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INTRODUCTION

The wave equation is a partial differential equation that describes how waves propagate in physical systems. It was first derived by D'Alembert in 1746 and has many applications in physics and engineering where mechanical vibrations are the oscillating responses of elastic bodies to disturbances. Various factors, including mechanical or physical sources, can cause such vibrations. The one-dimensional wave equation is a particular case of the wave equation that describes the motion of vibrating strings, sound waves, and water waves.

This thesis is dedicated to studying the vibrations of strings that are axially moving or have a variable length. Both phenomena can be modeled by a wave equation in a time-dependent interval, meaning that the endpoints of the interval can move with time. We investigate whether the solution of the wave equation has analogous properties when the length of the string varies in time. Such situations, where the spatial domain is time-dependent, arise in many different areas of physics, such as optics, electromagnetism, fluid dynamics, and quantum mechanics. For example, see [13, 31, 34, 53] and the survey paper [22].

After the present Introduction, we divided this thesis into four chapters. In *Chapter* 1, we introduce some preliminaries on the wave equation in time-independent interval and discuss some of its basic properties.

In *Chapter* 2, we study the small free vibrations of axially moving strings described by a wave equation in an interval with two endpoints moving in the same direction with a constant speed $\mathbf{v}$ and its length $L > 0$. The model is given by the following wave equation

$$\begin{cases} \phi_{tt} - \phi_{xx} = 0, & \text{for } x \in (\mathbf{v}t, L + \mathbf{v}t) \text{ and } t \in (0, T), \\ \phi(\mathbf{v}t, t) = \phi(L + \mathbf{v}t, t) = 0, & \text{for } t \in (0, T), \\ \phi(x, 0) = \phi^0(x), \quad \phi_t(x, 0) = \phi^1(x), & \text{for } x \in (0, L), \end{cases} \tag{WP$_1$}$$

where the subscripts t and x stand for the derivatives in time and space variables respectively, ϕ^0 is the initial shape of the string and ϕ^1 is its initial transverse speed. We assume that the speed $\mathbf{v}$ is strictly less then the speed of propagation of the wave (here normalized to $c = 1$), i.e.

$$0 \leq \mathbf{v} < 1. \tag{1}$$

The wave equation formulated above is a simple model to represent several mechanical systems such as plastic films, magnetic tapes, elevator cables, textile and fibre winding, see for example [3, 9, 21]. This model can be dated back to Skutch [45], its simplicity is only apparent and we should mention that the method of separation of variables cannot be applied to this Problem. Miranker's work [32] is one of the early influencing papers on the topic of axially moving media. He proposed two approaches to solve Problem (WP$_1$). The first one is to "freeze" the space interval by formulating the Problem in the interval $(0, L)$. Thus, introducing the variables $\eta = x - \mathbf{v}t$ and $\tau = t$, the first equation in (WP$_1$) becomes

$$\phi_{\tau\tau} - 2\mathbf{v}\phi_{\eta\tau} - \left(1 - \mathbf{v}^2\right)\phi_{\eta\eta} = 0, \quad \text{for } \eta \in (0, L)\,,\ \tau > 0. \tag{2}$$

The obtained Problem is more familiar and the vast majority of the literature on travelling strings follows this approach. Some important results in this direction are given by Wickert and Mote [55] where the authors write (2) as a first-order differential equation with matrix differential operators (a state space formulation) and obtained a closed form representation of the solution for arbitrary initial conditions.

The other approach is to solve (WP$_1$), i.e. keep the space interval depending on time. Using D'Alembert's method, Miranker writes the solution as the sum of two waves travelling in opposite directions

$$\phi(x,t) = f(t+x) + g\left(t-x\right),$$

then obtained a closed form of the solution by a series formulas (See page 39 in [32]). After few rearrangements, his formulas can be rewritten as

$$\phi(x,t) = \sum_{n \in \mathbb{Z}^*} c_n \left(e^{n\pi i(1-\mathbf{v})(t+x)/L} - e^{n\pi i(1+\mathbf{v})(t-x)/L}\right), \text{ for } x \in \left(\mathbf{v}t, L+\mathbf{v}t\right) \text{ and } t \in \left(0, T\right). \tag{3}$$

Despite the utility of such a formula for numerical and asymptotic approaches, it remained underexploited in the literature related to axially moving strings.

Since Miranker was not explicit on how to compute the coefficients c_n, we give in this chapter at hands a method to compute each c_n in function of the initial data ϕ^0 and ϕ^1, (see [15]). The idea is inspired from [44] where the second author obtained the exact solution of strings with two linearly moving endpoints at different speeds. Similar techniques where also used in [2, 42, 43] for a string with one moving endpoint. Then, we show that the series formulas (3) can be manipulated to establish the following results:

- *A conserved quantity.* The functional

  $$\mathcal{E}_{\mathbf{v}}\left(t\right) = \frac{1}{2}\int_{\mathbf{v}t}^{L+\mathbf{v}t} \left(\phi_t + \mathbf{v}\phi_x\right)^2 + \left(1 - \mathbf{v}^2\right)\phi_x^2 dx, \quad \text{for } t \geq 0, \tag{4}$$

  depending on $L, t, \mathbf{v}$ and the solution of (WP$_1$), is conserved in time. We give two different proofs for this fact. Note that $\phi_t + \mathbf{v}\phi_x = \frac{d}{dt}\left(\phi\left(x + \mathbf{v}t, t\right)\right)$ is the total (called also the material) derivative. Under the assumption (1), this functional is positive-definite and we will call it the ”energy” of the solution ϕ. Although there are many expressions of energy for axially moving strings, see for instance [41, 54].

- *Exact boundary observability and controllability.*
  - The wave equation (WP$_1$) is exactly observable at any endpoint $x = x_b + \mathbf{v}t$, where $x_b = 0$ or $x_b = L$. Due to the finite speed of propagation, the time of observability is expected to be positive and depends on the initial length L and the speed $\mathbf{v}$. We show that this time is exactly

    $$T_{\mathbf{v}} := \frac{2L}{1-\mathbf{v}^2}.$$
  - If we observe both endpoints, i.e. for $x = \mathbf{v}t$ and $x = L + \mathbf{v}t$, the time of observability is reduced to

    $$\tilde{T}_{\mathbf{v}} := \frac{L}{1-\mathbf{v}}.$$
  - Using the Hilbert uniqueness method (HUM), due to J-.L. Lions [26], the above observability results implies controllability result at one and both endpoints.

Let us also note that letting $\mathbf{v} \to 0$ in the above results, we recover some known facts for the wave equation in non-travelling intervals [24, 26, 57]. In particular, $E_0(t) = \frac{1}{2}\int_0^L \phi_t^2 + \phi_x^2 dx$ is known to be conserved and we get $T_0 = 2L, \tilde{T}_0 = L$ as sharp times for boundary observability. The results of this part have appeared in [15].

All mechanical systems in real life have nonconservative forces, like damping, which lead to the system’s energy dissipating. The energy exchange then proceeds until dampening during vibration dissipates all of the energy. More precisely, by looking at the reflected wave profile and the energetics of the model under consideration, one can make a conclusion about the efficiency of a placed boundary support as a vibration suppressor.

In *Chapter* 3, we study the small vibrations of a uniform string that travels at a constant speed $\mathbf{v}$ between two pulleys that are kept at a fixed distance L. The inlet is fixed, while the outlet is allowed to move transversely and is attached to a damping factor η.

$$\begin{cases} u_{\tau\tau} + 2\mathbf{v}u_{s\tau} - \left(1-\mathbf{v}^2\right)u_{ss} = 0, & \text{for } s \in (0,L) \text{ and } \tau > 0, \\ u(0,\tau) = 0, & \text{for } \tau > 0, \\ \left(1-\mathbf{v}^2\right)u_s(L,\tau) + (\eta - \mathbf{v})\,u_\tau(L,\tau) = 0, & \text{for } \tau > 0, \\ u(s,0) = u^0(s), \quad u_\tau(s,0) = u^1(s), & \text{for } s \in (0,L), \end{cases} \tag{5}$$

The functions u^0 and u^1 represents the initial shape and the initial transverse speed of the string, respectively. The existing approaches in the literature describe the above Problem in fixed space coordinates, see for instance [6–9, 25, 29, 48]. Gaiko and van Horssen [14] considered a simplified mathematical model describing the small vibration of the string with a mass-spring-dashpot damping at the outlet. Under the restriction that the speed $\mathbf{v}$ and the damping factor are small, i.e. $0 < \mathbf{v} \ll 1$ and $\eta \ll 1$, the authors in [14] obtained an asymptotic approximation of the solution by using a multiple scale approach.

We assume that $\eta \geq 0$ and $0 \leq \mathbf{v} < 1$. We introduce the variables $s = L - x + \mathbf{v}t$ and $\tau = t$. Rewriting Problem (5) in the new coordinates, we obtain the following (pure) wave equation with a damping at the moving boundary $x = \mathbf{v}t$

$$\begin{cases} \phi_{tt} - \phi_{xx} = 0, & \text{for } x \in (\mathbf{v}t, L + \mathbf{v}t) \text{ and } t > 0, \\ (1 - \eta\mathbf{v})\, \phi_x(\mathbf{v}t, t) - (\eta - \mathbf{v})\, \phi_t(\mathbf{v}t, t) = 0, & \text{for } t > 0, \\ \phi(L + \mathbf{v}t, t) = 0, & \text{for } t > 0, \\ \phi(x, 0) = \phi^0(x),\ \phi_t(x, 0) = \phi^1(x), & \text{for } x \in (0, L), \end{cases} \tag{DWP_1}$$

where $\phi^0 = u^0$ and $\phi^1 = u^1 - \mathbf{v}\phi^0_x$. After given the exact solution by the series formulas, we demonstrate how this series formulas can be used to achieve the following results:

- *For the undamped case, i.e.* $\eta = 0$, we show that the functional

  $$\mathcal{E}_{\mathbf{v}}(t) = \frac{1}{2}\int_{\mathbf{v}t}^{L+\mathbf{v}t} (\phi_t + \mathbf{v}\phi_x)^2 + (1 - \mathbf{v}^2)\, \phi_x^2 dx, \quad \text{for } t \geq 0,$$

  depending on $L, t, \mathbf{v}$ and the solution ϕ of (DWP_1), is conserved in time.

- *For the damped case* $\eta > 0$ *with* $\eta \neq 1$, the (usual) energy

  $$E_{\mathbf{v}}(t) = \frac{1}{2}\int_{\mathbf{v}t}^{L+\mathbf{v}t} \phi_t^2(x, t) + \phi_x^2(x, t)\, dx, \quad \text{for } t \geq 0, \tag{6}$$

  depending on $L, t, \mathbf{v}$ and the solution, satisfies

  $$\frac{1}{\gamma_\eta^2 \gamma_{\mathbf{v}}} E_{\mathbf{v}}(0)\, e^{-\frac{1-\mathbf{v}^2}{L}\ln|\gamma_\eta| t} \leq E_{\mathbf{v}}(t) \leq \gamma_\eta^2 \gamma_{\mathbf{v}} E_{\mathbf{v}}(0)\, e^{-\frac{1-\mathbf{v}^2}{L}\ln|\gamma_\eta| t}, \quad \text{for } t \geq 0. \tag{7}$$

  where γ_η, $\gamma_{\mathbf{v}}$ will be further stipulated in the next. In particular, it follows that the energy does not vanish in finite time. The results of this part have appeared in [16].

Next, in *Chapter* 4, we study small transversal vibrations of a uniform string with a time dependent length $\ell(t)$. One end of the string is fixed to a rigid wall, while the other end passes between two pulleys that can move horizontally and transversely. A dash-pot with a damping factor η is attached to the pulleys to stabilize the string, i.e., reduce or suppress its vibrations.

Denoting the displacement function by u, depending on the position x along the string and the time t, the equation model can be stated as follows

$$\begin{cases} u_{tt} - u_{xx} = 0, & \text{for } 0 < x < \ell(t) \text{ and } t > 0, \\ \left(1 + \eta\ell'\left(t\right)\right) u_x\left(\ell\left(t\right), t\right) \ + \left(\eta + \ell'\left(t\right)\right) u_t\left(\ell\left(t\right), t\right) = 0, & \text{for } t > 0, \\ u\left(0, t\right) = 0, & \text{for } t > 0, \\ u(x, 0) = u^0\left(x\right) \text{ and } u_t\left(x, 0\right) = u^1\left(x\right), & \text{for } 0 < x < L, \end{cases} \tag{DWP$_2$}$$

where the subscripts t and x stand for the derivatives in time and space variables respectively. The functions u^0 and u^1 represent the initial shape and the initial transverse speed of the string, respectively. The initial length of the string is denoted by $L = \ell(0)$.

We assume that $\ell \in C^1\left([0, +\infty[\right)$ and that

$$\left|\ell'\left(t\right)\right| < 1, \quad \text{for } t \geq 0, \tag{8}$$

we are mainly interested in the asymptotic behavior in time of the energy of the solution, defined as

$$E_\ell\left(t\right) := \frac{1}{2}\int_0^{\ell(t)} u_t^2\left(x, t\right) + u_x^2\left(x, t\right) dx, \quad \text{for } t \geq 0. \tag{9}$$

For the time-independent interval, i.e. when $\ell\left(t\right) = L$ for $t \geq 0$, it is well known that:

- If $\eta = 0$, then the energy is constant, i.e. $E_L\left(t\right) = E_L\left(0\right)$, for $t \geq 0$.
- If $\eta > 0$ with $\eta \neq 1$, then the energy decays exponentially

  $$E_L\left(t\right) \leq C E_L\left(0\right) e^{-\frac{1}{L}\ln\left|\gamma_\eta\right| t}, \quad \text{where } \gamma_\eta := \frac{1+\eta}{1-\eta}$$

  and C is a positive constant, see [10, 12, 39, 49]. In [16], the present authors showed that

  $$\frac{1}{\gamma_\eta^2} E_L\left(0\right) e^{-\frac{1}{L}\ln\left|\gamma_\eta\right| t} \leq E_L\left(t\right) \leq \gamma_\eta^2 E_L\left(0\right) e^{-\frac{1}{L}\ln\left|\gamma_\eta\right| t}, \quad \text{for } t \geq 0. \tag{10}$$

  The latter estimate turns out to be sharp as we will see.

The question of $E_\ell\left(t\right)$ behavior in time is more delicate if the interval depends on time. For instance, in the case of Dirichlet's boundary conditions, which corresponds to a rigid damper with $\eta = +\infty$ in (DWP$_2$), it is known that the energy is nonincreasing if the interval is expanding and nondecreasing if the interval is shrinking, see [4]. See also [42, 43, 47] when the variation of the length is uniform in time. Regarding the case with a velocity feedback at the moving endpoint $x = \ell\left(t\right)$. Gugat [18] considered the case $u_x\left(\ell\left(t\right), t\right) + c u_t\left(\ell\left(t\right), t\right) = 0$ where c is a constant. See also [28] for the particular case $\ell(t) = 1 + \mathbf{v}t$ where $\mathbf{v}$ is a constant satisfying

$0 < \mathbf{v} < 1$. Ammari et al. [1] considered the case $u_x(\ell(t), t) + f(t) u_t(\ell(t), t) = 0$, where $f \in L^\infty(0, +\infty)$ and $\ell(t)$ is periodic. See also [19] where $\ell(t)$ is not necessarily periodic. Mokhtari [33] considered the case with two moving endpoints.

For the special case $\eta = 1$, the boundary condition at $x = \ell(t)$ reads

$$u_x(\ell(t), t) + u_t(\ell(t), t) = 0.$$

This is a transparent condition, i.e. there is no reflections of waves from the moving endpoint and consequently all the initial disturbances leave the interval $(0, \ell(t))$ at most after a time

$$T_\ell := \beta^{-1}(L),$$

where $\beta(t) := t - \ell(t)$. Hence, whether the interval is expanding or shrinking, the linear velocity feedback $-u_t(\ell'(t), t)$ steers the solution to the zero state in the finite time T_ℓ. See for instance [18, 19], and for the particular case $\ell(t) = L$, see [12, 49].

After given the exact solution by the series formulas, we demonstrate how this series formulas can be used to achieve the following results:

- *For the undamped case, i.e.* $\eta = 0$, the energy of the solution satisfies

  $$\frac{m(t)}{M(t_0)} E_\ell(t_0) \leq E_\ell(t) \leq \frac{M(t)}{m(t_0)} E_\ell(t_0), \quad \text{for } 0 \leq t_0 < t,$$

  where

  $$m(t) := \min_{x \in [0, \ell(t)]} \left\{ \varphi'(t-x), \varphi'(t+x) \right\} \text{ and } M(t) := \max_{x \in [0, \ell(t)]} \left\{ \varphi'(t-x), \varphi'(t+x) \right\}.$$

- *For the damped case* $\eta > 0$, *with* $\eta \neq 1$, the energy $E_\ell(t)$ satisfies

  $$\frac{e^{\ln|\gamma_\eta|\varphi(t_0 - \ell(t_0))}}{M(t_0)} m(t) e^{-\ln|\gamma_\eta|\varphi(t+\ell(t))} E_\ell(t_0) \leq E_\ell(t) \leq \frac{e^{\ln|\gamma_\eta|\varphi(t_0 + \ell(t_0))}}{m(t_0)} M(t) e^{-\ln|\gamma_\eta|\varphi(t-\ell(t))} E_\ell(t_0),$$

  for $0 \leq t_0 < t$. The results of this part have appeared in [17].

Finally, an Appendix and a full list of References are included at the end of the manuscript.

1

PRELIMINARIES

In this chapter, we recall some results on the wave equation in time-independent interval. Then, we expose some notions about controllability and observability. The proofs can be found in [11, 24, 26, 57]

1.1 WAVE EQUATION IN TIME-INDEPENDENT INTERVALS

1.1.1 *Problem setting*

The wave equation is a simple mathematical model describing the transverse small vibrations of a homogeneous string under tension and constrained to move in a plane. Let $T > 0$. When the string endpoints are clamped and its length $L > 0$ is invariant with time, this model can be stated as

$$\begin{cases} w_{tt} - w_{xx} = 0, & \text{for } (x,t) \in (0,L) \times (0,T)\,, \\ w\,(0,t) = w\,(L,t) = 0, & \text{for } t \in (0,T)\,, \\ w(x,0) = w^0\,(x)\,, \quad w_t\,(x,0) = w^1\,(x)\,, & \text{for } x \in (0,L)\,. \end{cases} \tag{WP$_0$}$$

The function $x \to w(x,t)$, $x \in (0,L)$, describes the shape of the string at time t. It is well known that for initial data satisfying $w^0 \in H_0^1(0,L)$ and $w^1 \in L^2(0,L)$, the above Problem has a unique solution satisfying

$$w \in C\left([0,T]\,;H_0^1(0,L)\right) \text{ and } w_t \in C\left([0,T]\,;L^2(0,L)\right).$$

It is interesting to observe that the wave operator can be factored as follows $\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)$. All solutions to the wave equation are superpositions of "left-traveling" and "right-traveling" waves, $w\,(x,t) = f(t+x) + g(t-x)$. These are called left-traveling and right-traveling because while the overall shape of the wave remains constant, the wave translates to the left or right in time.

The wave equation is time-reversible because the wave operator $\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ is symmetric. This means that if we replace the time variable t with $T - t$ in (WP$_0$), we get the same equation. The above regularity result holds in both directions.

In particular, this solution enjoys the following proprieties (see [23, 26, 46, 57]):

- Let $w \in C\left([0,T]\,; H_0^1(0,L)\right)$, we define the total energy at time t by

$$E_0\,(t) = \frac{1}{2}\int_0^L w_x^2(x,t) + w_t^2(x,t)\,dx, \qquad t \in (0,T)\,, \tag{1.1}$$

where the first term corresponds to kinetic energy, the second to potential energy. As we know that the "total energy" is a conserved quantity in time, i.e. $E_0\,(t) = Constant$, $\forall t \geq 0$.

- Due to the finite speed of propagation, taken here equal to 1, the boundary observability at one endpoint (resp. at the two endpoints) holds iff the length of the time interval satisfies $T \geq 2L$ (resp. $T \geq L$).

- Using a control function acting at one of the endpoints, the wave equation is controllable iff $T \geq 2L$. If we use two controls, i.e. a control function at each endpoint, the controllability holds iff $T \geq L$.

1.1.2 *Deriving the 1-d wave equarion*

The vibrations of the string are transversal and small (compared to its length). This implies in particular that the slop u_s is also small. The string is uniform with a mass density ρ. The string is perfectly flexible and the effects due to gravity are neglected. This means that the tension **T** is constant. The string fixed, hence we have the boundary conditions $w\,(0,t) = 0$ and $w\,(L,t) = 0$, for $t \geq 0$. We apply the extended Hamilton's principle, over the finite time interval $t_1 \leq t \leq t_2$, in the following form

$$\delta \int_{t_1}^{t_2} \mathcal{L} dt = 0, \tag{1.2}$$

(see [30] and [5, Chapters 4 and 5]). Here δ denotes the variation in a given function, $\mathcal{L}$ is the Lagrangian and $\mathcal{W}$ is the virtual work performed by nonconservative forces. The Lagrangian is

$$\mathcal{L} = E_k - E_p,$$

where E_k is the kinetic energy and E_p is the potential energy of the string. The first one is given by

$$E_k = \frac{1}{2}\int_0^L \rho \left(\frac{dw}{dt}\right)^2 ds.$$

The potential energy E_p is equal to the work done to deform the string from its rest position. Hence, the potential energy per unit length of a differential element is given by

$$dE_p = \mathbf{T} \cdot \text{elongation} = \mathbf{T} \left(\sqrt{(ds)^2 + (dw)^2} - ds \right).$$

Since the slope of the string w_s is small, then

$$dE_p = \mathbf{T} \left(\sqrt{1 + (w_s)^2} - 1 \right) ds \approx \frac{\mathbf{T}}{2} w_s^2 ds,$$

hence, the potential energy of the string is given by

$$E_p = \frac{\mathbf{T}}{2} \int_0^L w_s^2 ds$$

and therefore

$$\mathcal{L} = \frac{\rho}{2} \int_0^L w_t^2 ds - \frac{\mathbf{T}}{2} \int_0^L w_s^2 ds.$$

Now, we are led to search the critical points of the functional,

$$I(w) := \frac{1}{2} \int_{t_1}^{t_2} \int_0^L \rho w_t^2 - \mathbf{T} w_s^2 \, dsdt, \tag{1.3}$$

when w is a critical point, we should have

$$\lim_{\theta \to 0} \frac{I(w + \theta v) - I(w)}{\theta} = 0, \tag{1.4}$$

where v is a smooth function such that

$$v(\cdot, t_1) = v(\cdot, t_2) = 0 \quad \text{and} \quad v(0, \cdot) = v(L, \cdot) = 0. \tag{1.5}$$

The function $w + \theta v$ is a small perturbation of w that does not affect the values of (the path) w at $t = t_1$ and $t = t_2$, as well as at $s = 0$. The limit (1.4) equals

$$\int_{t_1}^{t_2} \int_0^L \rho w_t v_t - \mathbf{T} w_s v_s dsdt = 0.$$

Integrating by parts, we infer that

$$\int_{t_1}^{t_2} \int_0^L \{\rho w_{tt} - \mathbf{T} w_{ss}\} \, v dsdt + \int_{t_1}^{t_2} \mathbf{T} w_s(L, t) \, v(L, t) \, dt = 0.$$

The other boundary terms vanishes due to (1.5). Since v can be chosen arbitrary, and in particular its value at $s = L$, the first integral implies that

$$w_{tt} - \frac{\mathbf{T}}{\rho} w_{ss} = 0, \ \text{ for } s \in (0, L) \ \text{ and } t_1 \leq t \leq t_2,$$

where the velocity $c = \sqrt{\mathbf{T}/\rho}$ varies for changing density.

1.2 SOME OBSERVABILITY AND CONTROLLABILITY RESULTS

A linear control system's controllability is identical to the surjectivity of a specific linear map $\mathcal{F}$ from one Hilbert space H_1 to another Hilbert space H_2. The surjectivity of $\mathcal{F}$ is identical to the existence of $c > 0$ such that

$$\left\|\mathcal{F}^*(x)\right\|_{H_1} \geq c\left\|x\right\|_{H_2}, \quad \text{for all } x \in H_2, \tag{1.6}$$

where $\mathcal{F}^*$ is the adjoint of $\mathcal{F}$. So, one first computes $\mathcal{F}^*$ and then proves (1.6). The inequality (1.6) is called the **observability inequality**. This method is now the most widely used to demonstrate the controllability of a linear partial differential equation. The most difficult aspect of this method is demonstrating the **observability inequality**. There are many methods to prove such an inequality.

We consider the following boundary controllability Problem

$$\begin{cases} y_{tt} - y_{xx} = 0, & \text{for } (x,t) \in (0,L) \times (0,T), \\ y(0,t) = 0, \quad y(L,t) = f(t), & \text{for } t \in (0,T), \\ y(x,0) = y^0(x), \quad y_t(x,0) = y^1(x), & \text{for } x \in (0,L), \end{cases} \tag{CWP$_0$}$$

where y is the state variable, f is the control variable, and any given initial value is represented by $\left(y^0, y^1\right) \in L^2(0,L) \times H^{-1}(0,L)$. Equation (CWP$_0$) may describe the motion of a string with fixed endpoints.

Definition 1.1 *Equation (CWP$_0$) is called exactly controllable at time T, if for any initial value $\left(y^0, y^1\right) \in L^2(0,L) \times H^{-1}(0,L)$ and any target $\left(y_T^0, y_T^1\right) \in L^2(0,L) \times H^{-1}(0,L)$, there exists a control function $f \in L^2(0,T)$, acting at one of the ends, say $x = L$, such that the corresponding solution y of (CWP$_0$) in the sense of a transposition satisfies final conditions*

$$y(x,T) = y_T^0(x), \quad y_t(x,T) = y_T^1(x), \quad \textit{for} \quad x \in (0,L).$$

Problem (CWP$_0$) can be reduced to a null-controllability one[1], as argued in [23, page 54-55], i.e., we can always assume that

$$y(T) = y_t(T) = 0, \quad \text{for} \quad x \in (0,L).$$

---

[1] Clearly, null-controllability is a particular case of exact controllability.

Let $t \in [0,T]$. Multiplying the first equation in (CWP$_0$) by w solution of (WP_0), and integrating by parts on $(0,L)$ and $(0,T)$ we obtain, at least formally, the following identity:

$$\int_0^L y\left(t\right) w_t\left(t\right) dx - \left\langle y_t\left(t\right), w\left(t\right)\right\rangle_{H_0^1(0,L)} - \int_0^L y\left(0\right) w^1 dx + \left\langle y_t\left(0\right), w^0\right\rangle_{H_0^1(0,L)} = \int_0^T f\left(s\right) w_x\left(L,s\right) ds, \quad (1.7)$$

where $\langle \cdot, \cdot \rangle_{H_0^1(0,L)}$ denotes the duality product between $H^{-1}\left(0,L\right)$ and $H_0^1\left(0,L\right)$. We adopt identity (1.7) as the definition of solutions of (CWP$_0$), in the sense of transposition. Thanks to (1.6) this definition can be justified by arguing as in [27]. Moreover, Problem (CWP$_0$) admits a unique solution $y \in C([0,T]; L^2\left(0,L\right)) \cap C^1([0,T]; H^{-1}\left(0,L\right))$ in the above sense.

