



PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH
UNIVERSITY MOHAMED BOUDIAF - M'SILA
Faculty of Mathematics and Computer Science
Department of Mathematics



Order Num :

THESIS

*Presented to obtain the degree of
Doctor of Science*

Specialty

Mathematics

Option

Applied Mathematics

By

Kheir Saadaoui

Title

Type-2 Fuzzy Sets Study and Application

Defended on . . / . . /2020 before the jury composed of :

Lemnaouar Zedam	Prof.	University of M'sila	President
Brahim Bouderah	Prof.	University of M'sila	Director
Ouarda Assas	M.A.A	University of Batna 2	Co-Director
Abdelaziz Amroune	Prof.	University of M'sila	Examiner
Djamel Benterki	Prof.	University of Setif I	Examiner
Bachir Merikhi	Prof.	University of Setif I	Examiner

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Introduction

The fuzzy logic theory was established by L. Zadeh in 1965 [76]. This logic allows the representation and processing of inaccurate or approximate knowledge. The number of applications based on this theory has in recent years increased considerably [21] [24] [7] [26] [71].

This is due to the fact that fuzzy logic is usually expressed by linguistic rules of the form IF THEN; it is used to solve control decision problems or to describe the dynamic behavior of an unknown or poorly defined system.

The fuzzy logic type-1 has been extended to get a new one fuzzy logic called the fuzzy logic type-2. Mendel and his students [30] [33] [35] [50] [73] worked in this field and developed the type-2 fuzzy sets. Type-2 fuzzy logic is very effective when it is very difficult to determine exact membership functions for a fuzzy system, so this new logic allows us to incorporate uncertainties into the rules which will act positively on the output of the given system. In type-1 fuzzy systems, There are at least three sources which are as follows:

1. The meaning of words used in the rules may be uncertain, that is, a word can be polysemic.
2. Measurements activating a type-1 fuzzy system can be noisy, which introduces uncertainties.
3. The data used to adjust the parameters of a fuzzy type-1 system can be noisy as well.

All of these uncertainties will be found in fuzzy membership functions. Type-1 fuzzy sets are therefore unable to model such uncertainties because their fuzzy membership functions are defined in a precise mathematical way. On the other hand, type-2 fuzzy sets are able to model such uncertainties because their membership functions are themselves indefinite. The fuzzy membership functions type-1 are bi-dimensional, however, the type-2 fuzzy membership functions are tri-dimensional. The new (third) dimension of type-2 fuzzy sets provides more freedom to support the modeling of uncertainties. The need for industrial robot in tasks like cutting, in welding, and painting and those requiring more precision, oblige researchers to have an exact mathematical description and mastery of this kind of processes in order to solve related industrial problems [9]-[11]. In this context, the fuzzy logic theory, firstly introduced by Zadeh [78] is gaining more and more attention from both researchers and industrial communities. The main advantage of fuzzy system is that it does not need the knowledge of the dynamic model system. This characteristic is important when dealing with nonlinear systems. Moreover, the dynamic modeling of robot manipulator shows a dependence on their dynamic parameters, function of lifetime variations (friction factors affected-by the abuse of joints), and on their dynamic parameters that vary with the completed task [11]-[9]. These characteristics give advantage to fuzzy controller on other nonlinear methods also seeing their robustness towards noises affecting the plant [64]. The major aim of this thesis is to study from theoretical and applying the type-1 and type-2 fuzzy sets on a mathematical point of view. secondly, applying these two control structures types to the applying side consists in working onto two trajectories (circle and LEAHY). Then, a proposed type reduction of Karnik-Mendel algorithm used in interval type-2 fuzzy

sets (IT2-FS) to increase the control of three DOF PUMA560 manipulator. It should be noted that there are many generalizations of the notion of fuzzy set, for example the notion of a multi fuzzy set, intuitionistic fuzzy set (IFS), L- fuzzy set (LFS) and others. In most engineering applications, we chose the notion of interval type-2 fuzzy set because they can improve certain kinds of inference better than do fuzzy sets with increasing imprecision, uncertainty, and fuzziness in information and this makes it easy for us to get the desired results.

This thesis contains four chapters.

The first chapter presents the basic concepts of Type-1 fuzzy sets and classification of fuzzy sets which include normal fuzzy set, subnormal fuzzy set, convex fuzzy set and fuzzy relations well as α -cut or strong α -level set.

The second one gives the basic concepts of a type-2 fuzzy sets, representation type-2 membership function, upper and lower membership functions, Interval type- 2 fuzzy set, Operation of Types-2 fuzzy sets, Type-2 fuzzy relations and their compositions, finally cartesian product and α -Planes.

In chapter 3, we define the concept of fuzzy logic control, the fuzzy rule base, fuzzy inference, the fuzzification, defuzzification, Puma560 robot dynamic modeling. It introduces the three degrees of freedom robot from PUMA560 and its dynamic model and state of art of fuzzy logic controller of a non-linear dynamic system.

The last chapter provides simulation results and a comparative evaluation of the type-1 and type-2 fuzzy logic controllers.

Chapter 1

Generalities of Type-1 Fuzzy Sets

The first publications in fuzzy set theory by (Zadeh, 1965) and (Goguen, 1967, 1969) display the authors intention to generalize the classical notion of a set and a proposition (statement) to accommodate fuzziness.

Let X be a space of objects and x be a generic element of X . A classical set A , $A \subseteq X$, is defined by a collection of elements or objects $x \in X$, and that each x can either belong or doesn't do to the set A . By defining a "characteristic function" for each element $x \in X$, we can represent a classical set A by a set of order pairs $(x, 0)$ or $(x, 1)$, which respectively indicates $x \notin A$ or $x \in A$.

1.1 Basic concepts of Type-1 fuzzy sets

This section contains the basic definitions and properties of type-1 fuzzy sets and several operations of this notion. Unlike the aforementioned conventional set, a fuzzy set [76] expresses the degree to which an element belongs to a set. Hence the characteristic function of a fuzzy set is allowed to have values between 0 and 1, which denotes the degree of membership of an element in a given set.

Definition 1.1.1. [76] *Let X be the collection of objects generically denoted by x , then a fuzzy set \underline{A} in X is defined as a set of ordered pairs:*

$$\underline{A} = \left\{ \left(x, \mu_{\underline{A}}(x) \right) \mid x \in X \right\} \quad (1.1)$$

Where $\mu_{\underline{A}}(x)$ is called the membership function (or MF for short) for the fuzzy set A . The MF maps, each element of X in terms of the membership function is permitted to have any values between 0 and 1. If the values of the membership function $\mu_{\underline{A}}(x)$ is a restricted grade (or membership value) between 0 and 1.

Obviously, the definition of a fuzzy set is a simple extension of the definition of a classical set in which the characteristic function is either 0 or 1, then A is reduced to a classical set and $\mu_{\underline{A}}(x)$ is the characteristic function of A .

When the universe of speech U is a continuous set (Exp: real numbers), the fuzzy subset A is written as follows:

$$A = \int \mu_{\underline{A}}(x) / x \quad (1.2)$$

When U is discrete, A becomes as follows:

$$A = \sum \mu_{\underline{A}}(x) / x \quad (1.3)$$



Figure 1.1: Function of membership

[45] Use the Matlab commands to show the triangular membership function

```

1           Step 1: First enter the x value
2           >> x = (0:0.2:10);
3           Step 2: enter triangular membership function
4           >> y1 = trimf(x,[3 4 5]);
5           Step 3: plot the curve
6           >>Plot(x,y1)
7

```

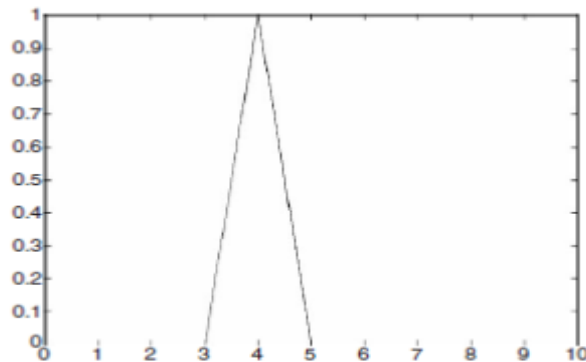


Figure 1.2: Triangular membership function

Fuzzy Set Operations [76]

Define three fuzzy sets \tilde{A} , \tilde{B} , and \tilde{C} on the universe X . For a given element x of the universe, the following function-theoretic operations for the set- union theoretical operations, intersection, and complement are defined for \tilde{A} , \tilde{B} , and \tilde{C} on X

Union

$$\mu_{\tilde{A} \cup \tilde{B}}(x) = \mu_{\tilde{A}}(x) \vee \mu_{\tilde{B}}(x) \quad (1.4)$$

Intersection

$$\mu_{\tilde{A} \cap \tilde{B}}(x) = \mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(x) \quad (1.5)$$

Complement

$$\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x) \quad (1.6)$$

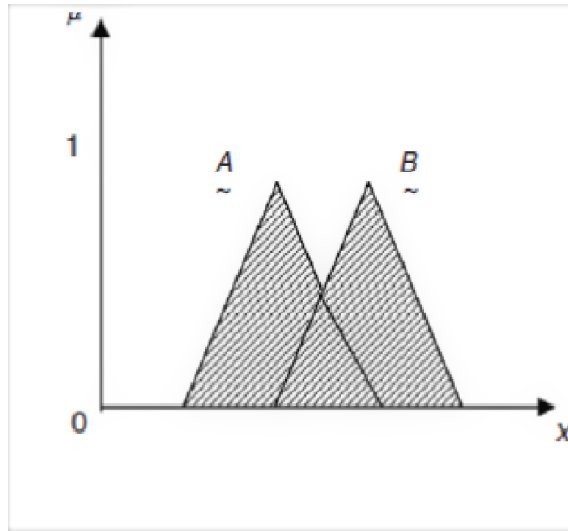


Figure 1.3: fuzzy sets union

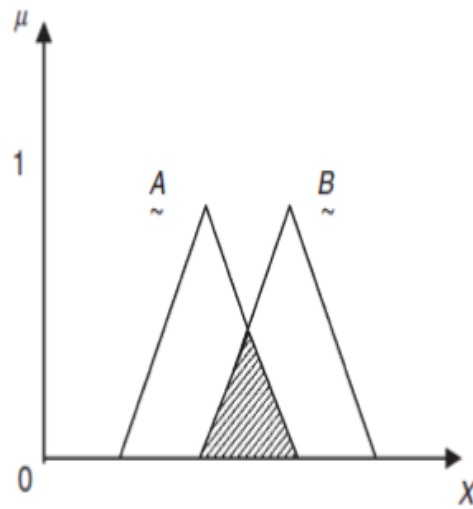


Figure 1.4: fuzzy sets intersection

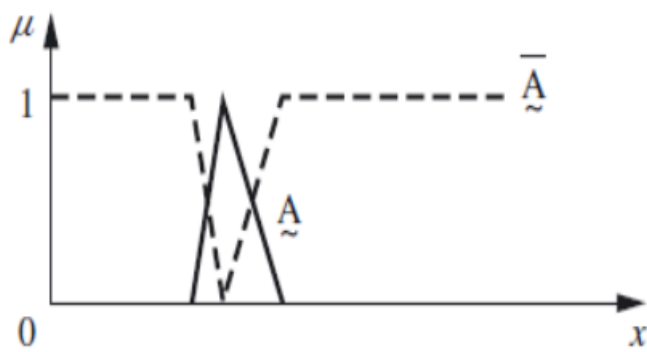


Figure 1.5: fuzzy sets complement \bar{A}

Fuzzy sets properties

De Morgan's laws: De Morgan's principles for classical sets also hold for fuzzy sets, as denoted by the following expressions :

$$\overline{\underline{A} \cap \underline{B}} = \overline{\underline{A}} \cup \overline{\underline{B}} \quad (1.7)$$

$$\overline{\underline{A} \cup \underline{B}} = \overline{\underline{A}} \cap \overline{\underline{B}} \quad (1.8)$$

$$\underline{A} \cup \overline{\underline{A}} \neq X, \underline{A} \cap \overline{\underline{A}} \neq \emptyset$$

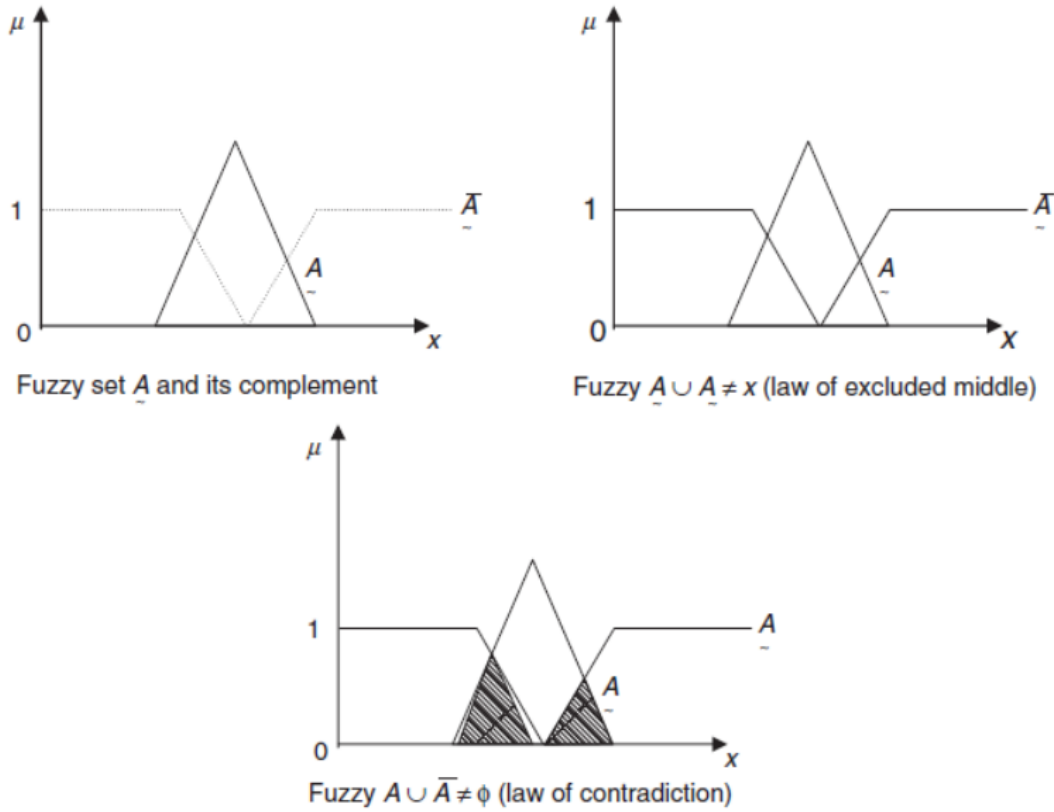


Figure 1.6: Law of excluded middle and law of contradiction

Commutativity

$$\underline{A} \cup \underline{B} = \underline{B} \cup \underline{A}$$

$$\underline{A} \cap \underline{B} = \underline{B} \cap \underline{A}$$

Associativity

$$\underline{A} \cup (\underline{B} \cup \underline{C}) = (\underline{A} \cup \underline{B}) \cup \underline{C}$$

$$\underline{A} \cap (\underline{B} \cap \underline{C}) = (\underline{A} \cap \underline{B}) \cap \underline{C}$$

Distributivity

$$\underline{A} \cup (\underline{B} \cap \underline{C}) = (\underline{A} \cup \underline{B}) \cap (\underline{A} \cup \underline{C})$$

$$\underline{A} \cap (\underline{B} \cup \underline{C}) = (\underline{A} \cap \underline{B}) \cup (\underline{A} \cap \underline{C})$$

Idempotency

$$\begin{aligned} \underline{\underline{A}} \cup \underline{\underline{A}} &= \underline{\underline{A}} \\ \underline{\underline{A}} \cap \underline{\underline{A}} &= \underline{\underline{A}} \end{aligned}$$

Identity

$$\begin{aligned} \underline{\underline{A}} \cup \phi &= \underline{\underline{A}} & \text{and} & & \underline{\underline{A}} \cap X &= \underline{\underline{A}} \\ \underline{\underline{A}} \cap \phi &= \phi & \text{and} & & \underline{\underline{A}} \cup X &= X \end{aligned}$$

Transitivity

$$\text{If } \underline{\underline{A}} \subset \underline{\underline{B}} \subset \underline{\underline{C}} \text{ then } \underline{\underline{A}} \subset \underline{\underline{C}}$$

Involution

$$\underline{\underline{\bar{\bar{A}}}} = \underline{\underline{A}}$$

Example 1.1.1. we have two discrete fuzzy sets, where we find Complement, Union, Intersection, Difference, and De Morgan's law.

$$\underline{\underline{A}} = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\} \quad \text{and} \quad \underline{\underline{B}} = \left\{ \frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\}$$

Intersection

$$\underline{\underline{A}} \cap \underline{\underline{B}} = \left\{ \frac{0.5}{2} + \frac{0.5}{3} + \frac{0.2}{4} + \frac{0.2}{5} \right\}$$

Union

$$\underline{\underline{A}} \cup \underline{\underline{B}} = \left\{ \frac{1}{2} + \frac{0.7}{3} + \frac{0.3}{4} + \frac{0.4}{5} \right\}$$

Complement

$$\begin{aligned} \bar{\underline{\underline{A}}} &= \left\{ \frac{0}{2} + \frac{0.5}{3} + \frac{0.7}{4} + \frac{0.8}{5} \right\} \\ \bar{\underline{\underline{B}}} &= \left\{ \frac{0.5}{2} + \frac{0.3}{3} + \frac{0.8}{4} + \frac{0.6}{5} \right\} \end{aligned}$$

Difference

$$\begin{aligned} \underline{\underline{A}} \setminus \underline{\underline{B}} &= \underline{\underline{A}} \cap \bar{\underline{\underline{B}}} = \left\{ \frac{0.5}{2} + \frac{0.3}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\} \\ \underline{\underline{B}} \setminus \underline{\underline{A}} &= \underline{\underline{B}} \cap \bar{\underline{\underline{A}}} = \left\{ \frac{0}{2} + \frac{0.5}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\} \end{aligned}$$

De morgan's principles

$$\begin{aligned} \overline{\underline{\underline{A}} \cup \underline{\underline{B}}} &= \bar{\underline{\underline{A}}} \cap \bar{\underline{\underline{B}}} = \left\{ \frac{0}{2} + \frac{0.3}{3} + \frac{0.7}{4} + \frac{0.6}{5} \right\} \\ \overline{\underline{\underline{A}} \cap \underline{\underline{B}}} &= \bar{\underline{\underline{A}}} \cup \bar{\underline{\underline{B}}} = \left\{ \frac{0.5}{2} + \frac{0.5}{3} + \frac{0.8}{4} + \frac{0.8}{5} \right\} \end{aligned}$$

Example 1.1.2. Consider the following fuzzy sets

$$\begin{aligned} A &= \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\} \\ B &= \left\{ \frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\} \end{aligned}$$

Calculate $A \cup B$, $A \cap B$, \bar{A} , \bar{B} by a Matlab program.

Solution: The Matlab program for the union, intersection, and complement is Program [45]

```

8      %enter the two matrix
9      u=input('enter the first matrix ');
10     v=input('enter the second matrix ');
11
12     option=input('enter the option');
13     %option 1 Union
14     %option 2 intersection
15     %option 3 complement
16     if (option==1)
17         end
18     if (option==2)
19         p=min(u,v)
20     end
21     if (option==3)
22         option1=input('enter wherthe to find complement for first matrix
23         or second matrix');
24         if (option1==1)
25             [m,n]=size(u);
26             q=ones(m)-u;
27         else
28             q=ones(m)-v;
29         end
30     end
31     output
32     (1) To find union of A and B
33     enter the first matrix [1 0.5 0.2 0.3]
34     enter the second matrix [0.5 0.7 0.2 0.4]
35     enter the option1
36     w =
37     1.0000 0.7000 0.2000 0.4000
38
39     (2) To find intersection of A and B is
40     enter the first matrix [1 0.5 0.2 0.3]
41     enter the second matrix [0.5 0.7 0.2 0.4]
42     enter the option2
43     p=
44     0.5000 0.5000 0.2000 0.3000
45     (3) To find complement of A
46     enter the first matrix [1 0.5 0.2 0.3]
47     enter the second matrix [0.5 .7 .2 .4]
48     enter the option3
49     enter the whether to find complement for first matrix or second matrix 2
50     q =
51     0.5000 0.3000 0.8000 0.6000

```

1.2 Characteristics of Type-1 fuzzy sets

Definition 1.2.1. (Support) Let A be a fuzzy set on a set X . The support of A is the crisp subset on X given by:

$$Supp(A) = \{x | \mu_A(x) > 0\} \quad (1.9)$$

Definition 1.2.2. (Kernel) Let A be a fuzzy set on a set X . The kernel or the core of A is the crisp subset on X given by:

$$Ker(A) = \{x | \mu_A(x) = 1\} \quad (1.10)$$

Definition 1.2.3. (Normality) A fuzzy set A is "normal" if its kernel is not empty. In other words, we can always find a point $x \in X$ such that $\mu_A(x) = 1$.

Definition 1.2.4. (Crossover points) A "crossover point" of a fuzzy set A is a point $x \in X$ at which $\mu_A(x) = 0.5$:

$$Cross(A) = \{x | \mu_A(x) = 0.5\}$$

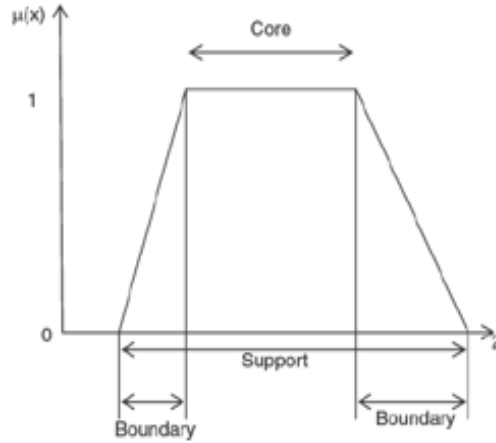


Figure 1.7: kernel and support of fuzzy set

Definition 1.2.5. (Height) Let A be a fuzzy set on a set X . The height of A is the highest value taken by its membership function given by:

$$H(A) = \text{Sup}_{x \in X} \mu_A(x)$$

Example 1.2.1. If $A = \left\{ \frac{0.2}{15} + \frac{1}{25} + \frac{0.8}{35} + \frac{1}{45} + \frac{0}{55} \right\}$

$$Ker(A) = \{25, 45\}$$

$$Cross(A) = \emptyset$$

$$Supp(A) = \{15, 25, 35, 45\}.$$

Definition 1.2.6. [77] λ -cut, strong λ -cut The " λ -cut" or " λ -level set" of a fuzzy set A is a crisp set defined by:

$$A_\lambda = \{x | \mu_A(x) \geq \lambda\} \quad (1.11)$$

Strong λ -cut or strong λ -level set are defined similarly:

$$A^\lambda = \{x | \mu_A(x) > \lambda\} \quad (1.12)$$

Using the notation for a level set, we can express the support and core of a fuzzy set A as $Supp(A) = A^\circ$ and $Ker(A) = A_1$

Example 1.2.2. $A = \left\{ \frac{1}{a} + \frac{0.9}{b} + \frac{0.6}{c} + \frac{0.3}{d} + \frac{0.01}{e} + \frac{0}{f} \right\}$

$$A_1 = \{a\}, \quad A_{0.9} = \{a, b\}$$

$$A_{0.6} = \{a, b, c\}, \quad A_{0.3} = \{a, b, c, d\}$$

$$A_{0+} = \{a, b, c, d, e\}, \quad A_0 = X$$

$\lambda = 1, 0.9, 0.6, 0.3, 0+, \text{ and } 0$

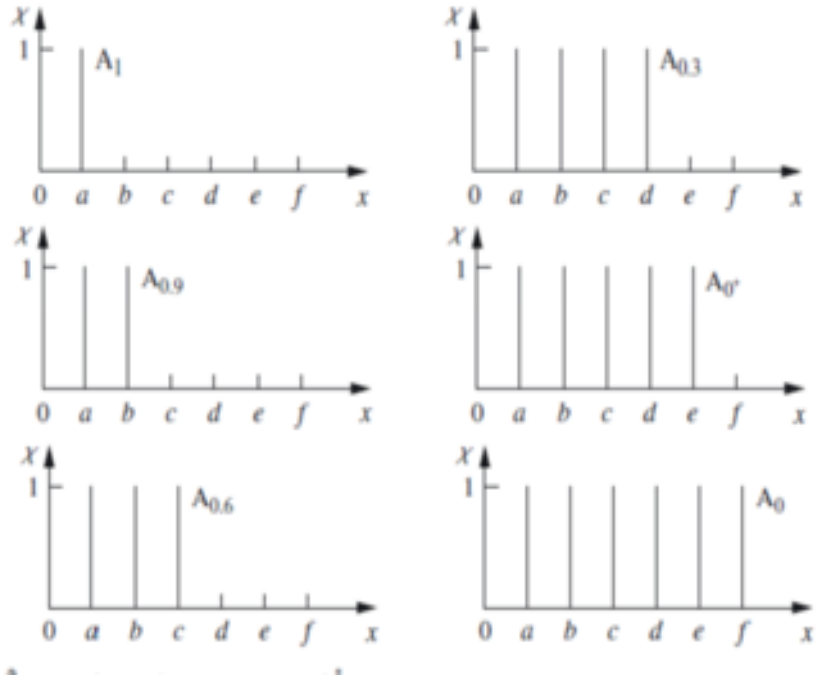


Figure 1.8: λ -cut"

λ -cut sets properties :

1. $(\tilde{A} \cup \tilde{B})_\lambda = \tilde{A}_\lambda \cup \tilde{B}_\lambda$
2. $(\tilde{A} \cap \tilde{B})_\lambda = \tilde{A}_\lambda \cap \tilde{B}_\lambda$
3. $(\overline{\tilde{A}})_\lambda \neq \overline{\tilde{A}_\lambda}$
4. For any $\lambda \leq \alpha$, where $0 \leq \alpha \leq 1$, it is true that $A_\alpha \subseteq A_\lambda$, where $A_0 = X$

Example 1.2.3. Two fuzzy sets \tilde{P} and \tilde{Q} are defined on x as follows :

$\mu(x)$	x_1	x_2	x_3	x_4	x_5
\tilde{P}	0.1	0.2	0.7	0.5	0.4
\tilde{Q}	0.9	0.6	0.3	0.2	0.8

- (a) $(\overline{\tilde{P}})_{0.2}$ (b) $(\tilde{Q})_{0.3}$ (c) $(\tilde{P} \cup \tilde{Q})_{0.5}$ (d) $(\tilde{P} \cap \tilde{Q})_{0.4}$ (e) $(\tilde{Q} \cup \overline{\tilde{P}})_{0.8}$

$$(f) \left(\underset{\sim}{P} \cup \underset{\sim}{\bar{P}} \right)_{0.2}$$

Solution :

Given Solution

$$\underset{\sim}{P} = \left\{ \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.7}{x_3} + \frac{0.5}{x_4} + \frac{0.4}{x_5} \right\}$$

$$\underset{\sim}{Q} = \left\{ \frac{0.9}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3} + \frac{0.2}{x_4} + \frac{0.8}{x_5} \right\}$$

Finding

$$\underset{\sim}{\bar{P}} = \left\{ \frac{0.9}{x_1} + \frac{0.8}{x_2} + \frac{0.3}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} \right\}$$

$$\underset{\sim}{\bar{Q}} = \left\{ \frac{0.1}{x_1} + \frac{0.4}{x_2} + \frac{0.7}{x_3} + \frac{0.8}{x_4} + \frac{0.2}{x_5} \right\}$$

$$(a) \left(\underset{\sim}{\bar{P}} \right)_{0.2} = \left\{ \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} \right\}$$

$$(b) \left(\underset{\sim}{\bar{Q}} \right)_{0.3} = \left\{ \frac{0}{x_1} + \frac{1}{x_2} + \frac{1}{x_2} + \frac{1}{x_4} + \frac{0}{x_5} \right\}$$

$$(c) \left(\underset{\sim}{P} \cup \underset{\sim}{Q} \right) = \left\{ \frac{0.9}{x_1} + \frac{0.6}{x_2} + \frac{0.7}{x_3} + \frac{0.5}{x_4} + \frac{0.8}{x_5} \right\}$$

$$\left(\underset{\sim}{P} \cup \underset{\sim}{Q} \right)_{0.4} = \left\{ \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} \right\}$$

$$(d) \left(\underset{\sim}{P} \cup \underset{\sim}{\bar{P}} \right) = \left\{ \frac{0.9}{x_1} + \frac{0.8}{x_2} + \frac{0.7}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} \right\}$$

$$\left(\underset{\sim}{P} \cup \underset{\sim}{\bar{P}} \right)_{0.8} = \left\{ \frac{1}{x_1} + \frac{1}{x_2} + \frac{0}{x_3} + \frac{0}{x_4} + \frac{0}{x_5} \right\}$$

$$(e) \left(\underset{\sim}{P} \cap \underset{\sim}{Q} \right) = \left\{ \frac{0.9}{x_1} + \frac{0.8}{x_2} + \frac{0.7}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} \right\}$$

$$\left(\underset{\sim}{P} \cap \underset{\sim}{Q} \right)_{0.4} = \left\{ \frac{0}{x_1} + \frac{0}{x_2} + \frac{0}{x_3} + \frac{0}{x_4} + \frac{1}{x_5} \right\}$$

$$(f) \left(\underset{\sim}{P} \cap \underset{\sim}{\bar{P}} \right) = \left\{ \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.3}{x_2} + \frac{0.5}{x_4} + \frac{0.4}{x_5} \right\}$$

$$\left(\underset{\sim}{P} \cap \underset{\sim}{\bar{P}} \right)_{0.8} = \left\{ \frac{0}{x_1} + \frac{1}{x_2} + \frac{1}{x_2} + \frac{1}{x_4} + \frac{1}{x_5} \right\}$$

Definition 1.2.7. For a finite fuzzy set $\underset{\sim}{A}$, the cardinality $|\underset{\sim}{A}|$ is defined as follows:

$$|\underset{\sim}{A}| = \sum_{x \in X} \mu_{\underset{\sim}{A}}(x) \text{ and } \|\underset{\sim}{A}\| = \frac{|\underset{\sim}{A}|}{|X|}$$

is Obviously called the relative cardinality of $\underset{\sim}{A}$. The relative cardinality of a fuzzy set depends on the cardinality of the universe. So you have to choose the same universe if you want to compare fuzzy sets by their relative cardinality.

Example 1.2.4. Consider a fuzzy subset of the set X , $X = \{a, b, c, d, e\}$ referred to

$$\underset{\sim}{A} = \left\{ \frac{0.6}{a}, \frac{0.9}{b}, \frac{0.1}{c}, \frac{0.3}{d}, \frac{0.2}{e} \right\}$$

$$|\underset{\sim}{A}| = \sum_{x \in X} \mu_{\underset{\sim}{A}}(x) = 2.1$$

Definition 1.2.8. (Fuzzy singleton) A fuzzy set whose support is a single point in X with $\mu_{\underset{\sim}{A}}(X) = 1$ is called a "fuzzy singleton".

1.3 Classification of Fuzzy sets

Fuzzy sets are classified according to their membership to:

Normal Fuzzy set

A fuzzy set A of X is called normal if there exists at least one element x in X such that $\mu_A(x) = 1$

Subnormal Fuzzy set

If all membership of the fuzzy set is affiliated fuzzy less than it is Convex Fuzzy set.

The latter is described by a membership function the membership of which values are strictly monotonically increasing or are strictly monotonically increasing then strictly monotonically decreasing with increasing values for elements in the universe. In other words, if for any elements x, y and z in a fuzzy set \tilde{A} , the relation $x < y < z$ implies that $\mu_{\tilde{A}}(y) \geq \min[\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(z)]$, then \tilde{A} is said to be a convex fuzzy set (Ross, 1995). Figure 1.9 shows a typical convex fuzzy set and a typical nonconvex one. Here, It is important to note that definition of convexity is different from other definitions of the same term in mathematics. In some areas of mathematics, the convexity of shape has to do with whether a straight line through any part of the shape goes outside the boundaries of that shape, This definition of convexity is not used here ; Figure succinctly summarize our definition or convexity.

Intersection of two convex sets

A special property of two convex fuzzy sets, say \tilde{A} and \tilde{B} , is that the intersection of these two convex fuzzy sets is also a convex fuzzy set, as shown in Figure 1.10. That is, for \tilde{A} and \tilde{B} . which are both convex, $\tilde{A} \cap \tilde{B}$ is also convex .

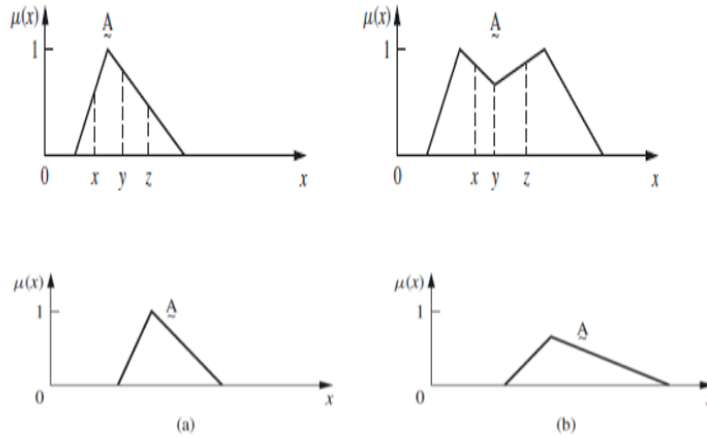


Figure 1.9: Fuzzy sets are that normal (a) and subnormal (b)

Definition 1.3.1. [76] An Interval-valued fuzzy set (IVFS), A , is characterized by a membership function σ_A ,

$$\sigma_A : X \rightarrow I = \{[a, b]; 0 \leq a \leq b \leq 1\}$$

where $\sigma_A(x)$ is the degree of membership of an element $x \in X$ in the set A . The set of all interval-valued fuzzy sets on X is denoted by $IVFS(X)$.

1.4 L-fuzzy sets type-1

In the previous sections, we have discussed the type -1 fuzzy sets in the unit real interval scale.

While in this section, we discuss these concepts on a lattice as a more general structure of truth values

Goguen generalized fuzzy sets to L -fuzzy sets in [20]. An L -fuzzy set φ on U is a mapping $\varphi: U \rightarrow L$, where L is a 'transitive partially ordered set'.

Definition 1.4.1. [32] Consider a bounded lattice $(L, \leq, 0_L, 1_L)$ and a universe X Goguen [20] introduced L -fuzzy sets as objects A characterized by some membership function $\mu_A: X \rightarrow L$. His approach has generalized the Cantorian (or crisp) sets (in which case $L = \{0, 1\}$ and μ_A is just the characteristic function $1_A: X \rightarrow \{0, 1\}$ of the Cantorian subset A of X), as well as the fuzzy sets proposed two years earlier by Zadeh [76] (here $L = [0, 1]$).

In the case of L -fuzzy sets, set relations and operations such as inclusion, union and intersection are defined pointwise by means of the ordering, supremum and infimum on L respectively. If the lattice L possesses a complement described by some unary operation $c: L \rightarrow L$, then the complement A^c

[20] We use the notation XY for the set-theoretic product of X and Y ; that is, $XY = \{(x, y) | x \in X, y \in Y\}$. $X + Y$ will denote the disjoint union of X and Y , and X^Y the set of all functions from Y to X . It is straightforward to verify the following set-theoretic isomorphisms.

Proposition 1.4.1. [20] $X + Y \cong Y + X$; $X + (Y + Z) \cong (X + Y) + Z$; $XY \cong YX$ $X(YZ) \cong (XY)Z$; $X(Y + Z) \cong XY + XZ$; $(X + Y)Z \cong XZ + YZ$ $X^{Y+Z} \cong X^Y X^Z$; $(XY)^Z \cong X^Z Y^Z$; $(X^Y)^Z \cong X^{YZ}$ There is no associative law for exponentiation, that is, $X^{(Y^Z)} \not\cong (X^Y)^Z$.

Definition 1.4.2. [20] An L -fuzzy set A on a set X is a function $A: X \rightarrow L$ Thus fuzzy sets are to be considered equal if and only if they are equal as functions. If the elements of X have a name (e.g., "characters"), A will be called an L -fuzzy set of such elements (e.g., "A is an L -fuzzy set of characters").

Definition 1.4.3. [56] For any L -fuzzy subsets A and B of X , define $A \subseteq B$ if and only if $A(x) \leq B(x)$ for all $x \in X$. Clearly \leq is a partial order on $FS_L(X)$. Also, for any crisp subsets S and T of X , we have $\chi_S \leq \chi_T \Leftrightarrow S \subseteq T$ The following is an easy verification.

Definition 1.4.4. [56] For an L -fuzzy set A and $\alpha \in L$, the α -cut A_α of A is defined as $A_\alpha = \{x | x \in X \text{ and } \alpha \leq A(x)\}$.

In the sequel, we assume that (L, \vee, \wedge) is a complete lattice with the least element 0 . The α -cuts of L -fuzzy sets have the following properties:

- (1) $\forall A \in F_L(X), A_0 = X$
- (2) $\forall \alpha \in L, \forall A \in F_L(X), A_\alpha = \bigcap_{\lambda \leq \alpha} A_\lambda$
- (3) $\forall \alpha_i \in L (i \in I), \forall A \in F_L(X), A_\alpha = \bigcap_{i \in I} A_{\alpha_i}$, where $\alpha = \bigvee_{i \in I} \alpha_i$

Let I be an arbitrary index set and $A_i \in F_L(X) (i \in I)$. The union $\bigcup_{i \in I} A_i$ and the intersection $\bigcap_{i \in I} A_i$ are respectively defined as follows: $\forall x \in X, (\bigcup_{i \in I} A_i)(x) = \bigvee_{i \in I} A_i(x), \forall x \in X, (\bigcap_{i \in I} A_i)(x) = \bigwedge_{i \in I} A_i(x)$

Proposition 1.4.2. [20] L^X can be given whatever operations L has, and these operations in L^X will obey any law valid in L which extends point by point. For example, if a binary operation $*$ is defined on L , it induces an operation $*$ on L^X by pointwise extension: $(A * B)(x) = A(x) * B(x)$, for $A, B \in L^X$ and $x \in X$. If some law, such as associative, commutative, or idempotent holds for $*$ on L , it also extends to L^X .

Definition 1.4.5. [20] A multiplicative lattice is a lattice M closed under an operation $*$ which satisfies the “distributive” laws, $\forall a, b \in M \ a * (b \vee c) = (a * b) \vee (a * c)$ and $(a \vee b) * c = (a * c) \vee (b * c)$. Identities involving $*$ ’s in multiplicative lattices will not remain identities when \wedge and \vee , and \geq and \leq are interchanged; thus the Principle of Duality is not valid. But there is a principle of left-right symmetry, as exhibited in the two “distributive” laws above.

Proposition 1.4.3. [20] Let a, b, c be elements of a multiplicative lattice M . Then:

- (1) $a \leq b \Rightarrow c * a \leq c * b$ and $a * c \Rightarrow b * c$;
- (2) $(a \wedge b) * (a \vee b) \leq (b * a) \wedge (a * b)$.

Proof. (1) $a \leq b \Rightarrow b * c = (a \vee b) * c = (a * c) \vee (b * c) \Rightarrow a * b \leq b * c$;

- (2) $(a \wedge b) * (a \vee b) = [(a \wedge b) * a] \vee [(a \wedge b) * b] \leq (b * a) \vee (a * b)$.

□

1.4.1 T–norm and t–conorm in lattices

Preliminaries

[47] A bounded lattice is a lattice (L, \leq) which has the top and bottom elements, which are written as: \top and \perp respectively, that is, there exist two elements $\top, \perp \in L$ such that $\perp \leq x \leq \top$, for all $x \in L$

Definition 1.4.6. [47] Given a bounded lattice (L, \leq, \perp, \top) , and $a, b \in L$, a subinterval $[a, b]$ of L , is a sublattice of L defined as

$$[a, b] = \{x \in L \mid a \leq x \leq b\}$$

The triangular norms on $[0, 1]$ are useful operators within the framework of fuzzy sets and fuzzy logic [[22], [38]]. since interest in lattices has been increasing in recent years, an adequate generalization of these operators should be given. A straightforward definition is presented in [63].

Definition 1.4.7. [47] Let (L_1, \leq, \perp, \top) be a bounded lattice. A binary operation $T: L \times L \rightarrow L$ is a triangular norm on L if the following conditions are fulfilled, for all $x, y, z \in L$:

1. $T(x, y) = T(y, x)$
2. If $x \leq y$, then $T(x, z) \leq T(y, z)$
3. $T(x, \top) = x$
4. $T(x, T(y, z)) = T(T(x, y), z)$

Example 1.4.1. [47] The following operators are examples of 1–norms in a bounded lattice. In any bounded lattice, we can define the meet (or infimum) t–norm. $T_M^L: L \times L \rightarrow L$, $T_M^L(x, y) = \inf\{x, y\}$ (as (L, \leq) is a lattice, there exists the infimum of two elements in L). Another interesting t–norm is the drastic product: $T_D^L: L \times L \rightarrow L$, defined as:

$$T_D^L(x, y) = \begin{cases} \inf\{x, y\} & \text{if } T \in \{x, y\} \\ \perp & \text{otherwise} \end{cases}$$

In a similar way that in $[0, 1]$, we see that T_D^L and T_M^L , are the smallest and greatest t–norms on a lattice.

T -distributivity

Definition 1.4.8. [59] Let $(L, \leq, 0, 1)$ be a bounded lattice and T_1 and T_2 every $x, y, z \in L$ such that at least one of the elements y, z is not

$$T_1(x, T_2(y, z)) = T_2(T_1(x, y), T_1(x, z))$$

is satisfied, then T_1 is called T_2 -distributive or we say that T_1 is

Example 1.4.2. [59] Let $(L = \{0, a, b, c, 1\}, \leq, 0, 1)$ be a bounded lattice w played in Figure 1 The functions T_1 and T_2 on the lattice L defined by

$$T_1(x, y) = \begin{cases} 0, & \text{if } x = a, y = a \\ b, & \text{if } x = c, y = c \\ x \wedge y, & \text{otherwise} \end{cases}$$

and

$$T_2(x, y) = \begin{cases} b, & \text{if } x = c, y = c \\ x \wedge y, & \text{otherwise} \end{cases}$$

are obviously t -norms on L such that T_1 is T_2 -distributive.

Proposition 1.4.4. [59] Let $(L, \leq, 0, 1)$ be a bounded lattice and T_1 and T_2 be two t -norms on L . If T_1 is T_2 -distributive, then T_1 is weaker than T_2 . Proof since all t -norms coincide on the boundary of L^2 , it is sufficient to show that $T_1 \leq T_2$ for all $x, y, z \in L \setminus \{0, 1\}$. By the T_2 -distributivity of T_1 , it is obtained that

$$T_1(x, y) = T_1(T_2(x, 1), y) = T_2(T_1(x, y), T_1(1, y)) = T_2(T_1(x, y), y) \leq T_2(x, y)$$

Thus, $T_1 \leq T_2$, i.e., T_1 is weaker than T_2

Remark 1.4.1. [59] The converse of Proposition 1.4.2. need not be true. Namely, for any two t -norms T_1 and T_2 , even if T_1 is weaker than T_2 , T_1 may not be T_2 -distributive. Now, let us investigate the following example.

Example 1.4.3. Consider the product t -norme $(T_P)(a, b) = a.b$ and the Łukasiewicz t -norm $T_L(a, b) = \max(0, a + b - 1)$. It is clear that $T_L < T_P$. since

$$T_L\left(\frac{4}{5}, T_P\left(\frac{7}{9}, \frac{1}{2}\right)\right) = T_L\left(\frac{4}{5}, \frac{7}{18}\right) = \frac{59}{90}$$

and

$$T_P\left(T_L\left(\frac{4}{5}, \frac{7}{9}\right), T_L\left(\frac{4}{5}, \frac{1}{2}\right)\right) = T_P\left(\frac{26}{45}, \frac{3}{10}\right) = \frac{78}{450} = \frac{13}{75}$$

T_L is not T_P -distributive.

Corollary 1.4.1. [59] Let L be a bounded lattice and T_1 and T_2 be any two t -norms on L . If both T_1 is T_2 -distributive and T_2 is T_1 -distributive, then $T_1 = T_2$

Proposition 1.4.5. [59] Let L be a bounded chain and T' be a t -norm on L . For every t -norm T , T is T' -distributive if and only if $T' = T_\wedge$.

Proof. \Rightarrow Let T be an arbitrary t -norm on L such that T' -distributive. By Proposition 1, it is obvious that $T \leq T'$ for any t -norm T . Thus, $T' = T_\wedge$

\Leftarrow : since L is a chain, for any $y, z \in L$, either $y \leq z$ or $z \leq y$. Suppose that $y \leq z$. Using the monotonicity of the t -norm T , we obtained that for any $x \in L$, $T(x, y) \leq T(x, z)$. Then

$$T(x, y) = T(x, y) \wedge T(x, z)$$

holds. Thus, for any $x, y, z \in L$

$$\begin{aligned} T(x, T_\wedge(y, z)) &= T(x, y) \\ &= T(x, y) \wedge T(x, z) \\ &= T_\wedge(T(x, y), T(x, z)) \end{aligned}$$

is satisfied, which shows that any t -norm T is T_\wedge -distributive. Reference document-1.pdf \square

Definition 1.4.9. [28] An operation $T: L^2 \rightarrow L$ (resp. $S: L^2 \rightarrow L$) is called a t -norm (resp. t -conorm) if it is commutative, associative, increasing with respect to both variables and has a neutral element $e = 1_L$ (resp. $e = 0_L$).

A pseudo- t -norm T (resp. pseudo- t -conorm S) [[13].[17]] on [0.1] is a t -norm (resp. t -conorm) [38] if it is commutative. We can extend the definition of pseudo- t -norms on bounded lattices.

[16]

Example 1.4.4. [16] Let $(L_1 \leq 0_L, 1_L)$ be a bounded lattice, a pseudo- t -norm $T_a^l: L^2 \rightarrow L$ and a pseudo- t -conorm $S_b^l: L^2 \rightarrow L$ be defined as follows, respectively.

$$T_a^l(x, y) = \begin{cases} (x \vee a) \wedge (y \vee a), & (x, y) \in I_a \times I_a, \\ x \wedge (y \vee a), & (x, y) \in [a, 1_L] \times I_a, \\ (x \vee a) \wedge y, & (x, y) \in I_a \times [0_L, a) \cup I_a \times [a, 1_L) \\ 0_L, & (x, y) \in [0_L, a) \times L \setminus \{1_L\} \\ x \wedge y, & \text{otherwise} \end{cases}$$

$$S_b^l(x, y) = \begin{cases} (x \wedge b) \vee (y \wedge b), & (x, y) \in I_b \times I_b \\ x \vee (y \wedge b), & (x, y) \in (0_L, b] \times I_b \\ (x \wedge b) \vee y, & (x, y) \in I_b \times (0_L, b] \cup I_b \times (b, 1_L], \\ 1_L, & (x, y) \in (b, 1_L] \times L \setminus \{0_L\} \\ x \vee y, & \text{otherwise} \end{cases}$$

where $a, b \in L \setminus \{0_L, 1_L\}$

It is easy to verify the associative law. monotonicity and neutral element in both operators. However, they do not satisfy the commutative law such as

$$T_a^l(x, y) = 0_L \neq T_a^l(y, x) = x, (x, y) \in [0_L, a) \times [a, 1_L)$$

Remark 1.4.2. All the notions and properties recalled before stay valid when we use bounded lattice instead of The unit real interval [0, 1].

1.5 Fuzzy relations

A fuzzy relation [77] \tilde{R} is a mapping from the Cartesian space $X \times Y$ to the interval $[0, 1]$, where the strength of the mapping is expressed by the membership function of the relation for ordered pairs from the two universes is $\mu_{\tilde{R}}(x, y)$.

1.5.1 Operations on Fuzzy Relations

Let \tilde{R} and \tilde{S} be fuzzy relations on the Cartesian space $X \times Y$. Then the following operations apply for the membership values for various set operations (these are similar to the same operations on crisp sets;

Union	$\mu_{\tilde{R} \cup \tilde{S}}(x, y) = \max(\mu_{\tilde{R}}(x, y), \mu_{\tilde{S}}(x, y))$
Intersection	$\mu_{\tilde{R} \cap \tilde{S}}(x, y) = \min(\mu_{\tilde{R}}(x, y), \mu_{\tilde{S}}(x, y))$
Complement	$\mu_{\tilde{R}^c}(x, y) = 1 - \mu_{\tilde{R}}(x, y)$
Containment	$\tilde{R} \subset \tilde{S} \Rightarrow \mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{S}}(x, y)$

Cardinality of Fuzzy Relations

Since the cardinality of fuzzy sets on any universe is infinity, the cardinality of a fuzzy relation between two or more universes is also infinity.

1.5.2 Properties of fuzzy Relations

Just as for crisp relations, the properties of commutativity, associativity, distributivity, involution, and idempotency all hold for fuzzy relations. Moreover, De Morgan's principles hold for fuzzy relations just as they do for crisp (classical) relations and the null relation O . And the complete relation, E . Are analogous to the null set and the whole set in set-theoretic form, respectively. Fuzzy relations are not constrained, as is the case for fuzzy sets in general, by the excluded middle axioms. Since a fuzzy relation \tilde{R} is also a fuzzy set. there is overlap between a relation and its complement; hence,

$$\tilde{R} \cup \tilde{R}^c \neq E$$

$$\tilde{R} \cap \tilde{R}^c \neq O$$

As seen in the foregoing expressions, the excluded middle axioms for fuzzy relations do not, generally result, in the null relation, O , or the complete relation, E .

1.5.3 Fuzzy Cartesian Product and Composition

Because fuzzy relations are fuzzy sets, we can define the Cartesian product to be a relation between two or more fuzzy sets. Let \tilde{A} be a fuzzy set on universe X and \tilde{B}

a fuzzy set on universe Y , then the Cartesian product between fuzzy sets \tilde{A} and \tilde{B} will result in a fuzzy relation \tilde{R} , which is contained within the full Cartesian product space, or

$$\tilde{A} \times \tilde{B} = \tilde{R} \subset X \times Y$$

where the fuzzy relation \tilde{R} has membership function

$$\mu_{\tilde{R}}(x, y) = \mu_{\tilde{A} \times \tilde{B}}(x, y) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y))$$

The cartesian product defined by $\tilde{A} \times \tilde{B} = \tilde{R}$ is implemented in the same fashion as is the cross product of two vectors. moreover, the cartesian product is not the same operation as the arithmetic product. In the case of bi-dimensional relations ($r = 2$), the former employs the idea of the pairing of elements among sets, whereas the latter uses actual arithmetic products between elements of sets. Each of the fuzzy sets could be thought of as a membership values vector; each value is associated with a particular element in each set. For example, for a fuzzy set (vector) \tilde{A} that has four elements, hence column vector of size 4×1 , and for a fuzzy set (vector) \tilde{B} that has live elements, hence a row vector size of 1×5 , the resulting fuzzy relation \tilde{R} will be represented by a matrix of size 4×5 , that is, \tilde{R} will have four rows and five columns. This idea is illustrated in the following example.

Example 1.5.1. Suppose we have two fuzzy sets, \tilde{A} defined on a universe of three discrete temperatures: $X = \{x_1, x_2, x_3\}$, \tilde{B} defined on a universe of two discrete pressures, and $Y = \{y_1, y_2\}$, then we want to find the fuzzy Cartesian product between them. Fuzzy set \tilde{A} could represent the "ambient" temperature and fuzzy set \tilde{B} the "near-optimum" pressure for a certain heat exchanger and the Cartesian product might represent the conditions (temperature-pressure pairs) of the exchanger that are associated with "efficient" operations. For example, let

$$\tilde{A} = \left\{ \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} \right\} \quad \text{and} \quad \tilde{B} = \left\{ \frac{0.3}{y_1} + \frac{0.9}{y_2} \right\}$$

Note that \tilde{A} can be represented as a column vector of size 3×1 and \tilde{B} can be represented by a row vector of 1×2 . Then the fuzzy Cartesian product results in a fuzzy relation \tilde{R} (of size 3×2) representing "efficient" conditions or

$$\tilde{A} \times \tilde{B} = \tilde{R} = \begin{matrix} & y_1 & y_2 \\ x_1 & \left[\begin{array}{cc} 0.2 & 0.2 \end{array} \right] \\ x_2 & \left[\begin{array}{cc} 0.3 & 0.5 \end{array} \right] \\ x_3 & \left[\begin{array}{cc} 0.3 & 0.9 \end{array} \right] \end{matrix}$$

The fuzzy composition can be defined just as it is for crisp (binary) relations. Suppose \tilde{R} is a fuzzy relation on the cartesian space $X \times Y$, \tilde{S} is a fuzzy relation on $Y \times Z$, and \tilde{T} is a fuzzy relation on $X \times Z$, the fuzzy max-min composition is defined in terms of the set-theoretic notation and membership function-theoretic notation as follows:

$$\tilde{T} = \tilde{R} \circ \tilde{S}$$

$$\mu_{\tilde{T}}(x, z) = \bigvee_{y \in Y} \left(\mu_{\tilde{R}}(x, y) \wedge \mu_{\tilde{S}}(y, z) \right)$$

and fuzzy max-product composition is defined in terms of the membership function-theoretic notation as

$$\mu_{\tilde{T}}(x, z) = \bigvee_{y \in Y} \left(\mu_{\tilde{R}}(x, y) \bullet \mu_{\tilde{S}}(y, z) \right)$$

It should be pointed out that, in general, neither crisp nor fuzzy compositions are commutative; that is,

$$\tilde{R} \circ \tilde{S} \neq \tilde{S} \circ \tilde{R}.$$

Example 1.5.2. Let us fuzzy relationships for $X \times Y$ (denoted by the fuzzy relation \tilde{R}) and $X \times Y$ (denoted by the fuzzy relation \tilde{S}). In this case, we change the elements of the universes to

$$X = \{x_1, x_2\}, \quad Y = \{y_1, y_2\}, \quad \text{and} \quad Z = \{z_1, z_2, z_3\}$$

Consider the following fuzzy relations:

$$\tilde{R} = \begin{array}{cc} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \end{array} \quad \text{and} \quad \tilde{S} = \begin{array}{cc} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \end{array} & \begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.1 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

Then, the resulting relation, \tilde{T} , which relates elements of universe X to elements of universe Z , that is, defined on Cartesian space $X \times Z$, can be found by max-min composition, for example,

$$\mu_{\tilde{T}}(x_1, z_1) = \max[\min(0.7, 0.9), \min(0.5, 0.1)] = 0.7$$

and the set

$$\tilde{T} = \begin{array}{ccc} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}$$

We now illustrate the use of relations with fuzzy sets of medicine, electrical engineering, and civil engineering.

Definition 1.5.1. [78] Each linguistic variable is fully characterized by a quintuple (v, T, X, g, m) in which v is the name of the variable. T is the set of the linguistic terms of v that refers to a base variable whose values range over the universal set X . g is a syntactic rule for generating linguistic terms. and m is a semantic rule that assigns to each linguistic term $t \in T$ its meaning, $m(t)$, which is an FS on X , that is $m: T \rightarrow F(X)$, where $F(X)$ denotes the set of all ordinary (i.e., Tl) FSs of X , one identical set for each $t \in T$,

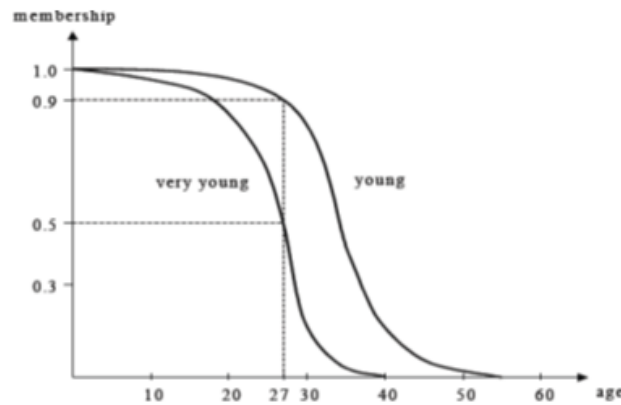


Figure 1.10: linguistic variable

1.6 t-norms and t-conorms

The notions of triangular-norms (*t-norms*) started with Menger [57] to construct metric spaces where probability distributions are used to describe the distance between two elements.

Definition 1.6.1. [57] A *t*-norm T on $[0, 1]$ is a function $T : [0, 1]^2 \rightarrow [0, 1]$ satisfies the following four axioms:

- 1 *Commutativity*: $(\forall x, y \in [0, 1])(T(x, y) = T(y, x))$;
- 2 *Associativity*: $(\forall x, y, z \in [0, 1])(T(x, T(y, z)) = T(T(x, y), z))$;
- 3 *Monotonicity*: $(\forall x, y, z \in [0, 1])(x \leq y \Rightarrow T(x, z) \leq T(y, z))$;
- 4 *Boundary condition*: $(\forall x \in [0, 1])(T(x, 1) = x)$.

Example 1.6.1. The following four operations are the most common *t*-norms:

- 1 *Minimum*: $T_M(x, y) = \min\{x, y\}$
- 2 *Product*: $T_P(x, y) = x \cdot y$
- 3 *Lukasiewicz*: $T_L(x, y) = \max\{x + y - 1, 0\}$
- 4 *Drastic product*:

$$T_D(x, y) = \begin{cases} 0 & \text{if } x, y \in [0, 1]^2. \\ \min(x, y) & \text{otherwise} \end{cases}$$

Definition 1.6.2. [57] A *t*-conorm is a function $S : [0, 1]^2 \rightarrow [0, 1]$ that for any $x, y, z \in [0, 1]$ satisfies (T1)-(T3) and the following boundary condition $S(x, 0) = S(0, x) = x$, $S(x, 1) = S(1, x) = 0$

Remark 1.6.1. Given a *t*-norm T , we find the associated dual *t*-conorm S by $S(x, y) = 1 - T(1 - x, 1 - y)$.

Example 1.6.2. The following four operations are the most common *t*-conorm:

- (1) *Maximum*: $S_M(x, y) = \max\{x, y\}$
- (2) *Probabilistic sum*: $S_P(x, y) = x + y - x \cdot y$
- (3) *Lukasiewicz*: $S_L(x, y) = \min\{x + y, 1\}$
- (4) *Drastic sum*:

$$S_D(x, y) = \begin{cases} 1, & \text{if } (x, y) \in]0, 1]^2 \\ \max\{x, y\}, & \text{otherwise} \end{cases}$$

Chapter 2

Type-2 fuzzy sets

The type-2 fuzzy set concept was first introduced by Zadeh (1975)[78] as an extension of a type-1 fuzzy set. A type-2 fuzzy set is characterized by a fuzzy membership function, i.e., the membership grade for each element of this set is a fuzzy set in $[0,1]$, unlike a type-1 set where the membership grade is a crisp number in $[0,1]$. Such sets can be used in situations where there is uncertainty about the membership grades themselves, e.g., uncertainty in the shape of the membership function or in some of its[5].

2.1 Type-2 Fuzzy Sets Concept

Definition 2.1.1. [79] A type m fuzzy set is a fuzzy set in X where the membership values of which are type $(m-1)$, $m > 1$ fuzzy sets on $[0, 1]$

Definition 2.1.2. [78],[50]. Let X be a finite and nonempty set, which is referred to as the universe a type-2 fuzzy set, denoted by \tilde{A} is characterized by a type-2 membership function $\mu_{\tilde{A}}(x, u)$ as $\mu_{\tilde{A}} : X \times [0, 1] \rightarrow [0, 1]^{J_x} = \text{map}(J_x, [0, 1])$, ($J_x \subseteq [0, 1]$), where $x \in X$ and $u \in J_x$ that is

$$\tilde{A} = \left\{ ((x, u), \mu_{\tilde{A}}(x, u)) / x \in X, \text{ and } u \in J_x \subseteq [0, 1], \text{ wher } 0 \leq \mu_{\tilde{A}}(x, u) \leq 1 \right\}$$

\tilde{A} can also be expressed as

$$\tilde{A} = \sum_{x \in X} \sum_{u \in J_x} \mu_{\tilde{A}}(x, u) / (x, u) = \sum_{x \in X} \sum_{u \in J_x} f_x(u) / u / x, J_x \subseteq [0, 1]$$

where $f_x(u) = \mu_{\tilde{A}}(x, u)$ and denotes union over all admissible x and u for continuous universes of discourse, $\sum^{\tilde{A}}$ is replaced by \int

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{A}}(x, u) / (x, u)$$

where x is the primary variable in the domain X ; u is the secondary variable in domain J_x at each $x \in X$. J_x is called the primary membership of x , and $\mu_{\tilde{A}}(x, u)$ the secondary membership grades of \tilde{A} , $J_x \subseteq [0, 1]$ and $\int \int$ denote union over all admissible x and u in definition 2.1.2, the first restriction that $\forall u \in J_x \subseteq [0, 1]$ is consistent with the type-1 constraint that $0 \leq \mu_A(x) \leq 1$, i.e., when uncertainties disappear a type-2 membership function must be reduced to a type-1 membership function, in which case the variable u equals $\mu_A(x)$ and $0 \leq \mu_A(x) \leq 1$. The second restriction that $0 \leq \mu_{\tilde{A}}(x, u) \leq 1$ is consistent with the fact that the amplitudes of a membership function should lie between or be equal to 0 and 1.

Type-2 membership function representation

$$\tilde{A} = \left\{ \left((x, u), \mu_{\tilde{A}}(x, u) \right) / x \in X, J_x \subseteq [0, 1] \right\}$$

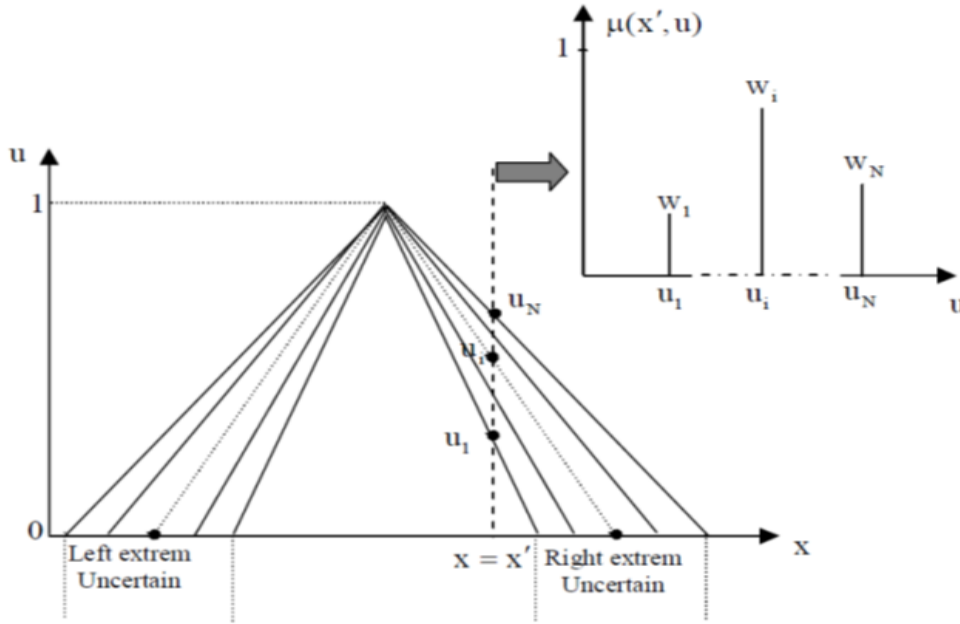


Figure 2.1: Triangular MFs when base end-points (l and r) have uncertainty intervals associated with them. [55]

Example 2.1.1. Suppose that $X = \text{Salima, Basma, Halima, Radhia, Aicha}$ is a set of women and that A is a fuzzy set of type 2 of beautiful women in X . Then we may have

$$A = \text{beautiful} = \text{middle/Salima} + \text{notlow/Basma} + \text{low/Halima} + \text{veryhigh/Radhia} + \text{high/Aicha}$$

where the fuzzy grades labeled middle, low, high are assumed to be fuzzy sets in $J = 0, 0.1, \dots, 0.9, 1 \subseteq [0, 1]$ and, for example, are expressed as follows.

$$\text{middle} = 0.3/0.3 + 0.7/0.4 + 1/0.5 + 0.7/0.6 + 0.3/0.7. \quad \text{low} = 1/0 + 0.9/0.1 + 0.7/0.2 + 0.4/0.3. \quad \text{high} = 0.4/0.7 + 0.7/0.8 + 0.9/0.9 + 1/1.$$

Moreover, fuzzy grades named not low and very high are defined from fuzzy grades low and high by using the linguistic hedges concept (Zadeh, 1972)

$$\text{not low} = 0.1/0.1 + 0.3/0.2 + 0.6/0.3 + 1/(0.4 + 0.5 + \dots + 1). \quad \text{very high} = 0.16/0.7 + 0.49/0.8 + 0.81/0.9 + 1/1.$$

Definition 2.1.3. [52] J_x , the primary membership of x , is :

$$J_x = \left\{ (x, u) \mid u \in [0, 1], \mu_{\tilde{A}}(x, u) > 0 \right\}$$

It can also be expressed as a subset of $\{x\} \times I_x$, i.e., $J_x = \{x\} \times I_x$ where

$$I_x = \left\{ u \in [0, 1] \mid \mu_{\tilde{A}}(x, u) > 0 \right\}$$

Definition 2.1.4. [52] The support of \tilde{A} comprise all $(x, u) \in X \times [0, 1]$ such that $\mu_{\tilde{A}}(x, u) > 0$, and is also called the domain of uncertainty of \tilde{A} , $DOU(\tilde{A})$, i.e.,

$$DOU(\tilde{A}) = \left\{ (x, u) \in X \times [0, 1] \mid \mu_{\tilde{A}}(x, u) > 0 \right\} = \bigcup_{x \in X} J_x$$

Definition 2.1.5. [52] When the support or the secondary MF, l_x , is closed for $x \in X$, .i.e.,

$$I_x = \left\{ u \in [0, 1] \mid \mu_{\tilde{A}}(x, u) > 0 \right\} = \left[\underline{\mu}_{\tilde{A}}(x), \bar{\mu}_{\tilde{A}}(x) \right] \quad (2.1)$$

where (Aisbett et al. 2010)

$$\bar{\mu}_{\tilde{A}}(x) = \sup \left\{ u \mid u \in [0, 1], \mu_{\tilde{A}}(x, u) > 0 \right\} \quad (2.2)$$

$$\underline{\mu}_{\tilde{A}}(x) = \inf \left\{ u \mid u \in [0, 1], \mu_{\tilde{A}}(x, u) > 0 \right\} \quad (2.3)$$

then the domain or uncertainty of \tilde{A} is called the footprint of uncertainty of \tilde{A} , $FOU(\tilde{A})$. .i.e.,

$$DOU(\tilde{A}) = FOU(\tilde{A}) = \left\{ (x, u) \mid x \in X \text{ and } u \in \left[\underline{\mu}_{\tilde{A}}(x), \bar{\mu}_{\tilde{A}}(x) \right] \right\} \quad (2.4)$$

Note that $\underline{\mu}_{\tilde{A}}(x)$ and $\bar{\mu}_{\tilde{A}}(x)$ are called the lower and upper MFs or $FOU(\tilde{A})$ [35], and are the lower and upper (type-1 fuzzy set) bounding function of the FOU, respectively. Commonly used abbreviations for the lower and upper MFs or \tilde{A} are $LMF(\tilde{A})$ and $UMF(\tilde{A})$.

[52] The class of all type-2 fuzzy sets of the universe X denoted by $\tilde{F}_{T_2}(X)$.

Type- 2 fuzzy universe set

Definition 2.1.6. A type-2 fuzzy universe set denoted \tilde{X} such that $\tilde{X} = \sum_{x \in X} \sum_{u \in [1,1]} \frac{1}{x}$

Upper and lower membership functions

Definition 2.1.7. [5]

an upper membership function and a lower membership function are two type-1 membership functions that are bounds for the FOU of a type-2 fuzzy set \tilde{A} . The upper membership function is associated with the upper bound of $FOU(\tilde{A})$. The lower membership function is associated with the lower bound of $FOU(\tilde{A})$. We illustrate the concept of upper and lower membership functions as well as the footprint of uncertainty in the following example.

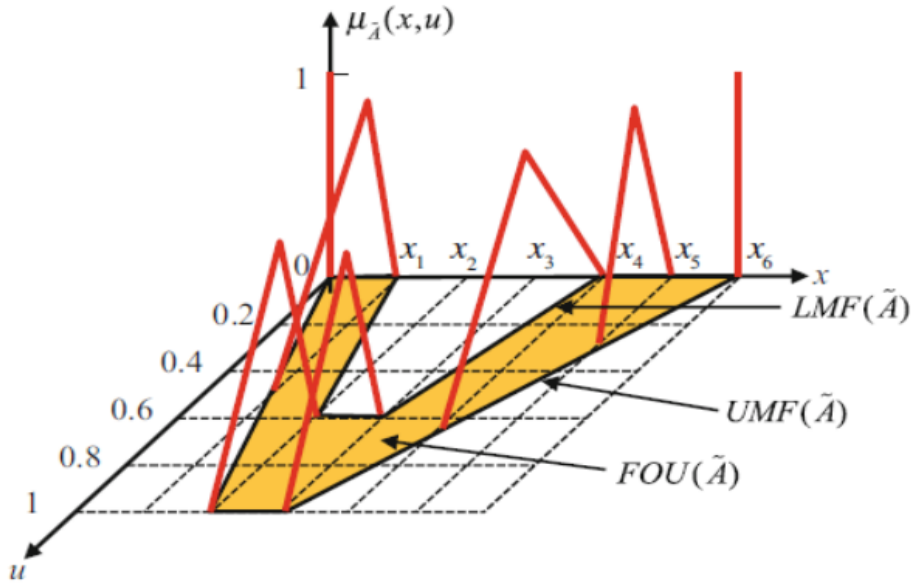


Figure 2.2: three-dimensional representation the of type-2 membership function [54]

2.2 Interval type- 2 fuzzy set

[48] When all the secondary grades of types $\tilde{\tilde{A}}$ are equal to 1, that is $\mu_{\tilde{\tilde{A}}}(x, u)$ for all $x \in X$ all $u \in J_x$, $\tilde{\tilde{A}}$ is an interval type-2 fuzzy set, $\tilde{\tilde{A}}$ is a special case of general type-2 fuzzy set where all of the secondary membership functions of $\tilde{\tilde{A}}$ are equal to 1. $\tilde{\tilde{A}}$ is an interval type-2 fuzzy set

$$\tilde{\tilde{A}} = \int_{x \in X} \int_{u \in J_x} \frac{1}{(x, u)} \quad (2.5)$$

where $J_x \subseteq [0, 1]$

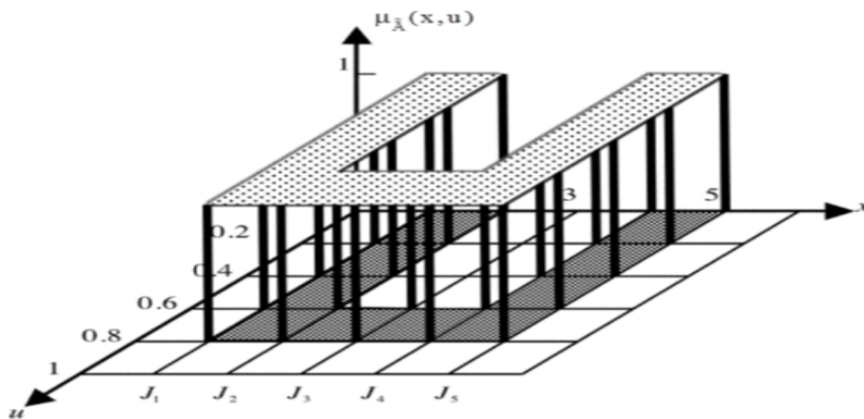


Figure 2.3: Example of an interval type-2 membership function for discrete universes of discourse [50]

Definition 2.2.1. [50]. At each value of x , say $x = x'$, the 2D plane whose axes are u and $\mu_{\tilde{A}}(x', u)$ is called a vertical slice of $\mu_{\tilde{A}}(x, u)$. A secondary membership function is a vertical slice of $\mu_{\tilde{A}}(x, u)$. It is $\mu_{\tilde{A}}(x = x', u)$ for $x' \in X$ and $\forall u \in J_{x'} \subseteq [0, 1]$, i.e., $\mu_{\tilde{A}}(x = x', u) \equiv \mu_{\tilde{A}}(x')$ $= \int_{u \in J_{x'}} f_{x'}(u)/u J_{x'} \subseteq [0, 1]$ (3) in which $0 \leq f_{x'}(u) \leq 1$. Because $\forall x' \in X$ we drop the prime notation on $\mu_{\tilde{A}}(x')$. and refer to $\mu_{\tilde{A}}(x)$ as a secondary membership function; it is a type-1 fuzzy set, which we also refer to as a secondary set.

Example 2.2.1. The type-2 membership function that is depicted in Fig 2.2 has five vertical slices associated with it. The secondary membership function at $x = 2$, e.g., is

$$\mu_{\tilde{A}}(2) = 0.5/0 + 0.35/0.2 + 0.35/0.4 + 0.2/0.6 + 0.5/0.8$$

2.3 Operation of Type-2 fuzzy sets

[49] In this section, we describe the type-2 fuzzy sets theoretical operations set. We are interested in the case of type-2 fuzzy sets, $\tilde{A}_i (i = 1, \dots, r)$, the secondary membership functions are type-1 fuzzy sets. To compute the union, the intersection, and the type-2 fuzzy sets complement, we need to extend the binary operations of minimum (or product) and maximum, and the unary operation of negation, from crisp numbers to type-1 fuzzy sets, because at each x , $\mu_{\tilde{A}_i}(x, u)$ is a function (unlike the type-1 case, where $\mu_{\tilde{A}_i}(x)$ is a crisp number). The tool for computing the union, intersection, and type-2 fuzzy sets complement is Zadeh's extension principle (Zadeh, 1975). We study operations set on type-2 fuzzy sets. We first discuss join and meet operations of membership grades of type-2 fuzzy sets under left continuous t-norms and derive distributive law of type-2 fuzzy sets [58].

Consider two type-2 fuzzy sets, \tilde{A} and \tilde{B} , in a universe X . Let $\mu_{\tilde{A}(x)}$ and $\mu_{\tilde{B}(x)}$ be the membership grades of these two sets, which are represented for each $x \in X$, $\mu_{\tilde{A}(x)} = \sum_{u \in J_x^u} f_x(u)/u$ and $\mu_{\tilde{B}(x)} = \sum_{w \in J_x^w} g_x(w)/w$, respectively, where $u \in J_x^u, w \in J_x^w$ indicate the primary memberships of x and $f_x(u), g_x(w) \in [0, 1]$ indicate the secondary memberships (grades) of x . The membership grades for the union, intersection, and complement of the type-2 fuzzy sets \tilde{A} and \tilde{B} have been defined as follows

Containment:

\tilde{A} is a subtype-2 fuzzy set of \tilde{B} denoted $\tilde{A} \subseteq \tilde{B}$ if $u < v$ and $f_x(u) \leq g_x(w)$ for every $x \in X$

Equality:

\tilde{A} and \tilde{B} are equal type-2 fuzzy sets, denoted $\tilde{A} = \tilde{B}$ if $u = w$ and $f_x(u) = \mu_{\tilde{A}}(x, u) = g_x(w) = \mu_{\tilde{B}}(x, w)$ for every $x \in X$

Two type-2 fuzzy sets union: [34]

$$\tilde{A} \cup \tilde{B} = \sum_{u \in J_x^u} \sum_{w \in J_x^w} (f_x(u) * g_x(w)) / (u \vee w)$$

Two type-2 fuzzy sets intersection: [34]

$$\tilde{A} \cap \tilde{B} = \sum_{u \in J_x^u} \sum_{w \in J_x^w} (f_x(u) * g_x(w)) / (u \wedge w)$$

where \vee represents the max t-conorm and \wedge indicates the minimum t-norm

Type-2 fuzzy set complement: [34]

Complement : $\tilde{A} \Leftrightarrow \mu_{\tilde{A}}(x) = \neg \mu_A(x) = \int_u f_x(u)/(1-u)$

Join and Meet under min t-norm

Definition 2.3.1. [49] Suppose that we have two convex, normal, type-1 real fuzzy sets \tilde{F} and \tilde{G} characterized by membership functions f and g respectively. Let $v_0 \in \mathbb{R}$ and $v_1 \in \mathbb{R}$ be such that $v_0 < v_1$ and $f(v_0) = g(v_1) = 1$. Then the membership functions of the join and meet of \tilde{F} and \tilde{G} , using max t-conorm and min t-norm, can be expressed as

$$\begin{aligned} \mu_{(\tilde{F} \cup \tilde{G})}(\theta) &= f(\theta) \wedge g(\theta); \theta < v_0 \\ &= g(\theta); v_0 \leq \theta \leq v_1 \\ &= f(\theta) \vee g(\theta); \theta > v_1 \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} \mu_{(\tilde{F} \cap \tilde{G})}(\theta) &= f(\theta) \vee g(\theta); \theta < v_0 \\ &= f(\theta); v_0 \leq \theta \leq v_1 \\ &= f(\theta) \wedge g(\theta); \theta > v_1 \end{aligned} \quad (2.7)$$

Definition 2.3.2. Type-2 fuzzy sets union

The union of \tilde{A}_1 and \tilde{A}_2 are another type-2 fuzzy set, just as the union of type-1 fuzzy sets A_1 and A_2 is another type-1 fuzzy set. More formally, we have the following expression

$$\tilde{A}_1 \cup \tilde{A}_2 = \int_{x \in X} \mu_{\tilde{A}_1 \cup \tilde{A}_2}(x) / x \quad (2.8)$$

Basically, the join between two secondary membership functions must be performed between every possible pair of primary memberships.

Let $\mu_A(x)$ and $\mu_B(x)$ be two fuzzy grades (that is, fuzzy sets in $J \subseteq [0, 1]$) of fuzzy sets of type 2, A and B , respectively, represented as

$$\begin{aligned} \mu_A(x) &= f(u_1) / u_1 + f(u_2) / u_2 + \dots + f(u_n) / u_n \\ &= \sum_i f(u_i) / u_i, \quad u_i \in J \end{aligned} \quad (2.9)$$

$$\begin{aligned} \mu_B(x) &= g(w_1) / w_1 + g(w_2) / w_2 + \dots + g(w_m) / w_m \\ &= \sum_j g(w_j) / w_j, \quad w_j \in J \end{aligned} \quad (2.10)$$

where the function f and g are membership functions of fuzzy grades (fuzzy sets in $J \subseteq [0, 1]$) $\mu_A(x)$ and $\mu_B(x)$ respectively, and the values $f(u_i)$ and $g(w_j)$ in $[0, 1]$ represent the grades for u_i and w_j in J , respectively.

Thus the operation type 2 fuzzy sets are expressed by the following.

Union

$$\begin{aligned} A \cup B \Leftrightarrow \mu_{A \cup B}(x) &= \mu_A(x) \sqcup \mu_B(x) \\ &= \left(\sum_i f(u_i) / u_i \right) \sqcup \left(\sum_j g(w_j) / w_j \right) \\ &= \sum_{i,j} (f(u_i) \wedge g(w_j)) / (u_i \vee w_j) \end{aligned} \quad (2.11)$$

Intersection

$$\begin{aligned}
 A \cap B &\Leftrightarrow \mu_{A \cap B}(x) = \mu_A(x) \sqcap \mu_B(x) \\
 &= \left(\sum_i f(u_i) / u_i \right) \sqcap \left(\sum_j g(w_j) / w_j \right) \\
 &= \sum_{i,j} (f(u_i) \wedge g(w_j)) / (u_i \wedge w_j)
 \end{aligned} \tag{2.12}$$

Complement

$$\tilde{A} \Leftrightarrow \mu_{\tilde{A}}(x) \text{ then the MF of the complement of } \tilde{A} \text{ is } \mu_{\tilde{A}}(x) = \int_u f_x(u) / (1 - u) \tag{2.13}$$

where \vee and \wedge represent max and min, respectively. We call the operations for fuzzy grades, that is, \sqcup like join, \sqcap as to meet, and \neg as negation hereafter. [58]. If more than one combination of pairs gives the same point, then in the join we keep the one with maximum membership grade. We will consider a simple example to illustrate the union operation. In Figure 3.6 we plot two type-2 Gaussian membership functions, and the union is shown in Figure 3.7.

