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Theme

Approximation properties in terms of Lipschitz maps

Presented by :
Beddiar Ilham

In front of the jury composed of :

Abdelmoumen TIAIBA	Prof,	University of Msila	**President**.
Rachid YAHI	MCB,	University of Msila	**Supervisor.**
Maatougui BELAALA	MCB,	University of Msila	**Examiner.**

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❖ Dedication ❖

To my dear parents **BEDDIAR Tayeb**

and **TRABELLSI Ouarda**, may Allah keep

you and grant you good health and a long life.

To my sister, the source of joy and happiness **Ines**.

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To my friends and my loved ones in this life.

To you, dear reader.

❄ Beddiar Ilham ❄

Abstract

ملخص

في هذه الأطروحة، تناولنا واحدة من أهم مواضيع التحليل الدالي، وهي خصائص التقريب في مفهوم المثاليات الخطية واللبيشيزية. . حاولنا قدر الإمكان، بجهد متواضع جدا، أن نتعرف على التعريفات الأكثر أهمية، والنظريات والخصائص في جميع فصول هذا البحث. في الفصل الأول، نستعرض أهم المفاهيم في الفضاءات الليبشتيزية: الفصل الثاني يتناول خاصية التقريب المتعلقة بفضاءات بناخ. وأخيرا، الفصل الثالث، وهو جوهر موضوعنا، نتناول خاصية التقريب المتعلقة بالمثاليات الليشيتيزية في الفضاءات المترية .

الكلمات المفتاحية: فضاءات باناخ ، مؤثر خطي ذو رتبة محدودة مؤثر ليشتيزي، خاصية التقريب – فضاء متري نقطي، مثالي ليبشتيزي.

Abstract

In this work , we dealt with one of the most important topics of functional analysis, which is the approximation properties related to the ideals of Lipschitz and linear operators.. We have tried, as much as possible , with a very modest effort to become familiar with the most important definitions, theorems and properties of all chapters of this memory . In the first chapter, we recall briefly the space of Lipschitz functions, Arens–Eells and Lipschitz operator ideals. Chapter two devoted to study the approximation property for Banach spaces. The third chapter devoted to study the approximation property for Lipschitz operator ideals.

Keywords : Arens–Eells space,Finite rank operator, Lipschitz mapping, Lipschitz operator ideal, the approximation property for Lipshitz operator ideals. .

Résumé

Dans ce travail, nous avons abordé l'un des sujets les plus importants de l'analyse fonctionnelle, c'est les propriétés d'approximation en termes des idéaux linéairs et Lipschitziens. Nous avons essayé, autant que possible, avec un effort très modeste, de nous familiariser avec les définitions les plus importantes, les théorèmes et les propriétés de tous les chapitres de ce mémoire. Dans le premier chapitre, nous rappelons les concepts dz l'espace des fonctions Lipschitziennes. Le deuxième chapitre traite une propriété d'approximation d'un espace de Banach. Enfin, le troisième chapitre, est le cœur de notre sujet, il aborde la propriété d'approximation pour ideaux lipschitziens.

Mots-clés: Espace de Arens-Eells, operateur lipschitzien, idéal d'opérateur lipschitzien, Les opérateurs lipschitziens p-sommants, propriété d'approximation.. .

Contents

Notaions

$\mathbb{K}$	The field of real or complex numbers.
$\mathbb{R}_+$	The field of non negative real numbers.
p'	The conjugate of the number p $(1 \leq p \leq \infty)$, that is $\frac{1}{p} + \frac{1}{p'} = 1$
E^*	The topological dual of E.
B_E	The closed unit ball of E
$L(E;F)$	The set of all linear operators.
$\mathcal{L}(E;F)$	The sets of all continuous linear operators.
$\mathcal{I}$	The ideal of all linear operator.
$\mathcal{W}$	The set of all weakly compact linear operators.
$\mathcal{L}_f$	The set of all finite rank linear operators.
Lip_{0K}	The set of all Lipschitz compact operators.
Lip_{0W}	The set of all Lipschitz weakly compact operators.
Lip_0	The set of all Lipschitz operators that vanish at 0.
$X^\#$	The Lipschitz dual of the pointed metric space X.
$\mathcal{M}(X)$	The linear space of all molecules on the metric space X.

$m_{xx'}$	The molecule defined by $m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$ for $x, x' \in X$, where
Æ(X)	The Arens-Eells space of X.
$\mathcal{F}(X)$	The Lipschitz free space of X.
T^*	The adjoint operator of T.
$T^{\#}$	The Lipschitz adjoint operator of T.
T_L	The linearization of the operator T.

Introduction

In 1953, Alexander Grothendieck [15] introduced the concept of the approximation property related to Banach space. A Banach space E has the approximation property (AP) if for every compact subset K of E and every $\varepsilon > 0$,there exists a finite rank operator S on E such that $\sup_{x\in K} \|S_x - x\| \leq \varepsilon$, briefly ,$id_E \in \overline{\mathcal{F}(E,E)}^{\tau_c}$. Then he put a characterization which states that E has the AP if and only if for every Banach space F every linear operator $T \in \mathcal{K}(F,E)$ can be approximated uniformly on bounded sets by finite rank operators . Roughly speaking, three are the main ingredients that play an important role in the approximation property: the approximating operators (the finite rank operators), the approximated operators (the linear or compact operators) and the bornology where the convergence is considered (compact or bounded sets).

Since $\mathcal{K} = (\mathcal{K}, \|.\|)$ is a Banach operators ideals. Oja in 2012 (see [22]) introduced the natural version of the AP related to an arbitrary Banach operator ideal $\mathcal{I}$. a Banach space E has the $\mathcal{I}$-approximation property if, for every Banach space F, $\overline{\mathcal{F}(F,E)}^{\|\cdot\|_{\mathcal{I}}} = \mathcal{I}(F,E)$.

To replace compact sets by another class of sets with some kind of compactness related to $\mathcal{I}$. The new class of sets is formed by $\mathcal{I}$-compact sets. This notion was introduced by Carl and Stephani [5] and the related approximation property has been studied in e.g. [11, 20, 21].

. In this work we will consider Lipschitz ideals $\mathcal{I}_{Lip}$ that are composition ideals, that is, $\mathcal{I}_{Lip} = I \circ Lip_0$ for some linear operator ideal $\mathcal{I}$.

The aim of this memory is to study the $\mathcal{I}$-Lipschitz approximation property, basing on the article of D. Achour, P. Rueda, E.A. Sánchez-Pérez and R. Yahi (see [1]) So our memory is organized into three chapters ,

The first chapter is a preliminaries which contains some basic concepts related to weak topology, the space of Lipschitz functions, Arens-Eells space, Lipschitz operators ideals,..etc which we need it on the sequel of this memory. In the second chapter we discussed the linear approximation property , The third chapter is devoted to study the $\mathcal{I}-$Lipschitz approximation property .