Let w be the solution of Problem (WP$_0$). We seek a control in the special form $f\left(t\right) = w_x\left(L,t\right)$, for a suitable choice of w^0, w^1. First, we consider the backward Problem

$$\begin{cases} \zeta_{tt} - \zeta_{xx} = 0, & \text{for } x \in (0,L) \text{ and } t \in (0,T), \\ \zeta\left(0,t\right) = 0, \ \ \zeta\left(L,t\right) = w_x\left(L,t\right), & \text{for } \ t \in (0,T), \\ \zeta\left(x,T\right) = 0, \ \ \zeta_t\left(x,T\right) = 0. & \text{for } \ x \in (0,L). \end{cases} \quad (1.8)$$

We obtain a linear map, that relates (w^0, w^1) to the initial data $\left(\zeta_t\left(0\right), -\zeta\left(0\right)\right)$,

$$\begin{aligned} \mathbf{\Lambda} : H_0^1\left(0,L\right) \times L^2\left(0,L\right) &\longrightarrow H^{-1}\left(0,L\right) \times L^2\left(0,L\right) \\ (w^0, w^1) &\longmapsto \left(\zeta_t\left(0\right), -\zeta\left(0\right)\right). \end{aligned}$$

The space $H_0^1\left(0,L\right) \times L^2\left(0,L\right)$ is equipped with the energy norm $(w^0, w^1) \longmapsto \left(E_0\left(t\right)\right)^{1/2}$. To show that (w^0, w^1) can be chosen such that $\left(\zeta_t\left(0\right), -\zeta\left(0\right)\right) = (y^1, -y^0)$, we argue as in [23].

Indeed, since the solution of (1.8) is taken in the transposition sense, it comes that

$$0 = \int_0^T \left\langle \left(\zeta_{tt} - \zeta_{xx}\right), w\right\rangle_{H_0^1(0,L)} dt = -\left\langle \zeta_t\left(0\right), w^0\right\rangle_{H_0^1(0,L)} + \int_{\mathbf{0}}^L \zeta\left(0\right) w^1 dx + \int_0^T \zeta\left(L,t\right) w_x\left(L,t\right) dt. \quad (1.9)$$

We can rewrite (1.9) as

$$\langle \mathbf{\Lambda}(w^0, w^1), (w^0, w^1)\rangle_{H_0^1(0,L)\times L^2(0,L)} = \int_0^T w_x^2\left(L,t\right) dt.$$

Thanks 1.6, we deduce that

$$CE_0\left(t\right) \leq \langle \mathbf{\Lambda}(w^0, w^1), (w^0, w^1)\rangle_{H_0^1(0,L)\times L^2(0,L)},$$

This means that $\mathbf{\Lambda}$ is an isomorphism and that (w^0, w^1) can be chosen so that the control $f = w_x\left(L,t\right)$ drives the solution of (CWP$_0$) from the initial data y^0, y^1 to the rest, i.e. $y\left(T\right) = y_t\left(T\right) = 0$.

2

FREE VIBRATIONS OF AXIALLY MOVING STRINGS

In this chapter, we study the small vibrations of axially moving strings described by a wave equation in an interval with two endpoints moving in the same direction with a constant speed. The solution is expressed by a series formula where the coefficients are explicitly computed in function of initial data. We also define an energy expression for the solution that is conserved in time. Moreover, using the Hilbert uniqueness method, we establish boundary observability and controllability results in a sharp time.

2.1 PROBLEM SETTING

We consider a small transverse vibrations of an infinite string moving axially with a constant speed. Two fixed supports, distanced by L as represented in Figure 2.1, prevent transversal displacements of the string at the supporting points while the axial motion remains unaffected.

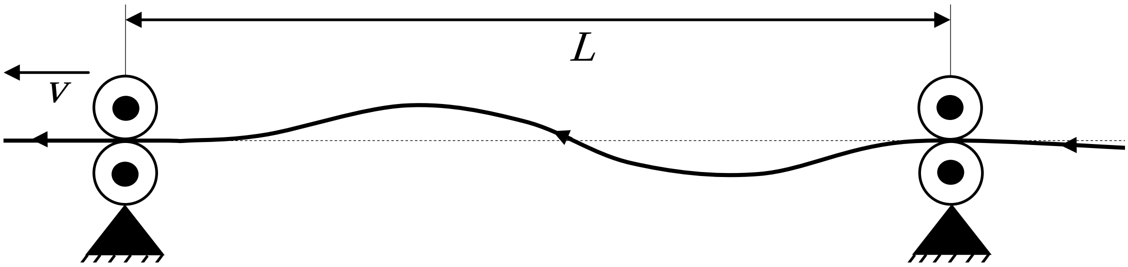

Figure 2.1: A string travelling to the left with a speed **v**.

We introduce a coordinate system (x, t), attached to the travelling string, where x coincides with the rest state axis of the string and t denotes the time. We denote transverse displacement of the string by $\phi(x, t)$, and we choose the position of the left support to coincide with $x = 0$. Assuming that the string travels to the left with a scalar speed $\mathbf{v}$, the positions of the left and right supports are $x = \mathbf{v}t$ and $x = L + \mathbf{v}t$ for $t \geq 0$, respectively. If we assume that the string travels to the right then it suffices to change $\mathbf{v}$ by $-\mathbf{v}$ in the remainder of this chapter.

For $T > 0$, we denote the interval

$$\mathbf{I}_t := (\mathbf{v}t, L + \mathbf{v}t), \quad \text{for } t \in (0, T).$$

A simplified model describing the free small transverse vibrations of this string is the following wave equation:

$$\begin{cases} \phi_{tt} - \phi_{xx} = 0, & \text{for } x \in \mathbf{I}_t \text{ and } t \in (0, T), \\ \phi(\mathbf{v}t, t) = \phi(L + \mathbf{v}t, t) = 0, & \text{for } t \in (0, T), \\ \phi(x, 0) = \phi^0(x), \quad \phi_t(x, 0) = \phi^1(x), & \text{for } x \in \mathbf{I}_0, \end{cases} \tag{WP}$$

where the subscripts t and x stand for the derivatives in time and space variables respectively, ϕ^0 is the initial shape of the string and ϕ^1 is its initial transverse speed. We assume that the speed $\mathbf{v}$ is strictly less than the speed of propagation of the wave (here normalized to $c = 1$).

$$0 < \mathbf{v} < 1 \tag{2.1}$$

If $\mathbf{v} \geq 1$, then the Problem is ill-posed, see for instance [40].

2.2 EXACT SOLUTION

To simplify some formulas, we introduce the notation

$$\gamma_{\mathbf{v}} := \frac{1 + \mathbf{v}}{1 - \mathbf{v}}, \qquad L_1 := \frac{1 - \mathbf{v}}{1 + \mathbf{v}} L \text{ and } L_2 := \frac{2}{1 - \mathbf{v}} L$$

since these constants will appear frequently in the sequel. Note that

$$1 < \gamma_{\mathbf{v}} < +\infty \quad \text{and} \quad 0 < L_1 < L < L_2/2, \quad \text{for } 0 < \mathbf{v} < 1.$$

For every initial data

$$\phi^0 \in H_0^1(\mathbf{I}_0), \quad \phi^1 \in L^2(\mathbf{I}_0), \tag{2.2}$$

we already know that if (2.1) holds the solution of Problem (WP) exists and satisfies

$$\phi \in C\left([0, T]; H_0^1(\mathbf{I}_t)\right) \quad \text{and} \quad \phi_t \in C\left([0, T]; L^2(\mathbf{I}_t)\right), \tag{2.3}$$

see for instance, previous studies [4, 38].

2.2.1 *Coefficients expressions*

Theorem 2.1 *Under the Assumptions (2.1) and (2.2), the solution of Problem (WP) is given by the series*

$$\phi(x,t) = \sum_{n\in\, \mathbb{Z}^*} c_n \left(e^{n\pi i(1-\mathbf{v})(t+x)/L} - e^{n\pi i(1+\mathbf{v})(t-x)/L} \right), \text{ for } x \in \mathbf{I}_t \text{ and } t \in (0,T)\,. \tag{2.4}$$

where the coefficients $c_n \in \mathbb{C}$ are given by any of the two following formulas

$$c_n = \frac{1}{4n\pi i} \int_0^{L_2} \left(\tilde{\phi}_x^0 + \tilde{\phi}^1 \right) e^{-n\pi i(1-\mathbf{v})x/L} dx, \tag{2.5}$$

$$= \frac{1}{4n\pi i} \int_{-L_1}^{L} \left(\tilde{\phi}_x^0 - \tilde{\phi}^1 \right) e^{n\pi i(1+\mathbf{v})x/L} dx, \quad \text{for } n \in \mathbb{Z}^*, \tag{2.6}$$

where $\tilde{\phi}_x^0$ and $\tilde{\phi}^1$ are extensions of the initial data $\phi^0$and ϕ^1 on the interval $(-L_1, L_2)$ given below by (2.8) and (2.9), respectively.

Before proceeding to the proof, let us describe how to extend the function ϕ, defined only on $\mathbf{I}_t = (\mathbf{v}t, L+\mathbf{v}t)$, to the intervals $(-L_1 + \mathbf{v}t, \mathbf{v}t)$ and $(L+\mathbf{v}t, L_2+\mathbf{v}t)$. On one hand, we set

$$\tilde{\phi}(x,t) = \begin{cases} -\phi\left(\gamma_{\mathbf{v}}\left(\mathbf{v}t - x\right) + \mathbf{v}t, t\right), & x \in (-L_1 + \mathbf{v}t, \mathbf{v}t)\,, \\ \phi\left(x,t\right), & x \in (\mathbf{v}t, L+\mathbf{v}t)\,, \\ -\phi\left(\frac{1}{\gamma_{\mathbf{v}}}\left(\mathbf{v}t - x\right) + \frac{2L}{1+\mathbf{v}} + \mathbf{v}t, t\right), & x \in (L+\mathbf{v}t, L_2+\mathbf{v}t)\,. \end{cases} \tag{2.7}$$

The obtained function is well defined since the first variable of ϕ remains in the interval $(\mathbf{v}t, L+\mathbf{v}t)$. In particular, $\tilde{\phi}(\mathbf{v}t,t) = \tilde{\phi}(L+\mathbf{v}t,t) = 0$, hence the homogeneous boundary conditions at $x = \mathbf{v}t$ and $x = L+\mathbf{v}t$ remain satisfied, for every $t \geq 0$ see Figure 2.2.

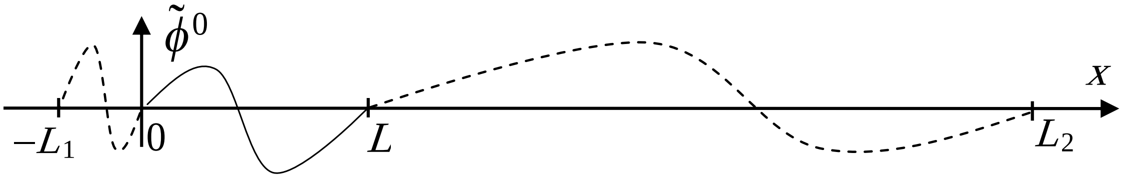

Figure 2.2: Extension of an initial data ϕ^0 .

Taking the derivative of (2.7) with respect to x, we obtain

$$\tilde{\phi}_x(x,t) = \begin{cases} \gamma_{\mathbf{v}}\phi_x\left(\gamma_{\mathbf{v}}\left(\mathbf{v}t - x\right) + \mathbf{v}t, t\right), & x \in (-L_1 + \mathbf{v}t, \mathbf{v}t)\,, \\ \phi_x\left(x,t\right), & x \in (\mathbf{v}t, L+\mathbf{v}t)\,, \\ \frac{1}{\gamma_{\mathbf{v}}}\phi_x\left(\frac{1}{\gamma_{\mathbf{v}}}\left(\mathbf{v}t - x\right) + \frac{2L}{1+\mathbf{v}} + \mathbf{v}t, t\right), & x \in (L+\mathbf{v}t, L_2+\mathbf{v}t)\,. \end{cases} \tag{2.8}$$

On the other hand, $\tilde{\phi}_t(x,t)$ is extended as follows:

$$\tilde{\phi}_t(x,t) = \begin{cases} -\gamma_{\mathbf{v}}\phi_t\left(\gamma_{\mathbf{v}}\left(\mathbf{v}t - x\right) + \mathbf{v}t, t\right), & x \in \left(-L_1 + \mathbf{v}t, \mathbf{v}t\right), \\ \phi_t\left(x,t\right), & x \in \left(\mathbf{v}t, L + \mathbf{v}t\right), \\ \frac{-1}{\gamma_{\mathbf{v}}}\phi_t\left(\frac{1}{\gamma_{\mathbf{v}}}\left(\mathbf{v}t - x\right) + \frac{2L}{1+\mathbf{v}} + \mathbf{v}t, t\right), & x \in \left(L + \mathbf{v}t, L_2 + \mathbf{v}t\right). \end{cases} \tag{2.9}$$

The extension $\tilde{\phi}_x$ (resp. $\tilde{\phi}_t$) satisfies some even-like (resp. odd-like) symmetry in the variable x with respect to the point $x = \mathbf{v}t$ on the interval $(-L_1 + \mathbf{v}t, L + \mathbf{v}t)$ and with respect to the point $x = L + \mathbf{v}t$ on the interval $(\mathbf{v}t, L_2 + \mathbf{v}t)$.

Remark 2.1 *If* $\mathbf{v} = 0$, *then* $L_1 = L$ *and* $L_2 = 2L$. *In this case, the functions* $\tilde{\phi}$ *and* $\tilde{\phi}_t$ *are odd on the intervals* $(-L, L)$ *and* $(0, 2L)$ *with respect to the middle of each interval. The extension* $\tilde{\phi}_x$ *is an even function on these intervals.*

Remark 2.2 *i) As shown in Figure 2.3, the forward characteristic line starting from the endpoint* $x = 0$ *(resp.* $x = L$*), hits the boundary* $x = L + \mathbf{v}t$ *(resp.* $x = \mathbf{v}t$*) at time* $t = \frac{L}{1-\mathbf{v}}$ *(resp.* $t = \frac{L}{1+\mathbf{v}}$*). ii) Consider* (x_1, t_1) *as the intersection of these two characteristic lines after one reflection on the boundary. We can check that, the two backward characteristic lines from this point intersect the* $x-$*axis precisely at* $x = -L_1$ *and* $x = L_2$.

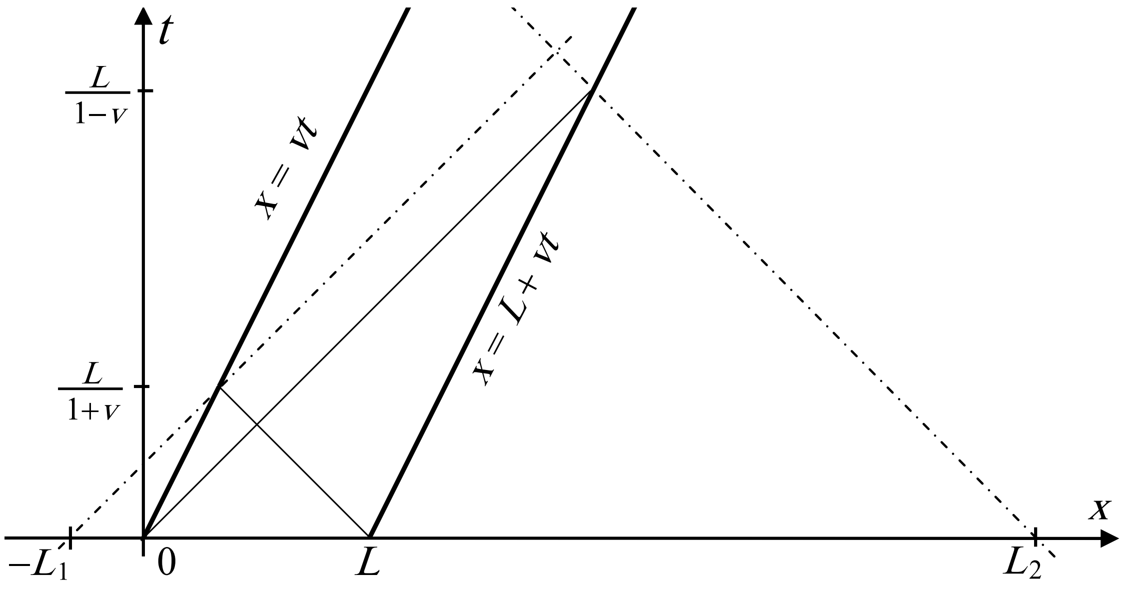

Figure 2.3: Relation between L_1, L_2 and the forward characteristics from endpoints.

Now we are ready to show the coefficients formulas.

Proof of Theorem 4.1. Thanks to (2.3), we can derive term-by-term the series (2.4), it comes that

$$\phi_x(x,t) = \frac{\pi i}{L}\sum_{n\in\,\mathbb{Z}^*} nc_n\left((1-\mathbf{v})\,e^{n\pi i(1-\mathbf{v})(t+x)/L} + (1+\mathbf{v})\,e^{n\pi i(1+\mathbf{v})(t-x)/L}\right), \tag{2.10}$$

$$\phi_t(x,t) = \frac{\pi i}{L}\sum_{n\in\,\mathbb{Z}^*} nc_n\left((1-\mathbf{v})\,e^{n\pi i(1-\mathbf{v})(t+x)/L} - (1+\mathbf{v})\,e^{n\pi i(1+\mathbf{v})(t-x)/L}\right), \tag{2.11}$$

where $t \geq 0$, $x \in (\mathbf{v}t, L+\mathbf{v}t)$. Combining this, with (2.8) and (2.9), the extensions $\tilde{\phi}_x$ and $\tilde{\phi}_t$ are given by

$$\tilde{\phi}_x(x,t) = \begin{cases} \frac{\gamma_{\mathbf{v}}\pi i}{L}\sum\limits_{n\in\,\mathbb{Z}^*} n\Big((1-\mathbf{v})\,c_n e^{\frac{n\pi i(1-\mathbf{v})}{L}((1+\mathbf{v})t+\gamma_{\mathbf{v}}(\mathbf{v}t-x))} \\ \qquad +(1+\mathbf{v})\,c_n e^{\frac{n\pi i(1+\mathbf{v})}{L}((1-\mathbf{v})t-\gamma_{\mathbf{v}}(\mathbf{v}t-x))}\Big), & \text{if } x\in(-L_1+\mathbf{v}t,\mathbf{v}t), \\ \frac{\pi i}{L}\sum\limits_{n\in\,\mathbb{Z}^*} n\Big((1-\mathbf{v})\,c_n e^{n\pi i(1-\mathbf{v})(t+x)/L} \\ \qquad +(1+\mathbf{v})\,c_n e^{n\pi i(1+\mathbf{v})(t-x)/L}\Big), & \text{if } x\in(\mathbf{v}t,L+\mathbf{v}t), \\ \frac{\pi i}{\gamma_{\mathbf{v}}L}\sum\limits_{n\in\,\mathbb{Z}^*} n\Big((1-\mathbf{v})\,c_n e^{\frac{n\pi i(1-\mathbf{v})}{L}\left((1+\mathbf{v})t+\frac{\mathbf{v}t-x}{\gamma_{\mathbf{v}}}+\frac{2L}{1+\mathbf{v}}\right)} \\ \qquad +(1+\mathbf{v})\,e^{\frac{n\pi i(1+\mathbf{v})}{L}\left((1-\mathbf{v})t-\frac{\mathbf{v}t-x}{\gamma_{\mathbf{v}}}-\frac{2L}{1+\mathbf{v}}\right)}\Big), & \text{if } x\in(L+\mathbf{v}t,L_2+\mathbf{v}t), \end{cases} \tag{2.12}$$

$$\tilde{\phi}_t(x,t) = \begin{cases} -\frac{\gamma_{\mathbf{v}}\pi i}{L}\sum\limits_{n\in\,\mathbb{Z}^*} n\Big((1-\mathbf{v})\,c_n e^{\frac{n\pi i(1-\mathbf{v})}{L}((1+\mathbf{v})t+\gamma_{\mathbf{v}}(\mathbf{v}t-x))} \\ \qquad -(1+\mathbf{v})\,c_n e^{\frac{n\pi i(1+\mathbf{v})}{L}((1-\mathbf{v})t-\gamma_{\mathbf{v}}(\mathbf{v}t-x))}\Big), & \text{if } x\in(-L_1+\mathbf{v}t,\mathbf{v}t), \\ \frac{\pi i}{L}\sum\limits_{n\in\,\mathbb{Z}^*} n\Big((1-\mathbf{v})\,c_n e^{n\pi i(1-\mathbf{v})(t+x)/L} \\ \qquad -(1+\mathbf{v})\,c_n e^{n\pi i(1+\mathbf{v})(t-x)/L}\Big), & \text{if } x\in(\mathbf{v}t,L+\mathbf{v}t), \\ -\frac{\pi i}{\gamma_{\mathbf{v}}L}\sum\limits_{n\in\,\mathbb{Z}^*} n\Big((1-\mathbf{v})\,c_n e^{\frac{n\pi i(1-\mathbf{v})}{L}\left((1+\mathbf{v})t+\frac{\mathbf{v}t-x}{\gamma_{\mathbf{v}}}+\frac{2L}{1+\mathbf{v}}\right)} \\ \qquad -(1+\mathbf{v})\,e^{\frac{n\pi i(1+\mathbf{v})}{L}\left((1-\mathbf{v})t-\frac{\mathbf{v}t-x}{\gamma_{\mathbf{v}}}-\frac{2L}{1+\mathbf{v}}\right)}\Big), & \text{if } x\in(L+\mathbf{v}t,L_2+\mathbf{v}t). \end{cases} \tag{2.13}$$

On one hand, taking the sum of (2.12) and (2.13) on the interval $(\mathbf{v}t, L_2+\mathbf{v}t)$, we get

$$\tilde{\phi}_x+\tilde{\phi}_t = \begin{cases} \frac{2\pi i}{L}\,(1-\mathbf{v})\sum\limits_{n\in\,\mathbb{Z}^*} nc_n e^{n\pi i(1-\mathbf{v})(t+x)/L}, & x\in(\mathbf{v}t,L+\mathbf{v}t), \\ \frac{2\pi i}{\gamma_{\mathbf{v}}L}\,(1+\mathbf{v})\sum\limits_{n\in\,\mathbb{Z}^*} nc_n e^{\frac{n\pi i(1+\mathbf{v})}{L}\left((1-\mathbf{v})t-\frac{\mathbf{v}t-x}{\gamma_{\mathbf{v}}}-\frac{2L}{1+\mathbf{v}}\right)}, & x\in(L+\mathbf{v}t,L_2+\mathbf{v}t). \end{cases}$$

Since $e^{\frac{n\pi i(1+\mathbf{v})}{L}\left((1-\mathbf{v})t-\frac{\mathbf{v}t-x}{\gamma_{\mathbf{v}}}-\frac{2L}{1+\mathbf{v}}\right)} = e^{n\pi i(1-\mathbf{v})(t+x)/L}$, we get the same expression on the two subintervals, i.e.

$$\tilde{\phi}_x+\tilde{\phi}_t = \frac{2\pi i}{L}\,(1-\mathbf{v})\sum_{n\in\,\mathbb{Z}^*} nc_n e^{n\pi i(1-\mathbf{v})(t+x)/L}, \quad \text{for } x\in(\mathbf{v}t,L_2+\mathbf{v}t). \tag{2.14}$$

Taking into account that $\left\{\sqrt{\frac{1-\mathbf{v}}{2L}}e^{n\pi i(1-\mathbf{v})(t+x)/L}\right\}_{n\in\,\mathbb{Z}}$ is an orthonormal basis for $L^2\left(\mathbf{v}t, L_2+\mathbf{v}t\right)$, for every $t\geq 0$ (see Appendix A), we rewrite (2.14) as

$$\frac{1}{4\pi i}\sqrt{\frac{2L}{1-\mathbf{v}}}\left(\tilde{\phi}_x+\tilde{\phi}_t\right)=\sum_{n\in\,\mathbb{Z}^*} nc_n\sqrt{\frac{1-\mathbf{v}}{2L}}e^{n\pi i(1-\mathbf{v})(t+x)/L}, \tag{2.15}$$

for $x\in\left(\mathbf{v}t, L_2+\mathbf{v}t\right)$. This means that nc_n is the n^{th} coefficient of the function

$$\frac{1}{4\pi i}\sqrt{\frac{2L}{1-\mathbf{v}}}\left(\tilde{\phi}_x+\tilde{\phi}_t\right)\in L^2\left(\mathbf{v}t, L_2+\mathbf{v}t\right). \tag{2.16}$$

By consequence,

$$nc_n=\frac{1}{4\pi i}\int\limits_{\mathbf{v}t}^{L_2+\mathbf{v}t}\left(\tilde{\phi}_x+\tilde{\phi}_t\right)e^{-n\pi i(1-\mathbf{v})(t+x)/L}dx,\quad \text{for } n\in\,\mathbb{Z}^*, \tag{2.17}$$

and (2.5) holds as claimed for $t=0$.

On the other hand, taking the difference of (2.12) and (2.13) on the interval $\left(-L_1+\mathbf{v}t, L+\mathbf{v}t\right)$ yields in particular

$$\tilde{\phi}_x-\tilde{\phi}_t=\begin{cases}\frac{2\pi i}{L}\gamma_{\mathbf{v}}\left(1-\mathbf{v}\right)\sum\limits_{n\in\,\mathbb{Z}^*} nc_n e^{n\pi i(1-\mathbf{v})((1+\mathbf{v})t+\gamma_{\mathbf{v}}(\mathbf{v}t-x))/L}, & x\in\left(-L_1+\mathbf{v}t, \mathbf{v}t\right),\\ \frac{2\pi i}{L}\left(1+\mathbf{v}\right)\sum\limits_{n\in\mathbb{Z}^*} nc_n e^{n\pi i(1+\mathbf{v})(t-x)/L}, & x\in\left(\mathbf{v}t, L+\mathbf{v}t\right).\end{cases}$$

After few rearrangement, we find

$$\tilde{\phi}_x-\tilde{\phi}_t=\frac{2\pi i}{L}\left(1+\mathbf{v}\right)\sum_{n\in\,\mathbb{Z}^*} nc_n e^{n\pi i(1+\mathbf{v})(t-x)/L},\quad \text{for } x\in\left(-L_1+\mathbf{v}t, L+\mathbf{v}t\right). \tag{2.18}$$

Since $\left\{\sqrt{\frac{1+\mathbf{v}}{2L}}e^{n\pi i(1+\mathbf{v})(t-x)/L}\right\}_{n\in\,\mathbb{Z}}$ is an orthonormal basis for $L^2\left(-L_1+\mathbf{v}t, L+\mathbf{v}t\right)$ (see Appendix A), we deduce that

$$nc_n=\frac{1}{4\pi i}\int\limits_{-L_1+\mathbf{v}t}^{L+\mathbf{v}t}\left(\tilde{\phi}_x-\tilde{\phi}_t\right)e^{-n\pi i(1+\mathbf{v})(t-x)/L}dx,\quad \text{for } n\in\,\mathbb{Z}^*. \tag{2.19}$$

For $t=0$, we obtain (2.6) and the theorem follows. ■

Remark 2.3 *An easy computation shows that the solution ϕ given by (2.4) satisfies the periodicity relation*

$$\phi(x+\mathbf{v}T_{\mathbf{v}}, t+T_{\mathbf{v}})=\phi(x,t), \tag{2.20}$$

i.e., after a time $T_{\mathbf{v}}=2L/\left(1-\mathbf{v}^2\right)$ the string travels a distance $\mathbf{v}T_{\mathbf{v}}$ and return to its original form at time t.