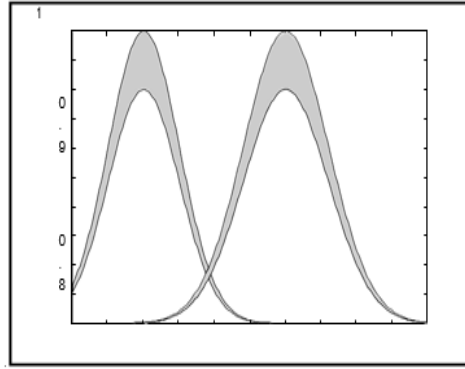


Figure 2.4: Type-2 Gaussian MFs

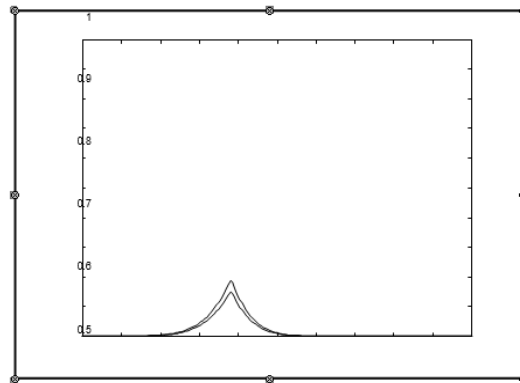


Figure 2.5: Intersection of two Interval Type-2 Fuzzy Sets

Example 2.3.1. [2] Let $X = \{x_1, x_2, x_3\}$ be a nonempty set, and let \tilde{A} and \tilde{B} are two type-2 fuzzy sets over the same universe X

$$\tilde{A} = \{(x_1, 0.1), 0.3), ((x_1, 0.5), 1), ((x_2, 0.5), 1), ((x_2, 0.6), 0.3), ((x_3, 0.8), 1)\}$$

$$\tilde{B} = \{(x_1, 0.1), 0.7), ((x_1, 0.2), 1), ((x_2, 0.6), 1), ((x_3, 0.5), 0.6), ((x_3, 0.9), 1)\}$$

$\tilde{A} \cup \tilde{B}$ for $x = x_1$ we get

$$\begin{aligned} \mu_{\tilde{A} \cup \tilde{B}}(x_1) &= \frac{0.3 \wedge 0.7}{0.1 \vee 0.1} + \frac{0.3 \wedge 1}{0.1 \vee 0.2} + \frac{1 \wedge 0.7}{0.5 \vee 0.1} + \frac{1 \wedge 1}{0.5 \vee 0.2} \\ &= \frac{0.3}{0.1} + \frac{0.3}{0.2} + \frac{0.7}{0.5} + \frac{1}{0.5} \\ &= \{(0.1, 0.3), (0.2, 0.3), (0.5, \max(0.7, 1))\} \end{aligned}$$

For $x = x_1$

$$\tilde{A} \cup \tilde{B} = \{(x_1, 0.1), 0.3), ((x_1, 0.2), 0.3), ((x_1, 0.5), 1)\}$$

$\tilde{A} \cup \tilde{B}$ for $x = x_2$ we get

$$\begin{aligned} \mu_{\tilde{A} \cup \tilde{B}}(x_2) &= \frac{1 \wedge 1}{0.5 \vee 0.6} + \frac{0.3 \wedge 1}{0.6 \vee 0.6} \\ &= \frac{1}{0.6} + \frac{0.3}{0.6} \\ &= \{(0.6, \max(1, 0.3))\} \end{aligned}$$

For $x = x_2$, $\tilde{A} \cup \tilde{B} = \{(x_2, 0.6), 1)\}$

$\tilde{A} \cup \tilde{B}$ for $x = x_3$ to get

$$\begin{aligned} \mu_{\tilde{A} \cup \tilde{B}}(x_3) &= \frac{1 \wedge 0.6}{0.8 \vee 0.5} + \frac{1 \wedge 1}{0.8 \vee 0.9} \\ &= \frac{0.6}{0.8} + \frac{1}{0.9} \\ &= \{(0.8, 0.6), (0.9, 1)\} \end{aligned}$$

For $x = x_3$,

$$\tilde{A} \cup \tilde{B} = \{(x_3, 0.6), 0.8), ((x_3, 0.9), 1)\}$$

Hence,

$$\tilde{A} \cup \tilde{B} = \{(x_1, 0.1), 0.3), ((x_1, 0.2), 0.3), ((x_1, 0.5), 1), ((x_2, 0.6), 1), ((x_3, 0.6), 0.8), ((x_3, 0.9), 1)\}$$

$\tilde{A} \cap \tilde{B}$ for $x = x_1$

$$\tilde{A} \cap \tilde{B} = \{(x_1, 0.1), 0.7), ((x_1, 0.2), 1)\}$$

$\tilde{A} \cap \tilde{B}$ for $x = x_2$

$$\begin{aligned} \mu_{\tilde{A} \cap \tilde{B}}(x_2) &= \frac{1 \wedge 1}{0.5 \wedge 0.6} + \frac{0.3 \wedge 1}{0.6 \wedge 0.6} \\ &= \frac{1}{0.5} + \frac{0.3}{0.6} \\ &= \{(0.5, 1), (0.6, 0.3)\} \end{aligned}$$

For $x = x_2$

$$\tilde{A} \cap \tilde{B} = \{(x_2, 0.5), 1), (x_2, 0.6), 0.3)\}$$

$\tilde{A} \cap \tilde{B}$ for $x = x_3$

$$\begin{aligned} \mu_{\tilde{A} \cap \tilde{B}}(x_3) &= \frac{1 \wedge 0.6}{0.8 \wedge 0.5} + \frac{1 \wedge 1}{0.8 \wedge 0.9} \\ &= \frac{0.6}{0.5} + \frac{1}{0.8} \\ &= \{(0.5, 0.6), (0.8, 1)\} \end{aligned}$$

$$\begin{aligned}
& \text{For } x = x_1, \quad \tilde{A} \cap \tilde{B} = \{(x_3, 0.5), 0.6\}, \{(x_3, 0.8), 1\} \\
& \tilde{A} \cap \tilde{B} = \{(x_1, 0.1), 0.7\}, \{(x_1, 0.2), 1\}, \{(x_2, 0.5), 1\}, \{(x_2, 0.6), 0.3\}, \{(x_3, 0.5), 0.6\}, \{(x_3, 0.8), 1\} \\
& \tilde{A} = \{(x_1, 0.1), 0.7\}, \{(x_1, 0.5), 0.5\}, \{(x_2, 0.5), 0.5\}, \{(x_2, 0.6), 0.7\}, \{(x_3, 0.8), 0.2\}
\end{aligned}$$

Theorem 2.3.1. [70] Let \tilde{A} , \tilde{B} , and \tilde{C} be two T2 fuzzy sets on X . If secondary membership functions $\mu_{\tilde{A}}^{\approx}(x)$, $\mu_{\tilde{B}}^{\approx}(x)$ and $\mu_{\tilde{C}}^{\approx}(x)$ are convex fuzzy sets for any $x \in X$, then we have

$$(\tilde{A} \cup \tilde{B}) \cap \tilde{C} = (\tilde{A} \cap \tilde{C}) \cup (\tilde{B} \cap \tilde{C}) \quad (2.14)$$

and

$$(\tilde{A} \cap \tilde{B}) \cup \tilde{C} = (\tilde{A} \cup \tilde{C}) \cap (\tilde{B} \cup \tilde{C}) \quad (2.15)$$

Proof 2.3.1. For any $x \in X$, since secondary membership functions $\mu_{\tilde{A}}^{\approx}(x)$, $\mu_{\tilde{B}}^{\approx}(x)$ and $\mu_{\tilde{C}}^{\approx}(x)$ are convex T1 fuzzy sets in $[0, 1]$, from [1], they satisfy distributive laws under \sqcup and \sqcap . Therefore, we obtain

$$\begin{aligned}
& \left[\mu_{\tilde{A}}^{\approx}(x) \sqcup \mu_{\tilde{B}}^{\approx}(x) \right] \sqcap \mu_{\tilde{C}}^{\approx}(x) \\
& = \left[\mu_{\tilde{A}}^{\approx}(x) \sqcap \mu_{\tilde{C}}^{\approx}(x) \right] \sqcup \left[\mu_{\tilde{B}}^{\approx}(x) \sqcap \mu_{\tilde{C}}^{\approx}(x) \right]
\end{aligned}$$

and

$$\begin{aligned}
& \left[\mu_{\tilde{A}}^{\approx}(x) \sqcap \mu_{\tilde{B}}^{\approx}(x) \right] \sqcup \mu_{\tilde{C}}^{\approx}(x) \\
& = \left[\mu_{\tilde{A}}^{\approx}(x) \sqcup \mu_{\tilde{C}}^{\approx}(x) \right] \sqcap \left[\mu_{\tilde{B}}^{\approx}(x) \sqcup \mu_{\tilde{C}}^{\approx}(x) \right]
\end{aligned}$$

2.4 Operations under type-2 fuzzy sets collection

[58] Let $\{\tilde{A}_i, i \in \mathbb{N}\}$ be an arbitrary collection of type-2 fuzzy sets of X such that \mathbb{N} is countable set, operations are possible under an arbitrary collection of type-2 fuzzy sets.

1) The union $\cup_{i \in \mathbb{N}} \tilde{A}_i$ is defined as

$$\left[\cup_{i \in \mathbb{N}} \tilde{A}_i \right] (x) = \sum_{x \in X} \sum_{u \in JX} \frac{\wedge_{i \in \mathbb{N}} (f_x(u))_i}{\vee_{i \in \mathbb{N}} (u)_i}$$

2) The intersection

$$\left[\cap_{i \in \mathbb{N}} \tilde{A}_i \right] (x) = \sum_{x \in X} \sum_{u \in JX} \frac{\wedge_{i \in \mathbb{N}} (f_x(u))_i}{\wedge_{i \in \mathbb{N}} (u)_i}$$

Proposition 2.4.1. [25] Let $\{\tilde{A}_i, i \in \mathbb{N}\}$ be an arbitrary collection of type-2 fuzzy sets of X such that \mathbb{N} is a countable set and \tilde{B} be another type-2 fuzzy set of X , then

$$1) \quad \tilde{B} \cap \left[\cup_{i \in \mathbb{N}} \tilde{A}_i \right] = \cup_{i \in \mathbb{N}} \left(\tilde{B} \cap \tilde{A}_i \right)$$

$$2) \quad \tilde{B} \cup \left[\cap_{i \in \mathbb{N}} \tilde{A}_i \right] = \cap_{i \in \mathbb{N}} \left(\tilde{B} \cup \tilde{A}_i \right)$$

$$3) \quad 1 - \left[\cup_{i \in \mathbb{N}} \tilde{A}_i \right] = \cap_{i \in \mathbb{N}} \left[1 - \tilde{A}_i \right]$$

$$4) \quad 1 - \left[\bigcap_{i \in \mathbb{N}} \tilde{A}_i \right] = \bigcup_{i \in \mathbb{N}} \left[1 - \tilde{A}_i \right]$$

Example 2.4.1. First, we will show the examples of fuzzy grades which do not satisfy the absorption laws, the distributive laws, the identity laws, and the complement laws.

Absorption laws failure. Let μ_A and μ_B be convex fuzzy grades as

$$\mu_A = 0.2/0.1 + 0.3/0.2 + 0.5/0.3 + 0.7/0.4 + 0.8/0.5$$

$$\mu_B = 0/0.1 + 0.1/0.2 + 0.2/0.3 + 0.3/0.2 + 0.4/0.5$$

where J is $\{0.1, 0.2, 0.3, 0.4, 0.5\}$. Then $\mu_A \sqcap (\mu_B = 0.2/0.1 + 0.3/0.2 + 0.4/0.3 + 0.4/0.4 + 0.4/0.5 \text{ and } \mu_A \sqcup \mu_B)$
 $= 0/0.1 + 0.1/0.2 + 0.2/0.3 + 0.3/0.4 + 0.4/0.5$

$$\mu_A \sqcap (\mu_A \sqcup \mu_B) = \mu_A \sqcup (\mu_A \sqcap \mu_B)$$

$$= 0.2/0.1 + 0.3/0.2 + 0.4/0.3 + 0.4/0.4 + 0.4/0.5 \neq \mu_A.$$

Distributive laws failure

$$\mu_A = 0.8/0.2 + 0.1/0.3 + 0/0.4 + 0.7/0.5$$

$$\mu_B = 0.3/0.2 + 0.4/0.3 + 0.5/0.4 + 0.2/0.5$$

$$\mu_C = 0.1/0.2 + 0.2/0.3 + 0.5/0.4 + 0.7/0.5$$

Then we have,

$$\mu_B \sqcup (\mu_C) = 0.1/0.2 + 0.2/0.3 + 0.5/0.4 + 0.5/0.5$$

$$\mu_A \sqcap (\mu_B \sqcup \mu_C) = 0.5/0.2 + 0.2/0.3 + 0.5/0.4 + 0.5/0.5$$

$$\mu_A \sqcap \mu_B = 0.5/0.2 + 0.4/0.3 + 0.5/0.4 + 0.2/0.5$$

$$\mu_A \sqcap \mu_C = 0.7/0.2 + 0.2/0.3 + 0.5/0.4 + 0.7/0.5$$

$$(\mu_A \sqcap \mu_B) \sqcup (\mu_A \sqcap \mu_C) = 0.5/0.2 + 0.4/0.3 + 0.5/0.4 + 0.2/0.5$$

Thus, we see that the distributive law is not satisfied. The same holds for the case in which \sqcup and \sqcap are interchanged.

A part of identity laws failure, let $\mu_A = 0.6/0.2 + 0.4/0.3 + 0.7/0.4$. Then, noting that the numbers 1 and 0 are represented as 1/1 and 1/0, respectively, we obtain

$$\mu_A \sqcup 1 = 0.7/1 \neq 1$$

$$\mu_A \sqcap 0 = 0.7/0 \neq 0$$

Complement laws failure. Let $\mu_A = 0.7/0.2 + 0.6/0.3 + 1/0.4$ then $\mu_{\tilde{A}}$ is given as

$$\mu_{\tilde{A}} = 0.7/0.8 + 0.6/0.7 + 1/0.6.$$

Thus we have

$$\mu_A \sqcup \mu_{\tilde{A}} = 1/0.6 + 0.6/0.7 + 0.7/0.8 \neq 1$$

$$\mu_A \sqcap \mu_{\tilde{A}} = 0.7/0.2 + 0.6/0.3 + 1/0.4 \neq 0$$

2.5 Type-2 fuzzy relations, their compositions, and cartesian product

Let X_1, X_2, \dots, X_n , be n universes. A crisp relation in $X_1 \times X_2 \times \dots \times X_n$ is a crisp subset of the product space. Similarly, a type-1 fuzzy relation in $X_1 \times X_2 \times \dots \times X_n$, is a type-1 fuzzy subset of the product space and a type-2 fuzzy relation in $X_1 \times X_2 \times \dots \times X_n$, is a type-2 fuzzy subset of the product space. We concentrate on binary type-2 fuzzy relations.

2.5.1 Type-2 FRs and their compositions

From type-1 fuzzy relations to type-2 ones

[75] A type-1 FR in $U \times V$ is a type-1 fuzzy subset of $U \times V$ and a type-2 FR in $U \times V$ is a type-2 fuzzy subset of $U \times V$.

Type-1 and type-2 fuzzy relations are usually expressed in matrix form as

$$R = \begin{bmatrix} \mu_R(u_1, v_1) & \mu_R(u_1, v_2) \cdots & \mu_R(u_1, v_n) \\ \mu_R(u_2, v_1) & \mu_R(u_2, v_2) \cdots & \mu_R(u_2, v_n) \\ \vdots & \vdots & \vdots \\ \mu_R(u_m, v_1) & \mu_R(u_m, v_2) \cdots & \mu_R(u_m, v_n) \end{bmatrix}$$

and

$$\tilde{R} = \begin{bmatrix} \mu_{\tilde{R}}(u_1, v_1) & \mu_{\tilde{R}}(u_1, v_2) \cdots & \mu_{\tilde{R}}(u_1, v_n) \\ \mu_{\tilde{R}}(u_2, v_1) & \mu_{\tilde{R}}(u_2, v_2) \cdots & \mu_{\tilde{R}}(u_2, v_n) \\ \vdots & \vdots & \vdots \\ \mu_{\tilde{R}}(u_m, v_1) & \mu_{\tilde{R}}(u_m, v_2) \cdots & \mu_{\tilde{R}}(u_m, v_n) \end{bmatrix}$$

where $\mu_R(u_i, v_j)$ are crisp numbers in $[0, 1]$, and $\mu_{\tilde{R}}(u_i, v_j)$ are fuzzy numbers (fuzzy sets) in $[0, 1]$.

As the extension from type-1 fuzzy sets to type-2 ones, we can obtain a type-2 fuzzy relation by adding additional uncertainty information to a type-1 one. The following example explains the extension

Example 2.5.1. Consider the type-1 FR $R \in \mathcal{F}(X \times Y) : 'x \text{ is close to } y'$, where $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ and

$$R = \begin{matrix} & y_1 & y_2 \\ x_1 & \begin{bmatrix} 0.5 & 0.2 \end{bmatrix} \\ x_2 & \begin{bmatrix} 0.8 & 0.1 \end{bmatrix} \\ x_3 & \begin{bmatrix} 0.2 & 0.7 \end{bmatrix} \end{matrix}$$

Consider another type-1 FR $S \in \mathcal{F}(Y \times Z) : 'y \text{ is much smaller than } z'$, where $Z = \{z_1, z_2, z_3\}$, and

$$S = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & \begin{bmatrix} 0.3 & 0.9 & 0.6 \end{bmatrix} \\ y_2 & \begin{bmatrix} 0.3 & 0.4 & 0.5 \end{bmatrix} \end{matrix}$$

Adding some additional uncertainties to these two type-1 FRs, we may obtain the following membership grades (For saving space, we sometimes represent a type-2 fuzzy relation in the following form, i.e., the solidus (/) are replaced by built-up fractions):

$$\tilde{R} = \begin{matrix} & y_1 & y_2 \\ x_1 & \begin{bmatrix} \frac{0.4}{0.4} + \frac{1}{0.5} + \frac{0.7}{0.6} & \frac{0.6}{0.5} + \frac{1}{0.1} + \frac{0.6}{0.2} \end{bmatrix} \\ x_2 & \begin{bmatrix} \frac{0.3}{0.6} + \frac{1}{0.7} + \frac{0.5}{0.8} & \frac{0.1}{0.6} + \frac{0.2}{1} + \frac{0.3}{0.4} \end{bmatrix} \\ x_3 & \begin{bmatrix} \frac{0.8}{0.8} + \frac{1}{0.9} + \frac{0.5}{0.1} & \frac{0.6}{0.7} + \frac{1}{0.8} + \frac{0.4}{0.9} \end{bmatrix} \end{matrix}$$

and

$$\tilde{S} = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & \begin{bmatrix} \frac{0.4}{0.8} + \frac{1}{0.9} + \frac{0.6}{0.6} & \frac{0.3}{0.5} + \frac{1}{0.6} + \frac{0.7}{0.7} & \frac{0.5}{0.8} + \frac{1}{0.1} + \frac{0.6}{0.2} \end{bmatrix} \\ y_2 & \begin{bmatrix} \frac{0.3}{0} + \frac{1}{0.1} + \frac{0.6}{0.2} & \frac{0.2}{0.5} + \frac{1}{0.6} + \frac{0.6}{0.7} & \frac{0.5}{0.8} + \frac{1}{0.9} + \frac{0.7}{1} \end{bmatrix} \end{matrix}$$

2.5.2 Composition of type-2 fuzzy relations

[49]

Definition 2.5.1. If R and S (or \tilde{R} and \tilde{S}) are two T1 (or T2) FRs on $U \times V$ and $V \times W$, respectively, the membership of any (u, w) , $u \in U, w \in W$, is non-zero if there was at least one $v \in V$ so that $\mu_R(u, v) \neq 0$ (or $\mu_{\tilde{R}}(u, v) \neq 1/0$) and $\mu_S(v, w) \neq 0$ (or $\mu_{\tilde{S}}(v, w) \neq 1/0$), in which $1/0$ denotes the concept of zero membership grades in the case of type-2 fuzzy sets.

$$\mu_{\tilde{R} \circ \tilde{S}}(u, w) = \sqcup v \in V [\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{S}}(v, w)] \quad (2.16)$$

where \sqcap and \sqcup are meet and join operations, respectively, which are defined as follows [58]

Example 2.5.2. Consider the fuzzy relations in Example 2.5.1.

It is known that the expression 'x is close to y and y is much smaller than z' indicates the composition $R \circ S$ (or $\tilde{R} \circ \tilde{S}$).

For the type-1 case, using the equation:

$$\begin{aligned} \mu_{R \circ S}(x_i, z_j) = & [\mu_R(x_i, y_1) \wedge \mu_S(y_1, z_j)] \vee [\mu_R(x_i, y_2) \\ & \wedge \mu_S(y_2, z_j)] \vee [\mu_R(x_i, y_3) \wedge \mu_S(y_3, z_j)] \end{aligned} \quad (2.17)$$

where $i = 1, 2, 3$ and $j = 1, 2, 3$. For type-2 fuzzy relation using equation:

$$\begin{aligned} \mu_{R \circ S}(x_i, z_j) = & [\mu_R(x_i, y_1) \sqcap \mu_S(y_1, z_j)] \sqcup [\mu_R(x_i, y_2) \\ & \sqcap \mu_S(y_2, z_j)] \end{aligned} \quad (2.18)$$

$$\begin{aligned} \mu_{R \circ S}(x_1, z_1) = & [\mu_R(x_1, y_1) \sqcap \mu_S(y_1, z_1)] \sqcup [\mu_R(x_1, y_2) \\ & \sqcap \mu_S(y_2, z_1)] \\ = & \left[\left(\frac{0.4}{0.4} + \frac{1}{0.5} + \frac{0.7}{0.6} \right) \sqcap \left(\frac{0.4}{0.8} + \frac{1}{0.9} + \frac{0.6}{1} \right) \right] \sqcup \left[\left(\frac{0.6}{0} + \frac{1}{0.1} + \frac{0.6}{0.2} \right) \right] \\ = & \left[\sqcap \left(\frac{0.5}{0} + \frac{1}{0.1} + \frac{0.6}{0.2} \right) \right] \end{aligned} \quad (2.19)$$

we obtain

$$\tilde{R} \circ \tilde{S} = \begin{bmatrix} \frac{0.4}{0.4} + \frac{1}{0.5} + \frac{0.7}{0.6} & \frac{0.4}{0.4} + \frac{1}{0.5} + \frac{0.7}{0.6} & \frac{0.5}{0} + \frac{1}{0.1} + \frac{0.6}{0.2} \\ \frac{0.3}{0.3} + \frac{1}{1} + \frac{0.5}{0.5} & \frac{0.3}{0.3} + \frac{1}{1} + \frac{0.7}{0.7} & \frac{0.5}{0.5} + \frac{1}{1} + \frac{0.7}{0.7} \\ \frac{0.6}{0.6} + \frac{0.7}{0.7} + \frac{0.8}{0.8} & \frac{0.5}{0.5} + \frac{0.6}{0.6} + \frac{0.7}{0.7} & \frac{0.1}{0.1} + \frac{0.2}{0.2} + \frac{0.3}{0.3} \\ \frac{0.8}{0.8} + \frac{1}{0.9} + \frac{1}{1} & \frac{0.2}{0.5} + \frac{1}{0.6} + \frac{0.5}{0.7} & \frac{0.6}{0.7} + \frac{1}{0.8} + \frac{0.4}{0.9} \end{bmatrix}$$

Properties[70]

Let \tilde{R} , \tilde{S} and \tilde{L} be T2 fuzzy relations defined on product space $U \times V$. If for any $(u, v) \in U \times V$, $\mu_{\tilde{R}}(u, v)$, $\mu_{\tilde{S}}(u, v)$ and $\mu_{\tilde{L}}(u, v)$ are convex T1 fuzzy sets, then we have

$$(\tilde{R} \cup \tilde{S}) \cap \tilde{L} = (\tilde{R} \cap \tilde{L}) \cup (\tilde{S} \cap \tilde{L})$$

and

$$(\tilde{R} \cap \tilde{S}) \cup \tilde{L} = (\tilde{R} \cup \tilde{L}) \cap (\tilde{S} \cup \tilde{L})$$

Theorem 2.5.1. [70] Suppose that \tilde{R} , \tilde{S} and \tilde{L} are T2 fuzzy relations on product spaces $U \times V$, $V \times W$ and $W \times Q$, respectively, and $\mu_{\tilde{R}}(u, v)$, $\mu_{\tilde{S}}(v, w)$ and $\mu_{\tilde{L}}(w, q)$ are convex T1 fuzzy sets. If the spaces V and W are finite, then

$$(\tilde{R} \circ \tilde{S}) \circ \tilde{L} = \tilde{R} \circ (\tilde{S} \circ \tilde{L})$$

Proof 2.5.1. From Definition 1, for any $(u, q) \in U \times Q$, we have

$$\begin{aligned} & \mu_{(\tilde{R} \circ \tilde{S}) \circ \tilde{L}}(u, q) \\ &= \sqcup_{w \in W} [\mu_{\tilde{R} \circ \tilde{S}}(u, w) \sqcap \mu_{\tilde{L}}(w, q)] \\ &= \sqcup_{w \in W} [\sqcup_{v \in V} [\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{S}}(v, w)] \sqcap \mu_{\tilde{L}}(w, q)] \end{aligned}$$

we know $\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{S}}(v, w)$ is a convex T1 fuzzy set in $[0, 1]$

$$\begin{aligned} & \mu_{\tilde{R} \circ (\tilde{S} \circ \tilde{L})}(u, q) \\ &= \sqcup_{v \in V} [\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{S} \circ \tilde{L}}(v, q)] \\ &= \sqcup_{v \in V} [\mu_{\tilde{R}}(u, v) \sqcap \sqcup_{w \in W} [\mu_{\tilde{S}}(v, w) \sqcap \mu_{\tilde{L}}(w, q)]] \\ &= \sqcup_{w \in W} [\sqcup_{v \in V} [\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{S}}(v, w) \sqcap \mu_{\tilde{L}}(w, q)]] \end{aligned}$$

implies $\mu_{(\tilde{R} \circ \tilde{S}) \circ \tilde{L}}(u, q) = \mu_{\tilde{R} \circ (\tilde{S} \circ \tilde{L})}(u, q)$ for any $(u, q) \in U \times Q$

Theorem 2.5.2. Let \tilde{R}, \tilde{S} be two T2 fuzzy relations defined on $U \times V$, and \tilde{L} be a T2 fuzzy relations on $V \times W$. If $\mu_{\tilde{R}}(u, v), \mu_{\tilde{S}}(u, v)$ and $\mu_{\tilde{L}}(v, w)$ are convex fuzzy sets for any $(u, v) \in U \times V$ and $(v, w) \in V \times W$, then we have $(\tilde{R} \cup \tilde{S}) \circ \tilde{L} = (\tilde{R} \circ \tilde{L}) \cup (\tilde{S} \circ \tilde{L})$

Proof. On the one hand, the LHS of Eq. (28) can be expressed as

$$\begin{aligned} & \mu_{(\tilde{R} \cup \tilde{S}) \circ \tilde{L}}(u, w) \\ &= \bigsqcup_{v \in V} [\mu_{\tilde{R} \cup \tilde{S}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)] \\ &= \bigsqcup_{v \in V} [[\mu_{\tilde{R}}(u, v) \sqcup \mu_{\tilde{S}}(u, v)] \sqcap \mu_{\tilde{L}}(v, w)] \end{aligned}$$

for any $(u, w) \in U \times W$ On the other hand, can be written

$$\mu_{(\tilde{R} \circ \tilde{L}) \cup (\tilde{S} \circ \tilde{L})}(u, w) = \mu_{\tilde{R} \circ \tilde{L}}(u, w) \sqcup \mu_{\tilde{S} \circ \tilde{L}}(u, w) = \bigsqcup_{v \in V} [\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)] \sqcup \bigsqcup_{v \in V} [\mu_{\tilde{S}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)]$$

since arbitrary fuzzy sets in $[0, 1]$ satisfy associative and commutative laws under \sqcup [1], we have

$$\begin{aligned} & \bigsqcup_{v \in V} [\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)] \sqcup \bigsqcup_{v \in V} [\mu_{\tilde{S}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)] \\ &= \bigsqcup_{v \in V} [[\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)] \sqcup [\mu_{\tilde{S}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)]] \end{aligned}$$

Furthermore, noting that $\mu_{\tilde{R}}(u, v), \mu_{\tilde{S}}(u, v)$ and $\mu_{\tilde{L}}(v, w)$ are convex fuzzy sets in $[0, 1]$, form [1], they satisfy distributive laws under \sqcup and \sqcap . Therefore, we get

$$\begin{aligned} & \bigsqcup_{v \in V} [[\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)] \sqcup [\mu_{\tilde{S}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)]] \\ &= \bigsqcup_{v \in V} [[\mu_{\tilde{R}}(u, v) \cup \mu_{\tilde{S}}(u, v)] \sqcap \mu_{\tilde{L}}(v, w)] \end{aligned}$$

we obtain

$$\mu_{(\tilde{R} \cup \tilde{S}) \circ \tilde{L}}(u, w) = \mu_{(\tilde{R} \circ \tilde{L}) \cup (\tilde{S} \circ \tilde{L})}(u, w)$$

for any $(u, w) \in U \times W$. Thus,

$$(\tilde{R} \cup \tilde{S}) \circ \tilde{L} = (\tilde{R} \circ \tilde{L}) \cup (\tilde{S} \circ \tilde{L})$$

Similarly, we have the following result. □

Theorem 2.5.3. Let \tilde{R}, \tilde{S} be two T_2 fuzzy relations defined on $U \times V$, and \tilde{L} be a T_2 fuzzy relations on $W \times U$. If $\mu_{\tilde{R}}(u, v), \mu_{\tilde{S}}(u, v)$ and $\mu_{\tilde{L}}(w, u)$ are convex T_1 fuzzy sets for any $(u, v) \in U \times V$ and $(w, u) \in W \times U$, then we have

$$\tilde{L} \circ (\tilde{R} \cup \tilde{S}) = (\tilde{L} \circ \tilde{R}) \cup (\tilde{L} \circ \tilde{S})$$

Remark 2.5.1. From Theorem 5 and 6, we know that T_2 fuzzy relations satisfy distributive law under \cup and \circ . However, the distributive law will not hold under \cap and \circ .

Consequence 2.5.1.

$$(\tilde{R} \cap \tilde{S}) \circ \tilde{L} \neq (\tilde{R} \circ \tilde{L}) \cap (\tilde{S} \circ \tilde{L})$$

2.6 Alpha Planes

An α plane for a generalized type-2 fuzzy set [18] \tilde{A} is denoted by \tilde{A}_α and can be defined as the union of all primary membership functions of \tilde{A} , where the secondary membership degrees are greater or equal to α ($0 \leq \alpha \leq 1$) and can be represented by (2.10) [42], [53], and can be graphically illustrated in Fig. 2.4.

$$\begin{aligned} \tilde{A}_\alpha &= \{(x, u), \mu_{\tilde{A}}(x, u) \geq \alpha \mid \forall x \in X, \forall u \in J_x \subseteq [0, 1]\} \\ &= \int_{\forall x \in X} \int_{\forall u \in J_x} \{(x, u) \mid \mu_{\tilde{A}}(x, u) \geq \alpha\} \end{aligned}$$

where the union of all the α alpha planes is represented by (2. 11). and $R_{\tilde{A}_\alpha}$ represents a horizontal slice. In Fig. 2.3 we can appreciate the α alpha planes made for generalized membership functions.

Also in Fig. 2.4 we can appreciate the representation of the three α planes in different points, the first α plane was realized in the point 0.17 (blue), the second α plane was in 0.46 (green), and the third in point 0.99 (red).

$$\tilde{A} = \bigcup_{\alpha \in [0,1]} R_{\tilde{A}_\alpha} \quad (2.20)$$

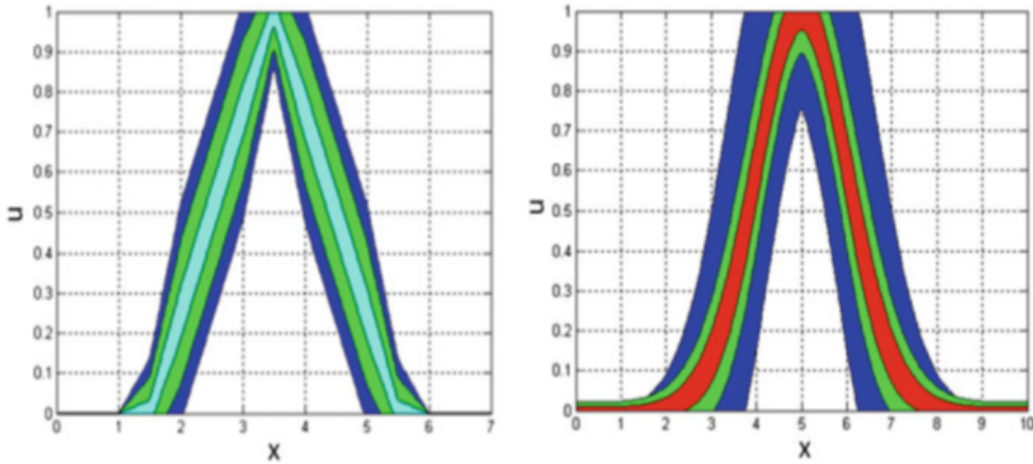


Figure 2.6: α plane in generalized type-2 membership functions[18]

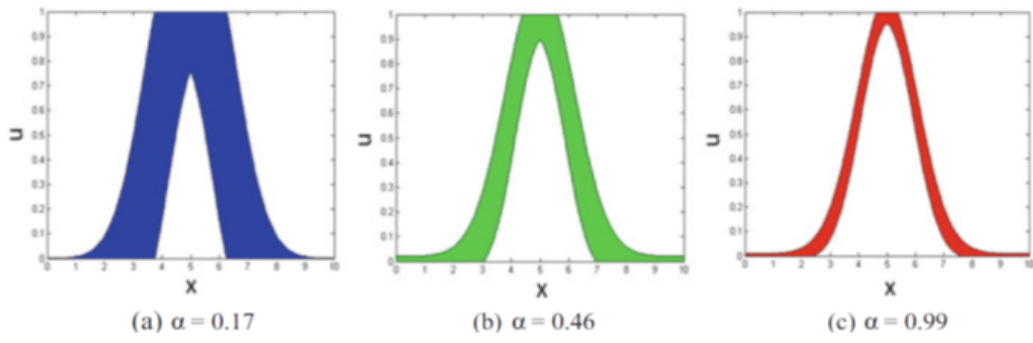


Figure 2.7: Cuts at certain points of the GMF[18]

Chapter 3

Fuzzy logic control of non-linear dynamic system

3.1 Introduction

The dynamic system is an aspect of systems theory used as a method to understand the dynamic behavior of complex systems. It has found application in a wide range of areas, for example population, agriculture, ecological and economic systems, which usually interact strongly with each other.

First, some notions are defined in this chapter which are the dynamic system and its control and the fuzzy control system. Then, we have presented Puma 560 robot modeling. In the end, a detailed study of some works of fuzzy logic controller of the non-linear dynamic system.

3.2 Dynamic system definition

3.2.1 General definition

A dynamic system is a system the state of which evolves with time over a state space according to a fixed rule. A dynamic system is a system or process in which motion occurs or includes active forces as opposed to static conditions with no motion. A dynamic system is a system that is constantly changing like the human body. The dynamic system stands to become static or at an equilibrium state. For example, if a car is assumed to be a dynamic system, then it requires fuel to continue moving forward or else it would come to a stop and become static. Dynamic systems output depends upon future and past values.

3.2.2 Formal definition of dynamical system

A dynamical system is formally defined as a state space X , a set of times T , and a rule R that specifies how the state evolves with time. The rule R is a function the domain of which is $X \times T$ and the codomain of which is X , i.e., $R : X \times T \rightarrow X$. The rule function R means that the R takes two inputs, $R = R(x, t)$, where $x \in X$ is the initial state (at time $t = 0$, for example) and $t \in T$ is a future time. In other words, $R(x, t)$ gives the state at time t knowing that the initial state was x

3.3 The dynamic system control

the control of a dynamic system was originally applied to a system that controls a mechane or motor. It is used to describe a control system in which a physical variable is required to follow or track.

3.4 Advantages of using Fuzzy Logic in Control Systems

A control system is an arrangement of physical components designed to alter another physical system so that it exhibits certain desired characteristics. Some reasons for using Fuzzy Logic in Control Systems are as follows:

- While applying traditional control, one needs to know about the model and the objective function formulated in precise terms. This makes it very difficult to apply in many cases.
- By applying fuzzy logic for control we can utilize the human expertise and experience for designing a controller.
- The fuzzy control rules, basically the IF-THEN rules, can be best utilized in designing a controller.

3.5 Fuzzy logic control

[4] The basic structure of a fuzzy logic control (FLC) system is shown in Figure 3.1. Fuzzy logic controller (FLC) used in this thesis is Mamdani.

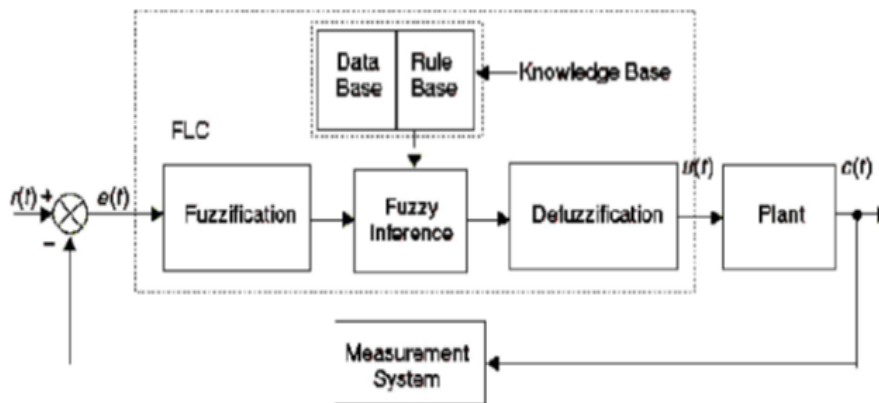


Figure 3.1: Block diagram of joint space control

3.5.1 The fuzzification process

Fuzzification is the process of mapping inputs to the FLC into fuzzy set membership values in the various input universes of discourse. Decisions need to be made regarding

- (a) Number of input

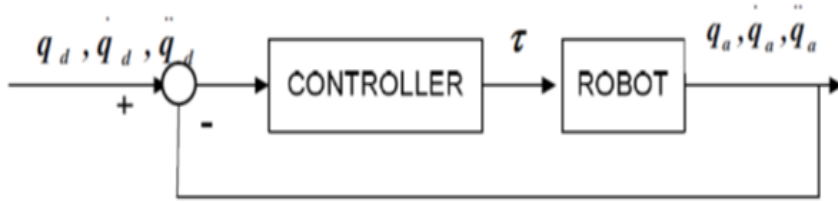


Figure 3.2: Fuzzy logic control
[60]

- (b) Size of universes of discourse
- (c) Number and shape of fuzzy sets.

A FLC that emulates a PD controller will be required to minimize the error $e(t)$ and the rate of change of error de/dt , or ce .

Each set is given a linguistic label to identify it, such as Positive Big (PB), Positive Medium (PM), Positive Small (PS), About Zero (Z), Negative Small (NS), Negative Medium (NM) and negative Big (NB). The seven set fuzzy input windows for e and ce are shown in Figure 3.3. If at a particular instant, $e(t) = 2.5$ and $de/dt = ce = -0.2$, then, from Figure 3.3, the input fuzzy set membership values are:

$$\begin{aligned} \mu_{PS}(e) &= 0.7 & \mu_{PM}(e) &= 0.4 \\ \mu_{NS}(ce) &= 0.6 & \mu_Z(ce) &= 0.3 \end{aligned} \quad (3.1)$$

3.5.2 The fuzzy rule base

The fuzzy rule base consists of a set of antecedent-consequent linguistic rules of the form

$$\text{IF } e \text{ is PS AND } ce \text{ is NS THEN } u \text{ is PS} \quad (3.2)$$

The size of the universes of discourse will depend upon the expected range (usually up to the saturation level) of the input variables. Assume for the system about to be considered that e has a range of ± 6 and ce a range of ± 1 .

The number and shape of fuzzy sets in a particular universe of discourse is a trade-off between the precision of control action and real-time computational complexity. In this example, seven triangular sets will be used.

This style of a fuzzy conditional statement is often called a Mamdani-type rule, after Mamdani (1976) who first used it in a fuzzy rule base to control steam plant.

The rule base is constructed using a priori knowledge from either one or all of the following sources:

- (a) Physical laws that govern the plant dynamics
- (b) Data from existing controllers
- (c) Imprecise heuristic knowledge obtained from experienced experts.

If (c) above is used, then knowledge of the plant mathematical model is not required. The two seven set fuzzy input windows shown in Figure 3.3 gives a possible 7×7 set of control rules of the form given in equation (3.2). It is convenient to tabulate the two-dimensional rule base as shown in table 3.1.

3.5.3 Fuzzy inference

[74] Table 3.1 assumes that the output window contains seven fuzzy sets with the same linguistic labels as the input fuzzy sets. If the universe of discourse for the control signal $u(t)$ is ± 9 , then the output window is as shown in Figure 3.4.

Assume that a certain rule in the rule base is given by equation (3.1)

$$\text{IF } e \text{ is } A \text{ AND } ce \text{ is } B \text{ THEN } u = C \quad (3.3)$$

From equation (1.4) the Boolean OR function becomes the fuzzy max operation, and from equation (1.5) the Boolean AND function becomes the fuzzy min operation. Hence equation (3.3) can be written in the following:

$$\mu_C(u) = \max[\min(\mu_A(e), \mu_B(ce))] \quad (3.4)$$

equation (3.4) is referred to as the max-min inference process or max-min fuzzy reasoning,

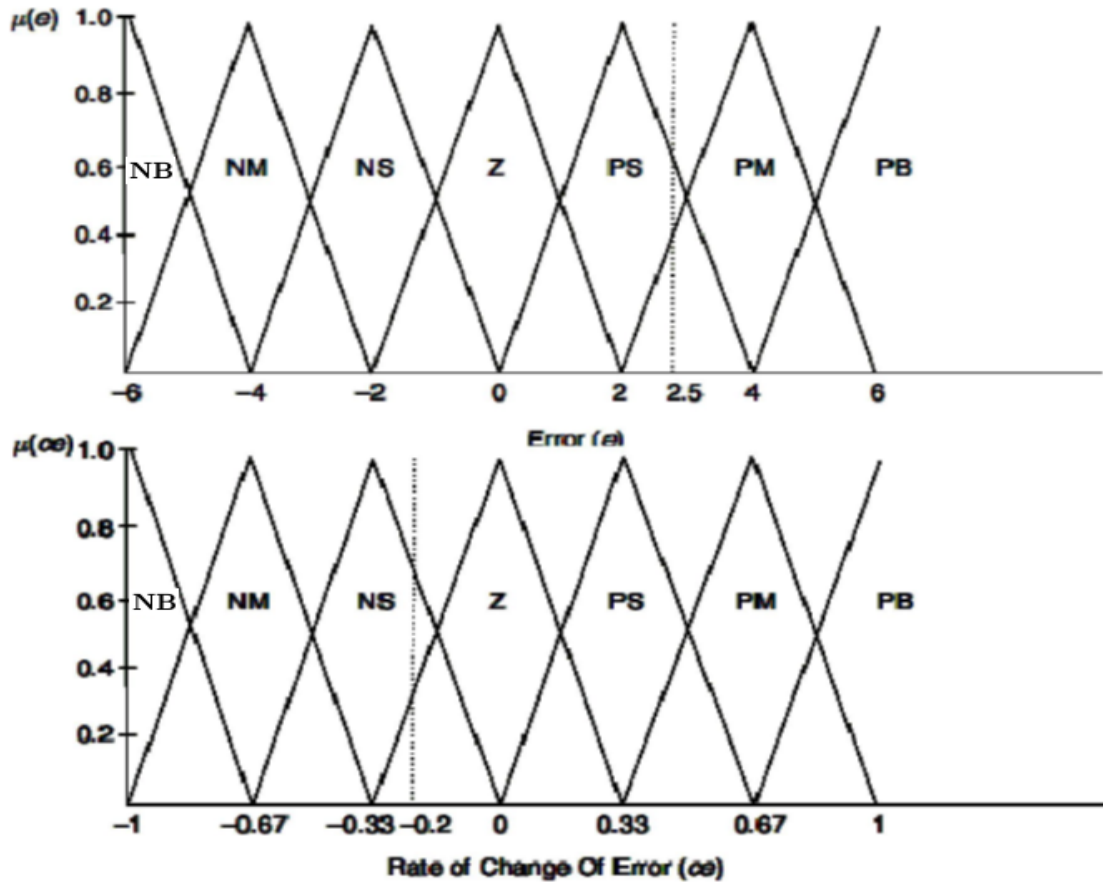


Figure 3.3: Seven set fuzzy input windows for error e and errata change of error (ce)

In Figure 3.3 and equation (3.1) the fuzzy sets that were 'hit' in the error input window when $e(t) = 2.5$ were PS and PM. In the rate of change input window when $ce = -0.2$, the fuzzy sets to be 'hit' were NS and Z. From table 3.1. the relevant rules that correspond to these 'hits' are:

$$\begin{aligned} &\dots \text{OR IF } e \text{ is } PS \text{ AND } ce \text{ is } NS \\ &\text{OR IF } e \text{ is } PS \text{ AND } ce \text{ is } Z \\ &\text{THEN } u = PS \end{aligned} \quad (3.5)$$

Table 3.1: Tabular structure of a linguistic fuzzy rulebase

$\begin{matrix} e \\ \backslash \\ ce \end{matrix}$	NB	NM	NS	Z	PS	PM	PB
NB	NB	NB	NB	NM	Z	PM	PB
NM	NB	NB	NB	NM	PS	PM	PB
NS	NB	NB	NM	NS	PS	PM	PB
Z	NB	NM	NS	Z	PS	PM	PB
PS	NB	NM	NS	PS	PM	PB	PB
PM	NB	NM	NS	PM	PB	PB	PB
PB	NB	NM	Z	PM	PB	PB	PB

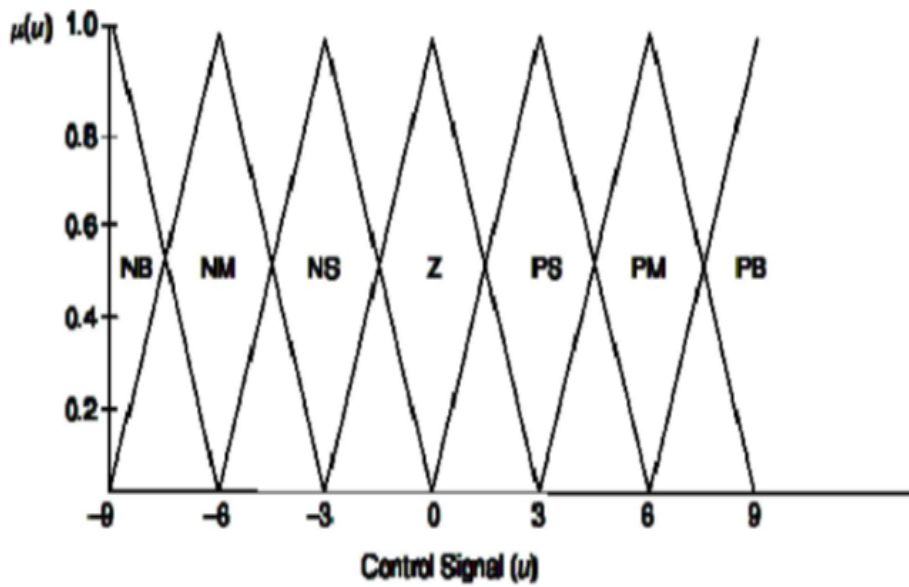


Figure 3.4: Seven set fuzzy output window for signal control (u)

$$\begin{aligned}
 & \dots \text{ OR IF } e \text{ is PM AND } ce \text{ is NS} \\
 & \text{ OR IF } e \text{ is PM AND } ce \text{ is Z} \\
 & \text{ THEN } u = \text{PM}
 \end{aligned} \tag{3.6}$$

Applying the max-min inference process to equation (3.5)

$$\mu_{PS}(u) = \max[\min(\mu_{PS}(e), \mu_{NS}(ce)), \min(\mu_{PS}(e), \mu_z(ce))] \tag{3.7}$$

Inserting values from equation (3.1)

$$\begin{aligned}
 \mu_{PS}(u) &= \max[\min(0.7, 0.6), \min(0.7, 0.3)] \\
 &= \max[0.6, 0.3] = 0.6
 \end{aligned} \tag{3.8}$$

Applying the max-min inference process to equation (3.6)

$$\mu_{PM}(u) = \max[\min(\mu_{PM}(e), \mu_{NS}(ce)), \min(\mu_{PM}(e), \mu_z(ce))] \tag{3.9}$$

Inserting values from equation (3.1)

$$\begin{aligned}
 \mu_{PM}(u) &= \max[\min(0.4, 0.6), \min(0.4, 0.3)] \\
 &= \max[0.4, 0.3] = 0.4
 \end{aligned} \tag{3.10}$$

Fuzzy inference is, therefore, the process of mapping membership values from the input windows, through the rule base, to the output window(s).

3.5.4 The defuzzification

Defuzzification is the procedure for mapping from a set of inferred fuzzy control signals contained within a fuzzy output window to a non-fuzzy (crisp) control signal. The center of area [68], [69], [12] method is the most famous defuzzification technique, which in linguistic terms can be expressed as follows:

$$\text{Crisp control signal} = \frac{\text{Sum of first moments of area}}{\text{Sum of areas}} \quad (3.11)$$

For a continuous system, equation (3.11) becomes:

$$u(t) = \frac{\int u\mu(u)du}{\int \mu(u)du} \quad (3.12)$$

Or alternatively, for a discrete system, equation (3.11) can be expressed as:

$$u(t) = \frac{\sum_{i=1}^n u_i \mu(u_i)}{\sum_{i=1}^n \mu(u_i)} \quad (3.13)$$

For the case when $e(t) = 2.5$ and $ce = -0.2$, as a result of the max-min inference process (equations (3.8) and (3.10)), the fuzzy output window in Figure 3.4 is 'clipped', and takes the form shown in Figure 3.5.

From Figure 3.5, using the equation for a trapezoid area:

$$\begin{aligned} \text{Area}_{PS} &= \frac{0.6(6+2.4)}{2} = 2.52 \\ \text{Area}_{PM} &= \frac{0.2(6+3.6)}{2} = 0.96 \end{aligned} \quad (3.14)$$

From equation (3.11)

$$u(t) = \frac{(2.52 \times 3) + (0.96 \times 6)}{2.52 + 0.96} = 3.83 \quad (3.15)$$

Hence, the error of 2.5, and an errata change of error of -0.2 , the control signal from the fuzzy controller is 3.83.

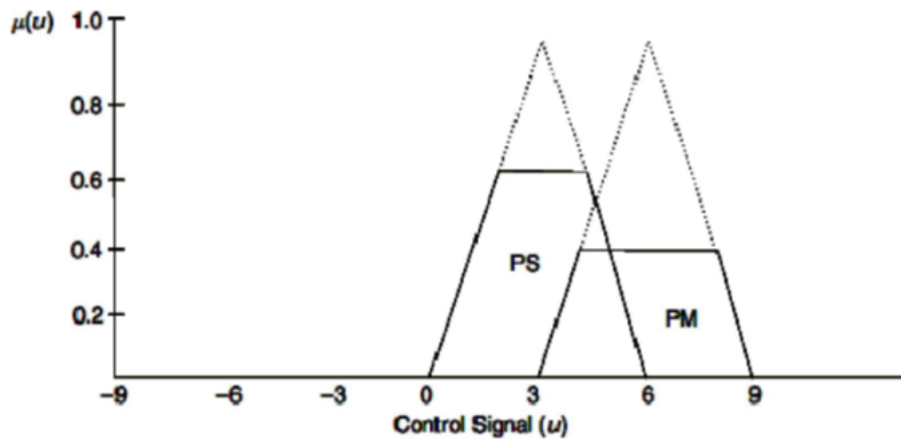


Figure 3.5: Clipped fuzzy output window due to fuzzy inference

Example 3.5.1. For the input and output fuzzy windows given in Figure 3.3 and 3.4, together with the fuzzy rule base shown in table 3.1, determine:

(a) The membership values of the input windows e and ce .

(b) The max-min fuzzy inference equation

(c) The crisp control signal $\mu(t)$.

when $e = -3$ and $ce = 0.3$

Solution

(a) When $e = -3$ and $ce = 0.3$ are mapped onto the input fuzzy windows, they refer to a fuzzy singletons. From Figure 3.3

$$e = -3 \quad \mu_{NS}(e) = 0.5 \quad \mu_{NM}(e) = 0.5 \quad (3.16)$$

$ce = 0.3$, using similar triangles :

$$\frac{1}{0.33} = \frac{\mu_z(ce)}{(0.33-0.3)} \quad (3.17)$$

$$\mu_z(ce) = 0.09$$

And

$$\frac{1}{0.33} = \frac{\mu_{PS}(ce)}{0.3} \quad (3.18)$$

$$\mu_{PS}(ce) = 0.91$$

(b) The rules that are 'hit' in the rule base in table 3.1 are

$$\begin{aligned} &\dots \text{OR IF } e \text{ is NS AND } ce \text{ is Z} \\ &\text{OR IF } e \text{ is NS AND } ce \text{ is PS} \\ &\text{THEN } u = \text{NS} \end{aligned} \quad (3.19)$$

$$\begin{aligned} &\dots \text{OR IF } e \text{ is NM AND } ce \text{ is Z} \\ &\text{OR IF } e \text{ is NM AND } ce \text{ is PS} \\ &\text{THEN } u = \text{NM} \end{aligned} \quad (3.20)$$

Applying max-min inference to equation (3.19)

$$\mu_{NS}(u) = \max[\min(\mu_{NS}(e), \mu_z(ce)), \min(\mu_{NS}(e), \mu_{PS}(ce))] \quad (3.21)$$

Inserting values into (3.21) :

$$\begin{aligned} \mu_{NS}(u) &= \max[\min(0.5, 0.09), \min(0.5, 0.91)] \\ &= \max[0.09, 0.5] = 0.5 \end{aligned} \quad (3.22)$$

and similarly with equation (3.20)

$$\begin{aligned} \mu_{NM}(u) &= \max[\min(\mu_{NM}(e), \mu_z(ce)), \min(\mu_{NM}(e), \mu_{PS}(ce))] \\ \mu_{NM}(u) &= \max[\min(0.5, 0.09), \min(0.5, 0.91)] \\ &= \max[0.09, 0.5] = 0.5 \end{aligned} \quad (3.23)$$

Using equations (3.22) and (3.23) to 'clip' the output window in Figure 3.4, the output window is now as illustrated in Figure 3.6.

(c) Due to the symmetry of the output window in Figure 3.6, from observation the crisp control signal is :

$$u(t) = -4,5$$

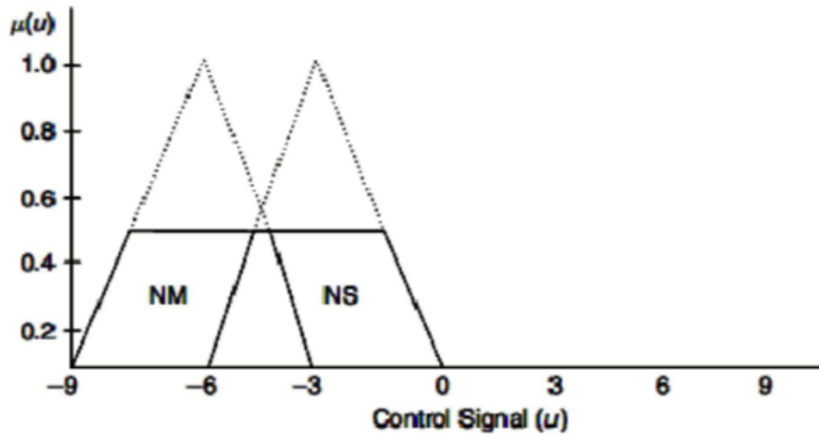


Figure 3.6: Fuzzy output window for to Example 3.5.1.

3.6 Puma560 robot dynamic modelling

This paragraph describes the PUMA 560 industrial robotic system that will be used in our research work as an example of dynamic system ; highlighting its main components , how they interact with each other and its dynamic modelisation.

3.6.1 General Overview of PUMA 560 Robot

[29]

The PUMA 560 robotic manipulator, produced by Unimate, is an old (> 25 years), but functional multi-joint robot and was used in this project, due to spare parts availability that are required for simulation of the different faults. It is a PC controlled serial manipulator designed for use in industrial applications, and has six revolute joints/degree of freedom (DOF) with three major axes of motion (X, Y , and Z) and resembles the human arm in function. Each of the robot's joints is actuated by a DC brushed permanent magnet servo motor. Positioning of the end effector, which is normally a device fixed at the end of the robotic arm, is achieved by the coordination of the first three joints named: waist (joint 1), shoulder (joint 2) and elbow (joint 3), allowing the robot to move the end effector into any position with maximum reach of 1 m. Orientation of the end effector is important when it approaches its final position and this is achieved using three wrist joints (4, 5, and 6) where the tool can be independently manoeuvred. This gives the robot six degrees of freedom as shown in Fig. 3.7 where the name of each joint with its maximum rotating range is indicated. Electromagnetic brakes are equipped with the first three joints (waist, shoulder, and elbow), which lock the motors to prevent collapsing when the power is removed from the robot (Potgieter et al. 2005). The control unit of the PUMA 560 is the most important part of the robotic system. It controls all operations of the manipulator and any connected device. There is two-way communication to each of the joints motors where sensory

information on position and speed is fed back to the control unit; outputs are sent to the motors related to the task's requirements. External digital input/output ports are available allowing other sensors and actuators to be controlled from this unit. There is a number of ways to position and orientate the manipulator end effector, but generally, one method is used at a time.

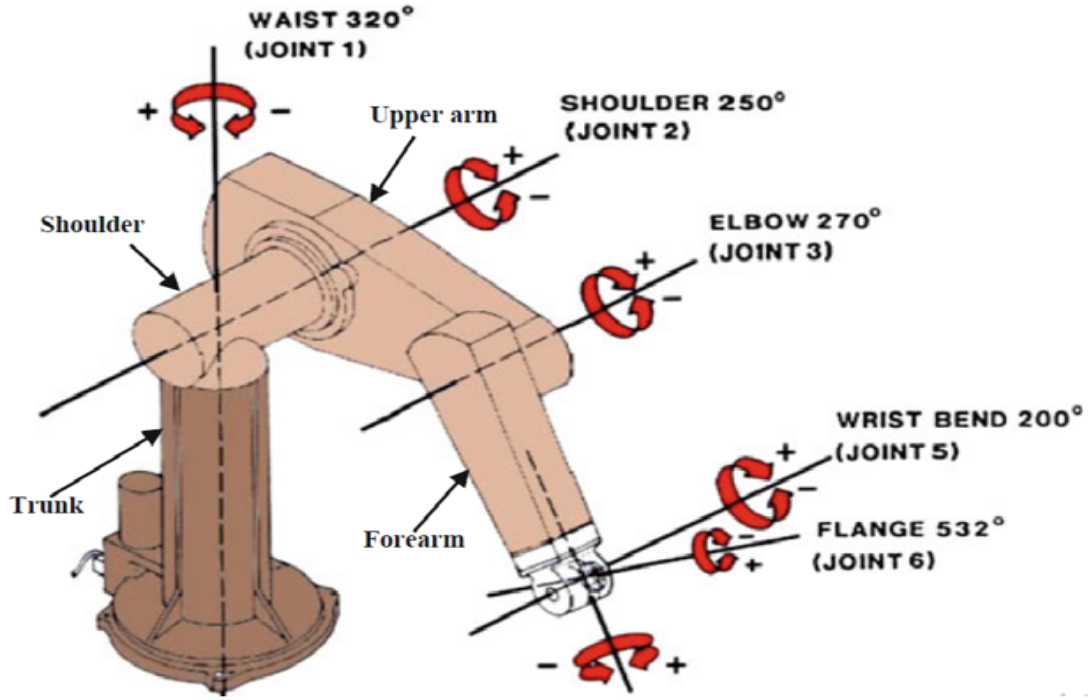


Figure 3.7: PUMA 560 robot member representations [61]

3.6.2 Puma 560 robot Dynamic Modelling

The dynamic model used for PUMA560 is taken from [3]. We have adopted a robot of three degrees of freedom and the configuration of space equation is given by:

$$M(q)\ddot{q} + B(q)[\dot{q}\dot{q}] + C(q)[\dot{q}^2] + G(q) = \Gamma \quad (1)$$

where:

$M(q)$: is 3×3 kinetic energy matrix.

$[\ddot{q}]$: 3×1 vector given by: $\ddot{q} = [\ddot{q}_1, \ddot{q}_2, \ddot{q}_3]^T$

$B(q)$: 3×1 matrix of Coriolis torques,

$[\dot{q}\dot{q}]$: 7×1 vector of joint velocity products given by:

$[\dot{q}\dot{q}] = [\dot{q}_1\dot{q}_2, \dot{q}_1\dot{q}_3, 0, 0, 0, \dot{q}_2\dot{q}_3, 0]^T$

$C(q)$: 3×1 vector given by: $[\dot{q}^2] = [\dot{q}_1^2, \dot{q}_2^2, \dot{q}_3^2]^T$

$G(q)$: is 3×1 vector of gravity torques,

Γ : 3×1 vector of joint torque.

\ddot{q} is the n-vector of accelerations;

3.7 Definition of a robot system controller

The controller is a device which can sense information from the linear or nonlinear system (e.g., robot manipulator) to improve the system's performance [9]. The main targets in designing control systems are stability, good disturbance rejection, and small tracking error [27]. Several industrial robot manipulators are controlled by linear methodologies (e.g., Proportional-Derivative (PD) controller, Proportional- Integral (PI) controller or Proportional- Integral-Derivative (PID) controller), but when the robot manipulator works with various payloads and has uncertainty in dynamic models this technique has limitations.

From the control point of view, uncertainty is divided into two main groups: uncertainty in unstructured inputs (e.g., noise, disturbance) and uncertainty in dynamics structure (e.g., payload, parameter variations). In some applications robot manipulators are used in an unknown and unstructured environment, therefore strong mathematical tools are used in new control methodologies to design nonlinear robust controller with an acceptable performance (e.g., minimum error, good trajectory, disturbance rejection). Joint space and operational space control are closed loop controllers that have been used to provide robustness and rejection of disturbance effect. The main target in the joint space controller is design a feedback controller that allows the actual motion ($q_a(t)$) tracking of the desired motion ($q_d(t)$) by inverse kinematics of robot manipulators [46]. Figure 3.8 shows the main block diagram of joint space controller. The main target in the operational space controller is to design a feedback controller to allow the actual end-effector motion $X_a(t)$ to track the desired end effector motion $X_d(t)$. This control methodology requires a greater algorithmic complexity and the inverse kinematics used in the feedback control loop as well. Direct measurement of operational space variables is very expensive that led to limitate the controller use in industrial robot manipulators[46]. Figure 3.9 shows the main block diagram of operational space control.

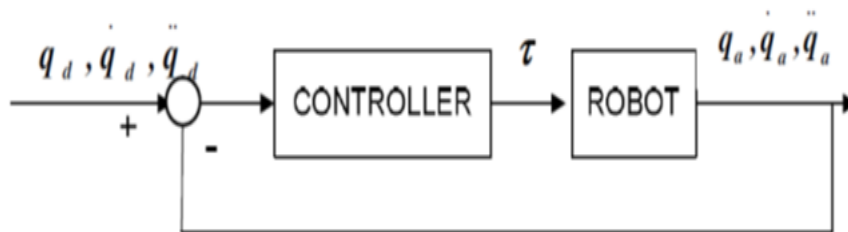


Figure 3.8: Block diagram of joint space control [60]

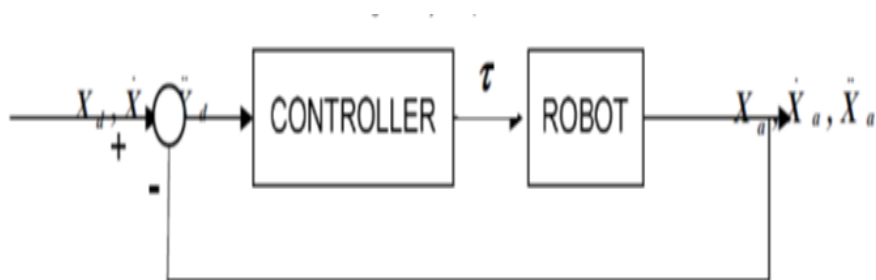


Figure 3.9: Block diagram of operational space control [60]

3.8 State of the art of fuzzy logic controller of nonlinear dynamic system

The subject of a fuzzy logic controller has attracted the attention of researchers, plenty of research works have been done.[15].-12 [39] The use of industrial robots became identifiable as a unique device in the 1960s. Since then, the field of application evolved from rather simple tasks like welding and painting to those requiring more precision, such as assembly tasks.

Control theory provides tools for designing and evaluating algorithms to achieve desired motions or force application. The methods of linear control and those of local linearization and moving linearization are not well suited for the control problem of robotic arms. This is due to the fact that robotic arms constantly move among widely separated regions of their workspace such that no valid linearization can be found in all regions. On the other hand, nonlinear control methods are progressing nowadays and different classes could be identified[65]: Trialand- Error, Feedback Linearization Control, Robust Control, Adaptive Control, and Gain-Scheduling. Nonlinear control methods used in robot arms' applications should, however, face the major difficulty resulting from the dynamic modeling of robots, i.e. the indetermination of their parameters [9]. Preferred methods are those which reduce or eliminate the undesired effects generated by this indetermination such as the Feedback Linearization control method [65], the Model-Reference Adaptive control method [[66],[65]] and the Self-Tuning method[65]. Another difficulty in robot arm control is the coupling effects of the Coriolis and centrifugal forces that might be canceled in a single axis mode operation where the joints are activated sequentially. Existing methods of nonlinear control are also used in robotics in order to eliminate the above-mentioned coupling effect like the Individual Joint PID control method [9] and the PD-plus-gravity control method [9]. Among the recent nonlinear control methods, fuzzy control methods grab nowadays the attention of many researchers. In fact, these methods do not require the knowledge of the dynamic model of the controlled system and this feature has become of major importance when dealing with complex nonlinear systems. Moreover, the dynamic modeling of robot arms shows a dependency on their mechanical parameters, subject to lifetime modifications (friction factors affected by the abuse of joints), and on their dynamical parameters that vary with the performed task (centers of gravity of the links affected by tool's replacements). These considerations also give advantage to fuzzy control methods on other nonlinear methods as a result of their robustness towards perturbations affecting the system.

The first fuzzy logic controller was introduced by Mamdani in 1974[44]. It is equivalent to two-input fuzzy PI controllers, where error and change of error were used as the inference system inputs. Mamdani's work also introduced the most common and robust fuzzy reasoning method called "Zadeh-Mamdani min-max gravity reasoning". Different comparative studies, like [27], prove that Zadeh-Mamdani min-max gravity scheme is the best reasoning scheme if the nonlinearity variation is the main concern. It also introduced the most common and robust fuzzy reasoning method, called Zadeh-Mamdani min-max gravity reasoning. [40] Robots are mechanized devices that have a certain degree to duplicate the working of humans, whenever they need continuous operations. Furthermore, they provide more accuracy, high strength and reduce the danger in the case of medical fields, industrial operations and nuclear plants (Spong & Vidyasagar, 2008). Controlling the motion of a manipulator robot's joints is a running issue the study of which is of great importance in order to make its terminal effectors follow a predefined trajectory with the minimum deviation (Spong & Fujita, 2011). Robotic manipulator systems are containing unavoidable uncertainties such as structured and unstructured uncertainties that degraded manipulators

performance. Correct dynamical model with variation parameters, sizes and mass distributions of payloads, difference in links properties of robotic manipulators are defined as structured uncertainty present in robotic manipulators. Unstructured uncertainty is defined as unmodelled dynamics, which contain the external disturbances, nonlinear frictions and random noise (Song, Yi, Zhao, & Li, 2005). To combat such challenges, it is essential to develop a robust controller for these systems. Controlling of robot manipulator requires the complex dynamic equation (mathematical modeling) and trajectory planning, which is computationally intensive. The conventional control theories such as PID have been successfully applied to areas where systems are well defined. However, when the system is complex, non-linear, ill-structured process with excessive variation parameter, the potency of conventional control becomes poor. With technology advancement, there are imperative needs to design controllers that are able to manage structured and unstructured uncertainty (Laxmidhar & Kar, 2010). The fuzzy set technique given by Zadeh has become a powerful modeling tool for solving severe real-world problems with an uncertain and unpredictable environment. The FLCs (fuzzy logic controllers) come in the category of intelligent control and as an intelligent controller, FLCs (type-1 fuzzy logic controllers i.e., T1FLCs) parameter is easily tuned by non-expertise person. The operation of T1FLCs is based on human expertise and the knowledge acquisition techniques to convert human expertise to appropriate if-then rules as well as a proper fuzzy membership function for each fuzzy variable. The T1FLC has some important advantages such as, [23] it provides a higher level of automation by incorporating expert knowledge, [10] it does not require exact knowledge of the controlled system dynamics model, [14] it plays a major role while controlling complex non-linear systems, [6] it reduces development and maintenance time. There are some drawbacks of T1FLCs, i.e. it can't fully deal with or handle the linguistic and numerical uncertainties connected with dynamic structured or unstructured environments. In addition to that, its performance is not satisfactory, because the ordinary T1FLCs have limited capabilities to directly handle data uncertainties (Mendel, 2007). There are generally five sources of uncertainties observed in T1FLC (Hagras, 2004, 2007; Mendel & John, 2002), which are listed here [23] Uncertainties in the inputs to the T1FLC are produced by noise and change in environmental conditions of sensors. These uncertainties are translated into uncertainties in the antecedents membership functions (MFs) and that the input sensors can be affected by the conditions of observation i.e., their characteristics can be changed by the environmental conditions such as wind, sunshine, humidity, rain, etc. [10] The output of T1FLC is applied to actuators to control the plant.

[15] Generation of a short and smooth path in three-dimensional space with obstacles for guiding an Unmanned Underwater Vehicle (UUV) without collision is investigated. This is done by utilizing the spline technique, in which the spline control points positions are determined by Imperialist Competitive Algorithm (ICA) in three-dimensional space such that the shortest possible path from the starting point to the target point without colliding with obstacles is achieved. Furthermore, for guiding the UUV in the generated path, an Interval Type-2 Fuzzy Logic Controller (IT2FLC), the coefficients of which are optimized by considering an objective function that includes quadratic terms of the input forces and state error of the system, is used. Selecting such objective function reduces the control error and also the force applied to the UUV, which consequently leads to a reduction of energy consumption. Therefore, by using a special method, desired signals of UUV state are obtained from a generated three-dimensional optimal path such that tracking these signals by the controller leads to the tracking of this path by UUV. In this thesis, the dynamical model of the UUV, entitled as "mUUV-WJ-1", is derived and its hydrodynamic coefficients are calculated by CFD in order to be used in the simulations. concerning simulation by the

method presented in this study, three environments with different obstacles are intended in order to check the performance of the IT2FLC controller in generating optimal paths for guiding the UUV. In this article, besides to ICA, Particle Swarm Optimization (PSO) and Artificial Bee Colony (ABC) are also used for the generation of the paths and the results are compared with each other. The results show the appropriate performance of ICA rather than ABC and PSO. Moreover, to evaluate the performance of the IT2FLC, optimal Type-1 Fuzzy Logic Controller (T1FLC), and Proportional Integrator Differentiator (PID) controller are designed and applied to the UUV then compared to each other. The simulation results show the superiority of the IT2FLC over the other two controllers.

[74] Interval type-2 fuzzy logic controllers (IT2 FLCs) have recently been attracting a lot of research attention. Many reported results have shown that IT2 FLCs are more capable to handle uncertainties than their type-1 (T1). For example, Hagra implemented a hierarchical IT2 FLC for different types of mobile robots navigating in indoor and outdoor environments. It outperformed a T1 FLC and had significantly fewer rules. Wu and Tan showed through both simulation and experiments that IT2 FLCs are more performing in coping with modeling uncertainties, and hence, IT2 FLCs optimized from simulations are more likely to perform well on the actual plant than T1 FLCs.

[19] After the inception of robotics, the robot manipulators are widely used as the promising device in the automated industrial applications including nuclear and medical fields. These manipulators are incorporated to enhance flexibility, productivity, and accuracy in accordance with reduced working cost. It also increased human working conditions where repetitive and hazardous works are executed [67], [8]. For exceptional utility, it is entailed to meticulously design efficient robust controllers that can command the robotic motion in a precise way. Some preliminary work had stated that the conventional proportional- integral-derivative (PID) controllers were used in industrial applications because of their elementary design, easy implementation, and low cost, whereas, recent studies in this field found that they are ineffective for complicated systems such as robotic manipulators with uncertainty and nonlinearity. Progressively, various control strategies and controller structures have been suggested in the past few decades to enhance the performance of systems under control, within this framework. The incorporation of fuzzy logic systems (FLS) in control theory, producing fuzzy logic controllers (FLC), has enhanced the flexibility of controller design and increased its applicability to control the complex, nonlinear, ill-structured, and uncertain systems. Thereafter, the FLC (later, called as type-1 fuzzy logic controllers T1-FLC) have received considerable interest and the control designers recommend T1- FLC over the conventional counterparts because of their unique characteristics such as (a) incorporation of knowledge based on human expertise, (b) no exact dynamic model of system is required hence useful when precise mathematical formulation is infeasible, (c) low development and maintenance cost, and importantly, (d) the ability of general framework of T1-FLC to handle uncertainty, etc. In literature, various authors proposed different T1-FLC structures and their extensions for numerous types of robotic applications. Subsequently, over the past few years, a growing number of researchers have proposed various designs and strategies of T1-FLCs and investigated them for numerous applications for developing effective controllers [37] The usage of the flexible joint manipulator (FJM) as an alternative with rigid body manipulators in the robotics industry such as service, space, medical and defense because the FJM provides faster movements, low manufacturing cost and, high weight to payload ratio as well as low energy consumption compared to the rigid-body structures. Despite these advantages, the flexibility of the joint cause vibrations (link deflections) and position errors. However, convenient control approaches may overcome these problems in the FJM. During the last decade, various linear and non-linear control methods using model-based

and model-free approaches have been presented for the trajectory and vibration control of the FJM. Proportional-Integral-Derivative (PID) state feedback, Linear Quadratic Regulator (LQR) and Linear Quadratic Gaussian (LQG), Intelligent Proportional-Integral (IPI) and distributed higher-order differential feedback controllers are some of the linear control methods applied in the literature. These linear controllers naturally showed overshoot and high link deflection because of their performance was limited by to only small regions around the equilibrium or operating points of the nonlinear FJM system. **conclusion**

Chapter 4

Fuzzy logic controller for puma560 robot: simulation and results discussion

4.1 Introduction

In this chapter, we will describe in details the fuzzy logic controllers (type1 fuzzy sets and interval type2 fuzzy sets) proposed to control the PUMA560 manipulator robot then a comparison of the simulation results of both approaches is presented. As a performance measure we will use the Integral of Square Error (ISE) and Integral of Square Torque (IST)[64] Fuzzy logic controller (FLC) which is used in this research is the Mamdani method, or known as the Max-Min method.

4.2 General description of the fuzzy logic system used in control of Puma560

The dynamic parameters of Puma560 are taken from [15], see the Appendix. The simulation control system diagram of PUMA560 is shown in Figure 4.1. To check the robustness of controller, a disturbance torque is applied on the robot and a cycloid trajectory test is performed. Recall that only three links of PUMA560 robot are used in this work. The three degrees of freedom PUMA560 robot has the same configuration space equation general form in [15] 6-DOF, in which the last three joints q_4 , q_5 and, q_6 are set to zero in equation cited in [15] so they keep their initial states while the robot is moving.

We can define a general equation that allows us to use PUMA560 robot as a 3-DOF robot and its general control system represented in Figure 4.1.

Note: All parameters and Matrix in (1) are taken from paper [15], data q_4 , q_5 and q_6 are to be replaced by zero and do simplification to get 3-DOF PUMA560 robot as in equation (1).

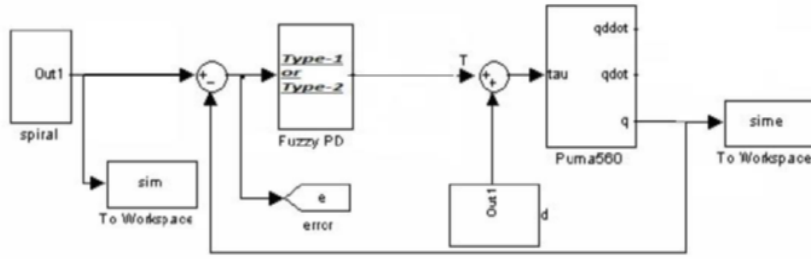


Figure 4.1: General diagram for fuzzy sets using in control of Puma 560

4.3 Controller type-1 fuzzy sets

The basic scheme of a fuzzy controller [74] is shown in Figure 4.2. The fuzzy controller has four parts; the first one is the fuzzification, which changes the crisp values to fuzzy value and determines membership degrees of the chosen PD, which needs two inputs, error and its change, and one output as a control signal. The position errors with their variations and the control signal quantized into five sets as in Figure 4.3: represented by a set of linguistic variables with triangle membership functions with $[-1, 1]$ as a universe of discourse [1], [5],[37] :

Negative Large (N-L), Negative Small (N-S), zero (Ze), Positive Small (P-S) and Positive Large (P-L).

We choose the output as singletons values $[N-L; N-S; Ze; P-S; P-L] = [-1 : -0.5; 0; 0.5; 1]$

The second part is the inference, which cannot function without rule base like that shown in Table 1 [39]. The max-min Mamdani fuzzy inference system [74] was adopted for our controller. The last part is the defuzzification in which the fuzzy inference result is changed to crisp value. There are several methods for defuzzification: in our work centroid of the area has been chosen [1], [5].

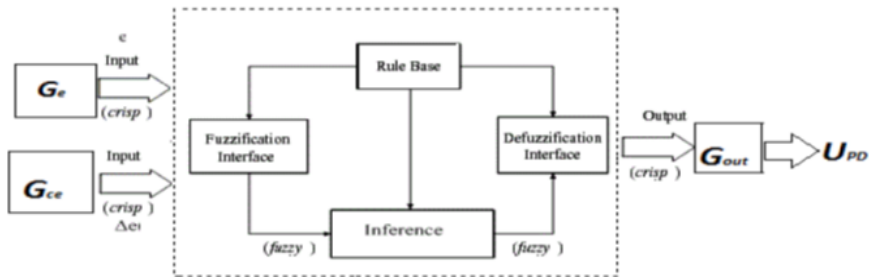


Figure 4.2: A type-1 fuzzy controller [5]

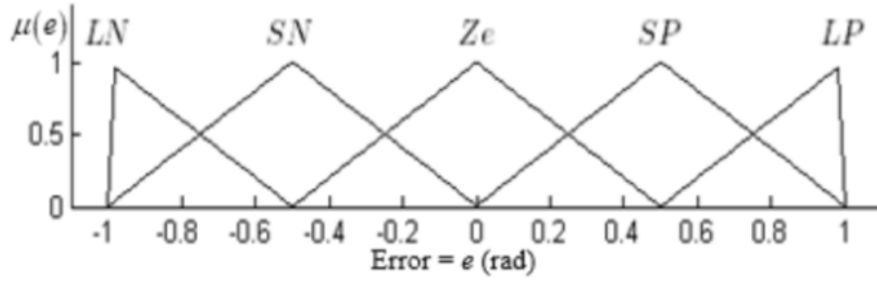


Figure 4.3: Fuzzy set for each articulation with triangle membership functions for error and its change

Table 4.1: Rulebase table[39]

		Velocity error				
		N-L	N-S	Ze	P-S	P-L
Position error	N-L	NL	NL	NL	NS	Ze
	N-S	NL	NL	NS	Ze	PS
	Ze	NL	NS	Ze	PS	PL
	P-S	NS	Ze	PS	PL	PL
	P-L	Ze	PS	PL	PL	PL

4.4 Controller type2 fuzzy sets

4.4.1 General type-2 fuzzy system

The basics of fuzzy logic do not change from type-1 to type-2 fuzzy sets, [33]. A higher-type number simply indicates a higher "degree of fuzziness", Since a higher type changes the nature of the membership functions, the operations that depend on the membership functions change; however, the basic principles of fuzzy logic are independent of the nature of membership functions and hence, do not change. Rules of inference like Generalized Modus Ponens or Generalized Modus Tollens continue to apply.