Chapter 1

Preliminaries

1.1 The weak topology $\sigma(E, E^*)$

Let E be a banach space and let $f \in E^*$. we denote by $\varphi_f : E \to \mathbb{R}$ the linear functional $\varphi_f = (f, x)$. As f runs through E^* we obtian a collection $(\varphi_f)_{f \in E^*}$ of maps from E into $\mathbb{R}$. We now ignore the usual topology on E (associated to $\|\|$) and define a new topology on the set E as follow:

Definition 1.1. *The weak topology $\sigma(E, E^*)$ on E is the coarset topology associated to the collection $(\varphi_f)_{f \in E^*}$.*

Note that every map φ_f is c ontinuous for the usual topology and therefore the weak topology is weaker the usual topology.

Proposition 1.1. *.The weak topology $\sigma(E, E^*)$ is Hausdorf*

Proposition 1.2. *When E is finite-dimensional, the weak topology $\sigma(E, E^*)$ and the usual topology are the same.In particular, a sequence (x_n) converges weakly if and only if it converges strongly.*

Theorem 1.1. *Let C be a convex subset of E.Then C is closed in the weak topology $\sigma(E, E^*)$ if and only if is closed in the strong topology.*

Corollary 1.1. ***(Mazur)***.*Assume (x_n) converges weakly to x .Then there exists a sequence (y_n) made up of convex combinations of x_n's that converges strongly to x.*

Corollary 1.2. *Assume that $\varphi \to (-\infty, \infty]$ is convex and l.s.c in the strong topology .Then φ is l.s.c in the weak topology $\sigma(E, E^*)$.*

Theorem 1.2. *Let E and F be tow banach spaces and let T be a linear opertor from E into F .Assume that T is continuous in the strong topology .Then T is continuous from E weak $\sigma(E, E^*)$ into F weak $\sigma(F, F^*)$ and conversely.*

Some consequences of Hahn-Banach Theorem. Let f be a continuous linear functional on a subspace G of a normed space E. Then there is $x^* \in E^*$ such that $\|f\| = \|x^*\|$ and $f(x) = \langle x, x^* \rangle$ for all $x \in G$.

Let E be a normed space and $x \in E, x \neq 0$. Then there exists $x^* \in E^*$ such that $\|x^*\| = 1$ and $\langle x, x^* \rangle = \|x\|$.

In particular, for every $x \in E$ we have

$$\|x\| = \sup_{x^* \in B_{E^*}} |\langle x, x^* \rangle|$$

1.2 The weak* Topology $\sigma(E^*, E)$

Definition 1.2. *The weak*topology,$\sigma(E^*, E)$, is the coarsest topology on E^* associated to the collection$(\varphi_x)_{x \in E^*}$.*

Since $E \subset E^{**}$ it is clear that the topology $\sigma(E^*, E)$ is coarser than the topology $\sigma(E^*, E^{**})$; i.e., the topology $\sigma(E^*, E)$ has fewer open sets (resp.closed sets) than the topology $\sigma(E^*, E^{**})$, which in turn has fewer open sets (resp. closed sets) than the strong topology.

Proposition 1.3. *The weak* topology is Hausdorff.*

Proposition 1.4. *Let $\varphi : E^* \to \mathbb{R}$ be a linear functional that is continous for the weak* topology. Then there exists some $x_0 \in E$ such that :*

$$\varphi(f) = \langle f, x_0 \rangle \quad \forall f \in E^*.$$

Theorem 1.3. ***(Banach-Alaoglu-Bourbaki)****The closed unit ball*

$$B_{E^*} = \{f \in E^*; \|f\| \leq 1\}.$$

is compact in the weak* topology $\sigma(E^*, E)$.

1.3 Reflexive Spaces

Definition 1.3. *Let E be a Banach space and let $J : E \to E^{**}$ be the canonical injection from E into E^{**}, The space E is said to be reflexive if J is surjective, i,e.. $J(E) = E^{**}$, When E is reflexive, E^{**} is usually identified with E.*

Theorem 1.4. ***(Kakutani)****Let E be a Banach space. Then E is reflexive if and only if*

$$B_E = \{x \in E; \|x\| \leq 1\}.$$

is compact in the weak topology $\sigma(E, E^*)$.

Lemma 1.1. ***(Goldstine)****Let E be any Banach space Then $J(B_E)$ is dense in $B_{E^{**}}$ with respect to the topology $\sigma(E^{**}, E^*)$,and consequently $J(E)$ is dense in E^{**} in the topology $\sigma(E^{**}, E^*)$.*

1.4 Linear operator ideals

Recall that a linear operator $T \in \mathcal{L}(E, F)$ is said to have finite rank if $T(E)$ is a finite dimensional subspace of F. The class of all finite rank linear operators between Banach spaces is denoted by

$\mathcal{L}_f(E,F)$. An operator has rank one if and only it has the form

$$x^* \otimes y : x \longmapsto \langle x, x^* \rangle \, y$$

i.e. if $u \in \mathcal{L}_f(E,F)$ we have

$$u = \sum_{i=1}^{n} x_i^* \otimes y_i,$$

where $(x_i^*)_{i=1}^n \subset E^*$ and $(y_i)_{i=1}^n \subset F$ (see [23, Page 25]).

Definition 1.4. *An operator ideal $\mathcal{I}$ is a subclass of the class $\mathcal{L}$ of all continuous linear operators between Banach spaces such that for all Banach spaces E and F its components $\mathcal{I}(E,F) := \mathcal{L}(E,F) \cap \mathcal{I}$ satisfy:*

(i) $\mathcal{I}(E,F)$ is a linear subspace of $\mathcal{L}(E,F)$ which contains the finite rank operators.

(ii) The ideal property: if $v \in \mathcal{L}(G,E)$, $u \in \mathcal{I}(E,F)$ and $w \in \mathcal{L}(F,H)$, then the composition $w \circ v \circ u$ is in $\mathcal{I}(G,H)$.

If $\|.\|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^+$ satisfies:

(i') $(\mathcal{I}(E,F), \|.\|_{\mathcal{I}})$ is a normed (Banach) space for all Banach spaces E and F,

(ii') $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1$,

(iii') If $v \in \mathcal{L}(G,E)$, $u \in \mathcal{I}(E,F)$ and $w \in \mathcal{L}(F,H)$,

$$\|w \circ u \circ v\|_{\mathcal{I}} \leq \|w\| \, \|v\|_{\mathcal{I}} \, \|u\|,$$

then $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is called a normed (Banach) operator ideal.