Remark 2.4 *Since we can take* $t = T$ *in* (2.17) *and in* (2.19)*, then we can solve Problem* (*WP*) *if the initial data are replaced by the (final) data*

$$\phi\left(x, T\right) = \phi_T^0 \in H_0^1\left(\mathbf{I}_T\right), \quad \phi_t\left(x, T\right) = \phi_T^1 \in L^2\left(\mathbf{I}_T\right).$$

Corollary 2.1 *Under the Assumptions* (2.1) *and* (2.2)*, the sum* $\sum_{n \in \mathbb{Z}^*} \left|nc_n\right|^2$ *is finite and is given by any of the two formulas, for* $t \geq 0$,

$$\sum_{n \in \mathbb{Z}^*} \left|nc_n\right|^2 = \frac{L}{8\pi^2\left(1 - \mathbf{v}\right)} \int\limits_{\mathbf{v}t}^{L_2 + \mathbf{v}t} \left(\tilde{\phi}_x + \tilde{\phi}_t\right)^2 dx \tag{2.21}$$

$$= \frac{L}{8\pi^2\left(1 + \mathbf{v}\right)} \int\limits_{-L_1 + \mathbf{v}t}^{L + \mathbf{v}t} \left(\tilde{\phi}_x - \tilde{\phi}_t\right)^2 dx. \tag{2.22}$$

Proof. Parseval's equality applied to the function given in (2.15) yields

$$\sum_{n \in\ \mathbb{Z}^*} \left|nc_n\right|^2 = \left|\frac{1}{4\pi i}\sqrt{\frac{2L}{1 - \mathbf{v}}}\right|^2 \int\limits_{\mathbf{v}t}^{L_2 + \mathbf{v}t} \left(\tilde{\phi}_x + \tilde{\phi}_t\right)^2 dx, \quad \text{for } t \geq 0.$$

Thus (2.21) holds as claimed. The identity (2.22) follows from (2.19) in a similar manner. ■

2.2.2 *A numerical example*

To illustrate the above results, we compute the solution of (*WP*) for

$$L = \pi, \ \ \phi^0\left(x\right) = \sin\left(x\right)/10, \ \ \phi^1\left(x\right) = 0$$

and use (2.21) for the first 40 frequencies in (2.4), i.e.

$$\phi(x, t) \simeq \sum_{\substack{n=-40,\\ n \neq 0}}^{40} c_n \left(e^{ni(1-\mathbf{v})(t+x)} - e^{ni(1+\mathbf{v})(t-x)}\right), \text{ for } x \in \left(\mathbf{v}t, \pi + \mathbf{v}t\right) \text{ and } t \in \left(0, T_{\mathbf{v}}\right), \tag{2.23}$$

with two values of speed $\mathbf{v} = 0.3$ and $\mathbf{v} = 0.7$.

2.3 ENERGY EXPRESSIONS AND ESTIMATES

In this section, we show that the energy $\mathcal{E}_{\mathbf{v}}\left(t\right)$, such that

$$\mathcal{E}_{\mathbf{v}}\left(t\right) = \frac{1}{2}\int_{\mathbf{I}_t} \left(\phi_t + \mathbf{v}\phi_x\right)^2 + \left(1 - \mathbf{v}^2\right)\phi_x^2 dx \tag{2.24}$$

depending on $L, t, \mathbf{v}$ of the solution of (*WP*), is conserved in time.

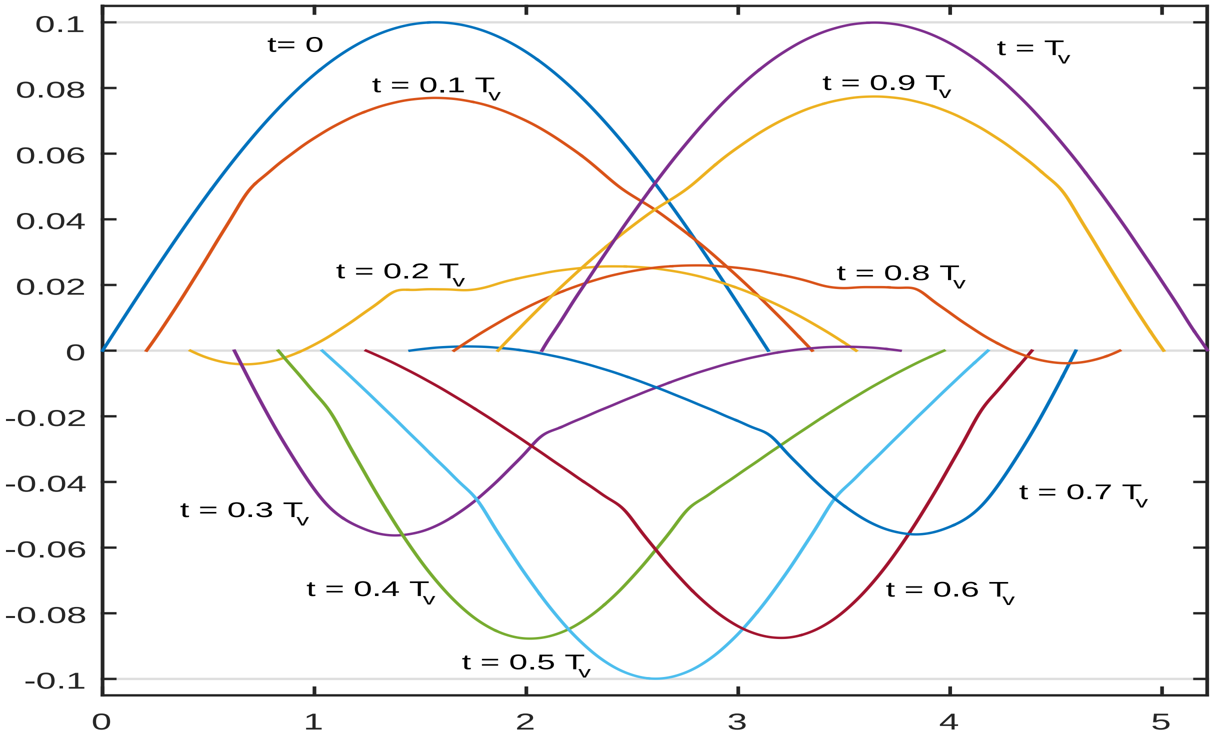

Figure 2.4: The solution ϕ for $\mathbf{v} = 0.3$ in the interval $(\mathbf{v}t, \pi + \mathbf{v}t)$ over one period $T_{\mathbf{v}} \simeq 6.91$.

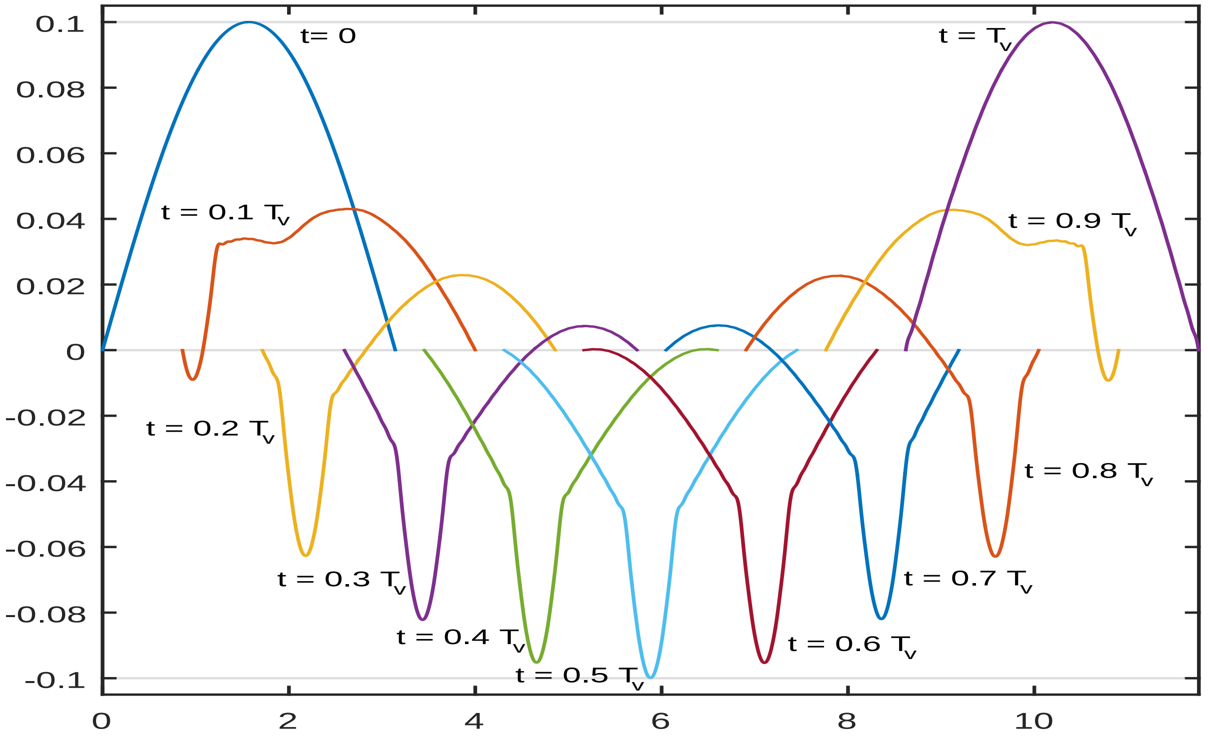

Figure 2.5: The solution ϕ for $\mathbf{v} = 0.7$ in the interval $(\mathbf{v}t, \pi + \mathbf{v}t)$ over one period $T_{\mathbf{v}} \simeq 12.32$.

Theorem 2.2 *Under the Assumptions* (2.1) *and* (2.2)*, the solution of Problem* (*WP*) *satisfies*

$$\mathcal{E}_{\mathbf{v}}(t)=\frac{2\pi^2\left(1-\mathbf{v}^2\right)}{L}\sum_{n\in\mathbb{Z}^*}|nc_n|^2,\qquad \text{for } t\geq 0. \tag{2.25}$$

(*the left-hand side is independent of* t).

Proof. The two identities (2.21) and (2.22) implies that

$$\frac{1}{1+\mathbf{v}}\int\limits_{-L_1+\mathbf{v}t}^{L+\mathbf{v}t}\left(\tilde{\phi}_x-\tilde{\phi}_t\right)^2dx=\frac{1}{1-\mathbf{v}}\int\limits_{\mathbf{v}t}^{L_2+\mathbf{v}t}\left(\tilde{\phi}_x+\tilde{\phi}_t\right)^2dx=\frac{8\pi^2}{L}\sum_{n\in\ \mathbb{Z}^*}|nc_n|^2. \tag{2.26}$$

Using the extensions (2.8), (2.9) and considering the change of variable $x=\frac{1}{\gamma_{\mathbf{v}}}\left(\mathbf{v}t-\xi\right)+\mathbf{v}t$, in $\left(-L_1+\mathbf{v}t,\mathbf{v}t\right)$, we obtain

$$\begin{aligned}\frac{1}{1+\mathbf{v}}\int\limits_{-L_1+\mathbf{v}t}^{\mathbf{v}t}\left(\tilde{\phi}_x(x,t)-\tilde{\phi}_t(x,t)\right)^2dx&=-\frac{1}{1+\mathbf{v}}\int\limits_{L+\mathbf{v}t}^{\mathbf{v}t}\gamma_{\mathbf{v}}\left(\phi_x(\xi,t)+\phi_t(\xi,t)\right)^2d\xi\\&=\frac{1}{1-\mathbf{v}}\int_{\mathbf{v}t}^{L+\mathbf{v}t}\left(\phi_x(\xi,t)+\phi_t(\xi,t)\right)^2d\xi.\end{aligned}$$

Taking $x=\gamma_{\mathbf{v}}\left(\mathbf{v}t-\xi\right)+\frac{2L}{\mathbf{v}-1}+\mathbf{v}t$, in $\left(L+\mathbf{v}t,L_2+\mathbf{v}t\right)$, we obtain

$$\frac{1}{1-\mathbf{v}}\int\limits_{L+\mathbf{v}t}^{L_2+\mathbf{v}t}\left(\tilde{\phi}_x(x,t)+\tilde{\phi}_t(x,t)\right)^2dx=\frac{1}{1+\mathbf{v}}\int_{\mathbf{v}t}^{L+\mathbf{v}t}\left(\phi_x(\xi,t)-\phi_t(\xi,t)\right)^2d\xi.$$

Then, taking (2.26) into account, it comes that

$$\begin{aligned}&\frac{1}{1-\mathbf{v}}\int\limits_{-L_1+\mathbf{v}t}^{L+\mathbf{v}t}\left(\tilde{\phi}_t+\tilde{\phi}_x\right)^2dx+\frac{1}{1+\mathbf{v}}\int\limits_{\mathbf{v}t}^{L_2+\mathbf{v}t}\left(\tilde{\phi}_x-\tilde{\phi}_t\right)^2dx\\&\quad=\frac{2}{1+\mathbf{v}}\int_{\mathbf{v}t}^{L+\mathbf{v}t}\left(\phi_x-\phi_t\right)^2dx+\frac{2}{1-\mathbf{v}}\int_{\mathbf{v}t}^{L+\mathbf{v}t}\left(\phi_t+\phi_x\right)^2dx=\frac{16\pi^2}{L}\sum_{n\in\ \mathbb{Z}^*}|nc_n|^2.\end{aligned}$$

Expanding $\left(\phi_x\pm\phi_t\right)^2$ and collecting similar terms, we get

$$\frac{1}{1-\mathbf{v}^2}\left(2\int_{\mathbf{v}t}^{L+\mathbf{v}t}\phi_x^2+\phi_t^2+4\mathbf{v}\phi_x\phi_t\,dx\right)=\frac{8\pi^2}{L}\sum_{n\in\ \mathbb{Z}^*}|nc_n|^2,\quad \text{for } t\geq 0. \tag{2.27}$$

Recalling that $\mathcal{E}_{\mathbf{v}}(t)$ is given by (2.24), this identity can be rewritten as in (2.25). This end the proof. ■

The fact that $\mathcal{E}_{\mathbf{v}}(t)$ is constant in time can be established by using only the identities $\phi_{tt}=\phi_{xx}$ and $\phi\left(\mathbf{v}t,t\right)=\phi\left(L+\mathbf{v}t,t\right)=0$ from (*WP*).

A second proof for the conservation of $\mathcal{E}_v(t)$. It suffices to show that $\frac{d}{dt}\mathcal{E}_{\mathbf{v}}(t)=0$.

First, the boundary conditions $\phi\left(\mathbf{v}t,t\right)=\phi\left(L+\mathbf{v}t,t\right)=0$ means that $\frac{d}{dt}\phi\left(\mathbf{v}t,t\right)=\frac{d}{dt}\phi\left(L+\mathbf{v}t,t\right)=0$, hence

$$\phi_t\left(\mathbf{v}t,t\right)+\mathbf{v}\phi_x\left(\mathbf{v}t,t\right)=\phi_t\left(L+\mathbf{v}t,t\right)+\mathbf{v}\phi_x\left(L+\mathbf{v}t,t\right)=0. \tag{2.28}$$

Since the limits of the integral in the expression of $\mathcal{E}_{\mathbf{v}}(t)$ are time-dependent, then Leibnitz's rule implies that

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_{\mathbf{v}}(t) &= \mathbf{v}\left(1-\mathbf{v}^2\right)\left(\phi_x^2(L+\mathbf{v}t,t)-\phi_x^2(\mathbf{v}t,t)\right) \\ &\quad + \int_{\mathbf{v}t}^{L+\mathbf{v}t} \frac{\partial}{\partial t}\left(\phi_t+\mathbf{v}\phi_x\right)^2\,dx + \left(1-\mathbf{v}^2\right)\frac{\partial}{\partial t}\left(\phi_x^2\right)dx. \end{aligned} \tag{2.29}$$

The remaining integral equals, after using $\phi_{tt}=\phi_{xx}$ then integrating by parts,

$$\begin{aligned} \int_{\mathbf{v}t}^{L+\mathbf{v}t} \left(\phi_t+\mathbf{v}\phi_x\right)\phi_{xx} + \left(\mathbf{v}\phi_t+\phi_x\right)\phi_{xt}dx &= \int_{\mathbf{v}t}^{L+\mathbf{v}t} -\left(\phi_{xt}+\mathbf{v}\phi_{xx}\right)\phi_x + \left(\mathbf{v}\phi_t+\phi_x\right)\phi_{xt}dx \\ &= \mathbf{v}\int_{\mathbf{v}t}^{L+\mathbf{v}t} -\phi_{xx}\phi_x+\phi_t\phi_{xt}dx, \end{aligned}$$

which is nothing but

$$\mathbf{v}\int_{\mathbf{v}t}^{L+\mathbf{v}t} \frac{\partial}{\partial x}\left(\phi_t^2-\phi_x^2\right)dx = -\mathbf{v}\left(1-\mathbf{v}^2\right)\left(\phi_x^2(L+\mathbf{v}t,t)-\phi_x^2(\mathbf{v}t,t)\right)$$

due to (2.28). Going back to (2.29), we infer that $\frac{d}{dt}\mathcal{E}_{\mathbf{v}}(t)=0$ as claimed. ■

Let us now compare $\mathcal{E}_{\mathbf{v}}(t)$ to the usual expression of energy for the wave equation

$$E_{\mathbf{v}}(t) := \frac{1}{2}\int_{\mathbf{v}t}^{L+\mathbf{v}t} \phi_t^2+\phi_x^2dx, \quad \text{for } t\geq 0.$$

In contrast with $\mathcal{E}_{\mathbf{v}}(t)$, the expression $E_{\mathbf{v}}(t)$ is not conserved in general. Due to the periodicity relation (2.20), we know that $E_{\mathbf{v}}$ is $T_{\mathbf{v}}-$periodic in time. Moreover we have

Corollary 2.2 *Under the Assumptions (2.1) and (2.2), the energy $E_{\mathbf{v}}(t)$ of the solution of Problem (WP) satisfies*

$$\frac{\mathcal{E}_{\mathbf{v}}(t)}{1+\mathbf{v}} \leq E_{\mathbf{v}}(t) \leq \frac{\mathcal{E}_{\mathbf{v}}(t)}{1-\mathbf{v}}, \quad \textit{for } t\geq 0 \tag{2.30}$$

and

$$\frac{1}{\gamma_{\mathbf{v}}}E_{\mathbf{v}}(0) \leq E_{\mathbf{v}}(t) \leq \gamma_{\mathbf{v}}E_{\mathbf{v}}(0), \quad \textit{for } t\geq 0. \tag{2.31}$$

Proof. We can write (2.27) as

$$E_{\mathbf{v}}(t) + \mathbf{v}\int_{\mathbf{v}t}^{L+\mathbf{v}t} \phi_x\phi_t\,dx = \mathcal{E}_{\mathbf{v}}(t), \quad \text{for } t\geq 0. \tag{2.32}$$

Thanks to the algebraic inequality $\pm ab \leq \left(a^2+b^2\right)/2$ we know that

$$\pm\int_{\mathbf{v}t}^{L+\mathbf{v}t} \phi_x\phi_t\,dx \leq E_{\mathbf{v}}(t), \quad \text{for } t\geq 0.$$

Then, it comes that

$$\mathcal{E}_{\mathbf{v}}(t) \leq (1+\mathbf{v})E_{\mathbf{v}}(t) \quad \text{and} \quad (1-\mathbf{v})E_{\mathbf{v}}(t) \leq \mathcal{E}_{\mathbf{v}}(t), \quad \text{for } t\geq 0. \tag{2.33}$$

This implies (2.30). Since (2.33) holds also for $t = 0$, then (2.31) follows by combining the two inequalities

$$\begin{aligned} &(1-\mathbf{v}) E_{\mathbf{v}}(t) \leq \mathcal{E}_{\mathbf{v}}(t) = \mathcal{E}_{\mathbf{v}}(0) \leq (1+\mathbf{v}) E_{\mathbf{v}}(0), \\ &(1-\mathbf{v}) E_{\mathbf{v}}(0) \leq \mathcal{E}_{\mathbf{v}}(0) = \mathcal{E}_{\mathbf{v}}(t) \leq (1+\mathbf{v}) E_{\mathbf{v}}(t), \end{aligned}$$

for $t \geq 0$. ■

Remark 2.5 *The equality in estimation (2.30) may hold for some $t \geq 0$. This is the case whenever $\phi_t(x,t) = \pm\phi_x(x,t)$, for $x \in \mathbf{I}_t$ and some $t \geq 0$. For instance, if the initial data satisfy $\phi^1 = \pm\phi_x^0$ we obtain from (2.32) that*

$$(1 \pm \mathbf{v}) E_{\mathbf{v}}(0) = E_{\mathbf{v}}(0) + \mathbf{v} \int_0^L \phi_x^0 \phi^1 \, dx = \mathcal{E}_{\mathbf{v}}(0),$$

i.e. $E_{\mathbf{v}}(0) = \mathcal{E}_{\mathbf{v}}(0) / (1 \pm \mathbf{v})$. By periodicity, we have also $E_{\mathbf{v}}(nT_{\mathbf{v}}) = \mathcal{E}_{\mathbf{v}}(0) / (1 \pm \mathbf{v})$, for $n \in \mathbb{Z}$. The + and – signs are used respectively.

Remark 2.6 *As $\mathbf{v} \to 1^-$, we have $\mathcal{E}_{\mathbf{v}}(0) \to \|\phi^1 + \phi_x^0\|_{L^2(0,L)}/2$. If the initial data satisfies $\phi^1 + \phi_x^0 \neq 0$, it follows from (2.30) that*

$$E_{\mathbf{v}}(t) \leq \frac{\mathcal{E}_{\mathbf{v}}(t)}{1-\mathbf{v}} = \frac{\mathcal{E}_{\mathbf{v}}(0)}{1-\mathbf{v}} \to +\infty, \text{ as } \mathbf{v} \to 1^-.$$

Taking the precedent remark into account, we may have large value for $E_{\mathbf{v}}(t)$, as $\mathbf{v}$ becomes close to the speed of propagation $c = 1$, even for small initial value $\mathcal{E}_{\mathbf{v}}(0)$. To see what happens to the string in this case, let us take $\mathbf{v} = 0.9$ in the precedent numerical example, see Figure 2.6. We observe a layer effect (i.e. a region in $\mathbf{I}_t$ where ϕ_x becomes very large) that travels from the left endpoint to the right one over one period $T_{\mathbf{v}}$. This phenomenon becomes more marked as $\mathbf{v}$ is closer to 1.

2.4 OBSERVABILITY AND CONTROLLABILITY AT AN ENDPOINT

In many applications, it is preferred that the sensors/controllers do not interfere with the vibrations of the string, so they are placed at the extremities. In addition, interior pointwise sensors/controllers are difficult to design and the system may become unobservable/uncontrollable depending on the sensors/controllers location. This fact was shown by Yang and Mote in [56] where they cast (2) in a state space form and use semi-group theory.

In this section, we show the observability of (WP) at each endpoint $x_b + \mathbf{v}t$ where

$$x_b = 0 \text{ or } x_b = L.$$

Then, by applying HUM we derive the exact boundary controllability at each endpoint.

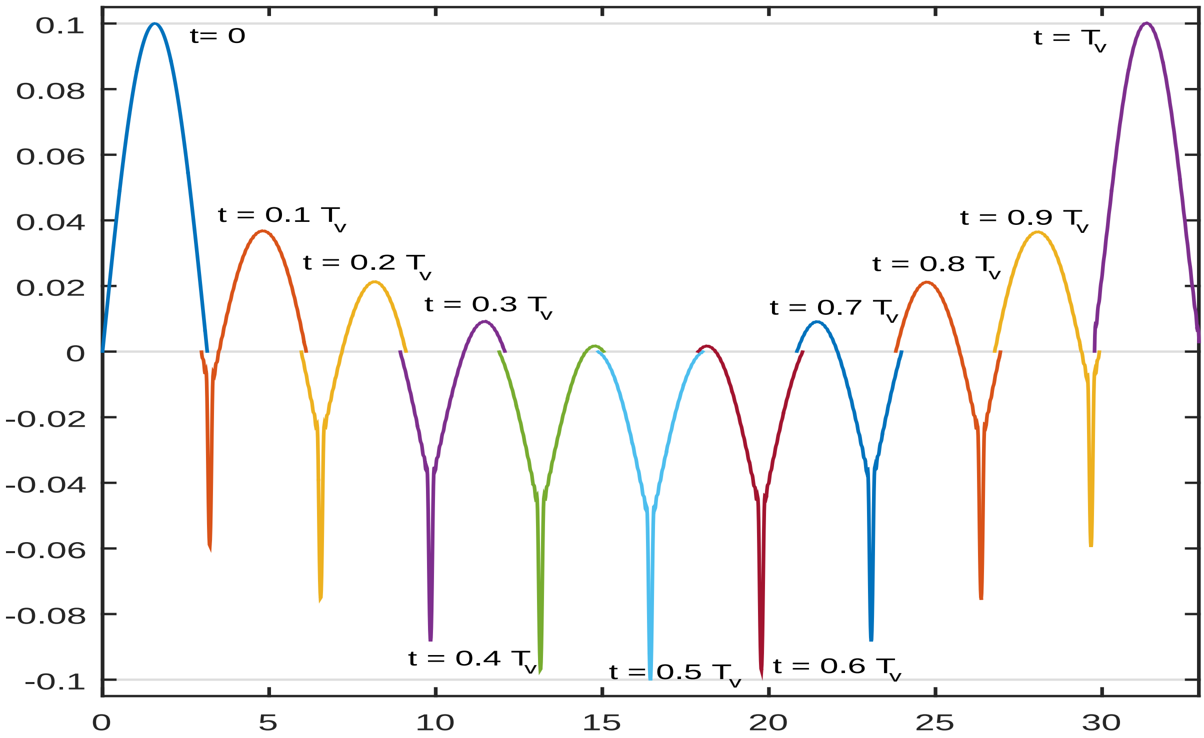

Figure 2.6: The solution ϕ for $\mathbf{v} = 0.9$ in the interval $(\mathbf{v}t, \pi + \mathbf{v}t)$ over one period $T_{\mathbf{v}} \simeq 33.07$.

2.4.1 *Observability at an endpoint*

This Problem of observability considered here can be stated as follows: To give sufficient conditions on the length T of the time interval such that there exists a constant $C(T) > 0$ for which the observability inequality[1]

$$\mathcal{E}_{\mathbf{v}}(0) \leq C(T) \int_0^T \phi_x^2(x_b + \mathbf{v}t, t)dt, \tag{2.34}$$

holds for all the solutions of (WP). This inequality is also called inverse inequality.

The next theorem shows that the minimal time of observability is $T_{\mathbf{v}} = 2L/\left(1 - \mathbf{v}^2\right)$.

Theorem 2.3 *Under the Assumptions* (2.1) *and* (2.2), *we have*

$$\int_0^{MT_{\mathbf{v}}} \phi_x^2(x_b + \mathbf{v}t, t)dt = \frac{4M}{(1 - \mathbf{v}^2)^2}\mathcal{E}_{\mathbf{v}}(0). \tag{2.35}$$

---

[1] One can replace $\mathcal{E}_{\mathbf{v}}(0)$ by $E_{\mathbf{v}}(0)$ in the left-hand side, but this does not matter since (2.30) holds under the assumption (2.1).