The structure of the type-2 fuzzy rules is the same as for the type-1 case because the distinction between type-2 and type-1 is associated with the nature of the membership functions. Hence, the only difference is that now some or all the sets involved in the rules are of type-2. In a type-1 fuzzy system, where the output sets are type-1 fuzzy sets, we perform defuzzification in order to get a number which is, in some sense, a crisp (type-0) representative of the combined output sets. In the type-2 case, the output sets are type-2, so we have to use extended versions of type-1 defuzzification methods. Since type-1 defuzzification gives a crisp number at the output of the fuzzy system, the extended defuzzification operation in the type-2 case gives a type-1 fuzzy set at the output. Since this operation takes us from the type-2 output sets of the fuzzy system to a type-1 set, we can call this operation "type reduction" and call the type-1 fuzzy set so obtained a "type-reduced set". The type-reduced fuzzy set may then be defuzzified to obtain a single crisp number; however, in many applications, the type-reduced set may be more important than a single crisp number [33] [55].

Type-2 sets can be used to convey the uncertainties in membership functions of type-1 fuzzy sets, due to the dependence of the membership functions on available linguistic and

numerical information. Linguistic information, in general, does not give any information about the shape of the membership functions. When membership functions are determined or tuned based on numerical data, the uncertainty in the numerical data, e.g., noise, translates into uncertainty in the membership functions. In all such cases, any available information about the linguistic numerical uncertainty can be incorporated in the type-2 framework. However, even with ailing advantages that fuzzy type-2 systems have, the literature on the applications of type-2 sets is scarce. We think that more applications of type-2 fuzzy systems will come in the near future as the area is gaining maturity and that theoretical results become more understandable for the general public in the fuzzy arena [36]

The general structure of a type-2 fuzzy controller (type-2 fuzzy controller: T2FC) is represented in Figure 4.4: This structure is similar to that of the type-1 fuzzy controller.

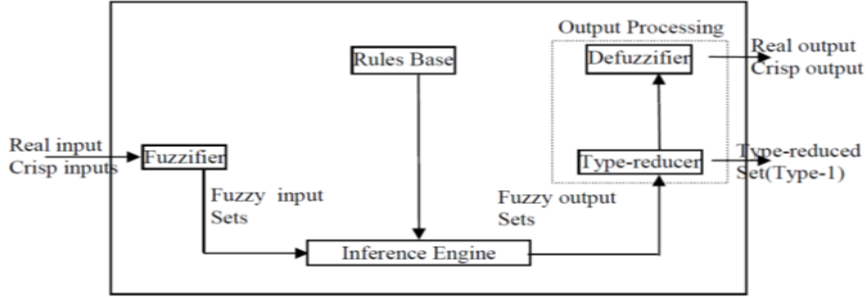


Figure 4.4: Type-2 Fuzzy Controller structure[55]

However, their differences in :

- The type of membership function used.
- Procedure of adopted defuzzifier. In a T2FC, a block of reduction of the type is essential to convert the type-2 fuzzy set to a type-1 fuzzy set.

a. Fuzzification

In this thesis, we will consider only singleton fuzzification (Crisp input), in the input fuzzy set.

b. Rules

In the type-1 case, we generally have "IF-THEN" rules, where the rule has the form

$$"R^l: \text{IF } x_1 \text{ is } F_1^l \text{ and } x_2 \text{ is } F_2^l \text{ and } \dots \text{ and } x_p \text{ is } F_p^l \text{ THEN } y \text{ is } G^l" \quad (4.1)$$

Where x_i 's are inputs, F_i^l 's are antecedent sets ($i = 1, \dots, p$), y is the output and G^l are consequent sets,

The distinction between type-1 FLS and type-2 FLS is associated with the nature of the membership function. which is not important while forming rules, hence, the structure of the rules remains exactly the same in the type-2 FLS case, the only difference being that now some or all of the sets involved are of type-2, so, the rule in a type-2 FLS has the form

$$"R^l: \text{IF } x_1 \text{ is } \tilde{F}_1^l \text{ and } x_2 \text{ is } \tilde{F}_2^l \text{ and } \dots \text{ and } x_p \text{ is } \tilde{F}_p^l \text{ THEN } y \text{ is } \tilde{G}^l" \quad (4.2)$$

Where x_i 's are inputs, \tilde{F}_i^l 's are antecedent sets ($i = 1, \dots, p$), y is the output, and \tilde{G}^l are consequent sets, It is not necessary that all the antecedents and the consequent need be type-2 fuzzy sets. As long as one antecedent or the consequent set is type-2. we will have a type-2 FLC.

c. Inference Engine

In general, the rules we use will have multiple antecedents connected by ands. Just as in the type-1 case. we can connect these multiple antecedents by the meet operation (corresponding to t-norm in the type-1 case). Different rules can be combined using the join operation (corresponding to t-norm in the type-1 case), or during defuzzification.

The output of the inference engine [33] [55] consists of the fired consequent fuzzy sets. Each one of these is modified from a consequent fuzzy set by a degree of firing. This degree of firing is obtained, In general, as a result of t-norm (meet) and t-norm (join) operations on membership grades of the inputs.

The relation (4.2) is interpreted as a fuzzy implication type-2 and defined by :

$$R^l : \tilde{F}_1^l \times \dots \times \tilde{F}_p^l \rightarrow \tilde{G}^l \quad (4.3)$$

This relation is described by the membership function in the same way as :

$$\mu_{R^l}(x_1, \dots, x_p, y) = \mu_{\tilde{F}_1^l} \times \dots \times \mu_{\tilde{F}_p^l} \rightarrow \tilde{G}^l(x_1, \dots, x_p, y) = \left[\bigcap_{i=1}^p \mu_{\tilde{F}_i^l}(x_i) \right] \cap \mu_{\tilde{G}^l}(y) \quad (4.4)$$

$\tilde{F}_1^l \times \dots \times \tilde{F}_p^l$ Denote the cartesian product of $\tilde{F}_1^l, \dots, \tilde{F}_p^l$

d. Type-Reduction

Observe, from figure 4.4, that the defuzzifier block in the type-1 FLC is replaced by two blocks: type-reducer and defuzzifier. We consider type-reduction in this subsection.

In a type-1 FLC. where the output sets are type-1 fuzzy sets, we perform defuzzification in order to get a number which is in some sense a crisp (type-0) representative of the combined output sets. In the type-2 case, the output sets are type-2, so we have to use "the extended principle" of type-1 defuzzification methods. Since type-1 defuzzification gives a crisp number at the output of the FLS, the extended defuzzification operation in the type-2 case gives a type-1 fuzzy set at the output. When this operation takes us from the type-2 output sets of the FLC to a type-1 set, we call this operation "type-reduction" [33], [36] and call the type-1 set so to obtain a single crisp number, however, in many application, the type reduced set may be more important than a single crisp number.

There exists a number of type-reduction kinds, such as centroid, center-of-sets, height and modified height, the details of which are given in [33] [36]. In this work, for illustrative purposes, we focus on center-of-sets type-reduction.

e. Center-of-sets type-reduction

In this method [36], each set of the consequence \tilde{G}^l is replaced by its centroid. If the set of output \tilde{G}^l is of type-2, its centroid $C_{\tilde{G}^l}$ is a type-1 fuzzy set. Then the weighted average of all the centroid is calculated, associating with each centroid $C_{\tilde{G}^l}$, a weight equals the degree of activation of the rule is given by $E_i(x) = \bigcap_{i=1}^p \mu_{\tilde{F}_i^l}(x)$. The procedure of calculation of the type- reduced set $Y_a(x)$ is:

1. Discretize the output space Y into a suitable number of points, and compute the centroid $C_{\tilde{G}^l}$ of each consequent set on the discretized output space. These consequent centroid sets can be computed ahead of time and stored for future use.
2. Compute the degree of firing, $E_l(x)$ associated with the consequent.

3. Discretize the domain of each $C_{\tilde{G}^l}$ into a suitable number of M_l points, $l = 1, \dots, M$.
4. Discretize the domain of each $E_l(x)$ into a suitable number of points, say $N_l, l = 1, \dots, M$.
5. Enumerate all the possible combinations $\{c_1, \dots, c_M, e_1, \dots, e_M\}$ such that $c_l \in C_{\tilde{G}^l}$, and $e_l \in E_l$.

The total number of combinations will be $\prod_{i=1}^M M_i N_i$

6. Compute the center-of-sums type-reduced set using (3.26).

$$Y_{ce}(x) = \int_{c_1} \dots \int_{c_M} \int_{e_1} \dots \int_{e_M} \sup \left[\mathfrak{S}_{l=1}^M \mu_{C_{\tilde{G}^l}}(c_l) * \mathfrak{S}_{l=1}^M \mu_{E_l}(e_l) \right] / \frac{\sum_{i=1}^M c_i e_i}{\sum_{i=1}^M e_i} \quad (4.5)$$

Where \mathfrak{S} and $*$ indicate the T-norm chosen

f. Defuzzification

To obtain a crisp output from the type-2 FLS, we can defuzzify the type-reduced set [33] [36]. The most natural way of doing this seems to be by finding the centroid of the type-reduced set, however, other possibilities exist, like choosing the unity membership point in the type-reduced set.

The defuzzification makes to transform the linguistic output of the type-reduction to numeric valued. Several methods of defuzzification were proposed in the literature [51]. But the largely used method is:

g. Centre of area

The defuzzification determines the X -coordinate of the center of gravity y_{CG}^* [33] [36] of the fuzzy function:

$$y_{CG}^* = \frac{\int_y y \mu_Y(y) dy}{\int_y \mu_Y(y) dy} \quad (4.6)$$

4.4.2 Interval type-2 Fuzzy controller

a. MEET and JOIN for Interval set

- The meet under minimum or product t-norms of n interval type-1 sets A_1, \dots, A_n having domains $[l_1, r_1], \dots, [l_n, r_n]$ respectively, where $[l_i, r_i] \subseteq [0, 1], i = 1, \dots, n$ is an interval set with domain $[l_1 * \dots * l_n, r_1 * \dots * r_n]$ such as [41] [43]:

$$\bigcap_{i=1}^n A_i = \int_{w \in [l_1 * \dots * l_n, r_1 * \dots * r_n]} 1/w \quad (4.7)$$

* chosen t-norms.

- The join under maximum t-norms of n interval type-1 sets $A_1, \dots, A_n, A_1, \dots, A_n$ having domains $[l_1, r_1], \dots, [l_n, r_n]$ respectively, where $[l_i, r_i] \subseteq [0, 1], i = 1, \dots, n$ is an interval set with domain $[l_1 \vee \dots \vee l_n, r_1 \vee \dots \vee r_n]$ such as:

$$\bigcup_{i=1}^n A_i = \int_{w \in [l_1 \vee \dots \vee l_n, r_1 \vee \dots \vee r_n]} 1/w \quad (4.8)$$

\vee denotes a max operation

b. Inference

In interval type-2 fuzzy system using the minimum or product t-norms operations, the /th activated rule $\bigcap_{i=1}^p \mu_{\tilde{F}_i^l} = F^l(x_1, \dots, x_p)$ gives us an interval determined by two extreme $f_-^l(x_1, \dots, x_n)$ and $\tilde{f}^l(x_1, \dots, x_n)$ same as [41] :

$$F^l(x_1, \dots, x_n) = [f_-^l(x_1, \dots, x_n), \tilde{f}^l(x_1, \dots, x_n)] \equiv [f_-^l, \tilde{f}^l] \quad (4.9)$$

With f_-^l and \tilde{f}^l are given as :

$$\begin{aligned} f_-^l &= \mu_{\tilde{F}_1^l}(x_1) * \dots * \mu_{\tilde{F}_p^l}(x_p) \\ \tilde{f}^l &= \bar{\mu}_{\tilde{F}_1^l}(x_1) * \dots * \bar{\mu}_{\tilde{F}_p^l}(x_p) \end{aligned} \quad (4.10)$$

The output set $\tilde{R}^l(y)$ of the /th activated rule R^l is type-2 fuzzy set :

$$\mu_{\tilde{B}^l}(y) = \int_{b^l \in [f_-^l * \underline{\mu}_{\tilde{G}^l}(y), \tilde{f}^l * \bar{\mu}_{\tilde{G}^l}(y)]} 1/b^l \quad , \quad y \in Y \quad (4.11)$$

$$\mu_{\tilde{B}}(y) = \bigcup_{l=1}^N \mu_{\tilde{B}^l}(y) \quad (4.12)$$

$\underline{\mu}_{\tilde{G}^l}(y)$ and $\bar{\mu}_{\tilde{G}^l}(y)$ represent upper and lower membership function of the set $\mu_{\tilde{G}^l}(y)$

c Type-reduction and Defuzzification

After fuzzification, fuzzy inference, type-reduction and defuzzification [41] [43] [33], we obtain a crisp output For an interval type-2 FLC, this crisp output is the center of the type-reduced set, we know that for an interval type-2 FLC, regardless of singleton or non-singleton fuzzification, and minimum or product t-norm, the result of input and antecedent operations (firing strength) is an interval type-1 set which is determined by its left-most and right-most points f_-^l and \tilde{f}^l . The fired output consequent set $\tilde{B}^l(y)$ of rule R^l can be obtained from the fired interval strength using (4.10). Then the fired combined output consequent set $\tilde{R}^l(y)$ can be computed using (4.12) There exist type-reduction different kinds, such as centroid, center-of-sets, height and modified height, the details of which are given in [78], [74] , [33]. In this memory, for illustrative purposes, we focus on center-of-sets type-reduction, which can be expressed as:

$$y_{\cos}(W^1, \dots, W^M, F^1, \dots, F^M) = [y_l, y_r] = \iint_{w^l \in [w_l^1, w_r^1], w^M \in [w_l^M, w_r^M]} \iint_{f^1 \in [f_-^1, \tilde{f}^1], f^M \in [f_-^M, \tilde{f}^M]} 1 / \frac{\sum_{i=1}^M f^i w^i}{\sum_{i=1}^M f^i} \quad (4.13)$$

Where Y_{cos} an interval set is determined by two end points (y_l and y_r), $f^l \in [f_-^l, \tilde{f}^l]$; $w^i \in W^i = [w_l^i, w_r^i]$, W^i is the centroid of the interval type-2 consequent set \tilde{G} (the centroid of a type-2 fuzzy set is described in Appendix C).

Observe that $Y_{cos}(W^1, \dots, W^M, F^1 \dots F^M)$ is an interval type-1 set. So, to find

$y_{cos}(W^1, \dots, W^M, F^1 \dots F^M)$ we just need to compute the two end-points of this interval. Unfortunately, no closed-form formula is available for Y_{cos} .

$$y = \frac{\sum_{i=1}^M f^i w^i}{\sum_{i=1}^M f^i} \quad (4.14)$$

The maximum value of y is y_r , and the minimum value of y is y_l from (4.14), we see that y is a monotonic increasing function with respect to w^i , so, y_r , and is only associated with w_r^i similarly y_l is only associated with w_l^i . In the COS type-reduction method, the two end-points of y_{cos} , (y_l and y_r) depend on a mixture of \underline{f}^i or \bar{f}^i values. In this case, (y_l and y_r) can be represented as :

$$y_l = \frac{\sum_{i=1}^M f_l^i w_l^i}{\sum_{i=1}^M f_l^i} \quad (4.15)$$

Where f_l^i denote the firing strength membership grade either (\underline{f}^i and \bar{f}^i) contributing to the right-most point y_l , similarly to y_r

$$y_r = \frac{\sum_{i=1}^M f_r^i w_r^i}{\sum_{i=1}^M f_r^i} \quad (4.16)$$

Where f_r^i denotes the firing strength membership grade either (\underline{f}^i or \bar{f}^i) contributing to the right-most point y_r .

In order to compute (y_l and y_r), we need to compute $\{f_l^i, i = 1, 2, \dots, M\}$ and $\{f_r^i, i = 1, 2, \dots, M\}$.

This can be done using the exact computational procedure given in [43] [33]. Here we briefly provide the computation procedure for y_r . Without loss of generality, assume the w_r^i 's are arranged in ascending order $w_r^1 \leq w_r^2 \leq \dots \leq w_r^M$.

1. Compute y in (4.16) by initially setting $f_r^i = \frac{\bar{f}^i + \underline{f}^i}{2}$ for $i = 1, \dots, M$ where (\underline{f}^i and \bar{f}^i) have been previously computed using (4.10) and let $y' = y$.
2. Find R ($1 \leq R \leq M - 1$) such that $w_r^R \leq y' \leq w_r^{R+1}$
3. Compute y in (4.14) when $f_r^i = \underline{f}^i$ for $i \leq R$, and $f_r^i = \bar{f}^i$ for $i > R$, then set $y'' = y_r$
4. If $y'' \neq y'$ then go to step 5. If $y'' = y'$ then stop and set $y_r = y''$
5. Set y' equal to y'' and return to step 2

This 5 step computation procedure has been proven to converge to the exact solution in no more than M iterations [43] [33]. Note that in this procedure, the number R is very important. For $i \leq R$, $f_r^i = \underline{f}^i$ and for $i > R$, $f_r^i = \bar{f}^i$, so y_r can be represented as:

$$y_r = y_r(\underline{f}^1, \dots, \underline{f}^R, \bar{f}^{R+1}, \dots, \bar{f}^M, w_r^1, \dots, w_r^M) \quad (4.17)$$

The procedure for computing y_l is very similar. Just replace w_r^i by w_l^i , and in step 2, find L ($1 \leq L \leq M - 1$), such that $w_l^L \leq y'_l \leq w_l^{L+1}$, and in step 3

$f_l^i = \bar{f}^i$ for $i \leq L$, and $f_l^i = \underline{f}^i$ for $i > L$. Then y_l can be represented as:

$$y_r = y_r \left(\bar{f}^1, \dots, \bar{f}^R, \underline{f}^{R+1}, \dots, \underline{f}^M, w_l^1, \dots, w_l^M \right) \quad (4.18)$$

Because Y_{cos} is an interval set, we defuzzify it using the average of (y_l and y_r), and hence, the defuzzified output of an interval type-2 FLC is: output of Centroid = $\frac{y_l + y_r}{2}$

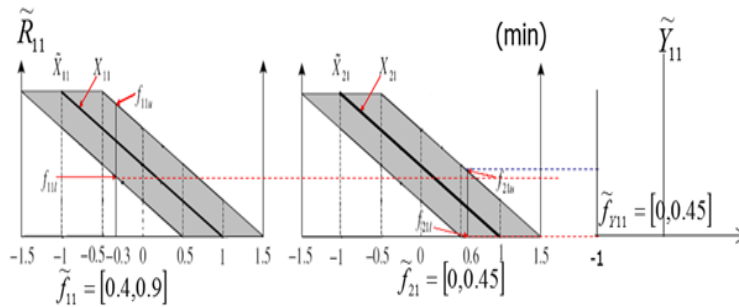
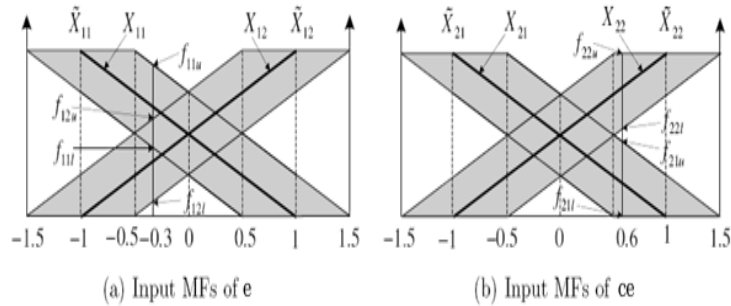
The control of dynamic system was originally applied to a system that control a mechanical position or motor. Used to describe a control system in which a physical variable is required to follow, or track.

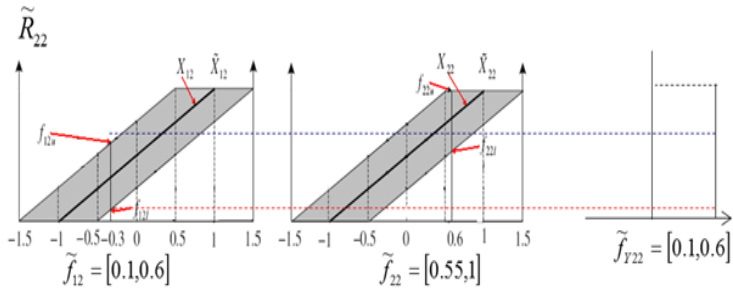
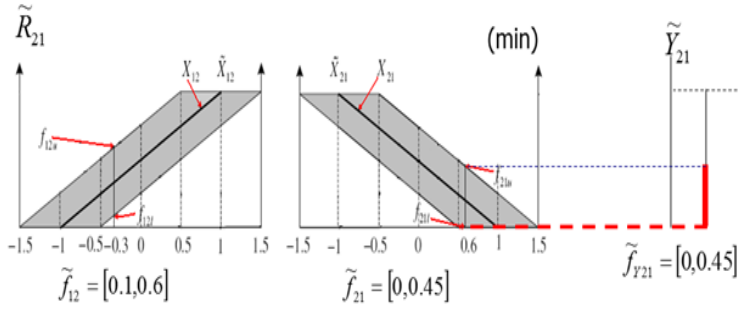
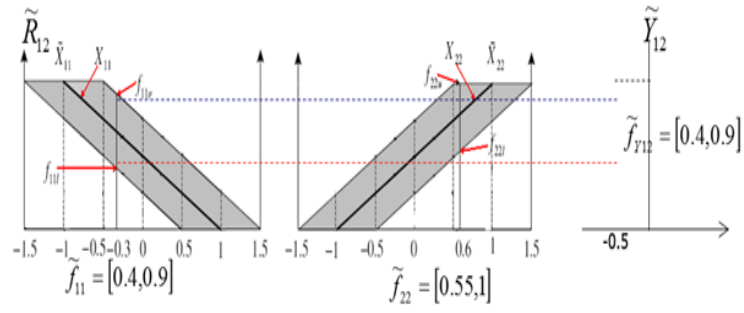
Example 4.4.1. to illustrate type2 FLS

\tilde{R}^{ij} : if e is \tilde{X}_{1i} and ce is \tilde{X}_{2j} , then y is \tilde{Y}_{ij} , $i, j = 1, 2$ Rule base and consequents of the

ce e	\tilde{X}_{21}	\tilde{X}_{22}
\tilde{X}_{11}	$\tilde{Y}_{11} = -1$	$\tilde{Y}_{12} = -0.5$
\tilde{X}_{12}	$\tilde{Y}_{21} = 0.5$	$\tilde{Y}_{22} = 1$

Interval type-2 FLS Consider an input vector $w = (e, ce) = (-0.3, 0.6)$





$$\tilde{R}_{11} : \tilde{f}_{Y_{11}} = [\min(0.4, 0), \min(0.9, 0.45)] = [0, 0.45] \rightarrow -1$$

$$\tilde{R}_{12} : \tilde{f}_{Y_{12}} = [\min(0.4, 0.55), \min(0.9, 1)] = [0.4, 0.9] \rightarrow -0.5$$

$$\tilde{R}_{21} : \tilde{f}_{Y_{21}} = [\min(0.1, 0), \min(0.6, 0.45)] = [0, 0.45] \rightarrow 0.5$$

$$\tilde{R}_{22} : \tilde{f}_{Y_{22}} = [\min(0.1, 0.55), \min(0.6, 1)] = [0.1, 0.6] \rightarrow 1$$

$$L = 1 \quad y_l = \frac{0.45 \times (-1) + 0.4 \times (-0.5) + 0 \times (0.5) + 0.1 \times (1)}{0.45 + 0.4 + 0 + 0.1} = -0.58$$

$$R = 2 \quad y_r = \frac{0 \times (-1) + 0.4 \times (-0.5) + 0.45 \times (0.5) + 0.6 \times (1)}{0 + 0.4 + 0.45 + 0.6} = 0.43$$

$$\text{Centroid} = \frac{y_l + y_r}{2} = -0.07$$

[74]

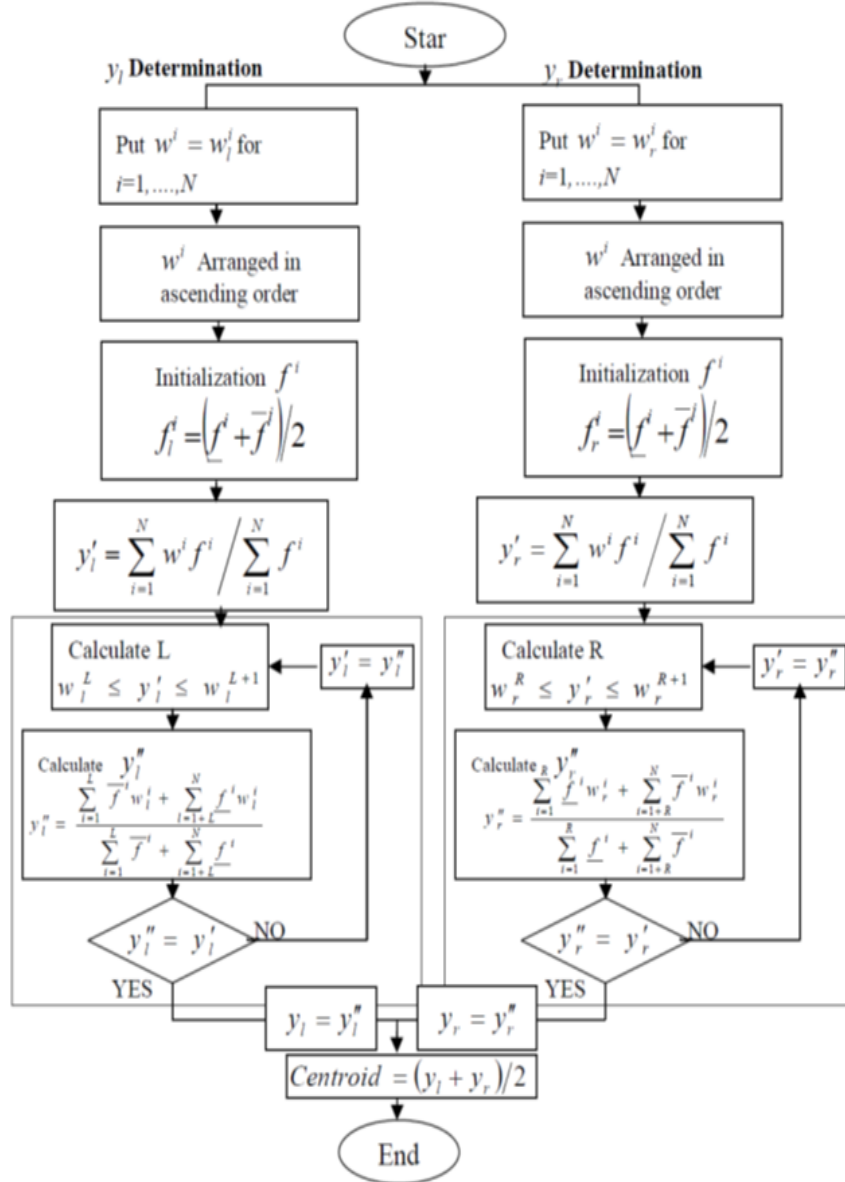


Figure 4.5: Karnik-Mendel Algorithms to locate Centroid on Interval type-2 set [36]

d. Detailed description of $K - M$ algorithm

In this work a center of sets type reduction approach is used the details are given as follows:

Step1. Determination of y_{cos} [y_l, y_r] where y_{cos} is interval set and y_r is the maximum value of y And y_l is the minimum value of y

➤ compute of y_l

- Compute w_l^i (for $i = 1, \dots, N$) where w_l^i is the lower fuzzy value of $w_i = (e, ce)$
- Put $w^i = w_l^i$ (for $i = 1, \dots, N$)
- Arranged w^i in ascending order $w_l^1 < w_l^2 < \dots < w_l^N$
- Compute the left most point and the right most point $(\underline{f}^i, \overline{f}^i)$ for $i = 1, \dots, N$
- Set initially: $(\underline{f}^i + \overline{f}^i) / 2$
- Compute: y_l' using (4 - 16)
 1. Find $R(1 \leq R \leq N - 1)$ such that $w_r^R \leq y_r' \leq w_r^{R+1}$
 2. Compute y in (4.14) when $f_r^i = \underline{f}^i$ for $i \leq R$, and $f_r^i = \overline{f}^i$ for $i > R$, then $y_r'' = y_r$
 3. If $y'' \neq y'$ then go to step 5 . If $y'' = y'$ then stop and set $y_r = y''$
 4. set $y' = y''$ and return to step 2
- Put $y_l = y_l''$

➤ Compute of y_r

- Compute w_r^i (for $i = 1, \dots, N$) where w_r^i is the lower fuzzy value of $w_i = (e, ce)$
- Put $w^i = w_r^i$ (for $i = 1, \dots, N$)
- Arranged w^i in ascending order $w_r^1 < w_r^2 < \dots < w_r^N$
- Compute the left most point and the right most point $(\underline{f}^i, \overline{f}^i)$ for $i = 1, \dots, N$
- Set initially: $(\underline{f}^i + \overline{f}^i) / 2$
- Compute: y_r' using (4 - 16)
 5. Find $L(1 \leq L \leq N - 1)$ such that $w_l^L \leq y_l' \leq w_l^{L+1}$
 6. compute y in (4.14) when $f_l^i = \underline{f}^i$ for $i \leq L$, and $f_l^i = T_i$ for $i > L$, then $y_l'' = y_l$
 7. If $y'' \neq y'$ then go to step 5 . If $y'' = y'$ then stop and set $y_l = y''$
 8. set $y' = y''$ and return to step 2
- Put $y_r = y_r''$

Step2. Compute the centeroid of interval set

$$\text{Centeroid} = (y_l + y_r) / 2$$

Table 4.2: Rulebase table[33]

		Velocity error				
Position error		NL	SN	Ze	SP	LP
	LN	LN	LN	LN	SN	Ze
	SN	LN	LN	SN	Ze	SP
	Ze	LN	SN	Ze	SP	LP
	SP	SN	Ze	SP	LP	LP
	LP	Ze	SP	LP	LP	LP

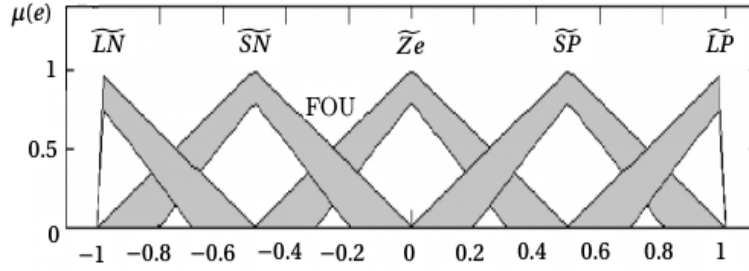


Figure 4.6: Fuzzy sets interval type-2 for each articulation and control signal

4.5 Interval type-2 Fuzzy control with PUMA560 3DOF

we show the structure interval type-2 FLC with PUM560 3DOF, the regulator we use is five classes, wich means that 25 rule base are involted, the rule base table in the table 4.2 and in Figure 4.7 fuzzy sets for error and change error and out put of control T. All the gains of Interval type-2 fuzzy controller we do tuning until get good positions with lower error in ideal case.

In this work and for illustrative purposes, we have used Karnik-Mendel algorithm of figure 4.5 to locate centroid on interval type-2 fuzzy set [36]. The proposed modification is carried out in the fourth step of this algorithm where Karnik-Mendel use arithmetic mean, but in the presented work it has been modified by harmonic mean, as in the nexte formulate:

$$y = \frac{\sum_{i=1}^n f_i}{\sum_i \frac{f_i}{w_i}}$$

4.6 Simulation results

For our model of equation (1) 3DOF PUMA560 robot, a cycloid trajectory test Figure 4.11 used the different articulation move respectively from position $(-50^\circ, -135^\circ, 135^\circ)$ to the position $(45, -85^\circ, 30^\circ)$ with three end time of movement equal to (1.5, 4, 8) seconds then will have three cycloid trajectory that differ in simulation time. This trajectory is described by

the following equation:

$$\theta_{di} = \begin{cases} \theta_{di}(0) + \frac{D_i}{2\pi} \left[2\pi \frac{t}{t_{end}} - \sin\left(2\pi \frac{t}{t_{end}}\right) \right] & \text{for } 0 \leq t \leq t_{end} \\ \theta_{di}(t_{end}) & \text{for } t_{end} < t \\ D_i = \theta_{di}(t_{end}) - \theta_{di}(0) & \end{cases} \quad (4.19)$$

Integral of Square Error (ISE) [40] is used as performance index in comparative study between type-1 and interval-type-2 fuzzy controller

$$ISE = \int_0^{t_{end}} [e(t)]^2 dt \quad (4.20)$$

In type-1 fuzzy PD controller of Figure 4.4, fuzzy set parameters take standard parameters as declared by default in MATLAB in the range of $[-1, 1]$; the triangle membership functions are symmetric fuzzy sets. The controller require three gains: two inputs gains and one output gain; these gain; are chosen using manual tuning until getting good positions responses with lower error, lower control energy and smooth control signal without chattering in ideal case.

In an Interval-type-2 fuzzy PD controller, the interval-type-2 fuzzy sets are programmed as type-1 fuzzy sets in the range of $[-1, 1]$ with upper and lower triangle membership function as shown in Figure 4.6. In addition to the length of FOU zone, the controller has also three gains to be designed: two input gains and one output gain. These gains and the FOU zone are chosen base on trial and error method until getting good position responses with lower error and lower control energy and smooth control signal without chattering in ideal case.

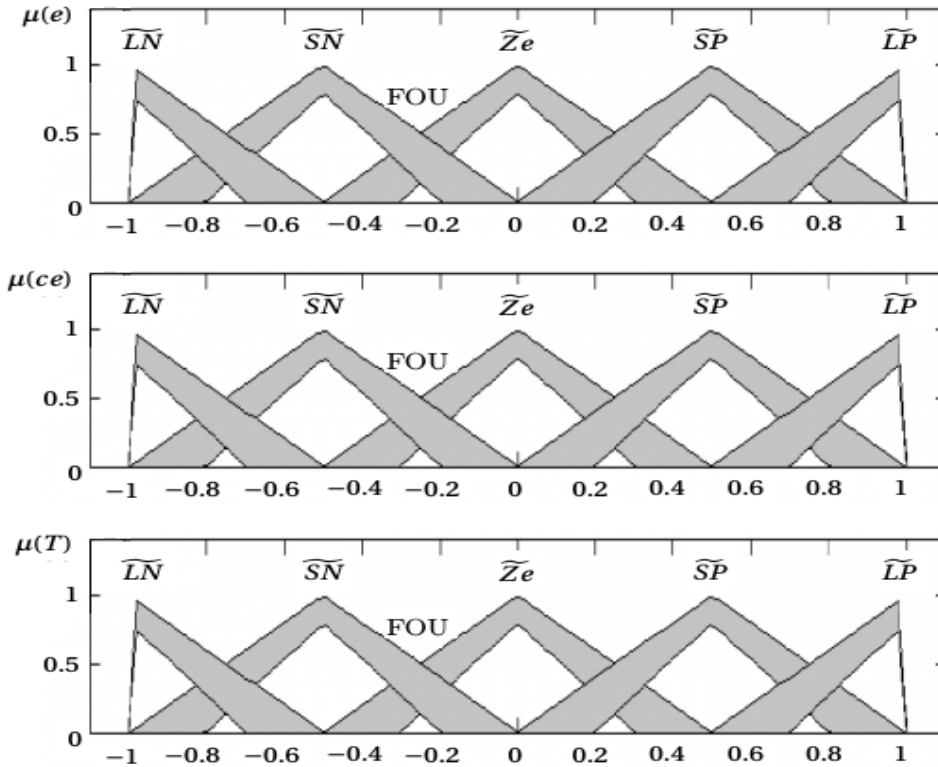


Figure 4.7: Fuzzy sets interval type-2 for each articulation and control signal

4.7 Result of simulation with two trajectory

We use same trajectory, a circle in space and LEAHY trajectory.

Result of simulation with circle trajectory

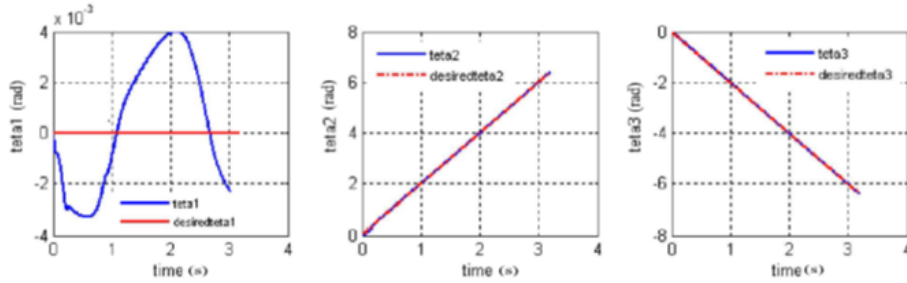


Figure 4.8: Position of joints 1,2,3 (rad) .

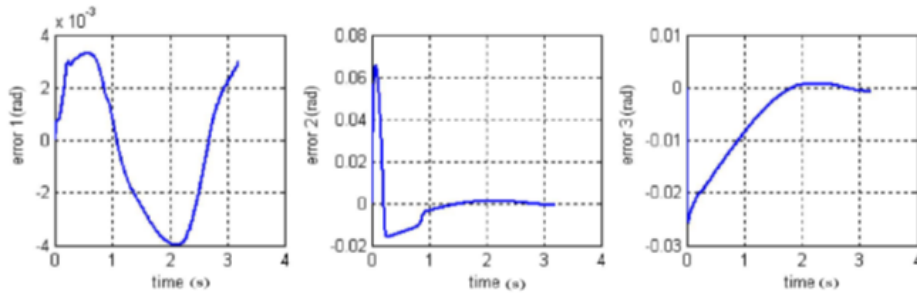


Figure 4.9: Position error of joints 1,2,3 (rad) .

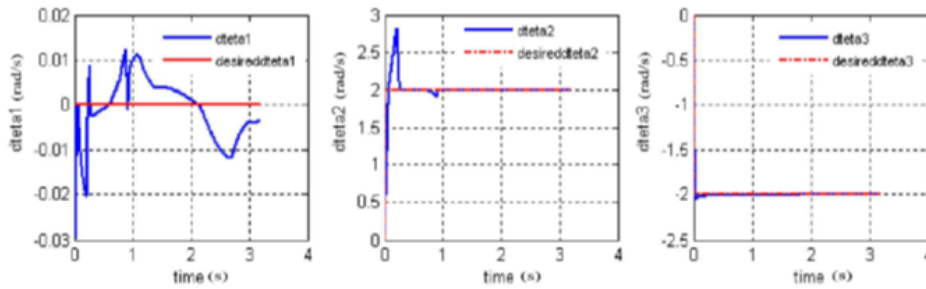


Figure 4.10: Velocity of joints 1,2,3 (rad/s)

In order to verify the robustness of the proposed controller and to help us to achieve a clear full comparison between the type-1 fuzzy controller and interval-type-2 fuzzy controller and to see the ability each one in handling uncertainties, two lands of noises are adopted: the first noise torque is $d = \sin(3t) + 1$ and the second noise torque is considered by taking 30 % of joint torque for each joint. [62]The simulation results are devoted to showing comparisons in the robot's responses when using both type-1 fuzzy controller and interval type-2 fuzzy controller. A set of three simulations in each one three cycloid trajectories are adopted by changing time with (1.5second, 4s, 8s). We start with type-1 fuzzy controller in ideal conditions without any kind of disturbances. The values of ISE of three joint angles for three

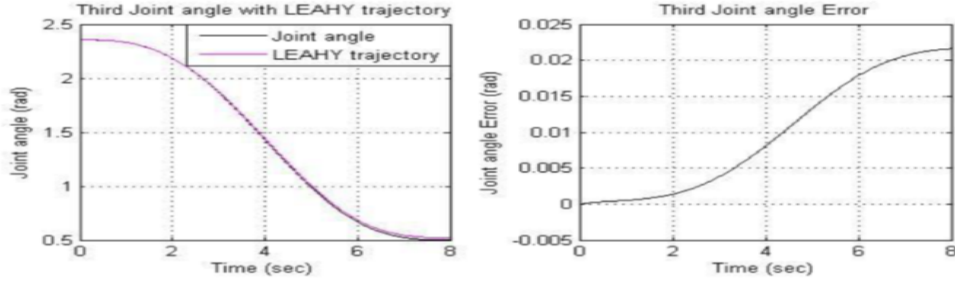


Figure 4.11: Third joint response to cycloid trajectory with noise and its tracking joint error

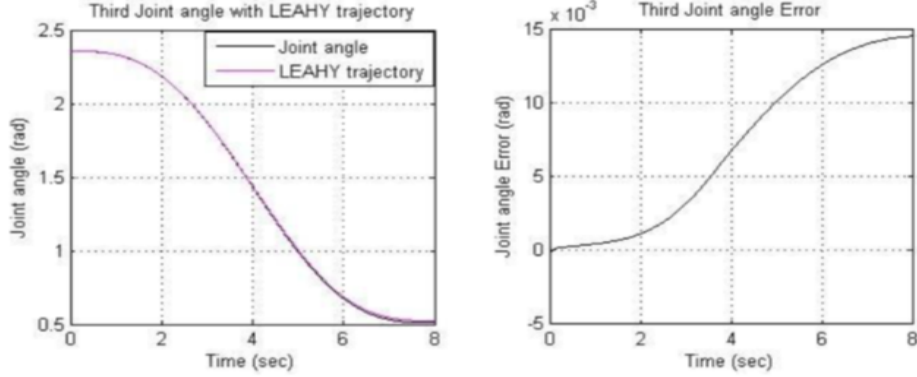


Figure 4.12: Third joint response with noise and its tracking joint error with modified type-reduction

trajectories are gathered in Table 4.3. Figure 4.11 shows the response obtained by using type-1 fuzzy controller with the presence of the noise torque taking $d = 30\%$ of the torque control added as noise to the main control torque to check the robustness of controller. Figure 4.12 shows the response of the interval type-2 fuzzy controller using modified type-reduction of Karnik-Mendel algorithm.

Comparison in terms of integral squared joint errors in ideal and disturbance cases ($d = \sin(3t) + 1$) is shown in Table 4.3. Table 4.4 represents comparison in terms of integral squared joint errors in ideal and disturbance $d = 30\%$ of joint signal control. The last column in Tables 4.3. and 4.4. is the factor (ISE_F) between the integral square joint error produced by using a type-1 fuzzy controller (ISE_T1) and integral square error produced by using interval type-2 fuzzy controller (ISE_IT2) in noisy case, calculated as (4.21):

$$ISE_F = ISE_T1 / ISE_IT2 \quad (4.21)$$

This factor gives an idea in how many energies of error in type-1 fuzzy set is bigger than energy error of interval type-2 fuzzy set case.

4.8 Results discussion

The membership functions of a type-1 fuzzy controller are type-1 fuzzy sets, see Figure 4.3, and their membership degrees are crisp values.

Table 4.3: ISE for different trajectory without and with disturbance $d = \sin(3t) + 1$

Second	Joint	ISE Type-1*		ISE Interval type-2*		ISE_F
		No d	With d	No d	with d	
1.5	J1	1.88	2.35	0.49	0.70	3.38
	J2	7.14	8.33	5.96	3.75	2.22
	J3	6.06	8.27	5.76	5.67	1.46
4	J1	0.23	0.54	0.24	0.15	3.49
	J2	32	36.9	18.71	16.84	2.19
	J3	19.7	30.2	17.48	24.91	1.21
8	J1	0.043	0.64	0.02	0.26	2.44
	J2	83.5	94	31.88	50.80	1.85
	J3	54	81	46.63	50.21	1.61

Note : * all values are multiplied by 10^{-5}

Table 4.4: ISE for different trajectory without and with disturbance in joint control $d = 30\%$ of each joint

Second	Joint	ISE Type-1*		ISE Interval type-2*		ISE_F
		No d	With d	No d	with d	
1.5s	J1	1.88	4.23	0.50	0.91	4.63
	J2	7.14	19.1	3.88	7.28	2.62
	J3	6.06	14.8	5.63	9.09	1.63
4s	J1	0.23	0.48	0.23	0.13	3.77
	J2	0.32	79.3	22.44	38.94	2.04
	J3	19.7	44.7	12.16	23.80	1.88
8s	J1	0.043	0.091	0.02	0.03	2.72
	J2	83.5	200	43.78	80.08	2.50
	J3	54	120	44.37	79.28	1.51

Note : * all values are multiplied by 10^{-5}

Interval type-2 fuzzy controller has FOU zone in which the membership degrees are an interval and not crisp values. Actually, this is the main part that improves the degree of dealing with uncertainties. In an interval-type-2 fuzzy controller, the outputs of the inference are interval-type-2 sets and a type-reducer is needed to convert them into a type-1 sets before defuzzification to generate a crisp output. This is the main structural difference between type-1 and interval-type-2 fuzzy systems. In Table 4.5, some differences between type-1 fuzzy controller and interval type-2 fuzzy controller are given.

When comparing results based on visual observation, the actual and desired joint angles are overlapping without tangible difference as illustrated in Figure 4.11. But in terms of ISE values, it is possible to measure the difference between the two fuzzy controllers. To perform this comparison, three simulations with different simulation tunes are realized, which give us three different trajectories for three robot joints with two disturbance torques to test robustness. All results are summed up in two tables; Table 4.3 for comparison between a type-1 fuzzy controller and modified interval type-2 fuzzy controller with disturbance $d = \sin(3t) + 1$ and Table 4.4 for comparison between type-1 fuzzy controller and modified interval type-2 fuzzy controller with disturbance $d = 30\%$ of control torque. It is very clear

that the average of ISE_F column around two times which means the tracking error angle for the type-1 fuzzy controller is twice bigger than error angle for modified interval-type-2 fuzzy controller.

Finally, when comparing all results in all figures, Table 4.3, and Table 4.4, it is very clear that interval-type-2 fuzzy controller is more performing than type-1 fuzzy controller in ideal or in noises cases.

Table 4.5: Some differences between type-1 and interval-type-2 fuzzy controller

	T1-FS	IT-FS
Grades of membership	Crisp	Fuzzy
footprint of uncertainty zone	Don't have	Have
Type-reducer	Don't need	Need
Computational	small	Little Large
Program	Easy	Little Hard

4.9 Conclusion

In this chapter, we have discussed in some detail the type-1 FLC and type-2 FLC then, we have applied them on PUMA560 3DOF. The results of simulations prove that Interval type-2 FLC has a low error and accepted control torques of the joints

General Conclusion

Fuzzy Type-2 logic has involved many researchers in recent years and has been applied in several areas such as video modeling, classification, diagnostics, artificial intelligence, transmission channels, adaptive filters, mobile robots, and other domains. In this thesis, a type-1 and Interval type-2 FLC are developed for the control of PUMA560 3DOF manipulators in the presence of dynamical modeling. To alleviate the naturally inherited high computational complexity of type-2 FLCs, interval membership functions are adopted. The controllers are also compared in similar operating conditions. Three simulations for three robot joints with and without noises are presented. Furthermore, a quantification of errors was achieved and documented in tables for the integral square error of trajectory tracking. It was shown that the lower errors were obtained using an interval-type-2 fuzzy controller where the amount of error is four times less than when using a type-1 fuzzy set to control a manipulator robot. The interval-type-2 fuzzy controller is much better because it can handle uncertainties and has more robustness than the type-1 fuzzy counterpart. Also, the proposed modification in type reduction of Karnik-Mendel algorithm by using harmonic mean performs better than the arithmetic mean used in the traditional Karnik-Mendel algorithm.

We are concluding that using type-2 FLC in real applications can be a better choice since the amount of uncertainty in real systems, mostly, is difficult to estimate. But when uncertainty in real system is negligible, we can consider it as an ideal system. In this case, it is a better choice to select a type-1 FLC since it works similar as type-2 FLC, and the size of type-1 FLC program is shorter than type-2 FLC, so it is faster and easier for programming and simulating.

Future work will be devoted to the study of ideal representations of the types-2 fuzzy lattices.

.1 APPENDIX

Matrix M is a symmetric 3×3 matrix :

$$M(q) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

where,

$$a_{11} = I_{m1} + I_1 + I_3 \cdot CC2 + I_7 \cdot SS23 + I_{10} \cdot SC23 + I_{11} \cdot SC2 + 2 \cdot [I_5 \cdot C2 \cdot S23]$$

$$a_{12} = I_4 \cdot S2 + I_8 \cdot C23 + I_9 \cdot C2 \quad a_{13} = I_8 \cdot C23 \quad a_{22} = I_{m2} + I_2 + I_6 + 2 \cdot [I_5 \cdot S3]$$

$$a_{23} = I_5 \cdot S3 + I_6$$

$$a_{33} = I_{m3} + I_6$$

$$a_{21} = a_{12}, a_{31} = a_{13} \text{ and } a_{32} = a_{23}$$

While matrix B is :

$$B(q) = \begin{bmatrix} b_{112} & b_{113} & 0 & 0 & 0 & b_{123} & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{223} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where,

$$b_{112} = 2 \cdot [-I_3 \cdot SC2 + I_5 \cdot C223 + I_7 \cdot SC23]$$

$$+ I_{10} \cdot (1 - 2 \cdot SS23) + I_{11} \cdot (1 - 2 \cdot SS2)$$

$$b_{113} = 2 \cdot [I_5 \cdot C2 \cdot C23 + I_7 \cdot SC23 -] + I_{10} \cdot (1 - 2 \cdot SS23)$$

$$b_{123} = 2 \cdot [-I_8 \cdot S23]$$

Matrix C is :

$$C(q) = \begin{bmatrix} 0 & c_{12} & c_{13} \\ c_{21} & 0 & c_{23} \\ c_{31} & c_{32} & 0 \end{bmatrix}$$

where,

$$c_{12} = I_4 \cdot C2 - I_8 \cdot S23 - I_9 \cdot S2$$

$$c_{13} = 0.5 \cdot b_{123} = -I_8 \cdot S23$$

$$c_{21} = -0.5 b_{112} = I_3 \cdot SC2 - I_5 \cdot C223 - I_7 \cdot SC23 - 0.5 I_{10} \cdot (1 - 2.5 S23) - 0.5 \cdot I_{11} \cdot (1 - 2 \cdot SS2)$$

$$c_{23} = 0.5 \cdot b_{223} = I_5 \cdot C3$$

$$c_{31} = -0.5 \cdot b_{113} = -I_5 \cdot C2 \cdot C23 - I_7 \cdot SC23 - 0.5 I_{10} \cdot (1 - 2 \cdot SS23)$$

$$c_{32} = -c_{23} = -I_5 \cdot C3$$

And matrix G is :

$$g(q) = \begin{bmatrix} 0 \\ g_2 \\ g_3 \end{bmatrix}$$

$$g_2 = g_1 \cdot C2 + g_2 \cdot S23 + g_3 \cdot S2$$

$$g_3 = g_2 \cdot S23$$

where,

$$Si = \sin(\theta i), Ci = \cos(\theta i), Cij = \cos(\theta i + \theta j), Sijk = \sin(\theta i + \theta j) + \theta k, \text{ with } i = 1, 2, 3$$

Table 6: Inertial constants ($kg.m^2$)

$I_1=1.43\pm 0.05$	$I_2 = 1.75 \pm 0.07$
$I_3=1.38\pm 0.05$	$I_4 = 0.69 \pm 0.02$
$I_5=0.372\pm 0.031$	$I_6 = 0.333 \pm 0.016$
$I_7=0.298\pm 0.029$	$I_8 = -0.134 \pm 0.014$
$I_9=0.0238\pm 0.012$	$I_{10} = -0.0213 \pm 0.0022$
$I_{m1} = 1.14 \pm 0.27$	$I_{m2} = 4.71 \pm 0.54$
$I_{m3} = 0.827 \pm 0.093$	

Table 7: Gravitational constants ($N.m$)

$g_1 = -37.2 \pm 0.5$	$g_2 = -8.44 \pm 0.20$
$g_3 = 1.02 \pm 0.50$	

Tables 4.5 and 6 contain the computed values for the constants appearing in the equations (1).

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ملخص

يركز البحث المقدم في أطروحة الدكتوراه هذه على مجالين: الدراسة النظرية على المجموعات الضبابية من النوع 1 والنوع 2 ، ثم تطبيقها للتحكم في نظام ديناميكي غير خطي وهو ذراع روبوت من الصنف بيما 560. لذلك ، قمنا بتطبيق كل من وحدات التحكم الضبابية: وحدات تحكم ضبابي من النوع 1 و المجال الضبابي من النوع 2 ، حيث تم اقتراح تعديل في خوارزمية كارنيك مندل. لحساب الناتج من النظام الضبابي نوع-2 ، هناك خطوة رئيسية تسمى تبسيط النوع ؛ تعتمد هذه العملية على خوارزمية كارنيك مندل ، والتي تستخدم المتوسط الحسابي لحساب مخرجات التحكم. في هذا العمل ، نقترح تغيير الوسط الحسابي بالوسط التوافقي. يُظهر التعديل المقترح في تبسيط نوع خوارزمية كارنيك مندل من أجل مجموعة غامضة من النوع 2 أفضل أداء. بالفعل، فإن مقدار الخطأ في حالة وحدة تحكم غامض تعديل نوع فاصل 2 أقل مرتين من تحكم غامض من النوع 1. الكلمات المفتاحية: المجموعات الضبابية نمط2، المجال الضبابي نمط2، التحكم الضبابي نمط1، التحكم بواسطة المجال الضبابي نمط2، الذراع الآلي بيوما 560 .

Abstract

The research presented in this doctoral thesis focuses on two areas: theoretical study on fuzzy sets type1 and type2, then their application for the control of a nonlinear dynamic system which is the manipulator robot PUMA560. For that, we implemented the both fuzzy controllers: type-1 and interval type-2 fuzzy controllers, in which a modification in Karnik-Mendel algorithm has been proposed. To calculate the output of interval type-2 fuzzy system there is a main step called type-reduced; this operation is based on Karnik-Mendel algorithm, which uses arithmetic mean to calculate the control output. In this work, we propose to change the arithmetic mean by harmonic one. The performances of modified interval type- 2 controller and type-1 fuzzy controller with and without noises are compared in terms of integral of squared error. The proposed modification in type reduction of Karnik-Mendel algorithm for interval type-2 fuzzy set shows best performance. Indeed, the amount of error in case of modified interval type-2 fuzzy controller is less two times than type-1 fuzzy controller.

Keywords: Type-2 fuzzy sets, Interval type-2 fuzzy sets, type-1 fuzzy controller, Interval type-2 fuzzy controller, puma 560 robot manipulator

Résumé

Les travaux de recherche présentés dans cette thèse de doctorat portent essentiellement sur deux axes : étude théorique sur les ensembles flous type1 et type2, puis leur application pour la commande d'un système dynamique non linéaire qui est le robot manipulateur PUMA560. Pour cela, nous avons implémenté les deux contrôleurs flous: les contrôleurs flous de type 1 et d'intervalle de type 2, dans lesquels une modification de l'algorithme de Karnik-Mendel a été proposée. Pour calculer la sortie du système à intervalle flous de type 2, il existe une étape principale appelée réduction de type2 cette opération est basée sur l'algorithme de Karnik-Mendel, qui utilise une moyenne arithmétique pour calculer la sortie de contrôle. Dans ce travail, nous proposons de changer la moyenne arithmétique par harmonique. Les performances du contrôleur d'intervalle modifié de type 2 et du contrôleur flou de type 1 et sans bruit sont comparées en termes d'intégrale d'erreur quadratique. La modification proposée dans la réduction de type de l'algorithme de Karnik-Mendel pour l'intervalle flou de type-2 montre les meilleures performances. En effet, la quantité d'erreur en cas d'intervalle modifié de contrôleur flou de type 2 est inférieure à deux fois à celle de contrôleur flou de type 1.

Les mots clés: les ensembles flous type-2, l'intervalle flou type-2, Contrôle fou type-1, Contrôle intervalle flou type-2, robot manipulateur Puma 560.



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Introduction

The fuzzy logic theory was established by L. Zadeh in 1965 [76]. This logic allows the representation and processing of inaccurate or approximate knowledge. The number of applications based on this theory has in recent years increased considerably [21] [24] [7] [26] [71].

This is due to the fact that fuzzy logic is usually expressed by linguistic rules of the form IF THEN; it is used to solve control decision problems or to describe the dynamic behavior of an unknown or poorly defined system.

The fuzzy logic type-1 has been extended to get a new one fuzzy logic called the fuzzy logic type-2. Mendel and his students [30] [33] [35] [50] [73] worked in this field and developed the type-2 fuzzy sets. Type-2 fuzzy logic is very effective when it is very difficult to determine exact membership functions for a fuzzy system, so this new logic allows us to incorporate uncertainties into the rules which will act positively on the output of the given system. In type-1 fuzzy systems, There are at least three sources which are as follows:

1. The meaning of words used in the rules may be uncertain, that is, a word can be polysemic.
2. Measurements activating a type-1 fuzzy system can be noisy, which introduces uncertainties.
3. The data used to adjust the parameters of a fuzzy type-1 system can be noisy as well.

All of these uncertainties will be found in fuzzy membership functions. Type-1 fuzzy sets are therefore unable to model such uncertainties because their fuzzy membership functions are defined in a precise mathematical way. On the other hand, type-2 fuzzy sets are able to model such uncertainties because their membership functions are themselves indefinite. The fuzzy membership functions type-1 are bi-dimensional, however, the type-2 fuzzy membership functions are tri-dimensional. The new (third) dimension of type-2 fuzzy sets provides more freedom to support the modeling of uncertainties. The need for industrial robot in tasks like cutting, in welding, and painting and those requiring more precision, oblige researchers to have an exact mathematical description and mastery of this kind of processes in order to solve related industrial problems [9]-[11]. In this context, the fuzzy logic theory, firstly introduced by Zadeh [78] is gaining more and more attention from both researchers and industrial communities. The main advantage of fuzzy system is that it does not need the knowledge of the dynamic model system. This characteristic is important when dealing with nonlinear systems. Moreover, the dynamic modeling of robot manipulator shows a dependence on their dynamic parameters, function of lifetime variations (friction factors affected-by the abuse of joints), and on their dynamic parameters that vary with the completed task [11]-[9]. These characteristics give advantage to fuzzy controller on other nonlinear methods also seeing their robustness towards noises affecting the plant [64]. The major aim of this thesis is to study from theoretical and applying the type-1 and type-2 fuzzy sets on a mathematical point of view. secondly, applying these two control structures types to the applying side consists in working onto two trajectories (circle and LEAHY). Then, a proposed type reduction of Karnik-Mendel algorithm used in interval type-2 fuzzy

sets (IT2-FS) to increase the control of three DOF PUMA560 manipulator. It should be noted that there are many generalizations of the notion of fuzzy set, for example the notion of a multi fuzzy set, intuitionistic fuzzy set (IFS), L- fuzzy set (LFS) and others. In most engineering applications, we chose the notion of interval type-2 fuzzy set because they can improve certain kinds of inference better than do fuzzy sets with increasing imprecision, uncertainty, and fuzziness in information and this makes it easy for us to get the desired results.

This thesis contains four chapters.

The first chapter presents the basic concepts of Type-1 fuzzy sets and classification of fuzzy sets which include normal fuzzy set, subnormal fuzzy set, convex fuzzy set and fuzzy relations well as α -cut or strong α -level set.

The second one gives the basic concepts of a type-2 fuzzy sets, representation type-2 membership function, upper and lower membership functions, Interval type- 2 fuzzy set, Operation of Types-2 fuzzy sets, Type-2 fuzzy relations and their compositions, finally cartesian product and α -Planes.

In chapter 3, we define the concept of fuzzy logic control, the fuzzy rule base, fuzzy inference, the fuzzification, defuzzification, Puma560 robot dynamic modeling. It introduces the three degrees of freedom robot from PUMA560 and its dynamic model and state of art of fuzzy logic controller of a non-linear dynamic system.

The last chapter provides simulation results and a comparative evaluation of the type-1 and type-2 fuzzy logic controllers.

Chapter 1

Generalities of Type-1 Fuzzy Sets

The first publications in fuzzy set theory by (Zadeh, 1965) and (Goguen, 1967, 1969) display the authors intention to generalize the classical notion of a set and a proposition (statement) to accommodate fuzziness.

Let X be a space of objects and x be a generic element of X . A classical set A , $A \subseteq X$, is defined by a collection of elements or objects $x \in X$, and that each x can either belong or doesn't do to the set A . By defining a "characteristic function" for each element $x \in X$, we can represent a classical set A by a set of order pairs $(x, 0)$ or $(x, 1)$, which respectively indicates $x \notin A$ or $x \in A$.

1.1 Basic concepts of Type-1 fuzzy sets

This section contains the basic definitions and properties of type-1 fuzzy sets and several operations of this notion. Unlike the aforementioned conventional set, a fuzzy set [76] expresses the degree to which an element belongs to a set. Hence the characteristic function of a fuzzy set is allowed to have values between 0 and 1, which denotes the degree of membership of an element in a given set.

Definition 1.1.1. [76] *Let X be the collection of objects generically denoted by x , then a fuzzy set \underline{A} in X is defined as a set of ordered pairs:*

$$\underline{A} = \left\{ (x, \mu_{\underline{A}}(x)) \mid x \in X \right\} \quad (1.1)$$

Where $\mu_{\underline{A}}(x)$ is called the membership function (or MF for short) for the fuzzy set A . The MF maps, each element of X in terms of the membership function is permitted to have any values between 0 and 1. If the values of the membership function $\mu_{\underline{A}}(x)$ is a restricted grade (or membership value) between 0 and 1.

Obviously, the definition of a fuzzy set is a simple extension of the definition of a classical set in which the characteristic function is either 0 or 1, then A is reduced to a classical set and $\mu_{\underline{A}}(x)$ is the characteristic function of A .

When the universe of speech U is a continuous set (Exp: real numbers), the fuzzy subset A is written as follows:

$$A = \int \mu_{\underline{A}}(x) / x \quad (1.2)$$

When U is discrete, A becomes as follows:

$$A = \sum \mu_{\underline{A}}(x) / x \quad (1.3)$$



Figure 1.1: Function of membership

[45] Use the Matlab commands to show the triangular membership function

```

1           Step 1: First enter the x value
2           >> x = (0:0.2:10);
3           Step 2: enter triangular membership function
4           >> y1 = trimf(x,[3 4 5 ]);
5           Step 3: plot the curve
6           >>Plot(x,y1)
7

```

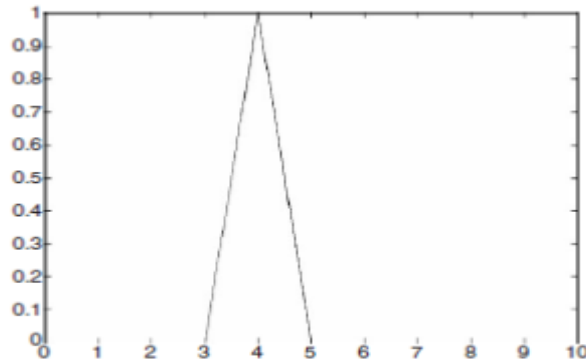


Figure 1.2: Triangular membership function

Fuzzy Set Operations [76]

Define three fuzzy sets \tilde{A} , \tilde{B} , and \tilde{C} on the universe X . For a given element x of the universe, the following function-theoretic operations for the set- union theoretical operations, intersection, and complement are defined for \tilde{A} , \tilde{B} , and \tilde{C} on X

Union

$$\mu_{\tilde{A} \cup \tilde{B}}(x) = \mu_{\tilde{A}}(x) \vee \mu_{\tilde{B}}(x) \quad (1.4)$$

Intersection

$$\mu_{\tilde{A} \cap \tilde{B}}(x) = \mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(x) \quad (1.5)$$

Complement

$$\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x) \quad (1.6)$$

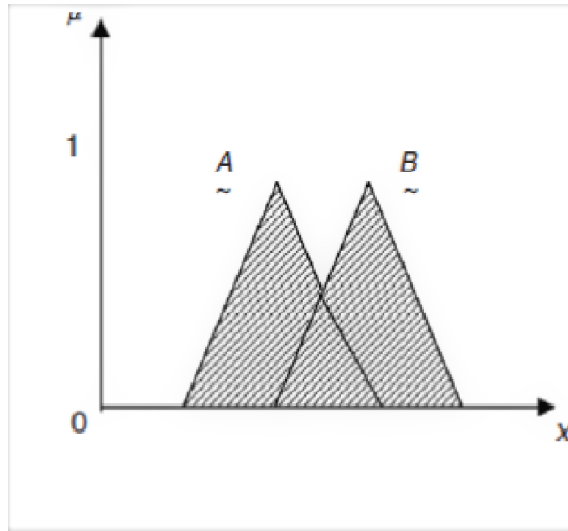


Figure 1.3: fuzzy sets union

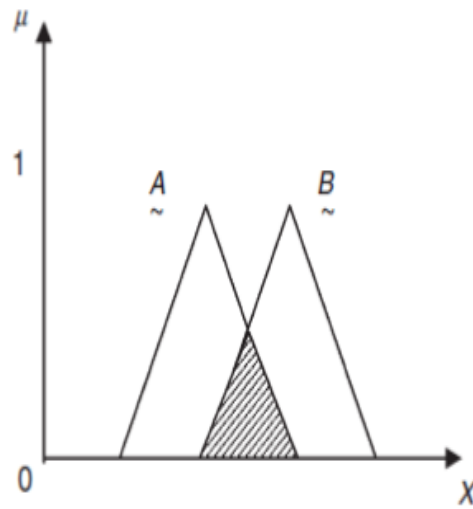


Figure 1.4: fuzzy sets intersection

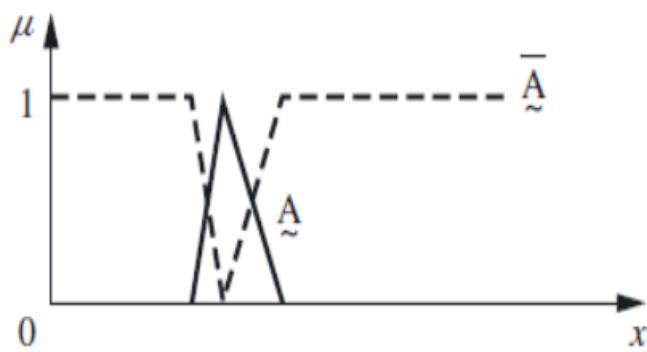


Figure 1.5: fuzzy sets complement \bar{A}

Fuzzy sets properties

De Morgan's laws: De Morgan's principles for classical sets also hold for fuzzy sets, as denoted by the following expressions :

$$\overline{\underline{A} \cap \underline{B}} = \overline{\underline{A}} \cup \overline{\underline{B}} \quad (1.7)$$

$$\overline{\underline{A} \cup \underline{B}} = \overline{\underline{A}} \cap \overline{\underline{B}} \quad (1.8)$$

$$\underline{A} \cup \overline{\underline{A}} \neq X, \underline{A} \cap \overline{\underline{A}} \neq \emptyset$$

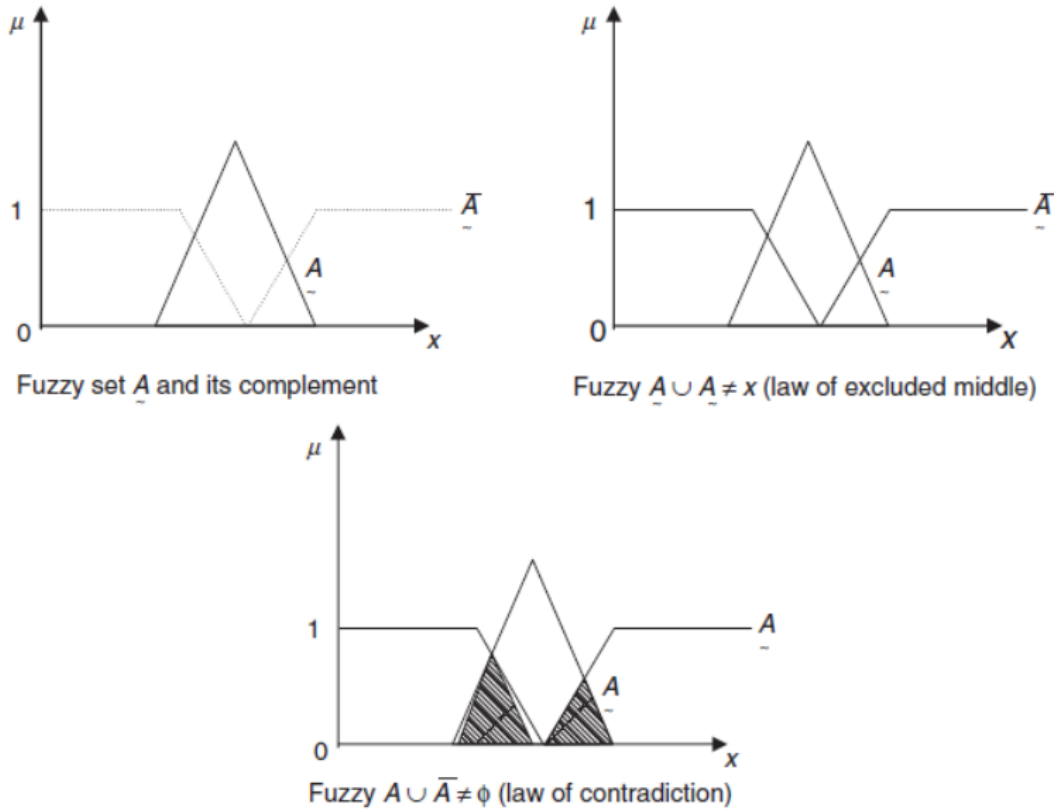


Figure 1.6: Law of excluded middle and law of contradiction

Commutativity

$$\underline{A} \cup \underline{B} = \underline{B} \cup \underline{A}$$

$$\underline{A} \cap \underline{B} = \underline{B} \cap \underline{A}$$

Associativity

$$\underline{A} \cup (\underline{B} \cup \underline{C}) = (\underline{A} \cup \underline{B}) \cup \underline{C}$$

$$\underline{A} \cap (\underline{B} \cap \underline{C}) = (\underline{A} \cap \underline{B}) \cap \underline{C}$$

Distributivity

$$\underline{A} \cup (\underline{B} \cap \underline{C}) = (\underline{A} \cup \underline{B}) \cap (\underline{A} \cup \underline{C})$$

$$\underline{A} \cap (\underline{B} \cup \underline{C}) = (\underline{A} \cap \underline{B}) \cup (\underline{A} \cap \underline{C})$$

Idempotency

$$\begin{aligned} \underline{\underline{A}} \cup \underline{\underline{A}} &= \underline{\underline{A}} \\ \underline{\underline{A}} \cap \underline{\underline{A}} &= \underline{\underline{A}} \end{aligned}$$

Identity

$$\begin{aligned} \underline{\underline{A}} \cup \phi &= \underline{\underline{A}} & \text{and} & & \underline{\underline{A}} \cap X &= \underline{\underline{A}} \\ \underline{\underline{A}} \cap \phi &= \phi & \text{and} & & \underline{\underline{A}} \cup X &= X \end{aligned}$$

Transitivity

$$\text{If } \underline{\underline{A}} \subset \underline{\underline{B}} \subset \underline{\underline{C}} \text{ then } \underline{\underline{A}} \subset \underline{\underline{C}}$$

Involution

$$\underline{\underline{\bar{\bar{A}}}} = \underline{\underline{A}}$$

Example 1.1.1. we have two discrete fuzzy sets, where we find Complement, Union, Intersection, Difference, and De Morgan's law.

$$\underline{\underline{A}} = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\} \quad \text{and} \quad \underline{\underline{B}} = \left\{ \frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\}$$

Intersection

$$\underline{\underline{A}} \cap \underline{\underline{B}} = \left\{ \frac{0.5}{2} + \frac{0.5}{3} + \frac{0.2}{4} + \frac{0.2}{5} \right\}$$

Union

$$\underline{\underline{A}} \cup \underline{\underline{B}} = \left\{ \frac{1}{2} + \frac{0.7}{3} + \frac{0.3}{4} + \frac{0.4}{5} \right\}$$

Complement

$$\begin{aligned} \bar{\underline{\underline{A}}} &= \left\{ \frac{0}{2} + \frac{0.5}{3} + \frac{0.7}{4} + \frac{0.8}{5} \right\} \\ \bar{\underline{\underline{B}}} &= \left\{ \frac{0.5}{2} + \frac{0.3}{3} + \frac{0.8}{4} + \frac{0.6}{5} \right\} \end{aligned}$$

Difference

$$\begin{aligned} \underline{\underline{A}} \setminus \underline{\underline{B}} &= \underline{\underline{A}} \cap \bar{\underline{\underline{B}}} = \left\{ \frac{0.5}{2} + \frac{0.3}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\} \\ \underline{\underline{B}} \setminus \underline{\underline{A}} &= \underline{\underline{B}} \cap \bar{\underline{\underline{A}}} = \left\{ \frac{0}{2} + \frac{0.5}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\} \end{aligned}$$

De morgan's principles

$$\begin{aligned} \overline{\underline{\underline{A}} \cup \underline{\underline{B}}} &= \bar{\underline{\underline{A}}} \cap \bar{\underline{\underline{B}}} = \left\{ \frac{0}{2} + \frac{0.3}{3} + \frac{0.7}{4} + \frac{0.6}{5} \right\} \\ \overline{\underline{\underline{A}} \cap \underline{\underline{B}}} &= \bar{\underline{\underline{A}}} \cup \bar{\underline{\underline{B}}} = \left\{ \frac{0.5}{2} + \frac{0.5}{3} + \frac{0.8}{4} + \frac{0.8}{5} \right\} \end{aligned}$$

Example 1.1.2. Consider the following fuzzy sets

$$\begin{aligned} A &= \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\} \\ B &= \left\{ \frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\} \end{aligned}$$

Calculate $A \cup B$, $A \cap B$, \bar{A} , \bar{B} by a Matlab program.

Solution: The Matlab program for the union, intersection, and complement is Program [45]

```

8      %enter the two matrix
9      u=input('enter the first matrix ');
10     v=input('enter the second matrix ');
11
12     option=input('enter the option');
13     %option 1 Union
14     %option 2 intersection
15     %option 3 complement
16     if (option==1)
17         end
18     if (option==2)
19         p=min(u,v)
20     end
21     if (option==3)
22         option1=input('enter wherthe to find complement for first matrix
23         or second matrix');
24         if (option1==1)
25             [m,n]=size(u);
26             q=ones(m)-u;
27         else
28             q=ones(m)-v;
29         end
30     end
31     output
32     (1) To find union of A and B
33     enter the first matrix [1 0.5 0.2 0.3]
34     enter the second matrix [0.5 0.7 0.2 0.4]
35     enter the option1
36     w =
37     1.0000 0.7000 0.2000 0.4000
38
39     (2) To find intersection of A and B is
40     enter the first matrix [1 0.5 0.2 0.3]
41     enter the second matrix [0.5 0.7 0.2 0.4]
42     enter the option2
43     p=
44     0.5000 0.5000 0.2000 0.3000
45     (3) To find complement of A
46     enter the first matrix [1 0.5 0.2 0.3]
47     enter the second matrix [0.5 .7 .2 .4]
48     enter the option3
49     enter the whether to find complement for first matrix or second matrix 2
50     q =
51     0.5000 0.3000 0.8000 0.6000

```

1.2 Characteristics of Type-1 fuzzy sets

Definition 1.2.1. (Support) Let A be a fuzzy set on a set X . The support of A is the crisp subset on X given by:

$$Supp(A) = \{x | \mu_A(x) > 0\} \quad (1.9)$$

Definition 1.2.2. (Kernel) Let A be a fuzzy set on a set X . The kernel or the core of A is the crisp subset on X given by:

$$Ker(A) = \{x | \mu_A(x) = 1\} \quad (1.10)$$

Definition 1.2.3. (Normality) A fuzzy set A is "normal" if its kernel is not empty. In other words, we can always find a point $x \in X$ such that $\mu_A(x) = 1$.

Definition 1.2.4. (Crossover points) A "crossover point" of a fuzzy set A is a point $x \in X$ at which $\mu_A(x) = 0.5$:

$$Cross(A) = \{x | \mu_A(x) = 0.5\}$$

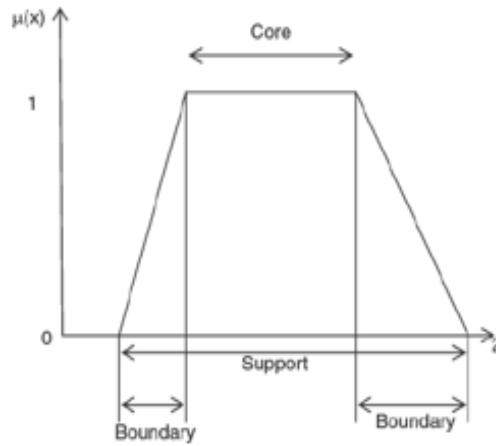


Figure 1.7: kernel and support of fuzzy set

Definition 1.2.5. (Height) Let A be a fuzzy set on a set X . The height of A is the highest value taken by its membership function given by:

$$H(A) = \text{Sup}_{x \in X} \mu_A(x)$$

Example 1.2.1. If $A = \left\{ \frac{0.2}{15} + \frac{1}{25} + \frac{0.8}{35} + \frac{1}{45} + \frac{0}{55} \right\}$

$$Ker(A) = \{25, 45\}$$

$$Cross(A) = \emptyset$$

$$Supp(A) = \{15, 25, 35, 45\}.$$

Definition 1.2.6. [77] λ -cut, strong λ -cut The " λ -cut" or " λ -level set" of a fuzzy set A is a crisp set defined by:

$$A_\lambda = \{x | \mu_A(x) \geq \lambda\} \quad (1.11)$$

Strong λ -cut or strong λ -level set are defined similarly:

$$A^\lambda = \{x | \mu_A(x) > \lambda\} \quad (1.12)$$

Using the notation for a level set, we can express the support and core of a fuzzy set A as $Supp(A) = A^\circ$ and $Ker(A) = A_1$

Example 1.2.2. $A = \left\{ \frac{1}{a} + \frac{0.9}{b} + \frac{0.6}{c} + \frac{0.3}{d} + \frac{0.01}{e} + \frac{0}{f} \right\}$

$$A_1 = \{a\}, \quad A_{0.9} = \{a, b\}$$

$$A_{0.6} = \{a, b, c\}, \quad A_{0.3} = \{a, b, c, d\}$$

$$A_{0+} = \{a, b, c, d, e\}, \quad A_0 = X$$

$\lambda = 1, 0.9, 0.6, 0.3, 0+, \text{ and } 0$

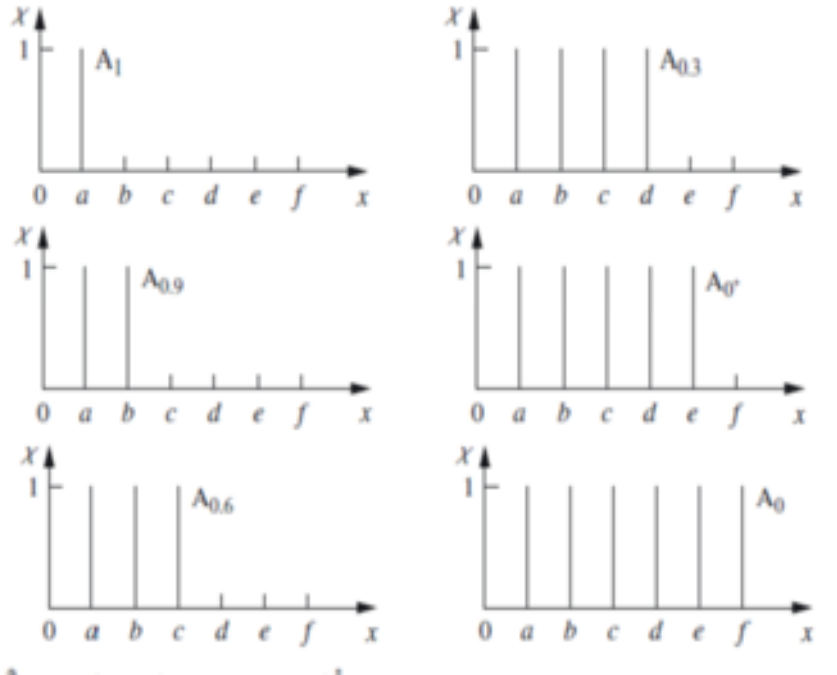


Figure 1.8: λ -cut"

λ -cut sets properties :

1. $(\tilde{A} \cup \tilde{B})_\lambda = \tilde{A}_\lambda \cup \tilde{B}_\lambda$
2. $(\tilde{A} \cap \tilde{B})_\lambda = \tilde{A}_\lambda \cap \tilde{B}_\lambda$
3. $(\overline{\tilde{A}})_\lambda \neq \overline{\tilde{A}_\lambda}$
4. For any $\lambda \leq \alpha$, where $0 \leq \alpha \leq 1$, it is true that $A_\alpha \subseteq A_\lambda$, where $A_0 = X$

Example 1.2.3. Two fuzzy sets \tilde{P} and \tilde{Q} are defined on x as follows :

$\mu(x)$	x_1	x_2	x_3	x_4	x_5
\tilde{P}	0.1	0.2	0.7	0.5	0.4
\tilde{Q}	0.9	0.6	0.3	0.2	0.8

- (a) $(\overline{\tilde{P}})_{0.2}$ (b) $(\tilde{Q})_{0.3}$ (c) $(\tilde{P} \cup \tilde{Q})_{0.5}$ (d) $(\tilde{P} \cap \tilde{Q})_{0.4}$ (e) $(\tilde{Q} \cup \overline{\tilde{P}})_{0.8}$

$$(f) \left(\underset{\sim}{P} \cup \underset{\sim}{\bar{P}} \right)_{0.2}$$

Solution :

Given Solution

$$\underset{\sim}{P} = \left\{ \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.7}{x_3} + \frac{0.5}{x_4} + \frac{0.4}{x_5} \right\}$$

$$\underset{\sim}{Q} = \left\{ \frac{0.9}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3} + \frac{0.2}{x_4} + \frac{0.8}{x_5} \right\}$$

Finding

$$\underset{\sim}{\bar{P}} = \left\{ \frac{0.9}{x_1} + \frac{0.8}{x_2} + \frac{0.3}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} \right\}$$

$$\underset{\sim}{\bar{Q}} = \left\{ \frac{0.1}{x_1} + \frac{0.4}{x_2} + \frac{0.7}{x_3} + \frac{0.8}{x_4} + \frac{0.2}{x_5} \right\}$$

$$(a) \left(\underset{\sim}{\bar{P}} \right)_{0.2} = \left\{ \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} \right\}$$

$$(b) \left(\underset{\sim}{\bar{Q}} \right)_{0.3} = \left\{ \frac{0}{x_1} + \frac{1}{x_2} + \frac{1}{x_2} + \frac{1}{x_4} + \frac{0}{x_5} \right\}$$

$$(c) \left(\underset{\sim}{P} \cup \underset{\sim}{Q} \right) = \left\{ \frac{0.9}{x_1} + \frac{0.6}{x_2} + \frac{0.7}{x_3} + \frac{0.5}{x_4} + \frac{0.8}{x_5} \right\}$$

$$\left(\underset{\sim}{P} \cup \underset{\sim}{Q} \right)_{0.4} = \left\{ \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} \right\}$$

$$(d) \left(\underset{\sim}{P} \cup \underset{\sim}{\bar{P}} \right) = \left\{ \frac{0.9}{x_1} + \frac{0.8}{x_2} + \frac{0.7}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} \right\}$$

$$\left(\underset{\sim}{P} \cup \underset{\sim}{\bar{P}} \right)_{0.8} = \left\{ \frac{1}{x_1} + \frac{1}{x_2} + \frac{0}{x_3} + \frac{0}{x_4} + \frac{0}{x_5} \right\}$$

$$(e) \left(\underset{\sim}{P} \cap \underset{\sim}{Q} \right) = \left\{ \frac{0.9}{x_1} + \frac{0.8}{x_2} + \frac{0.7}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} \right\}$$

$$\left(\underset{\sim}{P} \cap \underset{\sim}{Q} \right)_{0.4} = \left\{ \frac{0}{x_1} + \frac{0}{x_2} + \frac{0}{x_3} + \frac{0}{x_4} + \frac{1}{x_5} \right\}$$

$$(f) \left(\underset{\sim}{P} \cap \underset{\sim}{\bar{P}} \right) = \left\{ \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.3}{x_2} + \frac{0.5}{x_4} + \frac{0.4}{x_5} \right\}$$

$$\left(\underset{\sim}{P} \cap \underset{\sim}{\bar{P}} \right)_{0.8} = \left\{ \frac{0}{x_1} + \frac{1}{x_2} + \frac{1}{x_2} + \frac{1}{x_4} + \frac{1}{x_5} \right\}$$

Definition 1.2.7. For a finite fuzzy set $\underset{\sim}{A}$, the cardinality $|\underset{\sim}{A}|$ is defined as follows:

$$|\underset{\sim}{A}| = \sum_{x \in X} \mu_{\underset{\sim}{A}}(x) \text{ and } \|\underset{\sim}{A}\| = \frac{|\underset{\sim}{A}|}{|X|}$$

is Obviously called the relative cardinality of $\underset{\sim}{A}$. The relative cardinality of a fuzzy set depends on the cardinality of the universe. So you have to choose the same universe if you want to compare fuzzy sets by their relative cardinality.