The operator ideal $\mathcal{I}$ is said to be *closed* if each $\mathcal{I}(E,F)$ is a closed subspace of $\mathcal{L}(E,F)$ for the sup norm.

Definition 1.5. *(injective operator ideal)*

A normed operator ideal $(\mathcal{I}, \|.\|_{\mathcal{I}})$ *is said to be injective if for every metric injection* $i : F \hookrightarrow G$ *and every* $u \in \mathcal{L}(E, F)$ *it follows from* $i \circ u \in \mathcal{I}(E, G)$ *that* $u \in \mathcal{I}(E, F)$*. Moreover*

$$\|i \circ u\|_{\mathcal{I}} = \|u\|_{\mathcal{I}}\,,$$

The ideal $\mathcal{L}_f$ of finite rank linear operators is the smallest operator ideal and $\mathcal{L}$ the largest one [23, Theorem 1.2.2].

1.4.1 Ideal of compact and weakly compact linear operators.

We say that a bounded linear operator $T : E \to F$ is compact (weakly compact) if $T(B_E)$ is relatively compact (respectively, relatively weakly compact)in F. By $\mathcal{K}(E, F)$ and $\mathcal{W}(E, F)$ we denote the sets of compact linear operators and weakly compact linear operators from E to F, respectively.

Schauder theorem about the compactness of the adjoint of a compact linear operator between two Banach spaces is well known. We will remind it as we need in the sequel of this thesis.

Theorem 1.5. *Let* $T \in \mathcal{L}(E, F)$*. Then* T *is compact if and only if* T^* *is compact.*

The weakly version is due to Gantmacher:

Theorem 1.6. *Let* $T \in \mathcal{L}(E, F)$*. Then* T *is weakly compact if and only if* T^* *is weakly compact.*

Proposition 1.5. *([23]) The classes* $\mathcal{K}, \mathcal{W}$ *consitute closed injective Banach operator ideals, where the ideal norm is the operator norm.*

1.5 Arens-Eells space

Definition 1.6. *Let X be a metric space. A molecule on X is a scalar valued function m on X with finite support that satisfies $\sum_{x\in X} m(x) = 0$. We denote by $\mathcal{M}(X)$ the linear space of all molecules on X.*

For $x, x' \in X$ the molecule $m_{xx'}$ is defined by $m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$, where χ_A is the characteristic function of the set A. For $m \in \mathcal{M}(X)$ we can write

$$m = \sum_{j=1}^{n} \lambda_j m_{x_j x'_j}$$

for some suitable scalars λ_j, and we write

$$\|m\|_{\mathcal{M}(X)} = \inf\left\{ \sum_{j=1}^{n} |\lambda_j|\, d(x_j, x'_j),\ m = \sum_{j=1}^{n} \lambda_j m_{x_j x'_j} \right\},$$

where the infimum is taken over all representations of the molecule m. Denote by Æ$(X,)$ the completion of the normed space $(\mathcal{M}(X), \|.\|_{\mathcal{M}(X)})$. This space was first introduced by Arens and Eells [2] in 1956. The terminology *Arens-Eells space* Æ(X) is due to Weaver [25]. A different notation was used in [13] by Godefroy and Kalton. It is the Lipschitz-free space denoted by $\mathcal{F}(X)$

1. The map $\delta_X : X \longrightarrow$ Æ(X) defined by

   $$\delta_X(x) = m_{x0}$$

   is an isometric embedding of X into Æ(X) (see [25, Theorem 2.2.4]).

2. The map $Q_X : X^{\#} \longrightarrow Æ\,(X)^*$ defined by

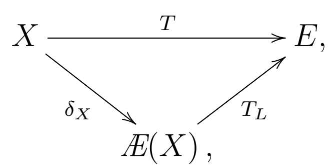
$$Q_X(f) = f_L, \text{ where } f_L(m) = \sum_{x \in X} f(x)m(x),$$

establish an isometric isomorphic between $X^{\#}$ and $Æ(X)^*$. On bounded subsets of $X^{\#}$ its weak * topology agrees with the topology of pointwise convergence (see [25, Theorem 2.2.2]).

The Banach space $Æ(X)$ has some remarkable properties. We mention the following ones.

Theorem 1.7. *([25, Theorem 2.2.4])*

Let X be a pointed metric space and E be a Banach space and let $T : X \longrightarrow E$ be a Lipschitz map which preserves the base point; that is $T(0) = 0$. Then there is a unique bounded linear map $T_L : Æ(X) \longrightarrow E$ such that $T = T_L \circ \delta_X$ that is, the diagram



commutes. Furthermore $\|T_L\| = Lip(T)$

The operator T_L is referred to as the linearization of T. The correspondence

$$T \longleftrightarrow T_L$$

establishes an isomorphism between the vector spaces $Lip_0(X, E)$ and $\mathcal{L}(0left(X, E)$.

Theorem 1.8. *([18, Lemma 3.1])*

Let X,Y two pointed metric spaces and let $T : X \longrightarrow Y$ be a Lipschitz map which preserves the *base point. Then there is a unique bounded linear map* $\hat{T} : Æ(X) \longrightarrow Æ(Y)$ *such that* $\widehat{T}\delta_X = \delta_Y T$ *that is, the diagram*

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \downarrow \delta_X & & \downarrow \delta_Y \\ Æ(X) & \xrightarrow{\hat{T}} & Æ(Y) \end{array}$$

commutes. Furthermore $\left\|\hat{T}\right\| = Lip(T)$.

Sawashima [24] defined the Lipschitz adjoint (or dual) of $T \in Lip_0(X, Y)$ as the continuous linear operator

$$\begin{array}{rccc} T^{\#}: & Y^{\#} & \longrightarrow & X^{\#} \\ & g & \longmapsto & T^{\#}(g) = g \circ T \end{array}$$

If $Y = E$ is a Banach space, the restriction of $T^{\#}$ to E^* is called the Lipschitz transpose map of T and is denoted here by T^t. The correspondence

$$T \longleftrightarrow T^t$$

establishes an isomorphism between the vector spaces $Lip_0(X, E)$ and $\mathcal{L}((E^*, w^*), (X^{\#}, w^*))$, where w^* denote the weak* topology (see [17, Theorem 3.1]).

1.6 Lipschitz operator ideal

Following [17] a mapping $T \in Lip_0(X, E)$ has Lipschitz finite dimensional rank if the linear hull of the set $\left\{\frac{T(x)-T(x')}{d(x,x')}, x, x' \in X, x \neq x'\right\}$ is a finite dimensional subspace of E. We denote by $Lip_{0\mathcal{F}}(X, E)$ the set of all Lipschitz finite rank mappings from X to E.