By consequence, the solution of (*WP*) *satisfies the direct inequality*

$$\int_0^T \phi_x^2(x_b+\mathbf{v}t,t)dt \leq K_1(\mathbf{v},T)\mathcal{E}_{\mathbf{v}}(0)\,, \text{for every } T\geq 0, \tag{2.36}$$

with a constant $K_1(\mathbf{v},T)$ *depending only on* $\mathbf{v}$ *and* T.

If $T\geq T_{\mathbf{v}}$, *Problem* (*WP*) *is observable at* $\xi(t)=x_b+\mathbf{v}t$ *and it holds that:*

$$\mathcal{E}_{\mathbf{v}}(0)\leq \frac{\left(1-\mathbf{v}^2\right)^2}{4}\int_0^T \phi_x^2(x_b+\mathbf{v}t,t)dt. \tag{2.37}$$

Proof. Thanks to (2.10), we can evaluate ϕ_x at the endpoint $x=x_b+\mathbf{v}t$. We obtain

$$\begin{aligned}\phi_x(x_b+\mathbf{v}t,t) &= \frac{\pi i}{L}\sum_{n\in\mathbb{Z}^*} nc_n\left((1-\mathbf{v})\,e^{\frac{n\pi i(1-\mathbf{v})}{L}((1+\mathbf{v})t+x_b)}+(1+\mathbf{v})\,e^{\frac{n\pi i(1+\mathbf{v})}{L}((1-\mathbf{v})t-x_b)}\right)\\ &= \frac{\pi i}{L}\sum_{n\in\mathbb{Z}^*} nc_n\left((1-\mathbf{v})\,e^{\frac{n\pi i(1-\mathbf{v})}{L}x_b}+(1+\mathbf{v})\,e^{-\frac{n\pi i(1+\mathbf{v})}{L}x_b}\right)e^{n\pi i\left(1-\mathbf{v}^2\right)t/L},\end{aligned}$$

which can be rewritten as

$$\phi_x(x_b+\mathbf{v}t,t)=\begin{cases}\dfrac{2\pi i}{L}\displaystyle\sum_{n\in\mathbb{Z}^*} nc_n e^{2n\pi it/T_{\mathbf{v}}}, & \text{if } x_b=0,\\ \dfrac{2\pi i}{L}\displaystyle\sum_{n\in\mathbb{Z}^*} nc_n e^{-n\pi i(1+\mathbf{v})}e^{2n\pi it/T_{\mathbf{v}}}, & \text{if } x_b=L.\end{cases}. \tag{2.38}$$

Let $M\in\mathbb{N}^*$. Since the set of functions $\left\{e^{2n\pi it/T_{\mathbf{v}}}/\sqrt{T_{\mathbf{v}}}\right\}_{n\in\mathbb{Z}}$ is complete and orthonormal in the space $L^2(mT_{\mathbf{v}},(m+1)\,T_{\mathbf{v}})$ for $m=0,...,M-1$, then Parseval's equality applied to the functions

$$\phi_x(x_b+\mathbf{v}t,t)\in L^2(mT_{\mathbf{v}},(m+1)\,T_{\mathbf{v}}),\quad \text{for } m=0,...,M-1,$$

yields, after summing up the integrals for all the subintervals of $(0,MT_{\mathbf{v}})$,

$$\frac{1}{T_{\mathbf{v}}}\int_0^{MT_{\mathbf{v}}} \phi_x^2(x_b+\mathbf{v}t,t)dt=\frac{4M\pi^2}{L^2}\sum_{n\in\mathbb{Z}^*}|nc_n|^2\,,$$

and (2.35) follows.

For every $T\geq 0$, we can take the integer M large enough to satisfy $MT_{\mathbf{v}}=M\frac{2L}{1-\mathbf{v}^2}\geq T$. Then, the identity (2.35) yields

$$\int_0^T \phi_x^2(x_b+\mathbf{v}t,t)dt\leq \int_0^{MT_{\mathbf{v}}} \phi_x^2(x_b+\mathbf{v}t,t)dt=\frac{4M}{\left(1-\mathbf{v}^2\right)^2}\mathcal{E}_{\mathbf{v}}(0)$$

i.e., (2.36) holds for $K_1(\mathbf{v},T):=4M/\left(1-\mathbf{v}^2\right)^2$. The inequality (2.37) follows from (2.35) with $M=1$. ∎

Remark 2.7 *Taking (2.28) into account, we have*

$$\phi_t^2(x_b + \mathbf{v}t, t) = \mathbf{v}^2 \phi_x^2(x_b + \mathbf{v}t, t), \quad \text{for } x_b = 0 \text{ or } x_b = L, \ \forall t \geq 0.$$

Then, the results of Theorem 2.3 hold if we replace $\phi_x(x_b + \mathbf{v}t, t)$ by $\phi_t(x_b + \mathbf{v}t, t)/\mathbf{v}^2$ with the same constants in the inequalities.

Remark 2.8 *The time of boundary observability $T_{\mathbf{v}}$ can be predicted by a simple argument, see Figure 2.7. An initial disturbances concentrated near $x = L + \mathbf{v}t$ may propagate to the left as t increases. It reaches the left boundary, when t is close to $\frac{L}{1+\mathbf{v}}$. Then travels back to reach the right boundary when t is close to $\frac{2L}{1-\mathbf{v}^2} = T_{\mathbf{v}}$, see Figure 2.7 (left). Figure 2.7 (right) shows that we need the same time $T_{\mathbf{v}}$ for an initial disturbance concentrated near $x = \mathbf{v}t$.*

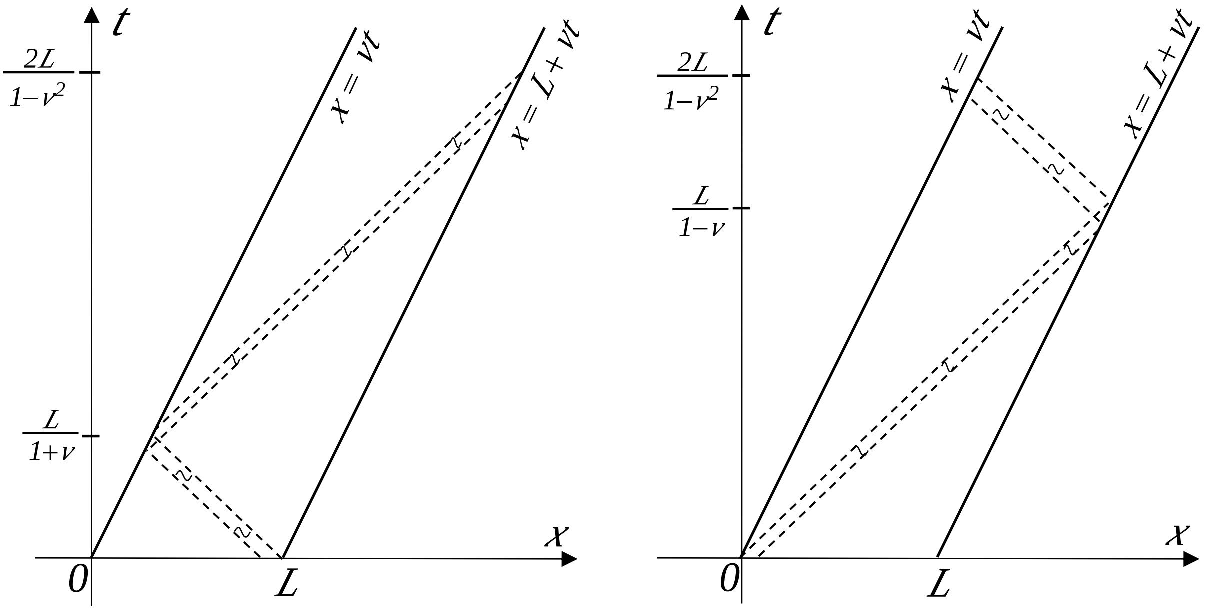

Figure 2.7: Propagation of small disturbances with support near an endpoint.

2.4.2 *Controllability at an endpoint*

We consider the following boundary controllability Problem: Given

$$(u^0, u^1) \in L^2(\mathbf{I}_0) \times H^{-1}(\mathbf{I}_0), \tag{2.39}$$
$$(u_T^0, u_T^1) \in L^2(\mathbf{I}_T) \times H^{-1}(\mathbf{I}_T),$$

find a control function $f \in L^2(0,T)$, acting at one of the endpoints, say $x = L + \mathbf{v}t$, such that the solution of the Problem

$$\begin{cases} u_{tt} - u_{xx} = 0, & \text{for } x \in \mathbf{I}_t, t \in (0,T), \\ u(\mathbf{v}t,t) = 0, \quad u(L+\mathbf{v}t,t) = f(t), & \text{for } t \in (0,T), \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), & \text{for } x \in \mathbf{I}_0, \end{cases} \tag{CWP}$$

satisfies also

$$u(x,T) = u_T^0(x), \quad u_t(x,T) = u_T^1(x), \quad \text{for} \quad x \in \mathbf{I}_T.$$

Arguing as in [4, page 135], Problem (CWP) can be reduced to a null-controllability one, i.e., we can always assume that

$$u(T) = u_t(T) = 0 \ \text{ on } \mathbf{I}_T. \tag{2.40}$$

Let $t \in [0,T]$. Multiplying the first equation in (CWP) by ϕ, solution of (WP), and integrating by parts on $\mathbf{I}_t$ and $(0,t)$ we obtain, at least formally, the following identity:

$$\int_{\mathbf{v}t}^{L+\mathbf{v}t} u(t)\,\phi_t(t)\,dx - \langle u_t(t),\phi(t)\rangle_{H_0^1(\mathbf{I}_t)}$$
$$= \int_0^L u(0)\,\phi^1 dx - \langle u_t(0),\phi^0\rangle_{H_0^1(\mathbf{I}_0)} + \int_0^t f(s)\,\phi_x(L+\mathbf{v}s,s)\,ds, \tag{2.41}$$

where $\langle\cdot,\cdot\rangle_{H_0^1(\mathbf{I}_t)}$ denotes the duality product between $H^{-1}(\mathbf{I}_t)$ and $H_0^1(\mathbf{I}_t)$. We adopt identity (2.41) as the definition of solutions of (CWP), in the sense of transposition. Thanks to Corollary 2.2 and to (2.36) this definition can be justified by arguing as in [27]. Moreover, Problem (CWP) admits a unique solution

$$u \in C([0,T];L^2(\mathbf{I}_t)) \cap C^1([0,T];H^{-1}(\mathbf{I}_t))$$

in the above sense.

Theorem 2.4 *Under the Assumptions (2.1) and (2.39), Problem (CWP) is exactly controllable at the endpoint $x = L + \mathbf{v}t$ for $T \geq T_{\mathbf{v}}$. Moreover, we can choose a control f satisfying*

$$\int_0^T f^2(t)\,dt \leq K_b(\mathbf{v},T)\,\mathcal{E}_{\mathbf{v}}(0), \tag{2.42}$$

where $K_b(\mathbf{v},T)$ is a constant depending on $\mathbf{v}$ and T.

Proof. Let ϕ be the solution of Problem (WP). We seek a control f in the special form

$$f = \phi_x(L+\mathbf{v}t,t) \in L^2(0,T),$$

for a suitable choice of ϕ^0 and ϕ^1. First, we consider the backward Problem

$$\begin{cases} \psi_{tt} - \psi_{xx} = 0, & \text{for } x \in \mathbf{I}_t, t \in (0,T), \\ \psi(\mathbf{v}t,t) = 0, \ \ \psi(L+\mathbf{v}t,t) = \phi_x(L+\mathbf{v}t,t), & \text{for } \ t \in (0,T), \\ \psi(x,T) = 0, \ \ \psi_t(x,T) = 0. & \text{for } \ \ x \in \mathbf{I}_T. \end{cases} \tag{2.43}$$

We obtain a linear map, that relates (ϕ^0, ϕ^1) to the initial data $(\psi_t(0), -\psi(0))$,

$$\begin{aligned} \Lambda_1 : H_0^1(\mathbf{I}_0) \times L^2(\mathbf{I}_0) &\longrightarrow H^{-1}(\mathbf{I}_0) \times L^2(\mathbf{I}_0) \\ (\phi^0, \phi^1) &\longmapsto (\psi_t(0), -\psi(0)). \end{aligned}$$

The space $H_0^1(\mathbf{I}_0) \times L^2(\mathbf{I}_0)$ is equipped with the energy norm $(\phi^0, \phi^1) \longmapsto (\mathcal{E}_\mathbf{v}(0))^{1/2}$. To show that (ϕ^0, ϕ^1) can be chosen such that $(\psi_t(0), -\psi(0)) = (u^1, -u^0)$, we argue as in [23].

Indeed, since the solution of (2.43) is taken in the transposition sense, it comes that

$$\begin{aligned} 0 = \int_0^T \langle (\psi_{tt} - \psi_{xx}), \phi \rangle_{H_0^1(\mathbf{I}_0)} dt = -\langle \psi_t(0), \phi^0 \rangle_{H_0^1(\mathbf{I}_0)} + \int_\mathbf{0}^L \psi(0) \phi^1 dx \\ + \int_0^T \psi(L+\mathbf{v}t,t) \phi_x(L+\mathbf{v}t,t) - \mathbf{v}\psi(L+\mathbf{v}t,t) \phi_t(L+\mathbf{v}t,t) dt. \end{aligned} \tag{2.44}$$

Using (2.28), we can rewrite (2.44) as

$$\langle \Lambda_1(\phi^0,\phi^1), (\phi^0,\phi^1) \rangle_{H_0^1(\mathbf{I}_0) \times L^2(\mathbf{I}_0)} = \left(1+\mathbf{v}^2\right) \int_0^T \phi_x^2(L+\mathbf{v}t,t) dt$$

Thanks to Theorem 2.5, we deduce that

$$\frac{4(1+\mathbf{v}^2)}{(1-\mathbf{v}^2)^2} \mathcal{E}_\mathbf{v}(0) \le \langle \Lambda_1(\phi^0,\phi^1), (\phi^0,\phi^1) \rangle_{H_0^1(\mathbf{I}_0) \times L^2(\mathbf{I}_0)} \le \left(1+\mathbf{v}^2\right) K_1(\mathbf{v},T) \mathcal{E}_\mathbf{v}(0),$$

for $T \ge T_\mathbf{v}$. This means that Λ_1 is an isomorphism for $T \ge T_\mathbf{v}$ and therefore (ϕ^0, ϕ^1) can be determined such that the control $f = \phi_x(L+\mathbf{v}t,t)$ drive the solution of (CWP) from the initial data u^0, u^1 to the rest, i.e. $u(T) = u_t(T) = 0$. ■

Remark 2.9 *A similar argument shows that the controllability also holds if the control f acts at the endpoint $\mathbf{v}t$ (i.e. $x_b = 0$). It suffices to choose $f = -\phi_x(\mathbf{v}t,t)$ in the application of HUM.*

Remark 2.10 *Arguing as in [23, 26, 57], we can show that the non observability of (WP) at one endpoint, for $0 < T < T_\mathbf{v}$ implies the non controllability of (CWP) for $0 < T < T_\mathbf{v}$. Moreover, the f constructed by HUM is the unique control which minimize $\int_0^T f^2(t) dt$ among all the controls driving the solution of (CWP) to the rest.*

2.5 OBSERVABILITY AND CONTROLLABILITY AT BOTH ENDPOINTS

If we place two sensors (resp. actuators) at both endpoints $x = \mathbf{v}t$ and $x = L + \mathbf{v}t$ of the interval $\mathbf{I}_t$, one expects a shorter time of observability (resp. controllability).

2.5.1 *Observability at both endpoints*

The next theorem shows that the observability, in this case, holds for $T \geq \tilde{T}_{\mathbf{v}} = L/\left(1-\mathbf{v}\right)$.

Theorem 2.5 *Under the assumption* (2.1) *and* (2.2), *we have:*

$$\int_0^{\frac{L}{1+\mathbf{v}}} \phi_x^2(\mathbf{v}t,t)dt + \int_0^{\frac{L}{1-\mathbf{v}}} \phi_x^2(L+\mathbf{v}t,t)dt = \frac{4}{\left(1-\mathbf{v}^2\right)^2}\mathcal{E}_{\mathbf{v}}\left(0\right). \tag{2.45}$$

By consequence, the solution of (*WP*) *satisfies the direct inequality*

$$\int_0^T \phi_x^2(\mathbf{v}t,t) + \phi_x^2(L+\mathbf{v}t,t)dt \leq K_2(\mathbf{v},T)\mathcal{E}_{\mathbf{v}}\left(0\right), \text{for every } T \geq 0, \tag{2.46}$$

with a constant $K_2(\mathbf{v},T)$ *depending only on* $\mathbf{v}$ *and* T.

If $T \geq \tilde{T}_{\mathbf{v}}$, *Problem* (*WP*) *is observable at both endpoints* $x = \mathbf{v}t$, $x = L+\mathbf{v}t$ *and it holds that*

$$\mathcal{E}_{\mathbf{v}}\left(0\right) \leq \frac{\left(1-\mathbf{v}^2\right)^2}{4}\int_0^T \phi_x^2(\mathbf{v}t,t) + \phi_x^2(L+\mathbf{v}t,t)dt. \tag{2.47}$$

Proof. First, we establish (2.45) for smooth initial data. Assume that ϕ_x^0 and ϕ^1 are continuous functions. This ensures in particular that their generalized Fourier series are absolutely converging. More precisely, the coefficients c_n, given by (2.5), satisfy

$$\sum_{n\in\ \mathbb{Z}^*} \left|nc_n\right| < +\infty. \tag{2.48}$$

Let $m \in \mathbb{Z}^*$. On one hand, taking $x_b = 0$ in (2.38), multiplying by $\overline{imc_m e^{2m\pi it/T_{\mathbf{v}}}}$ then integrating on $\left(0, L/\left(1+\mathbf{v}\right)\right)$, we obtain

$$\int_0^{\frac{L}{1+\mathbf{v}}} \phi_x(\mathbf{v}t,t)\ \overline{imc_m e^{2m\pi it/T_{\mathbf{v}}}}dt = \frac{2\pi}{L}m\bar{c}_m\int_0^{\frac{L}{1+\mathbf{v}}}\left(\sum_{n\in\ \mathbb{Z}^*} nc_n e^{2(n-m)\pi it/T_{\mathbf{v}}}\right)dt.$$

Since $\left|nc_n e^{(n-m)\pi i\left(1-\mathbf{v}^2\right)t/L}\right| = \left|nc_n\right|$, then due to (2.48) the series in the right hand side is absolutely converging. Applying Lebesgue's dominated convergence theorem and integrating term-by-term, we obtain

$$\int_0^{\frac{L}{1+\mathbf{v}}} \phi_x(\mathbf{v}t,t)\ \overline{imc_m e^{2m\pi it/T_{\mathbf{v}}}}dt = \frac{2\pi}{L}\sum_{n\in\ \mathbb{Z}^*} nmc_n\bar{c}_m\int_0^{\frac{L}{1+\mathbf{v}}} e^{2(n-m)\pi it/T_{\mathbf{v}}}dt = \sum_{n\in\ \mathbb{Z}^*} A_{nm}, \tag{2.49}$$

where

$$A_{nm} = \begin{cases} \dfrac{2\pi}{1+\mathbf{v}}\left|mc_m\right|^2 & \text{if } n = m, \\ \dfrac{2nmc_n\bar{c}_m}{i\left(n-m\right)\left(1-\mathbf{v}^2\right)}\left(e^{\pi i(n-m)(1-\mathbf{v})}-1\right) & \text{if } n \neq m. \end{cases}$$

On the other hand, taking $x_b = L$ in (2.38), multiplying by $\overline{imc_m e^{-m\pi i(1+\mathbf{v})} e^{2m\pi it/T_\mathbf{v}}}$ then integrating term-by-term on $(0, L/(1-\mathbf{v}))$, we end up with

$$\int_0^{\frac{L}{1-\mathbf{v}}} \phi_x(L+\mathbf{v}t,t)\, \overline{imc_m e^{-m\pi i(1+\mathbf{v})} e^{2m\pi it/T_\mathbf{v}}} dt = \sum_{n\in \mathbb{Z}^*} B_{nm}, \tag{2.50}$$

where

$$B_{nm} = \begin{cases} \dfrac{2\pi}{1-\mathbf{v}} \left| mc_m \right|^2 & \text{if } n = m, \\ \dfrac{2nmc_n \bar{c}_m}{i(n-m)(1-\mathbf{v}^2)} \left(1 - e^{-(n-m)\pi i(1+\mathbf{v})}\right) & \text{if } n \neq m. \end{cases}$$

Computing $A_{nm} + B_{nm}$ we obtain:

i) If $n = m$, then

$$A_{mm} + B_{mm} = 2\pi \left| mc_m \right|^2 \left(\frac{1}{1+\mathbf{v}} + \frac{1}{1-\mathbf{v}} \right) = \frac{4\pi}{1-\mathbf{v}^2} \left| mc_m \right|^2, \quad m \in \mathbb{Z}^*.$$

ii) If $n \neq m$, then

$$\begin{aligned} A_{nm} + B_{nm} &= \frac{2nmc_n \bar{c}_m}{i(n-m)(1-\mathbf{v}^2)} \left(e^{\pi i(n-m)(1-\mathbf{v})} - e^{-(n-m)\pi i(1+\mathbf{v})} \right) \\ &= \frac{2nmc_n \bar{c}_m}{i(n-m)(1-\mathbf{v}^2)} e^{-\pi i(n-m)(1-\mathbf{v})} \left(e^{(n-m)\pi i(1-\mathbf{v}+1+\mathbf{v})} - 1 \right), \end{aligned}$$

i.e.

$$A_{nm} + B_{nm} = 0 \quad \text{if } n \neq m, \quad n, m \in \mathbb{Z}^*.$$

By consequence, the sum of (2.49) and (2.50) is simply given by

$$\begin{aligned} &\int_0^{\frac{L}{1+\mathbf{v}}} \phi_x(\mathbf{v}t,t) \overline{imc_m e^{2m\pi it/T_\mathbf{v}}} dt \\ &\qquad + \int_0^{\frac{L}{1-\mathbf{v}}} \phi_x(L+\mathbf{v}t,t)\, \overline{imc_m e^{-m\pi i(1+\mathbf{v})} e^{2m\pi it/T_\mathbf{v}}} dt = \frac{4\pi}{1-\mathbf{v}^2} \left| mc_m \right|^2, \end{aligned}$$

for every $m \in \mathbb{Z}^*$. Taking the sum for $m \in \mathbb{Z}^*$, and applying Lebesgue's theorem to interchange summation and integration, it comes that

$$\begin{aligned} &\int_0^{\frac{L}{1+\mathbf{v}}} \phi_x(\mathbf{v}t,t) \left(\sum_{m=-\infty}^{+\infty} \overline{imc_m e^{2m\pi it/T_\mathbf{v}}} \right) dt \\ &\qquad + \int_0^{\frac{L}{1-\mathbf{v}}} \phi_x(L+\mathbf{v}t,t) \left(\sum_{m=-\infty}^{+\infty} \overline{imc_m e^{-m\pi i(1+\mathbf{v})} e^{2m\pi it/T_\mathbf{v}}} \right) dt = \frac{4\pi}{1-\mathbf{v}^2} \sum_{m=-\infty}^{+\infty} \left| mc_m \right|^2. \end{aligned}$$

Thanks to (2.38), we obtain

$$\frac{L}{2\pi} \left(\int_0^{\frac{L}{1+\mathbf{v}}} \phi_x^2(\mathbf{v}t,t) dt + \int_0^{\frac{L}{1-\mathbf{v}}} \phi_x^2(L+\mathbf{v}t,t) dt \right) = \frac{4\pi}{1-\mathbf{v}^2} \sum_{m=-\infty}^{+\infty} \left| mc_m \right|^2.$$

This shows (2.45) when ϕ_x^0 and ϕ^1 are continuous functions.

In the general case, i.e. $\phi^0 \in H_0^1(0,L)$ and $\phi^1 \in L^2(0,L)$, we use an argument of density. Consider two sequences

$$\phi_j^0 \in C^1([0,L]) \text{ and } \phi_j^1 \in C([0,L]), \quad j \in \mathbb{N},$$

such that

$$\left(\phi_j^0\right)_x \to \phi_x^0 \text{ and } \phi_j^1 \to \phi^1 \text{ in } L^2(0,L), \quad \text{as } j \to +\infty.$$

Then, denote by ϕ_j and $\mathcal{E}_j$ the solution and the energy associated with each data ϕ_j^0, ϕ_j^1. Taking into account (2.25) for $t = 0$, and using the precedent step of the proof, we have

$$\int_0^{\frac{L}{1+\mathbf{v}}} \left(\phi_j\right)_x^2 (\mathbf{v}t, t)dt + \int_0^{\frac{L}{1-\mathbf{v}}} \left(\phi_j\right)_x^2 (L + \mathbf{v}t, t)dt = \frac{4}{\left(1-\mathbf{v}^2\right)^2}\mathcal{E}_j(0).$$

Thanks to (2.36), the above inequality holds as $j \to +\infty$ and (2.45) follows as claimed.

Inequality (2.46) is a consequence of Theorem 2.3, it suffices to choose $x_b = \mathbf{v}t$ then $x_b = L + \mathbf{v}t$ in the direct inequality (2.36) and take the sum. The inequality (2.47) holds for $T = \max\left\{\frac{L}{1-\mathbf{v}}, \frac{L}{1+\mathbf{v}}\right\} = \tilde{T}_{\mathbf{v}}$ and therefore for every $T \geq \tilde{T}_{\mathbf{v}}$ as well. ∎

Remark 2.11 *On the time interval* $\left(\frac{L}{1+\mathbf{v}}, \frac{L}{1-\mathbf{v}}\right)$ *we may only observe a fraction of the initial energy of the wave. To see this, consider a wave composed of two disturbances with small initial supports near the two endpoints* $x = \mathbf{v}t$, $x = L + \mathbf{v}t$. *Then, one disturbance reaches the left boundary at a time close to* $\frac{L}{1+\mathbf{v}}$, *and the other one reaches the right boundary only at a time close to* $\frac{L}{1-\mathbf{v}}$, *see Figure 2.8 .*

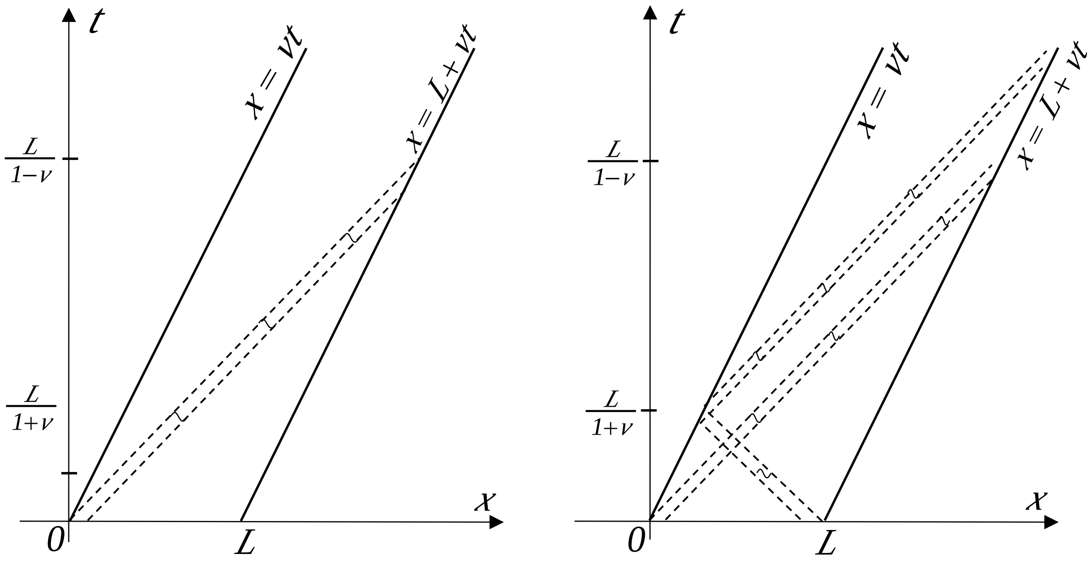

Figure 2.8: Propagation of small disturbances with supports near one or two ends.