Example 1.2.4. Consider a fuzzy subset of the set X , $X = a, b, c, d, e$ referred to

$$A = \left\{ \frac{0.6}{a}, \frac{0.9}{b}, \frac{0.1}{c}, \frac{0.3}{d}, \frac{0.2}{e} \right\}$$

$$|\underset{\sim}{A}| = \sum_{x \in X} \mu_{\underset{\sim}{A}}(x) = 2.1$$

Definition 1.2.8. (Fuzzy singleton) A fuzzy set whose support is a single point in X with $\mu_A(X) = 1$ is called a "fuzzy singleton".

1.3 Classification of Fuzzy sets

Fuzzy sets are classified according to their membership to:

Normal Fuzzy set

A fuzzy set A of X is called normal if there exists at least one element x in X such that $\mu_A(x) = 1$

Subnormal Fuzzy set

If all membership of the fuzzy set is affiliated fuzzy less than it is Convex Fuzzy set.

The latter is described by a membership function the membership of which values are strictly monotonically increasing or are strictly monotonically increasing then strictly monotonically decreasing with increasing values for elements in the universe. In other words, if for any elements x, y and z in a fuzzy set \tilde{A} , the relation $x < y < z$ implies that $\mu_{\tilde{A}}(y) \geq \min[\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(z)]$, then \tilde{A} is said to be a convex fuzzy set (Ross, 1995). Figure 1.9 shows a typical convex fuzzy set and a typical nonconvex one. Here, It is important to note that definition of convexity is different from other definitions of the same term in mathematics. In some areas of mathematics, the convexity of shape has to do with whether a straight line through any part of the shape goes outside the boundaries of that shape, This definition of convexity is not used here ; Figure succinctly summarize our definition or convexity.

Intersection of two convex sets

A special property of two convex fuzzy sets, say \tilde{A} and \tilde{B} , is that the intersection of these two convex fuzzy sets is also a convex fuzzy set, as shown in Figure 1.10. That is, for \tilde{A} and \tilde{B} . which are both convex, $\tilde{A} \cap \tilde{B}$ is also convex .

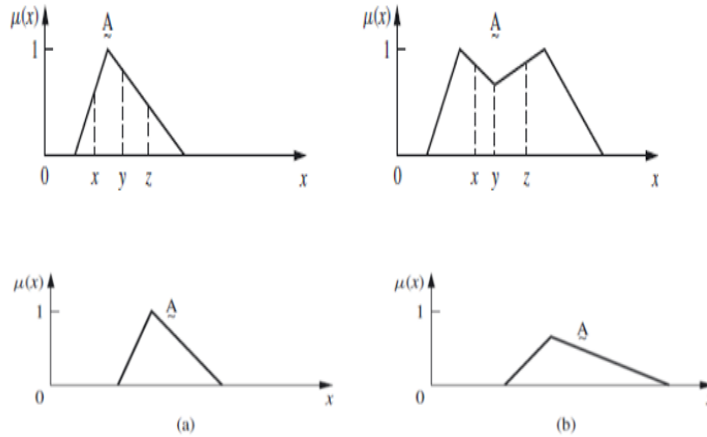


Figure 1.9: Fuzzy sets are that normal (a) and subnormal (b)

Definition 1.3.1. [76] An Interval-valued fuzzy set (IVFS), A , is characterized by a membership function σ_A ,

$$\sigma_A : X \rightarrow I = \{[a, b]; 0 \leq a \leq b \leq 1\}$$

where $\sigma_A(x)$ is the degree of membership of an element $x \in X$ in the set A . The set of all interval-valued fuzzy sets on X is denoted by $IVFS(X)$.

1.4 L-fuzzy sets type-1

In the previous sections, we have discussed the type -1 fuzzy sets in the unit real interval scale.

While in this section, we discuss these concepts on a lattice as a more general structure of truth values

Goguen generalized fuzzy sets to L -fuzzy sets in [20]. An L -fuzzy set φ on U is a mapping $\varphi: U \rightarrow L$, where L is a 'transitive partially ordered set'.

Definition 1.4.1. [32] Consider a bounded lattice $(L, \leq, 0_L, 1_L)$ and a universe X Goguen [20] introduced L -fuzzy sets as objects A characterized by some membership function $\mu_A: X \rightarrow L$. His approach has generalized the Cantorian (or crisp) sets (in which case $L = \{0, 1\}$ and μ_A is just the characteristic function $1_A: X \rightarrow \{0, 1\}$ of the Cantorian subset A of X), as well as the fuzzy sets proposed two years earlier by Zadeh [76] (here $L = [0, 1]$).

In the case of L -fuzzy sets, set relations and operations such as inclusion, union and intersection are defined pointwise by means of the ordering, supremum and infimum on L respectively. If the lattice L possesses a complement described by some unary operation $c: L \rightarrow L$, then the complement A^c

[20] We use the notation XY for the set-theoretic product of X and Y ; that is, $XY = \{(x, y) | x \in X, y \in Y\}$. $X + Y$ will denote the disjoint union of X and Y , and X^Y the set of all functions from Y to X . It is straightforward to verify the following set-theoretic isomorphisms.

Proposition 1.4.1. [20] $X + Y \cong Y + X$; $X + (Y + Z) \cong (X + Y) + Z$; $XY \cong YX$ $X(YZ) \cong (XY)Z$; $X(Y + Z) \cong XY + XZ$; $(X + Y)Z \cong XZ + YZ$ $X^{Y+Z} \cong X^Y X^Z$; $(XY)^Z \cong X^Z Y^Z$; $(X^Y)^Z \cong X^{YZ}$ There is no associative law for exponentiation, that is, $X^{(Y^Z)} \not\cong (X^Y)^Z$.

Definition 1.4.2. [20] An L -fuzzy set A on a set X is a function $A: X \rightarrow L$ Thus fuzzy sets are to be considered equal if and only if they are equal as functions. If the elements of X have a name (e.g., "characters"), A will be called an L -fuzzy set of such elements (e.g., "A is an L -fuzzy set of characters").

Definition 1.4.3. [56] For any L -fuzzy subsets A and B of X , define $A \subseteq B$ if and only if $A(x) \leq B(x)$ for all $x \in X$. Clearly \leq is a partial order on $FS_L(X)$. Also, for any crisp subsets S and T of X , we have $\chi_S \leq \chi_T \Leftrightarrow S \subseteq T$ The following is an easy verification.

Definition 1.4.4. [56] For an L -fuzzy set A and $\alpha \in L$, the α -cut A_α of A is defined as $A_\alpha = \{x | x \in X \text{ and } \alpha \leq A(x)\}$.

In the sequel, we assume that (L, \vee, \wedge) is a complete lattice with the least element 0 . The α -cuts of L -fuzzy sets have the following properties:

- (1) $\forall A \in F_L(X), A_0 = X$
- (2) $\forall \alpha \in L, \forall A \in F_L(X), A_\alpha = \bigcap_{\lambda \leq \alpha} A_\lambda$
- (3) $\forall \alpha_i \in L (i \in I), \forall A \in F_L(X), A_\alpha = \bigcap_{i \in I} A_{\alpha_i}$, where $\alpha = \bigvee_{i \in I} \alpha_i$

Let I be an arbitrary index set and $A_i \in F_L(X) (i \in I)$. The union $\bigcup_{i \in I} A_i$ and the intersection $\bigcap_{i \in I} A_i$ are respectively defined as follows: $\forall x \in X, (\bigcup_{i \in I} A_i)(x) = \bigvee_{i \in I} A_i(x), \forall x \in X, (\bigcap_{i \in I} A_i)(x) = \bigwedge_{i \in I} A_i(x)$

Proposition 1.4.2. [20] L^X can be given whatever operations L has, and these operations in L^X will obey any law valid in L which extends point by point. For example, if a binary operation $*$ is defined on L , it induces an operation $*$ on L^X by pointwise extension: $(A * B)(x) = A(x) * B(x)$, for $A, B \in L^X$ and $x \in X$. If some law, such as associative, commutative, or idempotent holds for $*$ on L , it also extends to L^X .

Definition 1.4.5. [20] A multiplicative lattice is a lattice M closed under an operation $*$ which satisfies the “distributive” laws, $\forall a, b \in M \ a * (b \vee c) = (a * b) \vee (a * c)$ and $(a \vee b) * c = (a * c) \vee (b * c)$. Identities involving $*$ ’s in multiplicative lattices will not remain identities when \wedge and \vee , and \geq and \leq are interchanged; thus the Principle of Duality is not valid. But there is a principle of left-right symmetry, as exhibited in the two “distributive” laws above.

Proposition 1.4.3. [20] Let a, b, c be elements of a multiplicative lattice M . Then:

- (1) $a \leq b \Rightarrow c * a \leq c * b$ and $a * c \Rightarrow b * c$;
- (2) $(a \wedge b) * (a \vee b) \leq (b * a) \wedge (a * b)$.

Proof. (1) $a \leq b \Rightarrow b * c = (a \vee b) * c = (a * c) \vee (b * c) \Rightarrow a * b \leq b * c$;

- (2) $(a \wedge b) * (a \vee b) = [(a \wedge b) * a] \vee [(a \wedge b) * b] \leq (b * a) \vee (a * b)$.

□

1.4.1 T–norm and t–conorm in lattices

Preliminaries

[47] A bounded lattice is a lattice (L, \leq) which has the top and bottom elements, which are written as: \top and \perp respectively, that is, there exist two elements $\top, \perp \in L$ such that $\perp \leq x \leq \top$, for all $x \in L$

Definition 1.4.6. [47] Given a bounded lattice (L, \leq, \perp, \top) , and $a, b \in L$, a subinterval $[a, b]$ of L , is a sublattice of L defined as

$$[a, b] = \{x \in L \mid a \leq x \leq b\}$$

The triangular norms on $[0, 1]$ are useful operators within the framework of fuzzy sets and fuzzy logic [[22], [38]]. since interest in lattices has been increasing in recent years, an adequate generalization of these operators should be given. A straightforward definition is presented in [63].

Definition 1.4.7. [47] Let (L_1, \leq, \perp, \top) be a bounded lattice. A binary operation $T: L \times L \rightarrow L$ is a triangular norm on L if the following conditions are fulfilled, for all $x, y, z \in L$:

1. $T(x, y) = T(y, x)$
2. If $x \leq y$, then $T(x, z) \leq T(y, z)$
3. $T(x, \top) = x$
4. $T(x, T(y, z)) = T(T(x, y), z)$

Example 1.4.1. [47] The following operators are examples of 1–norms in a bounded lattice. In any bounded lattice, we can define the meet (or infimum) t–norm. $T_M^L: L \times L \rightarrow L$, $T_M^L(x, y) = \inf\{x, y\}$ (as (L, \leq) is a lattice, there exists the infimum of two elements in L). Another interesting t–norm is the drastic product: $T_D^L: L \times L \rightarrow L$, defined as:

$$T_D^L(x, y) = \begin{cases} \inf\{x, y\} & \text{if } T \in \{x, y\} \\ \perp & \text{otherwise} \end{cases}$$

In a similar way that in $[0, 1]$, we see that T_D^L and T_M^L , are the smallest and greatest t–norms on a lattice.

T -distributivity

Definition 1.4.8. [59] Let $(L, \leq, 0, 1)$ be a bounded lattice and T_1 and T_2 every $x, y, z \in L$ such that at least one of the elements y, z is not

$$T_1(x, T_2(y, z)) = T_2(T_1(x, y), T_1(x, z))$$

is satisfied, then T_1 is called T_2 -distributive or we say that T_1 is

Example 1.4.2. [59] Let $(L = \{0, a, b, c, 1\}, \leq, 0, 1)$ be a bounded lattice w played in Figure 1 The functions T_1 and T_2 on the lattice L defined by

$$T_1(x, y) = \begin{cases} 0, & \text{if } x = a, y = a \\ b, & \text{if } x = c, y = c \\ x \wedge y, & \text{otherwise} \end{cases}$$

and

$$T_2(x, y) = \begin{cases} b, & \text{if } x = c, y = c \\ x \wedge y, & \text{otherwise} \end{cases}$$

are obviously t -norms on L such that T_1 is T_2 -distributive.

Proposition 1.4.4. [59] Let $(L, \leq, 0, 1)$ be a bounded lattice and T_1 and T_2 be two t -norms on L . If T_1 is T_2 -distributive, then T_1 is weaker than T_2 . Proof since all t -norms coincide on the boundary of L^2 , it is sufficient to show that $T_1 \leq T_2$ for all $x, y, z \in L \setminus \{0, 1\}$. By the T_2 -distributivity of T_1 , it is obtained that

$$T_1(x, y) = T_1(T_2(x, 1), y) = T_2(T_1(x, y), T_1(1, y)) = T_2(T_1(x, y), y) \leq T_2(x, y)$$

Thus, $T_1 \leq T_2$, i.e., T_1 is weaker than T_2

Remark 1.4.1. [59] The converse of Proposition 1.4.2. need not be true. Namely, for any two t -norms T_1 and T_2 , even if T_1 is weaker than T_2 , T_1 may not be T_2 -distributive. Now, let us investigate the following example.

Example 1.4.3. Consider the product t -norme $(T_P)(a, b) = a.b$ and the Łukasiewicz t -norm $T_L(a, b) = \max(0, a + b - 1)$. It is clear that $T_L < T_P$. since

$$T_L\left(\frac{4}{5}, T_P\left(\frac{7}{9}, \frac{1}{2}\right)\right) = T_L\left(\frac{4}{5}, \frac{7}{18}\right) = \frac{59}{90}$$

and

$$T_P\left(T_L\left(\frac{4}{5}, \frac{7}{9}\right), T_L\left(\frac{4}{5}, \frac{1}{2}\right)\right) = T_P\left(\frac{26}{45}, \frac{3}{10}\right) = \frac{78}{450} = \frac{13}{75}$$

T_L is not T_P -distributive.

Corollary 1.4.1. [59] Let L be a bounded lattice and T_1 and T_2 be any two t -norms on L . If both T_1 is T_2 -distributive and T_2 is T_1 -distributive, then $T_1 = T_2$

Proposition 1.4.5. [59] Let L be a bounded chain and T' be a t -norm on L . For every t -norm T , T is T' -distributive if and only if $T' = T_\wedge$.

Proof. \Rightarrow Let T be an arbitrary t -norm on L such that T' -distributive. By Proposition 1, it is obvious that $T \leq T'$ for any t -norm T . Thus, $T' = T_\wedge$

\Leftarrow : since L is a chain, for any $y, z \in L$, either $y \leq z$ or $z \leq y$. Suppose that $y \leq z$. Using the monotonicity of the t -norm T , we obtained that for any $x \in L$, $T(x, y) \leq T(x, z)$. Then

$$T(x, y) = T(x, y) \wedge T(x, z)$$

holds. Thus, for any $x, y, z \in L$

$$\begin{aligned} T(x, T_\wedge(y, z)) &= T(x, y) \\ &= T(x, y) \wedge T(x, z) \\ &= T_\wedge(T(x, y), T(x, z)) \end{aligned}$$

is satisfied, which shows that any t -norm T is T_\wedge -distributive. Reference document-1.pdf \square

Definition 1.4.9. [28] An operation $T: L^2 \rightarrow L$ (resp. $S: L^2 \rightarrow L$) is called a t -norm (resp. t -conorm) if it is commutative, associative, increasing with respect to both variables and has a neutral element $e = 1_L$ (resp. $e = 0_L$).

A pseudo- t -norm T (resp. pseudo- t -conorm S) [[13].[17]] on [0.1] is a t -norm (resp. t -conorm) [38] if it is commutative. We can extend the definition of pseudo- t -norms on bounded lattices.

[16]

Example 1.4.4. [16] Let $(L_1 \leq 0_L, 1_L)$ be a bounded lattice, a pseudo- t -norm $T_a^l: L^2 \rightarrow L$ and a pseudo- t -conorm $S_b^l: L^2 \rightarrow L$ be defined as follows, respectively.

$$T_a^l(x, y) = \begin{cases} (x \vee a) \wedge (y \vee a), & (x, y) \in I_a \times I_a, \\ x \wedge (y \vee a), & (x, y) \in [a, 1_L] \times I_a, \\ (x \vee a) \wedge y, & (x, y) \in I_a \times [0_L, a) \cup I_a \times [a, 1_L) \\ 0_L, & (x, y) \in [0_L, a) \times L \setminus \{1_L\} \\ x \wedge y, & \text{otherwise} \end{cases}$$

$$S_b^l(x, y) = \begin{cases} (x \wedge b) \vee (y \wedge b), & (x, y) \in I_b \times I_b \\ x \vee (y \wedge b), & (x, y) \in (0_L, b] \times I_b \\ (x \wedge b) \vee y, & (x, y) \in I_b \times (0_L, b] \cup I_b \times (b, 1_L], \\ 1_L, & (x, y) \in (b, 1_L] \times L \setminus \{0_L\} \\ x \vee y, & \text{otherwise} \end{cases}$$

where $a, b \in L \setminus \{0_L, 1_L\}$

It is easy to verify the associative law. monotonicity and neutral element in both operators. However, they do not satisfy the commutative law such as

$$T_a^l(x, y) = 0_L \neq T_a^l(y, x) = x, (x, y) \in [0_L, a) \times [a, 1_L)$$

Remark 1.4.2. All the notions and properties recalled before stay valid when we use bounded lattice instead of The unit real interval [0, 1].

1.5 Fuzzy relations

A fuzzy relation [77] \tilde{R} is a mapping from the Cartesian space $X \times Y$ to the interval $[0, 1]$, where the strength of the mapping is expressed by the membership function of the relation for ordered pairs from the two universes is $\mu_{\tilde{R}}(x, y)$.

1.5.1 Operations on Fuzzy Relations

Let \tilde{R} and \tilde{S} be fuzzy relations on the Cartesian space $X \times Y$. Then the following operations apply for the membership values for various set operations (these are similar to the same operations on crisp sets;

Union	$\mu_{\tilde{R} \cup \tilde{S}}(x, y) = \max(\mu_{\tilde{R}}(x, y), \mu_{\tilde{S}}(x, y))$
Intersection	$\mu_{\tilde{R} \cap \tilde{S}}(x, y) = \min(\mu_{\tilde{R}}(x, y), \mu_{\tilde{S}}(x, y))$
Complement	$\mu_{\tilde{R}^c}(x, y) = 1 - \mu_{\tilde{R}}(x, y)$
Containment	$\tilde{R} \subset \tilde{S} \Rightarrow \mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{S}}(x, y)$

Cardinality of Fuzzy Relations

Since the cardinality of fuzzy sets on any universe is infinity, the cardinality of a fuzzy relation between two or more universes is also infinity.

1.5.2 Properties of fuzzy Relations

Just as for crisp relations, the properties of commutativity, associativity, distributivity, involution, and idempotency all hold for fuzzy relations. Moreover, De Morgan's principles hold for fuzzy relations just as they do for crisp (classical) relations and the null relation O . And the complete relation, E . Are analogous to the null set and the whole set in set-theoretic form, respectively. Fuzzy relations are not constrained, as is the case for fuzzy sets in general, by the excluded middle axioms. Since a fuzzy relation \tilde{R} is also a fuzzy set. there is overlap between a relation and its complement; hence,

$$\tilde{R} \cup \tilde{R}^c \neq E$$

$$\tilde{R} \cap \tilde{R}^c \neq O$$

As seen in the foregoing expressions, the excluded middle axioms for fuzzy relations do not, generally result, in the null relation, O , or the complete relation, E .

1.5.3 Fuzzy Cartesian Product and Composition

Because fuzzy relations are fuzzy sets, we can define the Cartesian product to be a relation between two or more fuzzy sets. Let \tilde{A} be a fuzzy set on universe X and \tilde{B}

a fuzzy set on universe Y , then the Cartesian product between fuzzy sets \tilde{A} and \tilde{B} will result in a fuzzy relation \tilde{R} , which is contained within the full Cartesian product space, or

$$\tilde{A} \times \tilde{B} = \tilde{R} \subset X \times Y$$

where the fuzzy relation \tilde{R} has membership function

$$\mu_{\tilde{R}}(x, y) = \mu_{\tilde{A} \times \tilde{B}}(x, y) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y))$$

The cartesian product defined by $\tilde{A} \times \tilde{B} = \tilde{R}$ is implemented in the same fashion as is the cross product of two vectors. moreover, the cartesian product is not the same operation as the arithmetic product. In the case of bi-dimensional relations ($r = 2$), the former employs the idea of the pairing of elements among sets, whereas the latter uses actual arithmetic products between elements of sets. Each of the fuzzy sets could be thought of as a membership values vector; each value is associated with a particular element in each set. For example, for a fuzzy set (vector) \tilde{A} that has four elements, hence column vector of size 4×1 , and for a fuzzy set (vector) \tilde{B} that has live elements, hence a row vector size of 1×5 , the resulting fuzzy relation \tilde{R} will be represented by a matrix of size 4×5 , that is, \tilde{R} will have four rows and five columns. This idea is illustrated in the following example.

Example 1.5.1. Suppose we have two fuzzy sets, \tilde{A} defined on a universe of three discrete temperatures: $X = \{x_1, x_2, x_3\}$, \tilde{B} defined on a universe of two discrete pressures, and $Y = \{y_1, y_2\}$, then we want to find the fuzzy Cartesian product between them. Fuzzy set \tilde{A} could represent the "ambient" temperature and fuzzy set \tilde{B} the "near-optimum" pressure for a certain heat exchanger and the Cartesian product might represent the conditions (temperature-pressure pairs) of the exchanger that are associated with "efficient" operations. For example, let

$$\tilde{A} = \left\{ \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} \right\} \quad \text{and} \quad \tilde{B} = \left\{ \frac{0.3}{y_1} + \frac{0.9}{y_2} \right\}$$

Note that \tilde{A} can be represented as a column vector of size 3×1 and \tilde{B} can be represented by a row vector of 1×2 . Then the fuzzy Cartesian product results in a fuzzy relation \tilde{R} (of size 3×2) representing "efficient" conditions or

$$\tilde{A} \times \tilde{B} = \tilde{R} = \begin{matrix} & y_1 & y_2 \\ x_1 & \begin{bmatrix} 0.2 & 0.2 \end{bmatrix} \\ x_2 & \begin{bmatrix} 0.3 & 0.5 \end{bmatrix} \\ x_3 & \begin{bmatrix} 0.3 & 0.9 \end{bmatrix} \end{matrix}$$

The fuzzy composition can be defined just as it is for crisp (binary) relations. Suppose \tilde{R} is a fuzzy relation on the cartesian space $X \times Y$, \tilde{S} is a fuzzy relation on $Y \times Z$, and \tilde{T} is a fuzzy relation on $X \times Z$, the fuzzy max-min composition is defined in terms of the set-theoretic notation and membership function-theoretic notation as follows:

$$\tilde{T} = \tilde{R} \circ \tilde{S}$$

$$\mu_{\tilde{T}}(x, z) = \bigvee_{y \in Y} \left(\mu_{\tilde{R}}(x, y) \wedge \mu_{\tilde{S}}(y, z) \right)$$

and fuzzy max-product composition is defined in terms of the membership function-theoretic notation as

$$\mu_{\tilde{T}}(x, z) = \bigvee_{y \in Y} \left(\mu_{\tilde{R}}(x, y) \bullet \mu_{\tilde{S}}(y, z) \right)$$

It should be pointed out that, in general, neither crisp nor fuzzy compositions are commutative; that is,

$$\tilde{R} \circ \tilde{S} \neq \tilde{S} \circ \tilde{R}.$$

Example 1.5.2. Let us fuzzy relationships for $X \times Y$ (denoted by the fuzzy relation \tilde{R}) and $X \times Y$ (denoted by the fuzzy relation \tilde{S}). In this case, we change the elements of the universes to

$$X = \{x_1, x_2\}, \quad Y = \{y_1, y_2\}, \quad \text{and} \quad Z = \{z_1, z_2, z_3\}$$

Consider the following fuzzy relations:

$$\tilde{R} = \begin{array}{c} y_1 \quad y_2 \\ x_1 \begin{bmatrix} 0.7 & 0.5 \end{bmatrix} \\ x_2 \begin{bmatrix} 0.8 & 0.4 \end{bmatrix} \end{array} \quad \text{and} \quad \tilde{S} = \begin{array}{c} z_1 \quad z_2 \quad z_3 \\ y_1 \begin{bmatrix} 0.9 & 0.6 & 0.2 \end{bmatrix} \\ y_2 \begin{bmatrix} 0.1 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

Then, the resulting relation, \tilde{T} , which relates elements of universe X to elements of universe Z , that is, defined on Cartesian space $X \times Z$, can be found by max-min composition, for example,

$$\mu_{\tilde{T}}(x_1, z_1) = \max[\min(0.7, 0.9), \min(0.5, 0.1)] = 0.7$$

and the set

$$\tilde{T} = \begin{array}{c} z_1 \quad z_2 \quad z_3 \\ x_1 \begin{bmatrix} 0.7 & 0.6 & 0.5 \end{bmatrix} \\ x_2 \begin{bmatrix} 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}$$

We now illustrate the use of relations with fuzzy sets of medicine, electrical engineering, and civil engineering.

Definition 1.5.1. [78] Each linguistic variable is fully characterized by a quintuple (v, T, X, g, m) in which v is the name of the variable. T is the set of the linguistic terms of v that refers to a base variable whose values range over the universal set X . g is a syntactic rule for generating linguistic terms. and m is a semantic rule that assigns to each linguistic term $t \in T$ its meaning, $m(t)$, which is an FS on X , that is $m: T \rightarrow F(X)$, where $F(X)$ denotes the set of all ordinary (i.e., Tl) FSs of X , one identical set for each $t \in T$,

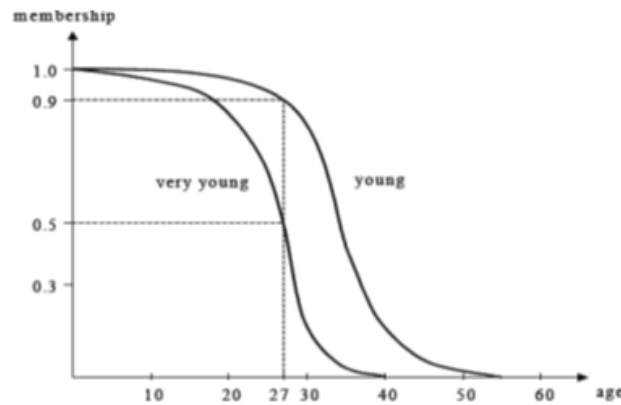


Figure 1.10: linguistic variable

1.6 t-norms and t-conorms

The notions of triangular-norms (*t-norms*) started with Menger [57] to construct metric spaces where probability distributions are used to describe the distance between two elements.

Definition 1.6.1. [57] A *t*-norm T on $[0, 1]$ is a function $T : [0, 1]^2 \rightarrow [0, 1]$ satisfies the following four axioms:

- 1 *Commutativity*: $(\forall x, y \in [0, 1])(T(x, y) = T(y, x))$;
- 2 *Associativity*: $(\forall x, y, z \in [0, 1])(T(x, T(y, z)) = T(T(x, y), z))$;
- 3 *Monotonicity*: $(\forall x, y, z \in [0, 1])(x \leq y \Rightarrow T(x, z) \leq T(y, z))$;
- 4 *Boundary condition*: $(\forall x \in [0, 1])(T(x, 1) = x)$.

Example 1.6.1. The following four operations are the most common *t*-norms:

- 1 *Minimum*: $T_M(x, y) = \min\{x, y\}$
- 2 *Product*: $T_P(x, y) = x \cdot y$
- 3 *Lukasiewicz*: $T_L(x, y) = \max\{x + y - 1, 0\}$
- 4 *Drastic product*:

$$T_D(x, y) = \begin{cases} 0 & \text{if } x, y \in [0, 1]^2. \\ \min(x, y) & \text{otherwise} \end{cases}$$

Definition 1.6.2. [57] A *t*-conorm is a function $S : [0, 1]^2 \rightarrow [0, 1]$ that for any $x, y, z \in [0, 1]$ satisfies (T1)-(T3) and the following boundary condition $S(x, 0) = S(0, x) = x$, $S(x, 1) = S(1, x) = 0$

Remark 1.6.1. Given a *t*-norm T , we find the associated dual *t*-conorm S by $S(x, y) = 1 - T(1 - x, 1 - y)$.

Example 1.6.2. The following four operations are the most common *t*-conorm:

- (1) *Maximum*: $S_M(x, y) = \max\{x, y\}$
- (2) *Probabilistic sum*: $S_P(x, y) = x + y - x \cdot y$
- (3) *Lukasiewicz*: $S_L(x, y) = \min\{x + y, 1\}$
- (4) *Drastic sum*:

$$S_D(x, y) = \begin{cases} 1, & \text{if } (x, y) \in]0, 1]^2 \\ \max\{x, y\}, & \text{otherwise} \end{cases}$$

Chapter 2

Type-2 fuzzy sets

The type-2 fuzzy set concept was first introduced by Zadeh (1975)[78] as an extension of a type-1 fuzzy set. A type-2 fuzzy set is characterized by a fuzzy membership function, i.e., the membership grade for each element of this set is a fuzzy set in $[0,1]$, unlike a type-1 set where the membership grade is a crisp number in $[0,1]$. Such sets can be used in situations where there is uncertainty about the membership grades themselves, e.g., uncertainty in the shape of the membership function or in some of its[5].

2.1 Type-2 Fuzzy Sets Concept

Definition 2.1.1. [79] A type m fuzzy set is a fuzzy set in X where the membership values of which are type $(m-1)$, $m > 1$ fuzzy sets on $[0,1]$

Definition 2.1.2. [78],[50]. Let X be a finite and nonempty set, which is referred to as the universe a type-2 fuzzy set, denoted by \tilde{A} is characterized by a type-2 membership function $\mu_{\tilde{A}}(x, u)$ as $\mu_{\tilde{A}} : X \times [0,1] \rightarrow [0,1]^{J_x} = \text{map}(J_x, [0,1])$, ($J_x \subseteq [0,1]$), where $x \in X$ and $u \in J_x$ that is

$$\tilde{A} = \left\{ ((x, u), \mu_{\tilde{A}}(x, u)) / x \in X, \text{ and } u \in J_x \subseteq [0,1], \text{ wher } 0 \leq \mu_{\tilde{A}}(x, u) \leq 1 \right\}$$

\tilde{A} can also be expressed as

$$\tilde{A} = \sum_{x \in X} \sum_{u \in J_x} \mu_{\tilde{A}}(x, u) / (x, u) = \sum_{x \in X} \sum_{u \in J_x} f_x(u) / u / x, J_x \subseteq [0,1]$$

where $f_x(u) = \mu_{\tilde{A}}(x, u)$ and denotes union over all admissible x and u for continuous universes of discourse, $\sum^{\tilde{A}}$ is replaced by \int

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{A}}(x, u) / (x, u)$$

where x is the primary variable in the domain X ; u is the secondary variable in domain J_x at each $x \in X$. J_x is called the primary membership of x , and $\mu_{\tilde{A}}(x, u)$ the secondary membership grades of \tilde{A} , $J_x \subseteq [0,1]$ and $\int \int$ denote union over all admissible x and u in definition 2.1.2, the first restriction that $\forall u \in J_x \subseteq [0,1]$ is consistent with the type-1 constraint that $0 \leq \mu_A(x) \leq 1$, i.e., when uncertainties disappear a type-2 membership function must be reduced to a type-1 membership function, in which case the variable u equals $\mu_A(x)$ and $0 \leq \mu_A(x) \leq 1$. The second restriction that $0 \leq \mu_{\tilde{A}}(x, u) \leq 1$ is consistent with the fact that the amplitudes of a membership function should lie between or be equal to 0 and 1.

Type-2 membership function representation

$$\tilde{A} = \left\{ (x, u), \mu_{\tilde{A}}(x, u) \mid x \in X, J_x \subseteq [0, 1] \right\}$$

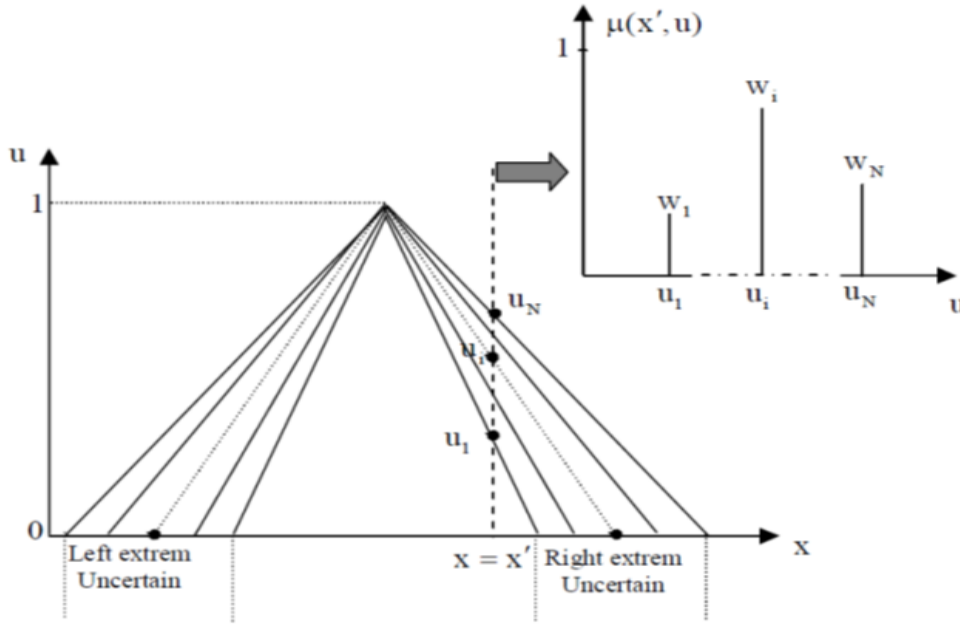


Figure 2.1: Triangular MFs when base end-points (l and r) have uncertainty intervals associated with them. [55]

Example 2.1.1. Suppose that $X = \text{Salima, Basma, Halima, Radhia, Aicha}$ is a set of women and that A is a fuzzy set of type 2 of beautiful women in X . Then we may have

$$A = \text{beautiful} = \text{middle/Salima} + \text{notlow/Basma} + \text{low/Halima} + \text{veryhigh/Radhia} + \text{high/Aicha}$$

where the fuzzy grades labeled middle, low, high are assumed to be fuzzy sets in $J = 0, 0.1, \dots, 0.9, 1 \subseteq [0, 1]$ and, for example, are expressed as follows.

$$\text{middle} = 0.3/0.3 + 0.7/0.4 + 1/0.5 + 0.7/0.6 + 0.3/0.7. \quad \text{low} = 1/0 + 0.9/0.1 + 0.7/0.2 + 0.4/0.3. \quad \text{high} = 0.4/0.7 + 0.7/0.8 + 0.9/0.9 + 1/1.$$

Moreover, fuzzy grades named not low and very high are defined from fuzzy grades low and high by using the linguistic hedges concept (Zadeh, 1972)

$$\text{not low} = 0.1/0.1 + 0.3/0.2 + 0.6/0.3 + 1/(0.4 + 0.5 + \dots + 1). \quad \text{very high} = 0.16/0.7 + 0.49/0.8 + 0.81/0.9 + 1/1.$$

Definition 2.1.3. [52] J_x , the primary membership of x , is :

$$J_x = \left\{ (x, u) \mid u \in [0, 1], \mu_{\tilde{A}}(x, u) > 0 \right\}$$

It can also be expressed as a subset of $\{x\} \times I_x$, i.e., $J_x = \{x\} \times I_x$ where

$$I_x = \left\{ u \in [0, 1] \mid \mu_{\tilde{A}}(x, u) > 0 \right\}$$

Definition 2.1.4. [52] The support of \tilde{A} comprise all $(x, u) \in X \times [0, 1]$ such that $\mu_{\tilde{A}}(x, u) > 0$, and is also called the domain of uncertainty of \tilde{A} , $DOU(\tilde{A})$, i.e.,

$$DOU(\tilde{A}) = \left\{ (x, u) \in X \times [0, 1] \mid \mu_{\tilde{A}}(x, u) > 0 \right\} = \bigcup_{x \in X} J_x$$

Definition 2.1.5. [52] When the support or the secondary MF, l_x , is closed for $x \in X$, .ie.,

$$I_x = \left\{ u \in [0, 1] \mid \mu_{\tilde{A}}(x, u) > 0 \right\} = \left[\underline{\mu}_{\tilde{A}}(x), \bar{\mu}_{\tilde{A}}(x) \right] \quad (2.1)$$

where (Aisbett et al. 2010)

$$\bar{\mu}_{\tilde{A}}(x) = \sup \left\{ u \mid u \in [0, 1], \mu_{\tilde{A}}(x, u) > 0 \right\} \quad (2.2)$$

$$\underline{\mu}_{\tilde{A}}(x) = \inf \left\{ u \mid u \in [0, 1], \mu_{\tilde{A}}(x, u) > 0 \right\} \quad (2.3)$$

then the domain or uncertainty of \tilde{A} is called the footprint of uncertainty of \tilde{A} , $FOU(\tilde{A})$. i.e.,

$$DOU(\tilde{A}) = FOU(\tilde{A}) = \left\{ (x, u) \mid x \in X \text{ and } u \in \left[\underline{\mu}_{\tilde{A}}(x), \bar{\mu}_{\tilde{A}}(x) \right] \right\} \quad (2.4)$$

Note that $\underline{\mu}_{\tilde{A}}(x)$ and $\bar{\mu}_{\tilde{A}}(x)$ are called the lower and upper MFs or $FOU(\tilde{A})$ [35], and are the lower and upper (type-1 fuzzy set) bounding function of the FOU, respectively. Commonly used abbreviations for the lower and upper MFs or \tilde{A} are $LMF(\tilde{A})$ and $UMF(\tilde{A})$.

[52] The class of all type-2 fuzzy sets of the universe X denoted by $\tilde{F}_{T_2}(X)$.

Type- 2 fuzzy universe set

Definition 2.1.6. A type-2 fuzzy universe set denoted \tilde{X} such that $\tilde{X} = \sum_{x \in X} \sum_{u \in [0, 1]} \frac{1}{x}$

Upper and lower membership functions

Definition 2.1.7. [5]

an upper membership function and a lower membership function are two type-1 membership functions that are bounds for the FOU of a type-2 fuzzy set \tilde{A} . The upper membership function is associated with the upper bound of $FOU(\tilde{A})$. The lower membership function is associated with the lower bound of $FOU(\tilde{A})$. We illustrate the concept of upper and lower membership functions as well as the footprint of uncertainty in the following example.

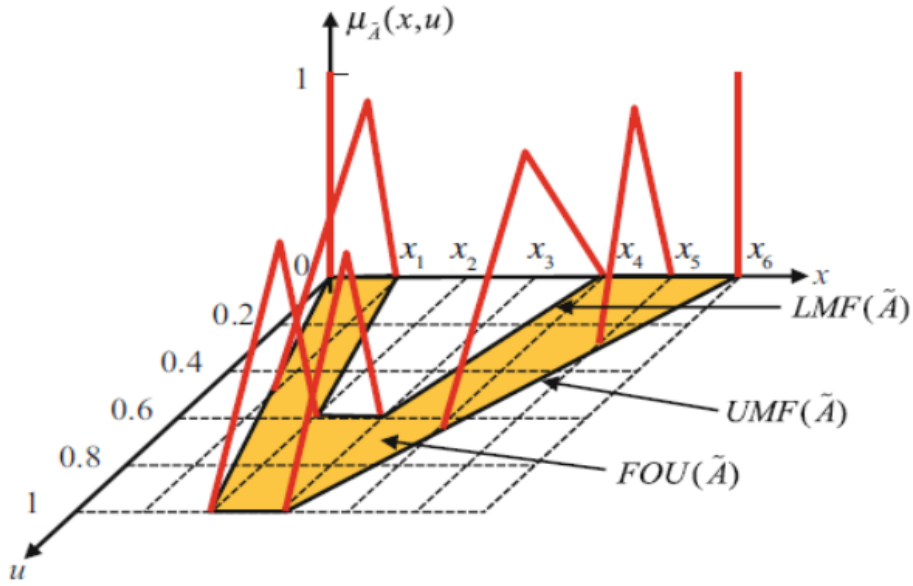


Figure 2.2: three-dimensional representation the of type-2 membership function [54]

2.2 Interval type- 2 fuzzy set

[48] When all the secondary grades of types $\tilde{\tilde{A}}$ are equal to 1, that is $\mu_{\tilde{\tilde{A}}}(x, u)$ for all $x \in X$ all $u \in J_x$, $\tilde{\tilde{A}}$ is an interval type-2 fuzzy set, $\tilde{\tilde{A}}$ is a special case of general type-2 fuzzy set where all of the secondary membership functions of $\tilde{\tilde{A}}$ are equal to 1. $\tilde{\tilde{A}}$ is an interval type-2 fuzzy set

$$\tilde{\tilde{A}} = \int_{x \in X} \int_{u \in J_x} \frac{1}{(x, u)} \quad (2.5)$$

where $J_x \subseteq [0, 1]$

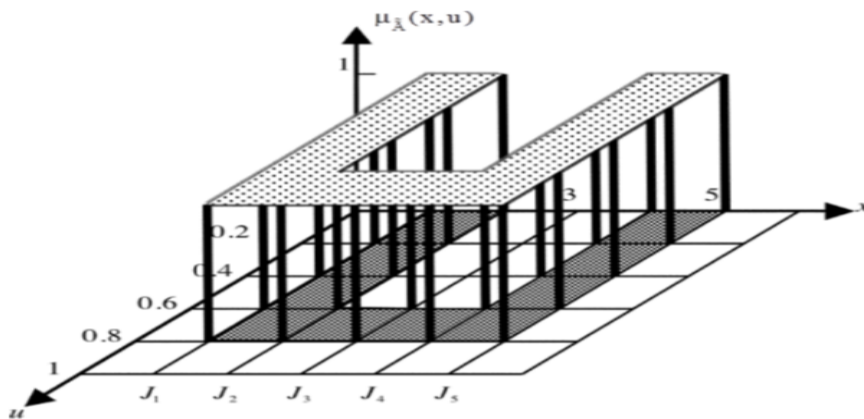


Figure 2.3: Example of an interval type-2 membership function for discrete universes of discourse [50]

Definition 2.2.1. [50]. At each value of x , say $x = x'$, the 2D plane whose axes are u and $\mu_{\tilde{A}}(x', u)$ is called a vertical slice of $\mu_{\tilde{A}}(x, u)$. A secondary membership function is a vertical slice of $\mu_{\tilde{A}}(x, u)$. It is $\mu_{\tilde{A}}(x = x', u)$ for $x' \in X$ and $\forall u \in J_{x'} \subseteq [0, 1]$, i.e., $\mu_{\tilde{A}}(x = x', u) \equiv \mu_{\tilde{A}}(x')$ $= \int_{u \in J_{x'}} f_{x'}(u)/u J_{x'} \subseteq [0, 1]$ (3) in which $0 \leq f_{x'}(u) \leq 1$. Because $\forall x' \in X$ we drop the prime notation on $\mu_{\tilde{A}}(x')$. and refer to $\mu_{\tilde{A}}(x)$ as a secondary membership function; it is a type-1 fuzzy set, which we also refer to as a secondary set.

Example 2.2.1. The type-2 membership function that is depicted in Fig 2.2 has five vertical slices associated with it. The secondary membership function at $x = 2$, e.g., is

$$\mu_{\tilde{A}}(2) = 0.5/0 + 0.35/0.2 + 0.35/0.4 + 0.2/0.6 + 0.5/0.8$$

2.3 Operation of Type-2 fuzzy sets

[49] In this section, we describe the type-2 fuzzy sets theoretical operations set. We are interested in the case of type-2 fuzzy sets, $\tilde{A}_i (i = 1, \dots, r)$, the secondary membership functions are type-1 fuzzy sets. To compute the union, the intersection, and the type-2 fuzzy sets complement, we need to extend the binary operations of minimum (or product) and maximum, and the unary operation of negation, from crisp numbers to type-1 fuzzy sets, because at each x , $\mu_{\tilde{A}_i}(x, u)$ is a function (unlike the type-1 case, where $\mu_{\tilde{A}_i}(x)$ is a crisp number). The tool for computing the union, intersection, and type-2 fuzzy sets complement is Zadeh's extension principle (Zadeh, 1975). We study operations set on type-2 fuzzy sets. We first discuss join and meet operations of membership grades of type-2 fuzzy sets under left continuous t-norms and derive distributive law of type-2 fuzzy sets [58].

Consider two type-2 fuzzy sets, \tilde{A} and \tilde{B} , in a universe X . Let $\mu_{\tilde{A}(x)}$ and $\mu_{\tilde{B}(x)}$ be the membership grades of these two sets, which are represented for each $x \in X$, $\mu_{\tilde{A}(x)} = \sum_{u \in J_x^u} f_x(u)/u$ and $\mu_{\tilde{B}(x)} = \sum_{w \in J_x^w} g_x(w)/w$, respectively, where $u \in J_x^u, w \in J_x^w$ indicate the primary memberships of x and $f_x(u), g_x(w) \in [0, 1]$ indicate the secondary memberships (grades) of x . The membership grades for the union, intersection, and complement of the type-2 fuzzy sets \tilde{A} and \tilde{B} have been defined as follows

Containment:

\tilde{A} is a subtype-2 fuzzy set of \tilde{B} denoted $\tilde{A} \subseteq \tilde{B}$ if $u < v$ and $f_x(u) \leq g_x(w)$ for every $x \in X$

Equality:

\tilde{A} and \tilde{B} are equal type-2 fuzzy sets, denoted $\tilde{A} = \tilde{B}$ if $u = w$ and $f_x(u) = \mu_{\tilde{A}}(x, u) = g_x(w) = \mu_{\tilde{B}}(x, w)$ for every $x \in X$

Two type-2 fuzzy sets union: [34]

$$\tilde{A} \cup \tilde{B} = \sum_{u \in J_x^u} \sum_{w \in J_x^w} (f_x(u) * g_x(w)) / (u \vee w)$$

Two type-2 fuzzy sets intersection: [34]

$$\tilde{A} \cap \tilde{B} = \sum_{u \in J_x^u} \sum_{w \in J_x^w} (f_x(u) * g_x(w)) / (u \wedge w)$$

where \vee represents the max t-conorm and \wedge indicates the minimum t-norm

Type-2 fuzzy set complement: [34]

Complement : $\tilde{A} \Leftrightarrow \mu_{\tilde{A}}(x) = \neg \mu_A(x) = \int_u f_x(u)/(1-u)$

Join and Meet under min t-norm

Definition 2.3.1. [49] Suppose that we have two convex, normal, type-1 real fuzzy sets \tilde{F} and \tilde{G} characterized by membership functions f and g respectively. Let $v_0 \in \mathbb{R}$ and $v_1 \in \mathbb{R}$ be such that $v_0 < v_1$ and $f(v_0) = g(v_1) = 1$. Then the membership functions of the join and meet of \tilde{F} and \tilde{G} , using max t-conorm and min t-norm, can be expressed as

$$\begin{aligned} \mu_{(\tilde{F} \cup \tilde{G})}(\theta) &= f(\theta) \wedge g(\theta); \theta < v_0 \\ &= g(\theta); v_0 \leq \theta \leq v_1 \\ &= f(\theta) \vee g(\theta); \theta > v_1 \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} \mu_{(\tilde{F} \cap \tilde{G})}(\theta) &= f(\theta) \vee g(\theta); \theta < v_0 \\ &= f(\theta); v_0 \leq \theta \leq v_1 \\ &= f(\theta) \wedge g(\theta); \theta > v_1 \end{aligned} \quad (2.7)$$

Definition 2.3.2. Type-2 fuzzy sets union

The union of \tilde{A}_1 and \tilde{A}_2 are another type-2 fuzzy set, just as the union of type-1 fuzzy sets A_1 and A_2 is another type-1 fuzzy set. More formally, we have the following expression

$$\tilde{A}_1 \cup \tilde{A}_2 = \int_{x \in X} \mu_{\tilde{A}_1 \cup \tilde{A}_2}(x) / x \quad (2.8)$$

Basically, the join between two secondary membership functions must be performed between every possible pair of primary memberships.

Let $\mu_A(x)$ and $\mu_B(x)$ be two fuzzy grades (that is, fuzzy sets in $J \subseteq [0, 1]$) of fuzzy sets of type 2, A and B , respectively, represented as

$$\begin{aligned} \mu_A(x) &= f(u_1) / u_1 + f(u_2) / u_2 + \dots + f(u_n) / u_n \\ &= \sum_i f(u_i) / u_i, \quad u_i \in J \end{aligned} \quad (2.9)$$

$$\begin{aligned} \mu_B(x) &= g(w_1) / w_1 + g(w_2) / w_2 + \dots + g(w_m) / w_m \\ &= \sum_j g(w_j) / w_j, \quad w_j \in J \end{aligned} \quad (2.10)$$

where the function f and g are membership functions of fuzzy grades (fuzzy sets in $J \subseteq [0, 1]$) $\mu_A(x)$ and $\mu_B(x)$ respectively, and the values $f(u_i)$ and $g(w_j)$ in $[0, 1]$ represent the grades for u_i and w_j in J , respectively.

Thus the operation type 2 fuzzy sets are expressed by the following.

Union

$$\begin{aligned} A \cup B \Leftrightarrow \mu_{A \cup B}(x) &= \mu_A(x) \sqcup \mu_B(x) \\ &= \left(\sum_i f(u_i) / u_i \right) \sqcup \left(\sum_j g(w_j) / w_j \right) \\ &= \sum_{i,j} (f(u_i) \wedge g(w_j)) / (u_i \vee w_j) \end{aligned} \quad (2.11)$$

Intersection

$$\begin{aligned}
 A \cap B &\Leftrightarrow \mu_{A \cap B}(x) = \mu_A(x) \sqcap \mu_B(x) \\
 &= \left(\sum_i f(u_i) / u_i \right) \sqcap \left(\sum_j g(w_j) / w_j \right) \\
 &= \sum_{i,j} (f(u_i) \wedge g(w_j)) / (u_i \wedge w_j)
 \end{aligned} \tag{2.12}$$

Complement

$$\tilde{A} \Leftrightarrow \mu_{\tilde{A}}(x) \text{ then the MF of the complement of } \tilde{A} \text{ is } \mu_{\tilde{A}}(x) = \int_u f_x(u) / (1 - u) \tag{2.13}$$

where \vee and \wedge represent max and min, respectively. We call the operations for fuzzy grades, that is, \sqcup like join, \sqcap as to meet, and \neg as negation hereafter. [58]. If more than one combination of pairs gives the same point, then in the join we keep the one with maximum membership grade. We will consider a simple example to illustrate the union operation. In Figure 3.6 we plot two type-2 Gaussian membership functions, and the union is shown in Figure 3.7.

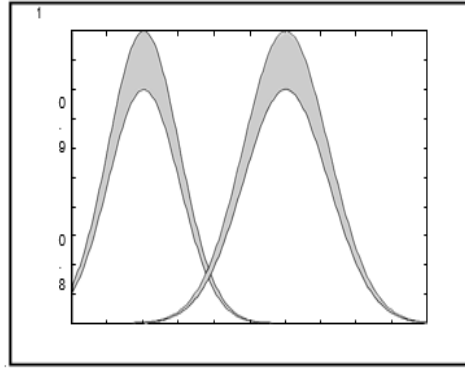


Figure 2.4: Type-2 Gaussian MFs

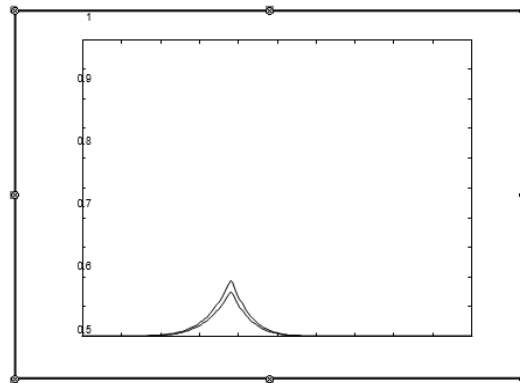


Figure 2.5: Intersection of two Interval Type-2 Fuzzy Sets

Example 2.3.1. [2] Let $X = \{x_1, x_2, x_3\}$ be a nonempty set, and let \tilde{A} and \tilde{B} are two type-2 fuzzy sets over the same universe X

$$\tilde{A} = \{(x_1, 0.1), 0.3), ((x_1, 0.5), 1), ((x_2, 0.5), 1), ((x_2, 0.6), 0.3), ((x_3, 0.8), 1)\}$$

$$\tilde{B} = \{(x_1, 0.1), 0.7), ((x_1, 0.2), 1), ((x_2, 0.6), 1), ((x_3, 0.5), 0.6), ((x_3, 0.9), 1)\}$$

$\tilde{A} \cup \tilde{B}$ for $x = x_1$ we get

$$\begin{aligned} \mu_{\tilde{A} \cup \tilde{B}}(x_1) &= \frac{0.3 \wedge 0.7}{0.1 \vee 0.1} + \frac{0.3 \wedge 1}{0.1 \vee 0.2} + \frac{1 \wedge 0.7}{0.5 \vee 0.1} + \frac{1 \wedge 1}{0.5 \vee 0.2} \\ &= \frac{0.3}{0.1} + \frac{0.3}{0.2} + \frac{0.7}{0.5} + \frac{1}{0.5} \\ &= \{(0.1, 0.3), (0.2, 0.3), (0.5, \max(0.7, 1))\} \end{aligned}$$

For $x = x_1$

$$\tilde{A} \cup \tilde{B} = \{(x_1, 0.1), 0.3), ((x_1, 0.2), 0.3), ((x_1, 0.5), 1)\}$$

$\tilde{A} \cup \tilde{B}$ for $x = x_2$ we get

$$\begin{aligned} \mu_{\tilde{A} \cup \tilde{B}}(x_2) &= \frac{1 \wedge 1}{0.5 \vee 0.6} + \frac{0.3 \wedge 1}{0.6 \vee 0.6} \\ &= \frac{1}{0.6} + \frac{0.3}{0.6} \\ &= \{(0.6, \max(1, 0.3))\} \end{aligned}$$

For $x = x_2$, $\tilde{A} \cup \tilde{B} = \{(x_2, 0.6), 1)\}$

$\tilde{A} \cup \tilde{B}$ for $x = x_3$ to get

$$\begin{aligned} \mu_{\tilde{A} \cup \tilde{B}}(x_3) &= \frac{1 \wedge 0.6}{0.8 \vee 0.5} + \frac{1 \wedge 1}{0.8 \vee 0.9} \\ &= \frac{0.6}{0.8} + \frac{1}{0.9} \\ &= \{(0.8, 0.6), (0.9, 1)\} \end{aligned}$$

For $x = x_3$,

$$\tilde{A} \cup \tilde{B} = \{(x_3, 0.6), 0.8), ((x_3, 0.9), 1)\}$$

Hence,

$$\tilde{A} \cup \tilde{B} = \{(x_1, 0.1), 0.3), ((x_1, 0.2), 0.3), ((x_1, 0.5), 1), ((x_2, 0.6), 1), ((x_3, 0.6), 0.8), ((x_3, 0.9), 1)\}$$

$\tilde{A} \cap \tilde{B}$ for $x = x_1$

$$\tilde{A} \cap \tilde{B} = \{(x_1, 0.1), 0.7), ((x_1, 0.2), 1)\}$$

$\tilde{A} \cap \tilde{B}$ for $x = x_2$

$$\begin{aligned} \mu_{\tilde{A} \cap \tilde{B}}(x_2) &= \frac{1 \wedge 1}{0.5 \wedge 0.6} + \frac{0.3 \wedge 1}{0.6 \wedge 0.6} \\ &= \frac{1}{0.5} + \frac{0.3}{0.6} \\ &= \{(0.5, 1), (0.6, 0.3)\} \end{aligned}$$

For $x = x_2$

$$\tilde{A} \cap \tilde{B} = \{(x_2, 0.5), 1), (x_2, 0.6), 0.3)\}$$

$\tilde{A} \cap \tilde{B}$ for $x = x_3$

$$\begin{aligned} \mu_{\tilde{A} \cap \tilde{B}}(x_3) &= \frac{1 \wedge 0.6}{0.8 \wedge 0.5} + \frac{1 \wedge 1}{0.8 \wedge 0.9} \\ &= \frac{0.6}{0.5} + \frac{1}{0.8} \\ &= \{(0.5, 0.6), (0.8, 1)\} \end{aligned}$$

$$\begin{aligned}
& \text{For } x = x_1, \quad \tilde{A} \cap \tilde{B} = \{(x_3, 0.5), 0.6\}, \{(x_3, 0.8), 1\}\} \\
& \tilde{A} \cap \tilde{B} = \{(x_1, 0.1), 0.7\}, \{(x_1, 0.2), 1\}, \{(x_2, 0.5), 1\}, \{(x_2, 0.6), 0.3\}, \{(x_3, 0.5), 0.6\}, \{(x_3, 0.8), 1\}\} \\
& \tilde{A} = \{(x_1, 0.1), 0.7\}, \{(x_1, 0.5), 0.5\}, \{(x_2, 0.5), 0.5\}, \{(x_2, 0.6), 0.7\}, \{(x_3, 0.8), 0.2\}\}
\end{aligned}$$

Theorem 2.3.1. [70] Let \tilde{A} , \tilde{B} , and \tilde{C} be two T2 fuzzy sets on X . If secondary membership functions $\mu_{\tilde{A}}^{\approx}(x)$, $\mu_{\tilde{B}}^{\approx}(x)$ and $\mu_{\tilde{C}}^{\approx}(x)$ are convex fuzzy sets for any $x \in X$, then we have

$$(\tilde{A} \cup \tilde{B}) \cap \tilde{C} = (\tilde{A} \cap \tilde{C}) \cup (\tilde{B} \cap \tilde{C}) \quad (2.14)$$

and

$$(\tilde{A} \cap \tilde{B}) \cup \tilde{C} = (\tilde{A} \cup \tilde{C}) \cap (\tilde{B} \cup \tilde{C}) \quad (2.15)$$

Proof 2.3.1. For any $x \in X$, since secondary membership functions $\mu_{\tilde{A}}^{\approx}(x)$, $\mu_{\tilde{B}}^{\approx}(x)$ and $\mu_{\tilde{C}}^{\approx}(x)$ are convex T1 fuzzy sets in $[0, 1]$, from [1], they satisfy distributive laws under \sqcup and \sqcap . Therefore, we obtain

$$\begin{aligned}
& \left[\mu_{\tilde{A}}^{\approx}(x) \sqcup \mu_{\tilde{B}}^{\approx}(x) \right] \sqcap \mu_{\tilde{C}}^{\approx}(x) \\
& = \left[\mu_{\tilde{A}}^{\approx}(x) \sqcap \mu_{\tilde{C}}^{\approx}(x) \right] \sqcup \left[\mu_{\tilde{B}}^{\approx}(x) \sqcap \mu_{\tilde{C}}^{\approx}(x) \right]
\end{aligned}$$

and

$$\begin{aligned}
& \left[\mu_{\tilde{A}}^{\approx}(x) \sqcap \mu_{\tilde{B}}^{\approx}(x) \right] \sqcup \mu_{\tilde{C}}^{\approx}(x) \\
& = \left[\mu_{\tilde{A}}^{\approx}(x) \sqcup \mu_{\tilde{C}}^{\approx}(x) \right] \sqcap \left[\mu_{\tilde{B}}^{\approx}(x) \sqcup \mu_{\tilde{C}}^{\approx}(x) \right]
\end{aligned}$$

2.4 Operations under type-2 fuzzy sets collection

[58] Let $\{\tilde{A}_i, i \in \mathbb{N}\}$ be an arbitrary collection of type-2 fuzzy sets of X such that \mathbb{N} is countable set, operations are possible under an arbitrary collection of type-2 fuzzy sets.

1) The union $\cup_{i \in \mathbb{N}} \tilde{A}_i$ is defined as

$$\left[\cup_{i \in \mathbb{N}} \tilde{A}_i \right] (x) = \sum_{x \in X} \sum_{u \in JX} \frac{\wedge_{i \in \mathbb{N}} (f_x(u))_i}{\vee_{i \in \mathbb{N}} (u)_i}$$

2) The intersection

$$\left[\cap_{i \in \mathbb{N}} \tilde{A}_i \right] (x) = \sum_{x \in X} \sum_{u \in JX} \frac{\wedge_{i \in \mathbb{N}} (f_x(u))_i}{\wedge_{i \in \mathbb{N}} (u)_i}$$

Proposition 2.4.1. [25] Let $\{\tilde{A}_i, i \in \mathbb{N}\}$ be an arbitrary collection of type-2 fuzzy sets of X such that \mathbb{N} is a countable set and \tilde{B} be another type-2 fuzzy set of X , then

$$1) \quad \tilde{B} \cap \left[\cup_{i \in \mathbb{N}} \tilde{A}_i \right] = \cup_{i \in \mathbb{N}} \left(\tilde{B} \cap \tilde{A}_i \right)$$

$$2) \quad \tilde{B} \cup \left[\cap_{i \in \mathbb{N}} \tilde{A}_i \right] = \cap_{i \in \mathbb{N}} \left(\tilde{B} \cup \tilde{A}_i \right)$$

$$3) \quad 1 - \left[\cup_{i \in \mathbb{N}} \tilde{A}_i \right] = \cap_{i \in \mathbb{N}} \left[1 - \tilde{A}_i \right]$$

$$4) \quad 1 - \left[\bigcap_{i \in \mathbb{N}} \tilde{A}_i \right] = \bigcup_{i \in \mathbb{N}} \left[1 - \tilde{A}_i \right]$$

Example 2.4.1. First, we will show the examples of fuzzy grades which do not satisfy the absorption laws, the distributive laws, the identity laws, and the complement laws.

Absorption laws failure. Let μ_A and μ_B be convex fuzzy grades as

$$\mu_A = 0.2/0.1 + 0.3/0.2 + 0.5/0.3 + 0.7/0.4 + 0.8/0.5$$

$$\mu_B = 0/0.1 + 0.1/0.2 + 0.2/0.3 + 0.3/0.2 + 0.4/0.5$$

where J is $\{0.1, 0.2, 0.3, 0.4, 0.5\}$. Then $\mu_A \sqcap (\mu_B = 0.2/0.1 + 0.3/0.2 + 0.4/0.3 + 0.4/0.4 + 0.4/0.5 \text{ and } \mu_A \sqcup \mu_B)$
 $= 0/0.1 + 0.1/0.2 + 0.2/0.3 + 0.3/0.4 + 0.4/0.5$

$$\mu_A \sqcap (\mu_A \sqcup \mu_B) = \mu_A \sqcup (\mu_A \sqcap \mu_B)$$

$$= 0.2/0.1 + 0.3/0.2 + 0.4/0.3 + 0.4/0.4 + 0.4/0.5 \neq \mu_A.$$

Distributive laws failure

$$\mu_A = 0.8/0.2 + 0.1/0.3 + 0/0.4 + 0.7/0.5$$

$$\mu_B = 0.3/0.2 + 0.4/0.3 + 0.5/0.4 + 0.2/0.5$$

$$\mu_C = 0.1/0.2 + 0.2/0.3 + 0.5/0.4 + 0.7/0.5$$

Then we have,

$$\mu_B \sqcup (\mu_C) = 0.1/0.2 + 0.2/0.3 + 0.5/0.4 + 0.5/0.5$$

$$\mu_A \sqcap (\mu_B \sqcup \mu_C) = 0.5/0.2 + 0.2/0.3 + 0.5/0.4 + 0.5/0.5$$

$$\mu_A \sqcap \mu_B = 0.5/0.2 + 0.4/0.3 + 0.5/0.4 + 0.2/0.5$$

$$\mu_A \sqcap \mu_C = 0.7/0.2 + 0.2/0.3 + 0.5/0.4 + 0.7/0.5$$

$$(\mu_A \sqcap \mu_B) \sqcup (\mu_A \sqcap \mu_C) = 0.5/0.2 + 0.4/0.3 + 0.5/0.4 + 0.2/0.5$$

Thus, we see that the distributive law is not satisfied. The same holds for the case in which \sqcup and \sqcap are interchanged.

A part of identity laws failure, let $\mu_A = 0.6/0.2 + 0.4/0.3 + 0.7/0.4$. Then, noting that the numbers 1 and 0 are represented as 1/1 and 1/0, respectively, we obtain

$$\mu_A \sqcup 1 = 0.7/1 \neq 1$$

$$\mu_A \sqcap 0 = 0.7/0 \neq 0$$

Complement laws failure. Let $\mu_A = 0.7/0.2 + 0.6/0.3 + 1/0.4$ then $\mu_{\tilde{A}}$ is given as

$$\mu_{\tilde{A}} = 0.7/0.8 + 0.6/0.7 + 1/0.6.$$

Thus we have

$$\mu_A \sqcup \mu_{\tilde{A}} = 1/0.6 + 0.6/0.7 + 0.7/0.8 \neq 1$$

$$\mu_A \sqcap \mu_{\tilde{A}} = 0.7/0.2 + 0.6/0.3 + 1/0.4 \neq 0$$

2.5 Type-2 fuzzy relations, their compositions, and cartesian product

Let X_1, X_2, \dots, X_n , be n universes. A crisp relation in $X_1 \times X_2 \times \dots \times X_n$ is a crisp subset of the product space. Similarly, a type-1 fuzzy relation in $X_1 \times X_2 \times \dots \times X_n$, is a type-1 fuzzy subset of the product space and a type-2 fuzzy relation in $X_1 \times X_2 \times \dots \times X_n$, is a type-2 fuzzy subset of the product space. We concentrate on binary type-2 fuzzy relations.

2.5.1 Type-2 FRs and their compositions

From type-1 fuzzy relations to type-2 ones

[75] A type-1 FR in $U \times V$ is a type-1 fuzzy subset of $U \times V$ and a type-2 FR in $U \times V$ is a type-2 fuzzy subset of $U \times V$.

Type-1 and type-2 fuzzy relations are usually expressed in matrix form as

$$R = \begin{bmatrix} \mu_R(u_1, v_1) & \mu_R(u_1, v_2) \cdots & \mu_R(u_1, v_n) \\ \mu_R(u_2, v_1) & \mu_R(u_2, v_2) \cdots & \mu_R(u_2, v_n) \\ \vdots & \vdots & \vdots \\ \mu_R(u_m, v_1) & \mu_R(u_m, v_2) \cdots & \mu_R(u_m, v_n) \end{bmatrix}$$

and

$$\tilde{R} = \begin{bmatrix} \mu_{\tilde{R}}(u_1, v_1) & \mu_{\tilde{R}}(u_1, v_2) \cdots & \mu_{\tilde{R}}(u_1, v_n) \\ \mu_{\tilde{R}}(u_2, v_1) & \mu_{\tilde{R}}(u_2, v_2) \cdots & \mu_{\tilde{R}}(u_2, v_n) \\ \vdots & \vdots & \vdots \\ \mu_{\tilde{R}}(u_m, v_1) & \mu_{\tilde{R}}(u_m, v_2) \cdots & \mu_{\tilde{R}}(u_m, v_n) \end{bmatrix}$$

where $\mu_R(u_i, v_j)$ are crisp numbers in $[0, 1]$, and $\mu_{\tilde{R}}(u_i, v_j)$ are fuzzy numbers (fuzzy sets) in $[0, 1]$.

As the extension from type-1 fuzzy sets to type-2 ones, we can obtain a type-2 fuzzy relation by adding additional uncertainty information to a type-1 one. The following example explains the extension

Example 2.5.1. Consider the type-1 FR $R \in \mathcal{F}(X \times Y) : 'x \text{ is close to } y'$, where $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ and

$$R = \begin{matrix} & y_1 & y_2 \\ x_1 & \begin{bmatrix} 0.5 & 0.2 \end{bmatrix} \\ x_2 & \begin{bmatrix} 0.8 & 0.1 \end{bmatrix} \\ x_3 & \begin{bmatrix} 0.2 & 0.7 \end{bmatrix} \end{matrix}$$

Consider another type-1 FR $S \in \mathcal{F}(Y \times Z) : 'y \text{ is much smaller than } z'$, where $Z = \{z_1, z_2, z_3\}$, and

$$S = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & \begin{bmatrix} 0.3 & 0.9 & 0.6 \end{bmatrix} \\ y_2 & \begin{bmatrix} 0.3 & 0.4 & 0.5 \end{bmatrix} \end{matrix}$$

Adding some additional uncertainties to these two type-1 FRs, we may obtain the following membership grades (For saving space, we sometimes represent a type-2 fuzzy relation in the following form, i.e., the solidus (/) are replaced by built-up fractions):

$$\tilde{R} = \begin{matrix} & y_1 & y_2 \\ x_1 & \begin{bmatrix} \frac{0.4}{0.4} + \frac{1}{0.5} + \frac{0.7}{0.6} & \frac{0.6}{0.5} + \frac{1}{0.1} + \frac{0.6}{0.2} \end{bmatrix} \\ x_2 & \begin{bmatrix} \frac{0.3}{0.6} + \frac{1}{0.7} + \frac{0.5}{0.8} & \frac{0.1}{0.6} + \frac{0.2}{1} + \frac{0.3}{0.4} \end{bmatrix} \\ x_3 & \begin{bmatrix} \frac{0.8}{0.8} + \frac{1}{0.9} + \frac{0.5}{0.1} & \frac{0.6}{0.7} + \frac{1}{0.8} + \frac{0.4}{0.9} \end{bmatrix} \end{matrix}$$

and

$$\tilde{S} = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & \begin{bmatrix} \frac{0.4}{0.8} + \frac{1}{0.9} + \frac{0.6}{0.6} & \frac{0.3}{0.5} + \frac{1}{0.6} + \frac{0.7}{0.7} & \frac{0.5}{0.8} + \frac{1}{0.1} + \frac{0.6}{0.2} \end{bmatrix} \\ y_2 & \begin{bmatrix} \frac{0.3}{0.5} + \frac{1}{0.1} + \frac{0.6}{0.2} & \frac{0.2}{0.5} + \frac{1}{0.6} + \frac{0.6}{0.7} & \frac{0.5}{0.8} + \frac{1}{0.9} + \frac{0.7}{1} \end{bmatrix} \end{matrix}$$

2.5.2 Composition of type-2 fuzzy relations

[49]

Definition 2.5.1. If R and S (or \tilde{R} and \tilde{S}) are two T1 (or T2) FRs on $U \times V$ and $V \times W$, respectively, the membership of any (u, w) , $u \in U, w \in W$, is non-zero if there was at least one $v \in V$ so that $\mu_R(u, v) \neq 0$ (or $\mu_{\tilde{R}}(u, v) \neq 1/0$) and $\mu_S(v, w) \neq 0$ (or $\mu_{\tilde{S}}(v, w) \neq 1/0$), in which $1/0$ denotes the concept of zero membership grades in the case of type-2 fuzzy sets.

$$\mu_{\tilde{R} \circ \tilde{S}}(u, w) = \sqcup v \in V [\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{S}}(v, w)] \quad (2.16)$$

where \sqcap and \sqcup are meet and join operations, respectively, which are defined as follows [58]

Example 2.5.2. Consider the fuzzy relations in Example 2.5.1.

It is known that the expression 'x is close to y and y is much smaller than z' indicates the composition $R \circ S$ (or $\tilde{R} \circ \tilde{S}$).

