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إهداء

الحمد لله الذي وهبني عقلا مفكرا، ولسانا ناطقا وأنار دربي، ويسر أمري لانهاء هذا العمل، والصلاة والسلام على رسول الله صلى الله عليه وسلم.

إذا كان الإهداء يعبر ولو بجزء من الوفاء، فأهدي ثمرة جهدي:

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إلى من قال فيهم الشاعر:

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Introduction

In the classical set theory, the crisp set is defined in such a way as to dichotomize the elements in some given universe of discourse into two groups: 1) Members, which belong in the set; 2) Non-members, which do not belong in the set i,.e, An element belongs to the set or does not belong to the set. Then distinction exists between the members and non-members of the set. However, many classification concepts that we commonly use and express in natural language describe sets that do not have this property, e.g. the set of tall people or expensive cars. In classical mathematics, problems could only be solved when Lotfi A. Zadeh presented a vague approach to set theory in 1965 [52]. The fuzzy set theory can be viewed as a generalization of the classical set theory. Because of this generalization, broad applications of fuzzy set theory have been found in various fields such as mathematics, computer science, artificial intelligence, medical sciences, economics, statistics, neural networks, etc . . .

Shortly thereafter, a further generalization in J. Goguen [32], about the unit of intervals $[0, 1]$ by an abstract set L (where L is a Lattice). Several authors then examined this new concept and its applications in many areas of modernity, see for example [53, 54, 33, 27, 28, 29, 5, 38].

The study of sets of objects known as lattices is called lattice theory. This theory was given a great boost by a series of papers and a subsequent textbook written by Birkhoff [13, 12].

In the same way, Yuan and Wu [15] studied the relationship between fuzzy ideals and fuzzy congruences on a distributive lattice and obtained that the lattice of fuzzy ideals is isomorphic to the lattice of fuzzy congruence on a generalized Boolean algebra. Ajmal and Thomas [2] defined and characterized a fuzzy sublattice as a fuzzy algebra. After a few years, Thomas and Nair [47, 48] studied intuitionistic fuzzy sublattice, intuitionistic fuzzy ideals, and intuitionistic fuzzy filters on a lattice, for more details we refer to [30, 1, 37, 36, 42, 47, 48].

In 1983 Atanassov [6] introduced a new notion called an intuitionistic fuzzy set as a generalization of fuzzy set [52]. In the fuzzy set theory, the degree of non-membership of an element x can be viewed as $\nu_A(x) = 1 - \mu_A(x)$. (Using a strong standard negation for the real interval $[0, 1]$), which is fixed, In the intuitionistic fuzzy environment, the degree of non-membership is a more or less independent degree: the only condition is that $\nu_A(x) \leq 1 - \mu_A(x)$. Certainly, fuzzy sets are intuitionistic Atanassov fuzzy sets where $\nu_A(x) = 1 - \mu_A(x)$.

In principle, models based on intuitionistic fuzzy sets may be suitable in situations where we are confronted with human testimonies, opinions, etc. Answers like yes or no do not apply. This concept is widespread and is discussed in a vague and intuitive sense. This concept is frequently used and discussed by various authors in fuzzy and fuzzy intuitionist environments [11, 41, 50, 49, 55].

Burillo and Bustince [17, 18] introduced the concept of intuitionistic fuzzy relation, in particular, they introduced the intuitionistic fuzzy order (or intuitionistic fuzzy ordered set) as a natural generalization of fuzzy order relation previously introduced by Zadeh [53]. Banerjee and Basnet in [10] studied the notion of intuitionistic fuzzy ideals on a ring, Akram and Dudek [3] introduced and studied the concept of intuitionistic fuzzy Lie ideals. Qin and Liu [40, 45] introduced and investigated the properties of intuitionistic fuzzy filters on a residuated lattice and Thomas and Nair [47, 48] considered intuitionistic fuzzy sublattices, intuitionistic fuzzy ideals, and intuitionistic fuzzy filters on a lattice.

In [51], Xu introduced the notion of interval-valued intuitionistic (T, S)-fuzzy filter on a lattice implication algebra subsequently, some basic properties of it were obtained. Because of the usefulness of these concepts in different structures, the first goal of this work is the representation and construction of the fuzzy preorder, and the weak orders are extended to the intuitionistic fuzzy case. Many fundamental representation results extending those of [16] are presented. The second goal is introducing the notion of a $\mathcal{T}$-intuitionistic fuzzy sublattice by associating the conditions mentioned in the definition of intuitionistic fuzzy sublattice [47, 48]. So a new equivalent definition is obtained which reduces the four conditions in only one. Thus, based on an intuitionistic fuzzy triangular norm, the study of intuitionistic fuzzy sublattices becomes so simple. Moreover, we extend the notion of an intuitionistic fuzzy ideal to a $\mathcal{T}$-intuitionistic fuzzy ideal with respect to the lattice operations and we investigate their various characterizations and properties.

This thesis is structured as follows.

Chapter 1, provides generalities on triangular norms, triangular conorms lattices, residuated lattices, fuzzy sets, fuzzy relations, intuitionistic fuzzy sets, intuitionistic fuzzy Triangular norms and intuitionistic fuzzy triangular conorms, that we need throughout this thesis. Chapter 2, is devoted to the representation and construction of the intuitionistic fuzzy $\mathcal{T}$-E-order. Chapter 3, gives some new results on intuitionistic fuzzy sublattices and their ideals.

Finally, general conclusions and future research are drawn.

Most parts of the results presented in this thesis have already been published.

Chapter 1

Preliminaries

In this chapter, we recall some useful notions, basic definitions and well-known results on a lattice, residuated lattices, fuzzy sets, and fuzzy relations.

1.1 Triangular norms and conorms

Triangular norms (t-norms), **triangular conorms** (t-conorms), and (strong) **negations** are classes of connectives for fuzzy logic analogous to the connectives in classical two-valued logic. The t-norms are models for "and," the t-conorms are models for "or," and negations are models for "not."

Definition 1.1.1

A binary operations $T\colon\ [0,1]^2 \to [0,1]$ *is a* ***t-norm*** *if it satisfies the following:*

1. *(Boundary condition)* $\forall x \in [0,1] : T(1,x) = x$
2. *(Commutativity)* $\forall x, y \in [0,1] : T(x,y) = T(y,x)$
3. *(Associativity)* $\forall x, y, z \in [0,1] : T(x, T(y,z)) = T(T(x,y), z)$
4. *(Monotonicity)* $\forall x, y, z, w \in [0,1] :$

   If $w \leq x$ *and* $y \leq z$ *then* $T(w,y) \leq T(x,z)$

Definition 1.1.2

A binary operations $S\colon\ [0,1]^2 \to [0,1]$ *is a* ***t-conorm*** *if it satisfies the following:*

1. *(Boundary condition)* $\forall x \in [0,1] : S\,(0,x) = x$
2. *(Commutativity)* $\forall x, y \in [0,1] : S\,(x,y) = S\,(y,x)$
3. *(Associativity)* $\forall x, y, z \in [0,1] : S\,(x, S\,(y,z)) = S\,(S\,(x,y)\,, z)$
4. *(Monotonicity)* $\forall x, y, z, w \in [0,1]$:

   If $w \le x$ *and* $y \le z$ *then* $S\,(w,y) \le S\,(x,z)$

The three basic t-norms (Prominent examples)

1. **Minimum and Maximum**

   t-norm: $T\,(x,y) = min(x,y)$

   t-conorm: $S\,(x,y) = max(x,y)$

(a) t-norm: $T\,(x,y) = min(x,y)$ (b) t-conorm: $S\,(x,y) = max(x,y)$

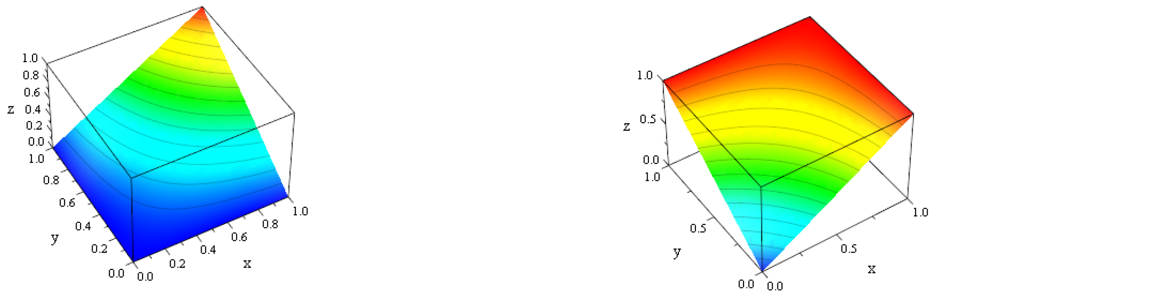
Figure 1.1: Minimum and Maximum.

2. **Algebraic product**

   The algebraic product is the prototypes for all strict t-norms and strict t-conorms, respectively.

   t-norm: $T\,(x,y) = xy$

   t-conorm: $S\,(x,y) = x + y - xy$



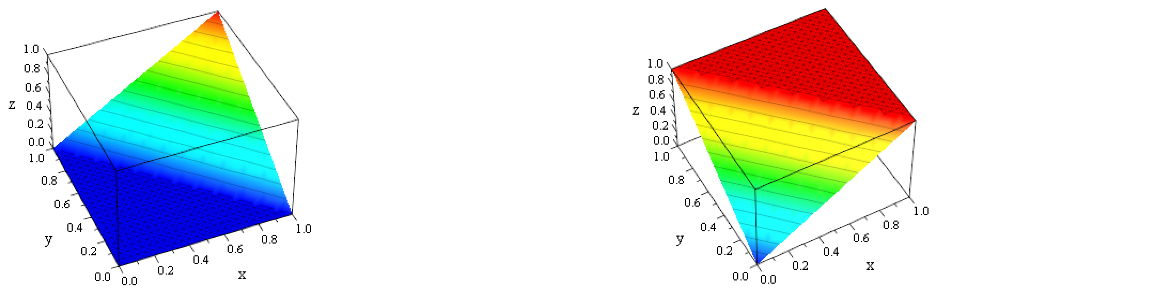
(a) t-norm: $T(x,y) = xy$ (b) t-conorm: $S(x,y) = x + y - xy$

Figure 1.2: Algebraic Product and Algebraic Sum

3. **Łukaciewisz t-norm and t-conorm**

$$\text{t-norm:} \quad T(x,y) = max((x+y-1),0)$$

$$\text{t-conorm:} \quad S(x,y) = min((x+y),1)$$



(a) t-norm: $max((x+y-1),0)$ (b) t-conorm: $min((x+y),1)$

Figure 1.3: Łukaciewisz t-norm and t-conorm

Definition 1.1.3 *A t-norm T is continuous if for all convergent sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ we have,*

$$T(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = \lim_{n\to\infty} T(x_n, y_n)$$

The structure of continuous t-norms is well known, see [39] for more details, especially Section 3.3 on ordinal sums.

Remark 1.1.4

Throughout this thesis, the term T denotes a t-norm that is at least left-continuous.

1.2 Lattices

Let $(L, \leq)$ be a partially ordered set (Poset), and $A \subseteq L$ is be arbitrary subset. An element $u \in L$ is said to be an **upper bound** of A if $a \leq u$ for each $a \in A$. An upper bound u of A is said to be **least upper bound**, or **join**, or **supremum**, if $u \leq x$ for each upper bound x of A. A set need not have a least upper bound, but it cannot have more than one. Dually, $l \in L$ is said to be a lower bound of A if $l \leq a$ for each $a \in A$. A lower bound l of A is said to be its greatest lower bound, or meet, or infimum, if $x \leq l$ for each lower bound x of A. A set may have many lower bounds, or none at all, but can have at most one greatest lower bound.

A partially ordered set $(L, \leq)$ is called a **join-semilattice** if each two-element subset $\{x, y\} \subseteq L$ has a join (i.e. least upper bound), and is called a **meet-semilattice** if each two-element subset has a meet (i.e. greatest lower bound), denoted by $x \vee y$ and $x \wedge y$ respectively. $(L, \leq)$ is called a **lattice** if it is both a join- and a meet-semilattice. This definition makes $\vee$ and $\wedge$ binary operations. Both operations are monotone with respect to the order: $x_1 \leq x_2$ and $y_1 \leq y_2$ implies that $x_1 \vee y_1 \leq x_2 \vee y_2$ and $x_1 \wedge y_1 \leq x_2 \wedge y_2$.

Moreover, If $\bigvee A$ and $\bigwedge A$ exist for any $A \subseteq L$, then $(L, \leq)$ is called a **complete lattice** [21].

1.2.1 Ideals and filters on a lattice

Ideals are of fundamental importance in algebra. Filters, the order duals of lattice ideals, have a variety of applications in logic and topology.

Definition 1.2.1 *[21]*

A nonempty subset I on a lattice L is called an ***ideal*** *of L if, for any $x, y \in L$, the following conditions are satisfied:*

1. *if $y \in I$ and $x \leq y$, then $x \in I$,*
2. *if $x, y \in I$ implies $x \vee y \in I$.*

The definition can be more compactly stated by declaring an ideal to be a non-empty down-set closed under join.

A dual ideal is called a filter. Specifically, a non-empty subset of L determined by the following definition.

Definition 1.2.2 *[21]*

A nonempty subset F on a lattice L is called a ***filter*** *if, for any $x, y \in L$, the following conditions are satisfied:*

1. *if $y \in F$ and $y \leq x$, then $x \in F$,*
2. *if $x, y \in F$ implies $x \wedge y \in F$.*

The set of all ideals (resp. filters) of L is denoted by $I(L)$ (resp. $F(L)$).

1.3 Residuated lattices

Residuated lattices are mostly found in algebraic structures associated with a variety of logical systems.

Definition 1.3.1 *[34, 35, 44]*

A ***residuated lattice*** *is an algebra* $L = (L, \wedge, \vee, *, \rightarrow, 0, 1)$ *of type* $(2, 2, 2, 2, 0, 0)$ *such that:*

1. $(L, \wedge, \vee, 0, 1)$ *is a bounded lattice,*

2. $(L, *, 1)$ *is a commutative monoid, and*

3. *The operation* $*$ *and* $\rightarrow$ *form an adjoint pair, i.e.,*

$$x * y \leq z \quad if\ and\ only\ if \quad x \leq y \rightarrow z \quad for\ all\ x, y, z \in L. \tag{1.1}$$

1.4 Fuzzy sets

Definition 1.4.1

If X *is a collection of objects denoted generically by* x*, then a* ***fuzzy set*** A *in* X *is a set of ordered pairs* $A = \{(x, \mu_A(x)) \mid x \in X\}$*, where* $\mu_A(x)$ *is called the membership function (i.e.,* $\mu_A \colon X \longrightarrow [0, 1]$*). A fuzzy set is represented solely by stating its membership function [43].*

The set of all fuzzy subsets of X *are called* ***fuzzy power sets*** *of* X *and denoted by the symbol* $\mathcal{F}(X)$*.*

Below we present some examples a variety of representable for a fuzzy sets

Example 1.4.2 *[58]*

A realtor wants to classify the house he offers to his clients. One indicator of the comfort of these houses is the number of bedrooms in it. Let $X = \{1, 2, 3, 4, ..., 10\}$ *be the set of available types of houses described by* $x =$*number of bedrooms in a house. Then the fuzzy set "comfortable type of house for a four-person family" may be described as*

$$A = \{(1, 0.2), (2, 0.5), (3, 0.8), (4, 1), (5, 0.7), (6, 0.3)\}$$

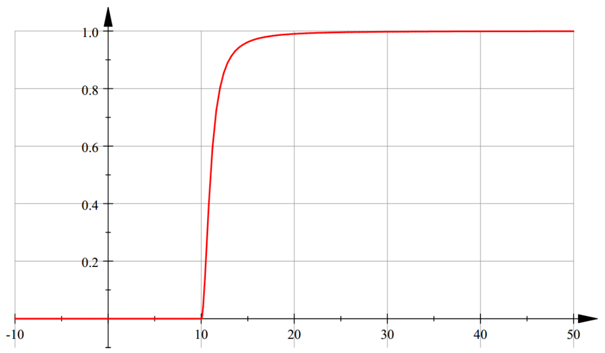
Example 1.4.3 *[58]*

$A =$*"real numbers considerably larger than* 10*"*

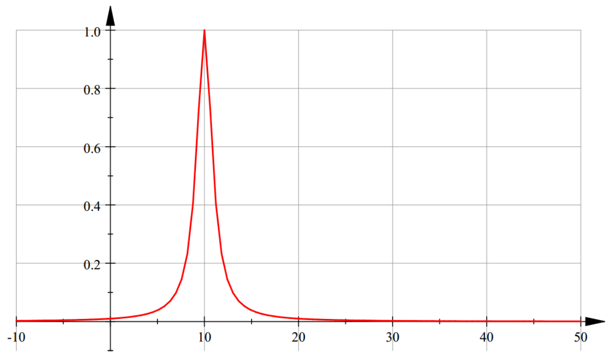
$$A = \{(x, \mu_A(x)) \mid x \in X\}$$

Where

$$\mu_A(x) = \begin{cases} 0, & x \leq 10 \\ \left(1 + (x - 10)^{-2}\right)^{-1}, & x > 10 \end{cases}$$



(a) $A =$"real numbers considerably larger than 10"



(b) $A =$"real numbers close to 10".

Figure 1.4: Representation of fuzzy sets

Remark 1.4.4

1. *In Example 1.4.2 we represented a fuzzy set by an ordered set of pairs, the first element of which designates the element and the second is the degree of belonging.*

2. *In Example 1.4.3 fuzzy set is represented solely by stating its membership function.*

1.4.1 Operations on fuzzy sets

Let $A, B \in \mathcal{F}(X)$. We have for all $x \in X$,

1. $A = \emptyset$ if and only if $\mu_A(x) = 0$.

2. $A = B$ if and only if $\mu_A(x) = \mu_B(x)$.

3. $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$.

4. **Intersection** (logical and): The membership function of the intersection of two fuzzy sets A and B is defined as $\mu_{A\cap B}(X) = \min(\mu_A(x), \mu_B(x))$.

5. **Union** (exclusive or): The membership function of the union is defined as: $\mu_{A\cup B}(X) = \max(\mu_A(x), \mu_B(x))$.

6. **Complement** (negation): The membership function of the complement of a normalized fuzzy set $\rceil A$, $\mu_{\rceil A}(x)$ is defined by $\mu_{\rceil A}(x) = 1 - \mu_A(x)$.