2.5.2 *Controllability at both endpoints*

As in Section 2.4.2, it suffices to consider the null controllability Problem at both endpoints. That is to say, for any

$$\left(y^0, y^1\right) \in L^2\left(\mathbf{I}_0\right) \times H^{-1}\left(\mathbf{I}_0\right), \tag{2.51}$$

find two control functions $f_1, f_2 \in L^2(0,T)$, for T large enough, such that the solution of

$$\begin{cases} y_{tt} - y_{xx} = 0, & \text{for } x \in \mathbf{I}_t, t \in (0,T), \\ y(\mathbf{v}t, t) = f_1(t), \quad y(L+\mathbf{v}t, t) = f_2(t), & \text{for } t \in (0,T), \\ y(x,0) = y^0(x), \quad y_t(x,0) = y^1(x), & \text{for } x \in \mathbf{I}_0, \end{cases} \tag{CWP2}$$

satisfies the final conditions

$$y(x,T) = 0, \quad y_t(x,T) = 0, \quad \text{for} \quad x \in \mathbf{I}_T.$$

The next theorem shows that the controllability of (CWP2) holds if $T \geq \tilde{T}_{\mathbf{v}}$.

Theorem 2.6 *Under the Assumptions (2.1) and (2.51), Problem (CWP2) is exactly controllable at the two endpoints $x = \mathbf{v}t, x = L + \mathbf{v}t$ for $T \geq \tilde{T}_{\mathbf{v}}$. Moreover, we can choose two controls f_1, f_2 satisfying*

$$\int_0^T f_1^2(t)\,dt, \quad \int_0^T f_2^2(t)\,dt \leq K_2(\mathbf{v},T)\,\mathcal{E}_{\mathbf{v}}(0), \tag{2.52}$$

where $K_2(\mathbf{v},T)$ is a constant depending on $\mathbf{v}$ and T.

Proof. We argue as in the proof of Theorem 2.4. Let η be the solution of the backward Problem

$$\begin{cases} \eta_{tt} - \eta_{xx} = 0, & \text{in } Q_T, \\ \eta(\mathbf{v}t,t) = -\phi_x(\mathbf{v}t,t), \quad \eta(L+\mathbf{v}t,t) = \phi_x(L+\mathbf{v}t,t), & \text{for } \ t \in (0,T), \\ \eta(x,T) = 0, \quad \eta_t(x,T) = 0. & \text{for } \ x \in \mathbf{I}_T. \end{cases} \tag{2.53}$$

We obtain then a linear map

$$\begin{aligned} \Lambda_2 : H_0^1(\mathbf{I}_0) \times L^2(\mathbf{I}_0) &\longrightarrow H^{-1}(\mathbf{I}_0) \times L^2(\mathbf{I}_0) \\ (\phi^0, \phi^1) &\longmapsto (\eta_t(0), -\eta(0)). \end{aligned}$$

The solution of (2.53), in the transposition sense, satisfies

$$\begin{aligned} &-\left\langle \eta_t(0), \phi^0 \right\rangle_{H_0^1(\mathbf{I}_0)} + \int_{\mathbf{0}}^L \eta(0)\,\phi^1 dx + \mathbf{v}\int_0^T \eta(\mathbf{v}t,t)\,\phi_t(\mathbf{v}t,t)\,dt \\ &- \mathbf{v}\int_0^T \eta(L+\mathbf{v}t,t)\,\phi_t(L+\mathbf{v}t,t)\,dt - \int_0^T \eta(\mathbf{v}t,t)\,\phi_x(\mathbf{v}t,t)\,dt \\ &\qquad\qquad\qquad\qquad + \int_0^T \eta(L+\mathbf{v}t,t)\,\phi_x(L+\mathbf{v}t,t)\,dt = 0. \end{aligned}$$

Taking into account (2.28), we can rewrite (2.53) as

$$0 = -\left\langle \eta_t\left(0\right), \phi^0 \right\rangle_{H_0^1(\mathbf{I}_0)} + \int_{\mathbf{0}}^{L} \eta\left(0\right)\phi^1 dx + \left(1+\mathbf{v}^2\right)\int_0^T \phi_x^2\left(\mathbf{v}t,t\right)dt$$
$$+ \left(1+\mathbf{v}^2\right)\int_0^T \phi_x^2\left(L+\mathbf{v}t,t\right)dt,$$

i.e.,

$$\begin{aligned}\langle \Lambda_2(\phi^0,\phi^1),(\phi^0,\phi^1)\rangle &= \left\langle \eta_t\left(0\right), \phi^0 \right\rangle_{H_0^1(\mathbf{I}_0)} - \int_{\mathbf{0}}^{L} \eta\left(0\right)\phi^1 dx \\ &= \left(1+\mathbf{v}^2\right)\int_0^T \phi_x^2\left(\mathbf{v}t,t\right) + \phi_x^2\left(L+\mathbf{v}t,t\right)dt.\end{aligned}$$

Thanks to Theorem 2.5, we have

$$4\frac{1+\mathbf{v}^2}{\left(1-\mathbf{v}^2\right)^2}\,\mathcal{E}_{\mathbf{v}}\left(0\right) \leq \langle \Lambda_2(\phi^0,\phi^1),(\phi^0,\phi^1)\rangle \leq \left(1+\mathbf{v}^2\right)K_2\left(\mathbf{v},T\right)\mathcal{E}_{\mathbf{v}}\left(0\right),$$

for $T \geq \tilde{T}_{\mathbf{v}}$ and some constant $K_2\left(\mathbf{v},T\right)$. This means that Λ_2 is an isomorphism for $T \geq \tilde{T}_{\mathbf{v}}$ and thus $\left(\phi^0,\phi^1\right)$ can be determined such that the control $f_1 = -\phi_x\left(\mathbf{v}t,t\right)$ and $f_2 = \phi_x\left(L+\mathbf{v}t,t\right)$ drive the solution of (CWP2) from the initial data y^0, y^1 to $y\left(T\right) = y_t\left(T\right) = 0$. ∎

Remark 2.12 *If one considers the Problem of boundary controllability to zero of the wave equation as a stabilization Problem, then the sharp values $T_{\mathbf{v}}$ and $\tilde{T}_{\mathbf{v}}$ for stabilization were mentioned in [25] but not achieved by the different control strategies proposed in that paper.*

3

VIBRATIONS OF AXIALLY TRAVELLING STRINGS DAMPED AT ONE END

In this chapter, we study the small vibrations of an axially travelling string with a dashpot damping at one end. The string is modelled by a wave equation in a time-dependent interval with two endpoints moving at a constant speed $\mathbf{v}$. For the undamped case, we obtain a conserved functional equivalent to the energy of the solution. We derive precise upper and lower exponentially decaying estimates for the energy with explicit constants. These estimates do not seem to be reported in the literature even for the non-travelling case $\mathbf{v} = 0$. Moreover, some numerical examples are given to illustrate the effectiveness of this damper.

3.1 PROBLEM SETTING

We consider small transversal vibrations of a uniform string travelling with a constant speed $\mathbf{v}$ between to two pulleys (inlet and outlet) kept at a fixed distance L. The mechanical setting is sketched in Figure 3.1 where the inlet is fixed while the outlet is allowed to move transversely and attached to a damping device (a dashpoint with a damping factor η).

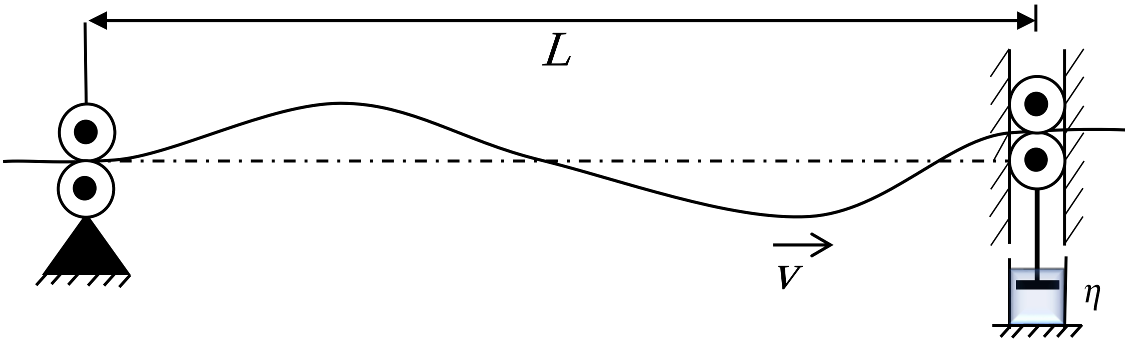

Figure 3.1: An axially travelling string with dash-point at the outlet pulley.

Many mechanical devices with axially moving continua, such as power transmission chains and belts, magnetic tapes, band saws and fibre winders, see for instance [9, 21, 29, 32, 35], are limited in their efficiency and utility due to unwanted vibrations. As a result, stabilization of axially moving systems is necessary to reduce or eliminate these vibrations and improve the overall performance and productivity of these mechanical systems.

Denoting the displacement function by u, depending on the position s along the string and the time τ, with only a dashpoint damping at the outlet, the model can be stated as follows

$$\begin{cases} u_{\tau\tau} + 2\mathbf{v} u_{s\tau} - \left(1-\mathbf{v}^2\right) u_{ss} = 0, & \text{for } s \in (0, L), \tau > 0, \\ u(0,\tau) = 0, & \text{for } \tau > 0, \\ \left(1-\mathbf{v}^2\right) u_s(L,\tau) + (\eta - \mathbf{v}) u_\tau(L,\tau) = 0, & \text{for } \tau > 0, \\ u(s,0) = u^0(s), \quad u_\tau(s,0) = u^1(s), & \text{for } s \in (0, L), \end{cases} \tag{3.1}$$

where the subscripts τ and s stand for the derivatives in time and space variables respectively. The functions u^0 and u^1 represent the initial shape and the initial transverse speed of the string, respectively.

Here, we do not consider the magnitudes of $\mathbf{v}$ and η as small ones. We only assume that $\eta \geq 0$ and that speed $\mathbf{v}$ is strictly less than the speed of propagation of the wave (here normalised to $c = 1$), i.e.

$$0 \leq \mathbf{v} < 1. \tag{3.2}$$

If the speed $\mathbf{v}$ approaches the critical speed $c = 1$, an instability will occur, as shown by [25, 35]. Another key difference with most of the existing works is that we consider the model in a moving space coordinates.

3.1.1 *Derivation of the model*

For the sake of completeness, we now derive the mathematical model (3.1) for the vibration of a travelling string subject to a boundary damping, as described in the Introduction. We refer to [14] for a model that considers more general boundary conditions.

We make the following assumptions:

- The vibrations of the string are transversal and small (compared to its length). This implies in particular that the slope u_s is also small.
- The string is uniform with a mass density ρ.

- The string travels with a constant speed v between to two massless pulleys (inlet and outlet) kept at a fixed distance L.
- The string is perfectly flexible and the effects due to gravity are neglected. This means that the tension $\mathbf{T}$ is constant.
- The inlet is not allowed to move transversely, hence we have the boundary condition

$$u(0,\tau) = 0, \text{ for } \tau \geq 0.$$

We apply the extended Hamilton's principle, over the finite time interval $\tau_1 \leq \tau \leq \tau_2$, in the following form

$$\delta \int_{\tau_1}^{\tau_2} \mathcal{L} d\tau + \int_{\tau_1}^{\tau_2} \delta \mathcal{W} d\tau = 0, \tag{3.3}$$

(see [30] and [5, Chapters 4 and 5]). Here δ denotes the variation in a given function, $\mathcal{L}$ is the Lagrangian and $\mathcal{W}$ is the virtual work performed by nonconservative forces, i.e. the boundary damping force in the present case. The Lagrangian is

$$\mathcal{L} = E_k - E_p,$$

where E_k is the kinetic energy and E_p is the potential energy of the string. The first one is given by

$$E_k = \frac{1}{2} \int_0^L \rho \left(\frac{du}{d\tau} \right)^2 ds$$

where $\frac{du}{d\tau}(s,\tau) = u_\tau(s,\tau) + v u_s(s,\tau)$ is the material velocity (also called the total derivative of u). The potential energy E_p is equal to the work done to deform the string from its rest position. Hence, the potential energy per unit length of a differential element is given by

$$dE_p = \mathbf{T} \cdot \text{elongation} = \mathbf{T} \left(\sqrt{(ds)^2 + (du)^2} - ds \right).$$

Since the slope of the string u_s is small, then

$$dE_p = \mathbf{T} \left(\sqrt{1 + (u_s)^2} - 1 \right) ds \approx \frac{\mathbf{T}}{2} u_s^2 ds,$$

hence the potential energy of the string is given by

$$E_p = \frac{\mathbf{T}}{2} \int_0^L u_s^2 ds$$

and therefore

$$\mathcal{L} = \frac{\rho}{2} \int_0^L (u_\tau + \mathbf{v} u_s)^2 ds - \frac{\mathbf{T}}{2} \int_0^L u_s^2 ds.$$

The damping force $\mathbf{F}_\eta$ of the dashpot at $s = L$ is contrary to the sense of movement and thus the work done by the dashpot is

$$\mathcal{W} = -\mathbf{F}_\eta \cdot \text{displacement} = -\mathbf{F}_\eta \, u\left(L, \tau\right).$$

Now, we are led to search the critical points of the functional

$$I\left(u\right) := \frac{1}{2}\int_{\tau_1}^{\tau_2}\int_0^L \rho\left(u_\tau + \mathbf{v}u_s\right)^2 - \mathbf{T}u_s^2 \, ds d\tau - \int_{\tau_1}^{\tau_2} \mathbf{F}_\eta u\left(L,\tau\right) d\tau. \tag{3.4}$$

When u is a critical point, we should have

$$\lim_{\theta\to 0} \frac{I\left(u+\theta v\right) - I\left(u\right)}{\theta} = 0, \tag{3.5}$$

where v is a smooth function such that

$$v\left(\cdot, \tau_1\right) = v\left(\cdot,\tau_2\right) = 0 \quad \text{and} \quad v\left(0,\cdot\right) = 0. \tag{3.6}$$

The function $u + \theta v$ is a small perturbation of u that does not affect the values of (the path) u at $\tau = \tau_1$ and $\tau = \tau_2$, as well as at $s = 0$. The limit (3.5) equals

$$\int_{\tau_1}^{\tau_2}\int_0^L \rho\left(\mathbf{v}^2 u_s v_s + \mathbf{v}u_\tau v_s + \mathbf{v}u_s v_\tau + u_\tau v_\tau\right) - \mathbf{T}u_s v_s ds d\tau - \int_{\tau_1}^{\tau_2} \mathbf{F}_\eta v\left(L,\tau\right) d\tau = 0.$$

Integrating by parts, we infer that

$$-\int_{\tau_1}^{\tau_2}\int_0^L \left\{\rho\left(u_{\tau\tau} + 2\mathbf{v}u_{s\tau} + \mathbf{v}^2 u_{ss}\right) - \mathbf{T}u_{ss}\right\} v ds d\tau$$
$$+ \int_{\tau_1}^{\tau_2} \left\{\rho\mathbf{v}u_\tau\left(L,\tau\right) + \left(\rho\mathbf{v}^2 - \mathbf{T}\right)u_s\left(L,\tau\right) - \mathbf{F}_\eta\right\} v\left(L,\tau\right) d\tau = 0.$$

The other boundary terms vanishes due to (3.6). Since v can be chosen arbitrary, and in particular its value at $s = L$, the first integral implies that

$$u_{\tau\tau} + 2\mathbf{v}u_{s\tau} + \left(\mathbf{v}^2 - \frac{\mathbf{T}}{\rho}\right)u_{ss} = 0, \quad \text{for } s \in (0, L) \text{ and } \tau_1 \le \tau \le \tau_2.$$

The quantity $c = \sqrt{\mathbf{T}/\rho}$ is the wave speed in the string.

The second integral implies that

$$\rho\mathbf{v}u_\tau\left(L,\tau\right) + \left(\mathbf{T} - \rho\mathbf{v}^2\right)u_s\left(L,\tau\right) - \mathbf{F}_\eta = 0, \quad \text{for } \tau_1 \le \tau \le \tau_2.$$

Assuming that the damping force $\mathbf{F}_\eta$ depends linearly on the velocity u_τ, i.e. $\mathbf{F}_\eta = \eta u_\tau$, we obtain the boundary condition

$$\left(\eta - \rho\mathbf{v}\right)u_\tau\left(L,\tau\right) + \left(\mathbf{T} - \rho\mathbf{v}^2\right)u_s\left(L,\tau\right) = 0, \quad \text{for } \tau_1 \le \tau \le \tau_2.$$

Rescaling the variables τ,v and η as follows

$$\tau \mapsto \tau\sqrt{\frac{\mathbf{T}}{\rho}}, \quad \mathbf{v} \mapsto \mathbf{v}\sqrt{\frac{\rho}{\mathbf{T}}} \text{ and } \eta \mapsto \frac{\eta}{\sqrt{\rho\mathbf{T}}},$$

we obtain the model (3.1) with a normalised speed $c = 1$.

3.1.2 *Governing equation in a time-dependent interval*

We introduce the variables

$$s = L - x + \mathbf{v}t \quad \text{and} \quad \tau = t,$$

hence

$$x \in \mathbf{I}_t := (\mathbf{v}t, L + \mathbf{v}t)\,, \text{ for } t \geq 0,$$

which is an interval travelling in the positive sense of the real axis (as in [15, 32]). It follows that

$$\partial_s = -\partial_x \quad \text{and} \quad \partial_\tau = \mathbf{v}\partial_x + \partial_t.$$

Rewriting Problem (3.1) in the new coordinates, we obtain the following (pure) wave equation with a damping at the moving boundary $x = \mathbf{v}t$,

$$\begin{cases} \phi_{tt} - \phi_{xx} = 0, & \text{for } x \in \mathbf{I}_t, t > 0, \\ (1 - \eta\mathbf{v})\,\phi_x\,(\mathbf{v}t, t) - (\eta - \mathbf{v})\,\phi_t\,(\mathbf{v}t, t) = 0, & \text{for } t > 0, \\ \phi\,(L + \mathbf{v}t, t) = 0, & \text{for } t > 0, \\ \phi(x, 0) = \phi^0\,(x)\,,\ \phi_t\,(x, 0) = \phi^1\,(x)\,, & \text{for } x \in \mathbf{I}_0, \end{cases} \tag{DWP_1}$$

where $\phi^0 = u^0$ and $\phi^1 = u^1 - \mathbf{v}\phi^0_x$.

Let us denote

$$\gamma_{\mathbf{v}} := \frac{1 + \mathbf{v}}{1 - \mathbf{v}}, \quad \gamma_\eta := \frac{1 + \eta}{1 - \eta} \quad \text{and} \quad \omega_n := \begin{cases} \frac{(2n+1)}{2} i\pi - \frac{1}{2} \ln \gamma_\eta, & \text{if } 0 \leq \eta < 1, \\ ni\pi - \frac{1}{2} \ln \left|\gamma_\eta\right|, & \text{if } \eta > 1. \end{cases}$$

Note that

- $\gamma_{\mathbf{v}} \geq 1$ for $0 \leq \mathbf{v} < 1$,
- $\left|\gamma_\eta\right| \geq 1$ and hence the real part of ω_n remains negative, for $\eta > 0$ with $\eta \neq 1$.

3.2 EXACT SOLUTION

As a first result, we derive a closed form for the solution of (DWP_1) given by the series formulas

$$\phi(x, t) = \sum_{n \in \mathbb{Z}} a_n \left(\gamma_\eta e^{\frac{1-\mathbf{v}}{L}\omega_n(t+x)} + e^{\frac{1+\mathbf{v}}{L}\omega_n(t-x)} \right), \quad \text{for } x \in \mathbf{I}_t \text{ and } t \geq 0, \tag{3.7}$$

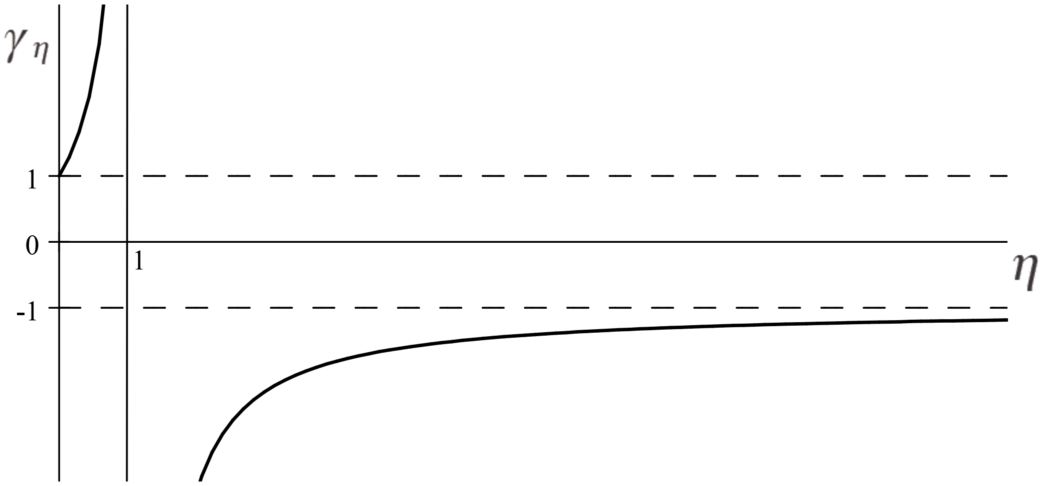

Figure 3.2: Variation of $\gamma_\eta = \frac{1+\eta}{1-\eta}$ in function of $\eta \geq 0$.

which is a sum of two waves travelling in opposite directions. The coefficients a_n are explicitly given in function of the initial data ϕ^0 and ϕ^1. To this end, we need to know the functions ϕ^0 and ϕ^1 on an interval larger than $\mathbf{I}_0 = (0, L)$. As in [15, 44], we introduce

$$L_2 := \frac{2L}{1-\mathbf{v}}$$

and extend ϕ to the interval $(L + \mathbf{v}t, L_2 + \mathbf{v}t)$ by setting

$$\tilde{\phi}(x,t) = \begin{cases} \phi\,(x,t)\,, & x \in (\mathbf{v}t, L + \mathbf{v}t)\,, \\ -\phi\left(\frac{1}{\gamma_{\mathbf{v}}}(\mathbf{v}t - x) + \frac{2L}{1+\mathbf{v}} + \mathbf{v}t, t\right), & x \in (L + \mathbf{v}t, L_2 + \mathbf{v}t)\,. \end{cases} \tag{3.8}$$

The obtained function is well defined since the first variable of ϕ remains in the interval $(\mathbf{v}t, L + \mathbf{v}t)$. In particular, the homogeneous boundary condition $\phi\,(L + \mathbf{v}t, t) = 0$ remains satisfied, for every $t \geq 0$.

Remark 3.1 *Clearly,* $0 < L \leq L_2/2$ *for* $0 \leq \mathbf{v} < 1$. *If* $\mathbf{v} = 0$, *then* $L_2 = 2L$ *and the function* $\tilde{\phi}$ *on* $(L, 2L)$ *is an odd function on* $(0, 2L)$, *with respect to* $x = L$. *If* $0 < \mathbf{v} < 1$, *then* ϕ *is extended as an odd function with an extra dilatation on the added interval* $(L + \mathbf{v}t, L_2 + \mathbf{v}t)$, *see Figure 3.3.*

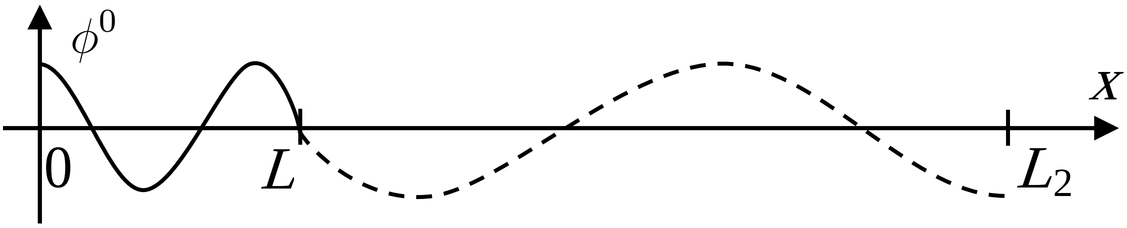

Figure 3.3: Example of the extension of an initial data ϕ^0 when $0 < \mathbf{v} < 1$.

Taking the derivative of (3.8) with respect to x, we obtain

$$\tilde{\phi}_x(x,t) = \begin{cases} \phi_x\left(x,t\right), & x \in \left(\mathbf{v}t, L+\mathbf{v}t\right), \\ \frac{1}{\gamma_{\mathbf{v}}}\phi_x\left(\frac{1}{\gamma_{\mathbf{v}}}\left(\mathbf{v}t - x\right) + \frac{2L}{1+\mathbf{v}} + \mathbf{v}t, t\right), & x \in \left(L+\mathbf{v}t, L_2+\mathbf{v}t\right). \end{cases} \tag{3.9}$$

On the other hand, the time derivative is extended as follows

$$\tilde{\phi}_t(x,t) = \begin{cases} \phi_t\left(x,t\right), & x \in \left(\mathbf{v}t, L+\mathbf{v}t\right), \\ \frac{-1}{\gamma_{\mathbf{v}}}\phi_t\left(\frac{1}{\gamma_{\mathbf{v}}}\left(\mathbf{v}t - x\right) + \frac{2L}{1+\mathbf{v}} + \mathbf{v}t, t\right), & x \in \left(L+\mathbf{v}t, L_2+\mathbf{v}t\right). \end{cases} \tag{3.10}$$

Let us introduce the following family of Hilbert spaces

$$\mathcal{H}_{L+\mathbf{v}t}\left(\mathbf{I}_t\right) := \left\{w \in H^1\left(\mathbf{I}_t\right), w\left(L+\mathbf{v}t\right) = 0\right\}, \quad \text{for } t \geq 0,$$

where $H^1\left(\mathbf{I}_t\right)$ is the Sobolev space defined on $\mathbf{I}_t$. We assume that the initial data satisfies

$$\phi^0 \in \mathcal{H}_L\left(\mathbf{I}_0\right), \quad \phi^1 \in L^2\left(\mathbf{I}_0\right). \tag{3.11}$$

Now, we are ready to state the following existence and uniqueness result for Problem (DWP_1).

