcal{L}(X)$, then*

(i) If $A \in \oplus^b(X)$ and $U \in \mathcal{F}^b(X)$, then $A+U \in \Phi^b(X)$ and $i(A+U) = i(A)$.

(ii) If $A \in \oplus^b_+(X)$ and $U \in \mathcal{F}^b+(X)$, then $A+U \in \Phi^b_+(X)$ and $i(A+U) = i(A)$.

(iii) If $A \in \oplus^b_-(X)$ and $U \in \mathcal{F}^b-(X)$, then $A+U \in \Phi^b_-(X)$ and $i(A+U) = i(A)$.

Definition 1.5.4 (*Riesz operators*) *Let X be a Banach space and $A \in \mathcal{L}(X)$, A is said to be a Riesz operator (and we note $A \in \mathcal{R}(X)$) if $\lambda \in \mathbb{C} \setminus \{0\}$ is Fredholm.*

(i) The family of Riesz operators is not an ideal of $\mathcal{L}(X)$.

(ii) It is proved that $\mathcal{F}(X)$ is the largest ideal of $\mathcal{L}(X)$ contained in the family of Riesz operators.

1.6 Spectrum

Definition 1.6.1 *Let A be a closable linear operator in a Banach space X. The resolvent set and the spectrum of A are, respectively, defined as*

$\rho(A) = \{\lambda \in \mathbb{C}$ such that $A - \lambda$ is injective and $(\lambda - A)^{-1} \in \mathcal{L}(X)\}$.

We recall $\sigma(A) = \mathbb{C} \setminus \rho(A)$ the specctrum of A and the point spectrum, continuous and the residual spectrum are defined as:

$\sigma_p(A) = \{\lambda \in \mathbb{C}$ such that A is not injective$\}$

$\sigma_c(A) = \left\{\lambda \in \mathbb{C} \text{ such that } A \text{ is injective } \overline{R(\lambda - A)} = X,\ R(\lambda - A) = X\right\}$

$\sigma_r(A) = \left\{\lambda \in \mathbb{C} \text{ such that } A \text{ is injective, } \overline{R(\lambda - A)} \neq X\right\}$

Note that, if $\rho(A) \neq 0$ then A is closed. In fact, if $\lambda \in \rho(A)$ then $(\lambda - A)^{-1}$ is closed, which is also valid for $\lambda - A$. Then, according to the closed graph we deduce that

$\rho(A) = \{\lambda \in \mathbb{C}$ such that A is bijective$\}$

And hence,

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$$

Let $(A, D(A))$ be a closed, densely defined and linear operator with a nonempty resolvent set $\rho(A)$ .For each $\lambda_0 \in \rho(A)$ we have

$$\sigma\left((\lambda_0 - A)^{-1}\right) = (\lambda_0 - \sigma(A))^{-1}$$

1.6.1 Essential Spectrum

Let X be a Banach space and let $A : X \to X$ be a closed linear operator, the essential spectre denoted by σ_{ess} :

$$\sigma_e(A) = \{\lambda \in \mathbb{C}; \lambda - A \text{ is not Fredholm } (A) \text{ indice } 0\}$$

There are sevral definitions of the essential spectrum, which are not equivalent:

$$\begin{aligned}
\sigma_{e1}(A) &= \{\lambda \in \mathbb{C}; \lambda - A \in \Phi_+(X)\} = \mathbb{C}\backslash\Phi_{+A} \\
\sigma_{e2}(A) &= \{\lambda \in \mathbb{C}; \lambda - A \in \Phi_-(X)\} = \mathbb{C}\backslash\Phi_{-A} \\
\sigma_{e3}(A) &= \{\lambda \in \mathbb{C}; \lambda\Phi_- A \in \Phi_\pm(X)\} = \mathbb{C}\backslash\Phi_{\pm A} \\
\sigma_{e4}(A) &= \{\blacksquare \in \mathbb{C}; \lambda - A \in \Phi(X)\} = \mathbb{C}\backslash\Phi_A \\
\sigma_{e5}(A) &= \bigcap_{K\in\mathcal{K}(X)} \sigma(A+K) \\
\sigma_{e6}(A) &= \mathbb{C}\backslash\rho_6(A) \\
\sigma_{e7}(A) &= \bigcap_{K\in\mathcal{K}(X)} \sigma_{ap}(A+K) \\
\sigma_{e8}(A) &= \bigcap_{K\in\mathcal{K}(X)} \sigma_\delta(A+K) \\
\sigma_{pa}(A) &= \{\lambda \in \mathbb{C}\backslash\ \lambda - A \text{ is not closed image}\}
\end{aligned}$$

with

$$\begin{aligned}
\sigma_{ap}(A) &= \{\lambda \in \mathbb{C}\backslash \inf_{\|x\|\leq 1} \|\lambda - A\| = 0\} \\
\sigma_\delta(A) &= \{\lambda \in \mathbb{C}\backslash\lambda - A \text{ not surjective }\} \\
\rho_6(A) &= \{\lambda \in \rho_5(A)\backslash \text{ all } \lambda \text{ in } \rho(A)\} \\
\rho_5(A) &= \{\lambda \in \Phi_A\backslash\ i(\lambda - A) = 0\}
\end{aligned}$$

Proposition 1.6.1 *We have $\sigma_{e1}(A)$ and $\sigma_{e2}(A)$, its is essential spectrum of Gustafson and Weidmann , $\sigma_{e3}(A)$ is essential spectrum of Kato, $\sigma_{e4}(A)$ is essential spectrum of Wolf, $\sigma_{e5}(A)$ is essential spectrum of Schechter or essential spectrum of Weyl, $\sigma_{e6}(A)$ is essential spectrum of Browder, $\sigma_{e7}(A)$ is essential spectrum of Rakovcèvic, $\sigma_{e8}(A)$ is essential spectrum introduced by Schmoeger.*

Proposition 1.6.2 *Let $\lambda \in \sigma_{ess}(A)$ if and only if, there existe a bounded sequence $(f_n)_n$ no compact such that $(A-\lambda \mathrm{I})f_n \to 0$. Let $\lambda \in \mathbb{C}$, for the sequence, we apply*

- $G = N(A-\lambda \mathrm{I})$.
- $A' = A_{|G^{\perp}}$.

Proposition 1.6.3 *we have:*

- $\sigma_{ess1}(A) \cap \sigma_{ess2}(A) = \sigma_{ess3}(A \subseteq \sigma_{ess4}(A) \subseteq \sigma_{ess5}(A) \subseteq \sigma_{ess6}(A)$.
- $\sigma_{ess5}(A) = \sigma_{ess7}(A) \cup \sigma_{ess8}(A)$.
- $\sigma_{ess1}(A) \subset \sigma_{ess7}(A)$.
- $\sigma_{ess2}(A) \subset \sigma_{ess8}(A)$.

1.6.2 Essential spectrum of wolf and weyl

Definition 1.6.2 *In Banach space X, the Wolf essential spectrum of the operator $A \in \mathcal{C}(X)$ is defined by*

$$\sigma_f(A) := \mathbb{C} \backslash \{\lambda \in \mathbb{C} : \lambda - A \in \Phi(X)\}$$

Definition 1.6.3 *The Weyl essential spectrum of the operator $A \in \mathcal{C}(X)$ is defined by*

$$\sigma_w(A) := \bigcap_{k \in \mathcal{K}(X)} \sigma(A+k)$$

Where $\mathcal{K}(X)$ stands for the ideal of all compact opaerator on X.