Example 1.4.5 *[58]*

Let A be the fuzzy set "comfortable type of house for a four-person family" from

Example 1.4.2 and B be the fuzzy set "large type of house" defined as

$$B = \{(3, 0.2), (4, 0.4), (5, 0.6), (6, 0.8), (7, 1), (8, 1)\}$$

The intersection $C = A \cap B$ is then

$$C = \{(3, 0.2), (4, 0.4), (5, 0.6), (6, 0.3)\}$$

The union $D = A \cup B$ is

$$D = \{(1, 0.2), (2, 0.5), (3, 0.8), (4, 1), (5, 0.7), (6, 0.8), (7, 1), (8, 1)\}$$

The complement $\rceil B$, which might be interpreted as "not large type of house," is

$$\rceil B = \{(1, 1), (2, 1), (3, 0.8), (4, 0.6), (5, 0.4), (6, 0.2), (9, 1), (10, 1)\}$$

Remark 1.4.6

It has to be noted that in contrast to the classical set theory, the intersection (resp. the union) of a fuzzy set and its complement does not result in the empty set (resp. in the universe). need that

$$\begin{aligned} B \cap \rceil B &= \{(3, 0.2), (4, 0.4), (5, 0.4), (6, 0.2)\} \neq \emptyset. \\ B \cup \rceil B &= \{(1, 1), (2, 1), (3, 0.8), (4, 0.6), (5, 0.6), (6, 0.8), (7, 1), (8, 1), (9, 1), (10, 1)\}, \\ &\neq X \end{aligned}$$

Example 1.4.7

Let us assume that $A =$ "x is considerable larger than 10," and $B =$"x is *approximately* 11*, " characterized by* $A = \{(x, \mu_A(x)) \mid x \in X\}$ *Where*

$$\mu_A(x) = \begin{cases} 0, & x \leq 10 \\ \left(1 + (x-10)^{-2}\right)^{-1}, & x > 10 \end{cases}$$

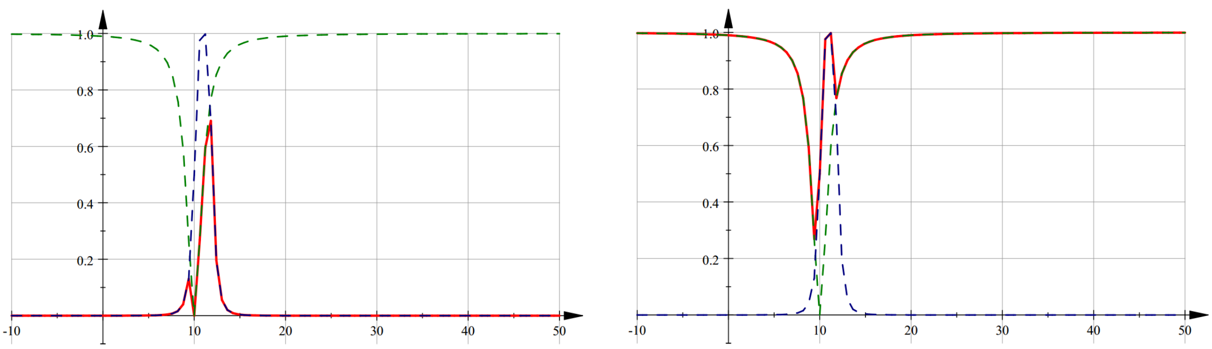
And, $B = \{(x, \mu_B(x)) \mid x \in X\}$*. Where,* $\mu_B(x) = \left(1 + (x-11)^4\right)^{-1}$*. Then,*

$$\mu_{A\cap B}(x) = \begin{cases} 0, & x \leq 10 \\ \min\left(\left(1 + (x-10)^{-2}\right)^{-1}, \left(1 + (x-11)^4\right)^{-1}\right), & x > 10 \end{cases}$$

(x is considerably larger than 10 *and approximately* 11*), and*

$$\mu_{A\cup B}(x) = \max\left(\left(1 + (x-10)^{-2}\right)^{-1}, \left(1 + (x-11)^4\right)^{-1}\right), x \in X$$

Figure 1.5a and 1.5b depict the allow.



(a) Intersection of Fuzzy set A and B. (b) Union of Fuzzy set A and B.

Figure 1.5: Intersection and union of Fuzzy sets.

1.5 Fuzzy relations

In this section, we introduce the definition of fuzzy relations [52] and their basic properties [52, 32].

Definition 1.5.1

Let X and Y be nonempty sets, a fuzzy relation R from X to Y, written $R\colon X \to Y$, is a function

$$R\colon X \times Y \to [0,1]$$

For $(x,y) \in X \times Y$ the value $R(x,y) \in [0,1]$ means the degree to which x and y are related under R.

The set of all fuzzy relations from X to Y will be denoted by $FR(X,Y)$. A fuzzy relation R is contained in a fuzzy relation S, written $R \subset S$, if $R(x,y) \leq S(x,y)$, for all $(x,y) \in X \times Y$.

Definition 1.5.2

A fuzzy relation R on a fixed set X is a function,

$$R : X \times X \to [0,1]$$

The set of all fuzzy relations on a fixed set X will be denoted by $FR(X)$.

1.5.1 L-Relations

Let $L = (L, \leq, \wedge, \vee, 0, 1)$ be a fixed complete distributive lattice.

Now recall some fundamentals on L-relations [32].

Definition 1.5.3

Let X and Y be non-empty sets, an L-relation R from X to Y is a function

$$R\colon X \times Y \to L.$$

The set of all L-relations from X into Y will be denoted by $L - FR(X, Y)$.

Definition 1.5.4

An L-relation R on a fixed set X is a function $R\colon\ X \times X \to L$.

The set of all L-relations on a fixed set X will be denoted by $L - FR(X)$.

Proposition 1.5.5

Let $R \in FR(X)$, R is called:

1. ***Reflexive**, if, for any $x \in X$, it holds that $R(x, x) = 1$.*

2. ***Symmetric**, if, for any $x, y \in X$, it holds that $R(x, y) = R(y, x)$.*

3. ***Antisymmetric**, if, for any $x, y \in X, x \neq y$ it holds that $R(x, y) > 0$ then $R(y, x) = 0$.*

4. ***Complete (strangle)**, if, for any $x, y \in X$ it holds that*

$$R(x, y) > 0 \text{ or } R(y, x) > 0$$

5. ***Transitive**, if, for any $x, z \in X$ it holds that*

$$R(x, z) \geq \max_{y \in X}(\min(R(x, y), R(y, z)))$$

Moreover, (Since $R^2 = R \circ R$ if, $R^2(x, z) = \max_{y \in X}(R(x, y), R(y, z)))$ then R is transitive if $R \circ R = R$, $(R \circ R \subseteq R)$ and $R^2 \subset R$ means that $R^2(x, y) \leq R(y, z)$.

Example 1.5.6

Let $X = \{x_1, x_2, x_3\}$, *we defined* $R : X \times X \to [0,1]$ *as follows,*

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 0.7 & 0.9 & 0.4 \\ 0.1 & 0.3 & 0.5 \\ 0.2 & 0.1 & 0 \end{pmatrix} \end{array}$$

A transitive relation ? Solution

$$R \circ R = \begin{pmatrix} 0.7 & 0.9 & 0.4 \\ 0.1 & 0.3 & 0.5 \\ 0.2 & 0.1 & 0 \end{pmatrix} \circ \begin{pmatrix} 0.7 & 0.9 & 0.4 \\ 0.1 & 0.3 & 0.5 \\ 0.2 & 0.1 & 0 \end{pmatrix} = \begin{pmatrix} 0.7 & 0.7 & 0.5 \\ 0.2 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.2 \end{pmatrix} = R^2$$

Since $R^2(x_i, x_j)$ *is not always less than or equal to* $R(x_i, x_j)$*, hence* R *is not transitive.*

Example 1.5.7

Let $X = \{x_1, x_2\}$, *we defined* $R : X \times X \to [0,1]$ *as follows,*

$$R = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{pmatrix} 0.4 & 0.2 \\ 0.7 & 0.3 \end{pmatrix} \end{array}$$

A transitive relation? Solution

$$R \circ R = \begin{pmatrix} 0.4 & 0.2 \\ 0.7 & 0.3 \end{pmatrix} \circ \begin{pmatrix} 0.4 & 0.2 \\ 0.7 & 0.3 \end{pmatrix}$$

Using max-min composition

$$R^2 = \begin{pmatrix} \max(\min(R(0.4,0.4), R(0.2,0.7))) & \max(\min(R(0.4,0.2), R(0.2,0.3))) \\ \max(\min(R(0.7,0.4), R(0.3,0.7))) & \max(\min(R(0.7,0.2), R(0.3,0.3))) \end{pmatrix}$$

$$R^2 = \begin{pmatrix} \max(0.4,0.2) & \max(0.2,0.2) \\ \max(0.4,0.3) & \max(0.2,0.3) \end{pmatrix}$$

$$R^2 = \begin{pmatrix} 0.4 & 0.2 \\ 0.4 & 0.3 \end{pmatrix}$$

Since $R^2(x_i, x_j)$ is less than or equal to Since $R(x_i, x_j)$, so R is transitive.

6. *T-**Transitive**, if, for any $x, y, z \in X$ it holds that $T(R(x,y), R(y,z)) \leq R(x,z)$ where T denotes a t-norm.*

1.5.2 Order fuzzy relation and equivalence fuzzy relation

Definition 1.5.8

Let T denotes a t-norm and let $R \in FR(X)$, R is called:

*(i) A **preorder (pseudo-order) fuzzy relation**, if it is reflexive and transitive.*

*(i') A T-**preorder fuzzy relation**, if it is reflexive and T-transitive.*

*(ii) An **order fuzzy relation**, if it is reflexive, antisymmetric, and transitive.*

*(ii') A T-**order fuzzy relation**, if it is reflexive, antisymmetric, and T-transitive.*

*(iii) A **total order fuzzy relation**, if it is reflexive, antisymmetric, transitive, and complete.*

*(iii') A **total** T**-order fuzzy relation**, if it is reflexive, antisymmetric, T-transitive, and complete.*

*(iv) A **equivalence fuzzy relation** if it is reflexive, symmetric, and transitive.*

*(iv') A T**-equivalence fuzzy relation** if it is reflexive, symmetric, and T-transitive.*

For more details about the order, strict order and equivalence relations we refer to [31].

1.5.3 Fuzzy sublattices and ideals

In this section, we recall some definitions and concepts needed in the sequel.

Definition 1.5.9 *[48]*

A fuzzy subset μ of L is called a fuzzy sublattice of L if,

(i) $\mu(x \vee y) \geq \min(\mu(x), \mu(y)\}$

(ii) $\mu(x \wedge y) \geq \min(\mu(x), \mu(y)\}$, *for all* $x, y \in L$.

Definition 1.5.10 *[48]*

A fuzzy subset μ of L is called a fuzzy ideal of L if,

(i) $\mu(x \vee y) \geq \min(\mu(x), \mu(y)\}$

(ii) $\mu(x \wedge y) \geq \max(\mu(x), \mu(y)\}$, *for all* $x, y \in L$.

Definition 1.5.11 *[48]*

A fuzzy subset μ of L is called a fuzzy filter of L if,

(i) $\mu(x \vee y) \geq \max(\mu(x), \mu(y)\}$

(ii) $\mu(x \wedge y) \geq \min(\mu(x), \mu(y)\}$, *for all* $x, y \in L$.

1.6 Intuitionistic fuzzy sets

Intuitionistic fuzzy sets (IFS) were introduced in 1983 by K. T. Atnassov [6] as a generalization of fuzzy sets [52]. He introduced a new component degree of nonmembership in addition to the degree of membership in the case of fuzzy sets with the requirement that their sum is less than or equal to one. The complement of the two degrees to one is regarded as a degree of uncertainty. Since then a great number of theoretical and practical results appeared in the area of Intuitionistic Fuzzy sets.

Definition 1.6.1 *[6, 8, 7]*

Let X *be a nonempty set. An intuitionistic fuzzy set (IFS, for short)* A *on* X *is an object of the form* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ *characterized by a membership function* $\mu_A : X \to [0,1]$ *and a non-membership function* $\nu_A : X \to [0,1]$ *which satisfy the condition:*

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \text{ for any } x \in X.$$

For any $x \in X$ *the number* $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ *is called the hesitation degree or the intuitionistic index of* x *to* A.

The class of intuitionistic fuzzy sets on X *is denoted by* $IFS(X)$.

Certainly, fuzzy sets are intuitionistic fuzzy sets by setting $\nu_A(x) = 1 - \mu_A(x)$.

Definition 1.6.2

Let $A, B \in IFS(X)$*, several operations are defined as follows (see, e.g., [49, 14, 6, 7, 9]*

(i) $A \subseteq B$ *if* $\mu_A(x) \leq \mu_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$*, for any* $x \in X$;

(ii) $A = B$ *if* $\mu_A(x) = \mu_B(x)$ *and* $\nu_A(x) = \nu_B(x)$, *for any* $x \in X$;

(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\}$;

(vi) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}$;

(v) $\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$;

Definition 1.6.3

Let $A \in IFS\,(X)$ *where* $A\,(x) = (\mu_A\,(x)\,, \nu_A\,(x))\,,$ *for* $x \in X$, *the sets*

1. $[A]\,(x) = (\mu_A\,(x)\,, \mu_A^c\,(x))$, *where* $\mu_A^c\,(x) = 1 - \mu_A\,(x)$,
2. $\langle A\rangle\,(x) = (\nu_A^c\,(x)\,, \nu_A\,(x))$, *where* $\nu_A^c\,(x) = 1 - \nu_A\,(x)$.

Are called, respectively, necessity and possibility operators.

Definition 1.6.4

Let A *be an intuitionistic fuzzy set on a set* X. *The support of* A *is the crisp subset on* X *given by*

$$Supp(A) = \{x \in X \mid 0_{L^*} <_{L^*} A(x)\}$$

Remark 1.6.5

Let A *be an intuitionistic fuzzy set on a set* X. *According to Definition 1.6.4, the support of* A *is the crisp subset on* X *given by,*

$$\{x \in X \mid \mu_A\,(x) > 0 \text{ or } \mu_A\,(x) = 0 \text{ and } \nu\,(x) < 1\}$$

In the following, we give some elementary notions and definition on intuitionistic fuzzy sets operations, intuitionistic fuzzy t-norms and intuitionistic fuzzy t-conorms.

1.7 Intuitionistic fuzzy triangular norms and triangular conorms

Berthold Schweizer and Abe Sklar in [46] gave an axiomatic approach to t-norms as they are used today. Deschrijver, Cornelis, and Kerre have been extended triangular norms to intuitionistic fuzzy triangular norms [22]. In [26] they have shown that intuitionistic fuzzy sets can also be seen as L-fuzzy sets in the sense of Goguen [33].

Lemma 1.7.1 *[25, 26]*
Consider the set $L^* = \{(x_1, x_2) \mid (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\}$, *and the operation* $\leqslant_{L^*}$ *defined by, for all* $(x_1, x_2), (y_1, y_2) \in L^*$:

$$(x_1, x_2) \leqslant_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2 \tag{1.2}$$

The structure $(L^*, \leqslant_{L^*})$ *is a complete lattice.*

The algebraic structure in Lemma 1.7.1 will be fundamental for our subsequent investigations. Deschrijver, Cornelis and Kerre have extended the notion of a triangular norm to the intuitionistic fuzzy case [24]. We denote its units by $0_{L^*} = (0, 1)$ *and* $1_{L^*} = (1, 0)$, *equivalently, this lattice can also be defined as an algebraic structure* $(L^*, \wedge_{L^*}, \vee_{L^*})$, *where the meet operator* $\wedge_{L^*}$ *and the join operator* $\vee_{L^*}$ *are defined as follows, for* $(x_1, x_2), (y_1, y_2) \in L^*$,

$$(x_1, x_2) \wedge_{L^*} (y_1, y_2) = (\min(x_1, y_1), \max(x_2, y_2))$$

$$(x_1, x_2) \vee_{L^*} (y_1, y_2) = (\max(x_1, y_1), \min(x_2, y_2))$$

Using this lattice, we easily see that with every intuitionistic fuzzy set $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ *corresponds to an* L^**-fuzzy set, i.e., a mapping* $A\colon\ X \longrightarrow L^*$*. In the sequel, we will use the same notation for an intuitionistic fuzzy set and its associated* L^**-fuzzy set. So for the intuitionistic fuzzy set* A *we will also use the notation*

$$A\,(x) = (\mu_A(x), \nu_A(x))\,.$$

Definition 1.7.2 *[23, 22]*

An intuitionistic fuzzy t-norm $\mathcal{T}$ *is a commutative, associative, and increasing mapping* $\mathcal{T}\colon\ (L^*)^2 \longrightarrow L^*$ *satisfying* $\mathcal{T}(1_{L^*}, x) = x$*, for all* $x \in L^*$*.*

Definition 1.7.3 *[23, 22]*

An intuitionistic fuzzy t-conorm is a commutative, associative, and increasing mapping $\mathcal{S}\colon\ (L^*)^2 \longrightarrow L^*$ *satisfying* $\mathcal{S}(0_{L^*}, x) = x$*, for all* $x \in L^*$*.*

Definition 1.7.4 *[23, 22]*

An intuitionistic fuzzy t-norm $\mathcal{T}$ *(resp. t-conorm* $\mathcal{S}$*) is called representable intuitionistic fuzzy t-norm if there exist a fuzzy t-norm* T *and a fuzzy t-conorm* S *(resp. a fuzzy t-conorm* S' *and fuzzy t-norm* T' *) on* $[0,1]$ *such that, for all* $x, y \in L^*$*,* $\mathcal{T}(x, y) = (T(x_1, y_1), S(x_2, y_2))$ *(resp.* $\mathcal{S}\,(x, y) = (S'\,(x_1, y_1)\,, T'\,(x_2, y_2)$*.* T *and* S *(resp.* S' *and* T'*) are called the representants of* $\mathcal{T}$ *(resp.* $\mathcal{S}$*).*

Also, in intuitionistic fuzzy set theory $\mathcal{S}$*-union and* $\mathcal{T}$*-intersection can be modeled by the newly defined intuitionistic fuzzy t-norms and t-conorms. We* *define, for all $x \in X$ and A, B intuitionistic fuzzy sets in X,*

$$(A \cap B)(x) = \mathcal{T}(A(x), B(x)), \quad (A \cup B)(x) = \mathcal{S}(A(x), B(x)).$$

The following theorem gives the condition which makes an intuitionistic fuzzy triangular norm a representable triangular norm.

Theorem 1.7.5 *[23, 22]*

Given a fuzzy t-norm T and a fuzzy t-conorm S (It's the dully of T) satisfying, for all $a, b \in [0,1]$, $T(a,b) \leq 1 - S(1-a, 1-b)$, the mappings $\mathcal{T}$ and $\mathcal{S}$ defined by $\mathcal{T}(x,y) = (T(x_1,y_1), S(x_2,y_2))$ and $\mathcal{S}(x,y) = (S(x_1,y_1), T(x_2,y_2))$ for $x, y \in L^*$ are a representable intuitionistic fuzzy t-norm and a representable intuitionistic fuzzy t-conorm respectively.

Remark 1.7.6

If $\mathcal{T}$ (resp. $\mathcal{S}$) is a representable intuitionistic fuzzy t-norm (resp. representable intuitionistic fuzzy t-conorm), we denoted by $\mathcal{T} = (T, S)$ (resp. $\mathcal{S} = (S, T)$), where

$$\text{For all}\,(a,b) \in [0,1] \text{ we have } S(a,b) \leq 1 - T(1-a, 1-b).$$

Now, we give some basic examples of representable intuitionistic fuzzy t-norm and representable intuitionistic fuzzy t-conorm (see also [22]). For all $x, y \in L^*$,

(i) $\mathcal{T}_M(x,y) = (\min(x_1,y_1), \max(x_2,y_2))$,

(ii) $\mathcal{T}_L(x,y) = (\max(0, x_1+y_1-1), \min(1, x_2+1-y_1, y_2+1-x_1))$.

Consequently, we can consider of the minimum intuitionistic fuzzy t-norm and the Lukasiewicz intuitionistic fuzzy t-norm, where theire duals are *(i')* $\mathcal{S}_M(x,y) = (\max(x_1,y_1), \min(x_2,y_2))$,

(ii') $\mathcal{S}_L(x,y) = (\min(1, x_1+1-y_2, y_1+1-x_2), \max(0, x_2+y_2-1))$.

Chapter 2

Representation and construction of intuitionistic fuzzy $\mathcal{T}$-preorders and intuitionistic fuzzy $\mathcal{T}$-orders

In this chapter, we consider the problem of representation and construction of intuitionistic fuzzy preorders and weak orders, where many fundamental representations result extending those of Ulrich Bodenhofer et al [16], are presented to the intuitionistic fuzzy $\mathcal{T}$-E-orders case where $\mathcal{T}$ is an intuitionistic fuzzy T-norm and we start with some of the results we need in this thesis and we give representation and construction of intuitionistic fuzzy $\mathcal{T}$-preorders and intuitionistic fuzzy $\mathcal{T}$-orders.