Theorem 3.1 *Let $T > 0$. Under the assumptions (3.2) and (3.11), the solution of Problem (DWP_1)*

$$\phi \in C\left(\left[0,T\right]; \mathcal{H}_{L+\mathbf{v}t}\left(\mathbf{I}_t\right)\right) \cap C^1\left(\left[0,T\right]; L^2\left(\mathbf{I}_t\right)\right), \tag{3.12}$$

is given by the series (3.7) where the coefficients $a_n \in \mathbb{C}$ are computed by the following formula

$$a_n = \frac{1}{4\gamma_\eta \omega_n} \int_0^{L_2} \left(\tilde{\phi}^0_x + \tilde{\phi}^1\right) e^{-\frac{1-\mathbf{v}}{L}\omega_n x} dx, \quad \textit{for } n \in \mathbb{Z}. \tag{3.13}$$

Moreover, we have

$$\sum_{n\in\mathbb{Z}} \left|\omega_n a_n\right|^2 = \frac{L}{8\left(1-\mathbf{v}\right)\gamma_\eta^2} \int_{\mathbf{v}t}^{L_2+\mathbf{v}t} e^{\frac{1-\mathbf{v}}{L}\ln\left|\gamma_\eta\right|\left(t+x\right)} \left(\tilde{\phi}_x + \tilde{\phi}_t\right)^2 dx < +\infty, \tag{3.14}$$

where $\tilde{\phi}^0_x$ and $\tilde{\phi}^1$ are extensions of the initial data $\phi^0$and ϕ^1 on the interval $\left(0, L_2\right)$ given above by (3.9) and (3.10) respectively.

Proof. The general solution of (DWP_1) is given by D'Alembert's formula

$$\phi(x,t) = f(t+x) + g\left(t-x\right),$$

where f and g are arbitrary continuous functions. Let us check the boundary conditions. On one hand, at the left endpoint we have

$$\left(1-\eta\mathbf{v}\right)f'(\left(1+\mathbf{v}\right)t) - g'\left(\left(1-\mathbf{v}\right)t\right) = \left(\eta - \mathbf{v}\right)\left(f'(\left(1+\mathbf{v}\right)t) + g'\left(\left(1-\mathbf{v}\right)t\right)\right).$$

Setting $z = (1 - \mathbf{v})\, t$, we obtain

$$(1 - \eta\mathbf{v} - \eta + \mathbf{v})\, f'\left(\gamma_{\mathbf{v}} z\right) = (1 - \eta\mathbf{v} + \eta - \mathbf{v})\, g'\left(z\right). \tag{3.15}$$

On the other hand, at the right endpoint, we infer that

$$f((1 + \mathbf{v})\, t + L) + g\left((1 - \mathbf{v})\, t - L\right) = 0.$$

Denoting $y = z - L$, we obtain

$$f(\gamma_{\mathbf{v}} y + \frac{2L}{1 - \mathbf{v}}) = -g\left(y\right). \tag{3.16}$$

Then, taking the derivative with respect to y, we get

$$\gamma_{\mathbf{v}} f'(\gamma_{\mathbf{v}} y + \frac{2L}{1 - \mathbf{v}}) = -g'\left(y\right). \tag{3.17}$$

Thanks to (3.15), (3.17) and taking $\xi = \gamma_{\mathbf{v}} y$, we deduce that f' satisfies

$$(1 - \eta\mathbf{v} + \eta - \mathbf{v})\, \gamma_{\mathbf{v}} f'\left(\xi + \frac{2L}{1 - \mathbf{v}}\right) = -\left(1 - \eta\mathbf{v} - \eta + \mathbf{v}\right) f'\left(\xi\right). \tag{3.18}$$

After few simplifications, this can be written as

$$f'\left(\xi + \frac{2L}{1 - \mathbf{v}}\right) = -\frac{f'\left(\xi\right)}{\gamma_\eta}. \tag{3.19}$$

This formula suggests that $f'(\xi) = e^{\beta\xi}$, for some parameter β to be determined later. Substituting the form $e^{\beta\xi}$ in (3.19), we infer that

$$e^{\beta\left(\xi + \frac{2L}{1-\mathbf{v}}\right)} = -e^{\beta\xi}/\gamma_\eta.$$

Then, it follows that:

- If $0 \leq \eta < 1$, then $\gamma_\eta \geq 1$ and we get

$$e^{\beta\left(\xi + \frac{2L}{1-\mathbf{v}}\right)} = -e^{-\ln\gamma_\eta} e^{\beta\xi}.$$

Solving this equation for β, we obtain a sequence of values $\beta_n = (1 - \mathbf{v})\, \omega_n / L, n \in \mathbb{Z}$, where

$$\omega_n = \frac{1}{2}\left(2n + 1\right) i\pi - \frac{1}{2} \ln\gamma_\eta.$$

- If $\eta > 1$, we have $\gamma_\eta < -1$ and we obtain another sequence of values $\beta_n = (1 - \mathbf{v})\, \omega_n / L, n \in \mathbb{Z}$, where this time

$$\omega_n = ni\pi - \frac{1}{2} \ln\left|\gamma_\eta\right|.$$

Note that in both cases we have $\ln\left|\gamma_\eta\right| \geq 1$ and the real part of ω_n is negative.

Due to the superposition principle, it follows that f' can be written as

$$f'(\xi) = \sum_{n\in\mathbb{Z}} c_n e^{\frac{1-\mathbf{v}}{L}\omega_n \xi}, \qquad c_n \in \mathbb{C},$$

where c_n are complex coefficients to be determined later. After integration, the function f can be written as

$$f(\xi) = c + \sum_{n\in\mathbb{Z}} \frac{Lc_n}{(1-\mathbf{v})\,\omega_n} e^{\frac{1-\mathbf{v}}{L}\omega_n \xi},$$

for some constant c. Using (3.16), we deduce that

$$g(\xi) = -c + \sum_{n\in\mathbb{Z}} \frac{Lc_n}{(1-\ \mathbf{v})\,\gamma_\eta \omega_n} e^{\frac{1+\mathbf{v}}{L}\omega_n \xi}, \tag{3.20}$$

where we have used the fact that $e^{2\omega_n} = -1/\gamma_\eta$ whether $0 \leq \eta < 1$ or $\eta > 1$.

Thanks to D'Alembert's formula, the solution of Problem (DWP_1) is given by the series

$$\phi(x,t) = \sum_{n\in\mathbb{Z}} \frac{L}{(1-\mathbf{v})\,\gamma_\eta \omega_n} c_n \left(\gamma_\eta e^{\frac{1-\mathbf{v}}{L}\omega_n(t+x)} + e^{\frac{1+\mathbf{v}}{L}\omega_n(t-x)}\right). \tag{3.21}$$

To obtain (3.7), we set

$$a_n := \frac{L}{(1-\mathbf{v})\,\gamma_\eta \omega_n} c_n. \tag{3.22}$$

The coefficient c_n are determined as follows. Going back to (3.21), we infer that

$$\phi_x(x,t) = \sum_{n\in\mathbb{Z}} c_n \left(e^{\frac{1-\mathbf{v}}{L}\omega_n(t+x)} - \frac{\gamma_{\mathbf{v}}}{\gamma_\eta} e^{\frac{1+\mathbf{v}}{L}\omega_n(t-x)}\right), \tag{3.23}$$

$$\phi_t(x,t) = \sum_{n\in\mathbb{Z}} c_n \left(e^{\frac{1-\mathbf{v}}{L}\omega_n(t+x)} + \frac{\gamma_{\mathbf{v}}}{\gamma_\eta} e^{\frac{1+\mathbf{v}}{L}\omega_n\ (t-x)}\right), \tag{3.24}$$

for $x \in (\mathbf{v}t, L+\mathbf{v}t)$ and $t \geq 0$. It follows from (3.9) and (3.10) that the extensions $\tilde{\phi}_x$ and $\tilde{\phi}_t$ are given by

$$\tilde{\phi}_x(x,t) = \begin{cases} \displaystyle\sum_{n\in\mathbb{Z}} c_n \left(e^{\frac{1-\mathbf{v}}{L}\omega_n(t+x)} - \frac{\gamma_{\mathbf{v}}}{\gamma_\eta} e^{\frac{1+\mathbf{v}}{L}\omega_n(t-x)}\right), & \text{if } x \in (\mathbf{v}t, L+\mathbf{v}t), \\ \displaystyle\frac{1}{\gamma_{\mathbf{v}}} \sum_{n\in\mathbb{Z}} c_n \left(e^{\frac{1-\mathbf{v}}{L}\omega_n((1+\mathbf{v})t + \frac{\mathbf{v}t-x}{\gamma_{\mathbf{v}}} + \frac{2L}{1+\mathbf{v}})}\right. & \\ \displaystyle\qquad\qquad \left. - \frac{\gamma_{\mathbf{v}}}{\gamma_\eta}\ e^{\frac{1+\mathbf{v}}{L}\omega_n((1-\mathbf{v})t - \frac{\mathbf{v}t-x}{\gamma_{\mathbf{v}}} - \frac{2L}{1+\mathbf{v}})}\right), & \text{if } x \in (L+\mathbf{v}t, L_2+\mathbf{v}t), \end{cases} \tag{3.25}$$

$$\tilde{\phi}_t(x,t) = \begin{cases} \displaystyle\sum_{n\in\mathbb{Z}} c_n \left(e^{\frac{1-\mathbf{v}}{L}\omega_n(t+x)} + \frac{\gamma_{\mathbf{v}}}{\gamma_\eta} e^{\frac{1+\mathbf{v}}{L}\omega_n(t-x)}\right), & \text{if } x \in (\mathbf{v}t, L+\mathbf{v}t), \\ \displaystyle\frac{-1}{\gamma_{\mathbf{v}}} \sum_{n\in\mathbb{Z}} c_n \left(e^{\frac{1-\mathbf{v}}{L}\omega_n((1+\mathbf{v})t + \frac{\mathbf{v}t-x}{\gamma_{\mathbf{v}}} + \frac{2L}{1+\mathbf{v}})}\right. & \\ \displaystyle\qquad\qquad \left. + \frac{\gamma_{\mathbf{v}}}{\gamma_\eta}\ e^{\frac{1+\mathbf{v}}{L}\omega_n((1-\mathbf{v})t - \frac{\mathbf{v}t-x}{\gamma_{\mathbf{v}}} - \frac{2L}{1+\mathbf{v}})}\right), & \text{if } x \in (L+\mathbf{v}t, L_2+\mathbf{v}t). \end{cases} \tag{3.26}$$

Taking the sum of (3.25) and (3.26), we get

$$\tilde{\phi}_x + \tilde{\phi}_t = \begin{cases} 2\displaystyle\sum_{n\in\mathbb{Z}} c_n e^{\frac{1-\mathbf{v}}{L}\omega_n(t+x)}, & x \in (\mathbf{v}t, L+\mathbf{v}t)\,, \\ \dfrac{-2}{\gamma_{\mathbf{v}}} \displaystyle\sum_{n\in\mathbb{Z}} c_n \dfrac{\gamma_{\mathbf{v}}}{\gamma_\eta} e^{-2\omega_n} e^{\frac{1-\mathbf{v}}{L}\omega_n(t+x)}, & x \in (L+\mathbf{v}t, L_2+\ \mathbf{v}t)\,. \end{cases}$$

Since $e^{-2\omega_n} = -\gamma_\eta$, then we have the unified expression

$$\tilde{\phi}_x + \tilde{\phi}_t = 2\sum_{n\in\mathbb{Z}} c_n e^{\frac{1-\mathbf{v}}{L}\omega_n(t+x)}, \quad \text{for } x \in (\mathbf{v}t, L_2+\mathbf{v}t) \text{ and } t \geq 0. \tag{3.27}$$

Using the definition of ω_n, we get

$$\tilde{\phi}_x + \tilde{\phi}_t = \begin{cases} 2e^{\frac{1-\mathbf{v}}{2L}\left(i\pi - \ln\gamma_\eta\right)(t+x)} \displaystyle\sum_{n\in\mathbb{Z}} c_n e^{\frac{1-\mathbf{v}}{L}ni\pi(t+x)}, & \text{if } 0 \leq \eta < 1, \\ 2e^{-\frac{1-\mathbf{v}}{2L}\ln\left|\gamma_\eta\right|(t+x)} \displaystyle\sum_{n\in\mathbb{Z}} c_n e^{\frac{1-\mathbf{v}}{L}ni\pi(t+x)}, & \text{if } \eta > 1. \end{cases} \tag{3.28}$$

Taking into account that $\left\{e^{\frac{n\pi i(1-\mathbf{v})}{L}(t+x)}/\sqrt{L_2}\right\}_{n\in\ \mathbb{Z}}$ is an orthonormal basis for $L^2\left(\mathbf{v}t, L_2+\mathbf{v}t\right)$, for every $t \geq 0$ (see Appendix A), we rewrite (3.27) as

$$\sum_{n\in\mathbb{Z}} c_n \frac{e^{\frac{1-\mathbf{v}}{L}ni\pi(t+x)}}{\sqrt{L_2}} = \begin{cases} \dfrac{1}{2\sqrt{L_2}} e^{-\frac{1-\mathbf{v}}{2L}\left(i\pi - \ln\gamma_\eta\right)(t+x)} \left(\tilde{\phi}_x + \tilde{\phi}_t\right), & \text{if } 0 \leq \eta < 1, \\ \dfrac{1}{2\sqrt{L_2}} e^{\frac{1-\mathbf{v}}{2L}\ln\left|\gamma_\eta\right|(t+x)} \left(\tilde{\phi}_x + \tilde{\phi}_t\right), & \text{if } \eta > 1. \end{cases} \tag{3.29}$$

By consequence,

$$c_n = \begin{cases} \dfrac{1}{2L_2} \displaystyle\int_{\mathbf{v}t}^{L_2+\mathbf{v}t} e^{-\frac{1-\mathbf{v}}{2L}\left(i\pi - \ln\gamma_\eta\right)(t+x)} \left(\tilde{\phi}_x + \tilde{\phi}_t\right) e^{-\frac{n\pi i(1-\mathbf{v})}{L}(t+x)} dx, & \text{if } 0 \leq \eta < 1, \\ \dfrac{1}{2L_2} \displaystyle\int_{\mathbf{v}t}^{L_2+\mathbf{v}t} e^{\frac{1-\mathbf{v}}{2L}\ln\left|\gamma_\eta\right|(t+x)} \left(\tilde{\phi}_x + \tilde{\phi}_t\right) e^{-\frac{n\pi i(1-\mathbf{v})}{L}(t+x)} dx, & \text{if } \eta > 1, \end{cases}$$

for $n \in \mathbb{Z}$. Whether $0 \leq \eta < 1$ or $\eta > 1$, in both cases, we have

$$c_n = \frac{1}{2L_2} \int_{\mathbf{v}t}^{L_2+\mathbf{v}t} \left(\tilde{\phi}_x + \tilde{\phi}_t\right) e^{-\frac{(1-\mathbf{v})}{L}\omega_n(t+x)} dx, \quad \text{for } n \in \mathbb{Z}.$$

For $t = 0$, and tacking (3.22) into account, we obtain (3.13) as claimed.

Moreover, as a consequence of Parseval's equality, it comes that

$$\sum_{n\in\mathbb{Z}} |c_n|^2 = \begin{cases} \dfrac{1}{4L_2} \displaystyle\int_{\mathbf{v}t}^{L_2+\mathbf{v}t} \left|e^{-\frac{1-\mathbf{v}}{2L}\left(i\pi - \ln\gamma_\eta\right)(t+x)}\right|^2 \left(\tilde{\phi}_x + \tilde{\phi}_t\right)^2 dx, & \text{if } 0 \leq \eta < 1, \\ \dfrac{1}{4L_2} \displaystyle\int_{\mathbf{v}t}^{L_2+\mathbf{v}t} \left|e^{\frac{1-\mathbf{v}}{2L}\ln\left|\gamma_\eta\right|(t+x)}\right|^2 \left(\tilde{\phi}_x + \tilde{\phi}_t\right)^2 dx, & \text{if } \eta > 1. \end{cases}$$

Whether $0 \leq \eta < 1$ or $\eta > 1$, it follows that

$$\sum_{n\in\mathbb{Z}} |\omega_n a_n|^2 = \frac{L^2}{\gamma_\eta^2 (1-\mathbf{v})^2} \sum_{n\in\mathbb{Z}} |c_n|^2 = \frac{L}{8\gamma_\eta^2 (1-\mathbf{v})} \int\limits_{\mathbf{v}t}^{L_2+\mathbf{v}t} e^{\frac{1-\mathbf{v}}{L}\ln|\gamma_\eta|(t+x)} \left(\tilde{\phi}_x + \tilde{\phi}_t\right)^2 dx.$$

Thanks to (3.11), ϕ^0 and ϕ^1 belongs to $L^2(0, L_2)$. Thus the integral at the right-hand side for $t = 0$ is finite and

$$\sum_{n\in\mathbb{Z}} |\omega_n a_n|^2 < +\infty.$$

Recalling that $|\omega_n|^2 = O\left(n^2\right)$, for large values of n, then

$$\sum_{n\in\mathbb{Z}} |n a_n|^2 < +\infty. \tag{3.30}$$

Let $T > 0$ and $t \in [0, T]$. Due to the continuity of the exponential function, we get

$$\left| a_n \left(\gamma_\eta e^{\frac{1-\mathbf{v}}{L}\omega_n(t+x)} + e^{\frac{1+\mathbf{v}}{L}\omega_n(t-x)} \right) \right| \leq C_T |a_n|,$$

where C_T is a constant depending only on v, η, L and T.

Going back to (3.23), (3.24) and due to (3.22), we can check that

$$\left| c_n \left(e^{\frac{1-\mathbf{v}}{L}\omega_n(t+x)} \pm \frac{\gamma_\mathbf{v}}{\gamma_\eta} e^{\frac{1+\mathbf{v}}{L}\omega_n(t-x)} \right) \right| \leq C'_T |n a_n|,$$

for some constant C'_T.

Taking (3.30) into account, we infer that $\phi(x,t)$, $\phi_x(x,t)$ and $\phi_t(x,t)$ belong to $L^2(\mathbf{I}_t)$, for $t \geq 0$. In particular, $\phi(x,t) \in \mathcal{H}_{L+\mathbf{v}t}(\mathbf{I}_t)$, for $t \geq 0$. The continuity in time of ϕ and ϕ_t as functions of t with values in $\mathcal{H}_{L+\mathbf{v}t}(\mathbf{I}_t)$ and $L^2(\mathbf{I}_t)$, respectively, follows as they are the sums of uniformly converging series of continuous functions. This shows (3.12). ∎

Next, we demonstrate how the series formulas (3.7) can be used to achieve the following results:

- *For the undamped case, i.e.* $\eta = 0$, *we show that the functional*[1]

  $$\mathcal{E}_\mathbf{v}(t) = \frac{1}{2} \int\limits_{\mathbf{v}t}^{L+\mathbf{v}t} \left(\phi_t + \mathbf{v}\phi_x\right)^2 + \left(1-\mathbf{v}^2\right)\phi_x^2 dx, \quad \text{for } t \geq 0, \tag{3.31}$$

  depending on L, t, $\mathbf{v}$ and the solution ϕ of (DWP_1), is conserved in time. See Theorem 3.2. Note that under the assumption (3.2), the functional $\mathcal{E}_\mathbf{v}$ is positive-definite and we will call it the "energy" of the solution ϕ.

---

[1] Here and in the sequel, the subscript $\mathbf{v}$ is used to emphasize the dependence on the speed $\mathbf{v}$.

- *For the damped case* $\eta > 0$ *with* $\eta \neq 1$, the (usual) energy

$$E_{\mathbf{v}}(t) = \frac{1}{2} \int\limits_{\mathbf{v}t}^{L+\mathbf{v}t} \phi_t^2(x,t) + \phi_x^2(x,t)\, dx, \quad \text{for } t \geq 0, \tag{3.32}$$

  depending on L, t, $\mathbf{v}$ and the solution of (DWP_1), satisfies

$$\frac{1}{\gamma_\eta^2 \gamma_{\mathbf{v}}} E_{\mathbf{v}}(0)\, e^{-\frac{1-\mathbf{v}^2}{L} \ln|\gamma_\eta| t} \leq E_{\mathbf{v}}(t) \leq \gamma_\eta^2 \gamma_{\mathbf{v}} E_{\mathbf{v}}(0)\, e^{-\frac{1-\mathbf{v}^2}{L} \ln|\gamma_\eta| t}, \quad \text{for } t \geq 0. \tag{3.33}$$

3.3 A CONSERVED QUANTITY FOR THE STRING WITH NO DAMPER

For the undamped case, i.e. $\eta = 0$ in Problem (DWP_1), we show that the energy $\mathcal{E}_{\mathbf{v}}$ given by (3.31) is conserved in time.

Theorem 3.2 *Under Assumptions* (3.2) *and* (3.11), *the solution of Problem* (DWP_1) *satisfies*

$$\mathcal{E}_{\mathbf{v}}(t) = \frac{\pi^2 (1-\mathbf{v}^2)}{2L} \sum_{n\in \mathbb{Z}} |(2n+1)\, a_n|^2, \quad \text{for } t \geq 0, \tag{3.34}$$

where the left-hand side is independent of t.

Proof. If $\eta = 0$, then $\omega_n = (2n+1)\, i\pi/2$ and the identity (3.14) becomes

$$\frac{1}{1-\mathbf{v}} \int\limits_{\mathbf{v}t}^{L_2+\mathbf{v}t} (\tilde{\phi}_x + \tilde{\phi}_t)^2\, dx = \frac{2\pi^2}{L} \sum_{n\in \mathbb{Z}} |(2n+1)\, a_n|^2. \tag{3.35}$$

Using the extensions (3.9), (3.10) and considering the change of variable

$$x = \gamma_{\mathbf{v}}(\mathbf{v}t - \xi) + \frac{2L}{1-\mathbf{v}} + \mathbf{v}t,$$

in $(L+\mathbf{v}t, L_2+\mathbf{v}t)$, then we have

$$\frac{1}{1-\mathbf{v}} \int\limits_{L+\mathbf{v}t}^{L_2+\mathbf{v}t} (\tilde{\phi}_x(x,t) + \tilde{\phi}_t(x,t))^2\, dx = \frac{1}{1+\mathbf{v}} \int\limits_{\mathbf{v}t}^{L+\mathbf{v}t} (\phi_x(\xi,t) - \phi_t(\xi,t))^2\, d\xi.$$

Taking (3.35) into account, it comes that

$$\begin{aligned} \frac{1}{1-\mathbf{v}} \int\limits_{\mathbf{v}t}^{L_2+\mathbf{v}t} (\tilde{\phi}_x + \tilde{\phi}_t)^2\, dx &= \frac{1}{1-\mathbf{v}} \int\limits_{\mathbf{v}t}^{L+\mathbf{v}t} (\phi_t + \phi_x)^2\, dx + \frac{1}{1+\mathbf{v}} \int\limits_{\mathbf{v}t}^{L+\mathbf{v}t} (\phi_x - \phi_t)^2\, dx \\ &= \frac{2\pi^2}{L} \sum_{n\in \mathbb{Z}} |(2n+1)\, a_n|^2. \end{aligned}$$

Expanding $(\phi_x \pm \phi_t)^2$ and collecting similar terms, we get

$$\frac{1}{1-\mathbf{v}^2} \left(\int\limits_{\mathbf{v}t}^{L+\mathbf{v}t} \phi_x^2 + \phi_t^2 + 2\mathbf{v}\phi_x\phi_t dx \right) = \frac{\pi^2}{L} \sum_{n\in \mathbb{Z}} |(2n+1)\, a_n|^2, \quad \text{for } t \geq 0. \tag{3.36}$$

The left-hand side is equal to $2\mathcal{E}_{\mathbf{v}}(t)/\left(1-\mathbf{v}^2\right)$ and (3.34) follows. ■

Let us check that $\frac{d}{dt}\mathcal{E}_{\mathbf{v}}(t)=0$ by using only the identities $\phi_{tt}=\phi_{xx}$, $\phi_x(\mathbf{v}t,t)+\mathbf{v}\phi_t(\mathbf{v}t,t)=0$ and $\phi(L+\mathbf{v}t,t)=0$ from (DWP_1).

A second proof for the conservation of $\mathcal{E}_v(t)$. First, when $\eta=0$ the boundary condition at $x=\mathbf{v}t$ reads

$$\phi_x(\mathbf{v}t,t)+\mathbf{v}\phi_t(\mathbf{v}t,t)=0. \tag{3.37}$$

At the other boundary, we have $\phi(L+\mathbf{v}t,t)=0$ for $t\geq 0$. This means that

$$\frac{d}{dt}\phi(L+\mathbf{v}t,t)=\mathbf{v}\phi_x(L+\mathbf{v}t,t)+\phi_t(L+\mathbf{v}t,t)=0. \tag{3.38}$$

It follows in particular that

$$\phi_x^2(\mathbf{v}t,t)=\mathbf{v}^2\phi_t^2(\mathbf{v}t,t)\ \text{ and }\ \mathbf{v}^2\phi_x^2(L+\mathbf{v}t,t)=\phi_t^2(L+\mathbf{v}t,t).$$

Expanding $\left(\phi_t+\mathbf{v}\phi_x\right)^2$ in the $\mathcal{E}_{\mathbf{v}}(t)$ expression, we get

$$\mathcal{E}_{\mathbf{v}}(t)=E_{\mathbf{v}}(t)+\mathbf{v}\int_{\mathbf{v}t}^{L+\mathbf{v}t}\phi_t\phi_x dx. \tag{3.39}$$

Let us differentiate each term separately. On one hand, the identities (3.37), (3.38) and Leibnitz's rule imply that

$$\frac{d}{dt}E_{\mathbf{v}}(t)=\frac{\mathbf{v}}{2}\left(1+\mathbf{v}^2\right)\left(\phi_x^2(L+\mathbf{v}t,t)-\phi_t^2(\mathbf{v}t,t)\right)+\int_{\mathbf{v}t}^{L+\mathbf{v}t}\left(\phi_x\phi_{tx}+\phi_{tt}\phi_t\right)dx. \tag{3.40}$$

Taking into account that $\phi_{tt}=\phi_{xx}$, the last integral equals

$$\int_{\mathbf{v}t}^{L+\mathbf{v}t}\left(\phi_x\phi_{tx}+\phi_{xx}\phi_t\right)dx=\int_{\mathbf{v}t}^{L+\mathbf{v}t}\left(\phi_x\phi_t\right)_x dx=-\mathbf{v}\left(\phi_x^2(L+\mathbf{v}t,t)-\phi_t^2(\mathbf{v}t,t)\right),$$

hence

$$\frac{d}{dt}E_{\mathbf{v}}(t)=-\frac{\mathbf{v}}{2}\left(1-\mathbf{v}^2\right)\left(\phi_x^2(L+\mathbf{v}t,t)-\phi_t^2(\mathbf{v}t,t)\right). \tag{3.41}$$

On the other hand

$$\frac{d}{dt}\int_{\mathbf{v}t}^{L+\mathbf{v}t}\phi_x\phi_t dx=-\mathbf{v}^2\left(\phi_x^2(L+\mathbf{v}t,t)-\phi_t^2(\mathbf{v}t,t)\right)+\int_{\mathbf{v}t}^{L+\mathbf{v}t}\phi_{tx}\phi_t+\phi_x\phi_{tt}dx.$$

Again, using $\phi_{tt}=\phi_{xx}$ and integrating by parts, the remaining integral equals

$$\int_{\mathbf{v}t}^{L+\mathbf{v}t}\phi_{tx}\phi_t+\phi_x\phi_{xx}dx=\frac{1}{2}\int_{\mathbf{v}t}^{L+\mathbf{v}t}\left(\phi_t^2+\phi_x^2\right)_x dx=\frac{1}{2}\left(1+\mathbf{v}^2\right)\left(\phi_x^2(L+\mathbf{v}t,t)-\phi_t^2(\mathbf{v}t,t)\right),$$

hence

$$\frac{d}{dt}\int\limits_{\mathbf{v}t}^{L+\mathbf{v}t} \phi_x\phi_t dx = \frac{1}{2}\left(1-\mathbf{v}^2\right)\left(\phi_x^2\left(L+\mathbf{v}t,t\right)-\phi_t^2\left(\mathbf{v}t,t\right)\right). \tag{3.42}$$

Due to (3.39),(3.41) and (3.42), we infer that $\frac{d}{dt}\mathcal{E}_{\mathbf{v}}\left(t\right)=0$ as claimed. ■

Remark 3.2 *The energy expression $\mathcal{E}_{\mathbf{v}}\left(t\right)$ is also shown to be conserved in time for the Dirichlet boundary conditions at both ends, see [15].*

Let us now compare $\mathcal{E}_{\mathbf{v}}\left(t\right)$ to the usual expression of energy $E_{\mathbf{v}}\left(t\right)$ for the wave equation.