Definition 1.7. *[1] A Lipschitz operator ideal* $\mathcal{I}_{Lip}$ *is a subclass of* Lip_0 *such that for every pointed* *metric space X and every Banach space E the components*

$$\mathcal{I}_{Lip}(X,E) := Lip_0(X,E) \cap \mathcal{I}_{Lip}$$

satisfy:

(i) $\mathcal{I}_{Lip}(X,E)$ is a linear subspace of $Lip_0(X,E)$.

(ii) $vg \in \mathcal{I}_{Lip}(X,E)$ for $v \in E$ and $g \in X^{\#}$.

(iii) The ideal property: if $S \in Lip_0(Y,X)$, $T \in \mathcal{I}_{Lip}(X,E)$ and $w \in \mathcal{L}(E,F)$, then the composition wTS is in $\mathcal{I}_{Lip}(Y,F)$.

A Lipschitz operator ideal $\mathcal{I}_{Lip}$ is a normed (Banach) Lipschitz operator ideal if there is $\|.\|_{\mathcal{I}_{Lip}} : \mathcal{I}_{Lip} \longrightarrow [0,+\infty[$ that satisfies

(i') For every pointed metric space X and every Banach space E, the pair $(\mathcal{I}_{Lip}(X,E), \|.\|_{\mathcal{I}_{Lip}})$ is a normed (Banach) space and $Lip(T) \leq \|T\|_{\mathcal{I}_{Lip}}$ for all $T \in \mathcal{I}_{Lip}(X,E)$.

(ii') $\|Id_{\mathbb{K}} : \mathbb{K} \longrightarrow \mathbb{K}, Id_{\mathbb{K}}(\lambda) = \lambda\|_{\mathcal{I}_{Lip}} = 1$.

(iii') If $S \in Lip_0(Y,X)$, $T \in \mathcal{I}_{Lip}(X,E)$ and $w \in \mathcal{L}(E,F)$, then

$$\|wTS\|_{\mathcal{I}_{Lip}} \leq Lip(S)\, \|T\|_{\mathcal{I}_{Lip}}\, \|w\|\,.$$

1.7 Composition ideal of Lipschitz mappings

Definition 1.8. *[1] (Composition Ideals) Given an operator ideal $\mathcal{I}$, a Lipschitz mapping $T \in Lip_0(X,E)$ belongs to the composition Lipschitz operator ideal $\mathcal{I} \circ Lip_0$, denoted $T \in \mathcal{I} \circ Lip_0(X,E)$, if there are a Banach space F, a Lipschitz operator $S \in Lip_0(X,F)$ and an operator $u \in \mathcal{I}(F,E)$ such that $T = u \circ S$. If $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is a normed operator ideal we write $\|T\|_{\mathcal{I} \circ Lip_0} = \inf \|u\|_{\mathcal{I}}\, Lip(S)$,* *where the infimum is taken over all u, S as above.*

Proposition 1.6. *[1] Let $\mathcal{I}$ be an operator ideal. The following are equivalent for $T \in Lip_0(X, E)$:*

(1) $T \in \mathcal{I} \circ Lip_0(X, E)$.

(2) $T_L \in \mathcal{I}(\text{Æ}(X), E)$.

If $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is a normed operator ideal, then

$$\|T\|_{\mathcal{I} \circ Lip_0} = \|T_L\|_{\mathcal{I}} .$$

Proof. (1) $\Rightarrow$ (2) Assume that $T \in \mathcal{I} \circ Lip_0(X, E)$. Then there is a Banach space F, a Lipschitz operator $S \in Lip_0(X, F)$ and an operator $u \in \mathcal{I}(F, E)$ such that $T = u \circ S$. Since $T_L = u \circ S_L$, the ideal property ensures that $T_L \in \mathcal{I}(\text{Æ}(X), E)$. If $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is normed then,

$$\|T_L\|_{\mathcal{I}} = \|u \circ S_L\|_{\mathcal{I}} \leq \|u\|_{\mathcal{I}} \|S_L\| = \|u\|_{\mathcal{I}} Lip(S).$$

Taking the infimum over all such factorizations we obtain $\|T_L\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I} \circ Lip_0}$.

(2) $\Rightarrow$ (1) Consider the factorization of T given by $T = T_L \circ \delta_X$. Since δ_X is Lipschitz and $T_L \in \mathcal{I}(\text{Æ}(X), E)$ then $T \in \mathcal{I} \circ Lip_0(X, E)$ and, if $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is normed we have

$$\|T\|_{\mathcal{I} \circ Lip_0} = \|T_L \circ \delta_X\|_{\mathcal{I} \circ Lip_0} \leq \|T_L\|_{\mathcal{I}} Lip(\delta_X) = \|T_L\|_{\mathcal{I}} .$$

$\square$

1.8 Lipschitz compact and weakly compact operators

Following [17] a Lipschitz map $T \in Lip_0(X, E)$ is *Lipschitz compact* (*Lipschitz weakly compact*) if the set

$$\left\{\frac{T(x)-T(x')}{d(x,x')}, x, x' \in X, x \neq x'\right\}$$

is relatively compact (respectively, relatively weakly compact) in E. Denote by $Lip_{0\mathcal{K}}(X, E)$ and $Lip_{0\mathcal{W}}(X, E)$ the sets of Lipschitz compact operators and Lipschitz weakly compact operators from X to E, respectively. In [17], the relationship between the compactness (weakly compactness) of a Lipschitz operator $T \in Lip_0(X, E)$ and the compactness (respectively, weakly compactness) of its linearization $T_L \in \mathcal{L}(Æ(X), E)$ has been established:

Proposition 1.7. *[17, Proposition 2.4] Let $T \in Lip_0(X, E)$. The following statements are equivalent:*

(1) T is compact Lipschitz.

(2) T_L is compact from $Æ(X)$ to E.

The Lipschitz versions of Schauder's theorem (respectively, Gantmacher's theorem) on the compactness (respectively, weak compactness) of the adjoint of a compact (respectively, weakly compact) linear operators are also studied in terms of the Lipschitz transpose map of a Lipschitz operator [17].

Proposition 1.8. *[17, Proposition 3.4](Gantmacher's theorem) Let $T \in Lip_0(X, E)$. The following statements are equivalent:*

(1) T is weakly compact Lipschitz.

(2) T^t is weakly compact from E^* to $X^\#$.

Proposition 1.9. *[17, Proposition 3.5](Schauder's theorem) Let $T \in Lip_0(X, E)$. The following statements are equivalent:*

(1) T is Lipschitz compact .

(2) T^t is compact from E^* to $X^\#$.