Proposition 1.6.4 *Let $A \in \mathcal{L}(X)$.Then $\lambda \notin \sigma_w(A)$ if and only if, $\lambda - A \in \Phi(X)$ and $i(\lambda - A) = 0$.*

Remark 1.6.1 (i) *The relation ship between the Weyl essential spectrum and the Wolf essential spectrum of A defined by*

$$\sigma_w(A) = \sigma_f(A) \cup \{\lambda \in \mathbb{C} : i(\lambda - A) \neq 0\}$$

(ii)We recall that for $\Theta \in \{\sigma_w(A), \sigma_f(A)\}$, $A \in \mathcal{R}(X)$ if and only if, $\Theta = \{0\}$.

Chaptre 2

Convergence and approximation of operators

A and A_n denote bounded linear operators on a complex Banach space X, that is $A, A_n \in \mathcal{L}(X)$ unless otherwise mentiond the convergence is as $n \to \infty$.

If (A_n) converges to A in some sense, the following questions arise naturally:

1. If $\lambda_n \in \sigma(A_n)$ and $\lambda_n \to \lambda$, does $\lambda \in \sigma(A)$?

2. If $\lambda \in \sigma(A)$, does there exist $\lambda_n \in \sigma(A_n)$ for each large n such that $\lambda_n \to \lambda$?

•We shall say that under a given mode of convergence:

·**Property U** holds if, whenever $A_n \to A$, λ_n belongs to $\sigma(A_n)$ and (λ_n) converges to λ, we have $\lambda \in \sigma(A)$.

·**Property L** holds if, whenever $A_n \to A$ and $\lambda \in \sigma(A)$, there exists some λ_n belonging to $\sigma(A_n)$ for each large enough n such that (λ_n) converges to λ.

Let us consider the following property:

$$\text{Whenever } A_n \to A, \sup\{dist(\mu; \sigma(A)) : \mu \in \sigma(A_n)\} \to 0$$

It is known as the upper semicontinuity of the spectrum under the given mode of convergence and Property U is a consequence of it. Property L is known as the lower semicontinuity of the spectrum under the given mode of convergence. It can also be stated as follows

$$\text{Whenever } A_n \to A \text{ and } \lambda \in \sigma(A),\ dist(\lambda; \sigma(A_n)) \to 0$$

The upper semicontinuity of the spectrum and the lower semicontinuity of the spectrum, taken togother give the continuity of the spectrum in the sense of Kuratowski.

2.1 Spectral approximation of the operators

Several problemes related to differentiel equations and partial derivativesthis axis is the following:

Give a bounded operator A and pertubation denoted by E, what will be the spectrum of A and the spectrum of $E + A$ relative.

But no respose can be except in special cases, this is caused by the absence of lower semi-continous properties of the spectrum of any operator A, the favorable fromwork was to consider compact pertubation, follous extend its studies to the case of the pertubation not compact.

For a linear operator bounded A on a Banach complex spaces, an approximate solition to the problem

$$A\Phi = \lambda\Phi \text{ for } \Phi \in X\setminus\{0\}$$

Is obtained by slove the problem approximates following:

$$A_n\Phi_n = \lambda_n\Phi_n \text{ for } \Phi_n \in X\setminus\{0\}$$

Several modes of convergence have been introduced for the approximation of the spectral elements of operator A, the collectively compact convergence, compact convergence, regular convergence and strongly stable convergence...

2.1.1 Strongly stable approximation

Let us consider there three well-know modes of convergence, let X be a Banach space:

Definition 2.1.1 *Let (A_n) be a sequence of operators in $\mathcal{L}(X, Y)$ and $\underset{n\to\infty}{lim} \|A_n - A\| = 0$ for some $A \in \mathcal{L}(X; Y)$, then we say that A_n converges uniformly to A or operator norm, topology on $\mathcal{L}(X; Y)$,denoted by $A_n \xrightarrow{n} A$,*

$$\|A_n - A\| \to 0 \text{ as } n \to \infty$$

Definition 2.1.2 *We can write a sequence $A_n \in \mathcal{L}(X, Y)$, and there is an operator $A : X \to Y$ such that. The pointwise convergence, denoted by $A_n \xrightarrow{p} A$*

$$\|A_n x - Ax\| \to 0 \textit{ for every } x \in X$$

Definition 2.1.3 *Let $\{A_n\}$be a sequence of operators in normed linear space converging to A by collectively compact convergence $A_n \xrightarrow{cc} A$ if and only if, $A_n \xrightarrow{p} A$ and for some positive integer n_0*

$$\bigcup_{n \geq n_{<à}} \{(A_n - A)x : x \in X; \|x\| \leq 1\}$$

Is a relatively compact subset of X. Every compact set of compact linear operators is collectively compact, then the latter condition is equivalent to the condition that for some positive integer n_0, the set

$$K := \bigcup_{n \geq n_0} \{A_n x : x \in X; \|x\| \leq 1\}$$

Is a relatively compact subset of X.

While pointwise convergence and norm convergence are classical concepts.

- As a partial converse, every collectively compact set of self adjoint or normal operators on a Hilbert space is totally bounded.

Remark 2.1.1 *Let X be a Banach space, the collectively compact convergence or the norm resuts the pointwise convergence.*

If $A_n \xrightarrow{cc} A$ or $A_n \xrightarrow{n} A$, then clearly $A_n \xrightarrow{p} A$. But the converse is not true.

Example 2.1.1 *In this example, $A_n \xrightarrow{p} \mathcal{I}$ but $A_n \stackrel{n}{\nrightarrow} \mathcal{I}$ and $A_n \stackrel{cc}{\nrightarrow} \mathcal{I}$.Consider $X := \ell^p, 1 < p < \infty$ For $n = 1, 2, \ldots$ and $x := \sum_{k=1}^{\infty} x(k) e_k$ in X. Let $A_n x := \sum_{k=1}^{\infty} x(k) e_k$*

Then each A_n is a bounded finite rank operators on X and $A_n \xrightarrow{p} \mathcal{I}$. but $A_n \stackrel{n}{\nrightarrow} \mathcal{I}$ since $\|A_n - \mathcal{I}\| = 1$ for each n and since $A_n \stackrel{cc}{\nrightarrow} \mathcal{I}$ given any positive integer n_0,

$$e_k \in \{(\mathcal{I} - A_n)\, x : x \in X, \|x\| \leq 1\} \quad \text{for } k = n_0 + 1,\, n_0 + 2, \ldots$$

But the sequence (e_k) has no convergent subsequence.

Remark 2.1.2 *If X is an infinite dimensional Banach space, then neither norm convergence nor collectively compact convergence is stronger than the other. For example*

If $A := \mathcal{I}$ and $A_n := c_n\mathcal{I}$, where $c_n \to 1$ in $\mathbb{C}$ but, $A_n \overset{cc}{\nrightarrow} A$ unless $c_n := 1$ for all large n. On the other hand, by a result of Josefson and Nissenzweig [8, (Chapter XII)] there is a sequence (f_n) in X^* such that$\|f_n\| = 1$ for each n and $\langle x; f_n\rangle \to 0$ for each $x \in X\setminus\{0\}$. For a fixed non zero $x_0 \in X$, let $A_n x := \langle x; f_n\rangle\, x_0$, $x \in X$ and $T := 0$. Then $A_n \overset{cc}{\to} A$, but $A_n \overset{n}{\nrightarrow} A$.

Under additional hypothesis, norm convergence may imply collectively compact convergence or collectively compact convergence may imply norm convergence.

Definition 2.1.4 *Let A be a compact linear operator and let X be a complexs an infinite dimensional Banach space and the sequence $(A_n)_n$ is a bounded linear operator in X*

$((A_n)$ is not required compact) such that:
$(a)\, \forall x \in X \quad \lim\limits_{n\to\infty} T_n x = Tx.$
$(b)\, \lim\limits_{n\to\infty} \|(T_n - T)T_n\| = 0.$
So, $(A_n)_n$ is Strongly stable approximation of A in the neighborhood of each non-zero eigenvalue.