Throughout this thesis, all intuitionistic fuzzy t-norms $\mathcal{T}$ (respectively intuitionistic fuzzy t-conorms $\mathcal{S}$) are representable intuitionistic fuzzy t-norm (resp, representable intuitionistic fuzzy t-conorm) and The structure $(L^*, \leqslant_{L^*})$ is a complete lattice.

2.1 Basic properties

In the following we consider $(L^*, \leqslant_{L^*})$ is a complete lattice and $\mathcal{T}$ is intuitionistic fuzzy t-norm T.

The following lemma will be used to prove some properties.

Lemma 2.1.1 *[57]*

Let $a \in L^*$. Then, $\mathcal{T}(a, 0_{L^*}) = 0_{L^*}$.

Proof.

Let $a \in L^*$. We have $\mathcal{T}(a, 0_{L^*}) \leqslant_{L^*} \mathcal{T}(1_{L^*}, 0_{L^*}) = 0_{L^*}$. Then, $\mathcal{T}(a, 0_{L^*}) = 0_{L^*}$.

■

Conclusion 2.1.2 *[57]*

Consider the lattice $L^*$and $0_{L^*} \in L^*$, then $\mathcal{T}(0_{L^*}, 0_{L^*}) = 0_{L^*}$.

Proposition 2.1.3 *[57]*

Let $a, b \in L^*$. Then,

1. *If $0_{L^*} <_{L^*} a$ and $0_{L^*} <_{L^*} b$ then $0_{L^*} <_{L^*} \mathcal{T}(a, b)$,*
2. *$0_{L^*} <_{L^*} \mathcal{T}_M(a, b)$ then $(0_{L^*} <_{L^*} a$ and $0_{L^*} <_{L^*} b)$,*
3. *$\mathcal{T}_M(a, a) = a$,*
4. *$a \leqslant_{L^*} \mathcal{S}(a, b)$,*
5. *$\mathcal{T}(a, b) \leqslant_{L^*} a$,*
6. *$a = \mathcal{T}_M(a, b)$ if and anly if $a \leqslant_{L^*} b$.*

Proof.

Let $a, b \in L^*$*.*

1. *Using the fact that* $\mathcal{T}$ *is monotone and Conclusion 2.1.2.*
2. *Assume that* $0_{L^*} = a$ *or* $0_{L^*} = b$*, then,* $\mathcal{T}_M(a, b) = 0_{L^*}$*.*
3. *Using the definition of* $\mathcal{T}_M = \wedge_M$*.*
4. *We have* $a = \mathcal{S}(a, 0_{L^*}) \leqslant_{L^*} \mathcal{S}(a, b)$*. Then,* $a \leqslant_{L^*} \mathcal{S}(a, b)$*.*
5. *Since* $\mathcal{T}(a, b) \leqslant_{L^*} \mathcal{T}(a, 1_{L^*}) = a$*, then,* $\mathcal{T}(a, b) \leqslant_{L^*} a$*.*
6. *Trivially.*

■

Definition 2.1.4 *[57]*

Let $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *be two representable intuitionistic fuzzy t-norms.* $\mathcal{T}_1$ *is said to dominate another t-norm* $\mathcal{T}_2$ *(briefly,* $\mathcal{T}_2 \ll \mathcal{T}_1$*) if and only if, for any quadruple* $(x = (x_1, x_2), y = (y_1, y_2), u = (u_1, u_2), v = (v_1, v_2)) \in (L^*)$*, the following holds:*

$$\mathcal{T}_2(\mathcal{T}_1(x, u), \mathcal{T}_1(y, v)) \leqslant_{L^*} \mathcal{T}_1(\mathcal{T}_2(x, y), \mathcal{T}_2(u, v))$$

The following lemma will be used to prove the subsequent results.

Lemma 2.1.5 *[57]*

For any representable intuitionistic fuzzy t-norm $\mathcal{T}$*, we have* $\mathcal{T} \ll \mathcal{T}_M$*.*

Proof.

Let $x, y, u, v \in L^*$*,*

$$\begin{cases} \mathcal{T}_M(x,y) \leqslant_{L^*} x \text{ and } \mathcal{T}_M(u,v) \leqslant_{L^*} u, \\ \mathcal{T}_M(x,y) \leqslant_{L^*} y \text{ and } \mathcal{T}_M(u,v) \leqslant_{L^*} v. \end{cases}$$

Then,

$$\begin{cases} \mathcal{T}(\mathcal{T}_M(x,y), \mathcal{T}_M(u,v)) \leqslant_{L^*} \mathcal{T}(x,u), \\ \mathcal{T}(\mathcal{T}_M(x,y), \mathcal{T}_M(u,v)) \leqslant_{L^*} \mathcal{T}(y,v). \end{cases}$$

Hence,

$$\mathcal{T}_M(\mathcal{T}(\mathcal{T}_M(x,y), \mathcal{T}_M(u,v)), \mathcal{T}(\mathcal{T}_M(x,y), \mathcal{T}_M(u,v))) \leqslant_{L^*}$$
$$\mathcal{T}_M(\mathcal{T}(x,u), \mathcal{T}(y,v)).$$

So,

$$\mathcal{T}(\mathcal{T}_M(x,y), \mathcal{T}_M(u,v)) \leqslant_{L^*} \mathcal{T}_M(\mathcal{T}(x,u), \mathcal{T}(y,v)).$$

■

Lemma 2.1.6 *[57]*

Any representable intuitionistic fuzzy t-norm $\mathcal{T}$ dominates itself, i.e., for any quadruple $(x,y,u,v) \in (L^*)^4$, we have,

$$\mathcal{T}(\mathcal{T}(x,u), \mathcal{T}(y,v)) = \mathcal{T}(\mathcal{T}(x,y), \mathcal{T}(u,v)).$$

Proof.

Let a representable intuitionistic fuzzy t-norm $\mathcal{T}$ *and* $(x,y,u,v) \in (L^*)^4$*. Then,*

$$\begin{aligned}\mathcal{T}(\mathcal{T}(x,u),\mathcal{T}(y,v)) &= \mathcal{T}(x,\mathcal{T}(u,\mathcal{T}(y,v))) \\ &= \mathcal{T}(x,\mathcal{T}(\mathcal{T}(y,v),u)) \\ &= \mathcal{T}(x,\mathcal{T}(y,\mathcal{T}(v,u))) \\ &= \mathcal{T}(\mathcal{T}(x,y),\mathcal{T}(v,u)) \\ &= \mathcal{T}(\mathcal{T}(x,y),\mathcal{T}(u,v))\end{aligned}$$

■

Remark 2.1.7 *[57]*

1. *The greatest representable intuitionistic fuzzy t-norm with respect to* $\leqslant_{L^*}$ *is* $\mathcal{T}_M$*, defined by* $\mathcal{T}_M(x,y) = x \wedge_{L^*} y$*.*

2. *The smallest representable intuitionistic fuzzy t-conorm with respect to* $\leqslant_{L^*}$ *is* $\mathcal{S}_M$*, defined by* $\mathcal{S}_M(x,y) = x \vee_{L^*} y$ *for all* $x,y \in L^*$*.*
   Moreover, $\mathcal{T}_M(x,y) \leqslant_{L^*} \mathcal{S}_M(x,y)$ *for all* $x,y \in L^*$*. Indeed, Let* x,$y \in L^*$*, where* $x = (x_1,x_2)$ *and* $y = (y_1,y_2)$*.*
   We have $\min(x_1,y_1) \leq \max(x_1,y_1)$ *and* $\max(x_2,y_2) \geq \min(x_2,y_2)$,
   then $(\min(x_1,y_1), \max(x_2,y_2)) \leqslant_{L^*} (\max(x_1,y_1), \min(x_2,y_2))$,
   then $\mathcal{T}_M(x,y) \leqslant_{L^*} \mathcal{S}_M(x,y)$*.*

Remark 2.1.8 *[57]*

It easy to see that $\mathcal{T}(x,y) \leqslant_{L^*} \mathcal{T}_M(x,y)$ *and* $\mathcal{S}_M(x,y) \leqslant_{L^*} \mathcal{S}(x,y)$ *for all* $x,y \in L^*$*. Indeed, let* $x,y \in L^*$*, then* $x = (x_1,x_2)$ *and* $y = (y_1,y_2)$*. We have* $T(x_1,y_1) \leq \min(x_1,y_1)$ *and* $S(x_2,y_2) \geq \max(x_2,y_2)$ *then* $\mathcal{T}(x,y) \leqslant_{L^*} \mathcal{T}_M(x,y)$ *and* $\mathcal{S}_M(x,y) \leqslant_{L^*} \mathcal{S}(x,y)$*.*
Finaly, $\mathcal{T}(x,y) \leqslant_{L^*} \mathcal{T}_M(x,y) \leqslant_{L^*} \mathcal{S}_M(x,y) \leqslant_{L^*} \mathcal{S}(x,y)$*.*

Remark 2.1.9 *[57]*

Note that it does not hold that for all $x, y \in L^*$*, neither* $\mathcal{T}_M(x,y) = x$ *nor* $\mathcal{T}_M(x,y) = y$*. For example,* $\mathcal{T}_M((0.5, 0.3), (0.2, 0.1)) = (0.2, 0.3)$.

2.1.1 The residuation principle

Definition 2.1.10 *[23]*

An intuitionistic fuzzy t-norm $\mathcal{T}$ *is said to be satisfied the* ***residuated principle*** *if and only if for all* $(a_1, a_2), (b_1, b_2), (c_1, c_2) \in L^*$,

$$(c_1, c_2) \leqslant_{L^*} \mathcal{I}_{\mathcal{T}}((a_1, a_2), (b_1, b_2)) \text{ if and only if } \mathcal{T}((a_1, a_2), (c_1, c_2)) \leqslant_{L^*} (b_1, b_2) \tag{2.1}$$

Where $\mathcal{I}_{\mathcal{T}}$ *denotes the* ***residual implicator generated*** *by* $\mathcal{T}$*, defined as*

$$\mathcal{I}_{\mathcal{T}}((a_1, a_2), (b_1, b_2)) = \sup\{(\alpha_1, \alpha_2) \in L^* \ / \ \mathcal{T}((a_1, a_2), (\alpha_1, \alpha_2)) \leqslant_{L^*} (b_1, b_2)\}$$

Theorem 2.1.11 *[23]*

Let F *be any increasing* $L^* \longrightarrow L^*$*-mapping. If*

$$\sup_{z \in Z} F(z) = F(\sup_{z \in Z} z),$$

for all non-empty subsets Z *of* L^**, then* F *called intuitionistic fuzzy left-continuous mapping.*

Theorem 2.1.12 *[25]*

Let $\mathcal{T}$ *be an intuitionistic fuzzy t-norm. If* $\mathcal{T}$ *satisfies the residuation principle, then the partial mappings of* $\mathcal{T}$ *are intuitionistic fuzzy left-continuous. If* $\mathcal{T}$ *is a t-representable, then the partial mappings of* $\mathcal{T}$ *are intuitionistic fuzzy* *left-continuous if and only if $\mathcal{T}$ satisfies the residuation principle.*

Remark 2.1.13 *[56, 23, 19]*

If an intuitionistic fuzzy t-norm $\mathcal{T}$ satisfies the residuation principle, then $\mathcal{T}$ is intuitionistic fuzzy left-continuous. In general, one can not derive an intuitionistic fuzzy t-norm that satisfies the residuation principle from intuitionistic fuzzy left-continuity.

Next, we recall the definition of the dominated of intuitionistic fuzzy t-norm.

Definition 2.1.14 *[57]*

An intuitionistic fuzzy t-norm $\mathcal{T}_1$ is said to dominate another intuitionistic fuzz t-norm $\mathcal{T}_2$ if and only if for any quadruple $((x_1, x_2), (y_1, y_2), (u_1, u_2), (v_1, v_2)) \in (L^*)^4$, the following holds:

$$\mathcal{T}_2(\mathcal{T}_1((x_1, x_2), (u_1, u_2)), \mathcal{T}_1((y_1, y_2), (v_1, v_2)))$$

$$\leqslant_{L^*} \mathcal{T}_1(\mathcal{T}_2((x_1, x_2), (y_1, y_2)), \mathcal{T}_2((u_1, u_2), (v_1, v_2)))$$

We need the following lemma to prove the main results.

Lemma 2.1.15 *[57]*

Any intuitionistic fuzzy t-norm $\mathcal{T}$ dominates itself.

Proof.

For any quadruple $((x_1, x_2), (y_1, y_2), (u_1, u_2), (v_1, v_2)) \in (L^*)^4$. We have,

$$
\begin{aligned}
& \mathcal{T}(\mathcal{T}((x_1,x_2),(u_1,u_2)),\mathcal{T}((y_1,y_2),(v_1,v_2))) \leqslant_{L^*} (*), \\
(*) \quad & \leqslant_{L^*} \mathcal{T}((x_1,x_2),\mathcal{T}((u_1,u_2),\mathcal{T}((y_1,y_2),(v_1,v_2)))), \\
& \leqslant_{L^*} \mathcal{T}((x_1,x_2),\mathcal{T}(\mathcal{T}((u_1,u_2),(y_1,y_2)),(v_1,v_2)), \\
& \leqslant_{L^*} \mathcal{T}((x_1,x_2),\mathcal{T}(\mathcal{T}((y_1,y_2),(u_1,u_2)),(v_1,v_2)), \\
& \leqslant_{L^*} \mathcal{T}((x_1,x_2),\mathcal{T}((y_1,y_2),\mathcal{T}((u_1,u_2),(v_1,v_2))), \\
& \leqslant_{L^*} \mathcal{T}(\mathcal{T}((x_1,x_2),(y_1,y_2)),\mathcal{T}((u_1,u_2),(v_1,v_2)).
\end{aligned}
$$

■

In the following, we characterize the set L^* on which this work is based.

Theorem 2.1.16 *[26]*

Consider the lattice $(L^*, \leqslant_{L^*})$ defined in Lemma 1.7.1. The algebraic structure $(L^*, \leqslant_{L^*}, \mathcal{T}, \mathcal{I}_{\mathcal{T}}, 0_{L^*}, 1_{L^*})$ is a residuated lattice.

The following lemma will be used to prove some results.

Lemma 2.1.17 *[57]*

For any $(z_1, z_2) \in L^*$, we have,

$$((a_1,a_2) \leqslant_{L^*} (z_1,z_2) \Leftrightarrow (b_1,b_2) \leqslant_{L^*} (z_1,z_2)) \text{ implies, } ((a_1,a_2) = (b_1,b_2)).$$

Dually, we get for any $(x_1, x_2) \in L^*$, the equivalence,

$$((x_1,x_2) \leqslant_{L^*} (a_1,a_2) \Leftrightarrow (x_1,x_2) \leqslant_{L^*} (b_1,b_2)) \text{ implies, } ((a_1,a_2) = (b_1,b_2)).$$

Proposition 2.1.18 *[57]*

The mapping $\mathcal{I}_{\mathcal{T}}$: $L^* \to L^*, (x_1,x_2) \mapsto \mathcal{I}_{\mathcal{T}}((x_1,x_2),(c_1,c_2))$, where (c_1,c_2) is a fixed element in L^*, changes all existing joints in the first argument $\mathcal{I}_{\mathcal{T}}$ in L^* to *meets, i.e.,*

$$\mathcal{I}_{\mathcal{T}}(\sup_{i\in I}(a_i,b_i),(c_1,c_2)) = \inf_{i\in I}\mathcal{I}_{\mathcal{T}}((a_i,b_i),(c_1,c_2)) \tag{2.2}$$

For any $(a_i,b_i),(c_1,c_2) \in L^*$ *and* $i \in I$.

Proof.

Let $(a_i,b_i),(c_1,c_2) \in L^*$ *and* $i \in I$. *And let* $(\alpha_1,\alpha_2) \in L^*$. *The following equivalences hold:*

$$(\alpha_1,\alpha_2) \leqslant_{L^*} \mathcal{I}_{\mathcal{T}}(\sup_{i\in I}(a_i,b_i),(c_1,c_2)).$$

Then, (Using Theoerm 2.1.11) we have, for any $i \in I$ *we have,*

$$\begin{aligned}
\mathcal{T}(\sup_{i\in I}(a_i,b_i),(\alpha_1,\alpha_2)) \leqslant_{L^*} (c_1,c_2) &\Leftrightarrow \sup_{i\in I}\mathcal{T}((a_i,b_i),(\alpha_1,\alpha_2)) \leqslant_{L^*} (c_1,c_2),\\
&\Leftrightarrow \mathcal{T}((a_i,b_i),(\alpha_1,\alpha_2)) \leqslant_{L^*} (c_1,c_2),\\
&\Leftrightarrow (\alpha_1,\alpha_2) \leqslant_{L^*} \mathcal{I}_{\mathcal{T}}((a_i,b_i),(c_1,c_2)),\\
&\Leftrightarrow (\alpha_1,\alpha_2) \leqslant_{L^*} \inf_{i\in I}\mathcal{I}_{\mathcal{T}}((a_i,b_i),(c_1,c_2)).
\end{aligned}$$

Hence,

$$\inf_{i\in I}\mathcal{I}_{\mathcal{T}}((a_i,b_i),(c_1,c_2)) = \mathcal{I}_{\mathcal{T}}\left(\sup_{i\in I}(a_i,b_i),(c_1,c_2)\right).$$

■

Proposition 2.1.19 *[57]*

The mapping $\mathcal{I}_{\mathcal{T}}$: $L^* \to L^*$, $(y_1,y_2) \mapsto \mathcal{I}_{\mathcal{T}}((a_1,a_2),(y_1,y_2))$ *preserves all existing meets of the second argument in* L^*, *i.e.,*

$$\mathcal{I}_{\mathcal{T}}((a_1,a_2),\inf_{i\in I}(b_i,b_j)) = \inf_{i\in I}\mathcal{I}_{\mathcal{T}}((a_1,a_2),(b_i,b_j))$$

For any $(a_1, a_2), (b_i, b_j) \in L^*$.

Proof.

Similarly to the Proposition 2.1.18. ■

Proposition 2.1.20 *[57]*

For any $(a_1, a_2), (b_i, b_j) \in L^*$ *we have*

$$\mathcal{T}(\inf_{i\in I}(a_i, b_i), \inf_{i\in I}(a'_i, b'_i)) \leqslant_{L^*} \inf_{i\in I}\mathcal{T}((a_i, b_i), (a'_i, b'_i)).$$

Proof.