For the type-1 case, using the equation:

$$\begin{aligned} \mu_{R \circ S}(x_i, z_j) = & [\mu_R(x_i, y_1) \wedge \mu_S(y_1, z_j)] \vee [\mu_R(x_i, y_2) \\ & \wedge \mu_S(y_2, z_j)] \vee [\mu_R(x_i, y_3) \wedge \mu_S(y_3, z_j)] \end{aligned} \quad (2.17)$$

where $i = 1, 2, 3$ and $j = 1, 2, 3$. For type-2 fuzzy relation using equation:

$$\begin{aligned} \mu_{R \circ S}(x_i, z_j) = & [\mu_R(x_i, y_1) \sqcap \mu_S(y_1, z_j)] \sqcup [\mu_R(x_i, y_2) \\ & \sqcap \mu_S(y_2, z_j)] \end{aligned} \quad (2.18)$$

$$\begin{aligned} \mu_{R \circ S}(x_1, z_1) = & [\mu_R(x_1, y_1) \sqcap \mu_S(y_1, z_1)] \sqcup [\mu_R(x_1, y_2) \\ & \sqcap \mu_S(y_2, z_1)] \\ = & \left[\left(\frac{0.4}{0.4} + \frac{1}{0.5} + \frac{0.7}{0.6} \right) \sqcap \left(\frac{0.4}{0.8} + \frac{1}{0.9} + \frac{0.6}{1} \right) \right] \sqcup \left[\left(\frac{0.6}{0} + \frac{1}{0.1} + \frac{0.6}{0.2} \right) \right] \\ = & \left[\sqcap \left(\frac{0.5}{0} + \frac{1}{0.1} + \frac{0.6}{0.2} \right) \right] \end{aligned} \quad (2.19)$$

we obtain

$$\tilde{R} \circ \tilde{S} = \begin{bmatrix} \frac{0.4}{0.4} + \frac{1}{0.5} + \frac{0.7}{0.6} & \frac{0.4}{0.4} + \frac{1}{0.5} + \frac{0.7}{0.6} & \frac{0.5}{0} + \frac{1}{0.1} + \frac{0.6}{0.2} \\ \frac{0.3}{0.3} + \frac{1}{1} + \frac{0.5}{0.5} & \frac{0.3}{0.3} + \frac{1}{1} + \frac{0.7}{0.7} & \frac{0.5}{0.5} + \frac{1}{1} + \frac{0.7}{0.7} \\ \frac{0.6}{0.6} + \frac{0.7}{0.7} + \frac{0.8}{0.8} & \frac{0.5}{0.5} + \frac{0.6}{0.6} + \frac{0.7}{0.7} & \frac{0.1}{0.1} + \frac{0.2}{0.2} + \frac{0.3}{0.3} \\ \frac{0.8}{0.8} + \frac{1}{0.9} + \frac{1}{1} & \frac{0.2}{0.5} + \frac{1}{0.6} + \frac{0.5}{0.7} & \frac{0.6}{0.7} + \frac{1}{0.8} + \frac{0.4}{0.9} \end{bmatrix}$$

Properties[70]

Let \tilde{R} , \tilde{S} and \tilde{L} be T2 fuzzy relations defined on product space $U \times V$. If for any $(u, v) \in U \times V$, $\mu_{\tilde{R}}(u, v)$, $\mu_{\tilde{S}}(u, v)$ and $\mu_{\tilde{L}}(u, v)$ are convex T1 fuzzy sets, then we have

$$(\tilde{R} \cup \tilde{S}) \cap \tilde{L} = (\tilde{R} \cap \tilde{L}) \cup (\tilde{S} \cap \tilde{L})$$

and

$$(\tilde{R} \cap \tilde{S}) \cup \tilde{L} = (\tilde{R} \cup \tilde{L}) \cap (\tilde{S} \cup \tilde{L})$$

Theorem 2.5.1. [70] Suppose that \tilde{R} , \tilde{S} and \tilde{L} are T2 fuzzy relations on product spaces $U \times V$, $V \times W$ and $W \times Q$, respectively, and $\mu_{\tilde{R}}(u, v)$, $\mu_{\tilde{S}}(v, w)$ and $\mu_{\tilde{L}}(w, q)$ are convex T1 fuzzy sets. If the spaces V and W are finite, then

$$(\tilde{R} \circ \tilde{S}) \circ \tilde{L} = \tilde{R} \circ (\tilde{S} \circ \tilde{L})$$

Proof 2.5.1. From Definition 1, for any $(u, q) \in U \times Q$, we have

$$\begin{aligned} & \mu_{(\tilde{R} \circ \tilde{S}) \circ \tilde{L}}(u, q) \\ &= \sqcup_{w \in W} [\mu_{\tilde{R} \circ \tilde{S}}(u, w) \sqcap \mu_{\tilde{L}}(w, q)] \\ &= \sqcup_{w \in W} [\sqcup_{v \in V} [\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{S}}(v, w)] \sqcap \mu_{\tilde{L}}(w, q)] \end{aligned}$$

we know $\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{S}}(v, w)$ is a convex T1 fuzzy set in $[0, 1]$

$$\begin{aligned} & \mu_{\tilde{R} \circ (\tilde{S} \circ \tilde{L})}(u, q) \\ &= \sqcup_{v \in V} [\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{S} \circ \tilde{L}}(v, q)] \\ &= \sqcup_{v \in V} [\mu_{\tilde{R}}(u, v) \sqcap \sqcup_{w \in W} [\mu_{\tilde{S}}(v, w) \sqcap \mu_{\tilde{L}}(w, q)]] \\ &= \sqcup_{w \in W} [\sqcup_{v \in V} [\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{S}}(v, w) \sqcap \mu_{\tilde{L}}(w, q)]] \end{aligned}$$

implies $\mu_{(\tilde{R} \circ \tilde{S}) \circ \tilde{L}}(u, q) = \mu_{\tilde{R} \circ (\tilde{S} \circ \tilde{L})}(u, q)$ for any $(u, q) \in U \times Q$

Theorem 2.5.2. Let \tilde{R}, \tilde{S} be two T2 fuzzy relations defined on $U \times V$, and \tilde{L} be a T2 fuzzy relations on $V \times W$. If $\mu_{\tilde{R}}(u, v), \mu_{\tilde{S}}(u, v)$ and $\mu_{\tilde{L}}(v, w)$ are convex fuzzy sets for any $(u, v) \in U \times V$ and $(v, w) \in V \times W$, then we have $(\tilde{R} \cup \tilde{S}) \circ \tilde{L} = (\tilde{R} \circ \tilde{L}) \cup (\tilde{S} \circ \tilde{L})$

Proof. On the one hand, the LHS of Eq. (28) can be expressed as

$$\begin{aligned} & \mu_{(\tilde{R} \cup \tilde{S}) \circ \tilde{L}}(u, w) \\ &= \bigsqcup_{v \in V} [\mu_{\tilde{R} \cup \tilde{S}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)] \\ &= \bigsqcup_{v \in V} [[\mu_{\tilde{R}}(u, v) \sqcup \mu_{\tilde{S}}(u, v)] \sqcap \mu_{\tilde{L}}(v, w)] \end{aligned}$$

for any $(u, w) \in U \times W$ On the other hand, can be written

$$\mu_{(\tilde{R} \circ \tilde{L}) \cup (\tilde{S} \circ \tilde{L})}(u, w) = \mu_{\tilde{R} \circ \tilde{L}}(u, w) \sqcup \mu_{\tilde{S} \circ \tilde{L}}(u, w) = \bigsqcup_{v \in V} [\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)] \sqcup \bigsqcup_{v \in V} [\mu_{\tilde{S}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)]$$

since arbitrary fuzzy sets in $[0, 1]$ satisfy associative and commutative laws under \sqcup [1], we have

$$\begin{aligned} & \bigsqcup_{v \in V} [\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)] \sqcup \bigsqcup_{v \in V} [\mu_{\tilde{S}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)] \\ &= \bigsqcup_{v \in V} [[\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)] \sqcup [\mu_{\tilde{S}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)]] \end{aligned}$$

Furthermore, noting that $\mu_{\tilde{R}}(u, v), \mu_{\tilde{S}}(u, v)$ and $\mu_{\tilde{L}}(v, w)$ are convex fuzzy sets in $[0, 1]$, form [1], they satisfy distributive laws under \sqcup and \sqcap . Therefore, we get

$$\begin{aligned} & \bigsqcup_{v \in V} [[\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)] \sqcup [\mu_{\tilde{S}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)]] \\ &= \bigsqcup_{v \in V} [[\mu_{\tilde{R}}(u, v) \cup \mu_{\tilde{S}}(u, v)] \sqcap \mu_{\tilde{L}}(v, w)] \end{aligned}$$

we obtain

$$\mu_{(\tilde{R} \cup \tilde{S}) \circ \tilde{L}}(u, w) = \mu_{(\tilde{R} \circ \tilde{L}) \cup (\tilde{S} \circ \tilde{L})}(u, w)$$

for any $(u, w) \in U \times W$. Thus,

$$(\tilde{R} \cup \tilde{S}) \circ \tilde{L} = (\tilde{R} \circ \tilde{L}) \cup (\tilde{S} \circ \tilde{L})$$

Similarly, we have the following result. □

Theorem 2.5.3. Let \tilde{R}, \tilde{S} be two T_2 fuzzy relations defined on $U \times V$, and \tilde{L} be a T_2 fuzzy relations on $W \times U$. If $\mu_{\tilde{R}}(u, v), \mu_{\tilde{S}}(u, v)$ and $\mu_{\tilde{L}}(w, u)$ are convex T_1 fuzzy sets for any $(u, v) \in U \times V$ and $(w, u) \in W \times U$, then we have

$$\tilde{L} \circ (\tilde{R} \cup \tilde{S}) = (\tilde{L} \circ \tilde{R}) \cup (\tilde{L} \circ \tilde{S})$$

Remark 2.5.1. From Theorem 5 and 6, we know that T_2 fuzzy relations satisfy distributive law under \cup and \circ . However, the distributive law will not hold under \cap and \circ .

Consequence 2.5.1.

$$(\tilde{R} \cap \tilde{S}) \circ \tilde{L} \neq (\tilde{R} \circ \tilde{L}) \cap (\tilde{S} \circ \tilde{L})$$

2.6 Alpha Planes

An α plane for a generalized type-2 fuzzy set [18] \tilde{A} is denoted by \tilde{A}_α and can be defined as the union of all primary membership functions of \tilde{A} , where the secondary membership degrees are greater or equal to α ($0 \leq \alpha \leq 1$) and can be represented by (2.10) [42], [53], and can be graphically illustrated in Fig. 2.4.

$$\begin{aligned} \tilde{A}_\alpha &= \{(x, u), \mu_{\tilde{A}}(x, u) \geq \alpha \mid \forall x \in X, \forall u \in J_x \subseteq [0, 1]\} \\ &= \int_{\forall x \in X} \int_{\forall u \in J_x} \{(x, u) \mid \mu_{\tilde{A}}(x, u) \geq \alpha\} \end{aligned}$$

where the union of all the α alpha planes is represented by (2. 11). and $R_{\tilde{A}_\alpha}$ represents a horizontal slice. In Fig. 2.3 we can appreciate the α alpha planes made for generalized membership functions.

Also in Fig. 2.4 we can appreciate the representation of the three α planes in different points, the first α plane was realized in the point 0.17 (blue), the second α plane was in 0.46 (green), and the third in point 0.99 (red).

$$\tilde{A} = \bigcup_{\alpha \in [0,1]} R_{\tilde{A}_\alpha} \quad (2.20)$$

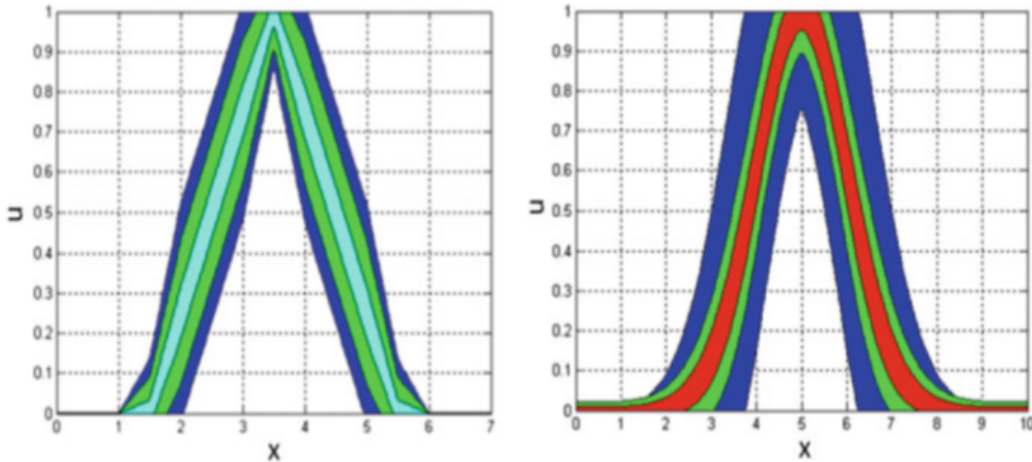


Figure 2.6: α plane in generalized type-2 membership functions[18]

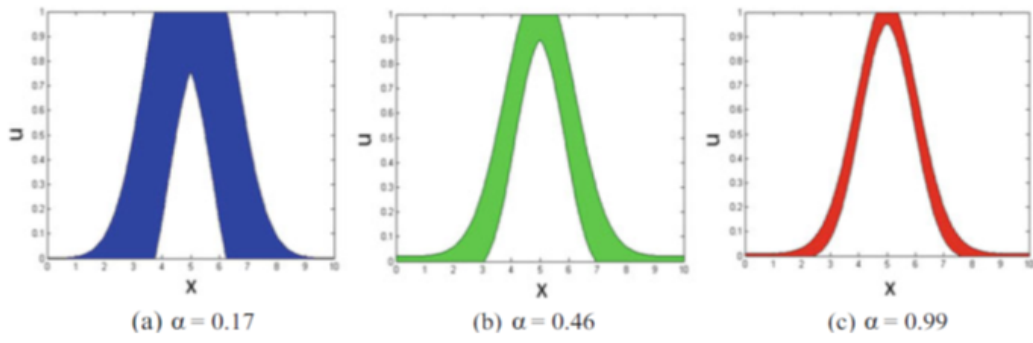


Figure 2.7: Cuts at certain points of the GMF[18]

Chapter 3

Fuzzy logic control of non-linear dynamic system

3.1 Introduction

The dynamic system is an aspect of systems theory used as a method to understand the dynamic behavior of complex systems. It has found application in a wide range of areas, for example population, agriculture, ecological and economic systems, which usually interact strongly with each other.

First, some notions are defined in this chapter which are the dynamic system and its control and the fuzzy control system. Then, we have presented Puma 560 robot modeling. In the end, a detailed study of some works of fuzzy logic controller of the non-linear dynamic system.

3.2 Dynamic system definition

3.2.1 General definition

A dynamic system is a system the state of which evolves with time over a state space according to a fixed rule. A dynamic system is a system or process in which motion occurs or includes active forces as opposed to static conditions with no motion. A dynamic system is a system that is constantly changing like the human body. The dynamic system stands to become static or at an equilibrium state. For example, if a car is assumed to be a dynamic system, then it requires fuel to continue moving forward or else it would come to a stop and become static. Dynamic systems output depends upon future and past values.

3.2.2 Formal definition of dynamical system

A dynamical system is formally defined as a state space X , a set of times T , and a rule R that specifies how the state evolves with time. The rule R is a function the domain of which is $X \times T$ and the codomain of which is X , i.e., $R : X \times T \rightarrow X$. The rule function R means that the R takes two inputs, $R = R(x, t)$, where $x \in X$ is the initial state (at time $t = 0$, for example) and $t \in T$ is a future time. In other words, $R(x, t)$ gives the state at time t knowing that the initial state was x

3.3 The dynamic system control

the control of a dynamic system was originally applied to a system that controls a mechane or motor. It is used to describe a control system in which a physical variable is required to follow or track.

3.4 Advantages of using Fuzzy Logic in Control Systems

A control system is an arrangement of physical components designed to alter another physical system so that it exhibits certain desired characteristics. Some reasons for using Fuzzy Logic in Control Systems are as follows:

- While applying traditional control, one needs to know about the model and the objective function formulated in precise terms. This makes it very difficult to apply in many cases.
- By applying fuzzy logic for control we can utilize the human expertise and experience for designing a controller.
- The fuzzy control rules, basically the IF-THEN rules, can be best utilized in designing a controller.

3.5 Fuzzy logic control

[4] The basic structure of a fuzzy logic control (FLC) system is shown in Figure 3.1. Fuzzy logic controller (FLC) used in this thesis is Mamdani.

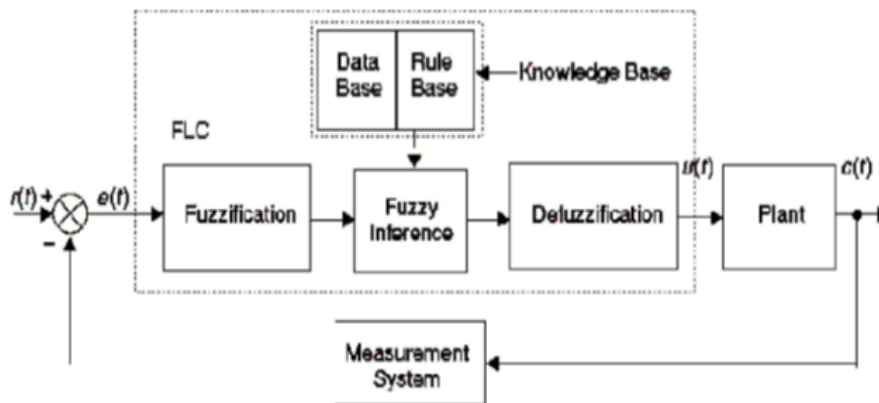


Figure 3.1: Block diagram of joint space control

3.5.1 The fuzzification process

Fuzzification is the process of mapping inputs to the FLC into fuzzy set membership values in the various input universes of discourse. Decisions need to be made regarding

- (a) Number of input

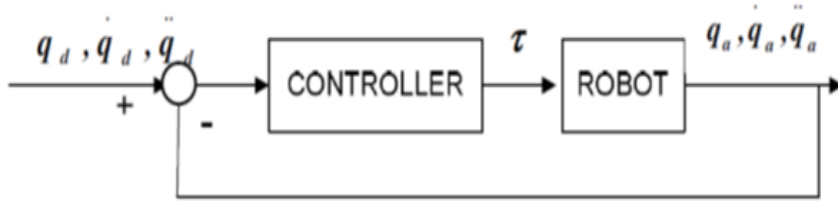


Figure 3.2: Fuzzy logic control
[60]

- (b) Size of universes of discourse
- (c) Number and shape of fuzzy sets.

A FLC that emulates a PD controller will be required to minimize the error $e(t)$ and the rate of change of error de/dt , or ce .

Each set is given a linguistic label to identify it, such as Positive Big (PB), Positive Medium (PM), Positive Small (PS), About Zero (Z), Negative Small (NS), Negative Medium (NM) and negative Big (NB). The seven set fuzzy input windows for e and ce are shown in Figure 3.3. If at a particular instant, $e(t) = 2.5$ and $de/dt = ce = -0.2$, then, from Figure 3.3, the input fuzzy set membership values are:

$$\begin{aligned} \mu_{PS}(e) &= 0.7 & \mu_{PM}(e) &= 0.4 \\ \mu_{NS}(ce) &= 0.6 & \mu_Z(ce) &= 0.3 \end{aligned} \quad (3.1)$$

3.5.2 The fuzzy rule base

The fuzzy rule base consists of a set of antecedent-consequent linguistic rules of the form

$$\text{IF } e \text{ is PS AND } ce \text{ is NS THEN } u \text{ is PS} \quad (3.2)$$

The size of the universes of discourse will depend upon the expected range (usually up to the saturation level) of the input variables. Assume for the system about to be considered that e has a range of ± 6 and ce a range of ± 1 .

The number and shape of fuzzy sets in a particular universe of discourse is a trade-off between the precision of control action and real-time computational complexity. In this example, seven triangular sets will be used.

This style of a fuzzy conditional statement is often called a Mamdani-type rule, after Mamdani (1976) who first used it in a fuzzy rule base to control steam plant.

The rule base is constructed using a priori knowledge from either one or all of the following sources:

- (a) Physical laws that govern the plant dynamics
- (b) Data from existing controllers
- (c) Imprecise heuristic knowledge obtained from experienced experts.

If (c) above is used, then knowledge of the plant mathematical model is not required. The two seven set fuzzy input windows shown in Figure 3.3 gives a possible 7×7 set of control rules of the form given in equation (3.2). It is convenient to tabulate the two-dimensional rule base as shown in table 3.1.

3.5.3 Fuzzy inference

[74] Table 3.1 assumes that the output window contains seven fuzzy sets with the same linguistic labels as the input fuzzy sets. If the universe of discourse for the control signal $u(t)$ is ± 9 , then the output window is as shown in Figure 3.4.

Assume that a certain rule in the rule base is given by equation (3.1)

$$\text{IF } e \text{ is } A \text{ AND } ce \text{ is } B \text{ THEN } u = C \quad (3.3)$$

From equation (1.4) the Boolean OR function becomes the fuzzy max operation, and from equation (1.5) the Boolean AND function becomes the fuzzy min operation. Hence equation (3.3) can be written in the following:

$$\mu_C(u) = \max[\min(\mu_A(e), \mu_B(ce))] \quad (3.4)$$

equation (3.4) is referred to as the max-min inference process or max-min fuzzy reasoning,

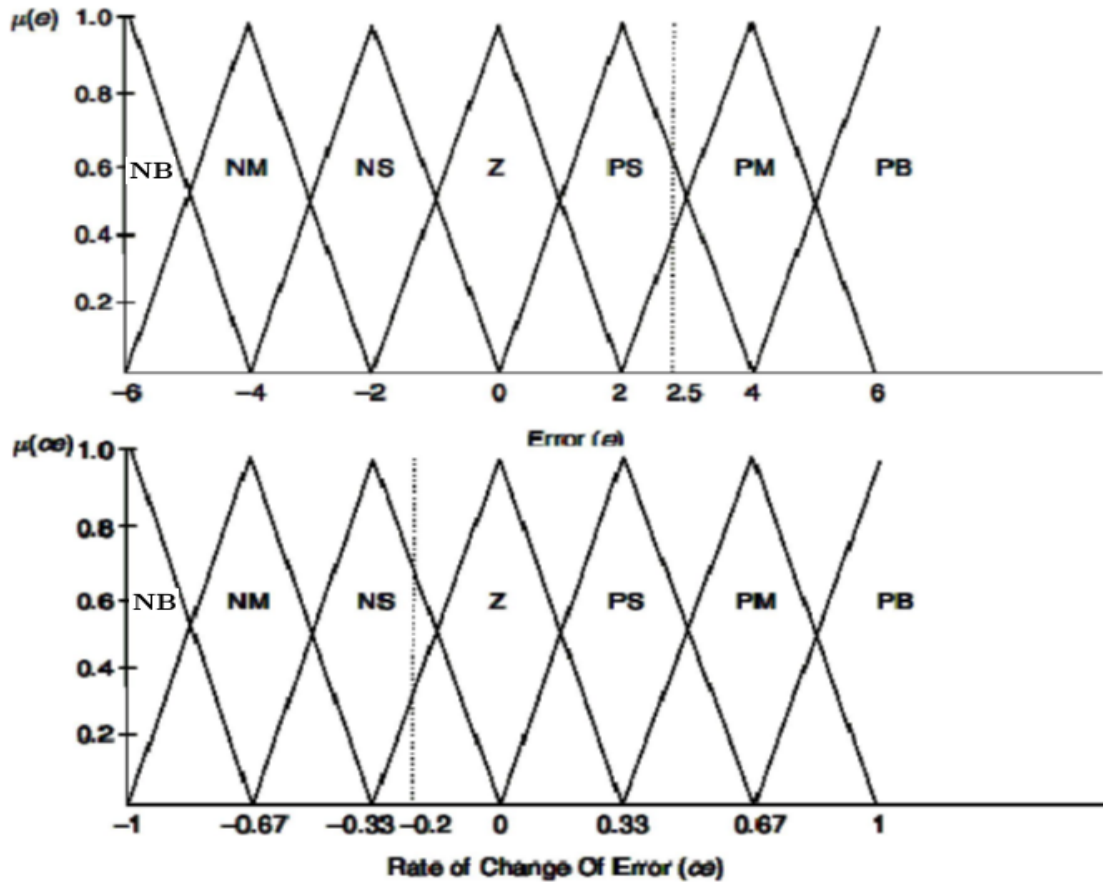


Figure 3.3: Seven set fuzzy input windows for error e and errata change of error (ce)

In Figure 3.3 and equation (3.1) the fuzzy sets that were 'hit' in the error input window when $e(t) = 2.5$ were PS and PM. In the rate of change input window when $ce = -0.2$, the fuzzy sets to be 'hit' were NS and Z. From table 3.1. the relevant rules that correspond to these 'hits' are:

$$\begin{aligned} &\dots \text{OR IF } e \text{ is } PS \text{ AND } ce \text{ is } NS \\ &\text{OR IF } e \text{ is } PS \text{ AND } ce \text{ is } Z \\ &\text{THEN } u = PS \end{aligned} \quad (3.5)$$

Table 3.1: Tabular structure of a linguistic fuzzy rulebase

$\begin{matrix} e \\ \backslash \\ ce \end{matrix}$	NB	NM	NS	Z	PS	PM	PB
NB	NB	NB	NB	NM	Z	PM	PB
NM	NB	NB	NB	NM	PS	PM	PB
NS	NB	NB	NM	NS	PS	PM	PB
Z	NB	NM	NS	Z	PS	PM	PB
PS	NB	NM	NS	PS	PM	PB	PB
PM	NB	NM	NS	PM	PB	PB	PB
PB	NB	NM	Z	PM	PB	PB	PB

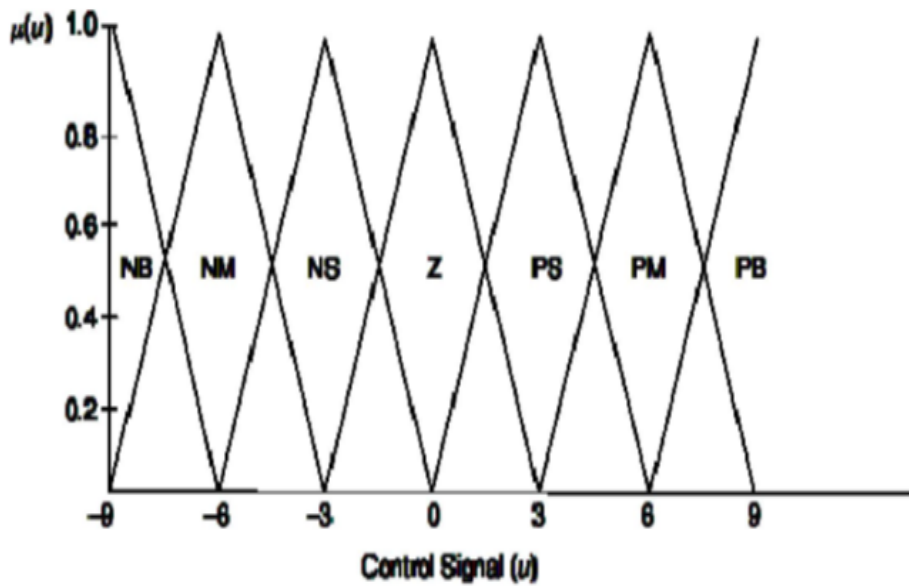


Figure 3.4: Seven set fuzzy output window for signal control (u)

$$\begin{aligned}
 & \dots \text{ OR IF } e \text{ is PM AND } ce \text{ is NS} \\
 & \text{ OR IF } e \text{ is PM AND } ce \text{ is Z} \\
 & \text{ THEN } u = \text{PM}
 \end{aligned} \tag{3.6}$$

Applying the max-min inference process to equation (3.5)

$$\mu_{PS}(u) = \max[\min(\mu_{PS}(e), \mu_{NS}(ce)), \min(\mu_{PS}(e), \mu_z(ce))] \tag{3.7}$$

Inserting values from equation (3.1)

$$\begin{aligned}
 \mu_{PS}(u) &= \max[\min(0.7, 0.6), \min(0.7, 0.3)] \\
 &= \max[0.6, 0.3] = 0.6
 \end{aligned} \tag{3.8}$$

Applying the max-min inference process to equation (3.6)

$$\mu_{PM}(u) = \max[\min(\mu_{PM}(e), \mu_{NS}(ce)), \min(\mu_{PM}(e), \mu_z(ce))] \tag{3.9}$$

Inserting values from equation (3.1)

$$\begin{aligned}
 \mu_{PM}(u) &= \max[\min(0.4, 0.6), \min(0.4, 0.3)] \\
 &= \max[0.4, 0.3] = 0.4
 \end{aligned} \tag{3.10}$$

Fuzzy inference is, therefore, the process of mapping membership values from the input windows, through the rule base, to the output window(s).

3.5.4 The defuzzification

Defuzzification is the procedure for mapping from a set of inferred fuzzy control signals contained within a fuzzy output window to a non-fuzzy (crisp) control signal. The center of area [68], [69], [12] method is the most famous defuzzification technique, which in linguistic terms can be expressed as follows:

$$\text{Crisp control signal} = \frac{\text{Sum of first moments of area}}{\text{Sum of areas}} \quad (3.11)$$

For a continuous system, equation (3.11) becomes:

$$u(t) = \frac{\int u\mu(u)du}{\int \mu(u)du} \quad (3.12)$$

Or alternatively, for a discrete system, equation (3.11) can be expressed as:

$$u(t) = \frac{\sum_{i=1}^n u_i \mu(u_i)}{\sum_{i=1}^n \mu(u_i)} \quad (3.13)$$

For the case when $e(t) = 2.5$ and $ce = -0.2$, as a result of the max-min inference process (equations (3.8) and (3.10)), the fuzzy output window in Figure 3.4 is 'clipped', and takes the form shown in Figure 3.5.

From Figure 3.5, using the equation for a trapezoid area:

$$\begin{aligned} \text{Area}_{PS} &= \frac{0.6(6+2.4)}{2} = 2.52 \\ \text{Area}_{PM} &= \frac{0.2(6+3.6)}{2} = 0.96 \end{aligned} \quad (3.14)$$

From equation (3.11)

$$u(t) = \frac{(2.52 \times 3) + (0.96 \times 6)}{2.52 + 0.96} = 3.83 \quad (3.15)$$

Hence, the error of 2.5, and an errata change of error of -0.2 , the control signal from the fuzzy controller is 3.83.

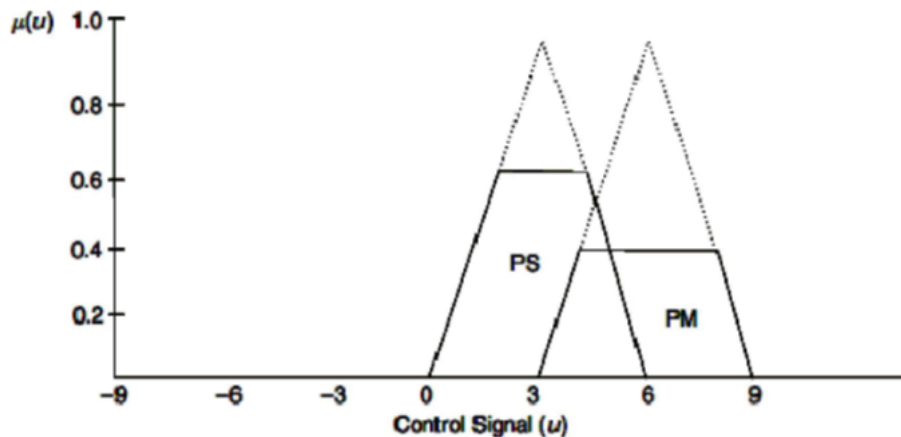


Figure 3.5: Clipped fuzzy output window due to fuzzy inference

Example 3.5.1. For the input and output fuzzy windows given in Figure 3.3 and 3.4, together with the fuzzy rule base shown in table 3.1, determine:

(a) The membership values of the input windows e and ce .

(b) The max-min fuzzy inference equation

(c) The crisp control signal $\mu(t)$.

when $e = -3$ and $ce = 0.3$

Solution

(a) When $e = -3$ and $ce = 0.3$ are mapped onto the input fuzzy windows, they refer to a fuzzy singletons. From Figure 3.3

$$e = -3 \quad \mu_{NS}(e) = 0.5 \quad \mu_{NM}(e) = 0.5 \quad (3.16)$$

$ce = 0.3$, using similar triangles :

$$\frac{1}{0.33} = \frac{\mu_z(ce)}{(0.33-0.3)} \quad (3.17)$$

$$\mu_z(ce) = 0.09$$

And

$$\frac{1}{0.33} = \frac{\mu_{PS}(ce)}{0.3} \quad (3.18)$$

$$\mu_{PS}(ce) = 0.91$$

(b) The rules that are 'hit' in the rule base in table 3.1 are

$$\begin{aligned} & \dots \text{OR IF } e \text{ is NS AND } ce \text{ is Z} \\ & \text{OR IF } e \text{ is NS AND } ce \text{ is PS} \\ & \text{THEN } u = \text{NS} \end{aligned} \quad (3.19)$$

$$\begin{aligned} & \dots \text{OR IF } e \text{ is NM AND } ce \text{ is Z} \\ & \text{OR IF } e \text{ is NM AND } ce \text{ is PS} \\ & \text{THEN } u = \text{NM} \end{aligned} \quad (3.20)$$

Applying max-min inference to equation (3.19)

$$\mu_{NS}(u) = \max[\min(\mu_{NS}(e), \mu_z(ce)), \min(\mu_{NS}(e), \mu_{PS}(ce))] \quad (3.21)$$

Inserting values into (3.21) :

$$\begin{aligned} \mu_{NS}(u) &= \max[\min(0.5, 0.09), \min(0.5, 0.91)] \\ &= \max[0.09, 0.5] = 0.5 \end{aligned} \quad (3.22)$$

and similarly with equation (3.20)

$$\begin{aligned} \mu_{NM}(u) &= \max[\min(\mu_{NM}(e), \mu_z(ce)), \min(\mu_{NM}(e), \mu_{PS}(ce))] \\ \mu_{NM}(u) &= \max[\min(0.5, 0.09), \min(0.5, 0.91)] \\ &= \max[0.09, 0.5] = 0.5 \end{aligned} \quad (3.23)$$

Using equations (3.22) and (3.23) to 'clip' the output window in Figure 3.4, the output window is now as illustrated in Figure 3.6.

(c) Due to the symmetry of the output window in Figure 3.6, from observation the crisp control signal is :

$$u(t) = -4,5$$

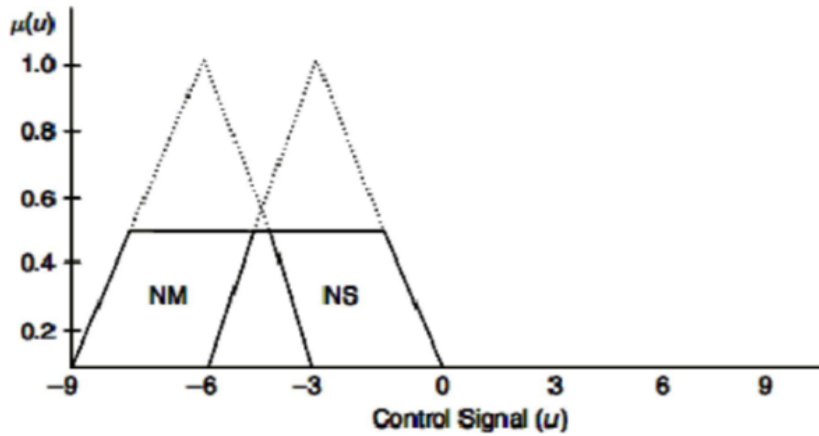


Figure 3.6: Fuzzy output window for to Example 3.5.1.

3.6 Puma560 robot dynamic modelling

This paragraph describes the PUMA 560 industrial robotic system that will be used in our research work as an example of dynamic system ; highlighting its main components , how they interact with each other and its dynamic modelisation.

3.6.1 General Overview of PUMA 560 Robot

[29]

The PUMA 560 robotic manipulator, produced by Unimate, is an old (> 25 years), but functional multi-joint robot and was used in this project, due to spare parts availability that are required for simulation of the different faults. It is a PC controlled serial manipulator designed for use in industrial applications, and has six revolute joints/degree of freedom (DOF) with three major axes of motion (X, Y , and Z) and resembles the human arm in function. Each of the robot's joints is actuated by a DC brushed permanent magnet servo motor. Positioning of the end effector, which is normally a device fixed at the end of the robotic arm, is achieved by the coordination of the first three joints named: waist (joint 1), shoulder (joint 2) and elbow (joint 3), allowing the robot to move the end effector into any position with maximum reach of 1 m. Orientation of the end effector is important when it approaches its final position and this is achieved using three wrist joints (4, 5, and 6) where the tool can be independently manoeuvred. This gives the robot six degrees of freedom as shown in Fig. 3.7 where the name of each joint with its maximum rotating range is indicated. Electromagnetic brakes are equipped with the first three joints (waist, shoulder, and elbow), which lock the motors to prevent collapsing when the power is removed from the robot (Potgieter et al. 2005). The control unit of the PUMA 560 is the most important part of the robotic system. It controls all operations of the manipulator and any connected device. There is two-way communication to each of the joints motors where sensory

information on position and speed is fed back to the control unit; outputs are sent to the motors related to the task's requirements. External digital input/output ports are available allowing other sensors and actuators to be controlled from this unit. There is a number of ways to position and orientate the manipulator end effector, but generally, one method is used at a time.

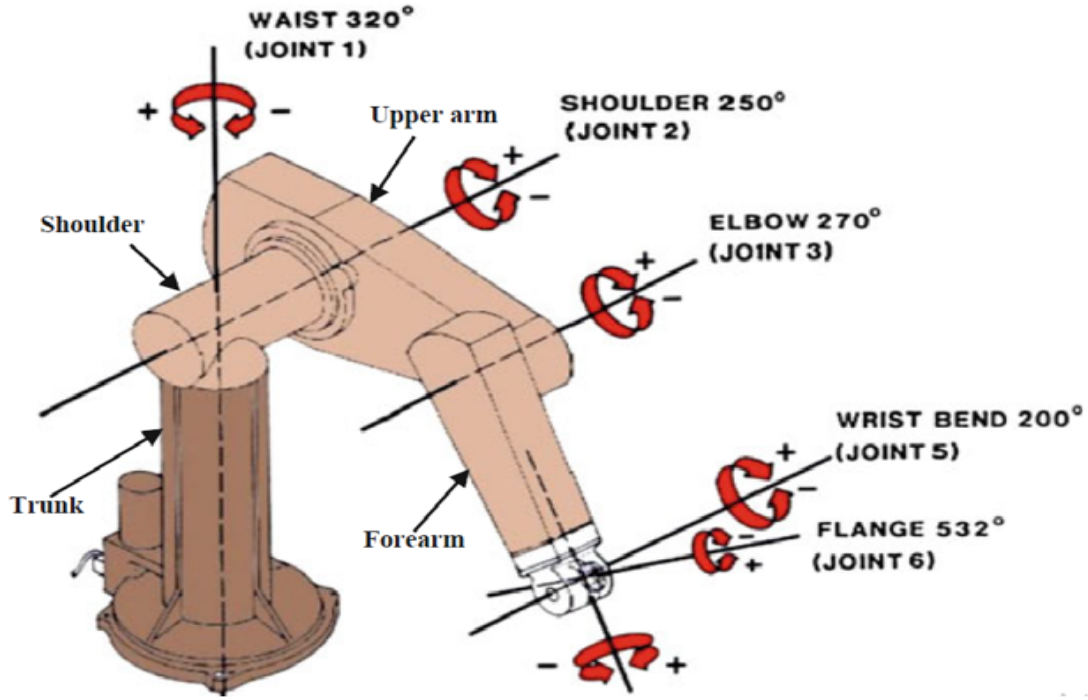


Figure 3.7: PUMA 560 robot member representations [61]

3.6.2 Puma 560 robot Dynamic Modelling

The dynamic model used for PUMA560 is taken from [3]. We have adopted a robot of three degrees of freedom and the configuration of space equation is given by:

$$M(q)\ddot{q} + B(q)[\dot{q}\dot{q}] + C(q)[\dot{q}^2] + G(q) = \Gamma \quad (1)$$

where:

$M(q)$: is 3×3 kinetic energy matrix.

$[\ddot{q}]$: 3×1 vector given by: $\ddot{q} = [\ddot{q}_1, \ddot{q}_2, \ddot{q}_3]^T$

$B(q)$: 3×1 matrix of Coriolis torques,

$[\dot{q}\dot{q}]$: 7×1 vector of joint velocity products given by:

$[\dot{q}\dot{q}] = [\dot{q}_1\dot{q}_2, \dot{q}_1\dot{q}_3, 0, 0, 0, \dot{q}_2\dot{q}_3, 0]^T$

$C(q)$: 3×1 vector given by: $[\dot{q}^2] = [\dot{q}_1^2, \dot{q}_2^2, \dot{q}_3^2]^T$

$G(q)$: is 3×1 vector of gravity torques,

Γ : 3×1 vector of joint torque.

\ddot{q} is the n-vector of accelerations;

3.7 Definition of a robot system controller

The controller is a device which can sense information from the linear or nonlinear system (e.g., robot manipulator) to improve the system's performance [9]. The main targets in designing control systems are stability, good disturbance rejection, and small tracking error [27]. Several industrial robot manipulators are controlled by linear methodologies (e.g., Proportional-Derivative (PD) controller, Proportional-Integral (PI) controller or Proportional-Integral-Derivative (PID) controller), but when the robot manipulator works with various payloads and has uncertainty in dynamic models this technique has limitations.

From the control point of view, uncertainty is divided into two main groups: uncertainty in unstructured inputs (e.g., noise, disturbance) and uncertainty in dynamics structure (e.g., payload, parameter variations). In some applications robot manipulators are used in an unknown and unstructured environment, therefore strong mathematical tools are used in new control methodologies to design nonlinear robust controller with an acceptable performance (e.g., minimum error, good trajectory, disturbance rejection). Joint space and operational space control are closed loop controllers that have been used to provide robustness and rejection of disturbance effect. The main target in the joint space controller is design a feedback controller that allows the actual motion ($q_a(t)$) tracking of the desired motion ($q_d(t)$) by inverse kinematics of robot manipulators [46]. Figure 3.8 shows the main block diagram of joint space controller. The main target in the operational space controller is to design a feedback controller to allow the actual end-effector motion $X_a(t)$ to track the desired end effector motion $X_d(t)$. This control methodology requires a greater algorithmic complexity and the inverse kinematics used in the feedback control loop as well. Direct measurement of operational space variables is very expensive that led to limitate the controller use in industrial robot manipulators[46]. Figure 3.9 shows the main block diagram of operational space control.

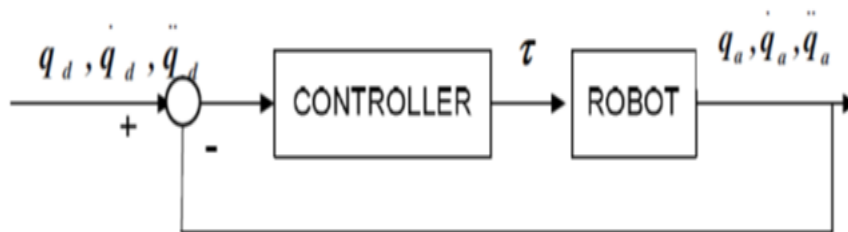


Figure 3.8: Block diagram of joint space control [60]

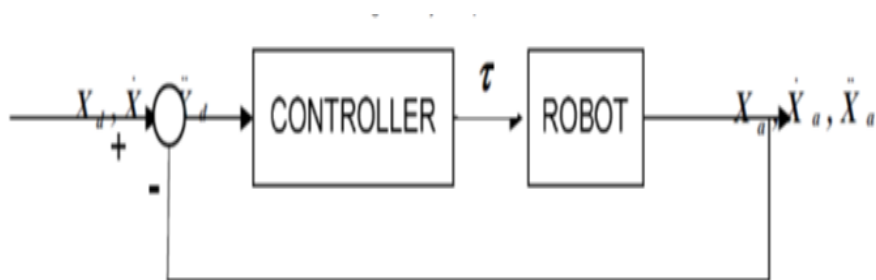


Figure 3.9: Block diagram of operational space control [60]

3.8 State of the art of fuzzy logic controller of nonlinear dynamic system

The subject of a fuzzy logic controller has attracted the attention of researchers, plenty of research works have been done.[15].-12 [39] The use of industrial robots became identifiable as a unique device in the 1960s. Since then, the field of application evolved from rather simple tasks like welding and painting to those requiring more precision, such as assembly tasks.

Control theory provides tools for designing and evaluating algorithms to achieve desired motions or force application. The methods of linear control and those of local linearization and moving linearization are not well suited for the control problem of robotic arms. This is due to the fact that robotic arms constantly move among widely separated regions of their workspace such that no valid linearization can be found in all regions. On the other hand, nonlinear control methods are progressing nowadays and different classes could be identified[65]: Trialand- Error, Feedback Linearization Control, Robust Control, Adaptive Control, and Gain-Scheduling. Nonlinear control methods used in robot arms' applications should, however, face the major difficulty resulting from the dynamic modeling of robots, i.e. the indetermination of their parameters [9]. Preferred methods are those which reduce or eliminate the undesired effects generated by this indetermination such as the Feedback Linearization control method [65], the Model-Reference Adaptive control method [[66],[65]] and the Self-Tuning method[65]. Another difficulty in robot arm control is the coupling effects of the Coriolis and centrifugal forces that might be canceled in a single axis mode operation where the joints are activated sequentially. Existing methods of nonlinear control are also used in robotics in order to eliminate the above-mentioned coupling effect like the Individual Joint PID control method [9] and the PD-plus-gravity control method [9]. Among the recent nonlinear control methods, fuzzy control methods grab nowadays the attention of many researchers. In fact, these methods do not require the knowledge of the dynamic model of the controlled system and this feature has become of major importance when dealing with complex nonlinear systems. Moreover, the dynamic modeling of robot arms shows a dependency on their mechanical parameters, subject to lifetime modifications (friction factors affected by the abuse of joints), and on their dynamical parameters that vary with the performed task (centers of gravity of the links affected by tool's replacements). These considerations also give advantage to fuzzy control methods on other nonlinear methods as a result of their robustness towards perturbations affecting the system.

The first fuzzy logic controller was introduced by Mamdani in 1974[44]. It is equivalent to two-input fuzzy PI controllers, where error and change of error were used as the inference system inputs. Mamdani's work also introduced the most common and robust fuzzy reasoning method called "Zadeh-Mamdani min-max gravity reasoning". Different comparative studies, like [27], prove that Zadeh-Mamdani min-max gravity scheme is the best reasoning scheme if the nonlinearity variation is the main concern. It also introduced the most common and robust fuzzy reasoning method, called Zadeh-Mamdani min-max gravity reasoning. [40] Robots are mechanized devices that have a certain degree to duplicate the working of humans, whenever they need continuous operations. Furthermore, they provide more accuracy, high strength and reduce the danger in the case of medical fields, industrial operations and nuclear plants (Spong & Vidyasagar, 2008). Controlling the motion of a manipulator robot's joints is a running issue the study of which is of great importance in order to make its terminal effectors follow a predefined trajectory with the minimum deviation (Spong & Fujita, 2011). Robotic manipulator systems are containing unavoidable uncertainties such as structured and unstructured uncertainties that degraded manipulators

performance. Correct dynamical model with variation parameters, sizes and mass distributions of payloads, difference in links properties of robotic manipulators are defined as structured uncertainty present in robotic manipulators. Unstructured uncertainty is defined as unmodelled dynamics, which contain the external disturbances, nonlinear frictions and random noise (Song, Yi, Zhao, & Li, 2005). To combat such challenges, it is essential to develop a robust controller for these systems. Controlling of robot manipulator requires the complex dynamic equation (mathematical modeling) and trajectory planning, which is computationally intensive. The conventional control theories such as PID have been successfully applied to areas where systems are well defined. However, when the system is complex, non-linear, ill-structured process with excessive variation parameter, the potency of conventional control becomes poor. With technology advancement, there are imperative needs to design controllers that are able to manage structured and unstructured uncertainty (Laxmidhar & Kar, 2010). The fuzzy set technique given by Zadeh has become a powerful modeling tool for solving severe real-world problems with an uncertain and unpredictable environment. The FLCs (fuzzy logic controllers) come in the category of intelligent control and as an intelligent controller, FLCs (type-1 fuzzy logic controllers i.e., T1FLCs) parameter is easily tuned by non-expertise person. The operation of T1FLCs is based on human expertise and the knowledge acquisition techniques to convert human expertise to appropriate if-then rules as well as a proper fuzzy membership function for each fuzzy variable. The T1FLC has some important advantages such as, [23] it provides a higher level of automation by incorporating expert knowledge, [10] it does not require exact knowledge of the controlled system dynamics model, [14] it plays a major role while controlling complex non-linear systems, [6] it reduces development and maintenance time. There are some drawbacks of T1FLCs, i.e. it can't fully deal with or handle the linguistic and numerical uncertainties connected with dynamic structured or unstructured environments. In addition to that, its performance is not satisfactory, because the ordinary T1FLCs have limited capabilities to directly handle data uncertainties (Mendel, 2007). There are generally five sources of uncertainties observed in T1FLC (Hagras, 2004, 2007; Mendel & John, 2002), which are listed here [23] Uncertainties in the inputs to the T1FLC are produced by noise and change in environmental conditions of sensors. These uncertainties are translated into uncertainties in the antecedents membership functions (MFs) and that the input sensors can be affected by the conditions of observation i.e., their characteristics can be changed by the environmental conditions such as wind, sunshine, humidity, rain, etc. [10] The output of T1FLC is applied to actuators to control the plant.

[15] Generation of a short and smooth path in three-dimensional space with obstacles for guiding an Unmanned Underwater Vehicle (UUV) without collision is investigated. This is done by utilizing the spline technique, in which the spline control points positions are determined by Imperialist Competitive Algorithm (ICA) in three-dimensional space such that the shortest possible path from the starting point to the target point without colliding with obstacles is achieved. Furthermore, for guiding the UUV in the generated path, an Interval Type-2 Fuzzy Logic Controller (IT2FLC), the coefficients of which are optimized by considering an objective function that includes quadratic terms of the input forces and state error of the system, is used. Selecting such objective function reduces the control error and also the force applied to the UUV, which consequently leads to a reduction of energy consumption. Therefore, by using a special method, desired signals of UUV state are obtained from a generated three-dimensional optimal path such that tracking these signals by the controller leads to the tracking of this path by UUV. In this thesis, the dynamical model of the UUV, entitled as "mUUV-WJ-1", is derived and its hydrodynamic coefficients are calculated by CFD in order to be used in the simulations. concerning simulation by the

method presented in this study, three environments with different obstacles are intended in order to check the performance of the IT2FLC controller in generating optimal paths for guiding the UUV. In this article, besides to ICA, Particle Swarm Optimization (PSO) and Artificial Bee Colony (ABC) are also used for the generation of the paths and the results are compared with each other. The results show the appropriate performance of ICA rather than ABC and PSO. Moreover, to evaluate the performance of the IT2FLC, optimal Type-1 Fuzzy Logic Controller (T1FLC), and Proportional Integrator Differentiator (PID) controller are designed and applied to the UUV then compared to each other. The simulation results show the superiority of the IT2FLC over the other two controllers.

[74] Interval type-2 fuzzy logic controllers (IT2 FLCs) have recently been attracting a lot of research attention. Many reported results have shown that IT2 FLCs are more capable to handle uncertainties than their type-1 (T1). For example, Hagra implemented a hierarchical IT2 FLC for different types of mobile robots navigating in indoor and outdoor environments. It outperformed a T1 FLC and had significantly fewer rules. Wu and Tan showed through both simulation and experiments that IT2 FLCs are more performing in coping with modeling uncertainties, and hence, IT2 FLCs optimized from simulations are more likely to perform well on the actual plant than T1 FLCs.

[19] After the inception of robotics, the robot manipulators are widely used as the promising device in the automated industrial applications including nuclear and medical fields. These manipulators are incorporated to enhance flexibility, productivity, and accuracy in accordance with reduced working cost. It also increased human working conditions where repetitive and hazardous works are executed [67], [8]. For exceptional utility, it is entailed to meticulously design efficient robust controllers that can command the robotic motion in a precise way. Some preliminary work had stated that the conventional proportional- integral-derivative (PID) controllers were used in industrial applications because of their elementary design, easy implementation, and low cost, whereas, recent studies in this field found that they are ineffective for complicated systems such as robotic manipulators with uncertainty and nonlinearity. Progressively, various control strategies and controller structures have been suggested in the past few decades to enhance the performance of systems under control, within this framework. The incorporation of fuzzy logic systems (FLS) in control theory, producing fuzzy logic controllers (FLC), has enhanced the flexibility of controller design and increased its applicability to control the complex, nonlinear, ill-structured, and uncertain systems. Thereafter, the FLC (later, called as type-1 fuzzy logic controllers T1-FLC) have received considerable interest and the control designers recommend T1- FLC over the conventional counterparts because of their unique characteristics such as (a) incorporation of knowledge based on human expertise, (b) no exact dynamic model of system is required hence useful when precise mathematical formulation is infeasible, (c) low development and maintenance cost, and importantly, (d) the ability of general framework of T1-FLC to handle uncertainty, etc. In literature, various authors proposed different T1-FLC structures and their extensions for numerous types of robotic applications. Subsequently, over the past few years, a growing number of researchers have proposed various designs and strategies of T1-FLCs and investigated them for numerous applications for developing effective controllers [37] The usage of the flexible joint manipulator (FJM) as an alternative with rigid body manipulators in the robotics industry such as service, space, medical and defense because the FJM provides faster movements, low manufacturing cost and, high weight to payload ratio as well as low energy consumption compared to the rigid-body structures. Despite these advantages, the flexibility of the joint cause vibrations (link deflections) and position errors. However, convenient control approaches may overcome these problems in the FJM. During the last decade, various linear and non-linear control methods using model-based

and model-free approaches have been presented for the trajectory and vibration control of the FJM. Proportional-Integral-Derivative (PID) state feedback, Linear Quadratic Regulator (LQR) and Linear Quadratic Gaussian (LQG), Intelligent Proportional-Integral (IPI) and distributed higher-order differential feedback controllers are some of the linear control methods applied in the literature. These linear controllers naturally showed overshoot and high link deflection because of their performance was limited by to only small regions around the equilibrium or operating points of the nonlinear FJM system. **conclusion**

Chapter 4

Fuzzy logic controller for puma560 robot: simulation and results discussion

4.1 Introduction

In this chapter, we will describe in details the fuzzy logic controllers (type1 fuzzy sets and interval type2 fuzzy sets) proposed to control the PUMA560 manipulator robot then a comparison of the simulation results of both approaches is presented. As a performance measure we will use the Integral of Square Error (ISE) and Integral of Square Torque (IST)[64] Fuzzy logic controller (FLC) which is used in this research is the Mamdani method, or known as the Max-Min method.

4.2 General description of the fuzzy logic system used in control of Puma560

The dynamic parameters of Puma560 are taken from [15], see the Appendix. The simulation control system diagram of PUMA560 is shown in Figure 4.1. To check the robustness of controller, a disturbance torque is applied on the robot and a cycloid trajectory test is performed. Recall that only three links of PUMA560 robot are used in this work. The three degrees of freedom PUMA560 robot has the same configuration space equation general form in [15] 6-DOF, in which the last three joints q_4 , q_5 and, q_6 are set to zero in equation cited in [15] so they keep their initial states while the robot is moving.

We can define a general equation that allows us to use PUMA560 robot as a 3-DOF robot and its general control system represented in Figure 4.1.

Note: All parameters and Matrix in (1) are taken from paper [15], data q_4 , q_5 and q_6 are to be replaced by zero and do simplification to get 3-DOF PUMA560 robot as in equation (1).

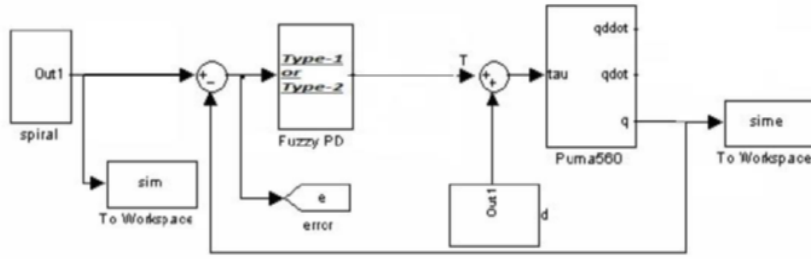


Figure 4.1: General diagram for fuzzy sets using in control of Puma 560

4.3 Controller type-1 fuzzy sets

The basic scheme of a fuzzy controller [74] is shown in Figure 4.2. The fuzzy controller has four parts; the first one is the fuzzification, which changes the crisp values to fuzzy value and determines membership degrees of the chosen PD, which needs two inputs, error and its change, and one output as a control signal. The position errors with their variations and the control signal quantized into five sets as in Figure 4.3: represented by a set of linguistic variables with triangle membership functions with $[-1, 1]$ as a universe of discourse [1], [5],[37] :

Negative Large (N-L), Negative Small (N-S), zero (Ze), Positive Small (P-S) and Positive Large (P-L).

We choose the output as singletons values $[N-L; N-S; Ze; P-S; P-L] = [-1; -0.5; 0; 0.5; 1]$

The second part is the inference, which cannot function without rule base like that shown in Table 1 [39]. The max-min Mamdani fuzzy inference system [74] was adopted for our controller. The last part is the defuzzification in which the fuzzy inference result is changed to crisp value. There are several methods for defuzzification: in our work centroid of the area has been chosen [1], [5].

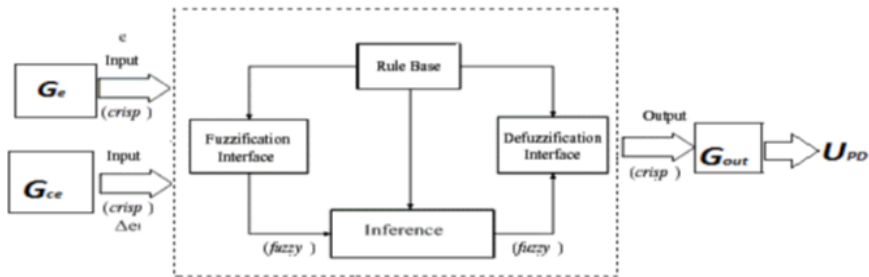


Figure 4.2: A type-1 fuzzy controller [5]

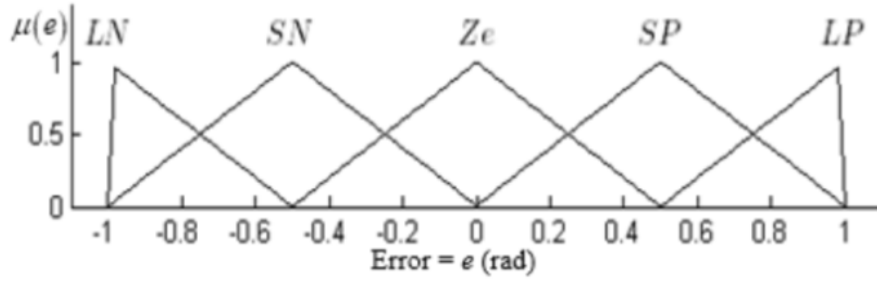


Figure 4.3: Fuzzy set for each articulation with triangle membership functions for error and its change

Table 4.1: Rulebase table[39]

		Velocity error				
		N-L	N-S	Ze	P-S	P-L
Position error	N-L	NL	NL	NL	NS	Ze
	N-S	NL	NL	NS	Ze	PS
	Ze	NL	NS	Ze	PS	PL
	P-S	NS	Ze	PS	PL	PL
	P-L	Ze	PS	PL	PL	PL

4.4 Controller type2 fuzzy sets

4.4.1 General type-2 fuzzy system

The basics of fuzzy logic do not change from type-1 to type-2 fuzzy sets, [33]. A higher-type number simply indicates a higher "degree of fuzziness", Since a higher type changes the nature of the membership functions, the operations that depend on the membership functions change; however, the basic principles of fuzzy logic are independent of the nature of membership functions and hence, do not change. Rules of inference like Generalized Modus Ponens or Generalized Modus Tollens continue to apply.

The structure of the type-2 fuzzy rules is the same as for the type-1 case because the distinction between type-2 and type-1 is associated with the nature of the membership functions. Hence, the only difference is that now some or all the sets involved in the rules are of type-2. In a type-1 fuzzy system, where the output sets are type-1 fuzzy sets, we perform defuzzification in order to get a number which is, in some sense, a crisp (type-0) representative of the combined output sets. In the type-2 case, the output sets are type-2, so we have to use extended versions of type-1 defuzzification methods. Since type-1 defuzzification gives a crisp number at the output of the fuzzy system, the extended defuzzification operation in the type-2 case gives a type-1 fuzzy set at the output. Since this operation takes us from the type-2 output sets of the fuzzy system to a type-1 set, we can call this operation "type reduction" and call the type-1 fuzzy set so obtained a "type-reduced set". The type-reduced fuzzy set may then be defuzzified to obtain a single crisp number; however, in many applications, the type-reduced set may be more important than a single crisp number [33] [55].

Type-2 sets can be used to convey the uncertainties in membership functions of type-1 fuzzy sets, due to the dependence of the membership functions on available linguistic and

numerical information. Linguistic information, in general, does not give any information about the shape of the membership functions. When membership functions are determined or tuned based on numerical data, the uncertainty in the numerical data, e.g., noise, translates into uncertainty in the membership functions. In all such cases, any available information about the linguistic numerical uncertainty can be incorporated in the type-2 framework. However, even with ailing advantages that fuzzy type-2 systems have, the literature on the applications of type-2 sets is scarce. We think that more applications of type-2 fuzzy systems will come in the near future as the area is gaining maturity and that theoretical results become more understandable for the general public in the fuzzy arena [36]

The general structure of a type-2 fuzzy controller (type-2 fuzzy controller: T2FC) is represented in Figure 4.4: This structure is similar to that of the type-1 fuzzy controller.

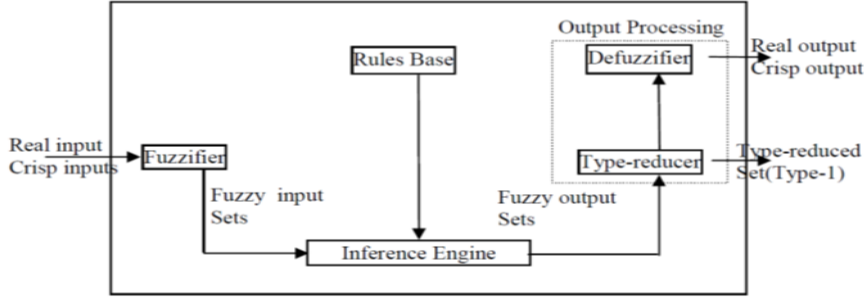


Figure 4.4: Type-2 Fuzzy Controller structure[55]

However, their differences in :

- The type of membership function used.
- Procedure of adopted defuzzifier. In a T2FC, a block of reduction of the type is essential to convert the type-2 fuzzy set to a type-1 fuzzy set.

a. Fuzzification

In this thesis, we will consider only singleton fuzzification (Crisp input), in the input fuzzy set.

b. Rules

In the type-1 case, we generally have "IF-THEN" rules, where the rule has the form

$$"R^l: \text{IF } x_1 \text{ is } F_1^l \text{ and } x_2 \text{ is } F_2^l \text{ and } \dots \text{ and } x_p \text{ is } F_p^l \text{ THEN } y \text{ is } G^l" \quad (4.1)$$

Where x_i 's are inputs, F_i^l 's are antecedent sets ($i = 1, \dots, p$), y is the output and G^l are consequent sets,

The distinction between type-1 FLS and type-2 FLS is associated with the nature of the membership function. which is not important while forming rules, hence, the structure of the rules remains exactly the same in the type-2 FLS case, the only difference being that now some or all of the sets involved are of type-2, so, the rule in a type-2 FLS has the form

$$"R^l: \text{IF } x_1 \text{ is } \tilde{F}_1^l \text{ and } x_2 \text{ is } \tilde{F}_2^l \text{ and } \dots \text{ and } x_p \text{ is } \tilde{F}_p^l \text{ THEN } y \text{ is } \tilde{G}^l" \quad (4.2)$$

Where x_i 's are inputs, \tilde{F}_i^l 's are antecedent sets ($i = 1, \dots, p$), y is the output, and \tilde{G}^l are consequent sets, It is not necessary that all the antecedents and the consequent need be type-2 fuzzy sets. As long as one antecedent or the consequent set is type-2. we will have a type-2 FLC.

c. Inference Engine

In general, the rules we use will have multiple antecedents connected by ands. Just as in the type-1 case. we can connect these multiple antecedents by the meet operation (corresponding to t-norm in the type-1 case). Different rules can be combined using the join operation (corresponding to t-norm in the type-1 case), or during defuzzification.

The output of the inference engine [33] [55] consists of the fired consequent fuzzy sets. Each one of these is modified from a consequent fuzzy set by a degree of firing. This degree of firing is obtained, In general, as a result of t-norm (meet) and t-norm (join) operations on membership grades of the inputs.

The relation (4.2) is interpreted as a fuzzy implication type-2 and defined by :

$$R^l : \tilde{F}_1^l \times \dots \times \tilde{F}_p^l \rightarrow \tilde{G}^l \quad (4.3)$$

This relation is described by the membership function in the same way as :

$$\mu_{R^l}(x_1, \dots, x_p, y) = \mu_{\tilde{F}_1^l \times \dots \times \tilde{F}_p^l \rightarrow \tilde{G}^l}(x_1, \dots, x_p, y) = \left[\bigcap_{i=1}^p \mu_{\tilde{F}_i^l}(x_i) \right] \cap \mu_{\tilde{G}^l}(y) \quad (4.4)$$

$\tilde{F}_1^l \times \dots \times \tilde{F}_p^l$ Denote the cartesian product of $\tilde{F}_1^l, \dots, \tilde{F}_p^l$

d. Type-Reduction

Observe, from figure 4.4, that the defuzzifier block in the type-1 FLC is replaced by two blocks: type-reducer and defuzzifier. We consider type-reduction in this subsection.

In a type-1 FLC. where the output sets are type-1 fuzzy sets, we perform defuzzification in order to get a number which is in some sense a crisp (type-0) representative of the combined output sets. In the type-2 case, the output sets are type-2, so we have to use "the extended principle" of type-1 defuzzification methods. Since type-1 defuzzification gives a crisp number at the output of the FLS, the extended defuzzification operation in the type-2 case gives a type-1 fuzzy set at the output. When this operation takes us from the type-2 output sets of the FLC to a type-1 set, we call this operation "type-reduction" [33], [36] and call the type-1 set so to obtain a single crisp number, however, in many application, the type reduced set may be more important than a single crisp number.

There exists a number of type-reduction kinds, such as centroid, center-of-sets, height and modified height, the details of which are given in [33] [36]. In this work, for illustrative purposes, we focus on center-of-sets type-reduction.

e. Center-of-sets type-reduction

In this method [36], each set of the consequence \tilde{G}^l is replaced by its centroid. If the set of output \tilde{G}^l is of type-2, its centroid $C_{\tilde{G}^l}$ is a type-1 fuzzy set. Then the weighted average of all the centroid is calculated, associating with each centroid $C_{\tilde{G}^l}$, a weight equals the degree of activation of the rule is given by $E_i(x) = \bigcap_{i=1}^p \mu_{\tilde{F}_i^l}(x)$. The procedure of calculation of the type- reduced set $Y_a(x)$ is:

1. Discretize the output space Y into a suitable number of points, and compute the centroid $C_{\tilde{G}^l}$ of each consequent set on the discretized output space. These consequent centroid sets can be computed ahead of time and stored for future use.
2. Compute the degree of firing, $E_l(x)$ associated with the consequent.

3. Discretize the domain of each $C_{\tilde{G}^l}$ into a suitable number of M_l points, $l = 1, \dots, M$.
4. Discretize the domain of each $E_l(x)$ into a suitable number of points, say $N_l, l = 1, \dots, M$.
5. Enumerate all the possible combinations $\{c_1, \dots, c_M, e_1, \dots, e_M\}$ such that $c_l \in C_{\tilde{G}^l}$, and $e_l \in E_l$.

The total number of combinations will be $\prod_{i=1}^M M_i N_i$

6. Compute the center-of-sums type-reduced set using (3.26).

$$Y_{ce}(x) = \int_{c_1} \dots \int_{c_M} \int_{e_1} \dots \int_{e_M} \sup \left[\mathfrak{S}_{l=1}^M \mu_{C_{\tilde{G}^l}}(c_l) * \mathfrak{S}_{l=1}^M \mu_{E_l}(e_l) \right] / \frac{\sum_{i=1}^M c_i e_i}{\sum_{i=1}^M e_i} \quad (4.5)$$

Where \mathfrak{S} and $*$ indicate the T-norm chosen

f. Defuzzification

To obtain a crisp output from the type-2 FLS, we can defuzzify the type-reduced set [33] [36]. The most natural way of doing this seems to be by finding the centroid of the type-reduced set, however, other possibilities exist, like choosing the unity membership point in the type-reduced set.

The defuzzification makes to transform the linguistic output of the type-reduction to numeric valued. Several methods of defuzzification were proposed in the literature [51]. But the largely used method is:

g. Centre of area

The defuzzification determines the X -coordinate of the center of gravity y_{CG}^* [33] [36] of the fuzzy function:

$$y_{CG}^* = \frac{\int_y y \mu_Y(y) dy}{\int_y \mu_Y(y) dy} \quad (4.6)$$

4.4.2 Interval type-2 Fuzzy controller

a. MEET and JOIN for Interval set

- The meet under minimum or product t-norms of n interval type-1 sets A_1, \dots, A_n having domains $[l_1, r_1], \dots, [l_n, r_n]$ respectively, where $[l_i, r_i] \subseteq [0, 1], i = 1, \dots, n$ is an interval set with domain $[l_1 * \dots * l_n, r_1 * \dots * r_n]$ such as [41] [43]:

$$\bigcap_{i=1}^n A_i = \int_{w \in [l_1 * \dots * l_n, r_1 * \dots * r_n]} 1/w \quad (4.7)$$

* chosen t-norms.

- The join under maximum t-norms of n interval type-1 sets $A_1, \dots, A_n, A_1, \dots, A_n$ having domains $[l_1, r_1], \dots, [l_n, r_n]$ respectively, where $[l_i, r_i] \subseteq [0, 1], i = 1, \dots, n$ is an interval set with domain $[l_1 \vee \dots \vee l_n, r_1 \vee \dots \vee r_n]$ such as:

$$\bigcup_{i=1}^n A_i = \int_{w \in [l_1 \vee \dots \vee l_n, r_1 \vee \dots \vee r_n]} 1/w \quad (4.8)$$

\vee denotes a max operation

b. Inference

In interval type-2 fuzzy system using the minimum or product t-norms operations, the /th activated rule $\bigcap_{i=1}^p \mu_{\tilde{F}_i^l} = F^l(x_1, \dots, x_p)$ gives us an interval determined by two extreme $f_-^l(x_1, \dots, x_n)$ and $\tilde{f}^l(x_1, \dots, x_n)$ same as [41] :

$$F^l(x_1, \dots, x_n) = [f_-^l(x_1, \dots, x_n), \tilde{f}^l(x_1, \dots, x_n)] \equiv [f_-^l, \tilde{f}^l] \quad (4.9)$$

With f_-^l and \tilde{f}^l are given as :

$$\begin{aligned} f_-^l &= \mu_{\tilde{F}_1^l}(x_1) * \dots * \mu_{\tilde{F}_p^l}(x_p) \\ \tilde{f}^l &= \bar{\mu}_{\tilde{F}_1^l}(x_1) * \dots * \bar{\mu}_{\tilde{F}_p^l}(x_p) \end{aligned} \quad (4.10)$$

The output set $\tilde{R}^l(y)$ of the /th activated rule R^l is type-2 fuzzy set :

$$\mu_{\tilde{B}^l}(y) = \int_{b^l \in [f_-^l * \mu_{\tilde{G}^l}(y), \tilde{f}^l * \bar{\mu}_{\tilde{G}^l}(y)]} 1/b^l \quad , \quad y \in Y \quad (4.11)$$

$$\mu_{\tilde{B}}(y) = \bigcup_{l=1}^N \mu_{\tilde{B}^l}(y) \quad (4.12)$$

$\mu_{\tilde{G}^l}(y)$ and $\bar{\mu}_{\tilde{G}^l}(y)$ represent upper and lower membership function of the set $\mu_{\tilde{G}^l}(y)$

c Type-reduction and Defuzzification

After fuzzification, fuzzy inference, type-reduction and defuzzification [41] [43] [33], we obtain a crisp output For an interval type-2 FLC, this crisp output is the center of the type-reduced set, we know that for an interval type-2 FLC, regardless of singleton or non-singleton fuzzification, and minimum or product t-norm, the result of input and antecedent operations (firing strength) is an interval type-1 set which is determined by its left-most and right-most points f_-^l and \tilde{f}^l . The fired output consequent set $\tilde{B}^l(y)$ of rule R^l can be obtained from the fired interval strength using (4.10). Then the fired combined output consequent set $\tilde{R}^l(y)$ can be computed using (4.12) There exist type-reduction different kinds, such as centroid, center-of-sets, height and modified height, the details of which are given in [78], [74] , [33]. In this memory, for illustrative purposes, we focus on center-of-sets type-reduction, which can be expressed as:

$$y_{\cos}(W^1, \dots, W^M, F^1, \dots, F^M) = [y_l, y_r] = \iint_{w^l \in [w_l^1, w_r^1], w^M \in [w_l^M, w_r^M]} \iint_{f^1 \in [f_-^1, \tilde{f}^1], f^M \in [f_-^M, \tilde{f}^M]} 1 / \frac{\sum_{i=1}^M f^i w^i}{\sum_{i=1}^M f^i} \quad (4.13)$$

Where Y_{cos} an interval set is determined by two end points (y_l and y_r), $f^l \in [f_-^l, \tilde{f}^l]$; $w^i \in W^i = [w_l^i, w_r^i]$, W^i is the centroid of the interval type-2 consequent set \tilde{G} (the centroid of a type-2 fuzzy set is described in Appendix C).

Observe that $Y_{cos}(W^1, \dots, W^M, F^1 \dots F^M)$ is an interval type-1 set. So, to find

$y_{cos}(W^1, \dots, W^M, F^1 \dots F^M)$ we just need to compute the two end-points of this interval. Unfortunately, no closed-form formula is available for Y_{cos} .

$$y = \frac{\sum_{i=1}^M f^i w^i}{\sum_{i=1}^M f^i} \quad (4.14)$$

The maximum value of y is y_r , and the minimum value of y is y_l from (4.14), we see that y is a monotonic increasing function with respect to w^i , so, y_r , and is only associated with w_r^i similarly y_l is only associated with w_l^i . In the COS type-reduction method, the two end-points of y_{cos} , (y_l and y_r) depend on a mixture of \underline{f}^i or \bar{f}^i values. In this case, (y_l and y_r) can be represented as :

$$y_l = \frac{\sum_{i=1}^M f_l^i w_l^i}{\sum_{i=1}^M f_l^i} \quad (4.15)$$

Where f_l^i denote the firing strength membership grade either (\underline{f}^i and \bar{f}^i) contributing to the right-most point y_l , similarly to y_r

$$y_r = \frac{\sum_{i=1}^M f_r^i w_r^i}{\sum_{i=1}^M f_r^i} \quad (4.16)$$

Where f_r^i denotes the firing strength membership grade either (\underline{f}^i or \bar{f}^i) contributing to the right-most point y_r .

In order to compute (y_l and y_r), we need to compute $\{f_l^i, i = 1, 2, \dots, M\}$ and $\{f_r^i, i = 1, 2, \dots, M\}$.

This can be done using the exact computational procedure given in [43] [33]. Here we briefly provide the computation procedure for y_r . Without loss of generality, assume the w_r^i 's are arranged in ascending order $w_r^1 \leq w_r^2 \leq \dots \leq w_r^M$.

1. Compute y in (4.16) by initially setting $f_r^i = \frac{\bar{f}^i + \underline{f}^i}{2}$ for $i = 1, \dots, M$ where (\underline{f}^i and \bar{f}^i) have been previously computed using (4.10) and let $y' = y$.
2. Find R ($1 \leq R \leq M - 1$) such that $w_r^R \leq y' \leq w_r^{R+1}$
3. Compute y in (4.14) when $f_r^i = \underline{f}^i$ for $i \leq R$, and $f_r^i = \bar{f}^i$ for $i > R$, then set $y'' = y_r$
4. If $y'' \neq y'$ then go to step 5. If $y'' = y'$ then stop and set $y_r = y''$
5. Set y' equal to y'' and return to step 2

This 5 step computation procedure has been proven to converge to the exact solution in no more than M iterations [43] [33]. Note that in this procedure, the number R is very important. For $i \leq R$, $f_r^i = \underline{f}^i$ and for $i > R$, $f_r^i = \bar{f}^i$, so y_r can be represented as:

$$y_r = y_r(\underline{f}^1, \dots, \underline{f}^R, \bar{f}^{R+1}, \dots, \bar{f}^M, w_r^1, \dots, w_r^M) \quad (4.17)$$

The procedure for computing y_l is very similar. Just replace w_r^i by w_l^i , and in step 2, find L ($1 \leq L \leq M - 1$), such that $w_l^L \leq y'_l \leq w_l^{L+1}$, and in step 3

$f_l^i = \bar{f}^i$ for $i \leq L$, and $f_l^i = \underline{f}^i$ for $i > L$. Then y_l can be represented as:

$$y_r = y_r \left(\bar{f}^1, \dots, \bar{f}^R, \underline{f}^{R+1}, \dots, \underline{f}^M, w_l^1, \dots, w_l^M \right) \quad (4.18)$$

Because Y_{cos} is an interval set, we defuzzify it using the average of (y_l and y_r), and hence, the defuzzified output of an interval type-2 FLC is: output of Centroid = $\frac{y_l + y_r}{2}$

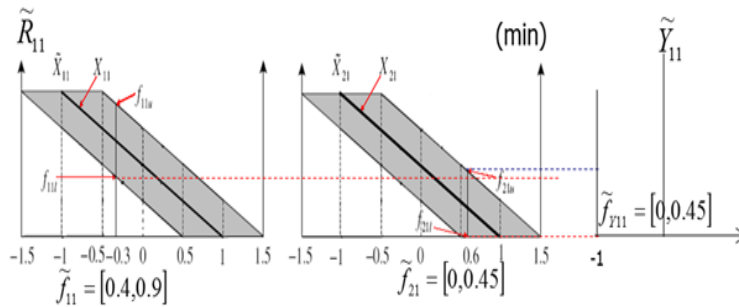
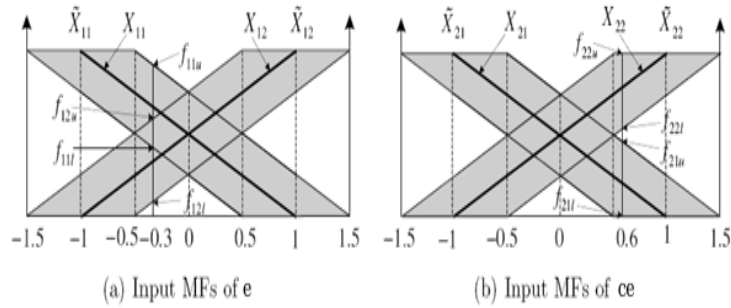
The control of dynamic system was originally applied to a system that control a mechanical position or motor. Used to describe a control system in which a physical variable is required to follow, or track.