Remark 2.1.3 *The conditions (a) and $(b)$is satisfied by uniform discretization and collectively compact but it is satisfied by norm perturbations of a compact approximation A it means $A_n = K_n + B_n$ such that B_n converges uniformly to 0 and K_n is approximation collectively compact of A.*

Theorem 2.1.1 *Let X be a complex Banach space and let $A \in \mathcal{L}(X)$, λ denoted either a nonzero isolated eigenvalue of A.*

A sequence$(A_n)_n$ of bounded linear operators in X if:
(a) A is compact operator.
(b) $\|(T - T_n)x\| \to 0$ for all$x \in X$.
(c) $\|(T - T_n)T_n\| \to 0$.
Then, $(A_n)_n$ is strongly stable approximation of A.

Theorem 2.1.2 *Let X be a complex Banach space and let $A \in \mathcal{L}(X)$, λ denoted either a nonzero isolated eigenvalue of A. such that A is not compact.*

A sequence$(A_n)_n$ of bounded linear operators in X if

(a) λ is algebraic multiplicity.

(b) $\|(A - A_n)\| \to 0$.

(c) $\|(A - A_n)x\| \to 0$ for all $x \in X$.

(d) $\|(A - A_n)A_n\| \to 0$.

Then, $(A_n)_n$ is strongly stable approximation of A.

2.2 ν-convergence

The concept of $\nu-$convergence emerged as a new mode of convergence introducedby M.Ahus. It is way to approximate the noncompact operators resorting to finite rank operators.

Definition 2.2.1 *A sequence (A_n) of bounded linear operaors mapping from X into X is said to be $\nu-$convergent to A, dentoed by $A_n \xrightarrow{\nu} A$, if*

- $(\|A_n\|)$ is bounded.
- $\|(A_n - A)\, A\| \to 0$.
- $\|(A_n - A)\, A_n\| \to 0$.

Remark 2.2.1 *The ν -convergence is 'pseudo-convergence' in the sense that it is possible to have $A_n \xrightarrow{\nu} A$ and $A_n \xrightarrow{\nu} U$, where $U \neq A$. However, if $A_n \xrightarrow{\nu} A$ and $A_n \xrightarrow{\nu} U$, then $\sigma(U) = \sigma(T)$ and $Ux = Ax$. Whenever x belongs to a spectral subspace of U corresponding to a spectral set not containing 0.*

Remark 2.2.2 (i) *Not that $\nu-$convergence is a 'pseudo-convergence'.In fact, it suffices to cosider the following examples:*

Let X be a Banach, $C \in \mathcal{L}(X), U_n, V_n \in \mathcal{L}(X)$ and U_n be a sequence of operators defined on $X \otimes X$ by

$$A_n = \begin{pmatrix} U_n & 0 \\ 0 & V_n \end{pmatrix} \text{ and let } B = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}.$$

If $U_n \to 0$ and $V_n \to 0$ then $\|A_n\| = \max\{\|U_n\|, \|V_n\|\} \to 0$.So, $A_n \to A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.This implies that $A_n \xrightarrow{\nu} A$.

(ii) Let X be a Banach space, A_n be a sequence of operator defined on $X \otimes X$ by

$$A_n = \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} \text{ for all } n \in \mathbb{N}\setminus\{0\} \text{,and } A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It is clear that $\|(A_n - A_0)A_0\| \to 0$ and $\|(A_n - A_0)A_n\| \to 0$.Except that A_n is not bounded.

(iii) If $A_n \xrightarrow{\nu} A$, this does not genaerally imply that $A_n \to A$. in fact, take $X = l_2(\mathbb{N}) = \left\{(x_j)_{j\geq 1} \text{ such that } x_j \in \mathbb{C} \text{ and } \sum_{j=1}^{+\infty} |x_j|^2 < \infty\right\}$ with cononical Hilbert basis $e_1, e_2, ...$,for $n = 1, 2, ...$

Let A_n be defined by $A_n x := x(x+1)e_n$ for $n = 1, 2, ...$ and $x = \sum_{k=1}^{+\infty} x(k)e_k \in X$ then $A_n \xrightarrow{\nu} 0$ but $A_n \nrightarrow 0$.

2.2.1 Link with the modes of convergence

We will give the links that can exist between $\nu-$ convergence and the modes of convergence mentioned above.

Lemma 2.2.1 *Let A_n be a bounded linear operators and $A \in \mathcal{C}(X)$*

(a) If $A_n \xrightarrow{n} A$, then $A_n \xrightarrow{\nu} A$. Conversely, if $0 \notin \sigma(A)$ and $A_n \xrightarrow{\nu} A$, then $A_n \xrightarrow{n} A$.

(b) Let $A_n \xrightarrow{\nu} A$ and $U_n \xrightarrow{n} U$. Then $A_n + U_n \xrightarrow{\nu} A + U$ if and only if $(A_n - A)U \to 0$. In particular,

(i) if $A_n \to A$ and $U_n \to 0$, then $A_n + U_n \to A$.

(ii) if $A_n \to 0$, $U_n \xrightarrow{n} U$ and $A_n U \xrightarrow{n} 0$, then $A_n + U_n \to U$.

(c) If $A_n \xrightarrow{cc} A$ and A is a compact operator, then $A_n \xrightarrow{\nu} A$.

(d) Let A_n ,$U_n \in \mathcal{L}(X)$, $A_n \xrightarrow{cc} A$, A be a compact operator, $U_n \xrightarrow{n} 0$ and $\hat{A} := A_n + U_n$. Then $\hat{A}_n \xrightarrow{\nu} A$. In addition, if $A_n \overset{n}{\nrightarrow} A$, then $\hat{A}_n \overset{n}{\nrightarrow} A$ and if $U_n \overset{cc}{\nrightarrow} 0$, then $\hat{A}_n \overset{cc}{\nrightarrow} A$.

Remark 2.2.3 · *Part (a) shows that norm convergence implies ν-convergence and $A_n \xrightarrow{\nu} A$ is equivalent to $A_n \xrightarrow{n} A$ if $0 \notin \sigma(A)$.*

· ν-convergence is stable under norm perturbations.

· Collectively compact convergence to a compact operator implies $\nu-$convergence. On the other hand, if we let $X = \ell^2$, $A = 0$, $A_n x := x_{n+1} e_n$ for $n = 1, 2, \ldots$, $x = \sum x_k e_k \in \ell^2$, then $A_n \xrightarrow{\nu} A, A_n \xrightarrow{p} A$ but $A_n \overset{n}{\nrightarrow} A$, $A_n \overset{cc}{\nrightarrow} A$. In this example, each A_n is of finite rank.

·ν-convergence can be available even in the absence of norm convergence and collectively compact convergence.

We note that if $A_n \xrightarrow{n} A$, or $A_n \xrightarrow{cc} A$ and each A_n is a compact operators, then A is necessarily compact. On the other hand, when $A_n \xrightarrow{\nu} A$ and

each A_n is a compact operator (or even a bounded nite rank operator), the operator A need not be compact.

Example 2.2.1 *In this example, $A_n \to A$, rank $(A_n) < 1$, but A is not compact:*

Consider $1 \leq p < \infty$. For $x := \sum_{k=1}^{\infty} x(2k)\, e_k \in \ell^p$, let

$$Ax := \sum_{k=1}^{\infty} x(2k)\, e_{2k-1}.$$

And for each positive integer n,

$$A_n x := \sum_{k=1}^{n} x(2k)\, e_{2k-1}\ .$$

Clearly, $A, A_n \in \mathcal{L}(\ell^p)$, $\|A_n\| = 1$ and $\text{rank}(A_n) = n$. Since for all $x \in \ell^{p\infty}$

$$(A_n - A)\, x = - \sum_{k=n+1}^{\infty} x(2k) e_{2k-1},$$

We see that $(A_n \to A)A = O = (A_n \to A)A_n$.