Proposition 1.10. *[1] Let X be a pointed metric space and E be a Banach space. We have*

1. $Lip_{0\mathcal{K}}(X, E) = \mathcal{K} \circ Lip_0(X, E)$ *isometrically.*

2. $Lip_{0\mathcal{W}}(X, E) = \mathcal{W} \circ Lip_0(X, E)$ *isometrically.*

Chapter 2

The approximation property of Banach spaces

The approximation property of Banach spaces was introduced by Alexander Grothendieck [15] in the fifties.

Definition 2.1 ([15]). *a Banach space E is said to have the approximation property (AP) if for every compact subset K of E and every $\varepsilon > 0$,there exists a finite rank operator S on E such that* $\sup_{x\in K} \|S_x - x\| \leq \varepsilon$ *, briefly ,$id_E \in \overline{\mathcal{F}(E,E)}^{\tau_c}$, where id_E is the identity map on E and τ_c is the topology of uniform convergence on compact subsets of E.*

Proposition 2.1. *E has the AP if and only if for every Banach space F every linear operator $T \in \mathcal{K}(F,E)$ can be approximated uniformly on bounded sets by finite rank operators.*

Proposition 2.2. *A Banach space E has the AP if and only if for every Banach space F and every operator $T \in \mathcal{L}(F,E)$, on has $T \in \overline{\{ST : S \in \mathcal{F}(E,E)\}}^{\tau_c}$.*

Definition 2.2. *Let $1 \leq \lambda < \infty$, a Banach space E is said to have the λ-bounded approximation property $(\lambda - BAP)$ if for every compact subset K of E and every $\varepsilon > 0$,there exists a finite rank operator S on E with $\|S\| \leq \lambda$ such that* $\sup_{x\in K} \|S_x - x\| \leq \varepsilon$.

Since $\mathcal{K} = (\mathcal{K}, \|.\|)$ is a Banach operators ideals. Oja in 2012 [22] introduced the following natural version of the AP related to an arbitrary Banach operator ideal $\mathcal{I}$.

Definition 2.3. *(Oja 2012 [22]) A Banach space E has the $\mathcal{I}$-approximation property ($\mathcal{I}$-AP) if, for every Banach space F, $\overline{\mathcal{F}(F,E)}^{\|\cdot\|_{\mathcal{I}}} = \mathcal{I}(F,E)$.*

Remark 2.1. *Clearly that the $\mathcal{K}$-AP is precisely the AP*

Given an operator ideal $\mathcal{I}$, Delgado and Piñeiro in [12] introduce the notion of approximation property depending on the operator ideal $\mathcal{I}$ by replacing the operator ideal $\mathcal{L}$ by $\mathcal{I}$.

Definition 2.4. *A Banach space E is said to have the approximation property with respect to the operator ideal $\mathcal{I}$ (for short, $AP_{\mathcal{I}}$) if fo every Banach space F and every operator $T \in \mathcal{I}(F,E)$, one has $T \in \overline{\{ST : S \in \mathcal{F}(E,E)\}}^{\tau_c}$.*

Proposition 2.3 ([12])**.** *A Banach space E has the $AP_{\mathcal{I}}$ if and only if $id_X \in \overline{\mathcal{F}(E,E)}^{\tau c(\mathcal{I})}$ where $\tau_c(\mathcal{I})$ is the topology of uniform convergence on $\mathcal{I}$-compact subsets of E.*

2.1 $\mathcal{I}$-uniform approximation property of Banach spaces

The situation of replacing the norm $\|\cdot\|_{\mathcal{I}}$ by the operator norm $\|\cdot\|$ in $\mathcal{L}$ is considered by S. Lassalle and P. Turco

Definition 2.5. *Definition (S. Lassalle and P. Turco(2013) [20]) A Banach space E has the $\mathcal{I}$-uniform approximation property if, for every Banach space F, $\overline{\mathcal{F}(F,E)}^{\|\cdot\|_{\mathcal{L}}} = \mathcal{I}(F,E)$*

Note that, whenever the ideal of compact operators $\mathcal{K}$ is considered, the $\mathcal{K}$-(uniform) approximation property is just the approximation property.

2.2 $\mathcal{K}_{\mathcal{I}}$-uniform approximation property

Recently new extensions of the approximation property for Banach spaces related to operator ideals have been introduced. For example, when we replace compact sets by another class of sets with some kind of compactness related to $\mathcal{I}$. The new class of sets is formed by $\mathcal{I}$-compact sets.This notion was introduced by Carl and Stephani [5]

Definition 2.6.

- *A subset B of a Banach space E is relatively $\mathcal{I}$ -compact if there is a Banach space G and an operator $S \in \mathcal{I}(G, E)$ such that $B \subseteq S(M)$, where M is a compact subset of G.*

- *A linear operator T between Banach spaces E and F is called $\mathcal{I}$-compact if $T(B_E)$ is a relatively $\mathcal{I}$-compact subset of F.*

- *The operator ideal formed by all linear $\mathcal{I}$-compact operators is denoted by $\mathcal{K}_{\mathcal{I}}$.*

The related approximation property for these sets has been studied by S. Lassalle and P. Turco in 2013.

Definition 2.7. *[20, S. Lassalle and P. Turco(2013)] A Banach space E has the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property if for every Banach space F, every $\mathcal{I}$-compact operator from F into E can be uniformly approximated by finite rank operators.*

S. Lassalle and P. Turco proved that

Proposition 2.4. *[20, S. Lassalle and P. Turco(2013)] The identity $Id_E : E \to E$ can be approximated uniformly on relatively $\mathcal{I}$-compact sets by finite rank operators if, and only if, for every Banach space F, every $\mathcal{I}$-compact operator from F into E can be uniformly approximated by finite rank operators.*

Chapter 3

The approximation property for Lipschitz operator ideals

This chapter based on the article of D. Achour, P. Rueda, E.A. Sánchez-Pérez and R. Yahi [1]

Definition 3.1. *Let X and Y be pointed metric spaces, and $\mathcal{I}_{Lip} = \mathcal{I} \circ Lip_0$ a composition Lipschitz operator ideal. A set $K \subseteq X$ is relatively $\mathcal{I}$-Lipschitz compact if there is a pointed metric space Y and a Lipschitz operator $S : Y \to Æ(X)$ in $\mathcal{I}_{Lip}$ such that $\delta_X(K) \subseteq S(M)$, where M is a compact subset of Y.*

Definition 3.2. *Let Y and X be pointed metric spaces. We say that a Lipschitz operator $\phi : Y \to X$ is $\mathcal{I}$-Lipschitz compact if $\phi(B_Y) = \phi(\{x \in Y : d(x, 0) \leq 1\})$ is relatively $\mathcal{I}$-Lipschitz compact.*