Let $(a_1, a_2), (b_i, b_j) \in L^*$. *We use the property of meet in* L^* *and (2.1), we obtain, for* $(z_1, z_2) \in L^\star$,

$\inf_{i\in I}\mathcal{T}((a_i, b_i), (a'_i, b'_i)) \leqslant_{L^*} \mathcal{T}((a_i, b_i), (a'_i, b'_i)), \forall i \in I$, *on put* $\mathcal{T}((a_i, b_i), (a'_i, b'_i)) \leqslant_{L^*} (z_1, z_2), \forall i \in I$, *We get,*

$$\begin{aligned}\mathcal{T}((a_i, b_i), (a'_i, b'_i)) \leqslant_{L^*} (z_1, z_2), \forall i \in I &\Leftrightarrow (a_i, b_i) \leqslant_{L^*} \mathcal{I}_{\mathcal{T}}((a'_i, b'_i), (z_1, z_2)), \forall i \in I, \\ &\Leftrightarrow \inf_{i\in I}(a_i, b_i) \leqslant_{L^*} \inf_{i\in I}\mathcal{I}_{\mathcal{T}}((a'_i, b'_i), (z_1, z_2)).\end{aligned}$$

From the fact that,

$$\inf_{i\in I}\mathcal{I}_{\mathcal{T}}((a'_i, b'_i), (z_1, z_2)) = \mathcal{I}_{\mathcal{T}}(\sup_{i\in I}(a'_i, b'_i), (z_1, z_2)).$$

And Proposition 2.1.18 we have,

$$\begin{aligned}\inf_{i\in I}(a_i, b_i) \leqslant_{L^*} \inf_{i\in I}\mathcal{I}_{\mathcal{T}}((a'_i, b'_i), (z_1, z_2)) &\Leftrightarrow \inf_{i\in I}(a_i, b_i) \leqslant_{L^*} \mathcal{I}_{\mathcal{T}}(\sup_{i\in I}(a'_i, b'_i), (z_1, z_2)), \\ &\Leftrightarrow \mathcal{T}(\inf_{i\in I}(a_i, b_i), \sup_{i\in I}(a'_i, b'_i)) \leqslant_{L^*} (z_1, z_2).\end{aligned}$$

Hence,

$$\inf_{i\in I}\mathcal{T}\left(\left(a_i,b_i\right),\left(a'_i,b'_i\right)\right)=\mathcal{T}(\inf_{i\in I}\left(a_i,b_i\right),\sup_{i\in I}\left(a'_i,b'_i\right)).$$

Note that,

$$\mathcal{T}(\inf_{i\in I}\left(a_i,b_i\right),\inf_{i\in I}\left(a'_i,b'_i\right))\leqslant_{L^*}\mathcal{T}(\inf_{i\in I}\left(a_i,b_i\right),\sup_{i\in I}\left(a'_i,b'_i\right)).$$

Hence,

$$\mathcal{T}(\inf_{i\in I}\left(a_i,b_i\right),\inf_{i\in I}\left(a'_i,b'_i\right))\leqslant_{L^*}\inf_{i\in I}\mathcal{T}\left(\left(a_i,b_i\right),\left(a'_i,b'_i\right)\right).$$

■

Proposition 2.1.21 *[57]*

For any $(a_1,a_2),(b_1,b_2)\in L^*$, we have,

$$(a_1,a_2)\leqslant_{L^*}(b_1,b_2)\ \textit{if and only if}\ \mathcal{I}_{\mathcal{T}}((a_1,a_2),(b_1,b_2))=(1,0).$$

Proof.

Let $(a_1,a_2),(b_1,b_2)\in L^*$. According to Theorem 2.1.16 and (2.1)

$$\begin{aligned}(a_1,a_2)\leqslant_{L^*}(b_1,b_2) &\Leftrightarrow \mathcal{T}\left((a_1,a_2),(1,0)\right)\leqslant_{L^*}(b_1,b_2),\\ &\Leftrightarrow (1,0)\leqslant_{L^*}\mathcal{I}_{\mathcal{T}}((a_1,a_2),(b_1,b_2)),\\ &\Leftrightarrow (1,0)=\mathcal{I}_{\mathcal{T}}((a_1,a_2),(b_1,b_2)).\end{aligned}$$

■

Proposition 2.1.22 *[57]*

Let $(a_1,a_2),(b_1,b_2),(c_1,c_2)\in L^*$. We have,

$\mathcal{T}\left(\mathcal{I}_{\mathcal{T}}((b_1,b_2),(c_1,c_2)),\mathcal{I}_{\mathcal{T}}((a_1,a_2),(b_1,b_2))\right)\leqslant_{L^*}\mathcal{I}_{\mathcal{T}}\left((a_1,a_2),(c_1,c_2)\right)$. *(i.e., $\mathcal{I}_{\mathcal{T}}$ is $\mathcal{T}$-transitivite)*

Proof.

Let $(a_1, a_2), (b_1, b_2), (c_1, c_2) \in L^*$. *According to (2.1) and proprieties of restudied lattice, we get,*

$$\mathcal{T}\left(\mathcal{T}[\mathcal{I}_{\mathcal{T}}((b_1, b_2), (c_1, c_2)), \mathcal{I}_{\mathcal{T}}((a_1, a_2), (b_1, b_2))], (a_1, a_2)\right) \leqslant_{L^*}$$

$$\leqslant_{L^*} \mathcal{T}\left(\mathcal{I}_{\mathcal{T}}((b_1, b_2), (c_1, c_2)), \mathcal{T}\left[(\mathcal{I}_{\mathcal{T}}((a_1, a_2), (b_1, b_2))), (a_1, a_2)\right]\right)$$

Which is true, since,

$$\mathcal{T}\left[(\mathcal{I}_{\mathcal{T}}((a_1, a_2), (b_1, b_2))), (a_1, a_2)\right] \leqslant_{L^*} (b_1, b_2)$$

Then,

$$\mathcal{T}\left(\mathcal{T}[\mathcal{I}_{\mathcal{T}}((b_1, b_2), (c_1, c_2)), \mathcal{I}_{\mathcal{T}}((a_1, a_2), (b_1, b_2))], (a_1, a_2)\right) \leqslant_{L^*} (*')$$

$$(*') \leqslant_{L^*} \mathcal{T}\left(\mathcal{I}_{\mathcal{T}}((b_1, b_2), (c_1, c_2)), (b_1, b_2)\right)$$

Hence,

$$\mathcal{T}\left(\mathcal{T}[\mathcal{I}_{\mathcal{T}}((b_1, b_2), (c_1, c_2)), \mathcal{I}_{\mathcal{T}}((a_1, a_2), (b_1, b_2))], (a_1, a_2)\right) \leqslant_{L^*} (c_1, c_2)$$

So,

$$\mathcal{T}\left(\mathcal{I}_{\mathcal{T}}((b_1, b_2), (c_1, c_2)), \mathcal{I}_{\mathcal{T}}((a_1, a_2), (b_1, b_2))\right) \leqslant_{L^*} \mathcal{I}_{\mathcal{T}}\left((a_1, a_2), (c_1, c_2)\right).$$

In another sense $\mathcal{I}_{\mathcal{T}}$ *is* $\mathcal{T}$*-transitivite.* ■

2.2 Intuitionistic fuzzy equivalence and order with respect to $\mathcal{T}$

The special types of intuitionistic fuzzy equivalence relations and intuitionistic fuzzy partially ordered relations have important applications in intuitionistic fuzzy subsets theory.

Definition 2.2.1 *[57]*

An intuitionistic fuzzy relation ρ on X is called,

1. *Reflexive, if* $\rho(x,x) = (1,0)$ *for all* $x \in X$, *i.e.,* $\mu_\rho(x,x) = 1$ *and* $\nu_\rho(x,x) = 0$ *for all* $x \in X$.

2. *Symmetric, if* $\rho(x,y) = \rho(y,x)$ *for all* $x,y \in X$, *i.e.,* $\mu_\rho(x,y) = \mu_\rho(y,x)$ *and* $\nu_\rho(x,y) = \nu_\rho(y,x)$ *for all* $x,y \in X$.

3. $\mathcal{T}$*-transitive, if* $\mathcal{T}(\rho(x,y), \rho(y,z)) \leqslant_{L^*} \rho(x,z)$ *for all* $x,y,z \in X$, *i.e.,*

$$\begin{cases} T(\mu_\rho(x,y), \mu_\rho(y,z)) \leq \mu_\rho(x,z), \\ S(\nu_\rho(x,y), \nu_\rho(y,z)) \geq \nu_\rho(x,z), \end{cases} \text{for all } x,y,z \in X.$$

4. *Separated, if* $\rho(x,y) = (1,0)$ *implies* $x = y$ *for all* $x,y \in X$.

5. *Strongly, if* $\rho(x,y) \vee_{L^*} \rho(y,x) = (1,0)$, *for all* $x,y \in X$.

   i.e., $\max(\mu_\rho(x,y), \mu_\rho(y,x)) = 1$ *and* $\min(\nu_\rho(x,y), \nu_\rho(y,x)) = 0$ *for all* $x,y \in X$.

- *If an intuitionistic fuzzy relation ρ on X satisfies only the conditions* (1) *and* (3) *then it called an intuitionistic fuzzy preordering with respect to $\mathcal{T}$, (intuitionistic fuzzy $\mathcal{T}$-preorder, for short).*

- *If an intuitionistic fuzzy relation ρ on X satisfies the conditions* (1), (2) *and* (3), *then it's called an intuitionistic fuzzy equivalence with respect to $\mathcal{T}$, (intuitionistic fuzzy $\mathcal{T}$-equivalence, for short).*

- *An intuitionistic fuzzy $\mathcal{T}$-equivalence on X satisfies the condition* (4), *is called an intuitionistic fuzzy equality with respect to $\mathcal{T}$, (intuitionistic fuzzy $\mathcal{T}$-equality, for short).*

- *An intuitionistic fuzzy relation ρ on X is called an intuitionistic fuzzy weak order relation with respect to an intuitionistic fuzzy t-norm $\mathcal{T}$ (intuitionistic fuzzy weak $\mathcal{T}$-order, for short) on X if it satisfies the conditions* (3) *and* (5).

Remark 2.2.2 *[57]*

As in the crisp case, an intuitionistic fuzzy weak $\mathcal{T}$-order is reflexive.

Conclusion 2.2.3 *[57]*

An intuitionistic fuzzy weak $\mathcal{T}$-order is a special kind of intuitionistic fuzzy $\mathcal{T}$-preorder.

Example 2.2.4 *[57]*

Let $X = \{a, b, c\}$, $\mathcal{T} = (\min, \max)$ *and the relation* $\rho = (\mu_\rho, \nu_\rho)$ *given by*

$$\mu_\rho\colon\ X \times X \longrightarrow [0;1]$$

μ_ρ	a	b	c
a	1	1	0.4
b	1	1	0.4
c	0.4	0.4	1

$$\nu_\rho\colon\ X \times X \longrightarrow [0;1]$$

ν_ρ	a	b	c
a	0	0	0.5
b	0	0	0.5
c	0.5	0.5	0

It is not difficult to see that ρ is an intuitionistic fuzzy $\mathcal{T}$-equivalence relation on X.

On the basis of the above definitions of intuitionistic fuzzy relations, we define a dominating class of x and the class dominated by x as follows.

Definition 2.2.5 *[57]*

Let ρ be an intuitionistic fuzzy $\mathcal{T}$-preordering defined on a set X. Then, for any element $x \in X$, we associate the dominating class of x denoted by $\rho_{x\uparrow}$ and is defined as

$$\rho_{x\uparrow} = \left\{ \left\langle y, \mu_{\rho_{x\uparrow}}(y), \nu_{\rho_{x\uparrow}}(y) \right\rangle \ \mid y \in X \right\}.$$

Where $\mu_{\rho_{x\uparrow}}(y) = \mu_\rho(x,y),\ \nu_{\rho_{x\uparrow}}(y) = \nu_\rho(x,y)$ for any $y \in X$.

Example 2.2.6 *[57]*

Let $X = \{a,b,c\}$, $\mathcal{T} = (\min, \max)$ and the intuitionistic relation $\rho = (\mu_\rho, \nu_\rho)$ given by

$$\mu_\rho\colon\ X \times X \longrightarrow [0,1],$$

$\mu_\rho(.,.)$	a	b	c
a	1	0.7	0
b	0	1	0
c	0.5	0.7	1

$$\nu_\rho: \ X \times X \longrightarrow [0;1]$$

$\nu_\rho(.,.)$	a	b	c
a	0	0	0.2
b	0.8	0	0
c	0.3	0.2	0

It is easy to see that ρ is an intuitionistic fuzzy $\mathcal{T}$ -preordering. We define the class of X dominated by a, b and c as follows

$$\rho_{z\uparrow} = \{(x, \langle \mu_\rho(z,x), \nu_\rho(z,x)\rangle), x \in X\},$$

$$\rho_{a\uparrow} = \{\langle a, 1, 0\rangle, \langle b, 0.7, 0\rangle, \langle c, 0, 0.2\rangle\},$$

$$\rho_{b\uparrow} = \{\langle a, 0, 0.8\rangle, \langle b, 1, 0\rangle, \langle c, 0, 0\rangle\},$$

$$\rho_{c\uparrow} = \{\langle a, 0.5, 0.3\rangle, \langle b, 0.7, 0.2\rangle, \langle c, 1, 0\rangle\}.$$

2.3 Intuitionistic fuzzy (resp. strongly) $\mathcal{T}$-E-order

Definition 2.3.1 *[57]*

Let X be a nonempty set, let $\mathcal{T}$ be an intuitionistic fuzzy t-norm and assume that E be an intuitionistic fuzzy $\mathcal{T}$-equivalence on X.

1. *An intuitionistic fuzzy relation ρ on X is called an intuitionistic fuzzy partial ordering w.r.t the intuitionistic fuzzy t-norm $\mathcal{T}$ and the intuitionistic fuzzy*

$\mathcal{T}$-equivalence (intuitionistic fuzzy $\mathcal{T}$-E-order, for short) on X if it is $\mathcal{T}$-transitive and additionally has the following two properties

(a) for all $x, y \in X$, $E(x,y) \leqslant_{L^*} \rho(x,y)$, i.e.,

$$\begin{cases} \mu_E(x,y) \leq \mu_\rho(x,y), \\ \nu_E(x,y) \geq \nu_\rho(x,y), \end{cases} \quad (E\text{-reflexivity}),$$

(b) for all $x, y \in X$, $\mathcal{T}(\rho(x,y), \rho(x,y)) \leqslant_{L^*} E(x,y)$, i.e.,

$$\begin{cases} T(\mu_\rho(x,y), \mu_\rho(y,x)) \leq \mu_E(x,y), \\ S(\nu_\rho(x,y), \nu_\rho(y,x)) \geq \nu_E(x,y). \end{cases} \quad (\mathcal{T}\text{-}E\text{-antisymmetry}),$$

2. *An intuitionistic fuzzy $\mathcal{T}$-E-order on X satisfies the condition (5) in Definition 2.2.1, is called an intuitionistic fuzzy strongly ordering w.r.t the intuitionistic fuzzy t-norm $\mathcal{T}$ (intuitionistic fuzzy strongly $\mathcal{T}$-E-order on X, for short).*

2.4 Representation and construction of intuitionistic fuzzy $\mathcal{T}$-preorders and intuitionistic fuzzy $\mathcal{T}$-orders

2.4.1 Representation and construction of intuitionistic fuzzy $\mathcal{T}$-preorders

The following lemma gives a great insight into the representation of intuitionistic fuzzy $\mathcal{T}$-preorders and we need this lemma in representation and construction of an intuitionistic fuzzy $\mathcal{T}$-orders

Lemma 2.4.1 *[57]*

Let X be a non-empty set and let $\mathcal{T}$ be an intuitionistic fuzzy t-representable *t-norm. Every intuitionistic fuzzy $\mathcal{T}$-preordering $\rho = (\mu_\rho, \nu_\rho)$ on X fulfills the following equality*

$$\rho(x, y) = \inf_{z \in X} \mathcal{T}_{\mathcal{I}}\left((\mu_\rho(z, x), \nu_\rho(z, x)), (\mu_\rho(z, y), \nu_\rho(z, y))\right), \text{ for any } x, y \in X.$$

Proof.

To prove the above equality, we use the $\mathcal{T}$-transitivity of ρ and the commutativity of $\mathcal{T}$.

For all $x, y \in X$, we have,

$$\mathcal{T}(\rho(z, x), \rho(x, y)) = \mathcal{T}(\rho(x, y), \rho(z, x)), \text{ for all } z \in X,$$

$$\mathcal{T}(\rho(x, y), \rho(z, x)) \leqslant_{L^*} \rho(z, y), \text{ for all } z \in X,$$

$$\Leftrightarrow \rho(x, y) \leqslant_{L^*} \mathcal{T}_{\mathcal{I}}(\rho(z, x), \rho(z, y)), \text{ for all } z \in X,$$

$$\Leftrightarrow \rho(x, y) \leqslant_{L^*} \mathcal{T}_{\mathcal{I}}\left((\mu_\rho(z, x), \nu_\rho(z, x)), (\mu_\rho(z, y), \nu_\rho(z, y))\right), \text{ for all } z \in X,$$

$$\Leftrightarrow \rho(x, y) \leqslant_{L^*} \inf_{z \in X} \mathcal{T}_{\mathcal{I}}\left((\mu_\rho(z, x), \nu_\rho(z, x)), (\mu_\rho(z, y), \nu_\rho(z, y))\right).$$

Setting $z = x$ in this inequality, we obtain:

$$\inf_{z \in X} \mathcal{T}_{\mathcal{I}}\left((\mu_\rho(z, x), \nu_\rho(z, x)), (\mu_\rho(z, y), \nu_\rho(z, y))\right)$$

$$\leqslant_{L^*} \mathcal{T}_{\mathcal{I}}\left((\mu_\rho(x, x), \nu_\rho(x, x)), (\mu_\rho(x, y), \nu_\rho(x, y))\right),$$

or

$$\mathcal{T}_{\mathcal{I}}\left((\mu_\rho(x, x), \nu_\rho(x, x)), (\mu_\rho(x, y), \nu_\rho(x, y))\right)$$

$$= (\mu_\rho(x, y), \nu_\rho(x, y)) = \rho(x, y).$$

Then,

$$\inf_{z \in X} \mathcal{T}_{\mathcal{I}}\left((\mu_\rho(z, x), \nu_\rho(z, x)), (\mu_\rho(z, y), \nu_\rho(z, y))\right) \leqslant_{L^*} \rho(x, y).$$

Which completes the proof of this lemma. ■

2.4.2 Representation and construction of intuitionistic fuzzy $\mathcal{T}$-orders

In the same way in the above Lemma 2.4.1, a new concept is given to representation and construction of intuitionistic fuzzy $\mathcal{T}$-orders, intuitionistic fuzzy strong $\mathcal{T}$-E-ordering and we extend the results [16, Theorem 1.1] to the intuitionistic fuzzy case.

***Theorem 2.4.2** [57]*

Let X be a non-empty set, let ρ be an intuitionistic fuzzy binary relation on X, and let $\mathcal{T}$ be an intuitionistic fuzzy t-representable t-norm. Then, ρ is an intuitionistic fuzzy weak $\mathcal{T}$-order relation if and only if there exists a non-empty domain Y, an intuitionistic fuzzy $\mathcal{T}$-equivalence relation E, an intuitionistic fuzzy strong $\mathcal{T}$-E-ordering F and a mapping $p\colon X \to Y$ such that the following equality holds for all $x, y \in X$:

$$\rho(x,y) = \left(\mu_F\left((p(x),p(y)\right),\nu_F(p(x),p(y))\right). \tag{2.3}$$

Proof.

To prove sufficiency, let Y be a non-empty domain equipped with an intuitionistic fuzzy $\mathcal{T}$-equivalence relation E, let F be an intuitionistic fuzzy strong $\mathcal{T}$-E-ordering and let p be a mapping from X to Y such that the representation in Equation (2.3) holds. As every strongly complete intuitionistic fuzzy $\mathcal{T}$-E -order is an intuitionistic fuzzy weak $\mathcal{T}$-order relation, ρ is trivially an intuitionistic fuzzy weak $\mathcal{T}$-order relation.

For the necessity, assume that ρ is an intuitionistic fuzzy weak $\mathcal{T}$-order relation. Define Y to be X and p to be the identity on X. Now we put $F(x,y) = \rho(x,y)$

and $E(x,y) = \mathcal{T}\left(\rho(x,y), \rho(y,x)\right).$

First, we prove that E is an intuitionistic fuzzy $\mathcal{T}$-equivalence relation. To prove the reflexivity we use the result of Remark 2.2.2. The symmetry of E is straightforward.