This follows since for the bounded sequence (e_{2k}) in ℓ^p, the sequence $(Ae_{2k}) = (e_{2k-1})$ does not have a convergent subsequence. As a result, we have $A_n \overset{n}{\nrightarrow} A$ and $A_n \overset{cc}{\nrightarrow} A$.

Remark 2.2.4 *(a) If $A_n \xrightarrow{n} A$, then $A_n \xrightarrow{\nu} A$. Conversely, if $0 \notin \sigma_s(A)$(surjective spectrum) and $A_n \xrightarrow{\nu} A$, then $A_n \xrightarrow{n} A$.*

(*b*) If $\|(A_n - A)A\| \to 0$ and $\|(A_n - U)U\| \to 0$, then for every $x \in R(A) \cap R(U)$, $Ax = Ux$.

Really, let $x \in R(A) \cap R(U)$. Then there exist $y, z \in X$ such that $x = Ay = Uz$ and

$$\begin{aligned} 0 \le \|Ux - Ax\| &= \|U^2 z - A^2 y\| = \|U^2 z - A_n Uz + A_n Ay - A^2 y\| \\ &\le \|(A_n - U)Uz\| + \|(A_n - A)Ay\| \underset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

(*c*) If $A_n \xrightarrow{\nu} A$, $A_n \xrightarrow{\nu} U$, $A_nA = AA_n$ and $A_nU = UA_n$ then $A^2 = U^2$. In fact, let $x \in X$, then

$$\begin{aligned} 0 \le \|A^2x - U^2x\| &= \|A^2x - A_nAx + A_nAx - A_n^2x + A_n^2x - A_nUx + A_nUx - U^2x\| \\ &\le \|(A_n - A)Ax\| + \|(A_n - A)A_nx\| + \|(A_n - U)A_nx\| + \|(A_n - U)Ux\| \underset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

.

(*d*) If $A_n \xrightarrow{\nu} A$ and $A_n \xrightarrow{\nu} U$, then $\sigma(A) = \sigma(U)$. This property solves the problem of pseudo-convergence and spectrum of $\nu-$limit of operators.

Example 2.2.2 *Let $A_n, n \in N \cup \{0\}$, be operators defined on $\mathbb{C} \oplus \mathbb{C}$ as*

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_n = \begin{bmatrix} 0 & \frac{1}{n} \\ 0 & 0 \end{bmatrix} \quad n \in \mathbb{N}$$

Then $\|A_n\| \le \frac{1}{n}$ for all $n \ge 1$, thus $A_n \xrightarrow{n} A_0$ and hence $A_n \xrightarrow{\nu} A_0$.

On the other hand, $\|(A_n - A_1)A_1\| = 0 = \|(A_n - A_1)A_n\|$, therefore $A_n \xrightarrow{\nu} A_1$.

Observe that $A_1 = A_0$.

These operators clearly satisfy $A_nA_1 = A_1A_n$ and $A_nA_0 = A_0A_n$.

Lemma 2.2.2 (*i*) *If $A_n \xrightarrow{\nu} A$ and $A_n \xrightarrow{\nu} A$ then $\sigma(A) = \sigma(U)$.*

(*ii*) If $A_n \to A$ then $A_n \xrightarrow{\nu} A$. conversely, if $0 \in \rho(A)$ and $A_n \xrightarrow{\nu} A$, then $A_n \to A$.

(*iii*) Let $A_n \xrightarrow{\nu} A$ and $U_n \to U$. Then $A_n + U_n \xrightarrow{\nu} A + U$ if and only if $(A_n - A)U \to 0$.

Chaptre 3

Spectral continuity

3.1 Continuity of the spectrum

Let S be the collection of all non-empty compact subsets of $\mathbb{C}$. It is well known that the convergence of a sequence in S with respect to the Hausdorff metric can be characterized through the concepts of limit inferior and superior.

Let $\{E_n\}$ be a sequence of arbitrary subsets of $\mathbb{C}$ and define the limits inferior and superior of $\{E_n\}$, denoted respectively by $\liminf E_n$ and $\limsup E_n$, as follows:

$\liminf\ E_n = \{\lambda \in \mathbb{C} :$ for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $B(\lambda, \varepsilon) \cap E_n \neq \emptyset$ for all $n \geq N$ $\}$.

$\limsup E_n = \{\lambda \in \mathbb{C} :$ for every $\varepsilon > 0$, there exists $J \subseteq \mathbb{N}$ infinite such that $\mathcal{B}(\lambda, \varepsilon) \cap E_n \neq \emptyset$ for all $n \in J\}$.

If $\liminf\ E_n = \limsup\ E_n$, then $\lim E_n$ is said to exist and is equal to this common limit.

Remark 3.1.1 *Let $\{E_n\}$ be a sequence of non-empty subsets of $\mathbb{C}$. The following properties of limit inferior and superior are known:*

(a) $\liminf\ E_n$ and $\limsup\ E_n$ are closed subsets of $\mathbb{C}$.

(b) $\lambda \in \limsup\ E_n$ if and only if, there exists an increasing sequence of natural numbers $n1 < n2 < n3 <\cdots$ and points $\lambda_{nk} \in E_{nk}$, for all $k \in N$, such that $\lim\ \lambda_{nk} = \lambda$.

$\limsup\ E_n := \{\lambda \in \mathbb{C} :$ there are $\lambda_{nk} \in E_n$ with $\lambda_{nk} \to \lambda\}$

(c) $\lambda \in \liminf E_n$ if and only if, there exists a sequence $\{\lambda_n\}$ such that $\lambda_n \in E_n$ for all $n \in N$, and $\lim \lambda_n = \lambda$.

$\liminf E_n := \{\lambda \in \mathbb{C} : \text{there are } \lambda_n \in E_n \text{ with } \lambda_n \to \lambda\}$

(d) Suppose $E, E_n \in S$ for all $n \in \mathbb{N}$ and there exists $K \in S$ such that $E_n \subseteq K$, for all $n \in \mathbb{N}$. Then $E_n \to E$ in the Hausdorff metric if and only if, $\limsup E_n \subseteq E$ and $E \subseteq \liminf E_n$.