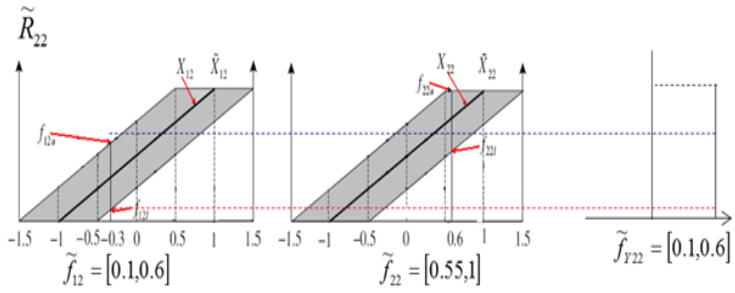
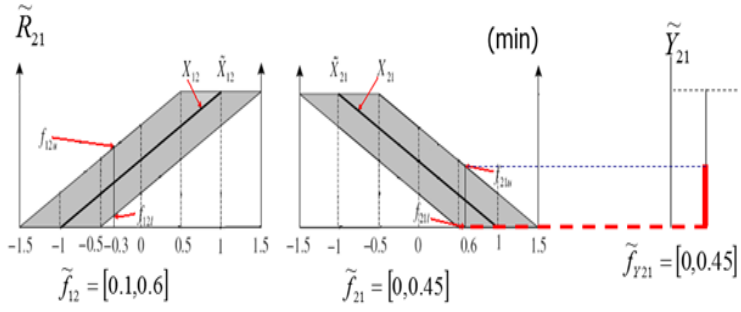
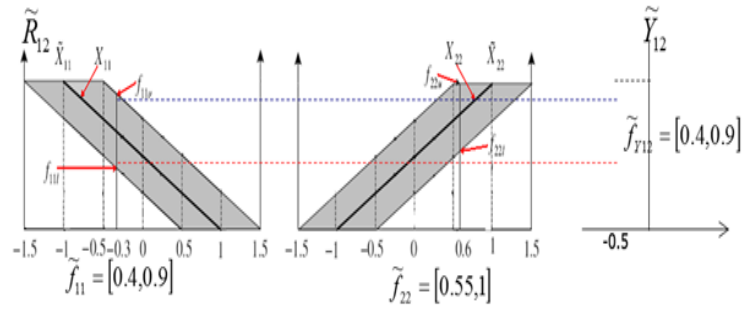
Example 4.4.1. to illustrate type2 FLS

\tilde{R}^{ij} : if e is \tilde{X}_{1i} and ce is \tilde{X}_{2j} , then y is \tilde{Y}_{ij} , $i, j = 1, 2$ Rule base and consequents of the

ce e	\tilde{X}_{21}	\tilde{X}_{22}
\tilde{X}_{11}	$\tilde{Y}_{11} = -1$	$\tilde{Y}_{12} = -0.5$
\tilde{X}_{12}	$\tilde{Y}_{21} = 0.5$	$\tilde{Y}_{22} = 1$

Interval type-2 FLS Consider an input vector $w = (e, ce) = (-0.3, 0.6)$





$$\tilde{R}_{11} : \tilde{f}_{Y_{11}} = [\min(0.4, 0), \min(0.9, 0.45)] = [0, 0.45] \rightarrow -1$$

$$\tilde{R}_{12} : \tilde{f}_{Y_{12}} = [\min(0.4, 0.55), \min(0.9, 1)] = [0.4, 0.9] \rightarrow -0.5$$

$$\tilde{R}_{21} : \tilde{f}_{Y_{21}} = [\min(0.1, 0), \min(0.6, 0.45)] = [0, 0.45] \rightarrow 0.5$$

$$\tilde{R}_{22} : \tilde{f}_{Y_{22}} = [\min(0.1, 0.55), \min(0.6, 1)] = [0.1, 0.6] \rightarrow 1$$

$$L = 1 \quad y_l = \frac{0.45 \times (-1) + 0.4 \times (-0.5) + 0 \times (0.5) + 0.1 \times (1)}{0.45 + 0.4 + 0 + 0.1} = -0.58$$

$$R = 2 \quad y_r = \frac{0 \times (-1) + 0.4 \times (-0.5) + 0.45 \times (0.5) + 0.6 \times (1)}{0 + 0.4 + 0.45 + 0.6} = 0.43$$

$$\text{Centroid} = \frac{y_l + y_r}{2} = -0.07$$

[74]

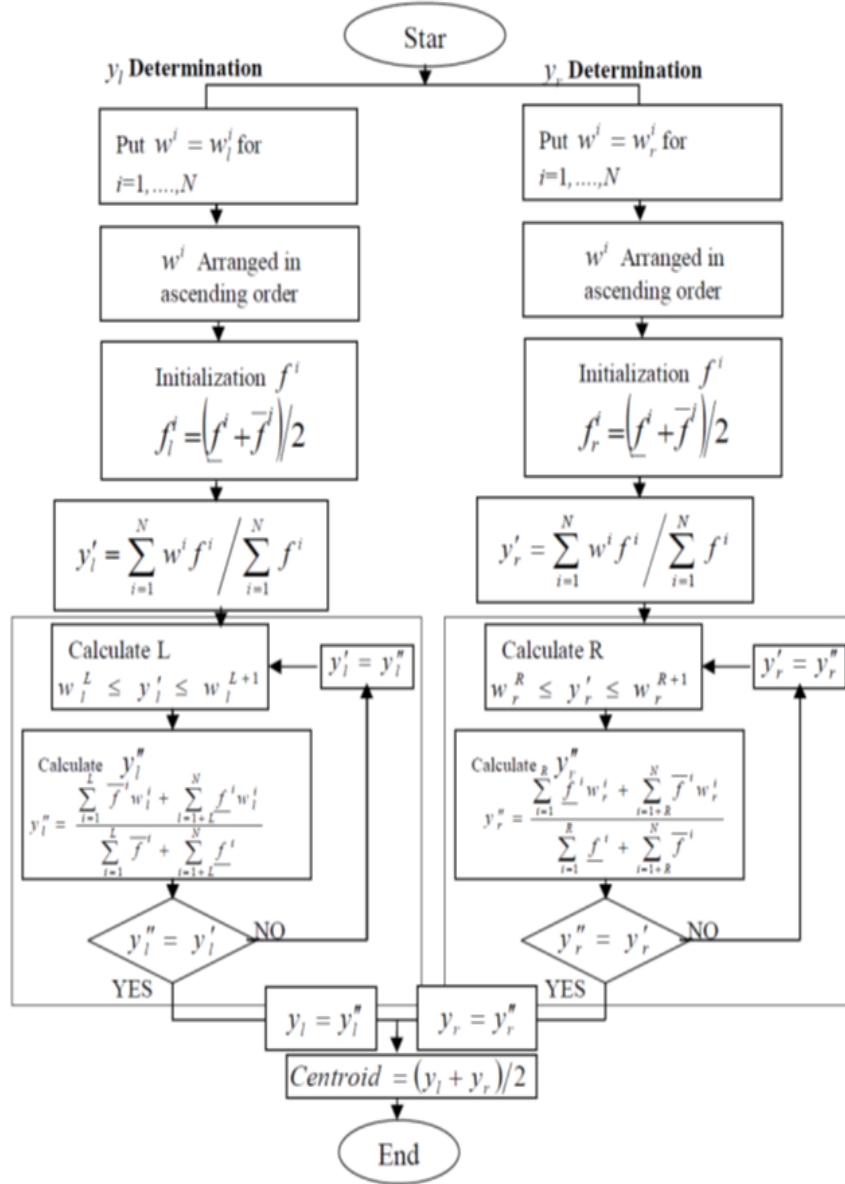


Figure 4.5: Karnik-Mendel Algorithms to locate Centroid on Interval type-2 set [36]

d. Detailed description of $K - M$ algorithm

In this work a center of sets type reduction approach is used the details are given as follows:

Step1. Determination of y_{cos} [y_l, y_r] where y_{cos} is interval set and y_r is the maximum value of y And y_l is the minimum value of y

➤ compute of y_l

- Compute w_l^i (for $i = 1, \dots, N$) where w_l^i is the lower fuzzy value of $w_i = (e, ce)$
- Put $w^i = w_l^i$ (for $i = 1, \dots, N$)
- Arranged w^i in ascending order $w_l^1 < w_l^2 < \dots < w_l^N$
- Compute the left most point and the right most point $(\underline{f}^i, \bar{f}^i)$ for $i = 1, \dots, N$
- Set initially: $(\underline{f}^i + \bar{f}^i) / 2$
- Compute: y_l' using (4 - 16)
 1. Find $R(1 \leq R \leq N - 1)$ such that $w_r^R \leq y_r' \leq w_r^{R+1}$
 2. Compute y in (4.14) when $f_r^i = \underline{f}^i$ for $i \leq R$, and $f_r^i = \bar{f}^i$ for $i > R$, then $y_r'' = y_r$
 3. If $y'' \neq y'$ then go to step 5 . If $y'' = y'$ then stop and set $y_r = y''$
 4. set $y' = y''$ and return to step 2
- Put $y_l = y_l''$

➤ Compute of y_r

- Compute w_r^i (for $i = 1, \dots, N$) where w_r^i is the lower fuzzy value of $w_i = (e, ce)$
- Put $w^i = w_r^i$ (for $i = 1, \dots, N$)
- Arranged w^i in ascending order $w_r^1 < w_r^2 < \dots < w_r^N$
- Compute the left most point and the right most point $(\underline{f}^i, \bar{f}^i)$ for $i = 1, \dots, N$
- Set initially: $(\underline{f}^i + \bar{f}^i) / 2$
- Compute: y_r' using (4 - 16)
 5. Find $L(1 \leq L \leq N - 1)$ such that $w_l^L \leq y_l' \leq w_l^{L+1}$
 6. compute y in (4.14) when $f_l^i = \underline{f}^i$ for $i \leq L$, and $f_l^i = T_i$ for $i > L$, then $y_l'' = y_l$
 7. If $y'' \neq y'$ then go to step 5 . If $y'' = y'$ then stop and set $y_l = y''$
 8. set $y' = y''$ and return to step 2
- Put $y_r = y_r''$

Step2. Compute the centeroid of interval set

$$\text{Centeroid} = (y_l + y_r) / 2$$

Table 4.2: Rulebase table[33]

		Velocity error				
Position error		NL	SN	Ze	SP	LP
	LN	LN	LN	LN	SN	Ze
	SN	LN	LN	SN	Ze	SP
	Ze	LN	SN	Ze	SP	LP
	SP	SN	Ze	SP	LP	LP
	LP	Ze	SP	LP	LP	LP

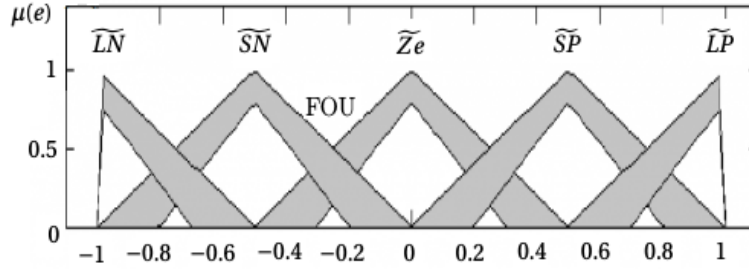


Figure 4.6: Fuzzy sets interval type-2 for each articulation and control signal

4.5 Interval type-2 Fuzzy control with PUMA560 3DOF

we show the structure interval type-2 FLC with PUM560 3DOF, the regulator we use is five classes, wich means that 25 rule base are involted, the rule base table in the table 4.2 and in Figure 4.7 fuzzy sets for error and change error and out put of control T. All the gains of Interval type-2 fuzzy controller we do tuning until get good positions with lower error in ideal case.

In this work and for illustrative purposes, we have used Karnik-Mendel algorithm of figure 4.5 to locate centroid on interval type-2 fuzzy set [36]. The proposed modification is carried out in the fourth step of this algorithm where Karnik-Mendel use arithmetic mean, but in the presented work it has been modified by harmonic mean, as in the nexte formulate:

$$y = \frac{\sum_{i=1}^n f_i}{\sum_i \frac{f_i}{w_i}}$$

4.6 Simulation results

For our model of equation (1) 3DOF PUMA560 robot, a cycloid trajectory test Figure 4.11 used the different articulation move respectively from position $(-50^\circ, -135^\circ, 135^\circ)$ to the position $(45, -85^\circ, 30^\circ)$ with three end time of movement equal to (1.5, 4, 8) seconds then will have three cycloid trajectory that differ in simulation time. This trajectory is described by

the following equation:

$$\theta_{di} = \begin{cases} \theta_{di}(0) + \frac{D_i}{2\pi} \left[2\pi \frac{t}{t_{end}} - \sin\left(2\pi \frac{t}{t_{end}}\right) \right] & \text{for } 0 \leq t \leq t_{end} \\ \theta_{di}(t_{end}) & \text{for } t_{end} < t \\ D_i = \theta_{di}(t_{end}) - \theta_{di}(0) & \end{cases} \quad (4.19)$$

Integral of Square Error (ISE) [40] is used as performance index in comparative study between type-1 and interval-type-2 fuzzy controller

$$ISE = \int_0^{t_{end}} [e(t)]^2 dt \quad (4.20)$$

In type-1 fuzzy PD controller of Figure 4.4, fuzzy set parameters take standard parameters as declared by default in MATLAB in the range of $[-1, 1]$; the triangle membership functions are symmetric fuzzy sets. The controller require three gains: two inputs gains and one output gain; these gain; are chosen using manual tuning until getting good positions responses with lower error, lower control energy and smooth control signal without chattering in ideal case.

In an Interval-type-2 fuzzy PD controller, the interval-type-2 fuzzy sets are programmed as type-1 fuzzy sets in the range of $[-1, 1]$ with upper and lower triangle membership function as shown in Figure 4.6. In addition to the length of FOU zone, the controller has also three gains to be designed: two input gains and one output gain. These gains and the FOU zone are chosen base on trial and error method until getting good position responses with lower error and lower control energy and smooth control signal without chattering in ideal case.

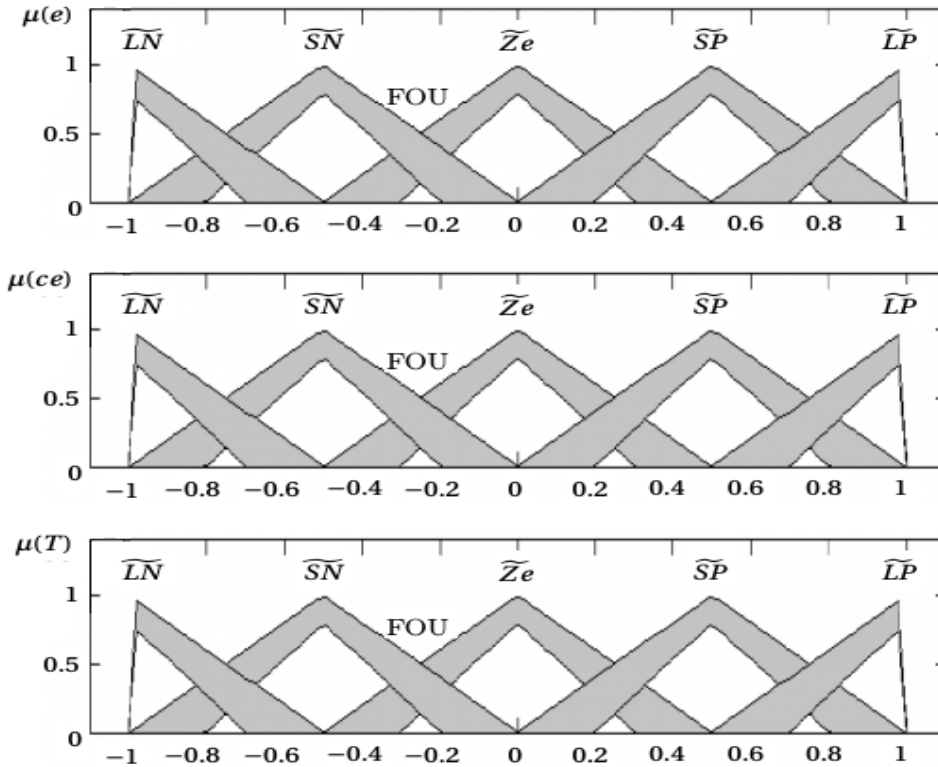


Figure 4.7: Fuzzy sets interval type-2 for each articulation and control signal

4.7 Result of simulation with two trajectory

We use same trajectory, a circle in space and LEAHY trajectory.

Result of simulation with circle trajectory

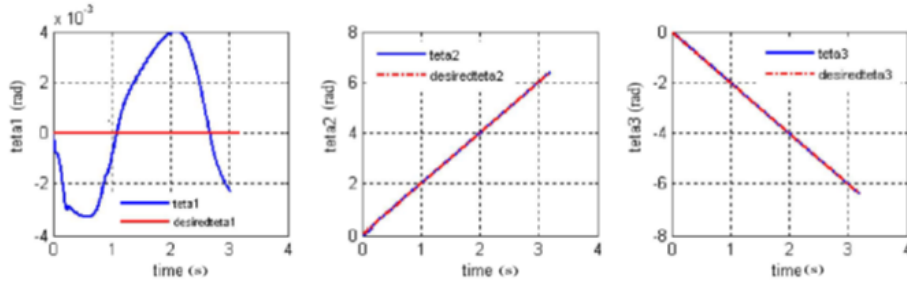


Figure 4.8: Position of joints 1,2,3 (rad) .

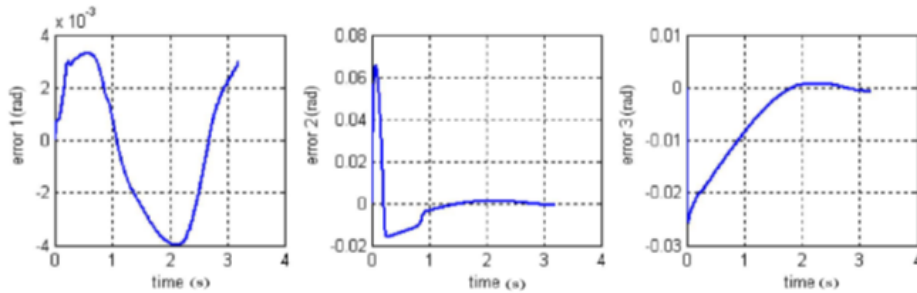


Figure 4.9: Position error of joints 1,2,3 (rad) .

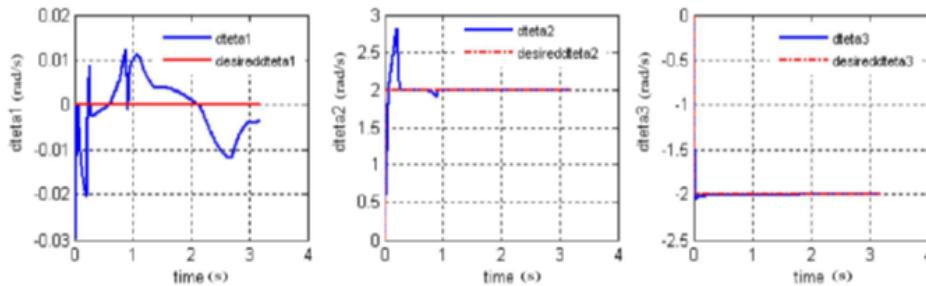


Figure 4.10: Velocity of joints 1,2,3 (rad/s)

In order to verify the robustness of the proposed controller and to help us to achieve a clear full comparison between the type-1 fuzzy controller and interval-type-2 fuzzy controller and to see the ability each one in handling uncertainties, two lands of noises are adopted: the first noise torque is $d = \sin(3t) + 1$ and the second noise torque is considered by taking 30 % of joint torque for each joint. [62]The simulation results are devoted to showing comparisons in the robot's responses when using both type-1 fuzzy controller and interval type-2 fuzzy controller. A set of three simulations in each one three cycloid trajectories are adopted by changing time with (1.5second, 4s, 8s). We start with type-1 fuzzy controller in ideal conditions without any kind of disturbances. The values of ISE of three joint angles for three

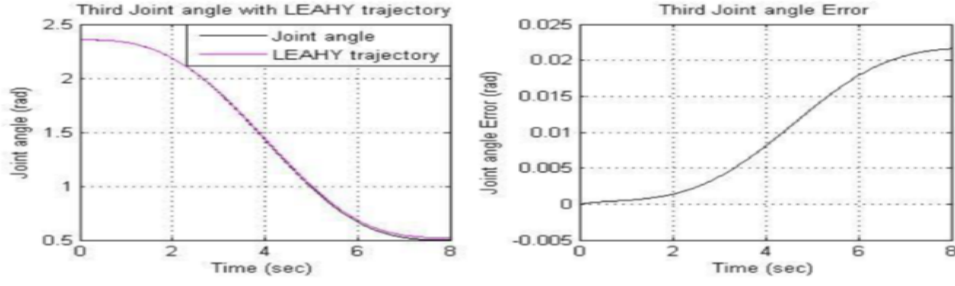


Figure 4.11: Third joint response to cycloid trajectory with noise and its tracking joint error

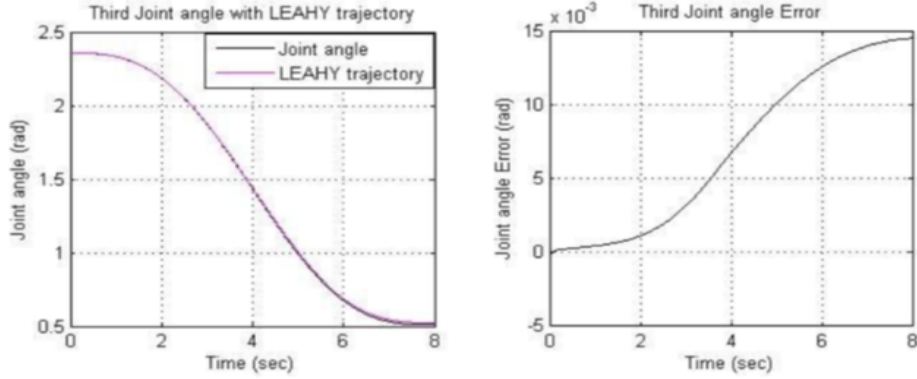


Figure 4.12: Third joint response with noise and its tracking joint error with modified type-reduction

trajectories are gathered in Table 4.3. Figure 4.11 shows the response obtained by using type-1 fuzzy controller with the presence of the noise torque taking $d = 30\%$ of the torque control added as noise to the main control torque to check the robustness of controller. Figure 4.12 shows the response of the interval type-2 fuzzy controller using modified type-reduction of Karnik-Mendel algorithm.

Comparison in terms of integral squared joint errors in ideal and disturbance cases ($d = \sin(3t) + 1$) is shown in Table 4.3. Table 4.4 represents comparison in terms of integral squared joint errors in ideal and disturbance $d = 30\%$ of joint signal control. The last column in Tables 4.3. and 4.4. is the factor (ISE_F) between the integral square joint error produced by using a type-1 fuzzy controller (ISE_T1) and integral square error produced by using interval type-2 fuzzy controller (ISE_IT2) in noisy case, calculated as (4.21):

$$ISE_F = ISE_T1 / ISE_IT2 \quad (4.21)$$

This factor gives an idea in how many energies of error in type-1 fuzzy set is bigger than energy error of interval type-2 fuzzy set case.

4.8 Results discussion

The membership functions of a type-1 fuzzy controller are type-1 fuzzy sets, see Figure 4.3, and their membership degrees are crisp values.

Table 4.3: ISE for different trajectory without and with disturbance $d = \sin(3t) + 1$

Second	Joint	ISE Type-1*		ISE Interval type-2*		ISE_F
		No d	With d	No d	with d	
1.5	J1	1.88	2.35	0.49	0.70	3.38
	J2	7.14	8.33	5.96	3.75	2.22
	J3	6.06	8.27	5.76	5.67	1.46
4	J1	0.23	0.54	0.24	0.15	3.49
	J2	32	36.9	18.71	16.84	2.19
	J3	19.7	30.2	17.48	24.91	1.21
8	J1	0.043	0.64	0.02	0.26	2.44
	J2	83.5	94	31.88	50.80	1.85
	J3	54	81	46.63	50.21	1.61

Note : * all values are multiplied by 10^{-5}

Table 4.4: ISE for different trajectory without and with disturbance in joint control $d = 30\%$ of each joint

Second	Joint	ISE Type-1*		ISE Interval type-2*		ISE_F
		No d	With d	No d	with d	
1.5s	J1	1.88	4.23	0.50	0.91	4.63
	J2	7.14	19.1	3.88	7.28	2.62
	J3	6.06	14.8	5.63	9.09	1.63
4s	J1	0.23	0.48	0.23	0.13	3.77
	J2	0.32	79.3	22.44	38.94	2.04
	J3	19.7	44.7	12.16	23.80	1.88
8s	J1	0.043	0.091	0.02	0.03	2.72
	J2	83.5	200	43.78	80.08	2.50
	J3	54	120	44.37	79.28	1.51

Note : * all values are multiplied by 10^{-5}

Interval type-2 fuzzy controller has FOU zone in which the membership degrees are an interval and not crisp values. Actually, this is the main part that improves the degree of dealing with uncertainties. In an interval-type-2 fuzzy controller, the outputs of the inference are interval-type-2 sets and a type-reducer is needed to convert them into a type-1 sets before defuzzification to generate a crisp output. This is the main structural difference between type-1 and interval-type-2 fuzzy systems. In Table 4.5, some differences between type-1 fuzzy controller and interval type-2 fuzzy controller are given.

When comparing results based on visual observation, the actual and desired joint angles are overlapping without tangible difference as illustrated in Figure 4.11. But in terms of ISE values, it is possible to measure the difference between the two fuzzy controllers. To perform this comparison, three simulations with different simulation tunes are realized, which give us three different trajectories for three robot joints with two disturbance torques to test robustness. All results are summed up in two tables; Table 4.3 for comparison between a type-1 fuzzy controller and modified interval type-2 fuzzy controller with disturbance $d = \sin(3t) + 1$ and Table 4.4 for comparison between type-1 fuzzy controller and modified interval type-2 fuzzy controller with disturbance $d = 30\%$ of control torque. It is very clear

that the average of ISE_F column around two times which means the tracking error angle for the type-1 fuzzy controller is twice bigger than error angle for modified interval-type-2 fuzzy controller.

Finally, when comparing all results in all figures, Table 4.3, and Table 4.4, it is very clear that interval-type-2 fuzzy controller is more performing than type-1 fuzzy controller in ideal or in noises cases.

Table 4.5: Some differences between type-1 and interval-type-2 fuzzy controller

	T1-FS	IT-FS
Grades of membership	Crisp	Fuzzy
footprint of uncertainty zone	Don't have	Have
Type-reducer	Don't need	Need
Computational	small	Little Large
Program	Easy	Little Hard

4.9 Conclusion

In this chapter, we have discussed in some detail the type-1 FLC and type-2 FLC then, we have applied them on PUMA560 3DOF. The results of simulations prove that Interval type-2 FLC has a low error and accepted control torques of the joints

General Conclusion

Fuzzy Type-2 logic has involved many researchers in recent years and has been applied in several areas such as video modeling, classification, diagnostics, artificial intelligence, transmission channels, adaptive filters, mobile robots, and other domains. In this thesis, a type-1 and Interval type-2 FLC are developed for the control of PUMA560 3DOF manipulators in the presence of dynamical modeling. To alleviate the naturally inherited high computational complexity of type-2 FLCs, interval membership functions are adopted. The controllers are also compared in similar operating conditions. Three simulations for three robot joints with and without noises are presented. Furthermore, a quantification of errors was achieved and documented in tables for the integral square error of trajectory tracking. It was shown that the lower errors were obtained using an interval-type-2 fuzzy controller where the amount of error is four times less than when using a type-1 fuzzy set to control a manipulator robot. The interval-type-2 fuzzy controller is much better because it can handle uncertainties and has more robustness than the type-1 fuzzy counterpart. Also, the proposed modification in type reduction of Karnik-Mendel algorithm by using harmonic mean performs better than the arithmetic mean used in the traditional Karnik-Mendel algorithm.

We are concluding that using type-2 FLC in real applications can be a better choice since the amount of uncertainty in real systems, mostly, is difficult to estimate. But when uncertainty in real system is negligible, we can consider it as an ideal system. In this case, it is a better choice to select a type-1 FLC since it works similar as type-2 FLC, and the size of type-1 FLC program is shorter than type-2 FLC, so it is faster and easier for programming and simulating.

Future work will be devoted to the study of ideal representations of the types-2 fuzzy lattices.

.1 APPENDIX

Matrix M is a symmetric 3×3 matrix :

$$M(q) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

where,

$$a_{11} = I_{m1} + I_1 + I_3 \cdot CC2 + I_7 \cdot SS23 + I_{10} \cdot SC23 + I_{11} \cdot SC2 + 2 \cdot [I_5 \cdot C2 \cdot S23]$$

$$a_{12} = I_4 \cdot S2 + I_8 \cdot C23 + I_9 \cdot C2 \quad a_{13} = I_8 \cdot C23 \quad a_{22} = I_{m2} + I_2 + I_6 + 2 \cdot [I_5 \cdot S3]$$

$$a_{23} = I_5 \cdot S3 + I_6$$

$$a_{33} = I_{m3} + I_6$$

$$a_{21} = a_{12}, a_{31} = a_{13} \text{ and } a_{32} = a_{23}$$

While matrix B is :

$$B(q) = \begin{bmatrix} b_{112} & b_{113} & 0 & 0 & 0 & b_{123} & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{223} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where,

$$b_{112} = 2 \cdot [-I_3 \cdot SC2 + I_5 \cdot C223 + I_7 \cdot SC23]$$

$$+ I_{10} \cdot (1 - 2 \cdot SS23) + I_{11} \cdot (1 - 2 \cdot SS2)$$

$$b_{113} = 2 \cdot [I_5 \cdot C2 \cdot C23 + I_7 \cdot SC23 -] + I_{10} \cdot (1 - 2 \cdot SS23)$$

$$b_{123} = 2 \cdot [-I_8 \cdot S23]$$

Matrix C is :

$$C(q) = \begin{bmatrix} 0 & c_{12} & c_{13} \\ c_{21} & 0 & c_{23} \\ c_{31} & c_{32} & 0 \end{bmatrix}$$

where,

$$c_{12} = I_4 \cdot C2 - I_8 \cdot S23 - I_9 \cdot S2$$

$$c_{13} = 0.5 \cdot b_{123} = -I_8 \cdot S23$$

$$c_{21} = -0.5 b_{112} = I_3 \cdot SC2 - I_5 \cdot C223 - I_7 \cdot SC23 - 0.5 I_{10} \cdot (1 - 2.5 S23) - 0.5 \cdot I_{11} \cdot (1 - 2 \cdot SS2)$$

$$c_{23} = 0.5 \cdot b_{223} = I_5 \cdot C3$$

$$c_{31} = -0.5 \cdot b_{113} = -I_5 \cdot C2 \cdot C23 - I_7 \cdot SC23 - 0.5 I_{10} \cdot (1 - 2 \cdot SS23)$$

$$c_{32} = -c_{23} = -I_5 \cdot C3$$

And matrix G is :

$$g(q) = \begin{bmatrix} 0 \\ g_2 \\ g_3 \end{bmatrix}$$

$$g_2 = g_1 \cdot C2 + g_2 \cdot S23 + g_3 \cdot S2$$

$$g_3 = g_2 \cdot S23$$

where,

$$Si = \sin(\theta i), Ci = \cos(\theta i), Cij = \cos(\theta i + \theta j), Sijk = \sin(\theta i + \theta j) + \theta k, \text{ with } i = 1, 2, 3$$

Table 6: Inertial constants ($kg.m^2$)

$I_1=1.43\pm 0.05$	$I_2 = 1.75 \pm 0.07$
$I_3=1.38\pm 0.05$	$I_4 = 0.69 \pm 0.02$
$I_5=0.372\pm 0.031$	$I_6 = 0.333 \pm 0.016$
$I_7=0.298\pm 0.029$	$I_8 = -0.134 \pm 0.014$
$I_9=0.0238\pm 0.012$	$I_{10} = -0.0213 \pm 0.0022$
$I_{m1} = 1.14 \pm 0.27$	$I_{m2} = 4.71 \pm 0.54$
$I_{m3} = 0.827 \pm 0.093$	

Table 7: Gravitational constants ($N.m$)

$g_1 = -37.2 \pm 0.5$	$g_2 = -8.44 \pm 0.20$
$g_3 = 1.02 \pm 0.50$	

Tables 4.5 and 6 contain the computed values for the constants appearing in the equations (1).

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ملخص

يركز البحث المقدم في أطروحة الدكتوراه هذه على مجالين: الدراسة النظرية على المجموعات الضبابية من النوع 1 والنوع 2 ، ثم تطبيقها للتحكم في نظام ديناميكي غير خطي وهو ذراع روبوت من الصنف بيما 560. لذلك ، قمنا بتطبيق كل من وحدات التحكم الضبابية: وحدات تحكم ضبابي من النوع 1 و المجال الضبابي من النوع 2 ، حيث تم اقتراح تعديل في خوارزمية كارنيك مندل. لحساب الناتج من النظام الضبابي نوع-2 ، هناك خطوة رئيسية تسمى تبسيط النوع ؛ تعتمد هذه العملية على خوارزمية كارنيك مندل ، والتي تستخدم المتوسط الحسابي لحساب مخرجات التحكم. في هذا العمل ، نقترح تغيير الوسط الحسابي بالوسط التوافقي. يُظهر التعديل المقترح في تبسيط نوع خوارزمية كارنيك مندل من أجل مجموعة غامضة من النوع 2 أفضل أداء. بالفعل، فإن مقدار الخطأ في حالة وحدة تحكم غامض تعديل نوع فاصل 2 أقل مرتين من تحكم غامض من النوع 1. الكلمات المفتاحية: المجموعات الضبابية نمط2، المجال الضبابي نمط2، التحكم الضبابي نمط1، التحكم بواسطة المجال الضبابي نمط2، الذراع الآلي بيوما 560 .

Abstract

The research presented in this doctoral thesis focuses on two areas: theoretical study on fuzzy sets type1 and type2, then their application for the control of a nonlinear dynamic system which is the manipulator robot PUMA560. For that, we implemented the both fuzzy controllers: type-1 and interval type-2 fuzzy controllers, in which a modification in Karnik-Mendel algorithm has been proposed. To calculate the output of interval type-2 fuzzy system there is a main step called type-reduced; this operation is based on Karnik-Mendel algorithm, which uses arithmetic mean to calculate the control output. In this work, we propose to change the arithmetic mean by harmonic one. The performances of modified interval type- 2 controller and type-1 fuzzy controller with and without noises are compared in terms of integral of squared error. The proposed modification in type reduction of Karnik-Mendel algorithm for interval type-2 fuzzy set shows best performance. Indeed, the amount of error in case of modified interval type-2 fuzzy controller is less two times than type-1 fuzzy controller.

Keywords: Type-2 fuzzy sets, Interval type-2 fuzzy sets, type-1 fuzzy controller, Interval type-2 fuzzy controller, puma 560 robot manipulator

Résumé

Les travaux de recherche présentés dans cette thèse de doctorat portent essentiellement sur deux axes : étude théorique sur les ensembles flous type1 et type2, puis leur application pour la commande d'un système dynamique non linéaire qui est le robot manipulateur PUMA560. Pour cela, nous avons implémenté les deux contrôleurs flous: les contrôleurs flous de type 1 et d'intervalle de type 2, dans lesquels une modification de l'algorithme de Karnik-Mendel a été proposée. Pour calculer la sortie du système à intervalle flous de type 2, il existe une étape principale appelée réduction de type2 cette opération est basée sur l'algorithme de Karnik-Mendel, qui utilise une moyenne arithmétique pour calculer la sortie de contrôle. Dans ce travail, nous proposons de changer la moyenne arithmétique par harmonique. Les performances du contrôleur d'intervalle modifié de type 2 et du contrôleur flou de type 1 et sans bruit sont comparées en termes d'intégrale d'erreur quadratique. La modification proposée dans la réduction de type de l'algorithme de Karnik-Mendel pour l'intervalle flou de type-2 montre les meilleures performances. En effet, la quantité d'erreur en cas d'intervalle modifié de contrôleur flou de type 2 est inférieure à deux fois à celle de contrôleur flou de type 1.

Les mots clés: les ensembles flous type-2, l'intervalle flou type-2, Contrôle fou type-1, Contrôle intervalle flou type-2, robot manipulateur Puma 560.

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Introduction

The fuzzy logic theory was established by L. Zadeh in 1965 [76]. This logic allows the representation and processing of inaccurate or approximate knowledge. The number of applications based on this theory has in recent years increased considerably [21] [24] [7] [26] [71].

This is due to the fact that fuzzy logic is usually expressed by linguistic rules of the form IF THEN; it is used to solve control decision problems or to describe the dynamic behavior of an unknown or poorly defined system.

The fuzzy logic type-1 has been extended to get a new one fuzzy logic called the fuzzy logic type-2. Mendel and his students [30] [33] [35] [50] [73] worked in this field and developed the type-2 fuzzy sets. Type-2 fuzzy logic is very effective when it is very difficult to determine exact membership functions for a fuzzy system, so this new logic allows us to incorporate uncertainties into the rules which will act positively on the output of the given system. In type-1 fuzzy systems, There are at least three sources which are as follows:

1. The meaning of words used in the rules may be uncertain, that is, a word can be polysemic.
2. Measurements activating a type-1 fuzzy system can be noisy, which introduces uncertainties.
3. The data used to adjust the parameters of a fuzzy type-1 system can be noisy as well.

All of these uncertainties will be found in fuzzy membership functions. Type-1 fuzzy sets are therefore unable to model such uncertainties because their fuzzy membership functions are defined in a precise mathematical way. On the other hand, type-2 fuzzy sets are able to model such uncertainties because their membership functions are themselves indefinite. The fuzzy membership functions type-1 are bi-dimensional, however, the type-2 fuzzy membership functions are tri-dimensional. The new (third) dimension of type-2 fuzzy sets provides more freedom to support the modeling of uncertainties. The need for industrial robot in tasks like cutting, in welding, and painting and those requiring more precision, oblige researchers to have an exact mathematical description and mastery of this kind of processes in order to solve related industrial problems [9]-[11]. In this context, the fuzzy logic theory, firstly introduced by Zadeh [78] is gaining more and more attention from both researchers and industrial communities. The main advantage of fuzzy system is that it does not need the knowledge of the dynamic model system. This characteristic is important when dealing with nonlinear systems. Moreover, the dynamic modeling of robot manipulator shows a dependence on their dynamic parameters, function of lifetime variations (friction factors affected-by the abuse of joints), and on their dynamic parameters that vary with the completed task [11]-[9]. These characteristics give advantage to fuzzy controller on other nonlinear methods also seeing their robustness towards noises affecting the plant [64]. The major aim of this thesis is to study from theoretical and applying the type-1 and type-2 fuzzy sets on a mathematical point of view. secondly, applying these two control structures types to the applying side consists in working onto two trajectories (circle and LEAHY). Then, a proposed type reduction of Karnik-Mendel algorithm used in interval type-2 fuzzy

sets (IT2-FS) to increase the control of three DOF PUMA560 manipulator. It should be noted that there are many generalizations of the notion of fuzzy set, for example the notion of a multi fuzzy set, intuitionistic fuzzy set (IFS), L- fuzzy set (LFS) and others. In most engineering applications, we chose the notion of interval type-2 fuzzy set because they can improve certain kinds of inference better than do fuzzy sets with increasing imprecision, uncertainty, and fuzziness in information and this makes it easy for us to get the desired results.

This thesis contains four chapters.

The first chapter presents the basic concepts of Type-1 fuzzy sets and classification of fuzzy sets which include normal fuzzy set, subnormal fuzzy set, convex fuzzy set and fuzzy relations well as α -cut or strong α -level set.

The second one gives the basic concepts of a type-2 fuzzy sets, representation type-2 membership function, upper and lower membership functions, Interval type- 2 fuzzy set, Operation of Types-2 fuzzy sets, Type-2 fuzzy relations and their compositions, finally cartesian product and α -Planes.

In chapter 3, we define the concept of fuzzy logic control, the fuzzy rule base, fuzzy inference, the fuzzification, defuzzification, Puma560 robot dynamic modeling. It introduces the three degrees of freedom robot from PUMA560 and its dynamic model and state of art of fuzzy logic controller of a non-linear dynamic system.

The last chapter provides simulation results and a comparative evaluation of the type-1 and type-2 fuzzy logic controllers.

Chapter 1

Generalities of Type-1 Fuzzy Sets

The first publications in fuzzy set theory by (Zadeh, 1965) and (Goguen, 1967, 1969) display the authors intention to generalize the classical notion of a set and a proposition (statement) to accommodate fuzziness.

Let X be a space of objects and x be a generic element of X . A classical set A , $A \subseteq X$, is defined by a collection of elements or objects $x \in X$, and that each x can either belong or doesn't do to the set A . By defining a "characteristic function" for each element $x \in X$, we can represent a classical set A by a set of order pairs $(x, 0)$ or $(x, 1)$, which respectively indicates $x \notin A$ or $x \in A$.

1.1 Basic concepts of Type-1 fuzzy sets

This section contains the basic definitions and properties of type-1 fuzzy sets and several operations of this notion. Unlike the aforementioned conventional set, a fuzzy set [76] expresses the degree to which an element belongs to a set. Hence the characteristic function of a fuzzy set is allowed to have values between 0 and 1, which denotes the degree of membership of an element in a given set.

Definition 1.1.1. [76] *Let X be the collection of objects generically denoted by x , then a fuzzy set \underline{A} in X is defined as a set of ordered pairs:*

$$\underline{A} = \left\{ \left(x, \mu_{\underline{A}}(x) \right) \mid x \in X \right\} \quad (1.1)$$

Where $\mu_{\underline{A}}(x)$ is called the membership function (or MF for short) for the fuzzy set A . The MF maps, each element of X in terms of the membership function is permitted to have any values between 0 and 1. If the values of the membership function $\mu_{\underline{A}}(x)$ is a restricted grade (or membership value) between 0 and 1.

Obviously, the definition of a fuzzy set is a simple extension of the definition of a classical set in which the characteristic function is either 0 or 1, then A is reduced to a classical set and $\mu_{\underline{A}}(x)$ is the characteristic function of A .

When the universe of speech U is a continuous set (Exp: real numbers), the fuzzy subset A is written as follows:

$$A = \int \mu_{\underline{A}}(x) / x \quad (1.2)$$

When U is discrete, A becomes as follows:

$$A = \sum \mu_{\underline{A}}(x) / x \quad (1.3)$$



Figure 1.1: Function of membership

[45] Use the Matlab commands to show the triangular membership function

```

1           Step 1: First enter the x value
2           >> x = (0:0.2:10);
3           Step 2: enter triangular membership function
4           >> y1 = trimf(x,[3 4 5 ]);
5           Step 3: plot the curve
6           >>Plot(x,y1)
7

```

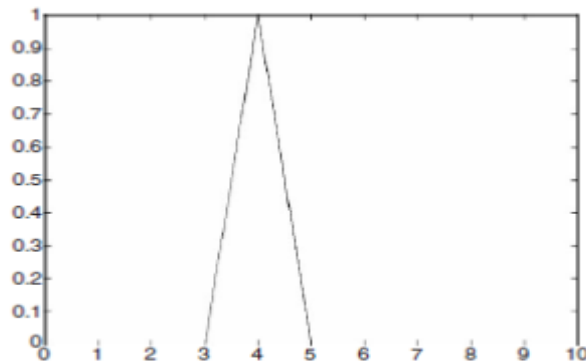


Figure 1.2: Triangular membership function

Fuzzy Set Operations [76]

Define three fuzzy sets \tilde{A} , \tilde{B} , and \tilde{C} on the universe X . For a given element x of the universe, the following function-theoretic operations for the set- union theoretical operations, intersection, and complement are defined for \tilde{A} , \tilde{B} , and \tilde{C} on X

Union

$$\mu_{\tilde{A} \cup \tilde{B}}(x) = \mu_{\tilde{A}}(x) \vee \mu_{\tilde{B}}(x) \quad (1.4)$$

Intersection

$$\mu_{\tilde{A} \cap \tilde{B}}(x) = \mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(x) \quad (1.5)$$

Complement

$$\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x) \quad (1.6)$$

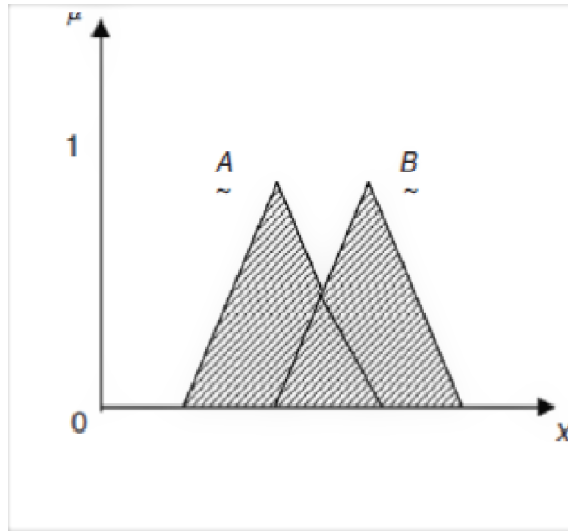


Figure 1.3: fuzzy sets union

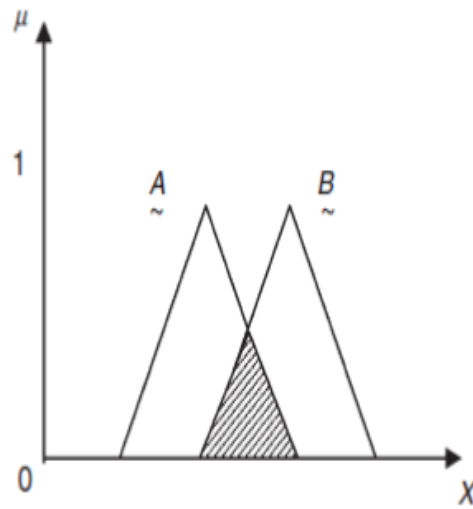


Figure 1.4: fuzzy sets intersection

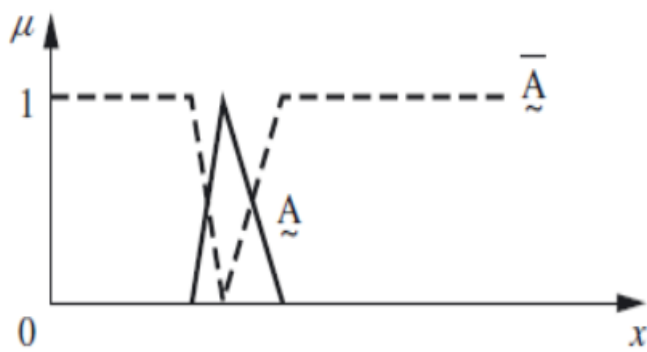


Figure 1.5: fuzzy sets complement \bar{A}

Fuzzy sets properties

De Morgan's laws: De Morgan's principles for classical sets also hold for fuzzy sets, as denoted by the following expressions :

$$\overline{\underline{A} \cap \underline{B}} = \overline{\underline{A}} \cup \overline{\underline{B}} \quad (1.7)$$

$$\overline{\underline{A} \cup \underline{B}} = \overline{\underline{A}} \cap \overline{\underline{B}} \quad (1.8)$$

$$\underline{A} \cup \overline{\underline{A}} \neq X, \underline{A} \cap \overline{\underline{A}} \neq \emptyset$$

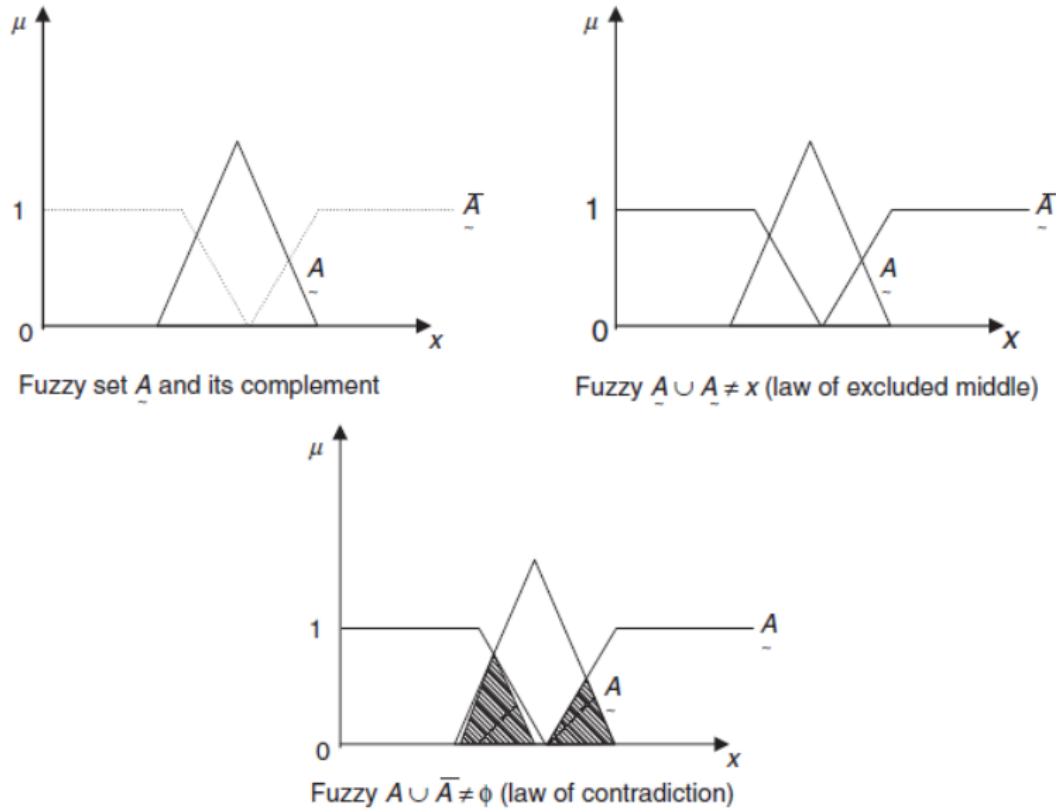


Figure 1.6: Law of excluded middle and law of contradiction

Commutativity

$$\underline{A} \cup \underline{B} = \underline{B} \cup \underline{A}$$

$$\underline{A} \cap \underline{B} = \underline{B} \cap \underline{A}$$

Associativity

$$\underline{A} \cup (\underline{B} \cup \underline{C}) = (\underline{A} \cup \underline{B}) \cup \underline{C}$$

$$\underline{A} \cap (\underline{B} \cap \underline{C}) = (\underline{A} \cap \underline{B}) \cap \underline{C}$$

Distributivity

$$\underline{A} \cup (\underline{B} \cap \underline{C}) = (\underline{A} \cup \underline{B}) \cap (\underline{A} \cup \underline{C})$$

$$\underline{A} \cap (\underline{B} \cup \underline{C}) = (\underline{A} \cap \underline{B}) \cup (\underline{A} \cap \underline{C})$$

Idempotency

$$\begin{aligned} \underline{\underline{A}} \cup \underline{\underline{A}} &= \underline{\underline{A}} \\ \underline{\underline{A}} \cap \underline{\underline{A}} &= \underline{\underline{A}} \end{aligned}$$

Identity

$$\begin{aligned} \underline{\underline{A}} \cup \phi &= \underline{\underline{A}} & \text{and} & & \underline{\underline{A}} \cap X &= \underline{\underline{A}} \\ \underline{\underline{A}} \cap \phi &= \phi & \text{and} & & \underline{\underline{A}} \cup X &= X \end{aligned}$$

Transitivity

$$\text{If } \underline{\underline{A}} \subset \underline{\underline{B}} \subset \underline{\underline{C}} \text{ then } \underline{\underline{A}} \subset \underline{\underline{C}}$$

Involution

$$\underline{\underline{\bar{\bar{A}}}} = \underline{\underline{A}}$$

Example 1.1.1. we have two discrete fuzzy sets, where we find Complement, Union, Intersection, Difference, and De Morgan's law.

$$\underline{\underline{A}} = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\} \quad \text{and} \quad \underline{\underline{B}} = \left\{ \frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\}$$

Intersection

$$\underline{\underline{A}} \cap \underline{\underline{B}} = \left\{ \frac{0.5}{2} + \frac{0.5}{3} + \frac{0.2}{4} + \frac{0.2}{5} \right\}$$

Union

$$\underline{\underline{A}} \cup \underline{\underline{B}} = \left\{ \frac{1}{2} + \frac{0.7}{3} + \frac{0.3}{4} + \frac{0.4}{5} \right\}$$

Complement

$$\begin{aligned} \bar{\underline{\underline{A}}} &= \left\{ \frac{0}{2} + \frac{0.5}{3} + \frac{0.7}{4} + \frac{0.8}{5} \right\} \\ \bar{\underline{\underline{B}}} &= \left\{ \frac{0.5}{2} + \frac{0.3}{3} + \frac{0.8}{4} + \frac{0.6}{5} \right\} \end{aligned}$$

Difference

$$\begin{aligned} \underline{\underline{A}} | \underline{\underline{B}} &= \underline{\underline{A}} \cap \bar{\underline{\underline{B}}} = \left\{ \frac{0.5}{2} + \frac{0.3}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\} \\ \underline{\underline{B}} | \underline{\underline{A}} &= \underline{\underline{B}} \cap \bar{\underline{\underline{A}}} = \left\{ \frac{0}{2} + \frac{0.5}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\} \end{aligned}$$

De morgan's principles

$$\begin{aligned} \overline{\underline{\underline{A}} \cup \underline{\underline{B}}} &= \bar{\underline{\underline{A}}} \cap \bar{\underline{\underline{B}}} = \left\{ \frac{0}{2} + \frac{0.3}{3} + \frac{0.7}{4} + \frac{0.6}{5} \right\} \\ \overline{\underline{\underline{A}} \cap \underline{\underline{B}}} &= \bar{\underline{\underline{A}}} \cup \bar{\underline{\underline{B}}} = \left\{ \frac{0.5}{2} + \frac{0.5}{3} + \frac{0.8}{4} + \frac{0.8}{5} \right\} \end{aligned}$$

Example 1.1.2. Consider the following fuzzy sets

$$\begin{aligned} A &= \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\} \\ B &= \left\{ \frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\} \end{aligned}$$

Calculate $A \cup B$, $A \cap B$, \bar{A} , \bar{B} by a Matlab program.

Solution: The Matlab program for the union, intersection, and complement is Program [45]

```

8      %enter the two matrix
9      u=input('enter the first matrix ');
10     v=input('enter the second matrix ');
11
12     option=input('enter the option');
13     %option 1 Union
14     %option 2 intersection
15     %option 3 complement
16     if (option==1)
17         end
18     if (option==2)
19         p=min(u,v)
20     end
21     if (option==3)
22         option1=input('enter wherthe to find complement for first matrix
23         or second matrix');
24         if (option1==1)
25             [m,n]=size(u);
26             q=ones(m)-u;
27         else
28             q=ones(m)-v;
29         end
30     end
31     output
32     (1) To find union of A and B
33     enter the first matrix [1 0.5 0.2 0.3]
34     enter the second matrix [0.5 0.7 0.2 0.4]
35     enter the option1
36     w =
37     1.0000 0.7000 0.2000 0.4000
38
39     (2) To find intersection of A and B is
40     enter the first matrix [1 0.5 0.2 0.3]
41     enter the second matrix [0.5 0.7 0.2 0.4]
42     enter the option2
43     p=
44     0.5000 0.5000 0.2000 0.3000
45     (3) To find complement of A
46     enter the first matrix [1 0.5 0.2 0.3]
47     enter the second matrix [0.5 .7 .2 .4]
48     enter the option3
49     enter the whether to find complement for first matrix or second matrix 2
50     q =
51     0.5000 0.3000 0.8000 0.6000

```

1.2 Characteristics of Type-1 fuzzy sets

Definition 1.2.1. (Support) Let A be a fuzzy set on a set X . The support of A is the crisp subset on X given by:

$$Supp(A) = \{x | \mu_A(x) > 0\} \quad (1.9)$$

Definition 1.2.2. (Kernel) Let A be a fuzzy set on a set X . The kernel or the core of A is the crisp subset on X given by:

$$Ker(A) = \{x | \mu_A(x) = 1\} \quad (1.10)$$

Definition 1.2.3. (Normality) A fuzzy set A is "normal" if its kernel is not empty. In other words, we can always find a point $x \in X$ such that $\mu_A(x) = 1$.

Definition 1.2.4. (Crossover points) A "crossover point" of a fuzzy set A is a point $x \in X$ at which $\mu_A(x) = 0.5$:

$$Cross(A) = \{x | \mu_A(x) = 0.5\}$$

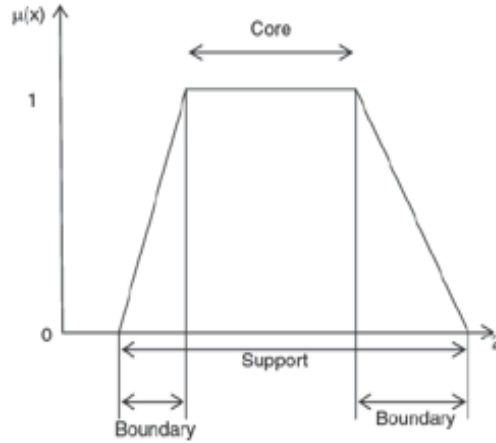


Figure 1.7: kernel and support of fuzzy set

Definition 1.2.5. (Height) Let A be a fuzzy set on a set X . The height of A is the highest value taken by its membership function given by:

$$H(A) = Sup_{x \in X} \mu_A(x)$$

Example 1.2.1. If $A = \left\{ \frac{0.2}{15} + \frac{1}{25} + \frac{0.8}{35} + \frac{1}{45} + \frac{0}{55} \right\}$

$$Ker(A) = \{25, 45\}$$

$$Cross(A) = \emptyset$$

$$Supp(A) = \{15, 25, 35, 45\}.$$

Definition 1.2.6. [77] λ -cut, strong λ -cut The " λ -cut" or " λ -level set" of a fuzzy set A is a crisp set defined by:

$$A_\lambda = \{x | \mu_A(x) \geq \lambda\} \quad (1.11)$$

Strong λ -cut or strong λ -level set are defined similarly:

$$A^\lambda = \{x | \mu_A(x) > \lambda\} \quad (1.12)$$

Using the notation for a level set, we can express the support and core of a fuzzy set A as $Supp(A) = A^\circ$ and $Ker(A) = A_1$

Example 1.2.2. $A = \left\{ \frac{1}{a} + \frac{0.9}{b} + \frac{0.6}{c} + \frac{0.3}{d} + \frac{0.01}{e} + \frac{0}{f} \right\}$

$$A_1 = \{a\}, \quad A_{0.9} = \{a, b\}$$

$$A_{0.6} = \{a, b, c\}, \quad A_{0.3} = \{a, b, c, d\}$$

$$A_{0+} = \{a, b, c, d, e\}, \quad A_0 = X$$

$\lambda = 1, 0.9, 0.6, 0.3, 0+, \text{ and } 0$

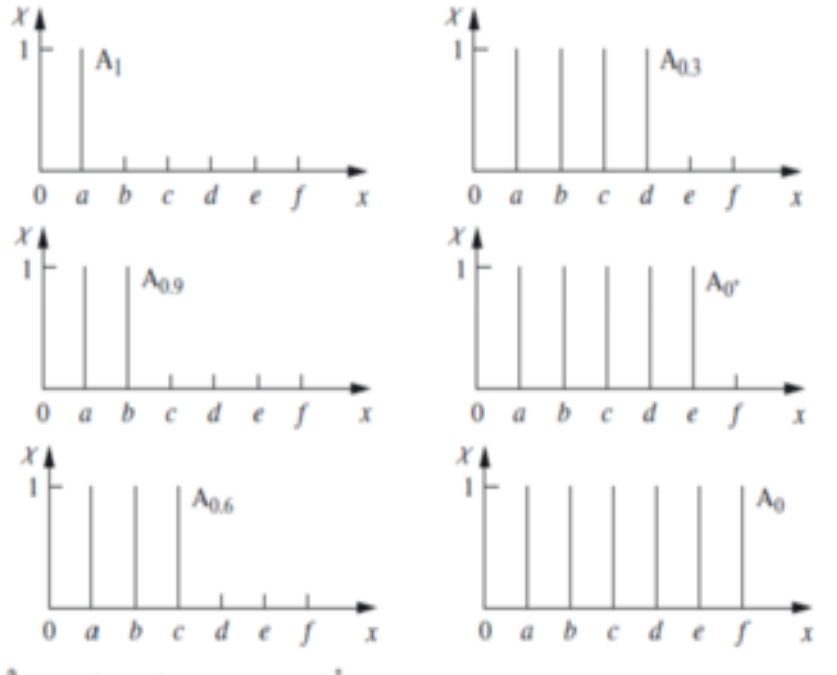


Figure 1.8: λ -cut"

λ -cut sets properties :

1. $(\tilde{A} \cup \tilde{B})_\lambda = \tilde{A}_\lambda \cup \tilde{B}_\lambda$
2. $(\tilde{A} \cap \tilde{B})_\lambda = \tilde{A}_\lambda \cap \tilde{B}_\lambda$
3. $(\overline{\tilde{A}})_\lambda \neq \overline{\tilde{A}_\lambda}$
4. For any $\lambda \leq \alpha$, where $0 \leq \alpha \leq 1$, it is true that $A_\alpha \subseteq A_\lambda$, where $A_0 = X$

Example 1.2.3. Two fuzzy sets \tilde{P} and \tilde{Q} are defined on x as follows :

$\mu(x)$	x_1	x_2	x_3	x_4	x_5
\tilde{P}	0.1	0.2	0.7	0.5	0.4
\tilde{Q}	0.9	0.6	0.3	0.2	0.8

- (a) $(\overline{\tilde{P}})_{0.2}$ (b) $(\tilde{Q})_{0.3}$ (c) $(\tilde{P} \cup \tilde{Q})_{0.5}$ (d) $(\tilde{P} \cap \tilde{Q})_{0.4}$ (e) $(\tilde{Q} \cup \overline{\tilde{P}})_{0.8}$

$$(f) \left(\underset{\sim}{P} \cup \underset{\sim}{\overline{P}} \right)_{0.2}$$

Solution :

Given Solution

$$\underset{\sim}{P} = \left\{ \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.7}{x_3} + \frac{0.5}{x_4} + \frac{0.4}{x_5} \right\}$$

$$\underset{\sim}{Q} = \left\{ \frac{0.9}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3} + \frac{0.2}{x_4} + \frac{0.8}{x_5} \right\}$$

Finding

$$\underset{\sim}{\overline{P}} = \left\{ \frac{0.9}{x_1} + \frac{0.8}{x_2} + \frac{0.3}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} \right\}$$

$$\underset{\sim}{\overline{Q}} = \left\{ \frac{0.1}{x_1} + \frac{0.4}{x_2} + \frac{0.7}{x_3} + \frac{0.8}{x_4} + \frac{0.2}{x_5} \right\}$$

$$(a) \left(\underset{\sim}{\overline{P}} \right)_{0.2} = \left\{ \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} \right\}$$

$$(b) \left(\underset{\sim}{\overline{Q}} \right)_{0.3} = \left\{ \frac{0}{x_1} + \frac{1}{x_2} + \frac{1}{x_2} + \frac{1}{x_4} + \frac{0}{x_5} \right\}$$

$$(c) \left(\underset{\sim}{P} \cup \underset{\sim}{Q} \right) = \left\{ \frac{0.9}{x_1} + \frac{0.6}{x_2} + \frac{0.7}{x_3} + \frac{0.5}{x_4} + \frac{0.8}{x_5} \right\}$$

$$\left(\underset{\sim}{P} \cup \underset{\sim}{Q} \right)_{0.4} = \left\{ \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} \right\}$$

$$(d) \left(\underset{\sim}{P} \cup \underset{\sim}{\overline{P}} \right) = \left\{ \frac{0.9}{x_1} + \frac{0.8}{x_2} + \frac{0.7}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} \right\}$$

$$\left(\underset{\sim}{P} \cup \underset{\sim}{\overline{P}} \right)_{0.8} = \left\{ \frac{1}{x_1} + \frac{1}{x_2} + \frac{0}{x_3} + \frac{0}{x_4} + \frac{0}{x_5} \right\}$$

$$(e) \left(\underset{\sim}{P} \cap \underset{\sim}{Q} \right) = \left\{ \frac{0.9}{x_1} + \frac{0.8}{x_2} + \frac{0.7}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} \right\}$$

$$\left(\underset{\sim}{P} \cap \underset{\sim}{Q} \right)_{0.4} = \left\{ \frac{0}{x_1} + \frac{0}{x_2} + \frac{0}{x_3} + \frac{0}{x_4} + \frac{1}{x_5} \right\}$$

$$(f) \left(\underset{\sim}{P} \cap \underset{\sim}{\overline{P}} \right) = \left\{ \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.3}{x_2} + \frac{0.5}{x_4} + \frac{0.4}{x_5} \right\}$$

$$\left(\underset{\sim}{P} \cap \underset{\sim}{\overline{P}} \right)_{0.8} = \left\{ \frac{0}{x_1} + \frac{1}{x_2} + \frac{1}{x_2} + \frac{1}{x_4} + \frac{1}{x_5} \right\}$$

Definition 1.2.7. For a finite fuzzy set $\underset{\sim}{A}$, the cardinality $|\underset{\sim}{A}|$ is defined as follows:

$$|\underset{\sim}{A}| = \sum_{x \in X} \mu_{\underset{\sim}{A}}(x) \text{ and } \|\underset{\sim}{A}\| = \frac{|\underset{\sim}{A}|}{|X|}$$

is Obviously called the relative cardinality of $\underset{\sim}{A}$. The relative cardinality of a fuzzy set depends on the cardinality of the universe. So you have to choose the same universe if you want to compare fuzzy sets by their relative cardinality.

Example 1.2.4. Consider a fuzzy subset of the set X , $X = a, b, c, d, e$ referred to

$$\underset{\sim}{A} = \left\{ \frac{0.6}{a}, \frac{0.9}{b}, \frac{0.1}{c}, \frac{0.3}{d}, \frac{0.2}{e} \right\}$$

$$|\underset{\sim}{A}| = \sum_{x \in X} \mu_{\underset{\sim}{A}}(x) = 2.1$$

Definition 1.2.8. (Fuzzy singleton) A fuzzy set whose support is a single point in X with $\mu_{\underset{\sim}{A}}(X) = 1$ is called a "fuzzy singleton".

1.3 Classification of Fuzzy sets

Fuzzy sets are classified according to their membership to:

Normal Fuzzy set

A fuzzy set A of X is called normal if there exists at least one element x in X such that $\mu_A(x) = 1$

Subnormal Fuzzy set

If all membership of the fuzzy set is affiliated fuzzy less than it is Convex Fuzzy set.

The latter is described by a membership function the membership of which values are strictly monotonically increasing or are strictly monotonically increasing then strictly monotonically decreasing with increasing values for elements in the universe. In other words, if for any elements x, y and z in a fuzzy set \tilde{A} , the relation $x < y < z$ implies that $\mu_{\tilde{A}}(y) \geq \min[\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(z)]$, then \tilde{A} is said to be a convex fuzzy set (Ross, 1995). Figure 1.9 shows a typical convex fuzzy set and a typical nonconvex one. Here, It is important to note that definition of convexity is different from other definitions of the same term in mathematics. In some areas of mathematics, the convexity of shape has to do with whether a straight line through any part of the shape goes outside the boundaries of that shape, This definition of convexity is not used here ; Figure succinctly summarize our definition or convexity.

Intersection of two convex sets

A special property of two convex fuzzy sets, say \tilde{A} and \tilde{B} , is that the intersection of these two convex fuzzy sets is also a convex fuzzy set, as shown in Figure 1.10. That is, for \tilde{A} and \tilde{B} . which are both convex, $\tilde{A} \cap \tilde{B}$ is also convex .

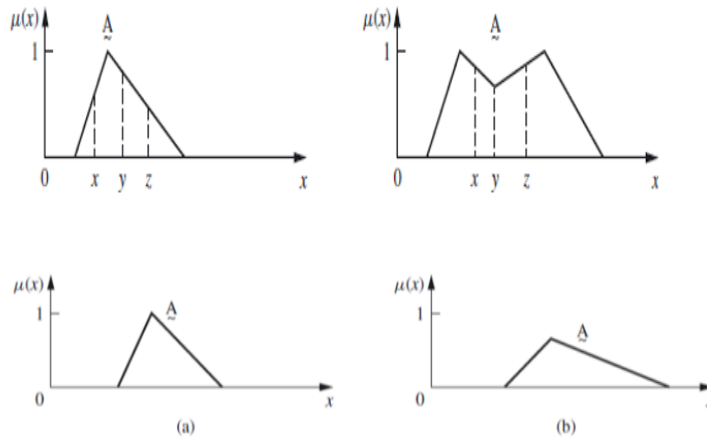


Figure 1.9: Fuzzy sets are that normal (a) and subnormal (b)

Definition 1.3.1. [76] An Interval-valued fuzzy set (IVFS), A , is characterized by a membership function σ_A ,

$$\sigma_A : X \rightarrow I = \{[a, b]; 0 \leq a \leq b \leq 1\}$$

where $\sigma_A(x)$ is the degree of membership of an element $x \in X$ in the set A . The set of all interval-valued fuzzy sets on X is denoted by $IVFS(X)$.

1.4 L-fuzzy sets type-1

In the previous sections, we have discussed the type -1 fuzzy sets in the unit real interval scale.

While in this section, we discuss these concepts on a lattice as a more general structure of truth values

Goguen generalized fuzzy sets to L -fuzzy sets in [20]. An L -fuzzy set φ on U is a mapping $\varphi: U \rightarrow L$, where L is a 'transitive partially ordered set'.

Definition 1.4.1. [32] Consider a bounded lattice $(L, \leq, 0_L, 1_L)$ and a universe X Goguen [20] introduced L -fuzzy sets as objects A characterized by some membership function $\mu_A: X \rightarrow L$. His approach has generalized the Cantorian (or crisp) sets (in which case $L = \{0, 1\}$ and μ_A is just the characteristic function $1_A: X \rightarrow \{0, 1\}$ of the Cantorian subset A of X), as well as the fuzzy sets proposed two years earlier by Zadeh [76] (here $L = [0, 1]$).

In the case of L -fuzzy sets, set relations and operations such as inclusion, union and intersection are defined pointwise by means of the ordering, supremum and infimum on L respectively. If the lattice L possesses a complement described by some unary operation $c: L \rightarrow L$, then the complement A^c

[20] We use the notation XY for the set-theoretic product of X and Y ; that is, $XY = \{(x, y) | x \in X, y \in Y\}$. $X + Y$ will denote the disjoint union of X and Y , and X^Y the set of all functions from Y to X . It is straightforward to verify the following set-theoretic isomorphisms.

Proposition 1.4.1. [20] $X + Y \cong Y + X$; $X + (Y + Z) \cong (X + Y) + Z$; $XY \cong YX$ $X(YZ) \cong (XY)Z$; $X(Y + Z) \cong XY + XZ$; $(X + Y)Z \cong XZ + YZ$ $X^{Y+Z} \cong X^Y X^Z$; $(XY)^Z \cong X^Z Y^Z$; $(X^Y)^Z \cong X^{YZ}$ There is no associative law for exponentiation, that is, $X^{(Y^Z)} \not\cong (X^Y)^Z$.

Definition 1.4.2. [20] An L -fuzzy set A on a set X is a function $A: X \rightarrow L$ Thus fuzzy sets are to be considered equal if and only if they are equal as functions. If the elements of X have a name (e.g., "characters"), A will be called an L -fuzzy set of such elements (e.g., "A is an L -fuzzy set of characters").

Definition 1.4.3. [56] For any L -fuzzy subsets A and B of X , define $A \subseteq B$ if and only if $A(x) \leq B(x)$ for all $x \in X$. Clearly \leq is a partial order on $FS_L(X)$. Also, for any crisp subsets S and T of X , we have $\chi_S \leq \chi_T \Leftrightarrow S \subseteq T$ The following is an easy verification.

Definition 1.4.4. [56] For an L -fuzzy set A and $\alpha \in L$, the α -cut A_α of A is defined as $A_\alpha = \{x | x \in X \text{ and } \alpha \leq A(x)\}$.

In the sequel, we assume that (L, \vee, \wedge) is a complete lattice with the least element 0 . The α -cuts of L -fuzzy sets have the following properties:

- (1) $\forall A \in F_L(X), A_0 = X$
- (2) $\forall \alpha \in L, \forall A \in F_L(X), A_\alpha = \bigcap_{\lambda \leq \alpha} A_\lambda$
- (3) $\forall \alpha_i \in L (i \in I), \forall A \in F_L(X), A_\alpha = \bigcap_{i \in I} A_{\alpha_i}$, where $\alpha = \bigvee_{i \in I} \alpha_i$

Let I be an arbitrary index set and $A_i \in F_L(X) (i \in I)$. The union $\bigcup_{i \in I} A_i$ and the intersection $\bigcap_{i \in I} A_i$ are respectively defined as follows: $\forall x \in X, (\bigcup_{i \in I} A_i)(x) = \bigvee_{i \in I} A_i(x), \forall x \in X, (\bigcap_{i \in I} A_i)(x) = \bigwedge_{i \in I} A_i(x)$

Proposition 1.4.2. [20] L^X can be given whatever operations L has, and these operations in L^X will obey any law valid in L which extends point by point. For example, if a binary operation $*$ is defined on L , it induces an operation $*$ on L^X by pointwise extension: $(A * B)(x) = A(x) * B(x)$, for $A, B \in L^X$ and $x \in X$. If some law, such as associative, commutative, or idempotent holds for $*$ on L , it also extends to L^X .

Definition 1.4.5. [20] A multiplicative lattice is a lattice M closed under an operation $*$ which satisfies the “distributive” laws, $\forall a, b \in M \ a * (b \vee c) = (a * b) \vee (a * c)$ and $(a \vee b) * c = (a * c) \vee (b * c)$. Identities involving $*$ ’s in multiplicative lattices will not remain identities when \wedge and \vee , and \geq and \leq are interchanged; thus the Principle of Duality is not valid. But there is a principle of left-right symmetry, as exhibited in the two “distributive” laws above.

Proposition 1.4.3. [20] Let a, b, c be elements of a multiplicative lattice M . Then:

- (1) $a \leq b \Rightarrow c * a \leq c * b$ and $a * c \Rightarrow b * c$;
- (2) $(a \wedge b) * (a \vee b) \leq (b * a) \wedge (a * b)$.

Proof. (1) $a \leq b \Rightarrow b * c = (a \vee b) * c = (a * c) \vee (b * c) \Rightarrow a * b \leq b * c$;

- (2) $(a \wedge b) * (a \vee b) = [(a \wedge b) * a] \vee [(a \wedge b) * b] \leq (b * a) \vee (a * b)$.

□

1.4.1 T–norm and t–conorm in lattices

Preliminaries

[47] A bounded lattice is a lattice (L, \leq) which has the top and bottom elements, which are written as: \top and \perp respectively, that is, there exist two elements $\top, \perp \in L$ such that $\perp \leq x \leq \top$, for all $x \in L$

Definition 1.4.6. [47] Given a bounded lattice (L, \leq, \perp, \top) , and $a, b \in L$, a subinterval $[a, b]$ of L , is a sublattice of L defined as

$$[a, b] = \{x \in L \mid a \leq x \leq b\}$$

The triangular norms on $[0, 1]$ are useful operators within the framework of fuzzy sets and fuzzy logic [[22], [38]]. since interest in lattices has been increasing in recent years, an adequate generalization of these operators should be given. A straightforward definition is presented in [63].

Definition 1.4.7. [47] Let (L_1, \leq, \perp, \top) be a bounded lattice. A binary operation $T: L \times L \rightarrow L$ is a triangular norm on L if the following conditions are fulfilled, for all $x, y, z \in L$:

1. $T(x, y) = T(y, x)$
2. If $x \leq y$, then $T(x, z) \leq T(y, z)$
3. $T(x, \top) = x$
4. $T(x, T(y, z)) = T(T(x, y), z)$

Example 1.4.1. [47] The following operators are examples of 1–norms in a bounded lattice. In any bounded lattice, we can define the meet (or infimum) t–norm. $T_M^L: L \times L \rightarrow L$, $T_M^L(x, y) = \inf\{x, y\}$ (as (L, \leq) is a lattice, there exists the infimum of two elements in L). Another interesting t–norm is the drastic product: $T_D^L: L \times L \rightarrow L$, defined as:

$$T_D^L(x, y) = \begin{cases} \inf\{x, y\} & \text{if } T \in \{x, y\} \\ \perp & \text{otherwise} \end{cases}$$

In a similar way that in $[0, 1]$, we see that T_D^L and T_M^L , are the smallest and greatest t–norms on a lattice.

T -distributivity

Definition 1.4.8. [59] Let $(L, \leq, 0, 1)$ be a bounded lattice and T_1 and T_2 every $x, y, z \in L$ such that at least one of the elements y, z is not

$$T_1(x, T_2(y, z)) = T_2(T_1(x, y), T_1(x, z))$$

is satisfied, then T_1 is called T_2 -distributive or we say that T_1 is

Example 1.4.2. [59] Let $(L = \{0, a, b, c, 1\}, \leq, 0, 1)$ be a bounded lattice w played in Figure 1 The functions T_1 and T_2 on the lattice L defined by

$$T_1(x, y) = \begin{cases} 0, & \text{if } x = a, y = a \\ b, & \text{if } x = c, y = c \\ x \wedge y, & \text{otherwise} \end{cases}$$

and

$$T_2(x, y) = \begin{cases} b, & \text{if } x = c, y = c \\ x \wedge y, & \text{otherwise} \end{cases}$$

are obviously t -norms on L such that T_1 is T_2 -distributive.

Proposition 1.4.4. [59] Let $(L, \leq, 0, 1)$ be a bounded lattice and T_1 and T_2 be two t -norms on L . If T_1 is T_2 -distributive, then T_1 is weaker than T_2 . Proof since all t -norms coincide on the boundary of L^2 , it is sufficient to show that $T_1 \leq T_2$ for all $x, y, z \in L \setminus \{0, 1\}$. By the T_2 -distributivity of T_1 , it is obtained that

$$T_1(x, y) = T_1(T_2(x, 1), y) = T_2(T_1(x, y), T_1(1, y)) = T_2(T_1(x, y), y) \leq T_2(x, y)$$

Thus, $T_1 \leq T_2$, i.e., T_1 is weaker than T_2

Remark 1.4.1. [59] The converse of Proposition 1.4.2. need not be true. Namely, for any two t -norms T_1 and T_2 , even if T_1 is weaker than T_2 , T_1 may not be T_2 -distributive. Now, let us investigate the following example.

Example 1.4.3. Consider the product t -norme $(T_P)(a, b) = a \cdot b$ and the Łukasiewicz t -norm $T_L(a, b) = \max(0, a + b - 1)$. It is clear that $T_L < T_P$. since

$$T_L\left(\frac{4}{5}, T_P\left(\frac{7}{9}, \frac{1}{2}\right)\right) = T_L\left(\frac{4}{5}, \frac{7}{18}\right) = \frac{59}{90}$$

and

$$T_P\left(T_L\left(\frac{4}{5}, \frac{7}{9}\right), T_L\left(\frac{4}{5}, \frac{1}{2}\right)\right) = T_P\left(\frac{26}{45}, \frac{3}{10}\right) = \frac{78}{450} = \frac{13}{75}$$

T_L is not T_P -distributive.

Corollary 1.4.1. [59] Let L be a bounded lattice and T_1 and T_2 be any two t -norms on L . If both T_1 is T_2 -distributive and T_2 is T_1 -distributive, then $T_1 = T_2$

Proposition 1.4.5. [59] Let L be a bounded chain and T' be a t -norm on L . For every t -norm T , T is T' -distributive if and only if $T' = T_\wedge$.

Proof. \Rightarrow Let T be an arbitrary t -norm on L such that T' -distributive. By Proposition 1, it is obvious that $T \leq T'$ for any t -norm T . Thus, $T' = T_\wedge$

\Leftarrow : since L is a chain, for any $y, z \in L$, either $y \leq z$ or $z \leq y$. Suppose that $y \leq z$. Using the monotonicity of the t -norm T , we obtained that for any $x \in L$, $T(x, y) \leq T(x, z)$. Then

$$T(x, y) = T(x, y) \wedge T(x, z)$$

holds. Thus, for any $x, y, z \in L$

$$\begin{aligned} T(x, T_\wedge(y, z)) &= T(x, y) \\ &= T(x, y) \wedge T(x, z) \\ &= T_\wedge(T(x, y), T(x, z)) \end{aligned}$$

is satisfied, which shows that any t -norm T is T_\wedge -distributive. Reference document-1.pdf \square

Definition 1.4.9. [28] An operation $T: L^2 \rightarrow L$ (resp. $S: L^2 \rightarrow L$) is called a t -norm (resp. t -conorm) if it is commutative, associative, increasing with respect to both variables and has a neutral element $e = 1_L$ (resp. $e = 0_L$).

A pseudo- t -norm T (resp. pseudo- t -conorm S) [[13].[17]] on [0.1] is a t -norm (resp. t -conorm) [38] if it is commutative. We can extend the definition of pseudo- t -norms on bounded lattices.

[16]

Example 1.4.4. [16] Let $(L_1 \leq 0_L, 1_L)$ be a bounded lattice, a pseudo- t -norm $T_a^l: L^2 \rightarrow L$ and a pseudo- t -conorm $S_b^l: L^2 \rightarrow L$ be defined as follows, respectively.

$$T_a^l(x, y) = \begin{cases} (x \vee a) \wedge (y \vee a), & (x, y) \in I_a \times I_a, \\ x \wedge (y \vee a), & (x, y) \in [a, 1_L] \times I_a, \\ (x \vee a) \wedge y, & (x, y) \in I_a \times [0_L, a) \cup I_a \times [a, 1_L) \\ 0_L, & (x, y) \in [0_L, a) \times L \setminus \{1_L\} \\ x \wedge y, & \text{otherwise} \end{cases}$$

$$S_b^l(x, y) = \begin{cases} (x \wedge b) \vee (y \wedge b), & (x, y) \in I_b \times I_b \\ x \vee (y \wedge b), & (x, y) \in (0_L, b] \times I_b \\ (x \wedge b) \vee y, & (x, y) \in I_b \times (0_L, b] \cup I_b \times (b, 1_L], \\ 1_L, & (x, y) \in (b, 1_L] \times L \setminus \{0_L\} \\ x \vee y, & \text{otherwise} \end{cases}$$

where $a, b \in L \setminus \{0_L, 1_L\}$

It is easy to verify the associative law. monotonicity and neutral element in both operators. However, they do not satisfy the commutative law such as

$$T_a^l(x, y) = 0_L \neq T_a^l(y, x) = x, (x, y) \in [0_L, a) \times [a, 1_L)$$

Remark 1.4.2. All the notions and properties recalled before stay valid when we use bounded lattice instead of The unit real interval [0, 1].

1.5 Fuzzy relations

A fuzzy relation [77] \tilde{R} is a mapping from the Cartesian space $X \times Y$ to the interval $[0, 1]$, where the strength of the mapping is expressed by the membership function of the relation for ordered pairs from the two universes is $\mu_{\tilde{R}}(x, y)$.

1.5.1 Operations on Fuzzy Relations

Let \tilde{R} and \tilde{S} be fuzzy relations on the Cartesian space $X \times Y$. Then the following operations apply for the membership values for various set operations (these are similar to the same operations on crisp sets;

Union	$\mu_{\tilde{R} \cup \tilde{S}}(x, y) = \max(\mu_{\tilde{R}}(x, y), \mu_{\tilde{S}}(x, y))$
Intersection	$\mu_{\tilde{R} \cap \tilde{S}}(x, y) = \min(\mu_{\tilde{R}}(x, y), \mu_{\tilde{S}}(x, y))$
Complement	$\mu_{\tilde{R}^c}(x, y) = 1 - \mu_{\tilde{R}}(x, y)$
Containment	$\tilde{R} \subset \tilde{S} \Rightarrow \mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{S}}(x, y)$

Cardinality of Fuzzy Relations

Since the cardinality of fuzzy sets on any universe is infinity, the cardinality of a fuzzy relation between two or more universes is also infinity.

1.5.2 Properties of fuzzy Relations

Just as for crisp relations, the properties of commutativity, associativity, distributivity, involution, and idempotency all hold for fuzzy relations. Moreover, De Morgan's principles hold for fuzzy relations just as they do for crisp (classical) relations and the null relation O . And the complete relation, E . Are analogous to the null set and the whole set in set-theoretic form, respectively. Fuzzy relations are not constrained, as is the case for fuzzy sets in general, by the excluded middle axioms. Since a fuzzy relation \tilde{R} is also a fuzzy set. there is overlap between a relation and its complement; hence,

$$\tilde{R} \cup \tilde{R}^c \neq E$$

$$\tilde{R} \cap \tilde{R}^c \neq O$$

As seen in the foregoing expressions, the excluded middle axioms for fuzzy relations do not, generally result, in the null relation, O , or the complete relation, E .

1.5.3 Fuzzy Cartesian Product and Composition

Because fuzzy relations are fuzzy sets, we can define the Cartesian product to be a relation between two or more fuzzy sets. Let \tilde{A} be a fuzzy set on universe X and \tilde{B}

a fuzzy set on universe Y , then the Cartesian product between fuzzy sets \tilde{A} and \tilde{B} will result in a fuzzy relation \tilde{R} , which is contained within the full Cartesian product space, or

$$\tilde{A} \times \tilde{B} = \tilde{R} \subset X \times Y$$

where the fuzzy relation \tilde{R} has membership function

$$\mu_{\tilde{R}}(x, y) = \mu_{\tilde{A} \times \tilde{B}}(x, y) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y))$$

The cartesian product defined by $\tilde{A} \times \tilde{B} = \tilde{R}$ is implemented in the same fashion as is the cross product of two vectors. moreover, the cartesian product is not the same operation as the arithmetic product. In the case of bi-dimensional relations ($r = 2$), the former employs the idea of the pairing of elements among sets, whereas the latter uses actual arithmetic products between elements of sets. Each of the fuzzy sets could be thought of as a membership values vector; each value is associated with a particular element in each set. For example, for a fuzzy set (vector) \tilde{A} that has four elements, hence column vector of size 4×1 , and for a fuzzy set (vector) \tilde{B} that has live elements, hence a row vector size of 1×5 , the resulting fuzzy relation \tilde{R} will be represented by a matrix of size 4×5 , that is, \tilde{R} will have four rows and five columns. This idea is illustrated in the following example.