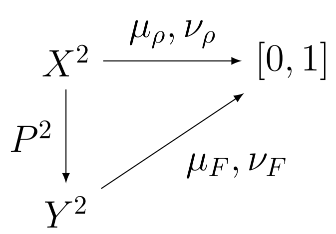
Finally, we prove the $\mathcal{T}$-transitivity of E i.e., $\mathcal{T}\left(E\left(x,y\right), E\left(y,z\right)\right) \leqslant_{L^*} E\left(x,z\right).$

Let x, y, z in X, using Lemma 2.1.15 we have :

$$\begin{aligned}
\mathcal{T}\left(E,\left(x,y\right), E\left(y,z\right)\right) &= \quad \mathcal{T}\left(\mathcal{T}\left(\rho(x,y), \rho(y,x)\right), \mathcal{T}\left(\rho(y,z), \rho(z,y)\right)\right), \\
&\leqslant_{L^*} \mathcal{T}\left(\mathcal{T}\left(\rho(x,y), \rho(y,z)\right), \mathcal{T}\left(\rho(y,x), \rho(z,y)\right)\right), \\
&\leqslant_{L^*} \mathcal{T}\left(\mathcal{T}\left(\rho(x,y), \rho(y,z)\right), \mathcal{T}\left(\rho(z,y), \rho(y,x)\right)\right), \\
&\leqslant_{L^*} \mathcal{T}\left(\rho(x,z), \rho(z,x)\right) = E\left(x,z\right).
\end{aligned}$$

Using Remark 2.2.2, it easy to see that F is a $\mathcal{T}$-transitive, E-reflexive, $\mathcal{T}$-E-antisymmetry and strongly complete on X. i.e., F is an intuitionistic fuzzy strong $\mathcal{T}$-E -ordering on X. Thus the proof is completed. ■

Theorem 2.4.2 is a natural generalization the results of U. Bodenhofer and all [16, Theorem 1.1] and it is a factorization of the intuitionistic fuzzy weak $\mathcal{T}$-order between two relations, intuitionistic fuzzy strong $\mathcal{T}$-E-ordering and intuitionistic fuzzy $\mathcal{T}$-equivalence relation.



Where,

$$p^2(x,y) = (p(x), p(y)),$$
$$\mu_\rho(x,y) = \mu_F(p(x), p(y)),$$
$$\nu_\rho(x,y) = \nu_F(p(x), p(y)).$$

2.4.3 Intuitionistic fuzzy set generator and $\mathcal{I}_\mathcal{T}$-representable

Definition 2.4.3 *[57]*

Let X be a non-empty set and let $\mathcal{T}$ be an intuitionistic fuzzy t-representable t-norm. Consider an intuitionistic fuzzy weak $\mathcal{T}$-order ρ. ρ is said to be $\mathcal{I}_\mathcal{T}$ -representable if there exists an intuitionistic fuzzy subset A called intuitionistic fuzzy set generator such that the equation

$$\rho(x,y) = \mathcal{I}_\mathcal{T}((\mu_A(x), \nu_A(x)), (\mu_A(y), \nu_A(y))) \tag{2.4}$$

holds.

Remark 2.4.4 *[57]*

Let X be a non-empty set and let $\mathcal{T}$ be an intuitionistic fuzzy t-representable t-norm. If $\mathcal{T} = (\min, \max)$. Then, for any intuitionistic fuzzy subset A on X, the intuitionistic fuzzy relation defined on X by Equality 2.4 is not, in general, an intuitionistic fuzzy weak $\mathcal{T}$-order. Indeed let $(a_1, a_2), (b_1, b_2) \in L^*$, define $\mathcal{I}_\mathcal{T}$ ([20]) as follows,

$$\mathcal{I}_\mathcal{T}((a_1,a_2),(b_1,b_2)) = \begin{cases} 1_{L^*} & \text{if } a_1 \le b_1 \text{ and } a_2 \ge b_2, \\ (1-b_2, b_2) & \text{if } a_1 \le b_1 \text{ and } a_2 < b_2, \\ (b_1, 0) & \text{if } a_1 > b_1 \text{ and } a_2 \ge b_2, \\ (b_1, b_2) & \text{if } a_1 > b_1 \text{ and } a_2 < b_2. \end{cases}$$

And consider A to be the intuitionistic fuzzy subset, $A = \{\langle x, 0.1, 0.2\rangle, \langle y, 0.3, 0.5\rangle\}$,

$$\begin{aligned}
\rho(x,y) \vee_{L^*} \rho(y,x) &= \begin{pmatrix} \mathcal{I}_{\mathcal{T}}((\mu_A(x), \nu_A(x)), (\mu_A(y), \nu_A(y))) \\ \vee_{L^*}\mathcal{I}_{\mathcal{T}}((\mu_A(y), \nu_A(y)), (\mu_A(x), \nu_A(x))) \end{pmatrix}, \\
&= \begin{pmatrix} \mathcal{I}_{\mathcal{T}}((0.1, 0.2), (0.3, 0.5)) \\ \vee_{L^*}\mathcal{I}_{\mathcal{T}}((0.3, 0.5), (0.1, 0.2)) \end{pmatrix}, \\
&= ((1-0.5), 0.5) \vee_{L^*} (0.1, 0.0), \\
&= (\max(0.5, 0.0), \min(0.5, 0.1)), \\
&= (0.5, 0.1) \neq (1, 0).
\end{aligned}$$

Then ρ is not strongly, hence ρ is not an intuitionistic fuzzy weak $\mathcal{T}$-order.

Conclusion 2.4.5 *[57]*

An intuitionistic fuzzy relation given by Equality 2.4 is not in general an intuitionistic fuzzy weak $\mathcal{T}$-order.

In other words, we can not give representation or construction of an intuitionistic fuzzy weak $\mathcal{T}$-order by Equality 2.4.

Moreover, the following theory provides another way to representation and construction of intuitionistic fuzzy $\mathcal{T}$-orders intuitionistic

Theorem 2.4.6 *[57]*

Let X be a non-empty set and let $\mathcal{T}$ be an intuitionistic fuzzy t-representable t-norm. For an intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ on X, the intuitionistic fuzzy relation ρ defined on X by Equality 2.4 is an intuitionistic fuzzy $\mathcal{T}$-preorder.

Proof.

Straightforward, according to Proposition 2.1.22. ■

The following gives a representation theorem for intuitionistic fuzzy $\mathcal{T}$-preorder by a family of intuitionistic fuzzy subsets. The following theory provides another way to representation and construction of intuitionistic fuzzy $\mathcal{T}$-preorder.

Theorem 2.4.7 *[57]*

Let ρ be an intuitionistic fuzzy relation on X. Then the following two statements are equivalent,

1. *ρ is an intuitionistic fuzzy $\mathcal{T}$-preordering.*
2. *There exists a non-empty family of intuitionistic fuzzy subsets $(A_i)_{i\in I}$ on X such that the following representation holds*

$$\rho(x,y) = \inf_{i\in I} \mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x), \nu_{A_i}(x)), (\mu_{A_i}(y), \nu_{A_i}(y))). \tag{2.5}$$

Proof.

For the sufficiency, assume that there exists a non-empty family of intuitionistic fuzzy subsets $(A_i)_{i\in I}$ on X such that the Equation (2.5) holds and proof that ρ is an intuitionistic fuzzy $\mathcal{T}$-preordering relation.

Firstly, from Proposition 2.1.21,

$$\rho(x,x) = \mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x), \nu_{A_i}(x)), (\mu_{A_i}(x), \nu_{A_i}(x))) = \inf_{i\in I}((1,0)) = (1,0),$$

hence ρ is reflexive.

Secondly, using Propriety 2.1.22,

$$\mathcal{T}(\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x), \nu_{A_i}(x)), (\mu_{A_i}(y), \nu_{A_i}(y)), \mathcal{I}_{\mathcal{T}}((\mu_{A_i}(y), \nu_{A_i}(y)), (\mu_{A_i}(z), \nu_{A_i}(z)))$$
$$\leqslant_{L^*} \mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x), \nu_{A_i}(x)), (\mu_{A_i}(z), \nu_{A_i}(z)), \text{ for any } x, y, z \in X, \text{ and } i \in I.$$

Then,

$$\inf_{i\in I}\mathcal{T}\left(\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x),\nu_{A_i}(x)),(\mu_{A_i}(y),\nu_{A_i}(y)),\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(y),\nu_{A_i}(y)),(\mu_{A_i}(z),\nu_{A_i}(z))\right)$$
$$\leqslant_{L^*} \inf_{i\in I}\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x),\nu_{A_i}(x)),(\mu_{A_i}(z),\nu_{A_i}(z)).$$

Put

$$\mathcal{T}(\inf_{i\in I}\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x),\nu_{A_i}(x)),(\mu_{A_i}(y),\nu_{A_i}(y)),\inf_{i\in I}\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(y),\nu_{A_i}(y)),(\mu_{A_i}(z),\nu_{A_i}(z))) = \lambda,$$

By Proposition 2.1.20, we have,

$$\lambda \leqslant_{L^*} \inf_{i\in I}\mathcal{T}\left(\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x),\nu_{A_i}(x)),(\mu_{A_i}(y),\nu_{A_i}(y)),\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(y),\nu_{A_i}(y)),(\mu_{A_i}(z),\nu_{A_i}(z))\right).$$

Then, if we put

$$\mathcal{T}(\inf_{i\in I}\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x),\nu_{A_i}(x)),(\mu_{A_i}(y),\nu_{A_i}(y)),\inf_{i\in I}\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(y),\nu_{A_i}(y)),(\mu_{A_i}(z),\nu_{A_i}(z))) = \lambda',$$

We obtain,

$$\lambda' \leqslant_{L^*} \inf_{i\in I}\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x),\nu_{A_i}(x)),(\mu_{A_i}(z),\nu_{A_i}(z)).$$

Thus,

$$\mathcal{T}\left(\rho(x,y),\rho(y,z)\right) \leqslant_{L^*} \rho(x,z).$$

Consequently, the intuitionistic fuzzy relation defined in Equation (2.5) is an intuitionistic fuzzy $\mathcal{T}$-preordering relation.

For the necessity, take $I = X$ and $A_z = \rho_{z\uparrow}$ the class dominated by z. Then *representation Equation (2.5) follows from Lemma 2.4.1.*

$$\begin{aligned}
\rho\left(x,y\right) &= \inf_{z\in X}\mathcal{T}_{\mathcal{I}}\left(\left(\mu_\rho(z,x),\nu_\rho(z,x)\right),\left(\mu_\rho(z,y),\nu_\rho(z,y)\right)\right),\\
&= \inf_{z\in X}\mathcal{T}_{\mathcal{I}}\left(\left(\mu_{\rho_{z\uparrow}}(x),\nu_{\rho_{z\uparrow}}(x)\right),\left(\mu_{\rho_{z\uparrow}}(y),\nu_{\rho_{z\uparrow}}(y)\right)\right),\\
&= \inf_{z\in X}\mathcal{I}_{\mathcal{T}}(\left(\mu_{A_z}(x),\nu_{A_z}(x)\right),\left(\mu_{A_z}(y),\nu_{A_z}(y)\right)).
\end{aligned}$$

■

Conclusion 2.4.8 *[57]*

Any intuitionistic fuzzy T-preordering relation is an intersection of representable based on a family of intuitionistic fuzzy subsets.

2.4.4 Construction of intuitionistic fuzzy $\mathcal{T}$-E-order

Theorem 2.4.9 *[57]*

Let ρ be an intuitionistic fuzzy $\mathcal{T}$-E -order on X. Then,

1. *The kernel relation $\trianglelefteq_\rho$ of ρ defined by $x \trianglelefteq_\rho y$ if and only if $\rho(x,y) = (1,0)$, for all $x, y \in X$, can be seen as a crisp preordering relation on X. Furthermore, $\trianglelefteq_\rho$ is a crisp partial ordering on X if and only if E is separated.*

2. *If E is separated, then $\trianglelefteq_\rho$ is a crisp linear ordering on X.*

Proof.

- *For the first assertion. The reflexivity of $\trianglelefteq_\rho$ follows directly from the E-reflexivity of ρ,*

$$\begin{cases} \mu_\rho(x,x) \geq \mu_E(x,x) = 1, \\ and \\ \nu_\rho(x,x) \leq \nu_E(x,x) = 0. \end{cases}$$

Hence,

$$\begin{cases} \mu_\rho(x,x) = 1, \\ and \\ \nu_\rho(x,x) = 0. \end{cases}$$

Then, $\rho(x,x) = (1,0)$. *Hence,* $x \trianglelefteq_\rho x$.

In order to prove the transitivity of $\trianglelefteq_\rho$, *consider the two equivalences,*

$$x \trianglelefteq_\rho y \quad \text{if and only if} \quad \rho(x,y) = (1,0).$$

$$y \trianglelefteq_\rho z \quad \text{if and only if} \quad \rho(y,z) = (1,0).$$

And $\mathcal{T}$*-transitivity entails,*

$$\begin{cases} 1 = T(\mu_\rho(x,y), \mu_\rho(y,z)) \leq \mu_\rho(x,z), \\ and \\ 0 = S(\nu_\rho(x,y), \nu_\rho(y,z)) \geq \nu_\rho(x,z). \end{cases}$$

$$\text{Thus,} \begin{cases} \mu_\rho(x,z) = 1, \\ and \\ \nu_\rho(x,z) = 0. \end{cases}$$

Hence, $x \trianglelefteq_\rho z$.

Assume that, E *is separated. For a pair* $(x,y) \in X^2$:

$$x \trianglelefteq_\rho y \text{ and } y \trianglelefteq_\rho x \Rightarrow \rho(x,y) = (1,0) \text{ and } \rho(y,x) = (1,0)$$

$$\Rightarrow \begin{cases} T(\mu_\rho(x,y), \mu_\rho(y,x)) \leq \mu_E(x,y), \\ \text{and} \\ S(\nu_\rho(x,y), \nu_\rho(y,x)) \geq \nu_E(x,y). \end{cases}$$

$$\Rightarrow \begin{cases} \mu_E(x,y) = 1, \\ \text{and} \\ \nu_E(x,y) = 0. \end{cases}$$

$$\Rightarrow x = y.$$

Conversely, suppose that $E(x,y) = (1,0)$. Since ρ is E-reflexive.

$$\text{We have} \begin{cases} \mu_E(x,y) = 1 \leq \mu_\rho(x,y), \\ \text{and} \\ \nu_E(x,y) = 0 \geq \nu_\rho(x,y). \end{cases}$$

Hence, $\rho(x,y) = (1,0)$, wich implies $x \trianglelefteq_\rho y$...... (1).

Similarly, $E(x,y) = (1,0)$ implies $y \trianglelefteq_\rho x$...... (2)

(1) and (2) gives $x = y$ ($\trianglelefteq_\rho$ is antisymmetric).

- *For the second assertion, assume that E is separated, we have for any arbitrary $x, y \in X$*

$$(x \trianglelefteq_\rho y \text{ or } y \trianglelefteq_\rho x) \quad \text{if and only if} \quad (\rho(x,y) = (1,0) \text{ or } \rho(y,x) = (1,0)).$$

Which completes the proof.

■

The results 2.4.10 and 2.4.12 characterize the intuitionistic fuzzy $\mathcal{T}$-E-order as intersections of representable intuitionistic fuzzy $\mathcal{T}$-E-orders generated by an *intuitionistic fuzzy subset that is monotonic with respect to the same crisp linear order.*

Corollary 2.4.10 *[57]*

Consider a binary intuitionistic fuzzy relation ρ: $X^2 \to [0,1]$ and an intuitionistic fuzzy $\mathcal{T}$-equality E on X. If ρ is an intuitionistic fuzzy $\mathcal{T}$-E -order, then there exists a crisp linear order and a non-empty family $(A_i)_{i\in I}$ of intuitionistic fuzzy subsets generators of ρ such that the representation (2.5) holds.

Proof.

Let ρ is an intuitionistic fuzzy $\mathcal{T}$-E-order and let E be an intuitionistic fuzzy $\mathcal{T}$-equality on X. Lemma 2.4.9 guarantees that the kernel relation $\lhd_\rho$ is a crisp linear ordering on X. Analogously to the proof of Theorem 2.4.7, take $I = X$ and $A_z = \rho_{z\uparrow}$ the class dominated by z. Lemma 2.4.1 ensures that the representation Equation (2.5) holds. ■

The following definition is inspired by [16].

Definition 2.4.11 *[57]*

Let X be a non-empty set, let ρ be an intuitionistic fuzzy $\mathcal{T}$-E-order on X and let B be an intuitionistic fuzzy subset of X. B is called increasing with respect to $\unlhd_\rho$ if and only if,

$$x \lhd_\rho y \Rightarrow B(x) \leqslant_{L^*} B(y).$$

2.4.5 Construction of the intuitionistic fuzzy weak $\mathcal{T}$-order

Theorem 2.4.12 *[57]*

Consider a binary intuitionistic fuzzy relation $\rho : X^2 \to [0,1]$. If the crisp order

$\trianglelefteq_\rho$ is linear and there exists a non-empty family $(A_i)_{i\in I}$ of intuitionistic fuzzy subsets generators of ρ such that A_i is increasing with respect to $\trianglelefteq_\rho$ for all $i \in I$ and the representation (2.5) holds, then ρ is an intuitionistic fuzzy weak $\mathcal{T}$-order.

Proof.

Theorem 2.4.7 states that ρ defined as in Equation (2.5) is a $\mathcal{T}$-preorder, it remains to prove that ρ is strongly. Since $\triangleleft_\rho$ is complete at least one of the two inequalities $x \triangleleft_\rho y$ and $y \triangleleft_\rho x$ holds. If we assume that $x \triangleleft_\rho y$ fulfills the increasingness of all A_i, this guarantees that $A_i(x) \leqslant_{L^*} A_i(y)$ holds for all $i \in I$. From Propriety 2.1.21, we can conclude that,

$$\mathcal{I}_\mathcal{T}((\mu_{A_i}(x), \nu_{A_i}(x)), (\mu_{A_i}(y), \nu_{A_i}(y))) = (1, 0) \text{ for all } i \in I.$$

Therefore,

$$\inf_{i\in I}\mathcal{I}_\mathcal{T}((\mu_{A_i}(x), \nu_{A_i}(x)), (\mu_{A_i}(y), \nu_{A_i}(y))) = (1, 0).$$

Then, $\rho(x, y) = (1, 0)$ we obtain that,

$$\mu_\rho(x, y) = 1 \text{ and } \nu_\rho(x, y) = 0.$$

Conversely, if we assume that $y \triangleleft_\rho x$, we obtain analogously that $\rho(y, x) = (1, 0)$. Hence, $\mu_\rho(y, x) = 1$ and $\nu_\rho(y, x) = 0$. Thus, in any case, we have,

$$\begin{cases} \max(\mu_\rho(x, y), \mu_\rho(y, x)) = 1, \\ \text{and} \\ \min(\nu_\rho(x, y), \nu_\rho(y, x)) = 0. \end{cases}$$

Thus, ρ is strongly, which completes the proof. ■

Chapter 3

Some new results on intuitionistic fuzzy sublattices and their ideals

In this chapter, we study the concept of intuitionistic fuzzy sublattices and intuitionistic fuzzy ideals with respect to an intuitionistic fuzzy t-norm on an adequate lattice. Some characterizations and properties of these intuitionistic fuzzy sublattices and ideals with respect to intuitionistic fuzzy t-norm are established.

3.1 Intuitionistic fuzzy lattices

In this section, we recall the basic definitions and properties of intuitionistic fuzzy lattices and some related notions that will be needed throughout the next chapters. The concept of an intuitionistic fuzzy lattice was introduced by Thomas and Nair [48] as an intuitionistic fuzzy set on a crisp lattice stable by the supremum and the infimum of the binary operations $\sqcap$ *and* $\sqcup$.