Corollary 3.1 *Under Assumptions (3.2) and (3.11), the energy $E_{\mathbf{v}}\left(t\right)$ of the solution of the undamped Problem (DWP_1) satisfies*

$$\frac{\mathcal{E}_{\mathbf{v}}\left(0\right)}{1+\mathbf{v}} \leq E_{\mathbf{v}}\left(t\right) \leq \frac{\mathcal{E}_{\mathbf{v}}\left(0\right)}{1-\mathbf{v}}, \qquad for\ t\geq 0 \tag{3.43}$$

and

$$\frac{1}{\gamma_{\mathbf{v}}}E_{\mathbf{v}}\left(0\right) \leq E_{\mathbf{v}}\left(t\right) \leq \gamma_{\mathbf{v}}E_{\mathbf{v}}\left(0\right), \qquad for\ t\geq 0. \tag{3.44}$$

Proof. It suffices to argue as in the proof of Corollary 2 in [15]. ■

Remark 3.3 *The solution ϕ given by (3.7), with $\eta=0$, satisfies the periodicity relation*

$$\phi(x+\mathbf{v}T_{\mathbf{v}},t+T_{\mathbf{v}}) = -\phi(x,t), \quad t\geq 0, \tag{3.45}$$

the previous relation can be rewritten as

$$\phi(x+n\mathbf{v}T_{\mathbf{v}},t+nT_{\mathbf{v}}) = \left(-1\right)^n\phi(x,t), \quad t\geq 0.$$

It follows in particular that the energy $E_{\mathbf{v}}$ is a $T_{\mathbf{v}}-$periodic function in time.

Remark 3.4 *The equality in (3.43) holds at least if $\phi_t\left(x,t_0\right)=\pm\phi_x\left(x,t_0\right)$, for $x\in\mathbf{I}_{t_0}$ and some $t_0\geq 0$. Indeed, we have*

$$\mathcal{E}_{\mathbf{v}}\left(t_0\right)=E_{\mathbf{v}}\left(t_0\right)\pm\mathbf{v}\int_{\mathbf{v}t_0}^{L+\mathbf{v}t_0}\phi_x\left(x,t_0\right)\phi_t\left(x,t_0\right)\,dx=\left(1\pm\mathbf{v}\right)E_{\mathbf{v}}\left(t_0\right),$$

i.e. $E_{\mathbf{v}}\left(t_0\right)=\mathcal{E}_{\mathbf{v}}\left(t_0\right)/\left(1\pm\mathbf{v}\right)$. The + and – signs are used respectively.

Remark 3.5 *Let $0<\mathbf{v}<1$ and $\eta=0$. Evaluating ϕ_x given by (3.23) at $x=\mathbf{v}t$ and $x=L+\mathbf{v}t$, we obtain*

$$\begin{aligned}
\phi_x(\mathbf{v}t,t) &= \frac{-2\mathbf{v}}{1-\mathbf{v}}\sum_{n\in\mathbb{Z}} c_n\, e^{\frac{1-\mathbf{v}^2}{2L}(2n+1)i\pi t}, \\
\phi_x(L+\mathbf{v}t,t) &= \frac{2}{1-\mathbf{v}}\sum_{n\in\mathbb{Z}}\left(c_n e^{\frac{1-\mathbf{v}}{2}(2n+1)i\pi}\right)e^{\frac{1-\mathbf{v}^2}{2L}(2n+1)i\pi t},
\end{aligned}$$

for $t \geq 0$*. Then, by Parseval's identity and* (3.34)*, we obtain*

$$\frac{1}{\mathbf{v}^2}\int_0^{T_\mathbf{v}} \phi_x^2(\mathbf{v}t,t)dt = \int_0^{T_\mathbf{v}} \phi_x^2(L+\mathbf{v}t,t)dt = \frac{\mathcal{E}_\mathbf{v}(0)}{(1-\mathbf{v}^2)^2}.$$

Using (3.43) *for* $t=0$*, we get in particular*

$$\frac{\mathbf{v}^2 E_\mathbf{v}(0)}{(1+\mathbf{v})(1-\mathbf{v}^2)} \leq \int_0^{T_\mathbf{v}} \phi_x^2(\mathbf{v}t,t)dt \quad \text{and} \quad \frac{E_\mathbf{v}(0)}{(1+\mathbf{v})(1-\mathbf{v}^2)} \leq \int_0^{T_\mathbf{v}} \phi_x^2(L+\mathbf{v}t,t)dt.$$

Each of these inequalities is called a boundary observability inequality. They ensure that if there is an initial disturbance of the string, i.e. $E_\mathbf{v}(0) > 0$*, it will be detected by a sensor placed at an endpoint at most after a time* $T \geq T_\mathbf{v} = 2L/\left(1-\mathbf{v}^2\right)$*. See [15, Section 4.1] and [44, Section 4.1] where the boundary observability of the Dirichlet case of the Problem is well developed.*

3.4 EXPONENTIAL STABILITY FOR THE STRING WITH A BOUNDARY DAMPER

In this section, we keep $0 \leq \mathbf{v} < 1$ and assume that $\eta > 0$ with $\eta \neq 1$.

Theorem 3.3 *Under Assumptions* (3.2) *and* (3.11)*, the solution of Problem* (DWP_1) *satisfies*

$$\frac{1}{1+\mathbf{v}}\int_{\mathbf{v}t}^{L+\mathbf{v}t} e^{\frac{1+\mathbf{v}}{L}\ln\left|\gamma_\eta\right|(t-x)}\left(\phi_x-\phi_t\right)^2 dx + \frac{1}{\gamma_\eta^2(1-\mathbf{v})}\int_{\mathbf{v}t}^{L+\mathbf{v}t} e^{\frac{1-\mathbf{v}}{L}\ln\left|\gamma_\eta\right|(t+x)}\left(\phi_t+\phi_x\right)^2 dx = \frac{8}{L}\sum_{n\in\mathbb{Z}}\left|\omega_n a_n\right|^2, \tag{3.46}$$

where the left-hand side is finite and independent of t*. Moreover, it holds that*

$$M_1 e^{-\frac{1-\mathbf{v}^2}{L}\ln\left|\gamma_\eta\right|t} \leq E_\mathbf{v}(t) \leq M_2 e^{-\frac{1-\mathbf{v}^2}{L}\ln\left|\gamma_\eta\right|t}, \text{ for } t \geq 0, \tag{3.47}$$

where

$$M_1 := \frac{2(1-\mathbf{v})}{L}\min\left\{\gamma_\mathbf{v}, \left|\gamma_\eta\right|^{(1+\mathbf{v})}\right\}\sum_{n\in\,\mathbb{Z}}\left|\omega_n a_n\right|^2,$$

$$M_2 := \frac{2(1-\mathbf{v})}{L}\max\left\{\left|\gamma_\eta\right|^{(1+\mathbf{v})}\gamma_\mathbf{v}, \gamma_\eta^2\right\}\sum_{n\in\,\mathbb{Z}}\left|\omega_n a_n\right|^2.$$

Proof. Let us split the integral in the identity (3.14) to the integrals

$$\int_{\mathbf{v}t}^{L_2+\mathbf{v}t} = \int_{\mathbf{v}t}^{L+\mathbf{v}t} + \int_{L+\mathbf{v}t}^{L_2+\mathbf{v}t}, \tag{3.48}$$

then considering the change of variable $x = \gamma_{\mathbf{v}}\left(\mathbf{v}t - \xi\right) + \frac{2L}{1-\mathbf{v}} + \mathbf{v}t$ in $\left(L + \mathbf{v}t, L_2 + \mathbf{v}t\right)$, we obtain

$$\begin{aligned}
&\frac{1}{\left(1-\mathbf{v}\right)\gamma_\eta^2}\int_{L+\mathbf{v}t}^{L_2+\mathbf{v}t} e^{\frac{1-\mathbf{v}}{L}\ln\left|\gamma_\eta\right|\left(t+x\right)}\left(\tilde{\phi}_x\left(x,t\right)+\tilde{\phi}_t\left(x,t\right)\right)^2 dx \\
&= -\frac{1}{\left(1-\mathbf{v}\right)\gamma_\eta^2}\int_{L+\mathbf{v}t}^{\mathbf{v}t}\frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}^2} e^{\frac{1-\mathbf{v}}{L}\ln\left|\gamma_\eta\right|\left(t+\gamma_{\mathbf{v}}\left(\mathbf{v}t-\xi\right)+\mathbf{v}t\right)} e^{2\ln\left|\gamma_\eta\right|}\left(\tilde{\phi}_x\left(\xi,t\right)-\tilde{\phi}_t\left(\xi,t\right)\right)^2 d\xi \\
&= \frac{1}{1+\mathbf{v}}\int_{\mathbf{v}t}^{L+\mathbf{v}t} e^{\frac{1+\mathbf{v}}{L}\ln\left|\gamma_\eta\right|\left(t-\xi\right)}\left(\phi_x\left(\xi,t\right)-\phi_t\left(\xi,t\right)\right)^2 d\xi. \qquad (3.49)
\end{aligned}$$

We used the definition of the extensions (3.9), (3.10) and the fact that $e^{2\ln\left|\gamma_\eta\right|} = \gamma_\eta^2$. Then, combining (3.48) and (3.49), we obtain (3.46).

Expanding $\left(\phi_x \pm \phi_t\right)^2$ and collecting similar terms, we get

$$\begin{aligned}
&\int_{\mathbf{v}t}^{L+\mathbf{v}t}\left(\frac{1}{1+\mathbf{v}} e^{\frac{1+\mathbf{v}}{L}\ln\left|\gamma_\eta\right|\left(t-x\right)} + \frac{1}{\gamma_\eta^2\left(1-\mathbf{v}\right)} e^{\frac{1-\mathbf{v}}{L}\ln\left|\gamma_\eta\right|\left(t+x\right)}\right)\left(\phi_t^2+\phi_x^2\right) dx \\
&-2\int_{\mathbf{v}t}^{L+\mathbf{v}t}\left(\frac{1}{1+\mathbf{v}} e^{\frac{1+\mathbf{v}}{L}\ln\left|\gamma_\eta\right|\left(t-x\right)} - \frac{1}{\gamma_\eta^2\left(1-\mathbf{v}\right)} e^{\frac{1-\mathbf{v}}{L}\ln\left|\gamma_\eta\right|\left(t+x\right)}\right)\phi_t\phi_x dx \\
&= \frac{8}{L}\sum_{n\in\mathbb{Z}}\left|\omega_n a_n\right|^2. \qquad (3.50)
\end{aligned}$$

For $\mathbf{v}t \le x \le L + \mathbf{v}t$ and $t \ge 0$, let us denote

$$A\left(x,t\right) = \frac{1}{1+\mathbf{v}} e^{\frac{1+\mathbf{v}}{L}\ln\left|\gamma_\eta\right|\left(t-x\right)} \quad\text{and}\quad B\left(x,t\right) = \frac{1}{\gamma_\eta^2\left(1-\mathbf{v}\right)} e^{\frac{1-\mathbf{v}}{L}\ln\left|\gamma_\eta\right|\left(t+x\right)}.$$

Then, we can rewrite (3.50) as

$$\int_{\mathbf{v}t}^{L+\mathbf{v}t}\left(A+B\right)\left(\phi_t^2+\phi_x^2\right) dx - 2\int_{\mathbf{v}t}^{L+\mathbf{v}t}\left(A-B\right)\phi_t\phi_x dx = \frac{8}{L}\sum_{n\in\mathbb{Z}}\left|\omega_n a_n\right|^2.$$

Using the algebraic inequality

$$-\left|A-B\right|\left(\phi_t^2+\phi_x^2\right) \le \pm 2\left(A-B\right)\phi_t\phi_x \le \left|A-B\right|\left(\phi_t^2+\phi_x^2\right),$$

we get

$$\begin{aligned}
&\int_{\mathbf{v}t}^{L+\mathbf{v}t}\left(\left(A+B\right)-\left|A-B\right|\right)\left(\phi_t^2+\phi_x^2\right) dx \le \frac{2}{L}\sum_{n\in\mathbb{Z}}\left|2\omega_n a_n\right|^2 \\
&\le \int_{\mathbf{v}t}^{L+\mathbf{v}t}\left(\left(A+B\right)+\left|A-B\right|\right)\left(\phi_t^2+\phi_x^2\right) dx.
\end{aligned}$$

Knowing that

$$(a+b)-|a-b|=2\min\{a,b\} \quad \text{and} \quad (a+b)+|a-b|=2\max\{a,b\},$$

for $a,b\in\mathbb{R}$, then the precedent estimation reads

$$\int_{\mathbf{v}t}^{L+\mathbf{v}t}\min\{A,B\}\left(\phi_t^2+\phi_x^2\right)dx\leq\frac{4}{L}\sum_{n\in\mathbb{Z}}|\omega_n a_n|^2\leq\int_{\mathbf{v}t}^{L+\mathbf{v}t}\max\{A,B\}\left(\phi_t^2+\phi_x^2\right)dx.$$

Since $\ln\left|\gamma_\eta\right|\geq 0$ and $\mathbf{v}t\leq x\leq L+\mathbf{v}t$, we have

$$\frac{1}{1+\mathbf{v}}e^{\frac{1+\mathbf{v}}{L}\ln\left|\gamma_\eta\right|(t-L-\mathbf{v}t)}\leq A(x,t)\leq\frac{1}{1+\mathbf{v}}e^{\frac{1+\mathbf{v}}{L}\ln\left|\gamma_\eta\right|(t-\mathbf{v}t)},$$

i.e.,

$$\frac{e^{-(1+\mathbf{v})\ln\left|\gamma_\eta\right|}}{1+\mathbf{v}}e^{\frac{1-\mathbf{v}^2}{L}\ln\left|\gamma_\eta\right|t}\leq A(x,t)\leq\frac{1}{1+\mathbf{v}}e^{\frac{1-\mathbf{v}^2}{L}\ln\left|\gamma_\eta\right|t}.$$

Similarly, we obtain

$$\frac{1}{\gamma_\eta^2(1-\mathbf{v})}e^{\frac{1-\mathbf{v}^2}{L}\ln\left|\gamma_\eta\right|t}\leq B(x,t)\leq\frac{e^{-(1+\mathbf{v})\ln\left|\gamma_\eta\right|}}{1-\mathbf{v}}e^{\frac{1-\mathbf{v}^2}{L}\ln\left|\gamma_\eta\right|t}.$$

It follows that

$$\begin{aligned}\min\left\{\frac{1}{\left|\gamma_\eta\right|^{(1+\mathbf{v})}(1+\mathbf{v})},\frac{1}{\gamma_\eta^2(1-\mathbf{v})}\right\}E_{\mathbf{v}}(t)&\leq\frac{2e^{-\frac{1-\mathbf{v}^2}{L}\ln\left|\gamma_\eta\right|t}}{L}\sum_{n\in\mathbb{Z}}|\omega_n a_n|^2\\&\leq\max\left\{\frac{1}{1+\mathbf{v}},\frac{1}{\left|\gamma_\eta\right|^{(1+\mathbf{v})}(1-\mathbf{v})}\right\}E_{\mathbf{v}}(t),\end{aligned}$$

hence

$$\begin{aligned}\left(\frac{2}{L}\sum_{n\in\mathbb{Z}}|\omega_n a_n|^2\right)\min\left\{1+\mathbf{v},\left|\gamma_\eta\right|^{(1+\mathbf{v})}(1-\mathbf{v})\right\}e^{-\frac{1-\mathbf{v}^2}{L}\ln\left|\gamma_\eta\right|t}&\leq E_{\mathbf{v}}(t)\\\leq\left(\frac{2}{L}\sum_{n\in\mathbb{Z}}|\omega_n a_n|^2\right)\max\left\{\left|\gamma_\eta\right|^{(1+\mathbf{v})}(1+\mathbf{v}),\gamma_\eta^2(1-\mathbf{v})\right\}&e^{-\frac{1-\mathbf{v}^2}{L}\ln\left|\gamma_\eta\right|t}.\end{aligned}$$

This shows (3.47) and the theorem follows. ∎

Remark 3.6 *If* $\mathbf{v}=0$ *in (3.47), then we get the estimate*

$$\left(\frac{2}{L}\sum_{n\in\mathbb{Z}}|\omega_n a_n|^2\right)e^{-\frac{1}{L}\ln\left|\gamma_\eta\right|t}\leq E_0(t)\leq\gamma_\eta^2\left(\frac{2}{L}\sum_{n\in\mathbb{Z}}|\omega_n a_n|^2\right)e^{-\frac{1}{L}\ln\left|\gamma_\eta\right|t},$$

which is sharper then the estimate (10), stated in the introduction.

Remark 3.7 *The solution ϕ given by (3.7), with $\eta \neq 1$, satisfies the periodicity relation*

$$\phi(x+\mathbf{v}T_{\mathbf{v}}, t+T_{\mathbf{v}}) = -\phi(x,t)/\gamma_\eta,$$

hence

$$E_{\mathbf{v}}\left(t+T_{\mathbf{v}}\right) = E_{\mathbf{v}}\left(t\right)/\gamma_\eta^2, \text{ for } t \geq 0.$$

Remark 3.8 *The constants in estimation (3.47) are (at least) asymptotically sharp in the sense that if $\eta \to 0$, we recover the estimation (3.43) with its sharp constants, see Remark 3.4.*

The next corollary compares $E_{\mathbf{v}}\left(t\right)$ to the initial energy $E_{\mathbf{v}}\left(0\right)$.

Corollary 3.2 *Under Assumptions* (3.2) *and* (3.11), *the energy of the solution of Problem* (DWP_1) *satisfies*

$$\frac{\min\left\{\gamma_{\mathbf{v}}, \left|\gamma_\eta\right|^{(1+\mathbf{v})}\right\}}{\max\left\{\gamma_{\mathbf{v}}\left|\gamma_\eta\right|^{(1+\mathbf{v})}, \gamma_\eta^2\right\}} E_{\mathbf{v}}\left(0\right) e^{-\frac{1-\mathbf{v}^2}{L}\ln\left|\gamma_\eta\right| t} \leq E_{\mathbf{v}}\left(t\right)$$
$$\leq \frac{\max\left\{\gamma_{\mathbf{v}}\left|\gamma_\eta\right|^{(1+\mathbf{v})}, \gamma_\eta^2\right\}}{\min\left\{\gamma_{\mathbf{v}}, \left|\gamma_\eta\right|^{(1+\mathbf{v})}\right\}} E_{\mathbf{v}}\left(0\right) e^{-\frac{1-\mathbf{v}^2}{L}\ln\left|\gamma_\eta\right| t}, \text{ for } t \geq 0. \quad (3.51)$$

Proof. Since (3.47) holds also for $t = 0$, then (3.51) follows by combining the two inequalities

$$\frac{e^{\frac{1-\mathbf{v}^2}{L}\ln\left|\gamma_\eta\right| t}}{\max\left\{\left|\gamma_\eta\right|^{(1+\mathbf{v})}(1+\mathbf{v}), \gamma_\eta^2(1-\mathbf{v})\right\}} E_{\mathbf{v}}\left(t\right) \leq \frac{2}{L}\sum_{n\in\mathbb{Z}}\left|\omega_n a_n\right|^2$$
$$\leq \frac{1}{\min\left\{1+\mathbf{v}, \left|\gamma_\eta\right|^{(1+\mathbf{v})}(1-\mathbf{v})\right\}} E_{\mathbf{v}}\left(0\right)$$

and

$$\frac{1}{\max\left\{\left|\gamma_\eta\right|^{(1+\mathbf{v})}(1+\mathbf{v}), \gamma_\eta^2(1-\mathbf{v})\right\}} E_{\mathbf{v}}\left(0\right) \leq \frac{2}{L}\sum_{n\in\mathbb{Z}}\left|\omega_n a_n\right|^2$$
$$\leq \frac{e^{\frac{1-\mathbf{v}^2}{L}\ln\left|\gamma_\eta\right| t}}{\min\left\{1+\mathbf{v}, \left|\gamma_\eta\right|^{(1+\mathbf{v})}(1-\mathbf{v})\right\}} E_{\mathbf{v}}\left(t\right),$$

for $t \geq 0$. ■

The next corollary gives more simple estimates, but less sharper than (3.47) and (3.51), for the energy $E_{\mathbf{v}}$.

Corollary 3.3 *Under Assumptions* (3.2) *and* (3.11), *the energy of the solution of Problem* (DWP_1) *satisfies*

$$(1-\mathbf{v})\left(\frac{2}{L}\sum_{n\in\mathbb{Z}}|\omega_n a_n|^2\right) e^{-\frac{1-\mathbf{v}^2}{L}\ln|\gamma_\eta| t} \leq E_{\mathbf{v}}(t)$$

$$\leq \gamma_\eta^2(1+\mathbf{v})\left(\frac{2}{L}\sum_{n\in\mathbb{Z}}|\omega_n a_n|^2\right) e^{-\frac{1-\mathbf{v}^2}{L}\ln|\gamma_\eta| t}, \quad \textit{for } t \geq 0 \qquad (3.52)$$

and

$$\frac{1}{\gamma_\eta^2\gamma_{\mathbf{v}}}E_{\mathbf{v}}(0)\,e^{-\frac{1-\mathbf{v}^2}{L}\ln|\gamma_\eta| t} \leq E_{\mathbf{v}}(t) \leq \gamma_\eta^2\gamma_{\mathbf{v}}E_{\mathbf{v}}(0)\,e^{-\frac{1-\mathbf{v}^2}{L}\ln|\gamma_\eta| t}, \quad \textit{for } t \geq 0. \qquad (3.53)$$

Proof. Since $0 \leq \mathbf{v} < 1$, then it suffices to simplify the constants in (3.47) and (3.51) using the fact that $1 \leq \left|\gamma_\eta\right| \leq \left|\gamma_\eta\right|^{(1+\mathbf{v})} < \gamma_\eta^2$. ■

3.5 SOME NUMERICAL EXAMPLES

In this section, we will compute an approximate solution of Problem (DWP_1), given by its series formula (3.7) for the first 20 frequencies, i.e.

$$\phi(x,t) \simeq \sum_{n=-20}^{n=20} a_n\left(\gamma_\eta e^{\frac{1-\mathbf{v}}{L}\omega_n(t+x)} + e^{\frac{1+\mathbf{v}}{L}\omega_n(t-x)}\right), \quad \text{for } x \in (\mathbf{v}t, \pi+\mathbf{v}t),$$

where the coefficient a_n are computed for the initial conditions

$$\phi^0(x) = (1+\cos x)/10 \ \text{ and } \ \phi^1(x) = 0,$$

hence $E_{\mathbf{v}}(0) \simeq 0.0079$. The values of $\mathbf{v}$ and η will be chosen to emphasise how the solution and its energy depend on these parameters.

3.5.1 *The undamped case* $\eta = 0$

We plot the solution for $t = 0, \frac{T_{\mathbf{v}}}{4}, \frac{T_{\mathbf{v}}}{2}, \frac{3T_{\mathbf{v}}}{4}, \ldots, 2T_{\mathbf{v}}$ and two different values of $\mathbf{v}$. The energy $E_{\mathbf{v}}(t)$ is plotted over two periods.

3.5.2 *The underdamped case* $0 < \eta < 1$

We plot the solution for $t = 0, \frac{T_{\mathbf{v}}}{4}, \frac{T_{\mathbf{v}}}{2}, \frac{3T_{\mathbf{v}}}{4}, \ldots, 3T_{\mathbf{v}}$, where $\mathbf{v} = 0.5, T_{\mathbf{v}} \simeq 8.38$ and two different values of η. The energy $E_{\mathbf{v}}(t)$ is plotted for $t \in [0, 3T_{\mathbf{v}}]$. The graphs show that the energy behave as predicted by estimations (3.47) and (3.53). In this case, the solution changes sign.

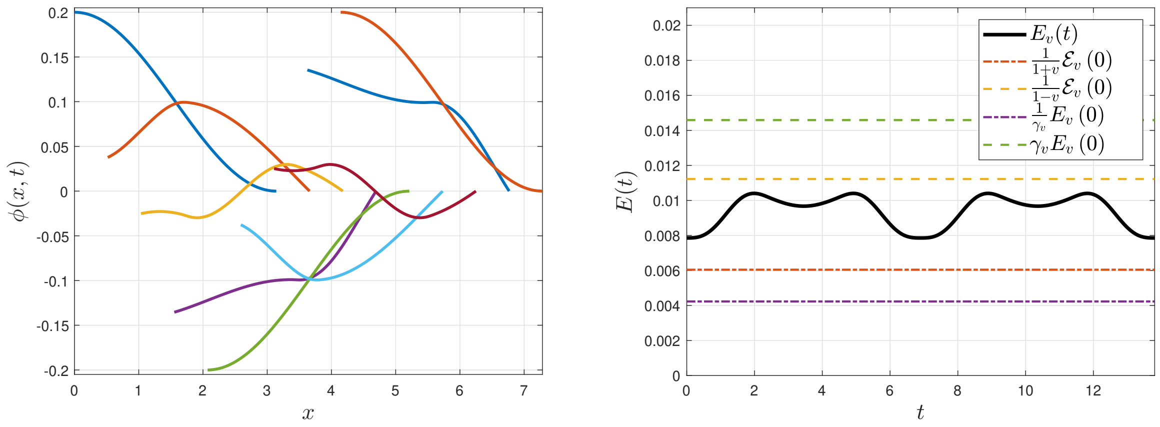

Figure 3.4: A string travelling at a speed $\mathbf{v} = 0.3$, and its energy $E_{\mathbf{v}}(t)$ plotted for $t \in [0, 2T_{\mathbf{v}}]$, where $T_{\mathbf{v}} \simeq 6.90$.