Remark 3.1. *Note that, if E is a Banach space then, a Lipschitz operator $T : X \to E$ is $\mathcal{I}$-Lipschitz compact if $T(\{x \in X : d(x, 0) \leq 1\})$ is a relatively $\mathcal{I}$-Lipschitz compact subset of E. We denote by $\mathcal{K}^L_{\mathcal{I}}(X, E)$ the class of all $\mathcal{I}$-Lipschitz compact operators from X to E, considered as a topological subspace of the Lipschitz operators*

Taking into account that the Lipschitz image of a relatively $\mathcal{I}$-Lipschitz compact set is again relatively $\mathcal{I}$-Lipschitz compact, we obtain the following fact:

Proposition 3.1. *Let $T : X \to Y$ is $\mathcal{I}$-Lipschitz compact and $R \in Lip_0(Y, Z)$, then $R \circ T : X \to Z$ is also $\mathcal{I}$-Lipschitz compact.*

Proposition 3.2. *Let X be a pointed metric space and let $A \subset X$. Then A is relatively $\mathcal{I}$-Lipschitz compact if, and only if, $\delta_X(A)$ is relatively $\mathcal{I}$-compact. Moreover, if E is a Banach space, every relatively $\mathcal{I}$-Lipschitz compact subset of E is relatively $\mathcal{I}$-compact.*

Proof. Assume that A is relatively $\mathcal{I}$-Lipschitz compact. There is a pointed metric space Y, $M \subset Y$ compact and $S \in \mathcal{I}_{Lip}(Y, \text{Æ}(X))$ such that $\delta_X(A) \subseteq S(M)$. Since we can write $S = S_L \circ \delta_Y$ with $S_L \in \mathcal{I}(\text{Æ}(Y), \text{Æ}(X))$, and $\delta_Y(M)$ is compact, we conclude that $\delta_X(A)$ is a relatively $\mathcal{I}$-compact set.

Now assume that $\delta_X(A)$ is relatively $\mathcal{I}$-compact. Let Z be a Banach space, K a compact subset of Z and $T \in \mathcal{I}(Z, \text{Æ}(X))$ such that $\delta_X(A) \subseteq T(K)$. This implies that $T \in \mathcal{I}_{Lip}(Z, \text{Æ}(X))$, which now does give the result.

Finally, assume that X is a Banach space E. Let us see that every relatively $\mathcal{I}$-Lipschitz compact subset of E is relatively $\mathcal{I}$-compact. Take a relatively $\mathcal{I}$-Lipschitz compact subset K of E. This means that there is a Lipschitz map $S : Y \to \text{Æ}(E)$ in $\mathcal{I}_{Lip}$ from a pointed metric space Y and a compact subset $M \subseteq Y$ such that $\delta_E(K) \subseteq S(M)$. We have the canonical factorization for S given by $S_L \circ \delta_Y$, and since $\mathcal{I}_{Lip}$ is a composition Lipschitz ideal, we have that S_L belongs to the linear ideal $\mathcal{I}$. Then the compact subset $M_1 := \delta_Y(M)$ of $\text{Æ}(Y)$ satisfies that there is a linear operator S_L in $\mathcal{I}$ such that $\delta_E(K) \subseteq S_L(M_1)$. Consider $(Id_E)_L : \text{Æ}(E) \to E$. Then we have that $K = (Id_E)_L(\delta_E(K)) \subseteq (Id_E)_L \circ S_L(M_1)$, where $(Id_E)_L \circ S_L$ is a linear map in $\mathcal{I}$, and so K is a relatively $\mathcal{I}$-compact subset of E.

□

From Proposition 3.2, we obtain the next

Corollary 3.1. *A linear map $T : F \to E$ between Banach spaces is $\mathcal{I}$-Lipschitz compact if, and only if, it is linear $\mathcal{I}$-compact.*

In what follows we will show more concrete information about the relation between the linear and the Lipschitz $\mathcal{I}$-compactness for operators. We start with a lemma that is already known. The linearization $\beta_E : Æ(E) \to E$ of the identity map in E is known as the barycentric map, and it is in fact a quotient map (see for example [**?**, p.25] and [13, Lemma 2.4], where this map is denoted by β). This gives a proof of the lemma; however, we write a direct proof for the aim of completeness.

Lemma 3.1. *Let E be a Banach space and let $\beta_E : Æ(E) \to E$ be the linearization of the identity map $Id^E : E \to E$. Then, $\beta_E(B_{Æ(E)}) = B_E$.*

Proof. Since $\beta_E \circ \delta_E(x) = Id^E(x) = x$ and $\delta_E(B_E) \subset B_{Æ(E)}$, the inclusion $B_E \subset \beta_E(B_{(E)})$ is trivial. Let us prove the other inclusion. It is enough to prove that $\beta_E(B_{\mathcal{M}(E)}) = B_E$. Assume that there is $m \in B_{\mathcal{M}(E)}$ such that $\beta_E(m)$ does not belong to B_E. The Hahn-Banach theorem gives a linear functional $\phi \in E^*$ of norm $\|\phi\| = 1$ such that $|\phi(\beta_E(m))| > 1$. Write $f := \phi \circ \beta_E \in \mathcal{M}(E)^*$ and take a real number a so that $1 < a < |f(m)|$. Let $\epsilon > 0$. Consider a representation of m, $m = \sum_{i=1}^n \lambda_i m_{x_i,x_i'}$ so that $\sum_{i=1}^n |\lambda_i| \|x_i - x_i'\| < 1 + \epsilon$. Then,

$$a < |f(m)| = |\sum_{i=1}^n \lambda_i \phi(\beta_E(\delta_E(x_i) - \delta_E(x_i')))| = |\phi(\sum_{i=1}^n \lambda_i(x_i - x_i'))| < 1 + \epsilon.$$

Letting $\epsilon \to 0$, we get a contradiction. □

The map $(\delta_E \circ T)_L$ appearing in (iii) of the next result is already known (see [13, Lemma 2.2]): it is the unique linear map $\hat{T} : Æ(F) \to Æ(E)$ such that $\hat{T} \circ \delta_F = \delta_E \circ T$.

Proposition 3.3. *Let F and E be Banach spaces, and $\mathcal{I}_{Lip} = \mathcal{I} \circ Lip_0$ a composition Lipschitz operator ideal. Consider a Lipschitz map $T : F \to E$.*

(i) If T is linear and $\mathcal{I}$-Lipschitz compact, then $T_L : Æ(F) \to E$ is (linear) $\mathcal{I}$-compact.

(ii) If $\delta_E \circ T_L : Æ(F) \to Æ(E)$ sends the unit ball to a relatively $\mathcal{I}$-compact set, then T is $\mathcal{I}$-Lipschitz compact.

(iii) If $(\delta_E \circ T)_L : Æ(F) \to Æ(E)$ is (linear) $\mathcal{I}$-compact, then T is $\mathcal{I}$-Lipschitz compact.