Definition 3.1.1 *An operator $T \in \mathcal{L}(X)$ is said to have the single-valued extension property (SVEP for short) at $\lambda \in \mathbb{C}$, if for every open neighborhood U_λ of λ, the only analytic function $f : U_\lambda \to X$ which satisfies the equation $(\mu - T)f(\mu) = 0$ for all $\mu \in U_\lambda$ is the function $f \equiv 0$.*

Theorem 3.1.1 *If $T \in \mathcal{L}(X)$ and $\{T_n\}$ is a sequence in $\mathcal{L}(X)$ such that $T_n \xrightarrow{\nu} T$, then $\limsup \sigma(T_n) \subseteq \sigma(T)$.*

Theorem 3.1.2 *Let $T \in \mathcal{L}(X)$ such that T^*has $SVEP$ at every $\beta \notin \sigma_{eap}(T)$. If $\{T_n\}$ is a sequence in $\mathcal{L}(X)$ such that $T_n \xrightarrow{\nu} T$, then $\limsup \sigma_{ap}(T_n) \subseteq \sigma_{ap}(T)$.*

Proof. Let $\lambda \in \limsup \sigma_{ap}(T_n)$. Since $\sigma_{ap}(T_n) \subseteq \sigma(T_n)$ for all $n \in \mathbb{N}$, then $\lambda \in \limsup \sigma(T_n)$ and hence, $\lambda \in \sigma(T)$. If $\lambda \notin \sigma_{ap}(T)$ then $\lambda - T$ is injective and $R(\lambda - T)$ has closed range. This implies that $\lambda - T \in \Phi_+(X)$ and $i(\lambda - T) \leq 0$, hence $\lambda \notin \blacksquare_{ea}(T)$ and so T^*has $SVEP$ at λ. Consequently, $i(\lambda - T) \geq 0$, i.e., $0 - \beta(\lambda - T) = i(\lambda - T) = 0$. Therefore $\lambda - T$ is invertible, **a contradiction**. ■

Proposition 3.1.1 *Let $\mathcal{R} \in \mathcal{L}(X)$ be a Riesz operator. If $\{T_n\}$ is a sequence in $\mathcal{L}(X)$ such that $T_n \xrightarrow{\nu} \mathcal{R}$, then $\sigma(T_n) \to \sigma(\mathcal{R})$.*

The class of nilpotent, quasinilpotent and compact operators. It is well known that if $T_n \xrightarrow{n} T$ and each T_n is a compact operators, then T is necessarily compact, however if $T_n \xrightarrow{\nu} T$ and each T_n is a compact operators (or even a finite rank operator), then the operator T need not be compact.

Example 3.1.1 *Let $\{e_n\}_n \in \mathbb{N}$ be an orthonormal basis for $\ell^2(\mathbb{N})$ and let T be the operator defined on $\ell^2(\mathbb{N})$ as:*

$$T(x_1, x_2, x_3, x_4, \cdots) = (x_2, 0, x_4, 0, x_6, 0, \cdots).$$

This operator is not compact, because for the bounded sequence $\{e_{2n}\}_{n\in\mathbb{N}}$ in $\ell^2(\mathbb{N})$, the sequence $\{Te_{2n}\}_{n\in\mathbb{N}}$ does not have a convergent subsequence. On the other hand, $T^2 = 0$, therefore T is nilpotent and so T is a Riesz operator. Now, for each $n \in \mathbb{N}$ define

$T(x_1, x_2, x_3, x_4 \cdots) = (x_2, 0, x_4, 0, x_6, 0, \cdots, x_{2n}, 0, 0 \cdots)$

Clearly $rank(T_n) = n$, $\|T_n\| = 1$ and $(T_n - T)T = 0 = (T_n - T)T_n$, so $T_n \xrightarrow{\nu} T$. Then, $\sigma(T_n) \to \sigma(T)$.

In this example, it is possible to approximate the spectrum of a non-compact operator through the spectrum of finite rank operators.

3.2 Spectral continuity and perturbations

In this section, we investigate the stability of points of spectral continuity as follows:

Given a sequence $\{K_n\}$ of compact operators and $\mathcal{R}$ a Riesz operator for which $K_n \xrightarrow{\nu} \mathcal{R}$, then $\sigma(K_n + S)$ converges to $\sigma(\mathcal{R} + S)$, when $S \in \mathcal{L}(X)$ satisfies some conditions.

Theorem 3.2.1 *Let $\{K_n\}$ be a sequence of compact operators, $\mathcal{R}$ be a Riesz operator and S be a not Weyl operator. If*

(a) S satisfies Browder's theorem

(b) $K_n \xrightarrow{\nu} \mathcal{R}$, $K_nS_n \to \mathcal{R}S$, and

(c) $\mathcal{R}S = S\mathcal{R}$,

Then $\sigma(K_n + S) \to \sigma(\mathcal{R} + S)$.

Example 3.2.1 *Let R, S and F_n be operators defined on $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ as*

R= $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, S= $\begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix}$, $F_n = \begin{pmatrix} 0 & \frac{I-UU^*}{n} \\ 0 & 0 \end{pmatrix}$

Where U is the unilateral shift on $\ell^2(\mathbb{N})$. Then $\mathcal{R}$ is a Riesz operator, $F_n, n \in \mathbb{N}$, are one-dimensional operators and $F_n + S \xrightarrow{\nu} \mathcal{R} + S$, but $\sigma(F_n + S) \nrightarrow \sigma(\mathcal{R} + S)$. Indeed, each $F_n + S$ is similar to $F_1 + S$ and $F_1 + S$ is an unitary operator, so for every n, $\sigma(F_n + S) = \sigma(F_1 + S) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, and $\sigma(\mathcal{R} + S) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$.

The sequence$\{F_n\}$in example above satisfies $F_n \xrightarrow{\nu} \mathcal{R}$, but $\sigma(F_n + S) \nrightarrow \sigma(\mathcal{R} + S)$. This same sequence also satisfies $F_n \xrightarrow{\nu} F_1$ and $F_1 \neq \mathcal{R}$, but in this case $\sigma(F_n + S) \to \sigma(F_1 + S)$. A natural question is under what conditions it holds $\sigma(T_n + S) \to \sigma(T + S)$ when $T_n \xrightarrow{\nu} T$.

Theorem 3.2.2 *Let $\{K_n\}$,$n \in \mathbb{N}$ be a sequence of compact operators, $\mathcal{R}$ be a Riesz operator and $S \in \mathcal{L}(X)$ such that $S \notin \Phi_+(X)$ or $i(S) > 0$. If*

(a) S^* has $SVEP$ at every $\lambda \notin \sigma_{eap}(S)$,

(b) $K_n \xrightarrow{\nu} \mathcal{R}$, $K_n S \xrightarrow{n} \mathcal{R}S$, and

(c) $\mathcal{R}S = S\mathcal{R}$,

Then $\sigma_{ap}(K_n + S) \to \sigma_{ap}(\mathcal{R} + S)$.

3.3 Properties of spectra essential in the ν-convergence

3.3.1 Preliminary theories of wolf and weyl essentiel spectrum

Theorem 3.3.1 *Let $A \in \Phi^b(X)$ and $B \in \mathcal{L}(X)$, there exists $\eta > 0$ such that if $\|A - B\| < \eta$ then $B \in \Phi^b(X)$ and $i(A) = i(B)$.*

Lemma 3.3.1 *Let $U, V \in \mathcal{L}(X)$. If there exists $n_0 \in \mathbb{N}\backslash\{0\}$ such that* $U^{n_0} = V^{n_0}$,

$E = R(U^{n_0-1}) \cap R(V^{n_0-1})$ is closed and $U\mid_E = V\mid_E$, then $\sigma_i(U) = \sigma_i(V)$ for $i = f, w$.

Proof. Let $\lambda \notin \sigma_w(U)$, we shall divide the proof into two cases:

1st case. If $\lambda \neq 0$, we have $\lambda - U \in \Phi^b(X)$ and $i(\lambda - U) = 0$. Since $U^{n_0} = V^{n_0}$, it follows that $U(E) \subseteq E, V(E) \subseteq E$,$U^{n_0}(X) \subseteq E$ and $V^{n_0}(X) \subseteq E$. We have $\lambda - U$ is a Fredholm operator if and only if, $\lambda - U|_E$ is a Fredholm operator and i $(\lambda - U) = i$ $(\lambda - U|_E) = 0$. Thus $\lambda - V|_E$ is a Fredholm operator and i $(\lambda - V|_E) = 0$. So, $\lambda - V$ is a Fredholm operator and i $(\lambda - V) = i(\lambda - V|_E) = 0$. consequently $\lambda \notin \sigma_w(V)$.