Definition 3.1.1 *[48]*

Let L *be a lattice and* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in L\}$ *be an IFS on* L. *Then* A *is called an intuitionistic fuzzy lattice if for any* $x, y \in L$, *the following conditions* *are satisfied:*

(i) $\mu_A(x \sqcup y) \geq \mu_A(x) \wedge \mu_A(y)$;

(ii) $\mu_A(x \sqcap y) \geq \mu_A(x) \wedge \mu_A(y)$;

(iii) $\nu_A(x \sqcup y) \leq \nu_A(x) \vee \nu_A(y)$;

(iv) $\nu_A(x \sqcap y) \leq \nu_A(x) \vee \nu_A(y)$.

Example 3.1.2 *[4]*

Figure 3.1 shows the Hasse diagram of a lattice $L = \{0, a, b, 1\}$. *The intuitionistic fuzzy set* A *on* L *given by* $A = \{< 0, 0.5, 0.1 >, < a, 0.4, 0.5 >, < b, 0.4, 0.3 >, < 1, 0.7, 0.3 >\}$ *is an intuitionistic fuzzy lattice.*

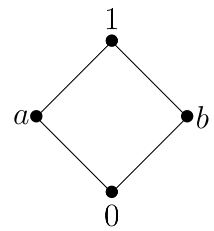


Figure 3.1: Hasse diagram of a lattice $(L, \leqslant, \sqcap, \sqcup)$ with $L = \{0, a, b, 1\}$.

For further details on intuitionistic fuzzy lattices, we refer to [42, 47, 48].

3.2 Intuitionistic fuzzy ideals and filters on a lattice

The notion of intuitionistic fuzzy ideal (resp. filter) on a lattice was first introduced by Thomas and Nair [48].

Definition 3.2.1 *[48]*

Let L *be a lattice and* $I = \{\langle x, \mu_I(x), \nu_I(x)\rangle \mid x \in L\}$ *be an IFS on* L. *Then* I *is called an intuitionistic fuzzy ideal on* L *(IF-ideal, for short) if for all* $x, y \in L$ *the following conditions are satisfied:*

(i) $\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$;

(ii) $\mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$;

(iii) $\nu_I(x \sqcup y) \leq \nu_I(x) \vee \nu_I(y)$;

(iv) $\nu_I(x \sqcap y) \leq \nu_I(x) \wedge \nu_I(y)$.

Example 3.2.2 *[4]*

Let L be the lattice given by the Hasse diagram in Figure 3.1. The intuitionistic fuzzy set I on L defined by $I = \{< 0, 0.5, 0.1 >, < a, 0.4, 0.3 >, < b, 0.1, 0.2 >, < 1, 0.1, 0.3 >\}$ *is an IF-ideal.*

Definition 3.2.3 *[48]*

Let L be a lattice and $F = \{\langle x, \mu_F(x), \nu_F(x)\rangle \mid x \in L\}$ *be an IFS on L. Then F is called an intuitionistic fuzzy filter on L (IF-filter, for short) if for any* $x, y \in L$, *the following conditions are satisfied:*

(i) $\mu_F(x \sqcup y) \geq \mu_F(x) \vee \mu_F(y)$;

(ii) $\mu_F(x \sqcap y) \geq \mu_F(x) \wedge \mu_F(y)$;

(iii) $\nu_F(x \sqcup y) \leq \nu_F(x) \wedge \nu_F(y)$;

(iv) $\nu_F(x \sqcap y) \leq \nu_F(x) \vee \nu_F(y)$.

Example 3.2.4 *[4]*

Let L be the lattice given by the Hasse diagram in Figure 3.1. The intuitionistic fuzzy set F on L defined by $F = \{< 0, 0.1, 0.6 >, < a, 0.2, 0.6 >, < b, 0.1, 0.5 >, < 1, 0.4, 0.3 >\}$ *is an IF-filter.*

Remark 3.2.5 *[4]*

Notice that every IF-ideal on L is an L-IF-lattice, but the converse is not true in general. Indeed, let L be the lattice given by the Hasse diagram in Figure 3.1 and $A \in IFS(L)$ *defined by* $A = \{< 0, 0.3, 0.1 >, < a, 0.4, 0.5 >, < b, 0.4, 0.3 >, < 1, 0.7, 0.3 >\}$. *Then A is an L-IF-lattice, but since* $\mu_A(a) = \mu_A(a \sqcap 1) = 0.4 \ngeq \max\{0.4; 0.7\}$, *then it holds that A is not an IF-ideal on L. As well since* $\mu_A(0) = \mu_A(a \sqcap b) = 0.3 \ngeq \min\{0.4; 0.4\}$, *then it holds that A is not an IF-filter on L.*

The following results will be needed throughout this chapter.

Proposition 3.2.6

Let L be a lattice, L^d *be its order-dual lattice, and* $A \in IFS(L)$. *Then it holds that A is an IF-ideal on L if and only if A is an IF-filter on* L^d *and conversely.*

Proposition 3.2.7 *[48]*

Let L be a lattice, A and B are two intuitionistic fuzzy sets on L. Then it holds that

(i) if A and B are two IF-ideals on L, then $A \cap B$ *is an IF-ideal on L;*

(ii) if A and B are two IF-filters on L, then $A \cap B$ *is an IF-filter on L.*

3.3 $\mathcal{T}$ -Intuitionistic fuzzy sublattices

In this section, we introduce and study the notion of an intuitionistic fuzzy sublattice w.r.t a representable intuitionistic fuzzy t-norm, intuitionistic fuzzy ideal w.r.t a representable intuitionistic fuzzy t-norm. Also, their characterizations w.r.t a representable intuitionistic fuzzy t-norm.

Definition 3.3.1 *[4]*

Let $\mathcal{T}$ be a representable intuitionistic fuzzy t-norm, L a crisp lattice, and A an intuitionistic fuzzy set. A is called an intuitionistic fuzzy sublattice on lattice L w.r.t the representable intuitionistic fuzzy t-norm $\mathcal{T}$ if and only if the following inequality holds

$$\mathcal{T}\left(A\left(x\right), A\left(y\right)\right) \leqslant_{L^*} A(x \vee y) \wedge_{L^*} A(x \wedge y) \tag{3.1}$$

Notation 3.3.2

The sets of all the intuitionistic fuzzy lattice on lattice L w.r.t representable intuitionistic fuzzy t-norm $\mathcal{T}$ will be denoted by $\mathcal{T}$-$IFL(L)$.

Remark 3.3.3 *[4]*

If $A \in \mathcal{T}_M$-$IFL(L)$, then A is an intuitionistic fuzzy lattice on lattice L, within the meaning of K. V. Thomas and L. S. Nair in [48]. So intuitionistic fuzzy lattice w.r.t the representable intuitionistic fuzzy t-norm $\mathcal{T}$ is a generalization of the intuitionistic fuzzy lattice.

Example 3.3.4 *[4]*

Consider the lattice $L = \{1, 2, 3, 6\}$ the of all divisors of 6. Let A be the intuitionistic fuzzy set given by $A\left(1\right) = (0.1, 0)$, $A\left(2\right) = (0.4, 0.3)$, $A\left(3\right) = (0.5, 0.5)$, $A\left(6\right) = (0.2, 0.3)$.

Then, A is a $\mathcal{T}_L$-$IFL\left(L\right)$. Indeed, it suffices to prove that if for $x, y \in L$ we get that

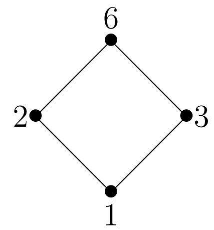


Figure 3.2: The Hasse diagram of $D(6)$

x	y	$A(x)$	$A(y)$	$x \wedge y$	$A(x \wedge y)$	$x \vee y$	$A(x \vee y)$
1	1	$(0.1, 0.2)$	$(0.1, 0.2)$	1	$(0.1, 0.2)$	1	$(0.1, 0.2)$
1	2	$(0.1, 0.2)$	$(0.4, 0.3)$	1	$(0.1, 0.2)$	2	$(0.4, 0.3)$
1	3	$(0.1, 0.2)$	$(0.5, 0.5)$	1	$(0.1, 0.2)$	3	$(0.5, 0.5)$
1	6	$(0.1, 0.2)$	$(0.2, 0.3)$	1	$(0.1, 0.2)$	6	$(0.2, 0.3)$
2	1	$(0.4, 0.3)$	$(0.1, 0.2)$	1	$(0.1, 0.2)$	2	$(0.4, 0.3)$
2	2	$(0.4, 0.3)$	$(0.4, 0.3)$	2	$(0.4, 0.3)$	2	$(0.4, 0.3)$
2	3	$(0.4, 0.3)$	$(0.5, 0.5)$	1	$(0.1, 0.2)$	6	$(0.2, 0.3)$
2	6	$(0.4, 0.3)$	$(0.2, 0.3)$	2	$(0.4, 0.3)$	6	$(0.2, 0.3)$
3	1	$(0.5, 0.5)$	$(0.1, 0.2)$	1	$(0.1, 0.2)$	3	$(0.5, 0.5)$
3	2	$(0.5, 0.5)$	$(0.4, 0.3)$	1	$(0.1, 0.2)$	6	$(0.4, 0.3)$
3	3	$(0.5, 0.5)$	$(0.5, 0.5)$	3	$(0.5, 0.5)$	3	$(0.5, 0.5)$
3	6	$(0.5, 0.5)$	$(0.2, 0.3)$	3	$(0.5, 0.5)$	6	$(0.2, 0.3)$
6	1	$(0.2, 0.3)$	$(0.1, 0.2)$	1	$(0.1, 0.2)$	6	$(0.2, 0.3)$
6	2	$(0.2, 0.3)$	$(0.4, 0.3)$	2	$(0.4, 0.3)$	6	$(0.2, 0.3)$
6	3	$(0.2, 0.3)$	$(0.5, 0.5)$	3	$(0.5, 0.5)$	6	$(0.2, 0.3)$
6	6	$(0.2, 0.3)$	$(0.2, 0.3)$	6	$(0.2, 0.3)$	6	$(0.2, 0.3)$

Table 3.1: Verification table of the fact that A is a $\mathcal{T}_L$-$IFL\,(L)$.

$\mathcal{T}_L(A(x),(A(y))$	$A(x \wedge y) \wedge_{L^*} A(x \vee y)$	The inequality 3.1
$(0, 0.4)$	$(0.1, 0.2)$	True
$(0, 0.5)$	$(0.1, 0.3)$	True
$(0, 0.7)$	$(0.1, 0.5)$	True
$(0, 0.5)$	$(0.1, 0.3)$	True
$(0, 0.5)$	$(0.1, 0.3)$	True
$(0, 0.6)$	$(0.4, 0.3)$	True
$(0, 0.8)$	$(0.1, 0.3)$	True
$(0, 0.6)$	$(0.2, 0.3)$	True
$(0, 0.7)$	$(0.1, 0.5)$	True
$(0, 0.8)$	$(0.5, 0.5)$	True
$(0, 1)$	$(0.2, 0.5)$	True
$(0, 0.8)$	$(0.1, 0.3)$	True
$(0, 0.5)$	$(0.2, 0.3)$	True
$(0, 0.6)$	$(0.2, 0.5)$	True
$(0, 0.8)$	$(0.2, 0.3)$	True
$(0, 0.6)$	$(0.1, 0.5)$	True

Table 3.2: Verification table of the fact that A is a $\mathcal{T}_L$-$IFL\,(L)$.

The following lemma is immediate and shows the generality of our work.

Lemma 3.3.5 *[4]*

If $A \in \mathcal{T}_M$-$IFL\,(L)$ then $A \in \mathcal{T}$-$IFL\,(L)$, the converse is not true.

Proof.

Assume that $A \in \mathcal{T}_M$-$IFL(L)$. Then,

$$\mathcal{T}_M\left(A\left(x\right), A\left(y\right)\right) \leqslant_{L^*} A(x \vee y) \wedge_{L^*} A(x \wedge y)$$

Using the Remark 2.1.8. Then,

$$\mathcal{T}\left(A\left(x\right), A\left(y\right)\right) \leqslant_{L^*} A(x \vee y) \wedge_{L^*} A(x \wedge y)$$

So, $A \in \mathcal{T}$-$IFL\,(L)$. ■

Remark 3.3.6 *Conversely is note true. Indeed, let L be the lattice in Example 3.3.4 and let A given by,*

$A\left(1\right) = (0.1, 0.2),\ A\left(2\right) = (0.4, 0.3),\ A\left(3\right) = (0.5, 0.5),\ A\left(6\right) = \left(0.2, 0.3\right).$

Then, A is an $\mathcal{T}_L$-$IFL\,(L)$ but A is note an $\mathcal{T}_M$-$IFL\,(L)$.

But, $\mathcal{T}_M(A(2), (A(3)) = (0.4, 0.5) \nleqslant_{L^*} (0.1, 0.3) = A(2 \wedge 3) \wedge_{L^*} A(2 \vee 3) = A(1) \wedge_{L^*} A(6)$.

Then, A is note an $\mathcal{T}_M$-$IFL\,(L)$.

The following theorems about the intersection of two $\mathcal{T}$-intuitionistic fuzzy lattices on lattice L.

Theorem 3.3.7 *[4]*

The $\mathcal{T}$ -intersection of two $\mathcal{T}$-intuitionistic fuzzy sublattices on a lattice L is a $\mathcal{T}$-intuitionistic fuzzy sublattice on the lattice L.

Proof.

Let $A, B \in \mathcal{T}\text{-}IFL(L)$, *i.e.,*

$$\begin{cases} \mathcal{T}(A(x), A(y)) \leqslant_{L^*} A(x \wedge y) \wedge_{L^*} A(x \vee y) \\ and \\ \mathcal{T}(B(x), B(y)) \leqslant_{L^*} B(x \wedge y) \wedge_{L^*} B(x \vee y) \end{cases}$$

We have for $x \in L : (A \cap B)(x) = \mathcal{T}(A(x), B(x))$. *Using the Lemma 2.1.5 and 2.1.6.*

We have,

$$\mathcal{T}\begin{pmatrix} \mathcal{T}(A(x), A(y)), \\ \mathcal{T}(B(x), B(y)) \end{pmatrix} \leqslant_{L^*} \mathcal{T}\begin{pmatrix} A(x \wedge y) \wedge_{L^*} A(x \vee y), \\ B(x \wedge y) \wedge_{L^*} B(x \vee y) \end{pmatrix}$$

Hence,

$$\mathcal{T}\begin{pmatrix} \mathcal{T}(A(x), B(x)), \\ \mathcal{T}(A(y), B(y)) \end{pmatrix} \leqslant_{L^*} \mathcal{T}(A(x \wedge y), B(x \wedge y)) \wedge_{L^*} \mathcal{T}(A(x \vee y), B(x \vee y))$$

Thus,

$$\mathcal{T}((A \cap B)(x), (A \cap B)(y)) \leqslant_{L^*} (A \cap B)(x \wedge y) \wedge_{L^*} (A \cap B)(x \vee y)$$

Then, $A \cap B \in \mathcal{T}\text{-}IFL(L)$. ■

Corollary 3.3.8 *[4]*

Let L *be a lattice. If* $A \in \mathcal{T}\text{-}IFL(L)$ *and* $B \in \mathcal{T}_M\text{-}IFL(L)$, *then* $A \cap B \in \mathcal{T}\text{-}IFL(L)$.

Proof.

Obviously according to Lemma 3.3.5. ■

The following theorem characterizes the intersection of two $\mathcal{T}$-intuitionistic fuzzy sublattices on lattice L, where the intersection of intuitionistic fuzzy sets is given w.r.t another representable intuitionistic fuzzy t-norm $\mathcal{T}$.

***Theorem 3.3.9** [4]*

Let L be a lattice, $A, B \in \mathcal{T}$-$IFL(L)$ and let $\mathcal{T}'$ a representable intuitionistic fuzzy t-norm dominates $\mathcal{T}$. Then, the $\mathcal{T}'$-intersection of A and B is a $\mathcal{T}'$-$IFL(L)$.

Proof.

Suppose that $\mathcal{T}$ is dominated by $\mathcal{T}'$.

Let A and B be in $\mathcal{T}$-$IFL(L)$ i.e.,

$\mathcal{T}(A(x), A(y)) \leqslant_{L^*} A(x \wedge y) \wedge_{L^*} A(x \vee y)$ *and*

$\mathcal{T}(B(x), B(y)) \leqslant_{L^*} B(x \wedge y) \wedge_{L^*} B(x \vee y)$ *for any $x, y \in L$.*

Then

$$\mathcal{T}'(\mathcal{T}(A(x), A(y)), \mathcal{T}(B(x), B(y))) \leqslant_{L^*} \mathcal{T}'\begin{pmatrix} A(x \wedge y) \wedge_{L^*} A(x \vee y), \\ B(x \wedge y) \wedge_{L^*} B(x \vee y) \end{pmatrix}$$

From $\mathcal{T} \ll \mathcal{T}'$, Lemma 2.1.5 and the transitivity of $\leqslant_{L^*}$ follow that

$$\mathcal{T}(\mathcal{T}'(A(x), B(x)), \mathcal{T}'(A(y), B(y))) \leqslant_{L^*} \mathcal{T}'\begin{pmatrix} (A(x \wedge y), B(x \wedge y)) \wedge_{L^*} \\ \mathcal{T}'(A(x \vee y), B(x \vee y)) \end{pmatrix}$$

As $(A \cap B)(x) = \mathcal{T}'(A(x), B(x))$ for any $x \in L$, then,

$$\mathcal{T}((A \cap B)(x), (A \cap B)(y)) \leqslant_{L^*} (A \cap B)(x \wedge y) \wedge_{L^*} (A \cap B)(x \vee y)$$

So, $A \cap B \in \mathcal{T}'$-$IFL(L)$. ■

***Remark 3.3.10** [4]*

The union of two $\mathcal{T}$-$IFLs$ need not be a $\mathcal{T}$-IFL. Indeed, consider the lattice given in Example 3.3.4 and define A, B by

$$A\,(1) = (0.7, 0.2)\,,\ A\,(2) = (0.4, 0.5)\,,\ A\,(3) = (0.1, 0.5)\,,\ A\,(6) = (0.2, 0.4)$$

And

$$B\,(1) = (0.6, 0.1)\,,\ B\,(2) = (0.1, 0.5)\,,\ B\,(3) = (0.3, 0.3)\,,\ B\,(6) = (0.2, 0.3)$$

It is easy to see that A and B are $\mathcal{T}$-$IFLs$ of L. If we put,

$$(A \cup B)\,(x) = \mathcal{S}_M\,(A(x), B(x)) \text{ for all } x \in L.$$

$$\begin{aligned} (A \cup B)\,(1) &= (0.7, 0.1)\,,\ (A \cup B)\,(2) = (0.4, 0.5)\,, \\ (A \cup B)\,(3) &= (0.3, 0.3)\,,\ (A \cup B)\,(6) = (0.2, 0.3)\,. \end{aligned}$$

But, $\mathcal{T}_M\,((A \cup B)\,(3)\,, (A \cup B)\,(2)) = (0.3, 0.5) \nleq_{L^*}$
$\mathcal{T}\,((A \cup B)\,(3 \wedge 2), (A \cup B)\,(3 \vee 2)) = \mathcal{T}\,((A \cup B)\,(1), (A \cup B)\,(6))$
$= (0.2, 0.3)\,.$
So $A \cup B$ is not a $\mathcal{T}$-IFL.

Now we show some properties of $\mathcal{T}$ -Intuitionistic fuzzy sublattices.

Proposition 3.3.11 *[4]*

Let L be a lattice. For any representable intuitionistic fuzzy t-norm $\mathcal{T}$, if A is a $\mathcal{T}$-$IFL\,(L)$ then $[A]$ and $\langle A\rangle$ are $\mathcal{T}$-$IFLs$ of L. Where $[A]$ and $\langle A\rangle$ are the Necessity and the possibility of A respectively that we knew in Definition 1.6.3.

Proof.

Let L be a lattice and assume that A is a $\mathcal{T}$-$IFL(L)$.