(iv) If T is linear, then T is $\mathcal{I}$-Lipschitz compact if, and only if, $T_L : Æ(F) \to E$ is (linear) $\mathcal{I}$-compact.

Proof. (i) Assume that T is linear and $\mathcal{I}$-Lipschitz compact. By definition, $T(B_F)$ is relatively Lipschitz $\mathcal{I}$-compact. By Proposition 3.2, $T(B_F)$ is relatively $\mathcal{I}$-compact. Note that since T is linear so is $T \circ \beta_F : Æ(F) \to E$ (where β_F is the barycentric map defined above), and

$$(T \circ \beta_F) \circ \delta_F = T \circ (\beta_F \circ \delta_F) = T.$$

But T_L is by definition the unique linear map $Æ(F) \to E$ such that $T = T_L \circ \delta_F$, so we obtain that $T_L = T \circ \beta_F$. Now, using Lemma 3.1,

$$T_L(B_{Æ(F)}) = T \circ \beta_F(B_{Æ(F)}) = T(B_F),$$

so $T_L(B_{Æ(F)})$ is a relatively $\mathcal{I}$-compact set and thus T_L is linear $\mathcal{I}$-compact.

(ii) This is an immediate consequence of Proposition 3.2. By hypothesis we have that $\delta_E \circ T_L(B_{Æ(F)})$ is relatively $\mathcal{I}$-compact. By Proposition 3.2, we have that $T_L(B_{Æ(F)})$ is a relatively $\mathcal{I}$-Lipschitz compact set. Since $\delta_F(B_F) \subset B_{Æ(F)}$ and $T = T_L \circ \delta_F$, it follows that $T(B_F)$ is relatively $\mathcal{I}$-Lipschitz compact and thus T is $\mathcal{I}$-Lipschitz compact.

(iii) If $(\delta_E \circ T)_L$ is $\mathcal{I}$-compact, then $(\delta_E \circ T)_L(B_{Æ(F)})$ is relatively $\mathcal{I}$-compact. Since $(\delta_E \circ T)(B_F) = ((\delta_E \circ T)_L) \circ \delta_F)(B_F) \subseteq (\delta_E \circ T)_L(B_{Æ(F)})$ it follows that $(\delta_E \circ T)(B_F)$ is relatively $\mathcal{I}$-compact and thus by Proposition 3.2 we conclude that $T(B_F)$ is relatively $\mathcal{I}$-Lipschitz compact.

(iv) We only need to prove ($\Longleftarrow$), since (i) gives the other implication.

Suppose that T_L is $\mathcal{I}$-compact. Then $T_L(B_{Æ(F)})$ is relatively $\mathcal{I}$-compact and thus it is relatively $\mathcal{I}$-Lipschitz compact by Proposition 3.2. Since $\delta_F(B_F) \subset B_{Æ(F)}$,

$$T(B_F) = (T_L \circ \delta_F)(B_F) \subset T_L(B_{Æ(F)}),$$

so $T(B_F)$ is relatively $\mathcal{I}$-Lipschitz compact, and thus T is $\mathcal{I}$-Lipschitz compact.

□

Proposition 3.3 gives that for linear operators, being (Lipschitz) compact and being $\mathcal{L}$-Lipschitz compact coincide. The first three statements of the following result have been proved in [17, Proposition 2.1].

Corollary 3.2. *Let F and E be Banach spaces and consider a linear operator $T : F \to E$. The following assertions are equivalent.*

(i) T is compact.

(ii) $T_L : Æ(F) \to E$ is compact.

(iii) T is Lipschitz compact as a Lipschitz map.

(iv) T is $\mathcal{L}$-Lipschitz compact as a Lipschitz map.

3.1 The $\mathcal{I}$-approximation property for Lipschitz operators.

Consider a linear operator ideal $\mathcal{I}$ and let $\mathcal{I}_{Lip} = \mathcal{I} \circ Lip_0$ be the associated composition Lipschitz operator ideal. On $Lip_0(X, E)$, we define the topology *Lipschitz-$\tau_{\mathcal{I}}$* of uniform convergence on $\mathcal{I}$-

Lipschitz compact sets in the space of operators $Lip_0(X, E)$ as the one generated by the seminorms

$$q_K(T) := \sup_{x \in K} \|T(x)\| = \sup_{m \in \delta_K(K)} \|T_L(m)\|_E,$$

where K is a relatively $\mathcal{I}$-Lipschitz compact set of X.

Note that this topology induces on the space $\mathcal{L}(F, E)$, of linear operators between Banach spaces F and E, the topology $\tau_{\mathcal{I}}$ of uniform convergence on $\mathcal{I}$-compact sets.

Definition 3.3. *Let X be a pointed metric space. Consider a class of operators $\mathcal{O}(X, Æ(X)) \subseteq Lip_0(X, Æ(X))$ with the operations inherited from this linear space. We say that X has the $\mathcal{I}$-Lipschitz approximation property with respect to $\mathcal{O}(X, Æ(X))$ if $\delta_X : X \to Æ(X)$ belongs to the Lipschitz-$\tau_{\mathcal{I}}$-closure of $\mathcal{O}(X, Æ(X))$.*

Of course, when looking for a genuine version of the approximation property for metric spaces, the elements of $\mathcal{O}$ must have some sort of finite-range-type property. In fact, the case $\mathcal{O} = Lip_{0\mathcal{F}}$ provides the main classical characterization of an approximation type property, as we will show in what follows. There are also two more interesting cases of sets of operators $\mathcal{O}$ that will be analyzed in the next section.

The first result is a natural extension of Proposition 3.6 of [21] for the Lipschitz case that can be obtained as a consequence of the factorization of the Lipschitz operators through $Æ(X)$.

Proposition 3.4. *Let X be a pointed metric space. The following assertions are equivalent.*

(i) X has the $\mathcal{I}$-Lipschitz approximation property with respect to $Lip_{0\mathcal{F}}(X, Æ(X))$.

(ii) For every Banach space E, $Lip_{0\mathcal{F}}(X, E)$ is Lipschitz-$\tau_{\mathcal{I}}$ dense in $Lip_0(X, E)$.

Proof. (i) $\Rightarrow$ (ii). Consider a Banach space E and $\phi \in Lip_0(X, E)$. Take $\varepsilon > 0$ and an $\mathcal{I}$-Lipschitz compact subset K of X. Let $g_\varepsilon \in Lip_0(X, Æ(X))$ satisfying that $\sup_{x \in K} \|\delta_X(x) - g_\varepsilon(x)\|_{Æ(X)} < \varepsilon$.