2nd case. If $\lambda = 0$, then U is a Fredholm operator and $i(U) = 0$. This implies that U^{n_0} is a Fredholm operator and i $(U^{n_0}) = n_0 i$ $(U) = 0$. Hence $V^{n_0}$is a Fredholm operator and $i(V^{n_0}) = 0$ which implies that V is a Fredholm and i $(V) = \frac{i(V^{n_0})}{n_0} = 0$. So, $0 \notin \sigma_w(V)$.Therefore $\sigma_w(V) \subseteq \sigma_w(U)$.The opposite inclusion is anslogous. By following the same reasning, we obtain the following equality $\sigma_f(U) = \sigma_f(V)$. ∎

Theorem 3.3.2 *Let $U, V \in \mathcal{L}(X)$ be two operators such that $R(U) \cap R(V)$ is closed and let (U_n) be a sequence of bounded operators commuting with both U and V.*

If $U_n \xrightarrow{\nu} U$ and $U_n \to V$ then

$$\sigma_i(U) = \sigma_i(V) \text{ , for } i = f, w$$

3.3.2 Contnuity of the wolf and weyl essential spectra

The goal of this section is to discuss the wolf and weyl essential spectra of a sequence of linear operatore ν-convergence on Banach space X.

The concept of $\nu-$convergence emerged as a new mode of convergence introduced by M.Ahues.

We should notice that if $U \in \mathcal{L}(X), \{U_n\} \subseteq \mathcal{L}(X), 0 \in \Phi_U^b$ and $U_n \to U$ then not necessarily $\sigma_f(U_n) \subseteq \sigma_f(U)$.In fact, it suffices to consider the following examples:

Let $R : l_2(\mathbb{N}) \to l_2(\mathbb{N})$ be the right shift operator defined by

$$R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots).$$

For each $n \in \mathbb{N}$, Let $U_n = (1 - \frac{1}{n})R$ and $U = R$.

Then $U, U_n \in \mathcal{L}(X), 0 \in \Phi_U^b$ and $U_n \xrightarrow{\nu} U$, but $\sigma_f(U_n) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{n-1}{n}\}$ and $\sigma_f(U) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Therefor $\sigma_f(U_n) \nsubseteq \sigma_f(U)$ for all $n \in \mathbb{N}$.

However, we have the following result.

Theorem 3.3.3 *Let $U \in \mathcal{L}(X)$ and (U_n) be a sequence in $\mathcal{L}(X)$ such that $0 \in \Phi_U^b$. If $U_n \xrightarrow{\nu} U$ and $\mathcal{O}$ is an open set of $\mathbb{C}$ with $0 \in \mathcal{O}$, then there exists $n_0 \in \mathbb{N}$ suth that for all $n \geq n_0$*

$$\sigma_i(U_n) \subseteq \sigma_i(U) + \mathcal{O}, \text{ for } i = f, w.$$

Proof. For $i = w$, assum that the assertion fails. Then by passing to a subsequence, it may be assumed that, for each $n \in \mathbb{N}$, there exists $\lambda_n \in \sigma_w(U_n)$ such that $\lambda_n \notin \sigma_w(U) + \mathcal{O}$. Since (λ_n) is bounded, we may assume that $\lim_{n \to +\infty} \lambda_n = \lambda$.This implies that $\lambda \notin \sigma_w(U) +$

$\mathcal{O}$.Using the fact that $0 \in \mathcal{O}$, we have $\lambda \notin \sigma_w(U)$ and therefore, $\lambda - U \in \Phi^b(X)$ and $i(\lambda - U) = 0$. Let

$$V_n = (\lambda_n - \lambda) U + (U - U_n) U. \tag{2.3.1}$$

It followes from Eq. (2.3.1), that the operator $(\lambda_n - U_n) U$ can be expressed in the form

$$(\lambda_n - U_n) U = (\lambda - U) U + V_n$$

Using the fact that $V_n \to 0$ and $(\lambda - U) U \in \Phi^b(X)$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $(\lambda_n - U_n) U \in \Phi^b(X)$ and $i[(\lambda_n - U_n) U] = i[(\lambda - U) U]$. Moreover $\beta(U) < \infty$,then we have $(\lambda_n - U_n) \in \Phi^b(X)$ with $i(\lambda_n - U_n) = i(\lambda - U)$.We get $(\lambda_n - U_n) \in \Phi^b(X)$ and $i(\lambda_n - U_n) = i(\lambda - U) = 0$. So $\lambda_n \notin \sigma_w(U_n)$, which is a contradiction. This enables us to conclude that, $\sigma_w(U_n) \subseteq \sigma_w(U) + O$, for all $n \geq n_0$. ■

As a direct consequence, we have the following corollary.

Corollary 3.3.1 *Let $U \in \mathcal{L}(X)$ and (U_n) be a sequence of linear operators in $\mathcal{L}(X)$ such that $0 \in \Phi_U^b$ and $U_n \xrightarrow{\nu} U$. If $\mathcal{O} \subseteq \mathbb{C}$ is an open set with $0 \in \mathcal{O}, V \in \mathcal{L}(X)$ and $VU \in \mathcal{F}^b(X)$ $\left(\text{or } UV \in \mathcal{F}^b(X)\right)$, then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,*

$$\sigma_i(U_n + V) \subseteq \sigma_i(U + V) + \mathcal{O}, \text{ for } i = f, w.$$

Proof. There exists $n_0 \in \mathbb{N}$ such that $\sigma_i(U_n) \subseteq \sigma_i(U) + \mathcal{O}$, for$i = f, w$. Now, we prove that $\sigma_i(U_n + V) \subseteq \sigma_i(U_n)$, for $i = f, w$. Since U is a Fredholm linear operator, then there exists $U_0 \in \mathcal{L}(X)$ such that $UU_0 = I - K$ where $K \in \mathcal{K}(X)$. In fact, that $VUU_0 = V(I - K) = V - VK \in \mathcal{F}^b(X)$ and $VK \in \mathcal{F}^b(X)$, we see that $V \in \mathcal{F}^b(X)$. Hence, $\sigma_i(U_n + V) \subseteq \sigma_i(U_n)$, for $i = f, w$.

For $UV \in \mathcal{F}^b(X)$, the proof can be cheked in the same way. ■

The aim of the following theorem is to extend this theorem to a sequence of closed linear operators.

Theorem 3.3.4 *Let V be a closed linear operator and let (V_n) be a sequence of closed linear operators on X and $\mathcal{O} \subseteq \mathbb{C}$ be an open set with $0 \in \mathcal{O}$. If for some $\lambda_0 \in \rho(V_n) \cap \rho(V)$, we have $(\lambda_0 - V_n)^{-1} \xrightarrow{\nu} (\lambda_0 - V)^{-1}$, then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,*

$$\sigma_i(V_n) \subseteq \sigma_i(V) + \mathcal{O}, \text{ for } i = f, w.$$

Proof. For $i = w$, take $\gamma_n \in \sigma_w(V_n)$ such that $\gamma_n \notin \sigma_w(V) + \mathcal{O}$. From σ is $\nu-$upper semi continuous at $(V - \lambda_0)^{-1}$, there exists $k > 0$ such that $k^{-1} \leq |\gamma_n - \lambda_0|^{-1}$, so (γ_n) is bounded. Therefore it can be assumed that $\gamma_n \to \gamma$. Then $\gamma \notin \sigma_w(V) + O$ and hence $\gamma \notin \sigma_w(V)$. This implies that $\gamma - \lambda_0 \notin \sigma_w(V - \lambda_0)$ and so $(\gamma - \lambda_0)^{-1} \notin \sigma_w\left((V - \lambda_0)^{-1}\right)$.