Example 1.5.1. Suppose we have two fuzzy sets, \tilde{A} defined on a universe of three discrete temperatures: $X = \{x_1, x_2, x_3\}$, \tilde{B} defined on a universe of two discrete pressures, and $Y = \{y_1, y_2\}$, then we want to find the fuzzy Cartesian product between them. Fuzzy set \tilde{A} could represent the "ambient" temperature and fuzzy set \tilde{B} the "near-optimum" pressure for a certain heat exchanger and the Cartesian product might represent the conditions (temperature-pressure pairs) of the exchanger that are associated with "efficient" operations. For example, let

$$\tilde{A} = \left\{ \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} \right\} \quad \text{and} \quad \tilde{B} = \left\{ \frac{0.3}{y_1} + \frac{0.9}{y_2} \right\}$$

Note that \tilde{A} can be represented as a column vector of size 3×1 and \tilde{B} can be represented by a row vector of 1×2 . Then the fuzzy Cartesian product results in a fuzzy relation \tilde{R} (of size 3×2) representing "efficient" conditions or

$$\tilde{A} \times \tilde{B} = \tilde{R} = \begin{matrix} & y_1 & y_2 \\ x_1 & \left[\begin{array}{cc} 0.2 & 0.2 \end{array} \right] \\ x_2 & \left[\begin{array}{cc} 0.3 & 0.5 \end{array} \right] \\ x_3 & \left[\begin{array}{cc} 0.3 & 0.9 \end{array} \right] \end{matrix}$$

The fuzzy composition can be defined just as it is for crisp (binary) relations. Suppose \tilde{R} is a fuzzy relation on the cartesian space $X \times Y$, \tilde{S} is a fuzzy relation on $Y \times Z$, and \tilde{T} is a fuzzy relation on $X \times Z$, the fuzzy max-min composition is defined in terms of the set-theoretic notation and membership function-theoretic notation as follows:

$$\tilde{T} = \tilde{R} \circ \tilde{S}$$

$$\mu_{\tilde{T}}(x, z) = \bigvee_{y \in Y} \left(\mu_{\tilde{R}}(x, y) \wedge \mu_{\tilde{S}}(y, z) \right)$$

and fuzzy max-product composition is defined in terms of the membership function-theoretic notation as

$$\mu_{\tilde{T}}(x, z) = \bigvee_{y \in Y} \left(\mu_{\tilde{R}}(x, y) \bullet \mu_{\tilde{S}}(y, z) \right)$$

It should be pointed out that, in general, neither crisp nor fuzzy compositions are commutative; that is,

$$\tilde{R} \circ \tilde{S} \neq \tilde{S} \circ \tilde{R}.$$

Example 1.5.2. Let us fuzzy relationships for $X \times Y$ (denoted by the fuzzy relation \tilde{R}) and $X \times Y$ (denoted by the fuzzy relation \tilde{S}). In this case, we change the elements of the universes to

$$X = \{x_1, x_2\}, \quad Y = \{y_1, y_2\}, \quad \text{and} \quad Z = \{z_1, z_2, z_3\}$$

Consider the following fuzzy relations:

$$\tilde{R} = \begin{array}{c} y_1 \quad y_2 \\ x_1 \begin{bmatrix} 0.7 & 0.5 \end{bmatrix} \\ x_2 \begin{bmatrix} 0.8 & 0.4 \end{bmatrix} \end{array} \quad \text{and} \quad \tilde{S} = \begin{array}{c} z_1 \quad z_2 \quad z_3 \\ y_1 \begin{bmatrix} 0.9 & 0.6 & 0.2 \end{bmatrix} \\ y_2 \begin{bmatrix} 0.1 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

Then, the resulting relation, \tilde{T} , which relates elements of universe X to elements of universe Z , that is, defined on Cartesian space $X \times Z$, can be found by max-min composition, for example,

$$\mu_{\tilde{T}}(x_1, z_1) = \max[\min(0.7, 0.9), \min(0.5, 0.1)] = 0.7$$

and the set

$$\tilde{T} = \begin{array}{c} z_1 \quad z_2 \quad z_3 \\ x_1 \begin{bmatrix} 0.7 & 0.6 & 0.5 \end{bmatrix} \\ x_2 \begin{bmatrix} 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}$$

We now illustrate the use of relations with fuzzy sets of medicine, electrical engineering, and civil engineering.

Definition 1.5.1. [78] Each linguistic variable is fully characterized by a quintuple (v, T, X, g, m) in which v is the name of the variable. T is the set of the linguistic terms of v that refers to a base variable whose values range over the universal set X . g is a syntactic rule for generating linguistic terms. and m is a semantic rule that assigns to each linguistic term $t \in T$ its meaning, $m(t)$, which is an FS on X , that is $m: T \rightarrow F(X)$, where $F(X)$ denotes the set of all ordinary (i.e., Tl) FSs of X , one identical set for each $t \in T$,

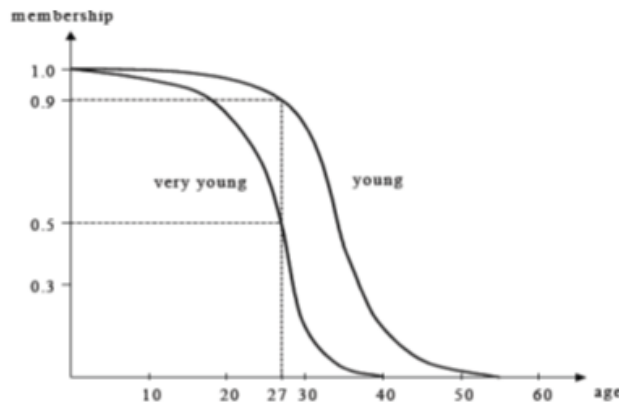


Figure 1.10: linguistic variable

1.6 t-norms and t-conorms

The notions of triangular-norms (*t-norms*) started with Menger [57] to construct metric spaces where probability distributions are used to describe the distance between two elements.

Definition 1.6.1. [57] A *t*-norm T on $[0, 1]$ is a function $T : [0, 1]^2 \rightarrow [0, 1]$ satisfies the following four axioms:

- 1 *Commutativity*: $(\forall x, y \in [0, 1])(T(x, y) = T(y, x))$;
- 2 *Associativity*: $(\forall x, y, z \in [0, 1])(T(x, T(y, z)) = T(T(x, y), z))$;
- 3 *Monotonicity*: $(\forall x, y, z \in [0, 1])(x \leq y \Rightarrow T(x, z) \leq T(y, z))$;
- 4 *Boundary condition*: $(\forall x \in [0, 1])(T(x, 1) = x)$.

Example 1.6.1. The following four operations are the most common *t*-norms:

- 1 *Minimum*: $T_M(x, y) = \min\{x, y\}$
- 2 *Product*: $T_P(x, y) = x \cdot y$
- 3 *Lukasiewicz*: $T_L(x, y) = \max\{x + y - 1, 0\}$
- 4 *Drastic product*:

$$T_D(x, y) = \begin{cases} 0 & \text{if } x, y \in [0, 1]^2. \\ \min(x, y) & \text{otherwise} \end{cases}$$

Definition 1.6.2. [57] A *t*-conorm is a function $S : [0, 1]^2 \rightarrow [0, 1]$ that for any $x, y, z \in [0, 1]$ satisfies (T1)-(T3) and the following boundary condition $S(x, 0) = S(0, x) = x$, $S(x, 1) = S(1, x) = 0$

Remark 1.6.1. Given a *t*-norm T , we find the associated dual *t*-conorm S by $S(x, y) = 1 - T(1 - x, 1 - y)$.

Example 1.6.2. The following four operations are the most common *t*-conorm:

- (1) *Maximum*: $S_M(x, y) = \max\{x, y\}$
- (2) *Probabilistic sum*: $S_P(x, y) = x + y - x \cdot y$
- (3) *Lukasiewicz*: $S_L(x, y) = \min\{x + y, 1\}$
- (4) *Drastic sum*:

$$S_D(x, y) = \begin{cases} 1, & \text{if } (x, y) \in]0, 1]^2 \\ \max\{x, y\}, & \text{otherwise} \end{cases}$$

Chapter 2

Type-2 fuzzy sets

The type-2 fuzzy set concept was first introduced by Zadeh (1975)[78] as an extension of a type-1 fuzzy set. A type-2 fuzzy set is characterized by a fuzzy membership function, i.e., the membership grade for each element of this set is a fuzzy set in $[0,1]$, unlike a type-1 set where the membership grade is a crisp number in $[0,1]$. Such sets can be used in situations where there is uncertainty about the membership grades themselves, e.g., uncertainty in the shape of the membership function or in some of its[5].

2.1 Type-2 Fuzzy Sets Concept

Definition 2.1.1. [79] A type m fuzzy set is a fuzzy set in X where the membership values of which are type $(m-1)$, $m > 1$ fuzzy sets on $[0, 1]$

Definition 2.1.2. [78],[50]. Let X be a finite and nonempty set, which is referred to as the universe a type-2 fuzzy set, denoted by \tilde{A} is characterized by a type-2 membership function $\mu_{\tilde{A}}(x, u)$ as $\mu_{\tilde{A}} : X \times [0, 1] \rightarrow [0, 1]^{J_x} = \text{map}(J_x, [0, 1])$, ($J_x \subseteq [0, 1]$), where $x \in X$ and $u \in J_x$ that is

$$\tilde{A} = \left\{ ((x, u), \mu_{\tilde{A}}(x, u)) / x \in X, \text{ and } u \in J_x \subseteq [0, 1], \text{ wher } 0 \leq \mu_{\tilde{A}}(x, u) \leq 1 \right\}$$

\tilde{A} can also be expressed as

$$\tilde{A} = \sum_{x \in X} \sum_{u \in J_x} \mu_{\tilde{A}}(x, u) / (x, u) = \sum_{x \in X} \sum_{u \in J_x} f_x(u) / u / x, J_x \subseteq [0, 1]$$

where $f_x(u) = \mu_{\tilde{A}}(x, u)$ and denotes union over all admissible x and u for continuous universes of discourse, $\sum^{\tilde{A}}$ is replaced by \int

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{A}}(x, u) / (x, u)$$

where x is the primary variable in the domain X ; u is the secondary variable in domain J_x at each $x \in X$. J_x is called the primary membership of x , and $\mu_{\tilde{A}}(x, u)$ the secondary membership grades of \tilde{A} , $J_x \subseteq [0, 1]$ and $\int \int$ denote union over all admissible x and u in definition 2.1.2, the first restriction that $\forall u \in J_x \subseteq [0, 1]$ is consistent with the type-1 constraint that $0 \leq \mu_A(x) \leq 1$, i.e., when uncertainties disappear a type-2 membership function must be reduced to a type-1 membership function, in which case the variable u equals $\mu_A(x)$ and $0 \leq \mu_A(x) \leq 1$. The second restriction that $0 \leq \mu_{\tilde{A}}(x, u) \leq 1$ is consistent with the fact that the amplitudes of a membership function should lie between or be equal to 0 and 1.

Type-2 membership function representation

$$\tilde{A} = \left\{ \left((x, u), \mu_{\tilde{A}}(x, u) \right) \mid x \in X, J_x \subseteq [0, 1] \right\}$$

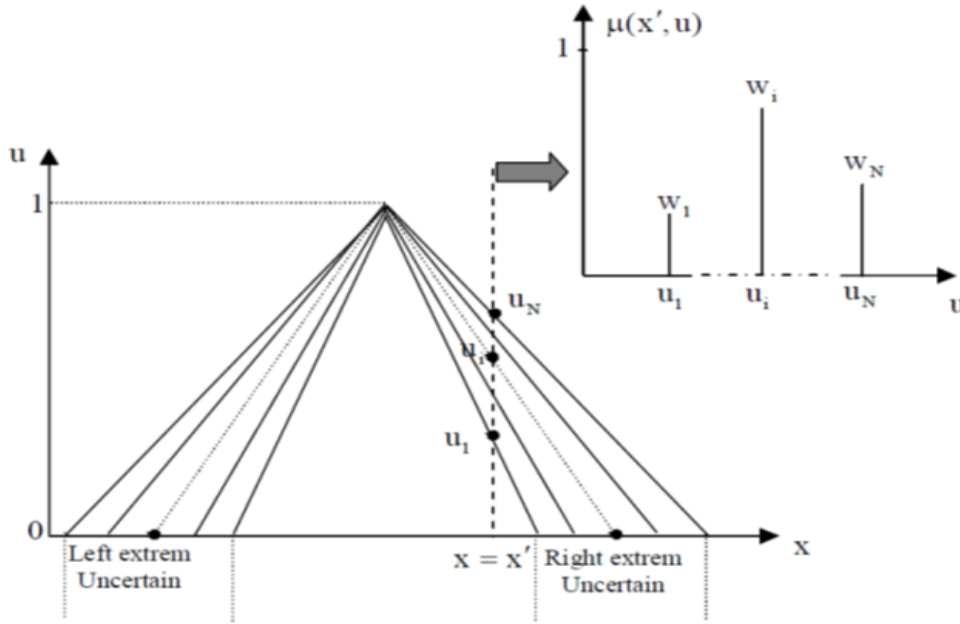


Figure 2.1: Triangular MFs when base end-points (l and r) have uncertainty intervals associated with them. [55]

Example 2.1.1. Suppose that $X = \text{Salima, Basma, Halima, Radhia, Aicha}$ is a set of women and that A is a fuzzy set of type 2 of beautiful women in X . Then we may have

$$A = \text{beautiful} = \text{middle/Salima} + \text{notlow/Basma} + \text{low/Halima} + \text{veryhigh/Radhia} + \text{high/Aicha}$$

where the fuzzy grades labeled middle, low, high are assumed to be fuzzy sets in $J = 0, 0.1, \dots, 0.9, 1 \subseteq [0, 1]$ and, for example, are expressed as follows.

$$\text{middle} = 0.3/0.3 + 0.7/0.4 + 1/0.5 + 0.7/0.6 + 0.3/0.7. \quad \text{low} = 1/0 + 0.9/0.1 + 0.7/0.2 + 0.4/0.3. \quad \text{high} = 0.4/0.7 + 0.7/0.8 + 0.9/0.9 + 1/1.$$

Moreover, fuzzy grades named not low and very high are defined from fuzzy grades low and high by using the linguistic hedges concept (Zadeh, 1972)

$$\text{not low} = 0.1/0.1 + 0.3/0.2 + 0.6/0.3 + 1/(0.4 + 0.5 + \dots + 1). \quad \text{very high} = 0.16/0.7 + 0.49/0.8 + 0.81/0.9 + 1/1.$$

Definition 2.1.3. [52] J_x , the primary membership of x , is :

$$J_x = \left\{ (x, u) \mid u \in [0, 1], \mu_{\tilde{A}}(x, u) > 0 \right\}$$

It can also be expressed as a subset of $\{x\} \times I_x$, i.e., $J_x = \{x\} \times I_x$ where

$$I_x = \left\{ u \in [0, 1] \mid \mu_{\tilde{A}}(x, u) > 0 \right\}$$

Definition 2.1.4. [52] The support of \tilde{A} comprise all $(x, u) \in X \times [0, 1]$ such that $\mu_{\tilde{A}}(x, u) > 0$, and is also called the domain of uncertainty of \tilde{A} , $DOU(\tilde{A})$, i.e.,

$$DOU(\tilde{A}) = \left\{ (x, u) \in X \times [0, 1] \mid \mu_{\tilde{A}}(x, u) > 0 \right\} = \bigcup_{x \in X} J_x$$

Definition 2.1.5. [52] When the support or the secondary MF, l_x , is closed for $x \in X$, .ie.,

$$I_x = \left\{ u \in [0, 1] \mid \mu_{\tilde{A}}(x, u) > 0 \right\} = \left[\underline{\mu}_{\tilde{A}}(x), \bar{\mu}_{\tilde{A}}(x) \right] \quad (2.1)$$

where (Aisbett et al. 2010)

$$\bar{\mu}_{\tilde{A}}(x) = \sup \left\{ u \mid u \in [0, 1], \mu_{\tilde{A}}(x, u) > 0 \right\} \quad (2.2)$$

$$\underline{\mu}_{\tilde{A}}(x) = \inf \left\{ u \mid u \in [0, 1], \mu_{\tilde{A}}(x, u) > 0 \right\} \quad (2.3)$$

then the domain or uncertainty of \tilde{A} is called the footprint of uncertainty of \tilde{A} , $FOU(\tilde{A})$. i.e.,

$$DOU(\tilde{A}) = FOU(\tilde{A}) = \left\{ (x, u) \mid x \in X \text{ and } u \in \left[\underline{\mu}_{\tilde{A}}(x), \bar{\mu}_{\tilde{A}}(x) \right] \right\} \quad (2.4)$$

Note that $\underline{\mu}_{\tilde{A}}(x)$ and $\bar{\mu}_{\tilde{A}}(x)$ are called the lower and upper MFs or $FOU(\tilde{A})$ [35], and are the lower and upper (type-1 fuzzy set) bounding function of the FOU, respectively. Commonly used abbreviations for the lower and upper MFs or \tilde{A} are $LMF(\tilde{A})$ and $UMF(\tilde{A})$.

[52] The class of all type-2 fuzzy sets of the universe X denoted by $\tilde{F}_{T_2}(X)$.

Type- 2 fuzzy universe set

Definition 2.1.6. A type-2 fuzzy universe set denoted \tilde{X} such that $\tilde{X} = \sum_{x \in X} \sum_{u \in [1,1]} \frac{1}{x}$

Upper and lower membership functions

Definition 2.1.7. [5]

an upper membership function and a lower membership function are two type-1 membership functions that are bounds for the FOU of a type-2 fuzzy set \tilde{A} . The upper membership function is associated with the upper bound of $FOU(\tilde{A})$. The lower membership function is associated with the lower bound of $FOU(\tilde{A})$. We illustrate the concept of upper and lower membership functions as well as the footprint of uncertainty in the following example.

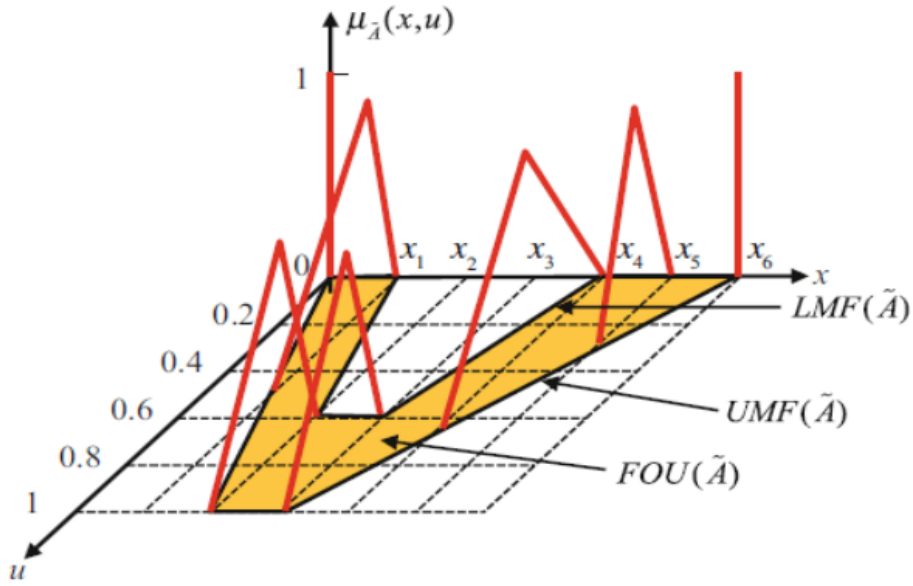


Figure 2.2: three-dimensional representation the of type-2 membership function [54]

2.2 Interval type- 2 fuzzy set

[48] When all the secondary grades of types $\tilde{\tilde{A}}$ are equal to 1, that is $\mu_{\tilde{\tilde{A}}}(x, u)$ for all $x \in X$ all $u \in J_x$, $\tilde{\tilde{A}}$ is an interval type-2 fuzzy set, $\tilde{\tilde{A}}$ is a special case of general type-2 fuzzy set where all of the secondary membership functions of $\tilde{\tilde{A}}$ are equal to 1. $\tilde{\tilde{A}}$ is an interval type-2 fuzzy set

$$\tilde{\tilde{A}} = \int_{x \in X} \int_{u \in J_x} \frac{1}{(x, u)} \quad (2.5)$$

where $J_x \subseteq [0, 1]$

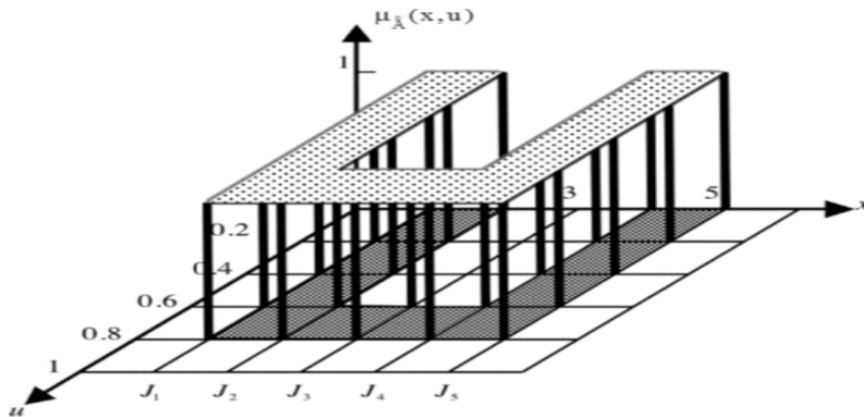


Figure 2.3: Example of an interval type-2 membership function for discrete universes of discourse [50]

Definition 2.2.1. [50]. At each value of x , say $x = x'$, the 2D plane whose axes are u and $\mu_{\tilde{A}}(x', u)$ is called a vertical slice of $\mu_{\tilde{A}}(x, u)$. A secondary membership function is a vertical slice of $\mu_{\tilde{A}}(x, u)$. It is $\mu_{\tilde{A}}(x = x', u)$ for $x' \in X$ and $\forall u \in J_{x'} \subseteq [0, 1]$, i.e., $\mu_{\tilde{A}}(x = x', u) \equiv \mu_{\tilde{A}}(x')$ $= \int_{u \in J_{x'}} f_{x'}(u)/u J_{x'} \subseteq [0, 1]$ (3) in which $0 \leq f_{x'}(u) \leq 1$. Because $\forall x' \in X$ we drop the prime notation on $\mu_{\tilde{A}}(x')$. and refer to $\mu_{\tilde{A}}(x)$ as a secondary membership function; it is a type-1 fuzzy set, which we also refer to as a secondary set.

Example 2.2.1. The type-2 membership function that is depicted in Fig 2.2 has five vertical slices associated with it. The secondary membership function at $x = 2$, e.g., is

$$\mu_{\tilde{A}}(2) = 0.5/0 + 0.35/0.2 + 0.35/0.4 + 0.2/0.6 + 0.5/0.8$$

2.3 Operation of Type-2 fuzzy sets

[49] In this section, we describe the type-2 fuzzy sets theoretical operations set. We are interested in the case of type-2 fuzzy sets, $\tilde{A}_i (i = 1, \dots, r)$, the secondary membership functions are type-1 fuzzy sets. To compute the union, the intersection, and the type-2 fuzzy sets complement, we need to extend the binary operations of minimum (or product) and maximum, and the unary operation of negation, from crisp numbers to type-1 fuzzy sets, because at each x , $\mu_{\tilde{A}_i}(x, u)$ is a function (unlike the type-1 case, where $\mu_{\tilde{A}_i}(x)$ is a crisp number). The tool for computing the union, intersection, and type-2 fuzzy sets complement is Zadeh's extension principle (Zadeh, 1975). We study operations set on type-2 fuzzy sets. We first discuss join and meet operations of membership grades of type-2 fuzzy sets under left continuous t-norms and derive distributive law of type-2 fuzzy sets [58].

Consider two type-2 fuzzy sets, \tilde{A} and \tilde{B} , in a universe X . Let $\mu_{\tilde{A}(x)}$ and $\mu_{\tilde{B}(x)}$ be the membership grades of these two sets, which are represented for each $x \in X$, $\mu_{\tilde{A}(x)} = \sum_{u \in J_x^u} f_x(u)/u$ and $\mu_{\tilde{B}(x)} = \sum_{w \in J_x^w} g_x(w)/w$, respectively, where $u \in J_x^u, w \in J_x^w$ indicate the primary memberships of x and $f_x(u), g_x(w) \in [0, 1]$ indicate the secondary memberships (grades) of x . The membership grades for the union, intersection, and complement of the type-2 fuzzy sets \tilde{A} and \tilde{B} have been defined as follows

Containment:

\tilde{A} is a subtype-2 fuzzy set of \tilde{B} denoted $\tilde{A} \subseteq \tilde{B}$ if $u < v$ and $f_x(u) \leq g_x(w)$ for every $x \in X$

Equality:

\tilde{A} and \tilde{B} are equal type-2 fuzzy sets, denoted $\tilde{A} = \tilde{B}$ if $u = w$ and $f_x(u) = \mu_{\tilde{A}}(x, u) = g_x(w) = \mu_{\tilde{B}}(x, w)$ for every $x \in X$

Two type-2 fuzzy sets union: [34]

$$\tilde{A} \cup \tilde{B} = \sum_{u \in J_x^u} \sum_{w \in J_x^w} (f_x(u) * g_x(w)) / (u \vee w)$$

Two type-2 fuzzy sets intersection: [34]

$$\tilde{A} \cap \tilde{B} = \sum_{u \in J_x^u} \sum_{w \in J_x^w} (f_x(u) * g_x(w)) / (u \wedge w)$$

where \vee represents the max t-conorm and \wedge indicates the minimum t-norm

Type-2 fuzzy set complement: [34]

Complement : $\tilde{A} \Leftrightarrow \mu_{\tilde{A}}(x) = \neg \mu_A(x) = \int_u f_x(u)/(1-u)$

Join and Meet under min t-norm

Definition 2.3.1. [49] Suppose that we have two convex, normal, type-1 real fuzzy sets \tilde{F} and \tilde{G} characterized by membership functions f and g respectively. Let $v_0 \in \mathbb{R}$ and $v_1 \in \mathbb{R}$ be such that $v_0 < v_1$ and $f(v_0) = g(v_1) = 1$. Then the membership functions of the join and meet of \tilde{F} and \tilde{G} , using max t-conorm and min t-norm, can be expressed as

$$\begin{aligned} \mu_{(\tilde{F} \cup \tilde{G})}(\theta) &= f(\theta) \wedge g(\theta); \theta < v_0 \\ &= g(\theta); v_0 \leq \theta \leq v_1 \\ &= f(\theta) \vee g(\theta); \theta > v_1 \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} \mu_{(\tilde{F} \cap \tilde{G})}(\theta) &= f(\theta) \vee g(\theta); \theta < v_0 \\ &= f(\theta); v_0 \leq \theta \leq v_1 \\ &= f(\theta) \wedge g(\theta); \theta > v_1 \end{aligned} \quad (2.7)$$

Definition 2.3.2. Type-2 fuzzy sets union

The union of \tilde{A}_1 and \tilde{A}_2 are another type-2 fuzzy set, just as the union of type-1 fuzzy sets A_1 and A_2 is another type-1 fuzzy set. More formally, we have the following expression

$$\tilde{A}_1 \cup \tilde{A}_2 = \int_{x \in X} \mu_{\tilde{A}_1 \cup \tilde{A}_2}(x) / x \quad (2.8)$$

Basically, the join between two secondary membership functions must be performed between every possible pair of primary memberships.

Let $\mu_A(x)$ and $\mu_B(x)$ be two fuzzy grades (that is, fuzzy sets in $J \subseteq [0, 1]$) of fuzzy sets of type 2, A and B , respectively, represented as

$$\begin{aligned} \mu_A(x) &= f(u_1) / u_1 + f(u_2) / u_2 + \dots + f(u_n) / u_n \\ &= \sum_i f(u_i) / u_i, \quad u_i \in J \end{aligned} \quad (2.9)$$

$$\begin{aligned} \mu_B(x) &= g(w_1) / w_1 + g(w_2) / w_2 + \dots + g(w_m) / w_m \\ &= \sum_j g(w_j) / w_j, \quad w_j \in J \end{aligned} \quad (2.10)$$

where the function f and g are membership functions of fuzzy grades (fuzzy sets in $J \subseteq [0, 1]$) $\mu_A(x)$ and $\mu_B(x)$ respectively, and the values $f(u_i)$ and $g(w_j)$ in $[0, 1]$ represent the grades for u_i and w_j in J , respectively.

Thus the operation type 2 fuzzy sets are expressed by the following.

Union

$$\begin{aligned} A \cup B \Leftrightarrow \mu_{A \cup B}(x) &= \mu_A(x) \sqcup \mu_B(x) \\ &= \left(\sum_i f(u_i) / u_i \right) \sqcup \left(\sum_j g(w_j) / w_j \right) \\ &= \sum_{i,j} (f(u_i) \wedge g(w_j)) / (u_i \vee w_j) \end{aligned} \quad (2.11)$$

Intersection

$$\begin{aligned}
 A \cap B &\Leftrightarrow \mu_{A \cap B}(x) = \mu_A(x) \sqcap \mu_B(x) \\
 &= \left(\sum_i f(u_i) / u_i \right) \sqcap \left(\sum_j g(w_j) / w_j \right) \\
 &= \sum_{i,j} (f(u_i) \wedge g(w_j)) / (u_i \wedge w_j)
 \end{aligned} \tag{2.12}$$

Complement

$$\tilde{A} \Leftrightarrow \mu_{\tilde{A}}(x) \text{ then the MF of the complement of } \tilde{A} \text{ is } \mu_{\tilde{A}}(x) = \int_u f_x(u) / (1 - u) \tag{2.13}$$

where \vee and \wedge represent max and min, respectively. We call the operations for fuzzy grades, that is, \sqcup like join, \sqcap as to meet, and \neg as negation hereafter. [58]. If more than one combination of pairs gives the same point, then in the join we keep the one with maximum membership grade. We will consider a simple example to illustrate the union operation. In Figure 3.6 we plot two type-2 Gaussian membership functions, and the union is shown in Figure 3.7.

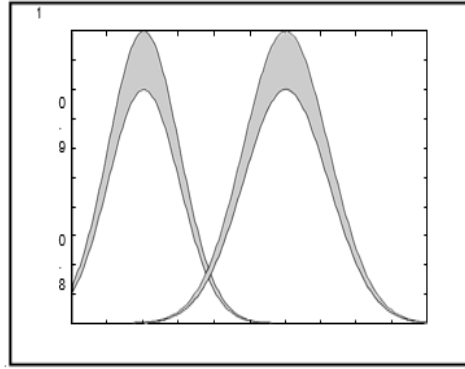


Figure 2.4: Type-2 Gaussian MFs

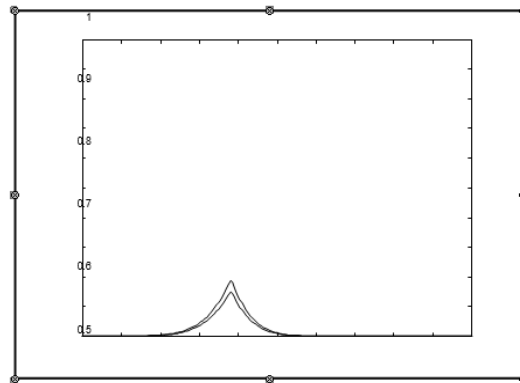


Figure 2.5: Intersection of two Interval Type-2 Fuzzy Sets

Example 2.3.1. [2] Let $X = \{x_1, x_2, x_3\}$ be a nonempty set, and let \tilde{A} and \tilde{B} are two type-2 fuzzy sets over the same universe X

$$\tilde{A} = \{(x_1, 0.1), 0.3), ((x_1, 0.5), 1), ((x_2, 0.5), 1), ((x_2, 0.6), 0.3), ((x_3, 0.8), 1)\}$$

$$\tilde{B} = \{(x_1, 0.1), 0.7), ((x_1, 0.2), 1), ((x_2, 0.6), 1), ((x_3, 0.5), 0.6), ((x_3, 0.9), 1)\}$$

$\tilde{A} \cup \tilde{B}$ for $x = x_1$ we get

$$\begin{aligned} \mu_{\tilde{A} \cup \tilde{B}}(x_1) &= \frac{0.3 \wedge 0.7}{0.1 \vee 0.1} + \frac{0.3 \wedge 1}{0.1 \vee 0.2} + \frac{1 \wedge 0.7}{0.5 \vee 0.1} + \frac{1 \wedge 1}{0.5 \vee 0.2} \\ &= \frac{0.3}{0.1} + \frac{0.3}{0.2} + \frac{0.7}{0.5} + \frac{1}{0.5} \\ &= \{(0.1, 0.3), (0.2, 0.3), (0.5, \max(0.7, 1))\} \end{aligned}$$

For $x = x_1$

$$\tilde{A} \cup \tilde{B} = \{(x_1, 0.1), 0.3), ((x_1, 0.2), 0.3), ((x_1, 0.5), 1)\}$$

$\tilde{A} \cup \tilde{B}$ for $x = x_2$ we get

$$\begin{aligned} \mu_{\tilde{A} \cup \tilde{B}}(x_2) &= \frac{1 \wedge 1}{0.5 \vee 0.6} + \frac{0.3 \wedge 1}{0.6 \vee 0.6} \\ &= \frac{1}{0.6} + \frac{0.3}{0.6} \\ &= \{(0.6, \max(1, 0.3))\} \end{aligned}$$

For $x = x_2$, $\tilde{A} \cup \tilde{B} = \{(x_2, 0.6), 1\}$

$\tilde{A} \cup \tilde{B}$ for $x = x_3$ to get

$$\begin{aligned} \mu_{\tilde{A} \cup \tilde{B}}(x_3) &= \frac{1 \wedge 0.6}{0.8 \vee 0.5} + \frac{1 \wedge 1}{0.8 \vee 0.9} \\ &= \frac{0.6}{0.8} + \frac{1}{0.9} \\ &= \{(0.8, 0.6), (0.9, 1)\} \end{aligned}$$

For $x = x_3$,

$$\tilde{A} \cup \tilde{B} = \{(x_3, 0.6), 0.8), ((x_3, 0.9), 1)\}$$

Hence,

$$\tilde{A} \cup \tilde{B} = \{(x_1, 0.1), 0.3), ((x_1, 0.2), 0.3), ((x_1, 0.5), 1), ((x_2, 0.6), 1), ((x_3, 0.6), 0.8), ((x_3, 0.9), 1)\}$$

$\tilde{A} \cap \tilde{B}$ for $x = x_1$

$$\tilde{A} \cap \tilde{B} = \{(x_1, 0.1), 0.7), ((x_1, 0.2), 1)\}$$

$\tilde{A} \cap \tilde{B}$ for $x = x_2$

$$\begin{aligned} \mu_{\tilde{A} \cap \tilde{B}}(x_2) &= \frac{1 \wedge 1}{0.5 \wedge 0.6} + \frac{0.3 \wedge 1}{0.6 \wedge 0.6} \\ &= \frac{1}{0.5} + \frac{0.3}{0.6} \\ &= \{(0.5, 1), (0.6, 0.3)\} \end{aligned}$$

For $x = x_2$

$$\tilde{A} \cap \tilde{B} = \{(x_2, 0.5), 1), (x_2, 0.6), 0.3)\}$$

$\tilde{A} \cap \tilde{B}$ for $x = x_3$

$$\begin{aligned} \mu_{\tilde{A} \cap \tilde{B}}(x_3) &= \frac{1 \wedge 0.6}{0.8 \wedge 0.5} + \frac{1 \wedge 1}{0.8 \wedge 0.9} \\ &= \frac{0.6}{0.5} + \frac{1}{0.8} \\ &= \{(0.5, 0.6), (0.8, 1)\} \end{aligned}$$

$$\begin{aligned}
& \text{For } x = x_1, \quad \tilde{A} \cap \tilde{B} = \{(x_3, 0.5), 0.6\}, \{(x_3, 0.8), 1\}\} \\
& \tilde{A} \cap \tilde{B} = \{(x_1, 0.1), 0.7\}, \{(x_1, 0.2), 1\}, \{(x_2, 0.5), 1\}, \{(x_2, 0.6), 0.3\}, \{(x_3, 0.5), 0.6\}, \{(x_3, 0.8), 1\}\} \\
& \tilde{A} = \{(x_1, 0.1), 0.7\}, \{(x_1, 0.5), 0.5\}, \{(x_2, 0.5), 0.5\}, \{(x_2, 0.6), 0.7\}, \{(x_3, 0.8), 0.2\}\}
\end{aligned}$$

Theorem 2.3.1. [70] Let \tilde{A} , \tilde{B} , and \tilde{C} be two T2 fuzzy sets on X . If secondary membership functions $\mu_{\tilde{A}}^{\approx}(x)$, $\mu_{\tilde{B}}^{\approx}(x)$ and $\mu_{\tilde{C}}^{\approx}(x)$ are convex fuzzy sets for any $x \in X$, then we have

$$(\tilde{A} \cup \tilde{B}) \cap \tilde{C} = (\tilde{A} \cap \tilde{C}) \cup (\tilde{B} \cap \tilde{C}) \quad (2.14)$$

and

$$(\tilde{A} \cap \tilde{B}) \cup \tilde{C} = (\tilde{A} \cup \tilde{C}) \cap (\tilde{B} \cup \tilde{C}) \quad (2.15)$$

Proof 2.3.1. For any $x \in X$, since secondary membership functions $\mu_{\tilde{A}}^{\approx}(x)$, $\mu_{\tilde{B}}^{\approx}(x)$ and $\mu_{\tilde{C}}^{\approx}(x)$ are convex T1 fuzzy sets in $[0, 1]$, from [1], they satisfy distributive laws under \sqcup and \sqcap . Therefore, we obtain

$$\begin{aligned}
& \left[\mu_{\tilde{A}}^{\approx}(x) \sqcup \mu_{\tilde{B}}^{\approx}(x) \right] \sqcap \mu_{\tilde{C}}^{\approx}(x) \\
& = \left[\mu_{\tilde{A}}^{\approx}(x) \sqcap \mu_{\tilde{C}}^{\approx}(x) \right] \sqcup \left[\mu_{\tilde{B}}^{\approx}(x) \sqcap \mu_{\tilde{C}}^{\approx}(x) \right]
\end{aligned}$$

and

$$\begin{aligned}
& \left[\mu_{\tilde{A}}^{\approx}(x) \sqcap \mu_{\tilde{B}}^{\approx}(x) \right] \sqcup \mu_{\tilde{C}}^{\approx}(x) \\
& = \left[\mu_{\tilde{A}}^{\approx}(x) \sqcup \mu_{\tilde{C}}^{\approx}(x) \right] \sqcap \left[\mu_{\tilde{B}}^{\approx}(x) \sqcup \mu_{\tilde{C}}^{\approx}(x) \right]
\end{aligned}$$

2.4 Operations under type-2 fuzzy sets collection

[58] Let $\{\tilde{A}_i, i \in \mathbb{N}\}$ be an arbitrary collection of type-2 fuzzy sets of X such that \mathbb{N} is countable set, operations are possible under an arbitrary collection of type-2 fuzzy sets.

1) The union $\cup_{i \in \mathbb{N}} \tilde{A}_i$ is defined as

$$\left[\cup_{i \in \mathbb{N}} \tilde{A}_i \right] (x) = \sum_{x \in X} \sum_{u \in JX} \frac{\wedge_{i \in \mathbb{N}} (f_x(u))_i}{\vee_{i \in \mathbb{N}} (u)_i}$$

2) The intersection

$$\left[\cap_{i \in \mathbb{N}} \tilde{A}_i \right] (x) = \sum_{x \in X} \sum_{u \in JX} \frac{\wedge_{i \in \mathbb{N}} (f_x(u))_i}{\wedge_{i \in \mathbb{N}} (u)_i}$$

Proposition 2.4.1. [25] Let $\{\tilde{A}_i, i \in \mathbb{N}\}$ be an arbitrary collection of type-2 fuzzy sets of X such that \mathbb{N} is a countable set and \tilde{B} be another type-2 fuzzy set of X , then

$$1) \quad \tilde{B} \cap \left[\cup_{i \in \mathbb{N}} \tilde{A}_i \right] = \cup_{i \in \mathbb{N}} \left(\tilde{B} \cap \tilde{A}_i \right)$$

$$2) \quad \tilde{B} \cup \left[\cap_{i \in \mathbb{N}} \tilde{A}_i \right] = \cap_{i \in \mathbb{N}} \left(\tilde{B} \cup \tilde{A}_i \right)$$

$$3) \quad 1 - \left[\cup_{i \in \mathbb{N}} \tilde{A}_i \right] = \cap_{i \in \mathbb{N}} \left[1 - \tilde{A}_i \right]$$

$$4) \quad 1 - \left[\bigcap_{i \in \mathbb{N}} \tilde{A}_i \right] = \bigcup_{i \in \mathbb{N}} \left[1 - \tilde{A}_i \right]$$

Example 2.4.1. First, we will show the examples of fuzzy grades which do not satisfy the absorption laws, the distributive laws, the identity laws, and the complement laws.

Absorption laws failure. Let μ_A and μ_B be convex fuzzy grades as

$$\mu_A = 0.2/0.1 + 0.3/0.2 + 0.5/0.3 + 0.7/0.4 + 0.8/0.5$$

$$\mu_B = 0/0.1 + 0.1/0.2 + 0.2/0.3 + 0.3/0.2 + 0.4/0.5$$

where J is $\{0.1, 0.2, 0.3, 0.4, 0.5\}$. Then $\mu_A \sqcap (\mu_B = 0.2/0.1 + 0.3/0.2 + 0.4/0.3 + 0.4/0.4 + 0.4/0.5 \text{ and } \mu_A \sqcup \mu_B)$
 $= 0/0.1 + 0.1/0.2 + 0.2/0.3 + 0.3/0.4 + 0.4/0.5$

$$\mu_A \sqcap (\mu_A \sqcup \mu_B) = \mu_A \sqcup (\mu_A \sqcap \mu_B)$$

$$= 0.2/0.1 + 0.3/0.2 + 0.4/0.3 + 0.4/0.4 + 0.4/0.5 \neq \mu_A.$$

Distributive laws failure

$$\mu_A = 0.8/0.2 + 0.1/0.3 + 0/0.4 + 0.7/0.5$$

$$\mu_B = 0.3/0.2 + 0.4/0.3 + 0.5/0.4 + 0.2/0.5$$

$$\mu_C = 0.1/0.2 + 0.2/0.3 + 0.5/0.4 + 0.7/0.5$$

Then we have,

$$\mu_B \sqcup (\mu_C) = 0.1/0.2 + 0.2/0.3 + 0.5/0.4 + 0.5/0.5$$

$$\mu_A \sqcap (\mu_B \sqcup \mu_C) = 0.5/0.2 + 0.2/0.3 + 0.5/0.4 + 0.5/0.5$$

$$\mu_A \sqcap \mu_B = 0.5/0.2 + 0.4/0.3 + 0.5/0.4 + 0.2/0.5$$

$$\mu_A \sqcap \mu_C = 0.7/0.2 + 0.2/0.3 + 0.5/0.4 + 0.7/0.5$$

$$(\mu_A \sqcap \mu_B) \sqcup (\mu_A \sqcap \mu_C) = 0.5/0.2 + 0.4/0.3 + 0.5/0.4 + 0.2/0.5$$

Thus, we see that the distributive law is not satisfied. The same holds for the case in which \sqcup and \sqcap are interchanged.

A part of identity laws failure, let $\mu_A = 0.6/0.2 + 0.4/0.3 + 0.7/0.4$. Then, noting that the numbers 1 and 0 are represented as 1/1 and 1/0, respectively, we obtain

$$\mu_A \sqcup 1 = 0.7/1 \neq 1$$

$$\mu_A \sqcap 0 = 0.7/0 \neq 0$$

Complement laws failure. Let $\mu_A = 0.7/0.2 + 0.6/0.3 + 1/0.4$ then $\mu_{\tilde{A}}$ is given as

$$\mu_{\tilde{A}} = 0.7/0.8 + 0.6/0.7 + 1/0.6.$$

Thus we have

$$\mu_A \sqcup \mu_{\tilde{A}} = 1/0.6 + 0.6/0.7 + 0.7/0.8 \neq 1$$

$$\mu_A \sqcap \mu_{\tilde{A}} = 0.7/0.2 + 0.6/0.3 + 1/0.4 \neq 0$$

2.5 Type-2 fuzzy relations, their compositions, and cartesian product

Let X_1, X_2, \dots, X_n , be n universes. A crisp relation in $X_1 \times X_2 \times \dots \times X_n$ is a crisp subset of the product space. Similarly, a type-1 fuzzy relation in $X_1 \times X_2 \times \dots \times X_n$, is a type-1 fuzzy subset of the product space and a type-2 fuzzy relation in $X_1 \times X_2 \times \dots \times X_n$, is a type-2 fuzzy subset of the product space. We concentrate on binary type-2 fuzzy relations.

2.5.1 Type-2 FRs and their compositions

From type-1 fuzzy relations to type-2 ones

[75] A type-1 FR in $U \times V$ is a type-1 fuzzy subset of $U \times V$ and a type-2 FR in $U \times V$ is a type-2 fuzzy subset of $U \times V$.

Type-1 and type-2 fuzzy relations are usually expressed in matrix form as

$$R = \begin{bmatrix} \mu_R(u_1, v_1) & \mu_R(u_1, v_2) \cdots & \mu_R(u_1, v_n) \\ \mu_R(u_2, v_1) & \mu_R(u_2, v_2) \cdots & \mu_R(u_2, v_n) \\ \vdots & \vdots & \vdots \\ \mu_R(u_m, v_1) & \mu_R(u_m, v_2) \cdots & \mu_R(u_m, v_n) \end{bmatrix}$$

and

$$\tilde{R} = \begin{bmatrix} \mu_{\tilde{R}}(u_1, v_1) & \mu_{\tilde{R}}(u_1, v_2) \cdots & \mu_{\tilde{R}}(u_1, v_n) \\ \mu_{\tilde{R}}(u_2, v_1) & \mu_{\tilde{R}}(u_2, v_2) \cdots & \mu_{\tilde{R}}(u_2, v_n) \\ \vdots & \vdots & \vdots \\ \mu_{\tilde{R}}(u_m, v_1) & \mu_{\tilde{R}}(u_m, v_2) \cdots & \mu_{\tilde{R}}(u_m, v_n) \end{bmatrix}$$

where $\mu_R(u_i, v_j)$ are crisp numbers in $[0, 1]$, and $\mu_{\tilde{R}}(u_i, v_j)$ are fuzzy numbers (fuzzy sets) in $[0, 1]$.

As the extension from type-1 fuzzy sets to type-2 ones, we can obtain a type-2 fuzzy relation by adding additional uncertainty information to a type-1 one. The following example explains the extension

Example 2.5.1. Consider the type-1 FR $R \in \mathcal{F}(X \times Y) : 'x \text{ is close to } y'$, where $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ and

$$R = \begin{matrix} & y_1 & y_2 \\ x_1 & \begin{bmatrix} 0.5 & 0.2 \end{bmatrix} \\ x_2 & \begin{bmatrix} 0.8 & 0.1 \end{bmatrix} \\ x_3 & \begin{bmatrix} 0.2 & 0.7 \end{bmatrix} \end{matrix}$$

Consider another type-1 FR $S \in \mathcal{F}(Y \times Z) : 'y \text{ is much smaller than } z'$, where $Z = \{z_1, z_2, z_3\}$, and

$$S = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & \begin{bmatrix} 0.3 & 0.9 & 0.6 \end{bmatrix} \\ y_2 & \begin{bmatrix} 0.3 & 0.4 & 0.5 \end{bmatrix} \end{matrix}$$

Adding some additional uncertainties to these two type-1 FRs, we may obtain the following membership grades (For saving space, we sometimes represent a type-2 fuzzy relation in the following form, i.e., the solidus (/) are replaced by built-up fractions):

$$\tilde{R} = \begin{matrix} & y_1 & y_2 \\ x_1 & \begin{bmatrix} \frac{0.4}{0.4} + \frac{1}{0.5} + \frac{0.7}{0.6} & \frac{0.6}{0.5} + \frac{1}{0.1} + \frac{0.6}{0.2} \end{bmatrix} \\ x_2 & \begin{bmatrix} \frac{0.3}{0.6} + \frac{1}{0.7} + \frac{0.5}{0.8} & \frac{0.1}{0.6} + \frac{0.2}{1} + \frac{0.3}{0.4} \end{bmatrix} \\ x_3 & \begin{bmatrix} \frac{0.8}{0.8} + \frac{1}{0.9} + \frac{0.5}{0.1} & \frac{0.6}{0.7} + \frac{1}{0.8} + \frac{0.4}{0.9} \end{bmatrix} \end{matrix}$$

and

$$\tilde{S} = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & \begin{bmatrix} \frac{0.4}{0.8} + \frac{1}{0.9} + \frac{0.6}{0.6} & \frac{0.3}{0.5} + \frac{1}{0.6} + \frac{0.7}{0.7} & \frac{0.5}{0.8} + \frac{1}{0.1} + \frac{0.6}{0.2} \end{bmatrix} \\ y_2 & \begin{bmatrix} \frac{0.3}{0.5} + \frac{1}{0.1} + \frac{0.6}{0.2} & \frac{0.2}{0.5} + \frac{1}{0.6} + \frac{0.6}{0.7} & \frac{0.5}{0.8} + \frac{1}{0.9} + \frac{0.7}{1} \end{bmatrix} \end{matrix}$$

2.5.2 Composition of type-2 fuzzy relations

[49]

Definition 2.5.1. If R and S (or \tilde{R} and \tilde{S}) are two T1 (or T2) FRs on $U \times V$ and $V \times W$, respectively, the membership of any (u, w) , $u \in U, w \in W$, is non-zero if there was at least one $v \in V$ so that $\mu_R(u, v) \neq 0$ (or $\mu_{\tilde{R}}(u, v) \neq 1/0$) and $\mu_S(v, w) \neq 0$ (or $\mu_{\tilde{S}}(v, w) \neq 1/0$), in which $1/0$ denotes the concept of zero membership grades in the case of type-2 fuzzy sets.

$$\mu_{\tilde{R} \circ \tilde{S}}(u, w) = \sqcup v \in V [\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{S}}(v, w)] \quad (2.16)$$

where \sqcap and \sqcup are meet and join operations, respectively, which are defined as follows [58]

Example 2.5.2. Consider the fuzzy relations in Example 2.5.1.

It is known that the expression 'x is close to y and y is much smaller than z' indicates the composition $R \circ S$ (or $\tilde{R} \circ \tilde{S}$).

For the type-1 case, using the equation:

$$\begin{aligned} \mu_{R \circ S}(x_i, z_j) = & [\mu_R(x_i, y_1) \wedge \mu_S(y_1, z_j)] \vee [\mu_R(x_i, y_2) \\ & \wedge \mu_S(y_2, z_j)] \vee [\mu_R(x_i, y_3) \wedge \mu_S(y_3, z_j)] \end{aligned} \quad (2.17)$$

where $i = 1, 2, 3$ and $j = 1, 2, 3$. For type-2 fuzzy relation using equation:

$$\begin{aligned} \mu_{R \circ S}(x_i, z_j) = & [\mu_R(x_i, y_1) \sqcap \mu_S(y_1, z_j)] \sqcup [\mu_R(x_i, y_2) \\ & \sqcap \mu_S(y_2, z_j)] \end{aligned} \quad (2.18)$$

$$\begin{aligned} \mu_{R \circ S}(x_1, z_1) = & [\mu_R(x_1, y_1) \sqcap \mu_S(y_1, z_1)] \sqcup [\mu_R(x_1, y_2) \\ & \sqcap \mu_S(y_2, z_1)] \\ = & \left[\left(\frac{0.4}{0.4} + \frac{1}{0.5} + \frac{0.7}{0.6} \right) \sqcap \left(\frac{0.4}{0.8} + \frac{1}{0.9} + \frac{0.6}{1} \right) \right] \sqcup \left[\left(\frac{0.6}{0} + \frac{1}{0.1} + \frac{0.6}{0.2} \right) \right] \\ = & \left[\sqcap \left(\frac{0.5}{0} + \frac{1}{0.1} + \frac{0.6}{0.2} \right) \right] \end{aligned} \quad (2.19)$$

we obtain

$$\tilde{R} \circ \tilde{S} = \begin{bmatrix} \frac{0.4}{0.4} + \frac{1}{0.5} + \frac{0.7}{0.6} & \frac{0.4}{0.4} + \frac{1}{0.5} + \frac{0.7}{0.6} & \frac{0.5}{0} + \frac{1}{0.1} + \frac{0.6}{0.2} \\ \frac{0.3}{0.3} + \frac{1}{1} + \frac{0.5}{0.5} & \frac{0.3}{0.3} + \frac{1}{1} + \frac{0.7}{0.7} & \frac{0.5}{0.5} + \frac{1}{1} + \frac{0.7}{0.7} \\ \frac{0.6}{0.6} + \frac{0.7}{0.7} + \frac{0.8}{0.8} & \frac{0.5}{0.5} + \frac{0.6}{0.6} + \frac{0.7}{0.7} & \frac{0.1}{0.1} + \frac{0.2}{0.2} + \frac{0.3}{0.3} \\ \frac{0.8}{0.8} + \frac{1}{0.9} + \frac{1}{1} & \frac{0.2}{0.5} + \frac{1}{0.6} + \frac{0.5}{0.7} & \frac{0.6}{0.7} + \frac{1}{0.8} + \frac{0.4}{0.9} \end{bmatrix}$$

Properties[70]

Let \tilde{R} , \tilde{S} and \tilde{L} be T2 fuzzy relations defined on product space $U \times V$. If for any $(u, v) \in U \times V$, $\mu_{\tilde{R}}(u, v)$, $\mu_{\tilde{S}}(u, v)$ and $\mu_{\tilde{L}}(u, v)$ are convex T1 fuzzy sets, then we have

$$(\tilde{R} \cup \tilde{S}) \cap \tilde{L} = (\tilde{R} \cap \tilde{L}) \cup (\tilde{S} \cap \tilde{L})$$

and

$$(\tilde{R} \cap \tilde{S}) \cup \tilde{L} = (\tilde{R} \cup \tilde{L}) \cap (\tilde{S} \cup \tilde{L})$$

Theorem 2.5.1. [70] Suppose that \tilde{R} , \tilde{S} and \tilde{L} are T2 fuzzy relations on product spaces $U \times V$, $V \times W$ and $W \times Q$, respectively, and $\mu_{\tilde{R}}(u, v)$, $\mu_{\tilde{S}}(v, w)$ and $\mu_{\tilde{L}}(w, q)$ are convex T1 fuzzy sets. If the spaces V and W are finite, then

$$(\tilde{R} \circ \tilde{S}) \circ \tilde{L} = \tilde{R} \circ (\tilde{S} \circ \tilde{L})$$

Proof 2.5.1. From Definition 1, for any $(u, q) \in U \times Q$, we have

$$\begin{aligned} & \mu_{(\tilde{R} \circ \tilde{S}) \circ \tilde{L}}(u, q) \\ &= \sqcup_{w \in W} [\mu_{\tilde{R} \circ \tilde{S}}(u, w) \sqcap \mu_{\tilde{L}}(w, q)] \\ &= \sqcup_{w \in W} [\sqcup_{v \in V} [\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{S}}(v, w)] \sqcap \mu_{\tilde{L}}(w, q)] \end{aligned}$$

we know $\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{S}}(v, w)$ is a convex T1 fuzzy set in $[0, 1]$

$$\begin{aligned} & \mu_{\tilde{R} \circ (\tilde{S} \circ \tilde{L})}(u, q) \\ &= \sqcup_{v \in V} [\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{S} \circ \tilde{L}}(v, q)] \\ &= \sqcup_{v \in V} [\mu_{\tilde{R}}(u, v) \sqcap \sqcup_{w \in W} [\mu_{\tilde{S}}(v, w) \sqcap \mu_{\tilde{L}}(w, q)]] \\ &= \sqcup_{w \in W} [\sqcup_{v \in V} [\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{S}}(v, w) \sqcap \mu_{\tilde{L}}(w, q)]] \end{aligned}$$

implies $\mu_{(\tilde{R} \circ \tilde{S}) \circ \tilde{L}}(u, q) = \mu_{\tilde{R} \circ (\tilde{S} \circ \tilde{L})}(u, q)$ for any $(u, q) \in U \times Q$

Theorem 2.5.2. Let \tilde{R}, \tilde{S} be two T2 fuzzy relations defined on $U \times V$, and \tilde{L} be a T2 fuzzy relations on $V \times W$. If $\mu_{\tilde{R}}(u, v), \mu_{\tilde{S}}(u, v)$ and $\mu_{\tilde{L}}(v, w)$ are convex fuzzy sets for any $(u, v) \in U \times V$ and $(v, w) \in V \times W$, then we have $(\tilde{R} \cup \tilde{S}) \circ \tilde{L} = (\tilde{R} \circ \tilde{L}) \cup (\tilde{S} \circ \tilde{L})$

Proof. On the one hand, the LHS of Eq. (28) can be expressed as

$$\begin{aligned} & \mu_{(\tilde{R} \cup \tilde{S}) \circ \tilde{L}}(u, w) \\ &= \bigsqcup_{v \in V} [\mu_{\tilde{R} \cup \tilde{S}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)] \\ &= \bigsqcup_{v \in V} [[\mu_{\tilde{R}}(u, v) \sqcup \mu_{\tilde{S}}(u, v)] \sqcap \mu_{\tilde{L}}(v, w)] \end{aligned}$$

for any $(u, w) \in U \times W$ On the other hand, can be written

$$\mu_{(\tilde{R} \circ \tilde{L}) \cup (\tilde{S} \circ \tilde{L})}(u, w) = \mu_{\tilde{R} \circ \tilde{L}}(u, w) \sqcup \mu_{\tilde{S} \circ \tilde{L}}(u, w) = \bigsqcup_{v \in V} [\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)] \sqcup \bigsqcup_{v \in V} [\mu_{\tilde{S}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)]$$

since arbitrary fuzzy sets in $[0, 1]$ satisfy associative and commutative laws under \sqcup [1], we have

$$\begin{aligned} & \bigsqcup_{v \in V} [\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)] \sqcup \bigsqcup_{v \in V} [\mu_{\tilde{S}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)] \\ &= \bigsqcup_{v \in V} [[\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)] \sqcup [\mu_{\tilde{S}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)]] \end{aligned}$$

Furthermore, noting that $\mu_{\tilde{R}}(u, v), \mu_{\tilde{S}}(u, v)$ and $\mu_{\tilde{L}}(v, w)$ are convex fuzzy sets in $[0, 1]$, form [1], they satisfy distributive laws under \sqcup and \sqcap . Therefore, we get

$$\begin{aligned} & \bigsqcup_{v \in V} [[\mu_{\tilde{R}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)] \sqcup [\mu_{\tilde{S}}(u, v) \sqcap \mu_{\tilde{L}}(v, w)]] \\ &= \bigsqcup_{v \in V} [[\mu_{\tilde{R}}(u, v) \cup \mu_{\tilde{S}}(u, v)] \sqcap \mu_{\tilde{L}}(v, w)] \end{aligned}$$

we obtain

$$\mu_{(\tilde{R} \cup \tilde{S}) \circ \tilde{L}}(u, w) = \mu_{(\tilde{R} \circ \tilde{L}) \cup (\tilde{S} \circ \tilde{L})}(u, w)$$

for any $(u, w) \in U \times W$. Thus,

$$(\tilde{R} \cup \tilde{S}) \circ \tilde{L} = (\tilde{R} \circ \tilde{L}) \cup (\tilde{S} \circ \tilde{L})$$

Similarly, we have the following result. □

Theorem 2.5.3. Let \tilde{R}, \tilde{S} be two T_2 fuzzy relations defined on $U \times V$, and \tilde{L} be a T_2 fuzzy relations on $W \times U$. If $\mu_{\tilde{R}}(u, v), \mu_{\tilde{S}}(u, v)$ and $\mu_{\tilde{L}}(w, u)$ are convex T_1 fuzzy sets for any $(u, v) \in U \times V$ and $(w, u) \in W \times U$, then we have

$$\tilde{L} \circ (\tilde{R} \cup \tilde{S}) = (\tilde{L} \circ \tilde{R}) \cup (\tilde{L} \circ \tilde{S})$$

Remark 2.5.1. From Theorem 5 and 6, we know that T_2 fuzzy relations satisfy distributive law under \cup and \circ . However, the distributive law will not hold under \cap and \circ .

Consequence 2.5.1.

$$(\tilde{R} \cap \tilde{S}) \circ \tilde{L} \neq (\tilde{R} \circ \tilde{L}) \cap (\tilde{S} \circ \tilde{L})$$

2.6 Alpha Planes

An α plane for a generalized type-2 fuzzy set [18] \tilde{A} is denoted by \tilde{A}_α and can be defined as the union of all primary membership functions of \tilde{A} , where the secondary membership degrees are greater or equal to α ($0 \leq \alpha \leq 1$) and can be represented by (2.10) [42], [53], and can be graphically illustrated in Fig. 2.4.

$$\begin{aligned} \tilde{A}_\alpha &= \{(x, u), \mu_{\tilde{A}}(x, u) \geq \alpha \mid \forall x \in X, \forall u \in J_x \subseteq [0, 1]\} \\ &= \int_{\forall x \in X} \int_{\forall u \in J_x} \{(x, u) \mid \mu_{\tilde{A}}(x, u) \geq \alpha\} \end{aligned}$$

where the union of all the α alpha planes is represented by (2. 11). and $R_{\tilde{A}_\alpha}$ represents a horizontal slice. In Fig. 2.3 we can appreciate the α alpha planes made for generalized membership functions.

Also in Fig. 2.4 we can appreciate the representation of the three α planes in different points, the first α plane was realized in the point 0.17 (blue), the second α plane was in 0.46 (green), and the third in point 0.99 (red).

$$\tilde{A} = \bigcup_{\alpha \in [0,1]} R_{\tilde{A}_\alpha} \quad (2.20)$$

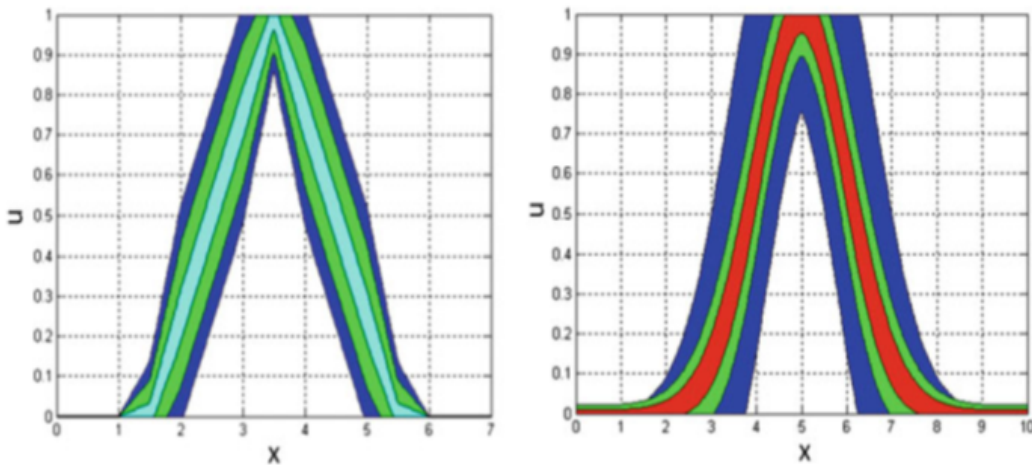


Figure 2.6: α plane in generalized type-2 membership functions[18]

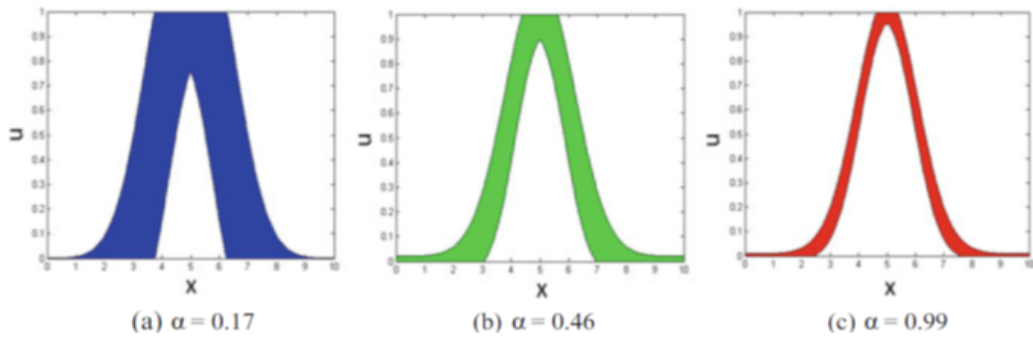


Figure 2.7: Cuts at certain points of the GMF[18]

Chapter 3

Fuzzy logic control of non-linear dynamic system

3.1 Introduction

The dynamic system is an aspect of systems theory used as a method to understand the dynamic behavior of complex systems. It has found application in a wide range of areas, for example population, agriculture, ecological and economic systems, which usually interact strongly with each other.

First, some notions are defined in this chapter which are the dynamic system and its control and the fuzzy control system. Then, we have presented Puma 560 robot modeling. In the end, a detailed study of some works of fuzzy logic controller of the non-linear dynamic system.

3.2 Dynamic system definition

3.2.1 General definition

A dynamic system is a system the state of which evolves with time over a state space according to a fixed rule. A dynamic system is a system or process in which motion occurs or includes active forces as opposed to static conditions with no motion. A dynamic system is a system that is constantly changing like the human body. The dynamic system stands to become static or at an equilibrium state. For example, if a car is assumed to be a dynamic system, then it requires fuel to continue moving forward or else it would come to a stop and become static. Dynamic systems output depends upon future and past values.

3.2.2 Formal definition of dynamical system

A dynamical system is formally defined as a state space X , a set of times T , and a rule R that specifies how the state evolves with time. The rule R is a function the domain of which is $X \times T$ and the codomain of which is X , i.e., $R : X \times T \rightarrow X$. The rule function R means that the R takes two inputs, $R = R(x, t)$, where $x \in X$ is the initial state (at time $t = 0$, for example) and $t \in T$ is a future time. In other words, $R(x, t)$ gives the state at time t knowing that the initial state was x

3.3 The dynamic system control

the control of a dynamic system was originally applied to a system that controls a mechane or motor. It is used to describe a control system in which a physical variable is required to follow or track.

3.4 Advantages of using Fuzzy Logic in Control Systems

A control system is an arrangement of physical components designed to alter another physical system so that it exhibits certain desired characteristics. Some reasons for using Fuzzy Logic in Control Systems are as follows:

- While applying traditional control, one needs to know about the model and the objective function formulated in precise terms. This makes it very difficult to apply in many cases.
- By applying fuzzy logic for control we can utilize the human expertise and experience for designing a controller.
- The fuzzy control rules, basically the IF-THEN rules, can be best utilized in designing a controller.

3.5 Fuzzy logic control

[4] The basic structure of a fuzzy logic control (FLC) system is shown in Figure 3.1. Fuzzy logic controller (FLC) used in this thesis is Mamdani.

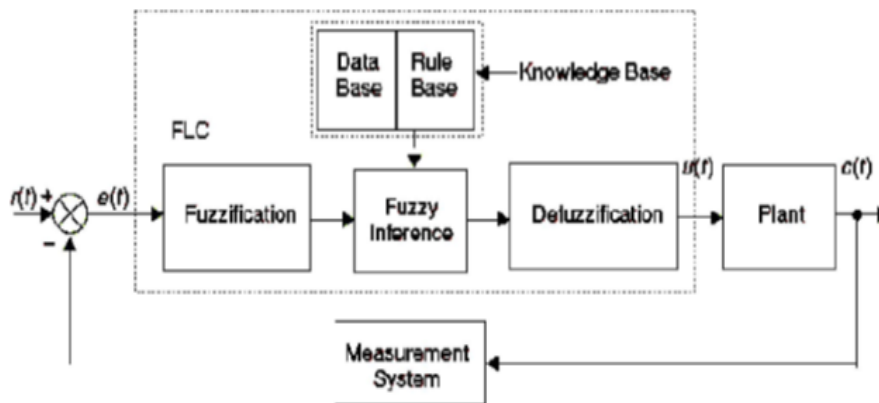


Figure 3.1: Block diagram of joint space control

3.5.1 The fuzzification process

Fuzzification is the process of mapping inputs to the FLC into fuzzy set membership values in the various input universes of discourse. Decisions need to be made regarding

- (a) Number of input

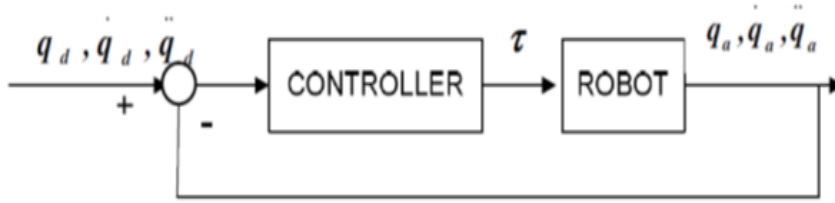


Figure 3.2: Fuzzy logic control
[60]

- (b) Size of universes of discourse
- (c) Number and shape of fuzzy sets.

A FLC that emulates a PD controller will be required to minimize the error $e(t)$ and the rate of change of error de/dt , or ce .

Each set is given a linguistic label to identify it, such as Positive Big (PB), Positive Medium (PM), Positive Small (PS), About Zero (Z), Negative Small (NS), Negative Medium (NM) and negative Big (NB). The seven set fuzzy input windows for e and ce are shown in Figure 3.3. If at a particular instant, $e(t) = 2.5$ and $de/dt = ce = -0.2$, then, from Figure 3.3, the input fuzzy set membership values are:

$$\begin{aligned} \mu_{PS}(e) &= 0.7 & \mu_{PM}(e) &= 0.4 \\ \mu_{NS}(ce) &= 0.6 & \mu_Z(ce) &= 0.3 \end{aligned} \quad (3.1)$$

3.5.2 The fuzzy rule base

The fuzzy rule base consists of a set of antecedent-consequent linguistic rules of the form

$$\text{IF } e \text{ is PS AND } ce \text{ is NS THEN } u \text{ is PS} \quad (3.2)$$

The size of the universes of discourse will depend upon the expected range (usually up to the saturation level) of the input variables. Assume for the system about to be considered that e has a range of ± 6 and ce a range of ± 1 .

The number and shape of fuzzy sets in a particular universe of discourse is a trade-off between the precision of control action and real-time computational complexity. In this example, seven triangular sets will be used.

This style of a fuzzy conditional statement is often called a Mamdani-type rule, after Mamdani (1976) who first used it in a fuzzy rule base to control steam plant.

The rule base is constructed using a priori knowledge from either one or all of the following sources:

- (a) Physical laws that govern the plant dynamics
- (b) Data from existing controllers
- (c) Imprecise heuristic knowledge obtained from experienced experts.

If (c) above is used, then knowledge of the plant mathematical model is not required. The two seven set fuzzy input windows shown in Figure 3.3 gives a possible 7×7 set of control rules of the form given in equation (3.2). It is convenient to tabulate the two-dimensional rule base as shown in table 3.1.

3.5.3 Fuzzy inference

[74] Table 3.1 assumes that the output window contains seven fuzzy sets with the same linguistic labels as the input fuzzy sets. If the universe of discourse for the control signal $u(t)$ is ± 9 , then the output window is as shown in Figure 3.4.

Assume that a certain rule in the rule base is given by equation (3.1)

$$\text{IF } e \text{ is } A \text{ AND } ce \text{ is } B \text{ THEN } u = C \quad (3.3)$$

From equation (1.4) the Boolean OR function becomes the fuzzy max operation, and from equation (1.5) the Boolean AND function becomes the fuzzy min operation. Hence equation (3.3) can be written in the following:

$$\mu_C(u) = \max[\min(\mu_A(e), \mu_B(ce))] \quad (3.4)$$

equation (3.4) is referred to as the max-min inference process or max-min fuzzy reasoning,

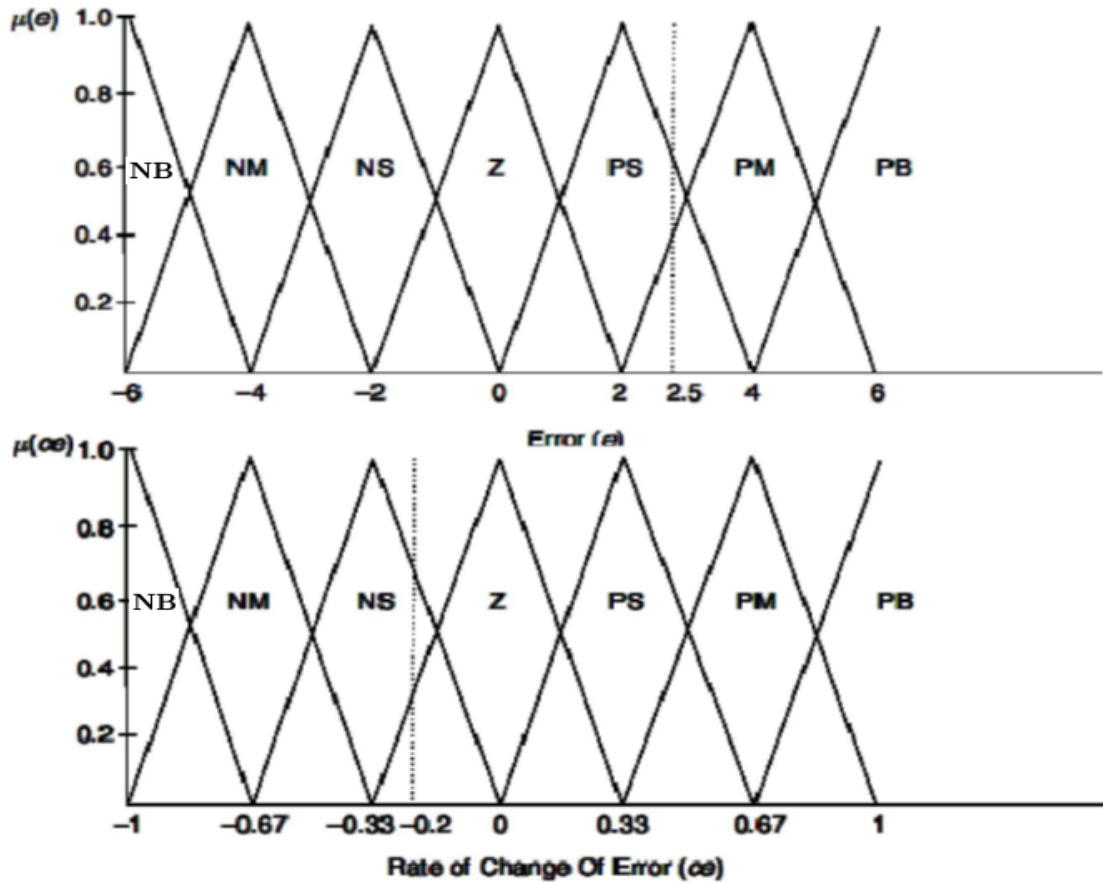


Figure 3.3: Seven set fuzzy input windows for error e and errata change of error (ce)

In Figure 3.3 and equation (3.1) the fuzzy sets that were 'hit' in the error input window when $e(t) = 2.5$ were PS and PM. In the rate of change input window when $ce = -0.2$, the fuzzy sets to be 'hit' were NS and Z. From table 3.1. the relevant rules that correspond to these 'hits' are:

$$\begin{aligned} &\dots \text{OR IF } e \text{ is } PS \text{ AND } ce \text{ is } NS \\ &\text{OR IF } e \text{ is } PS \text{ AND } ce \text{ is } Z \\ &\text{THEN } u = PS \end{aligned} \quad (3.5)$$

Table 3.1: Tabular structure of a linguistic fuzzy rulebase

$\begin{matrix} e \\ \backslash \\ ce \end{matrix}$	NB	NM	NS	Z	PS	PM	PB
NB	NB	NB	NB	NM	Z	PM	PB
NM	NB	NB	NB	NM	PS	PM	PB
NS	NB	NB	NM	NS	PS	PM	PB
Z	NB	NM	NS	Z	PS	PM	PB
PS	NB	NM	NS	PS	PM	PB	PB
PM	NB	NM	NS	PM	PB	PB	PB
PB	NB	NM	Z	PM	PB	PB	PB

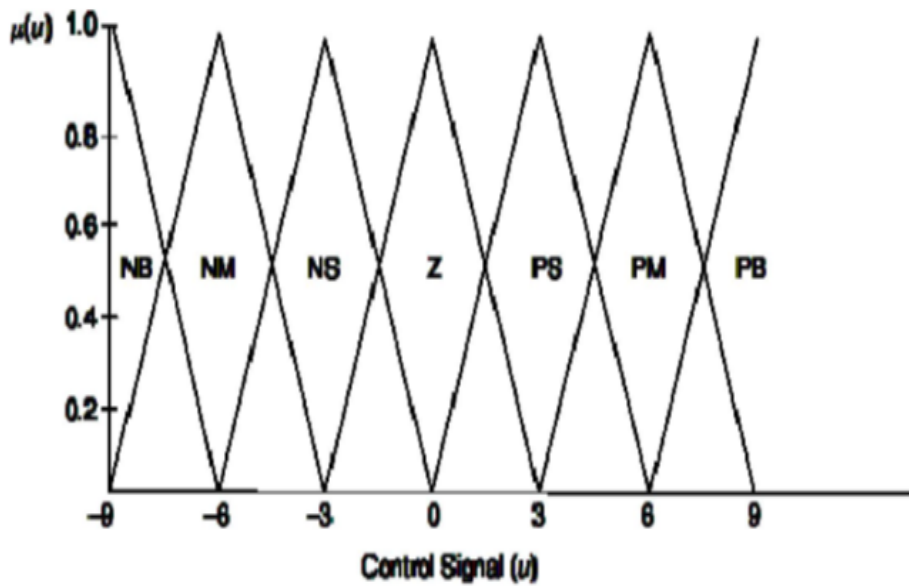


Figure 3.4: Seven set fuzzy output window for signal control (u)

$$\begin{aligned}
 & \dots \text{ OR IF } e \text{ is PM AND } ce \text{ is NS} \\
 & \text{ OR IF } e \text{ is PM AND } ce \text{ is Z} \\
 & \text{ THEN } u = \text{PM}
 \end{aligned} \tag{3.6}$$

Applying the max-min inference process to equation (3.5)

$$\mu_{PS}(u) = \max[\min(\mu_{PS}(e), \mu_{NS}(ce)), \min(\mu_{PS}(e), \mu_z(ce))] \tag{3.7}$$

Inserting values from equation (3.1)

$$\begin{aligned}
 \mu_{PS}(u) &= \max[\min(0.7, 0.6), \min(0.7, 0.3)] \\
 &= \max[0.6, 0.3] = 0.6
 \end{aligned} \tag{3.8}$$

Applying the max-min inference process to equation (3.6)

$$\mu_{PM}(u) = \max[\min(\mu_{PM}(e), \mu_{NS}(ce)), \min(\mu_{PM}(e), \mu_z(ce))] \tag{3.9}$$

Inserting values from equation (3.1)

$$\begin{aligned}
 \mu_{PM}(u) &= \max[\min(0.4, 0.6), \min(0.4, 0.3)] \\
 &= \max[0.4, 0.3] = 0.4
 \end{aligned} \tag{3.10}$$

Fuzzy inference is, therefore, the process of mapping membership values from the input windows, through the rule base, to the output window(s).

3.5.4 The defuzzification

Defuzzification is the procedure for mapping from a set of inferred fuzzy control signals contained within a fuzzy output window to a non-fuzzy (crisp) control signal. The center of area [68], [69], [12] method is the most famous defuzzification technique, which in linguistic terms can be expressed as follows:

$$\text{Crisp control signal} = \frac{\text{Sum of first moments of area}}{\text{Sum of areas}} \quad (3.11)$$

For a continuous system, equation (3.11) becomes:

$$u(t) = \frac{\int u\mu(u)du}{\int \mu(u)du} \quad (3.12)$$

Or alternatively, for a discrete system, equation (3.11) can be expressed as:

$$u(t) = \frac{\sum_{i=1}^n u_i\mu(u_i)}{\sum_{i=1}^n \mu(u_i)} \quad (3.13)$$

For the case when $e(t) = 2.5$ and $ce = -0.2$, as a result of the max-min inference process (equations (3.8) and (3.10)), the fuzzy output window in Figure 3.4 is 'clipped', and takes the form shown in Figure 3.5.