We have $[A] = \{\langle x, [A](x)\rangle \mid [A](x) \in L^*\}$, *where*

$[A](x) = (\mu_A(x), \mu_A^c(x))$ *and* $\mu_A(x) + \mu_A^c(x) = 1$. *Then for any* x,$y \in L$,

$A \in \mathcal{T}$-$IFL(L)$ *this implies* $\mathcal{T}(A(x), A(y)) \leqslant_{L^*} A(x \vee y) \wedge_{L^*} A(x \wedge y)$

$$\Rightarrow \; (T(\mu_A(x), \mu_A(y)), S(\nu_A(x), \nu_A(y))) \leqslant_{L^*} \begin{pmatrix} \min(\mu_A(x \vee y), \mu_A(x \wedge y)), \\ \max(\nu_A(x \vee y), \nu_A(x \wedge y)) \end{pmatrix},$$

$$\Rightarrow \; \begin{cases} T(\mu_A(x), \mu_A(y)) \leq \min(\mu_A(x \vee y), \mu_A(x \wedge y)) \\ S(\nu_A(x), \nu_A(y)) \geq \max(\nu_A(x \vee y), \nu_A(x \wedge y)) \end{cases},$$

$$\Rightarrow \; T(\mu_A(x), \mu_A(y)) \leq \min(\mu_A(x \vee y), \mu_A(x \wedge y)),$$

$$\Rightarrow \; 1 - T(\mu_A(x), \mu_A(y)) \geq 1 - \min(\mu_A(x \vee y), \mu_A(x \wedge y)),$$

$$\Rightarrow \; S(1 - \mu_A(x), 1 - \mu_A(y)) \geq \max(1 - \mu_A(x \vee y), 1 - \mu_A(x \wedge y)),$$

$$\Rightarrow \; S(\mu_A^c(x), \mu_A^c(y)) \geq \max(\mu_A^c(x \vee y), \mu_A^c(x \wedge y)).$$

Then

$$\begin{aligned} \mathcal{T}([A](x), [A](y)) &= \mathcal{T}((\mu_A(x), \mu_A^c(x)), (\mu_A(y), \mu_A^c(y))), \\ &= (T(\mu_A(x), \mu_A(y)), S(\mu_A^c(x), \mu_A^c(y))), \\ &\leqslant_{L^*} \begin{pmatrix} \min(\mu_A(x \vee y), \mu_A(x \wedge y)), \\ \max(\mu_A^c(x \vee y), \mu_A^c(x \wedge y)) \end{pmatrix}, \\ &= \begin{matrix} (\mu_A(x \vee y), \mu_A^c(x \vee y)) \\ \wedge_{L^*} (\mu_A(x \wedge y), \mu_A^c(x \wedge y)), \end{matrix} \end{aligned}$$

Hence $[A]$ *is a* $\mathcal{T}$-IFL *of* L.

Concerning the set $\langle A\rangle$, *we have for any* $x, y \in L$,

$$\begin{aligned}
\mathcal{T}\left(\langle A\rangle(x),\langle A\rangle(y)\right) &= \mathcal{T}\left(\left(\nu_A^c(x),\nu_A(x)\right),\left(\nu_A^c(y),\nu_A(y)\right)\right),\\
&= \left(T\left(\nu_A^c(x),\nu_A^c(y)\right),S\left(\nu_A(x),\nu_A(y)\right)\right),\\
&\leqslant_{L^*} \begin{pmatrix}\min\left(\nu_A^c(x\vee y),\nu_A^c(x\wedge y)\right),\\ \max\left(\nu_A(x\vee y),\nu_A(x\wedge y)\right)\end{pmatrix},\\
&= \left(\left(\nu_A^c(x\vee y),\nu_A(x\vee y)\right)\wedge_{L^*}\left(\nu_A^c(x\wedge y),\nu_A(x\wedge y)\right)\right),\\
&= \langle A\rangle(x\vee y)\wedge_{L^*}\langle A\rangle(x\wedge y).
\end{aligned}$$

Hence, $\langle A\rangle$ is a $\mathcal{T}$-IFL of L. ■

Proposition 3.3.12 *[4]*

Let L be a lattice. For any representable intuitionistic fuzzy t-norm $\mathcal{T}$, if A is a $\mathcal{T}$-$IFL(L)$ then $supp(A)$ is a crisp sublattice of L. Where $supp(A)$ that we knew in Definition 1.6.4.

Proof.

Let $x,y\in supp(A)$. Using Propriety 2.1.3. Then $0_{L^*}<_{L^*}A(x)$ and $0_{L^*}<_{L^*}A(y)$. Since $\mathcal{T}(A(x),A(y))\leqslant_{L^*}A(x\vee y)\wedge_{L^*}A(x\wedge y)$ it follows that $0_{L^*}<_{L^*}A(x\vee y)\wedge_{L^*}A(x\wedge y)$. So $0_{L^*}<_{L^*}A(x\vee y)$ and $0_{L^*}<_{L^*}A(x\wedge y)$. Hence $x\vee y\in supp(A)$ and $x\wedge y\in supp(A)$. Thus $supp(A)$ is a crisp sublattice of L. ■

Remark 3.3.13 *[4]*

The converse of the above Proposition 3.3.12 does not holds in general. Indeed, let L be the lattice in Example 3.3.4 and $A\in IFL(L)$ given by $A(1)=(0.4,0.3)$, $A(2)=(0.7,0.2)$, $A(3)=(0.2,0.2)$, $A(6)=(0.4,0.1)$. Obviously $suppA=\{1,2,3,6\}=L$ is a crisp lattice. Since $\mathcal{T}_M(A(2),A(3))=(0.2,0.2)$ and $\nleqslant_{L^*}A(2\vee 3)\wedge_{L^*}A(2\wedge 3)=A(6)\wedge_{L^*}A(1)=(0.4,0.3)$, it follows that A is not an intuitionistic fuzzy lattice of L.

3.4 $\mathcal{T}$-Intuitionistic fuzzy ideal

The notion of an intuitionistic fuzzy ideal on a lattice was first introduced by Thomas and Nair [48]. In this section, we give a new characterization of the intuitionistic fuzzy ideal on a lattice w.r.t a given intuitionistic fuzzy t-norm.

Definition 3.4.1 *[4]*

Let L be a lattice, $\mathcal{T}$ a representable intuitionistic fuzzy t-norm, $\mathcal{S}$ the intuitionistic triangular conorm associated with $\mathcal{T}$ and let I be an intuitionistic fuzzy set. I is called an intuitionistic fuzzy ideal on the lattice L w.r.t $\mathcal{T}$ if the following conditions are satisfied:

1. $\mathcal{T}\left(I\left(x\right), I\left(y\right)\right) \leqslant_{L^*} I(x \vee y)$,

2. $\mathcal{S}\left(I\left(x\right), I\left(y\right)\right) \leqslant_{L^*} I(x \wedge y)$.

Notation 3.4.2

The set of all the intuitionistic fuzzy ideals on lattice L w.r.t an intuitionistic fuzzy t-norm $\mathcal{T}$ will be denoted by $\mathcal{T}$-$IFI(L)$.

Example 3.4.3 *[4]*

Consider the lattice L of "all integer divisors of 18 ". That is $L = \{1, 2, 3, 6, 9, 18\}$. Let A be given by $A\left(1\right) = (0.7, 0.2)$, $A\left(2\right) = (0.5, 0.5)$, $A\left(3\right) = (0.6, 0.3)$, $A\left(6\right) = (0.4, 0.5)$, $A\left(9\right) = (0.5, 0.5)$, $A\left(18\right) = (0.4, 0.5)$. It easy to see that A is a $\mathcal{T}$-$IFI\left(L\right)$.

Remark 3.4.4 *[4]*

In general, a $\mathcal{T}_M$-$IFI\left(L\right)$ is not a $\mathcal{T}$ - $IFI\left(L\right)$. Indeed, let L be the lattice

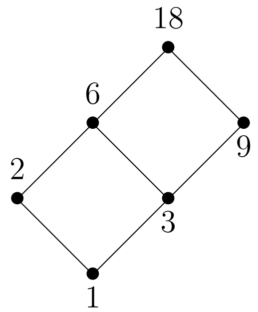


Figure 3.3: The Hasse diagram of $D(18)$

in Example 3.3.4 and $A \in IFS(L)$ *given by* $A(1) = (0.4, 0.2)$, $A(2) = (0.4, 0.3)$, $A(3) = (0.1, 0.3)$, $A(6) = (0.1, 0.3)$. *Obviously* $A \in \mathcal{T}_M\text{-}IFI(L)$. *Since* $\mathcal{S}_L(A(2), A(3)) = (0.8, 0) \nleqslant_{L^*} A(2 \wedge 3) = A(1) = (0.4, 0.2)$, *it follows that* A *is not* $\mathcal{T}_L\text{-}IFI(L)$.

Remark 3.4.5 *[48, 4]*

Every $\mathcal{T}_M\text{-}IFI(L)$ *is a* $\mathcal{T}_M\text{-}IFL(L)$. *Indeed, if* I *is* $\mathcal{T}_M\text{-}IFI(L)$, *then*

$$\begin{cases} \mathcal{T}_M(I(x), I(y)) \leqslant_{L^*} I((x \vee y) \\ \mathcal{S}_M(I(x), I(y)) \leqslant_{L^*} I((x \wedge y) \end{cases},$$

According with Remark 2.1.7, it is not difficult to see that,

$\mathcal{T}_M(I(x), I(y)) \leqslant_{L^*} I(x \vee y) \wedge_{L^*} I(x \wedge y)$, *then* I *is a* $\mathcal{T}_M\text{-}IFL(L)$.

The converse is not true as seen in the following example. Consider the lattice L *given in Example 3.3.4. Define* A *to be* $A(1) = (0.3, 0.2)$, $A(2) = (0.2, 0.4)$, $A(3) = (0.2, 0.6)$, $A(6) = (0.5, 0.4)$. *Hence* A *is a* $\mathcal{T}_M\text{-}IFL(L)$, *but not a* $\mathcal{T}_M\text{-}IFI(L)$, *because*

$\mathcal{S}_M(I(3), I(6)) = (0.5, 0.4) \nleqslant_{L^*} I(3 \wedge 6) = I(3) = (0.2, 0.6)$.

Lemma 3.4.6 *[4]*

If $I \in \mathcal{T}\text{-}IFI(L)$ *then* $I \in \mathcal{T}_M\text{-}IFL(L)$.

Proof.

Let L be a lattice and let I is $\mathcal{T}$*-*$IFI(L)$ *then* $\mathcal{T}(I(x), I(y)) \leqslant_{L^*} I(x \vee y)$ *and* $\mathcal{S}(I(x), I(y)) \leqslant_{L^*} I(x \wedge y)$*. According to the Lemma 2.1.8 and Propriety 2.1.3 we obtain the result.* ■

Remark 3.4.7 *[4]*

The converse of the Lemma 3.4.6 does not hold in general. Indeed, Let L be the lattice of all integers divisors of 6 *(see. Fig 3.2) and A an* $IFS(L)$ *defined by* $A(1) = (0.1, 0)$, $A(2) = (0.4, 0.3)$, $A(3) = (0.1, 0.2)$, $A(6) = (0.1, 0.3)$. *Trivially, that A is a* $\mathcal{T}_M$*-*$IFL(L)$*. But, since*

$\mathcal{S}_M(A(1), A(2)) = \mathcal{S}_M((0.1, 0), (0.4, 0.3)) = (0.4, 0) \nleqslant_{L^*} A(1 \vee 2) = A(6) = (0.1, 0.3)$ *. It holds that A is not a* $\mathcal{T}$*-*$IFI(L)$*.*

Theorem 3.4.8 *[4]*

For any $I \in \mathcal{T}$*-*$IFI(L)$*, then* $I \in \mathcal{T}$*-*$IFL(L)$*.*

Proof.

Using Lemma 3.4.6 and Lemma 3.3.5 the theorem holds. ■

Conclusion 3.4.9 *[4]*

Every $\mathcal{T}_M$*-*$IFI(L)$ *is a* $\mathcal{T}$*-*$IFL(L)$*.*

In this subsection, we characterize the notion of $\mathcal{T}$*-*$IFI(L)$*. We start with the key results.*

As a corollary, we obtain the following interesting theorem of $\mathcal{T}$*-*$IFI(L)$*.*

Lemma 3.4.10 *[4]*

Let L be a lattice and $I \in \mathcal{T}$*-*$IFI(L)$*. Then for any* $x, y \in L$*, if* $x \leq y$*, then* $I(y) \leqslant_{L^*} I(x)$*.*

Proof.

Direct from [42, Corollary 3.1]. ∎

In the following theorem, we provide a basic characterization of $\mathcal{T}_M$-$IFI(L)$.

Theorem 3.4.11 *[4]*

Let L be a lattice. Then it holds that I is a $\mathcal{T}_M$-$IFI(L)$ if and only if the following condition is satisfied:

$$I(x \vee y) = \mathcal{T}_M(I(x), I(y)). \tag{3.2}$$

Proof.

Suppose that I is a $\mathcal{T}_M$-$IFI(L)$. Then $\mathcal{T}_M(I(x), I(y)) \leqslant_{L^*} I(x \vee y)$.

Since,

$$\begin{cases} x \leq x \vee y, \\ y \leq x \vee y. \end{cases}$$

We obtain from the Lemma 3.4.10 that,

$$\begin{cases} I(x \vee y) \leqslant_{L^*} I(x), \\ I(x \vee y) \leqslant_{L^*} I(y). \end{cases}$$

Then,

$$\mathcal{T}_M(I(x \vee y), I(x \vee y)) \leqslant_{L^*} \mathcal{T}_M(I(x), I(y))$$

Hence,

$$I(x \vee y) \leqslant_{L^*} \mathcal{T}_M(I(x), I(y))$$

So,

$$I(x \vee y) = \mathcal{T}_M(I(x), I(y))$$

Conversely, suppose that $I(x \vee y) = \mathcal{T}_M(I(x), I(y))$, *for any* $x, y \in L$. *Then it is easy to see that* $\mathcal{T}_M(I(x), I(y)) \leqslant_{L^*} I(x \vee y)$, *for any* $x, y \in L^*$. *Next, we will show that* $\mathcal{S}_M(I(x), I(y)) \leqslant_{L^*} I(x \wedge y)$, *for any* $x, y \in L$. *Let* $x, y \in L$, *since* $x \vee (x \wedge y) = x$ *and* $y \vee (x \wedge y) = y$. *Then it holds that* $I(x \vee (x \wedge y)) = I(x)$ *and* $I(y \vee (x \wedge y)) = I(y)$. *From hypothes (3.2) is it follows that* $I(x) = \mathcal{T}_M(I(x), I(x \wedge y))$ *and* $I(y) = \mathcal{T}_M(I(y), I(x \wedge y))$. *Hence,* $I(x) \leqslant_{L^*} I(x \wedge y)$ *and* $I(y) \leqslant_{L^*} I(x \wedge y)$. *Thus,* $\mathcal{S}_M(I(x), I(y)) \leqslant_{L^*} I(x \wedge y)$, *for any* $x, y \in L^*$. *Therefore,* I *is* $\mathcal{T}_M$-$IFI(L)$. ■

The following proposition provides a basic characterization of the intuitionistic fuzzy ideal on a lattice.

Proposition 3.4.12 *[4]*

Let L *be a lattice. If* A *is a* $\mathcal{T}$-$IFI(L)$, *then* $[A]$ *and* $\langle A \rangle$ *are* $\mathcal{T}$-$IFIs$ *of* L.

Proof.

Assume that A *is a* $\mathcal{T}$-$IFI(L)$. *Then for any* $x, y \in L$,

$$\mathcal{T}(A(x), A(y)) \leqslant_{L^*} A(x \vee y)$$

Then,

$$(T(\mu_A(x), \mu_A(y)), S(\nu_A(x), \nu_A(y))) = (\mu_A(x \vee y), \nu_A(x \vee y))$$

Hence,

$$\begin{cases} T(\mu_A(x), \mu_A(y)) \leq \mu_A(x \vee y), \\ S(\nu_A(x), \nu_A(y)) \geq \nu_A(x \vee y). \end{cases}$$

So,

$$T(\mu_A(x), \mu_A(y)) \leq \mu_A(x \vee y)$$

This implies,

$$1 - T\left(\mu_A\left(x\right), \mu_A\left(y\right)\right) \geq 1 - \mu_A\left(x \vee y\right)$$

Since,

$$S\left(1 - \mu_A\left(x\right), 1 - \mu_A\left(y\right)\right) \geq 1 - \mu_A\left(x \vee y\right)$$

Finally,

$$S\left(\mu_A^c\left(x\right), \mu_A^c\left(y\right)\right) \geq \mu_A^c\left(x \vee y\right)$$

Then

$$\begin{aligned} \mathcal{T}\left(\left[A\right]\left(x\right), \left[A\right]\left(y\right)\right) &= \mathcal{T}\left(\left(\mu_A\left(x\right), \mu_A^c\left(x\right)\right), \left(\mu_A\left(y\right), \mu_A^c\left(y\right)\right)\right), \\ &= \left(T\left(\mu_A\left(x\right), \mu_A\left(y\right)\right), S\left(\mu_A^c\left(x\right), \mu_A^c\left(y\right)\right)\right), \\ &\leqslant_{L^*} \left(\mu_A\left(x \vee y\right), \mu_A^c\left(x \vee y\right)\right), \\ &= \left[A\right]\left(x \vee y\right) \end{aligned}$$

Now, for any $x, y \in L$,

$$\mathcal{S}\left(A\left(x\right), A\left(y\right)\right) \leqslant_{L^*} A(x \wedge y)$$

Then,

$$\left(S\left(\mu_A\left(x\right), \mu_A\left(y\right)\right), T\left(\nu_A\left(x\right), \nu_A\left(y\right)\right)\right) = \left(\mu_A\left(x \wedge y\right), \nu_A\left(x \wedge y\right)\right)$$

Hence,

$$\begin{cases} S\left(\mu_A\left(x\right), \mu_A\left(y\right)\right) \leq \mu_A\left(x \wedge y\right) \\ T\left(\nu_A\left(x\right), \nu_A\left(y\right)\right) \geq \nu_A\left(x \wedge y\right) \end{cases}$$

So,

$$S\left(\mu_A\left(x\right), \mu_A\left(y\right)\right) \leq \mu_A\left(x \wedge y\right)$$

This implies,

$$1 - S\left(\mu_A\left(x\right), \mu_A\left(y\right)\right) \geq 1 - \mu_A\left(x \wedge y\right)$$

Since,

$$T\left(1-\mu_A\left(x\right),1-\mu_A\left(y\right)\right)\geq 1-\mu_A\left(x\wedge y\right)$$

Finally,

$$T\left(\mu_A^c\left(x\right),\mu_A^c\left(y\right)\right)\geq \mu_A^c\left(x\wedge y\right)$$

Then,

$$\begin{aligned}
\mathcal{S}\left([A]\left(x\right),[A]\left(y\right)\right) &= \mathcal{S}\left(\left(\mu_A\left(x\right),\mu_A^c\left(x\right)\right),\left(\mu_A\left(y\right),\mu_A^c\left(y\right)\right)\right),\\
&= \left(S\left(\mu_A\left(x\right),\mu_A\left(y\right)\right),T\left(\mu_A^c\left(x\right),\mu_A^c\left(y\right)\right)\right),\\
&\leqslant_{L^*} \left(\mu_A\left(x\wedge y\right),\mu_A^c\left(x\wedge y\right)\right),\\
&= [A]\left(x\wedge y\right).
\end{aligned}$$

Hence $[A]$ is a $\mathcal{T}$-IFI of L. A similar proof for $\langle A\rangle$ is a $\mathcal{T}$-IFI. ■

The following proposition shows that the support of a $\mathcal{T}$-$IFI\,(L)$ is an ideal in this lattice.