We set $\lambda_n = (\gamma_n - \lambda_0)^{-1}$,$\lambda = (\gamma - \lambda_0)^{-1}$,$U_n = (V_n - \lambda_n)^{-1}$ and $U = (V - \lambda_0)^{-1}$, then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$. $\lambda_n \notin \sigma_w(U_n)$, i.e., $(\gamma_n - \lambda_0)^{-1} \notin \sigma_w\left((V_n - \lambda_0)^{-1}\right)$ which implies $(\gamma_n - \lambda_0) \notin \sigma_w(V_n - \lambda_0)$ and so, $\gamma_n \notin \sigma_w(V_n)$is a contradiction. The proof of the inclusion $\sigma_f(U_n) \subseteq \sigma_f(U) + \mathcal{O}$, follows the same reasoning. ■

3.3.3 Contnuity of the essential approximate point spectrum

Definition 3.3.1 *Let $\sigma_{ap}(U)$ denote the approximate point spectrum of $U \in \mathcal{L}(X)$, i.e.,*

$$\sigma_{ap}(U) = \{\lambda \in \mathbb{C} : \inf \|(\lambda I - U)x\| = 0\}$$

The essential approximate point spectrum $\sigma_{eap}(U)$ of U was introduced by *V. Rakočević* in as follows:

$$\sigma_{eap}(U) = \bigcap_{K \in \mathcal{K}(X)} \sigma_{ap}(U + K)$$

Lemma 3.3.2 *Let $U \in \Phi_+(X)$. Then the following statements are equivalent:*

(a) $i(U) \leq 0$.

(b) U can be expressed in the form $U = V + K$ where $K \in \mathcal{K}(X)$ and $V \in \mathcal{C}(X)$ an operator whith closed range and $\alpha(V) = 0$.

Proposition 3.3.1 *.Let $U \in \mathcal{C}(X)$.*

1· $\sigma_{eap}(U) \neq \emptyset$.

2· $\lambda \notin \sigma_{eap}(U)$ if and only if, $\lambda - U \in \Phi_+(X)$ and $i(\lambda I - U) \leq 0$.

3· $\sigma_{eap}(U)$ is compact, $\sigma_{eap}(U) \subseteq \sigma(U)$.

Proof. (1) and (3) see[16], ■

(2) ·Let $\lambda - U \in \Phi_+$ such that $i(\lambda - U) \leq 0$.Then $\lambda - U$ can be expressed in the form $\lambda - U = V + K$ where $K \in \mathcal{K}(X)$ and $V \in \mathcal{C}(X)$ an operator whith closed range and $\alpha(V) = 0$.

Hence there exists a constant $c > 0$ such that $\|Vx\| \geq c\|x\|$, for all $x \in \mathcal{D}(X)$.Thus $\lambda \notin \sigma_{ap}(U + K)$ and therefore $\lambda \notin \sigma_{eap}(U)$.Conversely, if $\lambda \notin \sigma_{eap}(U)$, then there exists $K \in \mathcal{K}(X)$ such that

$$\inf_{\|x\|=1, x \in \mathcal{D}(A)} \|(\lambda - U - K)x\| > 0.$$

The use of leads to $\lambda - U - K \in \Phi_+(X)$ and $\alpha(\lambda - U - K) = 0$, hence if followes that $\lambda - AU \in \Phi_+(X)$ and $i(\lambda - U) \leq 0$.

Lemma 3.3.3 *Let $U, V \in \mathcal{L}(X)$ satisfying $T^n = U^n$ for some positive integer n, then $0 \in \sigma_{ap}(U)$ if and only if $0 \in \sigma_{ap}(V)$. Moreover, if U and V have closed range $\alpha(R) < \infty$ and $U_{|R(U^{n-1}) \cap R(V^{n-1})} = V_{|R(U^{n-1}) \cap R(V^{n-1})}$, then $\sigma_{ap}(U) = \sigma_{ap}(V)$.*

Proof. Let $0 \in \sigma_{ap}(U)$, then by spectral mapping theorem we have $0 \in \sigma_{ap}(U^n)$ $= 0 \in \sigma_{ap}(V^n)$ and again $0 \in \sigma_{ap}(V)$. ■

Now, suppose that $U_{|R(U^{n-1}) \cap R(V^{n-1})} = V_{|R(U^{n-1}) \cap R(V^{n-1})}$. We set $E = R(U^{n-1}) \cap R(V^{n-1})$, clearly E is a closed subspace of X and since $U^n = V^n$, we have $U(E) \subseteq E$, $V(E) \subseteq E$, $U_n(X) \subseteq E$ and $V^n(X) \subseteq E$. Therefore by [19, Theorem6 (3)], for $\lambda = 0$, it follows that

$$\begin{aligned} \lambda - U \text{ is bounded below} &\iff \lambda - U_{|W} \text{ is bounded below} \\ &\iff \lambda - V_{|W} \text{ is bounded below} \\ &\iff \lambda - V \text{ is bounded below} \end{aligned}$$

Thus

$$\sigma_{ap}(U) \backslash \{0\} = \sigma_{ap}(V) \backslash \{0\}.$$

Theorem 3.3.5 *Let $U, V \in \mathcal{L}(X)$ be operators with closed range and let $\{U_n\}$ be a sequence of bounded operators commuting with both U and V. If $U_n \xrightarrow{\nu} U$ and $U_n \xrightarrow{\nu} V$, then $\sigma_{ap}(U) = \sigma_{ap}(V)$.*

Proof. It's a consequence of[19, (Remark1)] and [4, (Lemma 2.1)]. ■

Theorem 3.3.6 *Let $U \in \mathcal{L}(X)$ and (U_n) be a sequence in $\mathcal{L}(X)$ such that $0 \in \Phi_U^b$. If $U_n \xrightarrow{\nu} U$ and $\mathcal{O}$ is an open et of $\mathbb{C}$ with $0 \in \mathcal{O}$, then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$*

$$\sigma_i(U_n) \subseteq \sigma_i(U) + \mathcal{O} \text{ for } i = eap$$

Proof. For $i = eap$, assume that the assertion fails.Then by passing to a subsequence, it may be assumed that, for each $n \in \mathbb{N}$, there exists $\lambda_n \in \sigma_{eap}(U_n)$ such that $\lambda_n \notin \sigma_{eap}(U) + \mathcal{O}$. Since (λ_n) is bounded, we may assume that $\lim_{n\to+\infty} \lambda_n = \lambda$.This implies that $\lambda \notin \sigma_{eap}(U) + \mathcal{O}$. Using the fact that $0 \in \mathcal{O}$, we have $\lambda \notin \sigma_{eap}(U)$ and therefore, $\lambda - U \in \Phi_+^b(X)$ and $i(\lambda - U) \leq 0$, since $\Phi_+^b(X) = \Phi_+(X) \cap \mathcal{L}(X)$ and $\lambda_n - U_n \to \lambda - U$ whene $(n \to \infty)$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\lambda_n - U_n \in \Phi_+^b(X)$ and $i(\lambda_n - U_n) \leq 0$ then by proposition(1), we get $\lambda_n \notin \sigma_{eap}(U_n)$, which is **a contradiction**. ■

Corollary 3.3.2 *Let $U \in \mathcal{L}(X)$ and (U_n) be a sequence of linear operators in $\mathcal{L}(X)$ such that $0 \in \Phi_U^b$ and $U_n \to U$.If $\mathcal{O} \subseteq \mathbb{C}$ is an open set with $0 \in \mathcal{O}, V \in \mathcal{L}(X)$ and $VU \in \mathcal{F}_+^b(X)$ $\left(\text{or } UV \in \mathcal{F}_+^b(X)\right)$, then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,*

$$\sigma_i(U_n + V) \subseteq \sigma_i(U + V) + \mathcal{O}, \text{ for } i = eap.$$

3.4 v-continuity of Wolf and Weyl essential spectra

Inspired by the notion of ν-convergence, we examine the following definition.