Since there is a factorization for ϕ given by $\phi = \phi_L \circ \delta_X$, for each $x \in K$ we get that

$$\|\phi(x) - \phi_L \circ g_\varepsilon(x)\| \leq \|\phi_L\| \|\delta_X(x) - g_\varepsilon(x)\|_{Æ(X)} \leq \varepsilon \, Lip(\phi).$$

This gives the proof. (ii) ⇒ (i) is obvious.

□

The second main property related to the approximation property concerns the approximation of compact operators by finite rank ones. Let us show the Lipschitz version. If A is a subset of a Banach space E then $\overline{co}(A)$ denotes the closed convex hull of A.

Proposition 3.5. *Let X be a pointed metric space and $\mathcal{I}$ be an operator ideal. If X has the Lipschitz $\mathcal{I}$-approximation property with respect to $Lip_{0\mathcal{F}}(X, Æ(X))$ then, for any pointed metric space Z and any $\mathcal{I}$-Lipschitz compact mapping $\phi : Z \to X$, the mapping $\delta_X \circ \phi$ can be approximated by finite rank operators of $Lip_{0\mathcal{F}}(Z, ÆX))$ uniformly on B_Z.*

Proof. Assume that ϕ is Lipschitz $\mathcal{I}$-compact. There is a relatively $\mathcal{I}$-Lipschitz compact subset K of X such that $\phi(B_Z) \subseteq K$. Fix $n \in \mathbb{N}$. Then by the approximation property of X there is a finite rank Lipschitz map g_n such that $\sup_{x \in K} \|\delta_X(x) - g_n(x)\|_{(X)} < 1/n$. Consequently,

$$\sup_{z \in B_Z} \|\delta_X \circ \phi(z) - g_n \circ \phi(z)\|_{Æ(X)} \leq \sup_{x \in K} \|\delta_X(x) - g_n(x)\|_{Æ(X)} < \frac{1}{n}.$$

This gives a sequence $(g_n \circ \phi)_n$ of finite rank Lipschitz maps converging to $\delta_X \circ \phi$ uniformly on B_Z, and the result follows.

□

3.2 The relation between Lipschitz and linear approximation properties.

The purpose of this section is to show that the new concepts and results we have stated for the Lipschitz setting fit with the definitions and properties of the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property. To establish the connection, we consider the $\mathcal{I}$-Lipschitz approximation property with respect to the sets $\mathcal{O}_1(E, \text{Æ}(E)) = \delta_E \circ Lip_{0\mathcal{F}}(E,E)$ and $\mathcal{O}_2(X, \text{Æ}(X)) = Lip_{0\mathcal{F}}(\text{Æ}(X), \text{Æ}(X)) \circ \delta_X$. We will show that the choice of $\mathcal{O}_1$ or $\mathcal{O}_2$ depends on which version of the two canonical cases we want to get: either when X is a Banach space and has the approximation property or when Æ(X) has the approximation property.

Our first aim now is to prove that the $\mathcal{I}$-Lipschitz approximation property is weaker than the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property when they can be compared, that is, if a Banach space E has the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property, then it has the $\mathcal{I}$-Lipschitz approximation property with respect to the set $\delta_E \circ Lip_{0\mathcal{F}}(E,E)$ too. This clearly provides a lot of examples of our new approximation property for pointed metric spaces.

Proposition 3.6. *Let $\mathcal{I}$ be an operator ideal. Let E be a Banach space with the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property. Then it has the $\mathcal{I}$-Lipschitz approximation property as a metric space with respect to the set $\delta_E \circ Lip_{0\mathcal{F}}(E,E)$.*

Proof. Suppose that E has the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property. Then there is a sequence of finite rank operators $(T_n)_n$ that converges to Id_E in the $\tau_{\mathcal{I}}$ topology. Let us show that the sequence $(\hat{T}_n)_n$ of Lipschitz maps defined as $\hat{T}_n(x) = \delta_E(T_n(x)) \in \text{Æ}(E)$, $x \in E$, converges to δ_E in the Lipschitz-$\tau_{\mathcal{I}}$ topology. For a fixed relatively $\mathcal{I}$-Lipschitz compact subset K of E, consider the seminorm

$$q_K(R) := \sup_{x \in K} \|R(x)\|_{Æ(E)}, \quad R \in Lip_0(E, Æ(E)).$$

Recall that, by Proposition 3.2, every relatively $\mathcal{I}$-Lipschitz compact subset of E is relatively $\mathcal{I}$-compact. Thus, K is also a relatively $\mathcal{I}$-compact subset of E, and then this formula defines also a seminorm of the topology $\tau_{\mathcal{I}}$. We get

$$q_K(\delta_E - \hat{T}_n) = \sup_{x \in K} \|m_{x,0} - \delta_E(T_n(x))\|_{Æ(E)}$$

$$= \sup_{x \in K} \|m_{x,0} - m_{T_n(x),0}\|_{Æ(E)} \leq \sup_{x \in K} \|x - T_n(x)\|_E.$$

This proves the result. □

Some results are known about the standard approximation property for the free spaces $Æ(X)$ (see [8, 9, 14] and the references therein). In this direction, we can also show that under the hypothesis that $Æ(X)$ has the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property, we have that X has the $\mathcal{I}$-Lipschitz approximation property too with respect to a finite-range-type class of operators, showing that our definition can also be applied in these cases. Recall that $\mathcal{F}(E, E)$ denotes the space of all finite rank operators on the Banach space E.

Proposition 3.7. *Let $\mathcal{I}$ be an operator ideal and consider the associated composition Lipschitz operator ideal $\mathcal{I}_{Lip} = \mathcal{I} \circ Lip_0$. Let X be a pointed metric space. If $Æ(X)$ has the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property, then X has the $\mathcal{I}$-Lipschitz approximation property with respect to the class $\mathcal{F}(Æ(X), Æ(X)) \circ \delta_X$.*

Proof. Suppose that (X) has the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property as a Banach space. Fix a relatively $\mathcal{I}$-Lipschitz compact subset K of X. By Proposition 3.2 $\delta_X(K)$ is a relatively $\mathcal{I}$-compact set.

By [20, Proposition 3.2], there is a sequence of linear finite rank operators $T_n : Æ(X) \to Æ(X)$ such that T_n converges to $Id_{Æ(X)}$ uniformly on $\delta_X(K)$. Consider the finite rank Lipschitz maps $\tilde{T_n} := T_n \circ \delta_X$, that define Lipschitz operators from X to $Æ(X)$. It follows that

$$\sup_{x\in K} \|\delta_X(x) - \tilde{T_n}(x))\|_{Æ(X)} = \sup_{x\in K} \|\delta_X(x) - T_n \circ \delta_X(x)\|_{Æ(X)}$$

$$= \sup_{w\in\delta_X(K)} \|w - T_n(w)\|_{Æ(X)}$$

and this finishes the proof. □

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