Proposition 3.4.13 *[4]*

Let L be a lattice. The following holds

If I is a $\mathcal{T}$-$IFI\,(L)$, then Supp(I) is an ideal in L.

Proof.

Let L be a lattice. Suppose that I is a $\mathcal{T}$-$IFI\,(L)$ and show that Supp(I) is a crisp ideal in L.

Let $x\in$ Supp(I) and $y\leq x$, then it hold that $x\vee y=x$ and $0_{L^*}<_{L^*}I\left(x\right)=I(x\vee y)$. Thus $0_{L^*}<_{L^*}\mathcal{T}\left(I\left(x\right),I\left(y\right)\right)$. Using Propriety 2.1.3.

We obtain, $0_{L^*}<_{L^*}I\left(y\right)$ hence $y\in$ Supp(I).

For $x,y\in$ Supp(I), $0_{L^*}<_{L^*}I(x)$ and $0_{L^*}<_{L^*}I(y)$. Then $0_{L^*}<_{L^*}\mathcal{T}\left(I\left(x\right),I\left(y\right)\right)$.

Using Propriety 2.1.3, then it follows from Theorem 3.4.11 that $0_{L^*} <_{L^*} I(x \vee y)$ *hence* $x \vee y \in Supp(I)$*. Thus,* $Supp(I)$ *is an ideal on* L*.* ■

Remark 3.4.14 *[4]*

The converse of the Lemma 3.4.13 does not holds in general. Indeed, consider the lattice L *given in Example 3.3.4 and* $I \in IFS(L)$ *given by* $I(1) = (0.6, 0)$, $I(2) = (0.5, 0.4)$, $I(3) = (0.5, 0.2)$, $I(6) = (0.8, 0.1)$*. It is easy to verify that* $Supp(I) = L$ *is an ideal on* L*, but* I *is not a* $\mathcal{T}$-$IFI(L)$*.*

Theorem 3.4.15 *[4]*

The $\mathcal{T}$*-intersection of two* $\mathcal{T}$*-intuitionistic fuzzy ideals on a lattice* L *is a* $\mathcal{T}$ *-intuitionistic fuzzy ideal on the lattice* L*.*

Proof.

Let $I_1, I_2 \in \mathcal{T}$-$IFI(L)$*. Then,*

$$\begin{cases} \mathcal{T}\left(I_1(x), I_1(y)\right) \leqslant_{L^*} I_1(x \vee y) \\ \mathcal{T}\left(I_2(x), I_2(y)\right) \leqslant_{L^*} I_2(x \vee y) \end{cases}$$

Hence,

$\mathcal{T}\left(\mathcal{T}\left(I_1(x), I_1(y)\right), \mathcal{T}\left(I_2(x), I_2(y)\right)\right) \leqslant_{L^*} \mathcal{T}\left(I_1(x \vee y), I_2(x \vee y)\right).$

Using Lemma 2.1.6, we get,

$$\mathcal{T}\left(\mathcal{T}\left(I_1(x), I_2(x)\right), \mathcal{T}\left(I_1(y), I_2(y)\right)\right) \leqslant_{L^*} \mathcal{T}\left(I_1(x \vee y), I_2(x \vee y)\right),$$
$$\mathcal{T}\left(\left(I_1 \cap I_2\right)(x), \left(I_1 \cap I_2\right)(y)\right) \leqslant_{L^*} \left(I_1 \cap I_2\right)(x \vee y).$$

Then $I_1 \cap I_2$ *is an* $\mathcal{T}$-IFI *of* L*.* ■

Remark 3.4.16 *[4]*

The union of two $\mathcal{T}$-$IFIs$ *need not be a* $\mathcal{T}$-IFI*.*

Now, we give the definition a filter.

***Definition 3.4.17** [4]*

Let an intuitionistic fuzzy t-norm $\mathcal{T}$ and $\mathcal{S}$ it's your dully and let L be a lattice and $F = \{\langle x, F(x)\rangle \mid F(x) \in L^*\}$ be an IFS of L. F is called an intuitionistic fuzzy filter with respect Intuitionistic fuzzy t-norm $\mathcal{T}$ iff the followings conditions hold

1. $\mathcal{T}\left(F\left(x\right), F\left(y\right)\right) \leqslant_{L^*} F(x \wedge y)$,

2. $\mathcal{S}\left(F\left(x\right), F\left(y\right)\right) \leqslant_{L^*} F(x \vee y)$.

The following immediate proposition shows that all result on ideals is true on filters.

Proposition 3.4.18

Let $(L, \leq)$ be a lattice, $\left(L^d, \geq\right)$ its dual lattice and $A \in IFS(L)$. Then it holds that $A \in \mathcal{T}$-$IFI\left(L\right)$ if and only if $A \in \mathcal{T}$-$IFF\left(L^d\right)$ and conversely.

General conclusions and future research

In this thesis, the representation and construction for fuzzy preorder and weak orders are extended to the intuitionistic fuzzy case. Many fundamental representation results extending those of [16] are presented. Moreover, we have introduced the notion of a $\mathcal{T}$-intuitionistic fuzzy sublattice by associating the conditions mentioned in the definition of intuitionistic fuzzy sublattice [47, 48]. So a new equivalent definition is obtained which reduces the four conditions in only one. Thus, based on an intuitionistic fuzzy triangular norm, the study of intuitionistic fuzzy sublattices becomes so simple. Moreover, we extend the notion of an intuitionistic fuzzy ideal to a $\mathcal{T}$-intuitionistic fuzzy ideal w. r. t the lattice operations and we investigate their various characterizations and properties. Future work is anticipated in multiple directions. We think it makes sense to study the notions of intuitionistic fuzzy prime ideals and intuitionistic fuzzy filters for other types of lattices based on the intuitionistic fuzzy setting.

As open question

1. *What will happen for this study if the intuitionistic fuzzy t-norm is not t-representable?*
2. *It is possible to do such representation for an L-fuzzy weak order, where L is a complete lattice?*

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References

[1] Ahn, Y., Hur, K., and Kim, D. *The lattice of intuitionistic fuzzy ideals of a ring.* Journal of Applied Mathematics and Computing 19*, 1/2 (2005), 551.*

[2] Ajmal, N., and Thomas, K. *Fuzzy lattices.* Information sciences 79*, 3-4 (1994), 271–291.*

[3] Akram, M., and Dudek, W. A. *Interval-valued intuitionistic fuzzy lie ideals of lie algebras.* World Applied Sciences Journal 7*, 7 (2009), 812–819.*

[4] Amroune, A., and Ziane, B. *More on intuitionistic fuzzy sublattices and their ideals.* Facta Universitatis, Series: Mathematics and Informatics 35 *(2019), 871–888.*

[5] Atanassov, K., and Vassilev, P. *A new intuitionistic fuzzy definiteness norm.* Notes on Intuitionistic Fuzzy Sets 26*, 3 (2020), 52–60.*

[6] Atanassov, K. T. *Intuitionistic fuzzy sets, vii itkr's session, sofia deposed in central sci.* Technical Library of Bulg. Acad. of Sci 1697 *(1983), 84.*

[7] Atanassov, K. T. *Intuitionistic fuzzy sets.* Fuzzy Sets and Systems 20*, 1 (1986), 87–96.*

[8] Atanassov, K. T. *Intuitionistic fuzzy sets. In* Intuitionistic fuzzy sets. *Springer, 1999, pp. 1–137.*

[9] Atanassov, K. T. On intuitionistic fuzzy sets theory*, vol. 283. Springer, 2012.*

[10] Banerjee, B. *Intuitionistic fuzzy subrings and ideals.* J. Fuzzy Math. 11*, 1 (2003), 139–155.*

[11] Bělohlávek, R. *Concept lattices and order in fuzzy logic.* Annals of pure and applied logic 128*, 1-3 (2004), 277–298.*

[12] Birkhoff, G. Lattice theory*, vol. 25. American Mathematical Soc., 1940.*

[13] Birkhoff, G. *Lattice theory. 1967.* American Mathematical Society, Providence, RI *(1967).*

[14] Biswas, R. *On fuzzy sets and intuitionistic fuzzy sets.* Notes on Intuitionistic Fuzzy Sets 3*, 1 (1997).*

[15] Bo, Y., and Wangming, W. *Fuzzy ideals on a distributive lattice.* Fuzzy Sets and Systems 35*, 2 (1990), 231–240.*

[16] Bodenhofer, U., De Baets, B., and Fodor, J. *A compendium of fuzzy weak orders: Representations and constructions.* Fuzzy Sets and Systems 158*, 8 (2007), 811–829.*

[17] Burillo, P. J., and Bustince, H. *Intuitionistic fuzzy relations (part i).* Mathware and soft computing 2*, 1 (1995), 5–38.*

[18] Bustince, H., and Burillo, P. *Intuitionistic fuzzy relations (part ii).* Mathware and soft computing 2 *(1995), 117–148.*

[19] Cornelis, C., Deschrijver, G., De Cock, M., and Kerre, E. *Intuitionistic fuzzy relational calculus: an overview. In* Proceedings First International IEEE Symposium Intelligent Systems *(2002), vol. 1, IEEE, pp. 340–345.*

[20] Cornelis, C., Deschrijver, G., and Kerre, E. E. *Implication in intuitionistic fuzzy and interval-valued fuzzy set theory: construction, classification, application.* International journal of approximate reasoning 35*, 1 (2004), 55–96.*

[21] Davey, B. A., and Priestley, H. A. Introduction to lattices and order. *Cambridge university press, 2002.*

[22] Deschrijver, G., Cornelis, C., and Kerre, E. *Intuitionistic fuzzy connectives revisited. In* 9th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU 2002) *(2002), pp. 1839–1844.*

[23] Deschrijver, G., Cornelis, C., and Kerre, E. *On the representation of intuitionistic fuzzy t-norms and t-conorms.* Notes on Intuitionistic Fuzzy Sets 8*, 3 (2002), 1–10.*

[24] Deschrijver, G., Cornelis, C., and Kerre, E. E. *On the representation of intuitionistic fuzzy t-norms and t-conorms.* IEEE transactions on fuzzy systems 12*, 1 (2004), 45–61.*

[25] Deschrijver, G., and Kerre, E. E. *Classes of intuitionistic fuzzy t-norms satisfying the residuation principle.* International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 11*, 06 (2003), 691–709.*

[26] Deschrijver, G., and Kerre, E. E. *On the relationship between some extensions of fuzzy set theory.* Fuzzy Sets and Systems 133*, 2 (2003), 227–235.*

[27] Dubois, D., and Prade, H. *Fuzzy real algebra: some results.* Fuzzy Sets and Systems 2*, 4 (1979), 327–348.*

[28] Dubois, D., and Prade, H. *Towards fuzzy differential calculus part 1: Integration of fuzzy mappings.* Fuzzy Sets and Systems 8*, 1 (1982), 1–17.*

[29] Dubois, D., and Prade, H. *Towards fuzzy differential calculus part 2: integration on fuzzy intervals.* Fuzzy Sets and Systems 8*, 2 (1982), 105–116.*

[30] Dudek, W. *Intuitionistic fuzzy h-ideals of hemirings.* WSEAS Transactions on Mathematics 5*, 12 (2006), 1315.*

[31] Dushnik, B., and Miller, E. W. *Partially ordered sets.* American journal of mathematics 63*, 3 (1941), 600–610.*

[32] Goguen, J. A. *L-fuzzy sets.* Journal of mathematical analysis and applications 18*, 1 (1967), 145–174.*

[33] Goguen, J. A. *The logic of inexact concepts.* Synthese 19*, 3 (1969), 325–373.*

[34] Hájek, P. Metamathematics of Fuzzy Logic, Trends in Logic—Studia Logica Library*, vol. 4. Kluwer Academic Publishers, Dordrecht, 1998.*

[35] Hájek, P. Metamathematics of fuzzy logic*, vol. 4. Springer Science & Business Media, 2013.*

[36] Hur, K., Jang, S. Y., and Kang, H. W. *Intuitionistic fuzzy congruence on a lattice.* Journal of Appl. Math. Computing 18 *(2005), 465–486.*

[37] Hur, K., Jang, S. Y., and Kang, H. W. *The lattice of intuitionistic fuzzy congruences. In* International Mathematical Forum *(2006), vol. 1, pp. 211–236.*

[38] Kalina, M. *Constructions for t-conorms and t-norms on interval-valued and interval-valued intuitionistic fuzzy sets by paving.* Notes on Intuitionistic Fuzzy Sets 26*, 3 (2020), 1–12.*

[39] Klement, E. P., Mesiar, R., and Pap, E. *Triangular norms. 2000, 2000.*

[40] Liu, Y., Qin, X., and Xu, Y. *Interval-valued intuitionistic (t, s)-fuzzy filters theory on residuated lattices.* International Journal of Machine Learning and Cybernetics 5*, 5 (2014), 683–696.*

[41] Martinek, P. *Completely lattice l-ordered sets with and without l-equality.* Fuzzy Sets and Systems 166*, 1 (2011), 44–55.*

[42] MILLES, S., RAK, E., AND ZEDAM, L. *Intuitionistic fuzzy complete lattices. In* Novel Developments in Uncertainty Representation and Processing. *Springer, 2016, pp. 149–160.*

[43] NEGOITA, C., AND RALESCU, D. *Applications of fuzzy sets to systems analysis, interdisciplinary systems research series, vol. 11, 1975.*

[44] PICIU, D. Algebras of fuzzy logic*. Universitaria, 2007.*

[45] QIN, Y., AND LIU, Y. *Intuitionistic (t, s)-fuzzy filters on residuated lattices.* Gen 16*, 2 (2013), 55–65.*

[46] SCHWEIZER, B. *Associative funcrions and statistical triangle inequalities.* Puble. Math. Debrecen 8 *(1961), 169–186.*

[47] THOMAS, K., AND NAIR, L. S. *Quotient of ideals of an intuitionistic fuzzy lattice.* Advances in Fuzzy Systems 2010 *(2010).*

[48] THOMAS, K., AND NAIR, L. S. *Intuitionistic fuzzy sublattices and ideals.* Fuzzy Information and Engineering 3*, 3 (2011), 321.*

[49] WANG, G.-J., AND HE, Y.-Y. *Intuitionistic fuzzy sets and l-fuzzy sets.* Fuzzy Sets and Systems 110*, 2 (2000), 271–274.*

[50] WANG, K., AND ZHAO, B. *Join-completions of l-ordered sets.* Fuzzy Sets and Systems 199 *(2012), 92–107.*

[51] XU, J. *Some properties of intuitionistic (t, s)-fuzzy filters on lattice implication algebras.* Theoretical Mathematics and Applications 3 *(2013), 79–89.*

[52] Zadeh, L. A. *Fuzzy sets.* Information and control 8*, 3 (1965), 338–353.*

[53] Zadeh, L. A. *Similarity relations and fuzzy orderings.* Information sciences 3*, 2 (1971), 177–200.*

[54] Zadeh, L. A. *The concept of a linguistic variable and its application to approximate reasoning—ii.* Information sciences 8*, 4 (1975), 301–357.*

[55] Zhang, Q., Xie, W., and Fan, L. *Fuzzy complete lattices.* Fuzzy Sets and Systems 160*, 16 (2009), 2275–2291.*

[56] Zhang, X., Zhou, B., and Li, P. *A general frame for intuitionistic fuzzy rough sets.* Information Sciences 216 *(2012), 34–49.*

[57] Ziane, B., and Amroune, A. *Representation and construction of intuitionistic fuzzy $\mathcal{T}$-preorders and intuitionistic fuzzy $\mathcal{T}$-orders.* to appear in Discussiones Mathematicae - General Algebra and Applications. Vol. 41. *(2020).*

[58] Zimmermann, H.-J. Fuzzy control. fuzzy set theory—and its applications. *Springer Netherlands, 1996.*

List of Publications and Presentations

Publications

1. *A. Amroune and B. Ziane. More on intuitionistic fuzzy sublattices and their ideals, Facta Universitatis-Series Mathematics and Informatics, 2019,* ***vol. 35****, 871-888.*

2. *B. Ziane and A. Amroune. Ziane. Representation and construction of intuitionistic fuzzy $\mathcal{T}$-preorders and fuzzy weak $\mathcal{T}$- orders, to appear in Discussiones Mathematicae General Algebra and Applications, 2020,* ***vol.41.***

Conference Abstracts/Posters/Presentations

1. *Brahim Ziane, and Abdelaziz Amroune, relation d'ordre flou et ensemble partiellement flou ordonnée, Journées Internationales d'Algebre Appliquees, M'sila, Algeria, 29-30 Novembre et 01 Décembre 2011.*
2. *Brahim Ziane, Representation and construction of L-fuzzy Weak Orders, The 3RD Abu Dhabi University Annual International Conference: Mathematical Sciences and its Applications, Abu Dhabi, 27-30 December 2014.*
3. *Brahim Ziane, The representation and construction of L-fuzzy weak orders based by a family sub-sets, 2ND International Conference on Recent Advances in pure and applied Mathematics (ICRAPAM2015), the Istanbul Commerce University, Turkey, 3-6 June 2015.*
4. *Brahim Ziane, The representation and construction of L-fuzzy weak orders based by a family sub-sets, The 3rd International Conference on Applied Algebra (ICAA2015), M'sila, Algerien, 28-30 April 2015.*
5. *Brahim Ziane, Representation and construction of an intuitionistic fuzzy T-preorder and weak T-orders, Workshop on pure and Applied Mathematics, M'sila, Algeria, 17-18 December 2018.*

ملخص:

في هذه الأطروحة، نعتبر مشكلة تمثيل وبناء علاقة شبه ترتيب الضبابية الحدسية و الترتيب الضعيف، حيث يتم تقديم العديد من نتائج التمثيل الأساسي الممتد لتلك الموجودة في مقال للسادة *Ulrich Bodenhofer et al* . و بالإضافة إلى ذلك، درسنا مفهوم الشبكات الفرعية الضبابية الحدسية و المثاليات الحدسية الضبابية ودرسنا ايضا بعض خصائص هذه الشبكات الفرعية الحدسية الضبابية والمثاليات المتعلقة بالشبكة الحدسية الضبابية.

الكلمات المفتاحية:

المجموعات الضبابية الحدسية، العلاقات الضبابية الحدسية، الترتيب الضبابي الحدسي الضعيف، الشبكات الضبابية الحدسية.

Abstract:

In this thesis, we consider the problem of representation and construction of intuitionistic fuzzy preorders and weak orders, where many fundamental representations result extending those of Ulrich Bodenhofer et al are presented. Moreover, we study the concept of intuitionistic fuzzy sublattices and intuitionistic fuzzy ideals with respect to an intuitionistic fuzzy t-norm on an adequate lattice. Some characterizations and properties of these intuitionistic fuzzy sublattices and ideals with respect to an intuitionistic fuzzy t-norm are established.

Keywords:

Intuitionistic fuzzy set, intuitionistic fuzzy ordering relation, intuitionistic fuzzy equivalence relation, intuitionistic fuzzy weak order, intuitionistic fuzzy t-norm, residuated lattice.

Résumé:

Dans cette thèse, nous considérons le problème de représentation et construction de préordre flous intuitionnistes et d'ordres faibles, où de nombreux résultats de représentation fondamentale étendant ceux d'Ulrich Bodenhofer et al sont présentés. De plus, nous étudions le concept de treillis flous intuitionnistes et idéaux flous intuitionnistes par rapport à une t-norme floue intuitionniste sur un treillis. Quelques caractérisations et propriétés de ces treillis flous intuitionnistes et les idéaux par rapport a une t-norme floue intuitionniste sont établis.

Mots clés:

Ensemble flou intuitionniste, relation d'ordre flou intuitionniste, relation d'équivalence floue intuitionniste, ordre faible flou intuitionniste, t-norme floue intuitionniste, treillis résiduel.