From Figure 3.5, using the equation for a trapezoid area:

$$\begin{aligned} \text{Area}_{PS} &= \frac{0.6(6+2.4)}{2} = 2.52 \\ \text{Area}_{PM} &= \frac{0.2(6+3.6)}{2} = 0.96 \end{aligned} \quad (3.14)$$

From equation (3.11)

$$u(t) = \frac{(2.52 \times 3) + (0.96 \times 6)}{2.52 + 0.96} = 3.83 \quad (3.15)$$

Hence, the error of 2.5, and an errata change of error of -0.2 , the control signal from the fuzzy controller is 3.83.

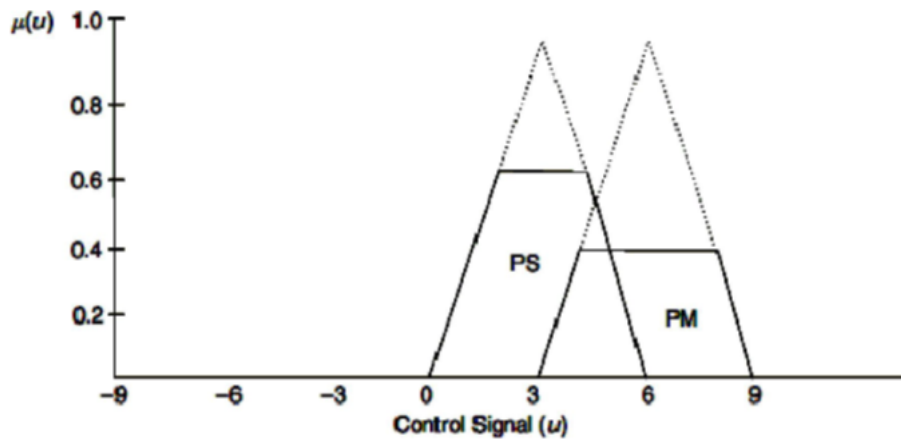


Figure 3.5: Clipped fuzzy output window due to fuzzy inference

Example 3.5.1. For the input and output fuzzy windows given in Figure 3.3 and 3.4, together with the fuzzy rule base shown in table 3.1, determine:

(a) The membership values of the input windows e and ce .

(b) The max-min fuzzy inference equation

(c) The crisp control signal $\mu(t)$.

when $e = -3$ and $ce = 0.3$

Solution

(a) When $e = -3$ and $ce = 0.3$ are mapped onto the input fuzzy windows, they refer to a fuzzy singletons. From Figure 3.3

$$e = -3 \quad \mu_{NS}(e) = 0.5 \quad \mu_{NM}(e) = 0.5 \quad (3.16)$$

$ce = 0.3$, using similar triangles :

$$\frac{1}{0.33} = \frac{\mu_z(ce)}{(0.33-0.3)} \quad (3.17)$$

$$\mu_z(ce) = 0.09$$

And

$$\frac{1}{0.33} = \frac{\mu_{PS}(ce)}{0.3} \quad (3.18)$$

$$\mu_{PS}(ce) = 0.91$$

(b) The rules that are 'hit' in the rule base in table 3.1 are

$$\begin{aligned} & \dots \text{OR IF } e \text{ is NS AND } ce \text{ is Z} \\ & \text{OR IF } e \text{ is NS AND } ce \text{ is PS} \\ & \text{THEN } u = \text{NS} \end{aligned} \quad (3.19)$$

$$\begin{aligned} & \dots \text{OR IF } e \text{ is NM AND } ce \text{ is Z} \\ & \text{OR IF } e \text{ is NM AND } ce \text{ is PS} \\ & \text{THEN } u = \text{NM} \end{aligned} \quad (3.20)$$

Applying max-min inference to equation (3.19)

$$\mu_{NS}(u) = \max[\min(\mu_{NS}(e), \mu_z(ce)), \min(\mu_{NS}(e), \mu_{PS}(ce))] \quad (3.21)$$

Inserting values into (3.21) :

$$\begin{aligned} \mu_{NS}(u) &= \max[\min(0.5, 0.09), \min(0.5, 0.91)] \\ &= \max[0.09, 0.5] = 0.5 \end{aligned} \quad (3.22)$$

and similarly with equation (3.20)

$$\begin{aligned} \mu_{NM}(u) &= \max[\min(\mu_{NM}(e), \mu_z(ce)), \min(\mu_{NM}(e), \mu_{PS}(ce))] \\ \mu_{NM}(u) &= \max[\min(0.5, 0.09), \min(0.5, 0.91)] \\ &= \max[0.09, 0.5] = 0.5 \end{aligned} \quad (3.23)$$

Using equations (3.22) and (3.23) to 'clip' the output window in Figure 3.4, the output window is now as illustrated in Figure 3.6.

(c) Due to the symmetry of the output window in Figure 3.6, from observation the crisp control signal is :

$$u(t) = -4,5$$

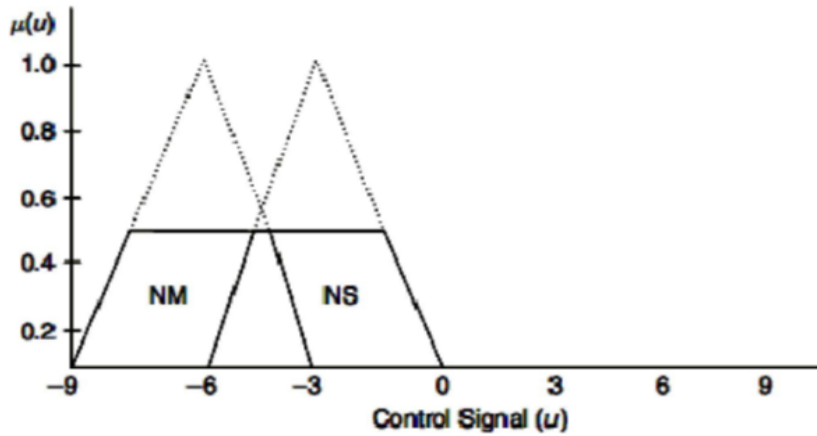


Figure 3.6: Fuzzy output window for to Example 3.5.1.

3.6 Puma560 robot dynamic modelling

This paragraph describes the PUMA 560 industrial robotic system that will be used in our research work as an example of dynamic system ; highlighting its main components , how they interact with each other and its dynamic modelisation.

3.6.1 General Overview of PUMA 560 Robot

[29]

The PUMA 560 robotic manipulator, produced by Unimate, is an old (> 25 years), but functional multi-joint robot and was used in this project, due to spare parts availability that are required for simulation of the different faults. It is a PC controlled serial manipulator designed for use in industrial applications, and has six revolute joints/degree of freedom (DOF) with three major axes of motion (X, Y , and Z) and resembles the human arm in function. Each of the robot's joints is actuated by a DC brushed permanent magnet servo motor. Positioning of the end effector, which is normally a device fixed at the end of the robotic arm, is achieved by the coordination of the first three joints named: waist (joint 1), shoulder (joint 2) and elbow (joint 3), allowing the robot to move the end effector into any position with maximum reach of 1 m. Orientation of the end effector is important when it approaches its final position and this is achieved using three wrist joints (4, 5, and 6) where the tool can be independently manoeuvred. This gives the robot six degrees of freedom as shown in Fig. 3.7 where the name of each joint with its maximum rotating range is indicated. Electromagnetic brakes are equipped with the first three joints (waist, shoulder, and elbow), which lock the motors to prevent collapsing when the power is removed from the robot (Potgieter et al. 2005). The control unit of the PUMA 560 is the most important part of the robotic system. It controls all operations of the manipulator and any connected device. There is two-way communication to each of the joints motors where sensory

information on position and speed is fed back to the control unit; outputs are sent to the motors related to the task's requirements. External digital input/output ports are available allowing other sensors and actuators to be controlled from this unit. There is a number of ways to position and orientate the manipulator end effector, but generally, one method is used at a time.

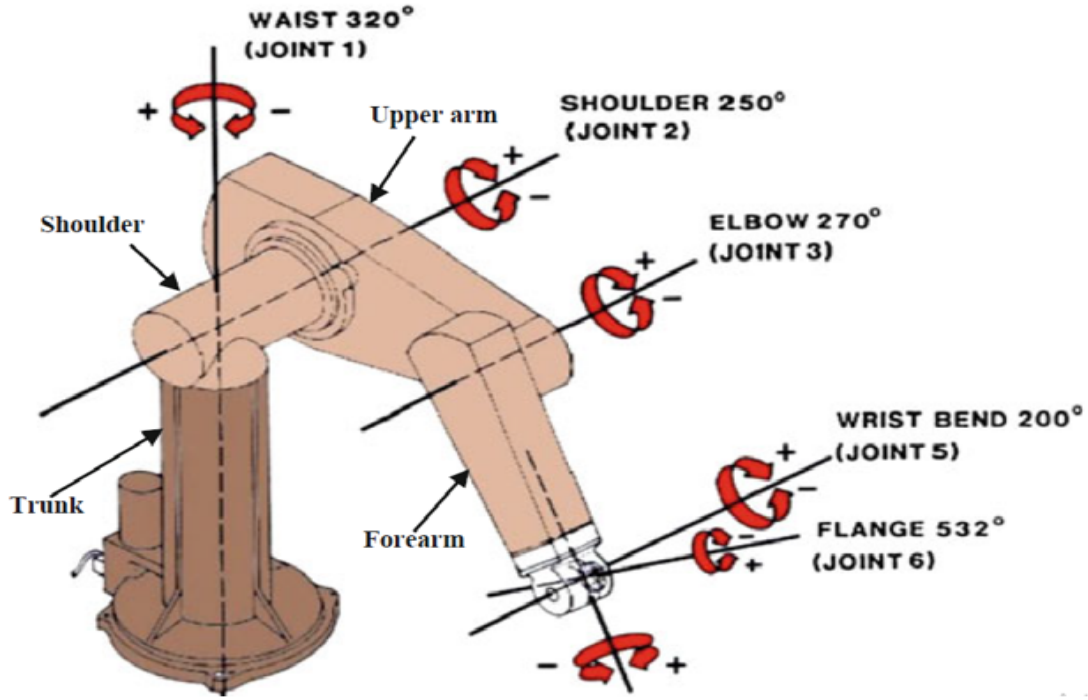


Figure 3.7: PUMA 560 robot member representations [61]

3.6.2 Puma 560 robot Dynamic Modelling

The dynamic model used for PUMA560 is taken from [3]. We have adopted a robot of three degrees of freedom and the configuration of space equation is given by:

$$M(q)\ddot{q} + B(q)[\dot{q}\dot{q}] + C(q)[\dot{q}^2] + G(q) = \Gamma \quad (1)$$

where:

$M(q)$: is 3×3 kinetic energy matrix.

$[\ddot{q}]$: 3×1 vector given by: $\ddot{q} = [\ddot{q}_1, \ddot{q}_2, \ddot{q}_3]^T$

$B(q)$: 3×1 matrix of Coriolis torques,

$[\dot{q}\dot{q}]$: 7×1 vector of joint velocity products given by:

$[\dot{q}\dot{q}] = [\dot{q}_1\dot{q}_2, \dot{q}_1\dot{q}_3, 0, 0, 0, \dot{q}_2\dot{q}_3, 0]^T$

$C(q)$: 3×1 vector given by: $[\dot{q}^2] = [\dot{q}_1^2, \dot{q}_2^2, \dot{q}_3^2]^T$

$G(q)$: is 3×1 vector of gravity torques,

Γ : 3×1 vector of joint torque.

\ddot{q} is the n-vector of accelerations;

3.7 Definition of a robot system controller

The controller is a device which can sense information from the linear or nonlinear system (e.g., robot manipulator) to improve the system's performance [9]. The main targets in designing control systems are stability, good disturbance rejection, and small tracking error [27]. Several industrial robot manipulators are controlled by linear methodologies (e.g., Proportional-Derivative (PD) controller, Proportional-Integral (PI) controller or Proportional-Integral-Derivative (PID) controller), but when the robot manipulator works with various payloads and has uncertainty in dynamic models this technique has limitations.

From the control point of view, uncertainty is divided into two main groups: uncertainty in unstructured inputs (e.g., noise, disturbance) and uncertainty in dynamics structure (e.g., payload, parameter variations). In some applications robot manipulators are used in an unknown and unstructured environment, therefore strong mathematical tools are used in new control methodologies to design nonlinear robust controller with an acceptable performance (e.g., minimum error, good trajectory, disturbance rejection). Joint space and operational space control are closed loop controllers that have been used to provide robustness and rejection of disturbance effect. The main target in the joint space controller is design a feedback controller that allows the actual motion ($q_a(t)$) tracking of the desired motion ($q_d(t)$) by inverse kinematics of robot manipulators [46]. Figure 3.8 shows the main block diagram of joint space controller. The main target in the operational space controller is to design a feedback controller to allow the actual end-effector motion $X_a(t)$ to track the desired end effector motion $X_d(t)$. This control methodology requires a greater algorithmic complexity and the inverse kinematics used in the feedback control loop as well. Direct measurement of operational space variables is very expensive that led to limitate the controller use in industrial robot manipulators[46]. Figure 3.9 shows the main block diagram of operational space control.

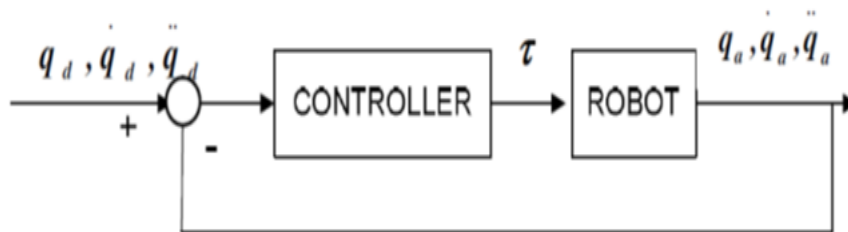


Figure 3.8: Block diagram of joint space control [60]

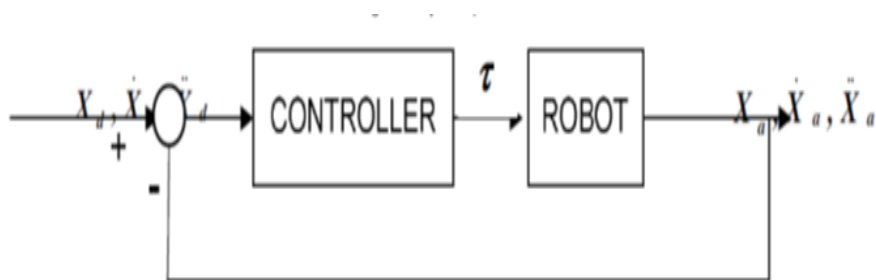


Figure 3.9: Block diagram of operational space control [60]

3.8 State of the art of fuzzy logic controller of nonlinear dynamic system

The subject of a fuzzy logic controller has attracted the attention of researchers, plenty of research works have been done.[15].-12 [39] The use of industrial robots became identifiable as a unique device in the 1960s. Since then, the field of application evolved from rather simple tasks like welding and painting to those requiring more precision, such as assembly tasks.

Control theory provides tools for designing and evaluating algorithms to achieve desired motions or force application. The methods of linear control and those of local linearization and moving linearization are not well suited for the control problem of robotic arms. This is due to the fact that robotic arms constantly move among widely separated regions of their workspace such that no valid linearization can be found in all regions. On the other hand, nonlinear control methods are progressing nowadays and different classes could be identified[65]: Trialand- Error, Feedback Linearization Control, Robust Control, Adaptive Control, and Gain-Scheduling. Nonlinear control methods used in robot arms' applications should, however, face the major difficulty resulting from the dynamic modeling of robots, i.e. the indetermination of their parameters [9]. Preferred methods are those which reduce or eliminate the undesired effects generated by this indetermination such as the Feedback Linearization control method [65], the Model-Reference Adaptive control method [[66],[65]] and the Self-Tuning method[65]. Another difficulty in robot arm control is the coupling effects of the Coriolis and centrifugal forces that might be canceled in a single axis mode operation where the joints are activated sequentially. Existing methods of nonlinear control are also used in robotics in order to eliminate the above-mentioned coupling effect like the Individual Joint PID control method [9] and the PD-plus-gravity control method [9]. Among the recent nonlinear control methods, fuzzy control methods grab nowadays the attention of many researchers. In fact, these methods do not require the knowledge of the dynamic model of the controlled system and this feature has become of major importance when dealing with complex nonlinear systems. Moreover, the dynamic modeling of robot arms shows a dependency on their mechanical parameters, subject to lifetime modifications (friction factors affected by the abuse of joints), and on their dynamical parameters that vary with the performed task (centers of gravity of the links affected by tool's replacements). These considerations also give advantage to fuzzy control methods on other nonlinear methods as a result of their robustness towards perturbations affecting the system.

The first fuzzy logic controller was introduced by Mamdani in 1974[44]. It is equivalent to two-input fuzzy PI controllers, where error and change of error were used as the inference system inputs. Mamdani's work also introduced the most common and robust fuzzy reasoning method called "Zadeh-Mamdani min-max gravity reasoning". Different comparative studies, like [27], prove that Zadeh-Mamdani min-max gravity scheme is the best reasoning scheme if the nonlinearity variation is the main concern. It also introduced the most common and robust fuzzy reasoning method, called Zadeh-Mamdani min-max gravity reasoning. [40] Robots are mechanized devices that have a certain degree to duplicate the working of humans, whenever they need continuous operations. Furthermore, they provide more accuracy, high strength and reduce the danger in the case of medical fields, industrial operations and nuclear plants (Spong & Vidyasagar, 2008). Controlling the motion of a manipulator robot's joints is a running issue the study of which is of great importance in order to make its terminal effectors follow a predefined trajectory with the minimum deviation (Spong & Fujita, 2011). Robotic manipulator systems are containing unavoidable uncertainties such as structured and unstructured uncertainties that degraded manipulators

performance. Correct dynamical model with variation parameters, sizes and mass distributions of payloads, difference in links properties of robotic manipulators are defined as structured uncertainty present in robotic manipulators. Unstructured uncertainty is defined as unmodelled dynamics, which contain the external disturbances, nonlinear frictions and random noise (Song, Yi, Zhao, & Li, 2005). To combat such challenges, it is essential to develop a robust controller for these systems. Controlling of robot manipulator requires the complex dynamic equation (mathematical modeling) and trajectory planning, which is computationally intensive. The conventional control theories such as PID have been successfully applied to areas where systems are well defined. However, when the system is complex, non-linear, ill-structured process with excessive variation parameter, the potency of conventional control becomes poor. With technology advancement, there are imperative needs to design controllers that are able to manage structured and unstructured uncertainty (Laxmidhar & Kar, 2010). The fuzzy set technique given by Zadeh has become a powerful modeling tool for solving severe real-world problems with an uncertain and unpredictable environment. The FLCs (fuzzy logic controllers) come in the category of intelligent control and as an intelligent controller, FLCs (type-1 fuzzy logic controllers i.e., T1FLCs) parameter is easily tuned by non-expertise person. The operation of T1FLCs is based on human expertise and the knowledge acquisition techniques to convert human expertise to appropriate if-then rules as well as a proper fuzzy membership function for each fuzzy variable. The T1FLC has some important advantages such as, [23] it provides a higher level of automation by incorporating expert knowledge, [10] it does not require exact knowledge of the controlled system dynamics model, [14] it plays a major role while controlling complex non-linear systems, [6] it reduces development and maintenance time. There are some drawbacks of T1FLCs, i.e. it can't fully deal with or handle the linguistic and numerical uncertainties connected with dynamic structured or unstructured environments. In addition to that, its performance is not satisfactory, because the ordinary T1FLCs have limited capabilities to directly handle data uncertainties (Mendel, 2007). There are generally five sources of uncertainties observed in T1FLC (Hagras, 2004, 2007; Mendel & John, 2002), which are listed here [23] Uncertainties in the inputs to the T1FLC are produced by noise and change in environmental conditions of sensors. These uncertainties are translated into uncertainties in the antecedents membership functions (MFs) and that the input sensors can be affected by the conditions of observation i.e., their characteristics can be changed by the environmental conditions such as wind, sunshine, humidity, rain, etc. [10] The output of T1FLC is applied to actuators to control the plant.

[15] Generation of a short and smooth path in three-dimensional space with obstacles for guiding an Unmanned Underwater Vehicle (UUV) without collision is investigated. This is done by utilizing the spline technique, in which the spline control points positions are determined by Imperialist Competitive Algorithm (ICA) in three-dimensional space such that the shortest possible path from the starting point to the target point without colliding with obstacles is achieved. Furthermore, for guiding the UUV in the generated path, an Interval Type-2 Fuzzy Logic Controller (IT2FLC), the coefficients of which are optimized by considering an objective function that includes quadratic terms of the input forces and state error of the system, is used. Selecting such objective function reduces the control error and also the force applied to the UUV, which consequently leads to a reduction of energy consumption. Therefore, by using a special method, desired signals of UUV state are obtained from a generated three-dimensional optimal path such that tracking these signals by the controller leads to the tracking of this path by UUV. In this thesis, the dynamical model of the UUV, entitled as "mUUV-WJ-1", is derived and its hydrodynamic coefficients are calculated by CFD in order to be used in the simulations. concerning simulation by the

method presented in this study, three environments with different obstacles are intended in order to check the performance of the IT2FLC controller in generating optimal paths for guiding the UUV. In this article, besides to ICA, Particle Swarm Optimization (PSO) and Artificial Bee Colony (ABC) are also used for the generation of the paths and the results are compared with each other. The results show the appropriate performance of ICA rather than ABC and PSO. Moreover, to evaluate the performance of the IT2FLC, optimal Type-1 Fuzzy Logic Controller (T1FLC), and Proportional Integrator Differentiator (PID) controller are designed and applied to the UUV then compared to each other. The simulation results show the superiority of the IT2FLC over the other two controllers.

[74] Interval type-2 fuzzy logic controllers (IT2 FLCs) have recently been attracting a lot of research attention. Many reported results have shown that IT2 FLCs are more capable to handle uncertainties than their type-1 (T1). For example, Hagraas implemented a hierarchical IT2 FLC for different types of mobile robots navigating in indoor and outdoor environments. It outperformed a T1 FLC and had significantly fewer rules. Wu and Tan showed through both simulation and experiments that IT2 FLCs are more performing in coping with modeling uncertainties, and hence, IT2 FLCs optimized from simulations are more likely to perform well on the actual plant than T1 FLCs.

[19] After the inception of robotics, the robot manipulators are widely used as the promising device in the automated industrial applications including nuclear and medical fields. These manipulators are incorporated to enhance flexibility, productivity, and accuracy in accordance with reduced working cost. It also increased human working conditions where repetitive and hazardous works are executed [67], [8]. For exceptional utility, it is entailed to meticulously design efficient robust controllers that can command the robotic motion in a precise way. Some preliminary work had stated that the conventional proportional- integral-derivative (PID) controllers were used in industrial applications because of their elementary design, easy implementation, and low cost, whereas, recent studies in this field found that they are ineffective for complicated systems such as robotic manipulators with uncertainty and nonlinearity. Progressively, various control strategies and controller structures have been suggested in the past few decades to enhance the performance of systems under control, within this framework. The incorporation of fuzzy logic systems (FLS) in control theory, producing fuzzy logic controllers (FLC), has enhanced the flexibility of controller design and increased its applicability to control the complex, nonlinear, ill-structured, and uncertain systems. Thereafter, the FLC (later, called as type-1 fuzzy logic controllers T1-FLC) have received considerable interest and the control designers recommend T1- FLC over the conventional counterparts because of their unique characteristics such as (a) incorporation of knowledge based on human expertise, (b) no exact dynamic model of system is required hence useful when precise mathematical formulation is infeasible, (c) low development and maintenance cost, and importantly, (d) the ability of general framework of T1-FLC to handle uncertainty, etc. In literature, various authors proposed different T1-FLC structures and their extensions for numerous types of robotic applications. Subsequently, over the past few years, a growing number of researchers have proposed various designs and strategies of T1-FLCs and investigated them for numerous applications for developing effective controllers [37] The usage of the flexible joint manipulator (FJM) as an alternative with rigid body manipulators in the robotics industry such as service, space, medical and defense because the FJM provides faster movements, low manufacturing cost and, high weight to payload ratio as well as low energy consumption compared to the rigid-body structures. Despite these advantages, the flexibility of the joint cause vibrations (link deflections) and position errors. However, convenient control approaches may overcome these problems in the FJM. During the last decade, various linear and non-linear control methods using model-based

and model-free approaches have been presented for the trajectory and vibration control of the FJM. Proportional-Integral-Derivative (PID) state feedback, Linear Quadratic Regulator (LQR) and Linear Quadratic Gaussian (LQG), Intelligent Proportional-Integral (IPI) and distributed higher-order differential feedback controllers are some of the linear control methods applied in the literature. These linear controllers naturally showed overshoot and high link deflection because of their performance was limited by to only small regions around the equilibrium or operating points of the nonlinear FJM system. **conclusion**

Chapter 4

Fuzzy logic controller for puma560 robot: simulation and results discussion

4.1 Introduction

In this chapter, we will describe in details the fuzzy logic controllers (type1 fuzzy sets and interval type2 fuzzy sets) proposed to control the PUMA560 manipulator robot then a comparison of the simulation results of both approaches is presented. As a performance measure we will use the Integral of Square Error (ISE) and Integral of Square Torque (IST)[64] Fuzzy logic controller (FLC) which is used in this research is the Mamdani method, or known as the Max-Min method.

4.2 General description of the fuzzy logic system used in control of Puma560

The dynamic parameters of Puma560 are taken from [15], see the Appendix. The simulation control system diagram of PUMA560 is shown in Figure 4.1. To check the robustness of controller, a disturbance torque is applied on the robot and a cycloid trajectory test is performed. Recall that only three links of PUMA560 robot are used in this work. The three degrees of freedom PUMA560 robot has the same configuration space equation general form in [15] 6-DOF, in which the last three joints q_4 , q_5 and, q_6 are set to zero in equation cited in [15] so they keep their initial states while the robot is moving.

We can define a general equation that allows us to use PUMA560 robot as a 3-DOF robot and its general control system represented in Figure 4.1.

Note: All parameters and Matrix in (1) are taken from paper [15], data q_4 , q_5 and q_6 are to be replaced by zero and do simplification to get 3-DOF PUMA560 robot as in equation (1).

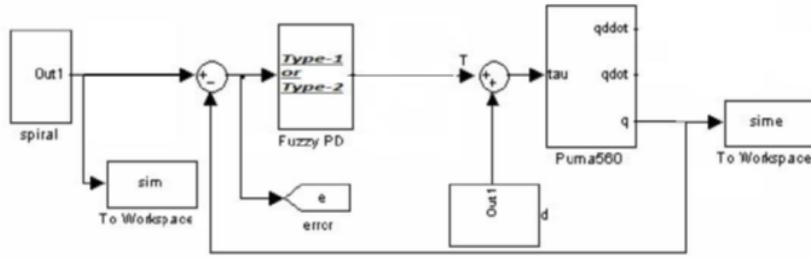


Figure 4.1: General diagram for fuzzy sets using in control of Puma 560

4.3 Controller type-1 fuzzy sets

The basic scheme of a fuzzy controller [74] is shown in Figure 4.2. The fuzzy controller has four parts; the first one is the fuzzification, which changes the crisp values to fuzzy value and determines membership degrees of the chosen PD, which needs two inputs, error and its change, and one output as a control signal. The position errors with their variations and the control signal quantized into five sets as in Figure 4.3: represented by a set of linguistic variables with triangle membership functions with $[-1, 1]$ as a universe of discourse [1], [5],[37] :

Negative Large (N-L), Negative Small (N-S), zero (Ze), Positive Small (P-S) and Positive Large (P-L).

We choose the output as singletons values $[N-L; N-S; Ze; P-S; P-L] = [-1 : -0.5; 0; 0.5; 1]$

The second part is the inference, which cannot function without rule base like that shown in Table 1 [39]. The max-min Mamdani fuzzy inference system [74] was adopted for our controller. The last part is the defuzzification in which the fuzzy inference result is changed to crisp value. There are several methods for defuzzification: in our work centroid of the area has been chosen [1], [5].

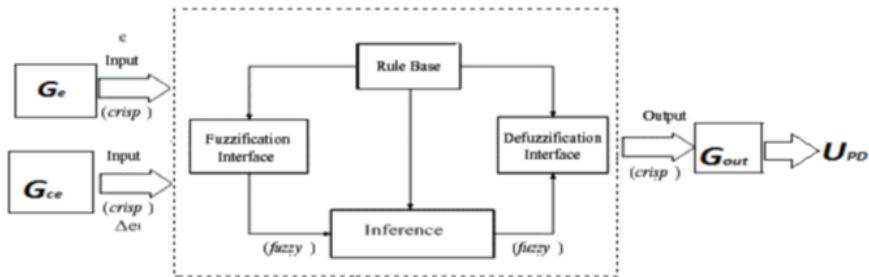


Figure 4.2: A type-1 fuzzy controller [5]

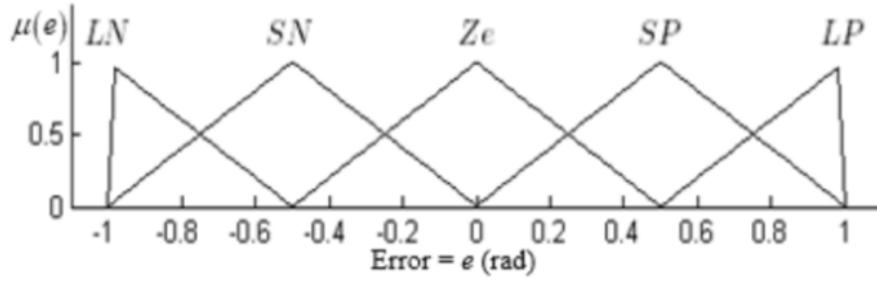


Figure 4.3: Fuzzy set for each articulation with triangle membership functions for error and its change

Table 4.1: Rulebase table[39]

		Velocity error				
		N-L	N-S	Ze	P-S	P-L
Position error	N-L	NL	NL	NL	NS	Ze
	N-S	NL	NL	NS	Ze	PS
	Ze	NL	NS	Ze	PS	PL
	P-S	NS	Ze	PS	PL	PL
	P-L	Ze	PS	PL	PL	PL

4.4 Controller type2 fuzzy sets

4.4.1 General type-2 fuzzy system

The basics of fuzzy logic do not change from type-1 to type-2 fuzzy sets, [33]. A higher-type number simply indicates a higher "degree of fuzziness", Since a higher type changes the nature of the membership functions, the operations that depend on the membership functions change; however, the basic principles of fuzzy logic are independent of the nature of membership functions and hence, do not change. Rules of inference like Generalized Modus Ponens or Generalized Modus Tollens continue to apply.

The structure of the type-2 fuzzy rules is the same as for the type-1 case because the distinction between type-2 and type-1 is associated with the nature of the membership functions. Hence, the only difference is that now some or all the sets involved in the rules are of type-2. In a type-1 fuzzy system, where the output sets are type-1 fuzzy sets, we perform defuzzification in order to get a number which is, in some sense, a crisp (type-0) representative of the combined output sets. In the type-2 case, the output sets are type-2, so we have to use extended versions of type-1 defuzzification methods. Since type-1 defuzzification gives a crisp number at the output of the fuzzy system, the extended defuzzification operation in the type-2 case gives a type-1 fuzzy set at the output. Since this operation takes us from the type-2 output sets of the fuzzy system to a type-1 set, we can call this operation "type reduction" and call the type-1 fuzzy set so obtained a "type-reduced set". The type-reduced fuzzy set may then be defuzzified to obtain a single crisp number; however, in many applications, the type-reduced set may be more important than a single crisp number [33] [55].

Type-2 sets can be used to convey the uncertainties in membership functions of type-1 fuzzy sets, due to the dependence of the membership functions on available linguistic and

numerical information. Linguistic information, in general, does not give any information about the shape of the membership functions. When membership functions are determined or tuned based on numerical data, the uncertainty in the numerical data, e.g., noise, translates into uncertainty in the membership functions. In all such cases, any available information about the linguistic numerical uncertainty can be incorporated in the type-2 framework. However, even with ailing advantages that fuzzy type-2 systems have, the literature on the applications of type-2 sets is scarce. We think that more applications of type-2 fuzzy systems will come in the near future as the area is gaining maturity and that theoretical results become more understandable for the general public in the fuzzy arena [36]

The general structure of a type-2 fuzzy controller (type-2 fuzzy controller: T2FC) is represented in Figure 4.4: This structure is similar to that of the type-1 fuzzy controller.

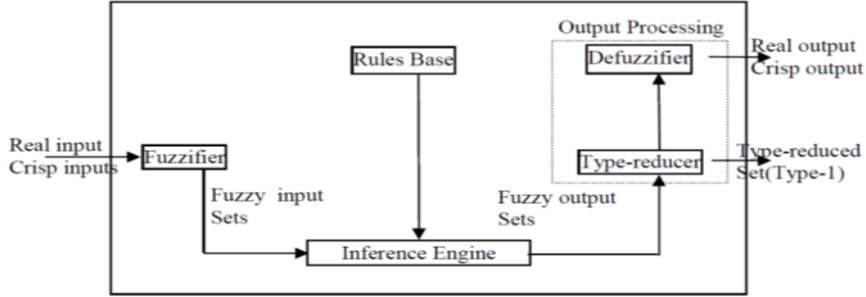


Figure 4.4: Type-2 Fuzzy Controller structure[55]

However, their differences in :

- The type of membership function used.
- Procedure of adopted defuzzifier. In a T2FC, a block of reduction of the type is essential to convert the type-2 fuzzy set to a type-1 fuzzy set.

a. Fuzzification

In this thesis, we will consider only singleton fuzzification (Crisp input), in the input fuzzy set.

b. Rules

In the type-1 case, we generally have "IF-THEN" rules, where the rule has the form

$$"R^l: \text{IF } x_1 \text{ is } F_1^l \text{ and } x_2 \text{ is } F_2^l \text{ and } \dots \text{ and } x_p \text{ is } F_p^l \text{ THEN } y \text{ is } G^l" \quad (4.1)$$

Where x_i 's are inputs, F_i^l 's are antecedent sets ($i = 1, \dots, p$), y is the output and G^l are consequent sets,

The distinction between type-1 FLS and type-2 FLS is associated with the nature of the membership function. which is not important while forming rules, hence, the structure of the rules remains exactly the same in the type-2 FLS case, the only difference being that now some or all of the sets involved are of type-2, so, the rule in a type-2 FLS has the form

$$"R^l: \text{IF } x_1 \text{ is } \tilde{F}_1^l \text{ and } x_2 \text{ is } \tilde{F}_2^l \text{ and } \dots \text{ and } x_p \text{ is } \tilde{F}_p^l \text{ THEN } y \text{ is } \tilde{G}^l" \quad (4.2)$$

Where x_i 's are inputs, \tilde{F}_i^l 's are antecedent sets ($i = 1, \dots, p$), y is the output, and \tilde{G}^l are consequent sets, It is not necessary that all the antecedents and the consequent need be type-2 fuzzy sets. As long as one antecedent or the consequent set is type-2. we will have a type-2 FLC.

c. Inference Engine

In general, the rules we use will have multiple antecedents connected by ands. Just as in the type-1 case. we can connect these multiple antecedents by the meet operation (corresponding to t-norm in the type-1 case). Different rules can be combined using the join operation (corresponding to t-norm in the type-1 case), or during defuzzification.

The output of the inference engine [33] [55] consists of the fired consequent fuzzy sets. Each one of these is modified from a consequent fuzzy set by a degree of firing. This degree of firing is obtained, In general, as a result of t-norm (meet) and t-norm (join) operations on membership grades of the inputs.

The relation (4.2) is interpreted as a fuzzy implication type-2 and defined by :

$$R^l : \tilde{F}_1^l \times \dots \times \tilde{F}_p^l \rightarrow \tilde{G}^l \quad (4.3)$$

This relation is described by the membership function in the same way as :

$$\mu_{R^l}(x_1, \dots, x_p, y) = \mu_{\tilde{F}_1^l} \times \dots \times \mu_{\tilde{F}_p^l} \rightarrow \tilde{G}^l(x_1, \dots, x_p, y) = \left[\bigcap_{i=1}^p \mu_{\tilde{F}_i^l}(x_i) \right] \cap \mu_{\tilde{G}^l}(y) \quad (4.4)$$

$\tilde{F}_1^l \times \dots \times \tilde{F}_p^l$ Denote the cartesian product of $\tilde{F}_1^l, \dots, \tilde{F}_p^l$

d. Type-Reduction

Observe, from figure 4.4, that the defuzzifier block in the type-1 FLC is replaced by two blocks: type-reducer and defuzzifier. We consider type-reduction in this subsection.

In a type-1 FLC. where the output sets are type-1 fuzzy sets, we perform defuzzification in order to get a number which is in some sense a crisp (type-0) representative of the combined output sets. In the type-2 case, the output sets are type-2, so we have to use "the extended principle" of type-1 defuzzification methods. Since type-1 defuzzification gives a crisp number at the output of the FLS, the extended defuzzification operation in the type-2 case gives a type-1 fuzzy set at the output. When this operation takes us from the type-2 output sets of the FLC to a type-1 set, we call this operation "type-reduction" [33], [36] and call the type-1 set so to obtain a single crisp number, however, in many application, the type reduced set may be more important than a single crisp number.

There exists anumber of type-reduction kinds, such as centroid, center-of-sets, height and modified height, the details of which are given in [33] [36]. In this work, for illustrative purposes, we focus on center-of-sets type-reduction.

e. Center-of-sets type-reduction

In this method [36], each set of the consequence \tilde{G}^l is replaced by its centroid. If the set of output \tilde{G}^l is of type-2, its centroid $C_{\tilde{G}^l}$ is a type-1 fuzzy set. Then the weighted average of all the centroid is calculated, associating with each centroid $C_{\tilde{G}^l}$, a weight equals the degree of activation of the rule is given by $E_i(x) = \bigcap_{i=1}^p \mu_{\tilde{F}_i^l}(x)$. The procedure of calculation of the type- reduced set $Y_a(x)$ is:

1. Discretize the output space Y into a suitable number of points, and compute the centroid $C_{\tilde{G}^l}$ of each consequent set on the discretized output space. These consequent centroid sets can be computed ahead of time and stored for future use.
2. Compute the degree of firing, $E_l(x)$ associated with the consequent.

3. Discretize the domain of each $C_{\tilde{G}^l}$ into a suitable number of M_l points, $l = 1, \dots, M$.
4. Discretize the domain of each $E_l(x)$ into a suitable number of points, say $N_l, l = 1, \dots, M$.
5. Enumerate all the possible combinations $\{c_1, \dots, c_M, e_1, \dots, e_M\}$ such that $c_l \in C_{\tilde{G}^l}$, and $e_l \in E_l$.

The total number of combinations will be $\prod_{i=1}^M M_i N_i$

6. Compute the center-of-sums type-reduced set using (3.26).

$$Y_{ce}(x) = \int_{c_1} \dots \int_{c_M} \int_{e_1} \dots \int_{e_M} \sup \left[\mathfrak{S}_{l=1}^M \mu_{C_{\tilde{G}^l}}(c_l) * \mathfrak{S}_{l=1}^M \mu_{E_l}(e_l) \right] / \frac{\sum_{i=1}^M c_i e_i}{\sum_{i=1}^M e_i} \quad (4.5)$$

Where \mathfrak{S} and $*$ indicate the T-norm chosen

f. Defuzzification

To obtain a crisp output from the type-2 FLS, we can defuzzify the type-reduced set [33] [36]. The most natural way of doing this seems to be by finding the centroid of the type-reduced set, however, other possibilities exist, like choosing the unity membership point in the type-reduced set.

The defuzzification makes to transform the linguistic output of the type-reduction to numeric valued. Several methods of defuzzification were proposed in the literature [51]. But the largely used method is:

g. Centre of area

The defuzzification determines the X -coordinate of the center of gravity y_{CG}^* [33] [36] of the fuzzy function:

$$y_{CG}^* = \frac{\int_y y \mu_Y(y) dy}{\int_y \mu_Y(y) dy} \quad (4.6)$$

4.4.2 Interval type-2 Fuzzy controller

a. MEET and JOIN for Interval set

- The meet under minimum or product t-norms of n interval type-1 sets A_1, \dots, A_n having domains $[l_1, r_1], \dots, [l_n, r_n]$ respectively, where $[l_i, r_i] \subseteq [0, 1], i = 1, \dots, n$ is an interval set with domain $[l_1 * \dots * l_n, r_1 * \dots * r_n]$ such as [41] [43]:

$$\bigcap_{i=1}^n A_i = \int_{w \in [l_1 * \dots * l_n, r_1 * \dots * r_n]} 1/w \quad (4.7)$$

* chosen t-norms.

- The join under maximum t-norms of n interval type-1 sets $A_1, \dots, A_n, A_1, \dots, A_n$ having domains $[l_1, r_1], \dots, [l_n, r_n]$ respectively, where $[l_i, r_i] \subseteq [0, 1], i = 1, \dots, n$ is an interval set with domain $[l_1 \vee \dots \vee l_n, r_1 \vee \dots \vee r_n]$ such as:

$$\bigcup_{i=1}^n A_i = \int_{w \in [l_1 \vee \dots \vee l_n, r_1 \vee \dots \vee r_n]} 1/w \quad (4.8)$$

\vee denotes a max operation

b. Inference

In interval type-2 fuzzy system using the minimum or product t-norms operations, the /th activated rule $\bigcap_{i=1}^p \mu_{\tilde{F}_i^l} = F^l(x_1, \dots, x_p)$ gives us an interval determined by two extreme $f_-^l(x_1, \dots, x_n)$ and $\tilde{f}^l(x_1, \dots, x_n)$ same as [41] :

$$F^l(x_1, \dots, x_n) = [f_-^l(x_1, \dots, x_n), \tilde{f}^l(x_1, \dots, x_n)] \equiv [f_-^l, \tilde{f}^l] \quad (4.9)$$

With f_-^l and \tilde{f}^l are given as :

$$\begin{aligned} f_-^l &= \mu_{\tilde{F}_1^l}(x_1) * \dots * \mu_{\tilde{F}_p^l}(x_p) \\ \tilde{f}^l &= \bar{\mu}_{\tilde{F}_1^l}(x_1) * \dots * \bar{\mu}_{\tilde{F}_p^l}(x_p) \end{aligned} \quad (4.10)$$

The output set $\tilde{R}^l(y)$ of the /th activated rule R^l is type-2 fuzzy set :

$$\mu_{\tilde{B}^l}(y) = \int_{b^l \in [f_-^l * \mu_{\tilde{G}^l}(y), \tilde{f}^l * \bar{\mu}_{\tilde{G}^l}(y)]} 1/b^l \quad , \quad y \in Y \quad (4.11)$$

$$\mu_{\tilde{B}}(y) = \bigcup_{l=1}^N \mu_{\tilde{B}^l}(y) \quad (4.12)$$

$\mu_{\tilde{G}^l}(y)$ and $\bar{\mu}_{\tilde{G}^l}(y)$ represent upper and lower membership function of the set $\mu_{\tilde{G}^l}(y)$

c Type-reduction and Defuzzification

After fuzzification, fuzzy inference, type-reduction and defuzzification [41] [43] [33], we obtain a crisp output For an interval type-2 FLC, this crisp output is the center of the type-reduced set, we know that for an interval type-2 FLC, regardless of singleton or non-singleton fuzzification, and minimum or product t-norm, the result of input and antecedent operations (firing strength) is an interval type-1 set which is determined by its left-most and right-most points f_-^l and \tilde{f}^l . The fired output consequent set $\tilde{B}^l(y)$ of rule R^l can be obtained from the fired interval strength using (4.10). Then the fired combined output consequent set $\tilde{R}^l(y)$ can be computed using (4.12) There exist type-reduction different kinds, such as centroid, center-of-sets, height and modified height, the details of which are given in [78], [74] , [33]. In this memory, for illustrative purposes, we focus on center-of-sets type-reduction, which can be expressed as:

$$y_{\cos}(W^1, \dots, W^M, F^1, \dots, F^M) = [y_l, y_r] = \iint_{w^l \in [w_l^1, w_r^1], w^M \in [w_l^M, w_r^M]} \iint_{f^1 \in [f_-^1, \tilde{f}^1], f^M \in [f_-^M, \tilde{f}^M]} 1 / \frac{\sum_{i=1}^M f^i w^i}{\sum_{i=1}^M f^i} \quad (4.13)$$

Where Y_{cos} an interval set is determined by two end points (y_l and y_r), $f^l \in [f_-^l, \tilde{f}^l]$; $w^i \in W^i = [w_l^i, w_r^i]$, W^i is the centroid of the interval type-2 consequent set \tilde{G} (the centroid of a type-2 fuzzy set is described in Appendix C).

Observe that $Y_{cos}(W^1, \dots, W^M, F^1 \dots F^M)$ is an interval type-1 set. So, to find

$y_{cos}(W^1, \dots, W^M, F^1 \dots F^M)$ we just need to compute the two end-points of this interval. Unfortunately, no closed-form formula is available for Y_{cos} .

$$y = \frac{\sum_{i=1}^M f^i w^i}{\sum_{i=1}^M f^i} \quad (4.14)$$

The maximum value of y is y_r , and the minimum value of y is y_l from (4.14), we see that y is a monotonic increasing function with respect to w^i , so, y_r , and is only associated with w_r^i similarly y_l is only associated with w_l^i . In the COS type-reduction method, the two end-points of y_{cos} , (y_l and y_r) depend on a mixture of \underline{f}^i or \bar{f}^i values. In this case, (y_l and y_r) can be represented as :

$$y_l = \frac{\sum_{i=1}^M f_l^i w_l^i}{\sum_{i=1}^M f_l^i} \quad (4.15)$$

Where f_l^i denote the firing strength membership grade either (\underline{f}^i and \bar{f}^i) contributing to the right-most point y_l , similarly to y_r

$$y_r = \frac{\sum_{i=1}^M f_r^i w_r^i}{\sum_{i=1}^M f_r^i} \quad (4.16)$$

Where f_r^i denotes the firing strength membership grade either (\underline{f}^i or \bar{f}^i) contributing to the right-most point y_r .

In order to compute (y_l and y_r), we need to compute $\{f_l^i, i = 1, 2, \dots, M\}$ and $\{f_r^i, i = 1, 2, \dots, M\}$.

This can be done using the exact computational procedure given in [43] [33]. Here we briefly provide the computation procedure for y_r . Without loss of generality, assume the w_r^i 's are arranged in ascending order $w_r^1 \leq w_r^2 \leq \dots \leq w_r^M$.

1. Compute y in (4.16) by initially setting $f_r^i = \frac{\bar{f}^i + \underline{f}^i}{2}$ for $i = 1, \dots, M$ where (\underline{f}^i and \bar{f}^i) have been previously computed using (4.10) and let $y' = y$.
2. Find R ($1 \leq R \leq M - 1$) such that $w_r^R \leq y' \leq w_r^{R+1}$
3. Compute y in (4.14) when $f_r^i = \underline{f}^i$ for $i \leq R$, and $f_r^i = \bar{f}^i$ for $i > R$, then set $y'' = y_r$
4. If $y'' \neq y'$ then go to step 5. If $y'' = y'$ then stop and set $y_r = y''$
5. Set y' equal to y'' and return to step 2

This 5 step computation procedure has been proven to converge to the exact solution in no more than M iterations [43] [33]. Note that in this procedure, the number R is very important. For $i \leq R$, $f_r^i = \underline{f}^i$ and for $i > R$, $f_r^i = \bar{f}^i$, so y_r can be represented as:

$$y_r = y_r(\underline{f}^1, \dots, \underline{f}^R, \bar{f}^{R+1}, \dots, \bar{f}^M, w_r^1, \dots, w_r^M) \quad (4.17)$$

The procedure for computing y_l is very similar. Just replace w_r^i by w_l^i , and in step 2, find L ($1 \leq L \leq M - 1$), such that $w_l^L \leq y'_l \leq w_l^{L+1}$, and in step 3

$f_l^i = \bar{f}^i$ for $i \leq L$, and $f_l^i = \underline{f}^i$ for $i > L$. Then y_l can be represented as:

$$y_r = y_r(\bar{f}^1, \dots, \bar{f}^R, \underline{f}^{R+1}, \dots, \underline{f}^M, w_l^1, \dots, w_l^M) \quad (4.18)$$

Because Y_{cos} is an interval set, we defuzzify it using the average of (y_l and y_r), and hence, the defuzzified output of an interval type-2 FLC is: output of Centroid = $\frac{y_l + y_r}{2}$

The control of dynamic system was originally applied to a system that control a mechanical position or motor. Used to describe a control system in which a physical variable is required to follow, or track.

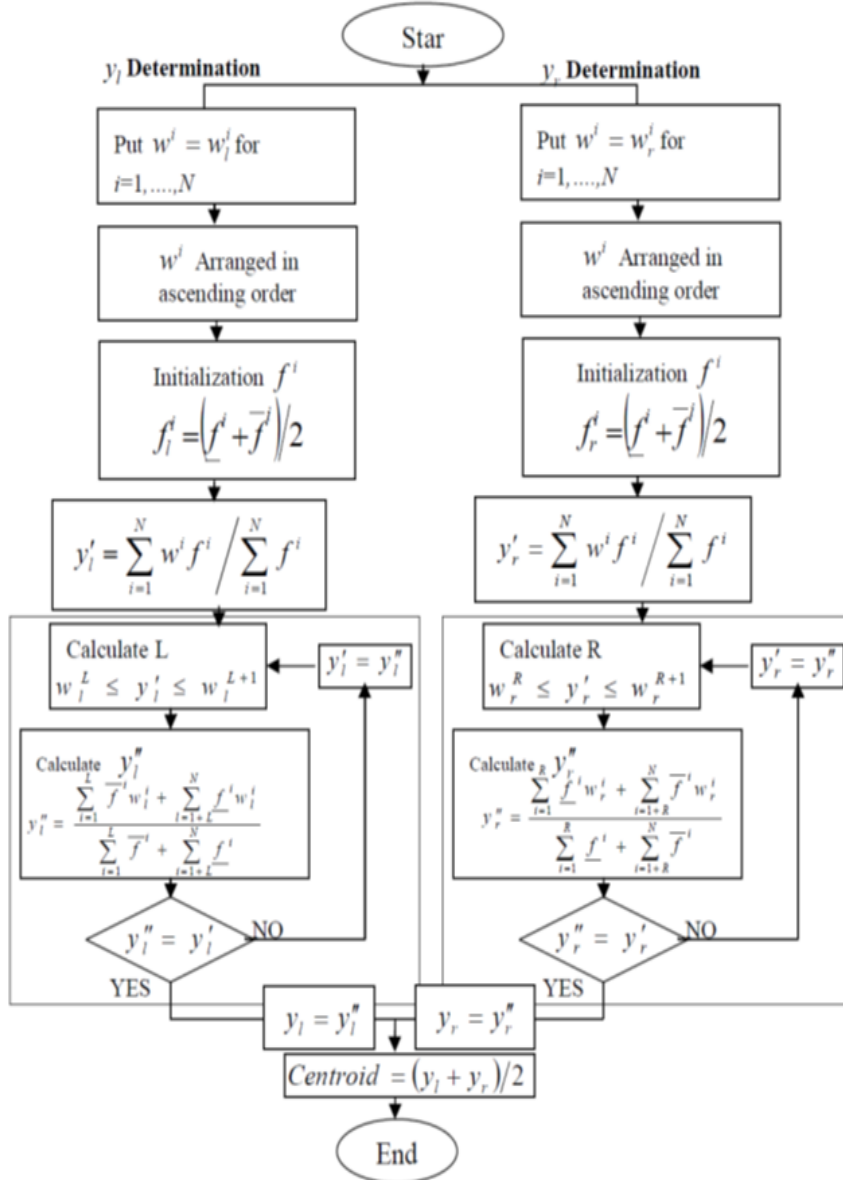


Figure 4.5: Karnik-Mendel Algorithms to locate Centroid on Interval type-2 set [36]

d. Detailed description of $K - M$ algorithm

In this work a center of sets type reduction approach is used the details are given as follows:

Step1. Determination of y_{cos} [y_l, y_r] where y_{cos} is interval set and y_r is the maximum value of y And y_l is the minimum value of y

➤ compute of y_l

- Compute w_l^i (for $i = 1, \dots, N$) where w_l^i is the lower fuzzy value of $w_i = (e, ce)$
- Put $w^i = w_l^i$ (for $i = 1, \dots, N$)
- Arranged w^i in ascending order $w_l^1 < w_l^2 < \dots < w_l^N$
- Compute the left most point and the right most point $(\underline{f}^i, \overline{f}^i)$ for $i = 1, \dots, N$
- Set initially: $(\underline{f}^i + \overline{f}^i) / 2$
- Compute: y'_l using (4-16)
 1. Find $L(1 \leq L \leq N-1)$ such that $w_l^L \leq y'_l \leq w_l^{L+1}$
 2. Compute y in (4.14) when $f_l^i = \overline{f}^i$ for $i \leq L$, and $f_l^i = \underline{f}^i$ for $i > R$, then $y''_l = y_l$
 3. If $y'' \neq y'$ then go to step 5 . If $y'' = y'$ then stop and set $y_l = y''$
 4. set $y' = y''$ and return to step 2
- Put $y_l = y''_l$

➤ Compute of y_r

- Compute w_r^i (for $i = 1, \dots, N$) where w_r^i is the upper fuzzy value of $w_i = (e, ce)$
- Put $w^i = w_r^i$ (for $i = 1, \dots, N$)
- Arranged w^i in ascending order $w_r^1 < w_r^2 < \dots < w_r^N$
- Compute the left most point and the right most point $(\underline{f}^i, \overline{f}^i)$ for $i = 1, \dots, N$
- Set initially: $(\underline{f}^i + \overline{f}^i) / 2$
- Compute: y'_r using (4-16)
 5. Find $R(1 \leq R \leq N-1)$ such that $w_r^R \leq y'_r \leq w_r^{R+1}$
 6. compute y in (4.14) when $f_r^i = \underline{f}^i$ for $i \leq R$, and $f_r^i = \overline{f}^i$ for $i > R$, then $y''_r = y_r$
 7. If $y'' \neq y'$ then go to step 5 . If $y'' = y'$ then stop and set $y_r = y''$
 8. set $y' = y''$ and return to step 2
- Put $y_r = y''_r$

Step2. Compute the centeroid of interval set

$$\text{Centeroid} = (y_l + y_r) / 2$$

Table 4.2: Rulebase table[33]

		Velocity error				
Position error		NL	SN	Ze	SP	LP
	LN	LN	LN	LN	SN	Ze
	SN	LN	LN	SN	Ze	SP
	Ze	LN	SN	Ze	SP	LP
	SP	SN	Ze	SP	LP	LP
	LP	Ze	SP	LP	LP	LP

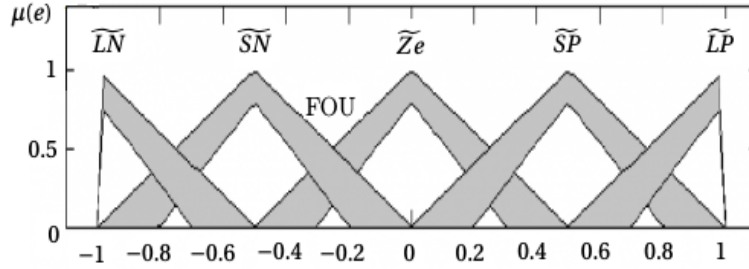


Figure 4.6: Fuzzy sets interval type-2 for each articulation and control signal

4.5 Interval type-2 Fuzzy control with PUMA560 3DOF

we show the structure interval type-2 FLC with PUM560 3DOF, the regulator we use is five classes, wich means that 25 rule base are involted, the rule base table in the table 4.2 and in Figure 4.7 fuzzy sets for error and change error and out put of control T. All the gains of Interval type-2 fuzzy controller we do tuning until get good positions with lower error in ideal case.

In this work and for illustrative purposes, we have used Karnik-Mendel algorithm of figure 4.5 to locate centroid on interval type-2 fuzzy set [36]. The proposed modification is carried out in the fourth step of this algorithm where Karnik-Mendel use arithmetic mean, but in the presented work it has been modified by harmonic mean, as in the nexte formulate:

$$y = \frac{\sum_{i=1}^n f_i}{\sum_i \frac{f_i}{w_i}}$$

4.6 Simulation results

For our model of equation (1) 3DOF PUMA560 robot, a cycloid trajectory test Figure 4.11 used the different articulation move respectively from position $(-50^\circ, -135^\circ, 135^\circ)$ to the position $(45, -85^\circ, 30^\circ)$ with three end time of movement equal to (1.5, 4, 8) seconds then will have three cycloid trajectory that differ in simulation time. This trajectory is described by

the following equation:

$$\theta_{di} = \begin{cases} \theta_{di}(0) + \frac{D_i}{2\pi} \left[2\pi \frac{t}{t_{end}} - \sin\left(2\pi \frac{t}{t_{end}}\right) \right] & \text{for } 0 \leq t \leq t_{end} \\ \theta_{di}(t_{end}) & \text{for } t_{end} < t \\ D_i = \theta_{di}(t_{end}) - \theta_{di}(0) & \end{cases} \quad (4.19)$$

Integral of Square Error (ISE) [40] is used as performance index in comparative study between type-1 and interval-type-2 fuzzy controller

$$ISE = \int_0^{t_{end}} [e(t)]^2 dt \quad (4.20)$$

In type-1 fuzzy PD controller of Figure 4.4, fuzzy set parameters take standard parameters as declared by default in MATLAB in the range of $[-1, 1]$; the triangle membership functions are symmetric fuzzy sets. The controller require three gains: two inputs gains and one output gain; these gain; are chosen using manual tuning until getting good positions responses with lower error, lower control energy and smooth control signal without chattering in ideal case.

In an Interval-type-2 fuzzy PD controller, the interval-type-2 fuzzy sets are programmed as type-1 fuzzy sets in the range of $[-1, 1]$ with upper and lower triangle membership function as shown in Figure 4.6. In addition to the length of FOU zone, the controller has also three gains to be designed: two input gains and one output gain. These gains and the FOU zone are chosen base on trial and error method until getting good position responses with lower error and lower control energy and smooth control signal without chattering in ideal case.

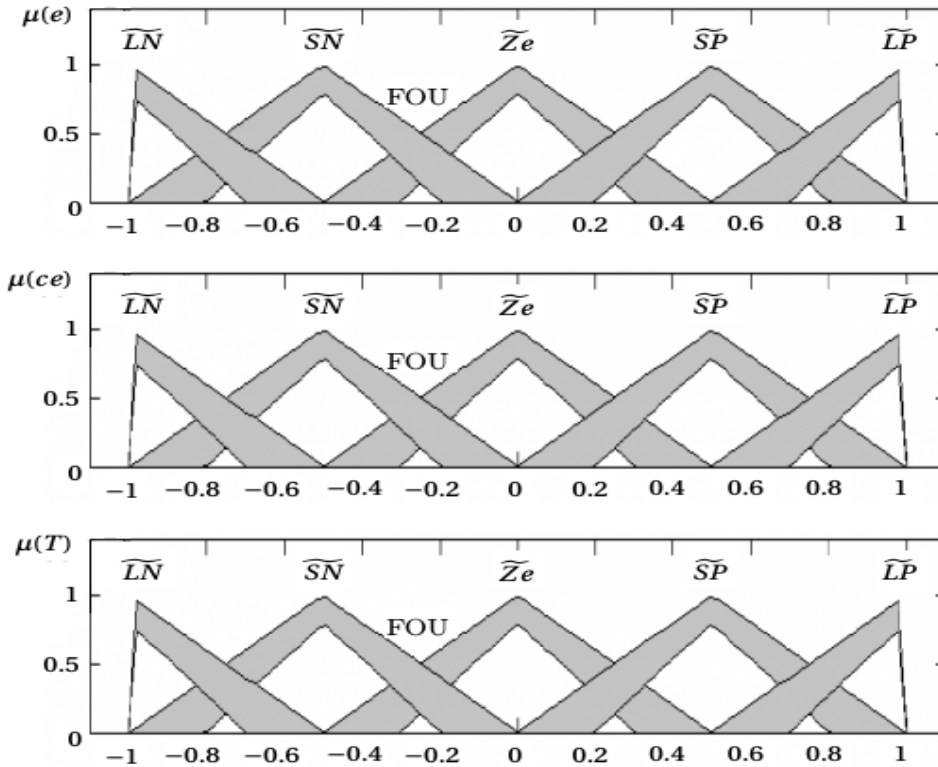


Figure 4.7: Fuzzy sets interval type-2 for each articulation and control signal

4.7 Result of simulation with two trajectory

We use same trajectory, a circle in space and LEAHY trajectory.

Result of simulation with circle trajectory

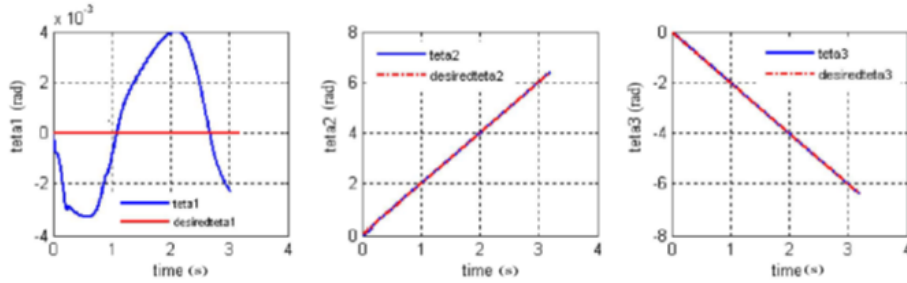


Figure 4.8: Position of joints 1,2,3 (rad) .

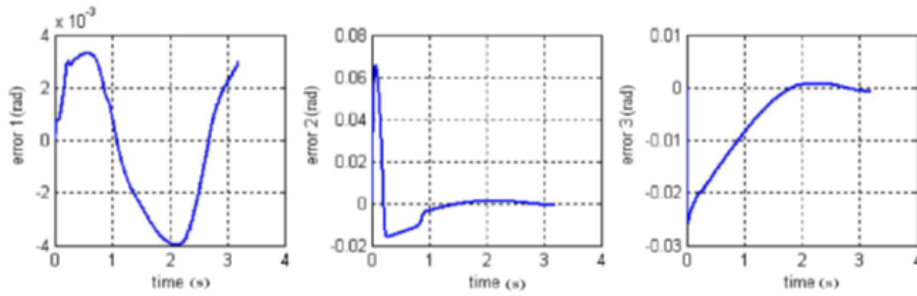


Figure 4.9: Position error of joints 1,2,3 (rad) .

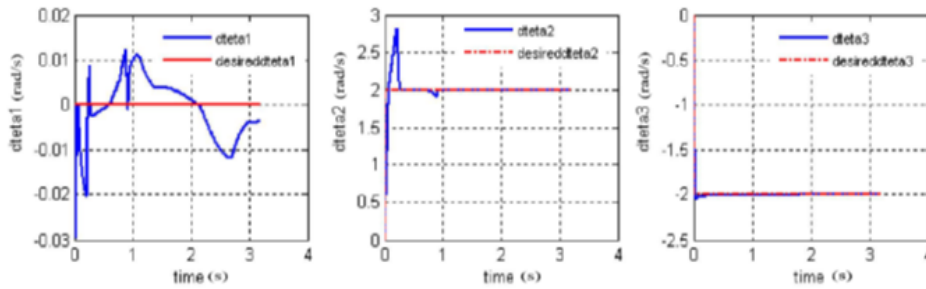


Figure 4.10: Velocity of joints 1,2,3 (rad/s)

In order to verify the robustness of the proposed controller and to help us to achieve a clear full comparison between the type-1 fuzzy controller and interval-type-2 fuzzy controller and to see the ability each one in handling uncertainties, two lands of noises are adopted: the first noise torque is $d = \sin(3t) + 1$ and the second noise torque is considered by taking 30 % of joint torque for each joint. [62]The simulation results are devoted to showing comparisons in the robot's responses when using both type-1 fuzzy controller and interval type-2 fuzzy controller. A set of three simulations in each one three cycloid trajectories are adopted by changing time with (1.5second, 4s, 8s). We start with type-1 fuzzy controller in ideal conditions without any kind of disturbances. The values of ISE of three joint angles for three

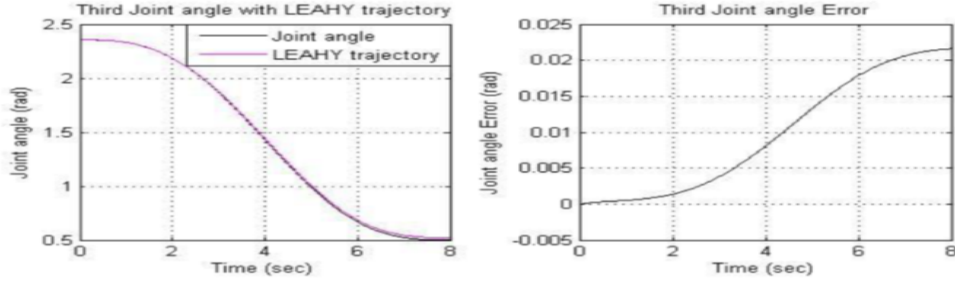


Figure 4.11: Third joint response to cycloid trajectory with noise and its tracking joint error

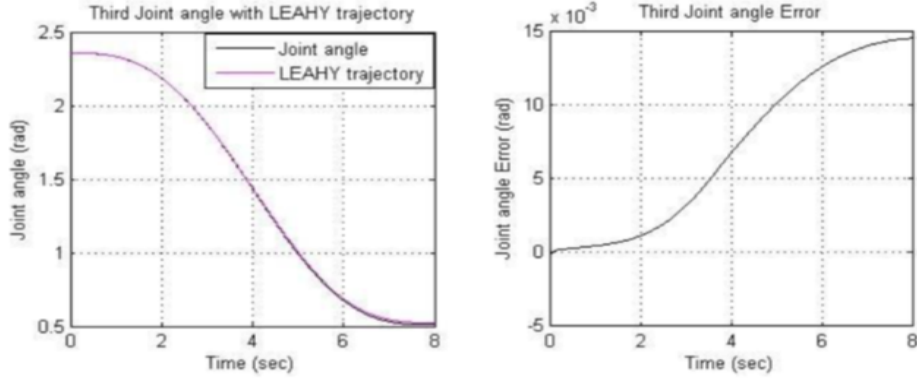


Figure 4.12: Third joint response with noise and its tracking joint error with modified type-reduction

trajectories are gathered in Table 4.3. Figure 4.11 shows the response obtained by using type-1 fuzzy controller with the presence of the noise torque taking $d = 30\%$ of the torque control added as noise to the main control torque to check the robustness of controller. Figure 4.12 shows the response of the interval type-2 fuzzy controller using modified type-reduction of Karnik-Mendel algorithm.

Comparison in terms of integral squared joint errors in ideal and disturbance cases ($d = \sin(3t) + 1$) is shown in Table 4.3. Table 4.4 represents comparison in terms of integral squared joint errors in ideal and disturbance $d = 30\%$ of joint signal control. The last column in Tables 4.3. and 4.4. is the factor (ISE_F) between the integral square joint error produced by using a type-1 fuzzy controller (ISE_T1) and integral square error produced by using interval type-2 fuzzy controller (ISE_IT2) in noisy case, calculated as (4.21):

$$ISE_F = ISE_T1 / ISE_IT2 \quad (4.21)$$

This factor gives an idea in how many energies of error in type-1 fuzzy set is bigger than energy error of interval type-2 fuzzy set case.

4.8 Results discussion

The membership functions of a type-1 fuzzy controller are type-1 fuzzy sets, see Figure 4.3, and their membership degrees are crisp values.

Table 4.3: ISE for different trajectory without and with disturbance $d = \sin(3t) + 1$

Second	Joint	ISE Type-1*		ISE Interval type-2*		ISE_F
		No d	With d	No d	with d	
1.5	J1	1.88	2.35	0.49	0.70	3.38
	J2	7.14	8.33	5.96	3.75	2.22
	J3	6.06	8.27	5.76	5.67	1.46
4	J1	0.23	0.54	0.24	0.15	3.49
	J2	32	36.9	18.71	16.84	2.19
	J3	19.7	30.2	17.48	24.91	1.21
8	J1	0.043	0.64	0.02	0.26	2.44
	J2	83.5	94	31.88	50.80	1.85
	J3	54	81	46.63	50.21	1.61

Note : * all values are multiplied by 10^{-5}

Table 4.4: ISE for different trajectory without and with disturbance in joint control $d = 30\%$ of each joint

Second	Joint	ISE Type-1*		ISE Interval type-2*		ISE_F
		No d	With d	No d	with d	
1.5s	J1	1.88	4.23	0.50	0.91	4.63
	J2	7.14	19.1	3.88	7.28	2.62
	J3	6.06	14.8	5.63	9.09	1.63
4s	J1	0.23	0.48	0.23	0.13	3.77
	J2	0.32	79.3	22.44	38.94	2.04
	J3	19.7	44.7	12.16	23.80	1.88
8s	J1	0.043	0.091	0.02	0.03	2.72
	J2	83.5	200	43.78	80.08	2.50
	J3	54	120	44.37	79.28	1.51

Note : * all values are multiplied by 10^{-5}

Interval type-2 fuzzy controller has FOU zone in which the membership degrees are an interval and not crisp values. Actually, this is the main part that improves the degree of dealing with uncertainties. In an interval-type-2 fuzzy controller, the outputs of the inference are interval-type-2 sets and a type-reducer is needed to convert them into a type-1 sets before defuzzification to generate a crisp output. This is the main structural difference between type-1 and interval-type-2 fuzzy systems. In Table 4.5, some differences between type-1 fuzzy controller and interval type-2 fuzzy controller are given.

When comparing results based on visual observation, the actual and desired joint angles are overlapping without tangible difference as illustrated in Figure 4.11. But in terms of ISE values, it is possible to measure the difference between the two fuzzy controllers. To perform this comparison, three simulations with different simulation tunes are realized, which give us three different trajectories for three robot joints with two disturbance torques to test robustness. All results are summed up in two tables; Table 4.3 for comparison between a type-1 fuzzy controller and modified interval type-2 fuzzy controller with disturbance $d = \sin(3t) + 1$ and Table 4.4 for comparison between type-1 fuzzy controller and modified interval type-2 fuzzy controller with disturbance $d = 30\%$ of control torque. It is very clear

that the average of ISE_F column around two times which means the tracking error angle for the type-1 fuzzy controller is twice bigger than error angle for modified interval-type-2 fuzzy controller.

Finally, when comparing all results in all figures, Table 4.3, and Table 4.4, it is very clear that interval-type-2 fuzzy controller is more performing than type-1 fuzzy controller in ideal or in noises cases.

Table 4.5: Some differences between type-1 and interval-type-2 fuzzy controller

	T1-FS	IT-FS
Grades of membership	Crisp	Fuzzy
footprint of uncertainty zone	Don't have	Have
Type-reducer	Don't need	Need
Computational	small	Little Large
Program	Easy	Little Hard

4.9 Conclusion

In this chapter, we have discussed in some detail the type-1 FLC and type-2 FLC then, we have applied them on PUMA560 3DOF. The results of simulations prove that Interval type-2 FLC has a low error and accepted control torques of the joints

General Conclusion

Fuzzy Type-2 logic has involved many researchers in recent years and has been applied in several areas such as video modeling, classification, diagnostics, artificial intelligence, transmission channels, adaptive filters, mobile robots, and other domains. In this thesis, a type-1 and Interval type-2 FLC are developed for the control of PUMA560 3DOF manipulators in the presence of dynamical modeling. To alleviate the naturally inherited high computational complexity of type-2 FLCs, interval membership functions are adopted. The controllers are also compared in similar operating conditions. Three simulations for three robot joints with and without noises are presented. Furthermore, a quantification of errors was achieved and documented in tables for the integral square error of trajectory tracking. It was shown that the lower errors were obtained using an interval-type-2 fuzzy controller where the amount of error is four times less than when using a type-1 fuzzy set to control a manipulator robot. The interval-type-2 fuzzy controller is much better because it can handle uncertainties and has more robustness than the type-1 fuzzy counterpart. Also, the proposed modification in type reduction of Karnik-Mendel algorithm by using harmonic mean performs better than the arithmetic mean used in the traditional Karnik-Mendel algorithm.

We are concluding that using type-2 FLC in real applications can be a better choice since the amount of uncertainty in real systems, mostly, is difficult to estimate. But when uncertainty in real system is negligible, we can consider it as an ideal system. In this case, it is a better choice to select a type-1 FLC since it works similar as type-2 FLC, and the size of type-1 FLC program is shorter than type-2 FLC, so it is faster and easier for programming and simulating.

Future work will be devoted to the study of ideal representations of the types-2 fuzzy lattices.

.1 APPENDIX

Matrix M is a symmetric 3×3 matrix :

$$M(q) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

where,

$$a_{11} = I_{m1} + I_1 + I_3 \cdot CC2 + I_7 \cdot SS23 + I_{10} \cdot SC23 + I_{11} \cdot SC2 + 2 \cdot [I_5 \cdot C2 \cdot S23]$$

$$a_{12} = I_4 \cdot S2 + I_8 \cdot C23 + I_9 \cdot C2 \quad a_{13} = I_8 \cdot C23 \quad a_{22} = I_{m2} + I_2 + I_6 + 2 \cdot [I_5 \cdot S3]$$

$$a_{23} = I_5 \cdot S3 + I_6$$

$$a_{33} = I_{m3} + I_6$$

$$a_{21} = a_{12}, a_{31} = a_{13} \text{ and } a_{32} = a_{23}$$

While matrix B is :

$$B(q) = \begin{bmatrix} b_{112} & b_{113} & 0 & 0 & 0 & b_{123} & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{223} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where,

$$b_{112} = 2 \cdot [-I_3 \cdot SC2 + I_5 \cdot C223 + I_7 \cdot SC23]$$

$$+ I_{10} \cdot (1 - 2 \cdot SS23) + I_{11} \cdot (1 - 2 \cdot SS2)$$

$$b_{113} = 2 \cdot [I_5 \cdot C2 \cdot C23 + I_7 \cdot SC23 -] + I_{10} \cdot (1 - 2 \cdot SS23)$$

$$b_{123} = 2 \cdot [-I_8 \cdot S23]$$

Matrix C is :

$$C(q) = \begin{bmatrix} 0 & c_{12} & c_{13} \\ c_{21} & 0 & c_{23} \\ c_{31} & c_{32} & 0 \end{bmatrix}$$

where,

$$c_{12} = I_4 \cdot C2 - I_8 \cdot S23 - I_9 \cdot S2$$

$$c_{13} = 0.5 \cdot b_{123} = -I_8 \cdot S23$$

$$c_{21} = -0.5 b_{112} = I_3 \cdot SC2 - I_5 \cdot C223 - I_7 \cdot SC23 - 0.5 I_{10} \cdot (1 - 2 \cdot SS23) - 0.5 \cdot I_{11} \cdot (1 - 2 \cdot SS2)$$

$$c_{23} = 0.5 \cdot b_{223} = I_5 \cdot C3$$

$$c_{31} = -0.5 \cdot b_{113} = -I_5 \cdot C2 \cdot C23 - I_7 \cdot SC23 - 0.5 I_{10} \cdot (1 - 2 \cdot SS23)$$

$$c_{32} = -c_{23} = -I_5 \cdot C3$$

And matrix G is :

$$g(q) = \begin{bmatrix} 0 \\ g_2 \\ g_3 \end{bmatrix}$$

$$g_2 = g_1 \cdot C2 + g_2 \cdot S23 + g_3 \cdot S2$$

$$g_3 = g_2 \cdot S23$$

where,

$$S_i = \sin(\theta_i), C_i = \cos(\theta_i), C_{ij} = \cos(\theta_i + \theta_j), S_{ijk} = \sin(\theta_i + \theta_j + \theta_k), \text{ with } i = 1, 2, 3$$

Table 6: Inertial constants ($kg.m^2$)

$I_1=1.43\pm 0.05$	$I_2 = 1.75 \pm 0.07$
$I_3=1.38\pm 0.05$	$I_4 = 0.69 \pm 0.02$
$I_5=0.372\pm 0.031$	$I_6 = 0.333 \pm 0.016$
$I_7=0.298\pm 0.029$	$I_8 = -0.134 \pm 0.014$
$I_9=0.0238\pm 0.012$	$I_{10} = -0.0213 \pm 0.0022$
$I_{m1} = 1.14 \pm 0.27$	$I_{m2} = 4.71 \pm 0.54$
$I_{m3} = 0.827 \pm 0.093$	

Table 7: Gravitational constants ($N.m$)

$g_1 = -37.2 \pm 0.5$	$g_2 = -8.44 \pm 0.20$
$g_3 = 1.02 \pm 0.50$	

Tables 4.5 and 6 contain the computed values for the constants appearing in the equations (1).

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ملخص

يركز الدكتوراه هذا على مجالين: الدراسة النظرية على المجموعات الضبابية من النوع 1 والنوع 2 ، ثم تطبيقها للتحكم في نظام ديناميكي غير خطي وهو ذراع روبوت من الصنف بيما 560. لذلك، قمنا بتطبيق كل من وحدات التحكم الضبابية: وحدات تحكم ضبابي من النوع 1 والمجال الضبابي من النوع 2 حيث تم اقتراح تعديل في خوارزمية كارنيك مندل. لحساب الناتج من النظام الضبابي نوع-2، هناك خطوة رئيسية تسمى تبسيط النوع، تعتمد هذه العملية هذه العملية على خوارزمية كارنيك مندل، والتي تستخدم المتوسط الحسابي لحساب مخرجات التحكم. في هذا العمل نقترح تغيير الوسط الحسابي بالوسط التوافقي. يُظهر التعديل المقترح في تبسيط نوع خوارزمية كارنيك مندل من أجل مجموعة غامضة من النوع 2 أفضل أداء. بالفعل، فإن مقدار الخطأ في حالة وحدة تحكم مجال ضبابي نوع 2 أقل مرتين من الخطأ بواسطة التحكم الضبابي من النوع 1. الكلمات المفتاحية: المجموعات الضبابية نمط-2، المجموعات الضبابية، المجموعات الضبابية في شبكة ، المجال الضبابي نمط-2، التحكم بواسطة مجال ضبابي نمط-2، الذراع الآلي بيما 560.

Abstract

The research presented in this doctoral thesis focuses on two areas: theoretical study on fuzzy sets type1 and type2, then their application for the control of a nonlinear dynamic system which is the manipulator robot PUMA560. For that, we implemented the both fuzzy controllers: type-1 and interval type-2 fuzzy controllers, in which a modification in Karnik-Mendel algorithm has been proposed. To calculate the output of interval type-2 fuzzy system there is a main step called type-reduced; this operation is based on Karnik-Mendel algorithm, which uses arithmetic mean to calculate the control output. In this work, we propose to change the arithmetic mean by harmonic one. The performances of modified interval type- 2 controller and type-1 fuzzy controller with and without noises are compared in terms of integral of squared error. The proposed modification in type reduction of Karnik-Mendel algorithm for interval type-2 fuzzy set shows best performance. Indeed, the amount of error in case of modified interval type-2 fuzzy controller is less two times than type-1 fuzzy controller.

Keywords: L-fuzzy sets, Type-2 fuzzy sets, Interval type-2 fuzzy sets, type-1 fuzzy controller Interval type-2 fuzzy controller, puma 560 robot manipulator

Résumé

Les travaux de recherche présentés dans cette thèse de doctorat portent essentiellement sur deux axes : étude théorique sur les ensembles flous type1 et type2, puis leur application pour la commande d'un système dynamique non linéaire qui est le robot manipulateur PUMA560. Pour cela, nous avons implémenté les deux contrôleurs flous: les contrôleurs flous de type 1 et d'intervalle de type 2, dans lesquels une modification de l'algorithme de Karnik-Mendel a été proposée. Pour calculer la sortie du système à intervalle flous de type 2, il existe une étape principale appelée réduction de type2 cette opération est basée sur l'algorithme de Karnik-Mendel, qui utilise une moyenne arithmétique pour calculer la sortie de contrôle. Dans ce travail, nous proposons de changer la moyenne arithmétique par harmonique. Les performances du contrôleur d'intervalle modifié de type 2 et du contrôleur flou de type 1 et sans bruit sont comparées en termes d'intégrale d'erreur quadratique. La modification proposée dans la réduction de type de l'algorithme de Karnik-Mendel pour l'intervalle flou de type-2 montre les meilleures performances. En effet, la quantité d'erreur en cas d'intervalle modifié de contrôleur flou de type 2 est inférieure à deux fois à celle de contrôleur flou de type 1.

Les mots clés: Les ensemble flou dans un treille, les ensembles flous type-2, l'intervalle flou type-2, Contrôle fou type-1, Contrôle intervalle flou type-2, robot manipulateur Puma 560.