Definition 3.4.1 *A mapping τ on $\mathcal{L}(X)$ whose values are compact subset of $\mathbb{C}$ is said to be $\nu-$upper semi continuous at U when*

$U_n \xrightarrow{\nu} U \Longrightarrow \limsup \tau(U_n) \subset \tau(U)$,

And to be $\nu-$lower semi continuous at U when

$U_n \xrightarrow{\nu} U \Longrightarrow \tau(U) \subset \liminf \tau(U_n)$

If τ is both $\nu-$upper and $\nu-$lower semi continuous, then it is said to be $\nu-$continuous.

Remark 3.4.1 *(a) If τ is $\nu-$lower semi continuous at U, then τ is lower semi continuous at U :*

(b) If τ is $\nu-$upper semi continuous at U, then τ is upper semi continuous at U.

(c) If τ is bounded on $\nu-$convergent sequence and τ is $\nu-$continuous at U, then τ is continuous in the Hausdorff metric at U.

(d) Genarally, the converse of (a), (b) and (c) is not true. But if $0 \in \rho(U)$ we have:

(d_1) τ is $\nu-$lower semi continuous at U if and only if, τ is lower semi continuous at U.

(d_2) τ is $\nu-$upper semi continuous at U if and only if,τ is upper semi continuous at U.

(d_3) If τ is bounded on $\nu-$convergent sequence then, τ is $\nu-$continuous at U if and only if,τ is continuous in the Hausdorff metric at U.

(e) It is worth noticing that σ is not $\nu-$continuous in general.

Theorem 3.4.1 *Let $U \in \mathcal{L}(X)$ such that $0 \in \Phi_U^b$, then σ_i is $\nu-$upper semi continuous at U for $i = f, w$.*

Furthermore, σ_w is continuous at U in each one of the following cases:

(a) σ_f is $\nu-$continuous at U.

(b) If there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $\sigma_f(U) \subset \sigma_w(U_n)$.

Proof. Let $\lambda \in \limsup(\sigma_i(U))$ for $i = f, w$.Then we may suppose that there exists a sequence (λ_n) such that $\lambda_n \in \sigma_i(U_n)$ for all $n \in \mathbb{N}$ and $\lambda_n \to \lambda$. For each $p \in \mathbb{N}\setminus\{0\}$, there exists $n_p \in \mathbb{N}$ such that $\sigma_i(U_n) \subset \sigma_i(U) + \mathcal{B}\left(0, \frac{1}{p}\right)$ for $i = f, w$ and for all $n \geq n_p$. This implies that for each $p \geq 1$, there exists $\beta \in \sigma_i\ (U)$ and $\gamma_p \in \mathcal{B}\left(0, \frac{1}{p}\right)$ such that $\lambda_{np} = \beta_p + \gamma_p$.Therefore, $\lim \beta_p = \lim \lambda_{np} - \lim \gamma_p = \lambda$.Since $\sigma_i(U)$ is a closed set, it follows that $\lambda \in \sigma_i(U)$ for $i = w, f$. We conclude that $\limsup(\sigma_i(U_n)) \subseteq \sigma_i(U)$ for $i = f, w$.

(a) Now, we prove that $\sigma_w(U) \subseteq \liminf(\sigma_i(U_n))$. Suppose that $\lambda \notin \liminf(\sigma_w(U_n))$ then there is a neighbourhood $\mathcal{V}$ of λ which does not intersect infinitely many $\sigma_w(U_n)$. Since $\sigma_f(U_n) \subset \sigma_w(U_n)$, then $\mathcal{V}$ does not intersect infinitely with many $\sigma_f(U_n)$. Hence $\lambda \notin \liminf(\sigma_f(U_n))$. Now using σ_f is $\nu-$continuous at U, then $\liminf(\sigma_f(U_n)) = \sigma_f(U)$.This implies that $\lambda \notin \sigma_f(U)$.This shows that

$$\lambda - U \in \Phi^b(X) \tag{3.2}$$

Now, we prove that $i(\lambda - U) = 0$.Since $\lambda \notin \liminf(\sigma_w(U_n))$ there exists an increasing sequence of natural numbers $n_1 < n_2 < \cdots$ such that $\lambda \notin \sigma_w\left(U_{n_j}\right)$ for all $j \in \mathbb{N}$.Then $\lambda - U_{n_j} \in \Phi^b(X)$ and $i\left(\lambda - U_{n_j}\right) = 0$.There exists $j_0 \in \mathbb{N}$ such that $i\left(\lambda - U_{n_j}\right) = i(\lambda - U)$ for all $j \geq j_0$.Therefore ■

$$i(\lambda - U) = 0 \tag{3.3}$$

It follows from eqs (3.2) , (3.3), that $\lambda \notin \sigma_w(U)$.So we get that $\sigma_w(U) \subset \liminf(\sigma_w(U_n))$.

(b) Assume that the assertion does not hold. Then there exists $\lambda \in \sigma_w(U)$ such that $\lambda \notin \liminf(\sigma_w(U_n))$. We discuss two cases.

<u>1st case</u>. If $\lambda \in \sigma_w(U) \backslash \sigma_f(U)$ then by using a simlilar reasoning of (a) we have $i(\lambda - U) = 0$

which is a contradiction .

<u>2nd case</u>. If $\lambda \in \sigma_f(U)$,since $\lambda \notin \liminf(\sigma_w(U_n))$. Arguing as above, there exists an increasing sequence of natural numbers $n_1 < n_2 < \cdots$,and $\varepsilon > 0$ such that for all $j \in \mathbb{N}$ we have $\mathcal{B}(\lambda, \varepsilon) \cap \sigma_w\left(U_{n_j}\right) = \emptyset$.

Then $\mathcal{B}(\lambda, \varepsilon) \cap \sigma_f(U) = \emptyset$ which is a contradiction.

Remark 3.4.2 *It was known byK. K. Oberai [15], σ_w is upper semi continuous. Moreover, if $0 \in \Phi_U^b$,we have σ_w is $\nu-$upper semi continuous at U.*

Corollary 3.4.1 *Let $U \in \mathcal{L}(X)$ such that $0 \in \Phi_U^b$, and $U_n \xrightarrow{\nu} U$.If $\Phi_{U_n}^b$ is connected then* $\lim \sigma_f(U_n) = \sigma_f(U)$ *if and only if,* $\lim \sigma_w(U_n) = \sigma_w(U)$.

Proof. First, we have that $\lim \sigma_f(U_n) = \sigma_f(U)$ implies $\lim \sigma_w(U_n) = \sigma_w(U)$ .Conversely, σ_f is $\nu-$upper semi continuous at U. It is sufficient to show that σ_f is $\nu-$lower semi continuous at U. We suppose that $\lambda \notin \liminf(\sigma_f(U_n))$ so that there is a neighbourhood $\mathcal{V}$ of λ which does not intersect infinitely many $\sigma_f(U_n)$ .Note that,$\Phi^b_{U_n}$ is connected. Then , $\sigma_f(U_n) = \sigma_w(U_n)$ .Hence, $\mathcal{V}$ does not intersect infinitely whith many $\sigma_w(U_n)$ .Hence , $\lambda \notin \liminf(\sigma_w(U_n)) = \sigma_w(U)$ .This implies that $\lambda \notin \sigma_w(U)$ .Hence, $\lambda \notin \sigma_f(U)$ .So,$\sigma_f(U) \subset \liminf(\sigma_f(U_n))$. ■

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