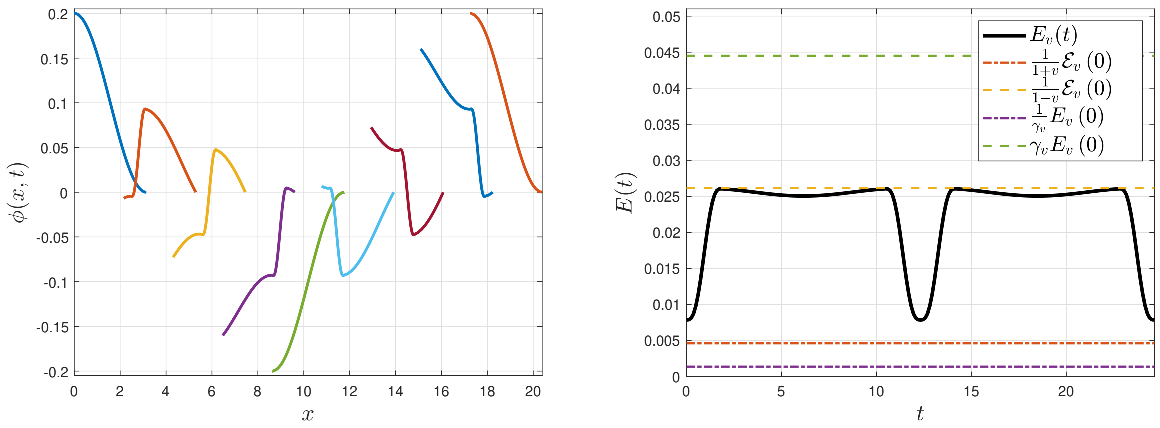

Figure 3.5: A string travelling at a speed $\mathbf{v} = 0.7$, and its energy $E_{\mathbf{v}}(t)$ plotted over two periods, where $T_{\mathbf{v}} \simeq 12.32$.

3.5.3 *The overdamped case $\eta > 1$*

We retain the speed value $\mathbf{v} = 0.5$ and take two values of $\eta > 1$. The solution do not change sign this time.

Remark 3.9 *Observe that $\gamma_{(1/\eta)} = -\gamma_\eta$. Thus $\sum_{n\in\mathbb{Z}} |\omega_n a_n|^2$ remains unchanged if we replace η by $1/\eta$ since the left-hand side of (3.14) remains unchanged. By consequence, the identity (3.46), M_1, M_2 and the constants in the energy estimates of Section 3.4 remain the same for η and $1/\eta$. For instance, taking $\mathbf{v} = 0.5$, the upper and lower estimates for $E_{\mathbf{v}}(t)$ with $\eta = 0.1$ are identical to those of Figure 3.9 for $\eta = 10$.*

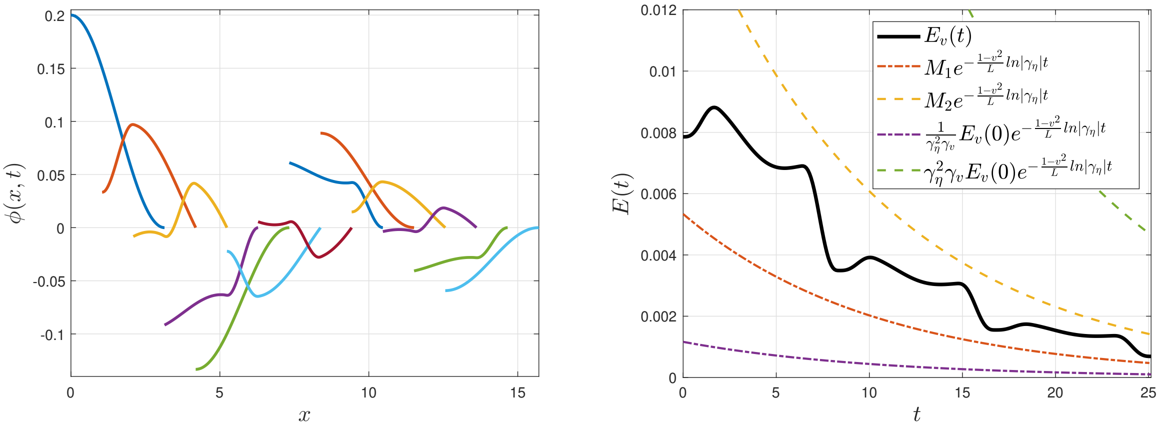

Figure 3.6: A string travelling at a speed $\mathbf{v} = 0.5$, and it energy $E_{\mathbf{v}}(t)$, for a damping factor $\eta = 0.2$.

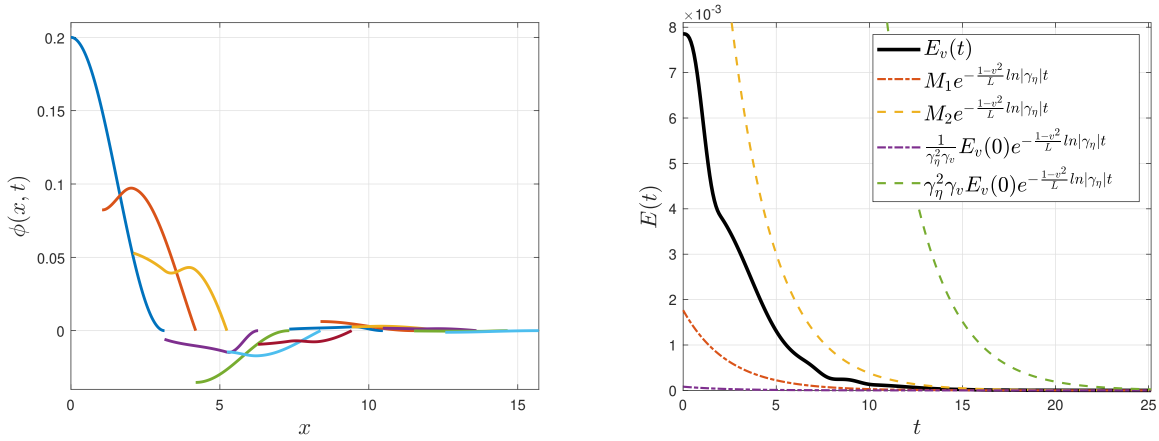

Figure 3.7: A string travelling at a speed $\mathbf{v} = 0.5$, and its energy $E_{\mathbf{v}}(t)$, for a damping factor $\eta = 0.7$.

Remark 3.10 *The case with a dashpot damping at the inlet pulley can be easily investigated by replacing* $\mathbf{v}$ *by* $-\mathbf{v}$ *in the above sections. In Corollary 3.1 we have to change* $\mathbf{v}$ *by* $|\mathbf{v}|$*, i.e.*

$$\frac{\mathcal{E}_{\mathbf{v}}(0)}{1+|\mathbf{v}|} \le E_{\mathbf{v}}(t) \le \frac{\mathcal{E}_{\mathbf{v}}(0)}{1-|\mathbf{v}|} \text{ and } \frac{E_{\mathbf{v}}(0)}{\gamma_{|\mathbf{v}|}} \le E_{\mathbf{v}}(t) \le \gamma_{|\mathbf{v}|} E_{\mathbf{v}}(0), \text{ for } t \ge 0.$$

More importantly, we still have the same exponential $e^{-\frac{1-\mathbf{v}^2}{L}\ln|\gamma_\eta| t}$ *in the estimates of Section 3.4 when* $\eta > 0$*. The analogue of estimations (3.52) is*

$$\frac{2(1-|\mathbf{v}|)}{L} \sum_{n \in \mathbb{Z}} |\omega_n a_n|^2 \, e^{-\frac{1-\mathbf{v}^2}{L}\ln|\gamma_\eta| t} \le E_{\mathbf{v}}(t)$$
$$\le \frac{2\gamma_\eta^2 (1+|\mathbf{v}|)}{L} \sum_{n \in \mathbb{Z}} |\omega_n a_n|^2 \, e^{-\frac{1-\mathbf{v}^2}{L}\ln|\gamma_\eta| t}, \text{ for } t \ge 0$$

and $\gamma_{\mathbf{v}}$ *is replaced by* $\gamma_{|\mathbf{v}|}$ *in (3.53).*

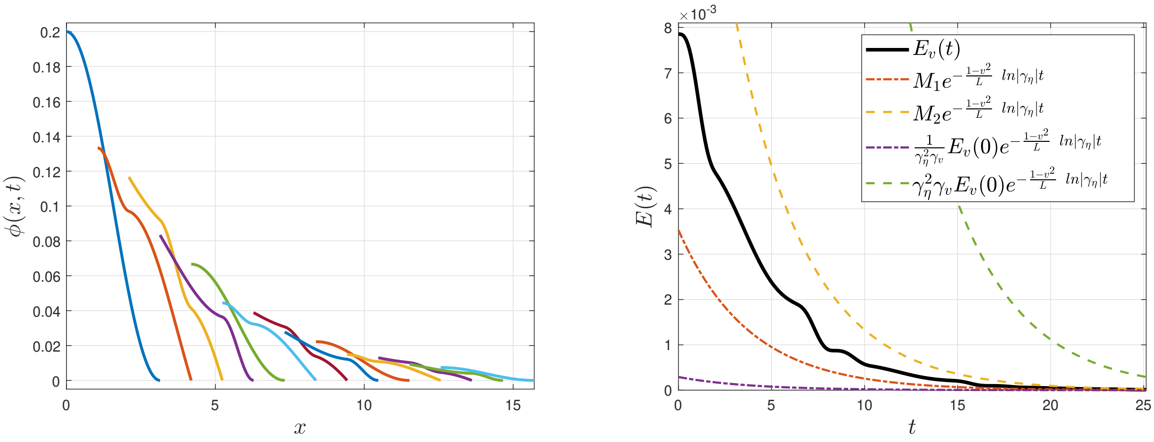

Figure 3.8: A string travelling at a speed $\mathbf{v} = 0.5$, and its energy $E_{\mathbf{v}}(t)$, for a damping factor $\eta = 2$.

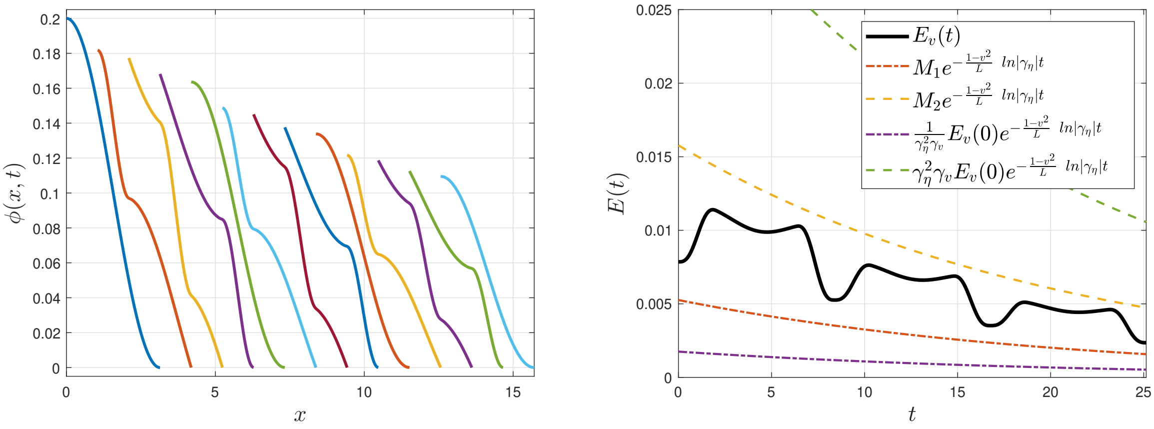

Figure 3.9: A string travelling at a speed $\mathbf{v} = 0.5$, and its energy $E_{\mathbf{v}}(t)$, for damping factor $\eta = 10$.

4

VIBRATIONS OF STRINGS WITH VARIABLE LENGTH DAMPED AT ONE END

In this chapter, we study small vibrations of a string with a length $\ell(t)$ varying in time at a speed less than the speed of propagation of vibrations. We establish lower and upper estimates for the energy of the string when a dash-pot of constant damping factor η is placed at the moving boundary. The estimates depend explicitly on $\ell(t)$, η and a function φ that solves the functional equation $\varphi\left(t+\ell\left(t\right)\right)-\varphi\left(t-\ell\left(t\right)\right)=2$.

4.1 PROBLEM SETTING

We consider small transversal vibrations of a uniform string with a time dependent length $\ell(t)$. One end of the string is fixed to a rigid wall, while the other end passes between two pulleys that can move horizontally and transversely. A dash-pot with a damping factor η is attached to the pulleys to stabilize the string, i.e., reduce or suppress its vibrations.

Denoting the displacement function by u, depending on the position x along the string and the time t, the equation model can be stated as follows

$$\begin{cases} u_{tt}-u_{xx}=0, & \text{for } 0<x<\ell(t) \text{ and } t>0, \\ \left(1+\eta\ell'\left(t\right)\right)u_x\left(\ell\left(t\right),t\right)+\left(\eta+\ell'\left(t\right)\right)u_t\left(\ell\left(t\right),t\right)=0, & \text{for } t>0, \\ u\left(0,t\right)=0, & \text{for } t>0, \\ u(x,0)=u^0\left(x\right) \quad \text{and } u_t\left(x,0\right)=u^1\left(x\right), & \text{for } 0<x<L, \end{cases} \tag{DWP$_2$}$$

where the subscripts t and x stand for the derivatives in time and space variables respectively. The functions u^0 and u^1 represent the initial shape and the initial transverse speed of the string, respectively. The initial length of the string is denoted by $L=\ell(0)$.

The above model is a special case from a model given in [51, Page 88] where we considered the damping at both ends. We were not able to find a derivation of the above model in the literature. Hence, we will derive the equation (DWP2) using the extended Hamilton principal, see the appendix.

We assume that $\ell \in C^1([0,+\infty[)$ and that

$$\left|\ell'(t)\right| < 1, \quad \text{for } t \geq 0, \tag{4.1}$$

which means that the length variation speed of the string is strictly less than the speed of propagation of the wave (here simplified to $c = 1$).

4.2 DERIVATION OF THE MODEL

We make the following assumptions:

- The vibrations of the string are transversal and small. This implies in particular that the slop u_x is also small.
- The string is uniform with a constant mass density ρ. The effects due to gravity are neglected and the string is perfectly flexible meaning that the tension $\mathbf{T}$ is constant.
- The string is fixed to a rigid wall at $x = 0$, which gives the boundary condition

  $$u(0,t) = 0, \text{ for } t \geq 0.$$

- The other end of the string passes through two pulleys moving horizontally with time. Hence, the part of the string between the wall and the pulleys has a variable length $\ell(t)$.
- The pulleys are supposed mass-less and allowed to move also transversely while attached to a dash-pot of constant damping factor η. The vibration are reflected from the pulleys and do not pass through them.

We apply the extended Hamilton's principle (see [30] and [5, Chapters 4 and 5]), over the finite time interval $t_1 \leq t \leq t_2$, in the following form

$$\delta \int_{t_1}^{t_2} \mathcal{L} dt + \int_{t_1}^{t_2} \delta \mathcal{W} dt = 0, \tag{4.2}$$

Here δ denotes the variation in a given function, $\mathcal{L}$ is the Lagrangian and $\mathcal{W}$ is the virtual work performed by non-conservative forces. The Lagrangian is defined by

$$\mathcal{L} = E_k - E_p,$$

where E_k and E_p are respectively the kinetic and potential energy of the string, defined by

$$E_k = \frac{1}{2}\int_0^{\ell(t)} \rho u_t^2 dx \quad \text{and} \quad E_p = \frac{1}{2}\int_0^{\ell(t)} \mathbf{T} u_x^2 dx.$$

The damping force $\mathbf{F}_\eta$ of the dash-pot at $x = \ell(t)$ is contrary to the sense of movement and thus the work done by the dash-pot is

$$\mathcal{W} = -\mathbf{F}_\eta \cdot \text{displacement} = -\mathbf{F}_\eta \; u(\ell(t), t).$$

Hamilton's principal states that u, describing the string vibrations, must be a critical (or a stationary) point for the functional

$$I(u) := \frac{1}{2}\int_{t_1}^{t_2}\int_0^{\ell(t)} \rho u_t^2 - \mathbf{T} u_x^2 \, dxdt - \int_{t_1}^{t_2} \mathbf{F}_\eta u(\ell(t), t)\, dt. \tag{4.3}$$

That is to say

$$\lim_{\varepsilon\to 0} \frac{I(u+\varepsilon v) - I(u)}{\varepsilon} = 0. \tag{4.4}$$

The function $u + \varepsilon v$ is a small perturbation of u that does not affect the values of (the path) u at $t = t_1$ and $t = t_2$. Thus, we take $\mathbf{v}$ to be a smooth function satisfying

$$v(\cdot, t_1) = v(\cdot, t_2) = 0 \quad \text{and} \quad v(0, \cdot) = 0. \tag{4.5}$$

After few computations, the limit (4.4) implies

$$\int_{t_1}^{t_2}\int_0^{\ell(t)} \rho u_t v_t - \mathbf{T} u_x v_x dxdt - \int_{t_1}^{t_2} \mathbf{F}_\eta v(\ell(t), t)\, dt = 0. \tag{4.6}$$

We need to apply Green's formula on the double integral. Denoting by Q_{t_1,t_2} the noncylindrical domain

$$Q_{t_1,t_2} = \left\{(x,t) \in \mathbb{R}^2 \,|\, t_1 < t < t_2 \text{ and } 0 < x < \ell(t)\right\},$$

it comes that

$$\int_{Q_{t_1,t_2}} \rho u_t v_t - \mathbf{T} u_x v_x dxdt = \int_{\partial Q_{t_1,t_2}} (-\mathbf{T} u_x n_x + \rho u_t n_t)\, v d\sigma - \int_{Q_{t_1,t_2}} (\rho u_{tt} - \mathbf{T} u_{xx})\, v dxdt, \tag{4.7}$$

where $d\sigma$ is Lebesgue's measure on $\partial Q_{t_1,t_2}$ and $n = (n_x, n_t)$ is the the unit outward normal vector at (x,t) on $\partial Q_{t_1,t_2}$. Due to (4.5), all the boundary integrals vanish accept the one on the lateral boundary

$$\Sigma_{t_1,t_2} = \left\{(x,t) \in \mathbb{R}^2 \,|\, t_1 < t < t_2 \text{ and } x = \ell(t)\right\}.$$

Since $(dx/dt, 1) = (\ell'(t), 1)$ is tangent to Σ_{t_1,t_2} at $(\ell(t), t)$, then

$$n = \left(\frac{1}{\sqrt{1 + (\ell'(t))^2}}, \frac{-\ell'(t)}{\sqrt{1 + (\ell'(t))^2}} \right).$$

Taking into account that $d\sigma = \sqrt{1 + (\ell'(t))^2} dt$, the boundary integral in (4.7) equals

$$\int_{t_1}^{t_2} \left(-\mathbf{T} u_x \left(\ell \left(t \right), t \right) - \rho \ell'(t) u_t \left(\ell \left(t \right), t \right) \right) \; v \left(\ell \left(t \right), t \right) dt.$$

Substituting this in (4.7), we can rewrite (4.6) as

$$\int_{t_1}^{t_2} \int_0^{\ell(t)} \left(\rho u_{tt} - \mathbf{T} u_{xx} \right) v dx dt \\ + \int_{t_1}^{t_2} \left(\rho \ell' \left(t \right) u_t \left(\ell \left(t \right), t \right) + \mathbf{T} u_x \left(\ell \left(t \right), t \right) + \mathbf{F}_\eta \right) v \left(\ell \left(t \right), t \right) dt = 0. \quad (4.8)$$

Since $\mathbf{v}$ can be chosen arbitrary, the first integral implies that

$$u_{tt} - \frac{\mathbf{T}}{\rho} u_{xx} = 0, \quad \text{for } x \in (0, \ell(t)) \text{ and } t_1 \leq t \leq t_2.$$

The quantity $c = \sqrt{\mathbf{T}/\rho}$ is the wave speed in the string.

Since the value of $v(\ell(t), t)$ can also be chosen arbitrary, the last integral in (4.8) yields

$$\rho \ell' \left(t \right) u_t \left(\ell \left(t \right), t \right) + \mathbf{T} u_x \left(\ell \left(t \right), t \right) + \mathbf{F}_\eta = 0, \quad \text{for } t_1 \leq t \leq t_2.$$

We assume that the damping force $\mathbf{F}_\eta$ depends linearly on the total derivative of $u(\ell(t), t)$, i.e.

$$\mathbf{F}_\eta = \eta \frac{du}{dt},$$

where $\frac{d}{dt} u = \ell'(t) u_x + u_t$ (also called the material derivative). We end up with the boundary condition

$$\left(\rho \ell' \left(t \right) u_t + \mathbf{T} u_x \right) \left(\ell \left(t \right), t \right) + \eta \left(u_t + \ell' \left(t \right) u_x \right) \left(\ell \left(t \right), t \right) = 0, \quad \text{for } t_1 \leq t \leq t_2.$$

Rescaling the variables t, $\ell'(t)$ and η as follows

$$\eta \mapsto \frac{\eta}{\sqrt{\rho \mathbf{T}}} \quad \text{and} \quad x \mapsto x \sqrt{\frac{\rho}{\mathbf{T}}},$$

we obtain the model (DWP$_2$) with a normalized speed $c = 1$.

4.3 EXACT SOLUTION

In this section, we derive the exact solution of (DWP$_2$), that is given by the series

$$u(x,t) = \sum_{n \in \mathbb{Z}} c_n \left(e^{\omega_n \varphi(t+x)} - e^{\omega_n \varphi(t-x)} \right), \quad \text{for } 0 < x < \ell(t) \text{ and } t \geq 0, \quad (4.9)$$

where the coefficients c_n can be explicitly computed in function of the initial data u^0, u^1. We denoted by ω_n the sequence of complex numbers

$$\omega_n := -\frac{1}{2}\ln\left|\gamma_\eta\right| + \begin{cases} \dfrac{2n+1}{2}i\pi, & \text{if } 0 \leq \eta < 1, \\ ni\pi, & \text{if } \eta > 1. \end{cases} \tag{4.10}$$

Observe that since $\left|\gamma_\eta\right| \geq 1$ for every $\eta \geq 0$, the real part of ω_n is nonpositive.

By φ, we denoted a real function satisfying the functional equation

$$\varphi\left(t + \ell\left(t\right)\right) - \varphi\left(t - \ell\left(t\right)\right) = 2. \tag{4.11}$$

Although this equation was already known in [20, 36], it is often called Moor's equation following his paper [34]. We will assume that φ is differentiable and increasing. More precisely,

$$\varphi \in C^1\left([-L, +\infty[\right) \quad \text{and} \quad \varphi' > 0, \quad \text{for } t \geq -L. \tag{4.12}$$

The assumption $\varphi' > 0$ is needed in the sequel since φ' will serve as a weight for an L^2 space. Besides, a decreasing φ can not satisfy (4.11) since $\ell\left(t\right) > 0$. We will also see from the examples given in the last section that Assumption (4.12) is a reasonable one. See [19] for further discussions on the regularity of the solutions of (4.11).

Let us introduce the following family of Hilbert spaces

$$V_0\left(0, \ell(t)\right) := \left\{w \in H^1\left(0, \ell(t)\right), w\left(0\right) = 0\right\}, \quad \text{for } t \geq 0,$$

where $H^1\left(0, \ell(t)\right)$ is the Sobolev space defined on $\left(0, \ell(t)\right)$. We assume that the initial data satisfies

$$u^0 \in V_0\left(0, L\right) \quad \text{and} \quad u^1 \in L^2\left(0, L\right). \tag{4.13}$$

Let $T > 0$. Then, we have the following existence result for Problem (DWP$_2$).

Theorem 4.1 *Under the assumptions (4.1), (4.12) and (4.13), Problem (DWP$_2$) has a unique solution satisfying*

$$u \in C\left([0, T]; V_0\left(0, \ell(t)\right)\right) \cap C^1\left([0, T]; L^2\left(0, \ell(t)\right)\right), \tag{4.14}$$

given by the series (4.9) where the coefficients $c_n \in \mathbb{C}$ are computed as follows

$$c_n = \frac{1}{4\omega_n}\int_{-L}^{L}\left(\tilde{u}_x^0 + \tilde{u}^1\right)e^{-\omega_n\varphi(x)}dx, \quad \text{for } n \in \mathbb{Z}, \tag{4.15}$$

where $\tilde{u}_x^0$ is an even (resp. $\tilde{u}^1$ is an odd) extension of the initial data u^0 (resp. u^1) defined on the interval $(-L, L)$. Moreover,

$$\sum_{n\in\mathbb{Z}}\left|\omega_n c_n\right|^2 = \frac{1}{8}\int_{-L}^{L}\left(\tilde{u}_x^0 + \tilde{u}^1\right)^2 e^{\ln\left|\gamma_\eta\right|\varphi(x)}\frac{dx}{\varphi'(x)} < +\infty. \tag{4.16}$$

Proof. We will split the proof to several steps.

• *The exact solution:* This part of the proof is slightly different from the approach in [50] where the author considered $1/\eta$ instead of η in the boundary condition at $x = \ell(t)$. We include it here for the sake clarity. The general solution of (DWP$_2$) is given by D'Alembert's formula

$$u(x,t) = f(t+x) + g(t-x), \tag{4.17}$$

where f and g are arbitrary continuous functions. The boundary conditions at $x = 0$ yields

$$f(t) = -g(t).$$

The condition at $x = \ell(t)$ implies that

$$\left(1+\eta\ell'(t)\right)\left[f'(t+\ell(t)) - g'(t-\ell(t))\right] = -\left(\eta+\ell'(t)\right)\left[f'(t+\ell(t)) + g'(t-\ell(t))\right],$$

hence

$$\left[1+\eta\ell'(t)+\eta+\ell'(t)\right] f'(\alpha(t)) = -\left[1+\eta\ell'(t)-\eta-\ell'(t)\right] f'(\beta(t)), \tag{4.18}$$

where β is still defined by $\beta(t) = t - \ell(t)$ and $\alpha(t) := t + \ell(t)$. Then, observing that

$$\frac{1+\eta\ell'(t)-\eta-\ell'(t)}{1+\eta\ell'(t)+\eta+\ell'(t)} = \frac{1}{\gamma_\eta}\frac{\beta'(t)}{\alpha'(t)}, \tag{4.19}$$

we can rewrite (4.18) as

$$\alpha'(t) f'(\alpha(t)) = -\frac{1}{\gamma_\eta}\beta'(t) f'(\beta(t)). \tag{4.20}$$

By integration, it follows that

$$f(\alpha(t)) = -\frac{1}{\gamma_\eta} f(\beta(t)) + C. \tag{4.21}$$

Let us assume for the moment that $C = 0$. Then, it is convenient to search for f in the form $f(\xi) = e^{\omega\varphi(\xi)}$, for some constant ω and function φ. Substituting $e^{\omega\varphi(\xi)}$ in (4.21), we get

$$e^{\omega[\varphi(\alpha(t))-\varphi(\beta(t))]} = -1/\gamma_\eta.$$

Assuming that φ satisfies (4.11), we are led to the following cases:

- If $0 \le \eta < 1$, then $\gamma_\eta \ge 1$ and we get $e^{2\omega} = e^{(2n+1)i\pi - \ln\gamma_\eta}$. Solving this equation for ω, we obtain a sequence of values $\omega_n, n \in \mathbb{Z}$, where

$$\omega_n = \frac{2n+1}{2}i\pi - \frac{1}{2}\ln\gamma_\eta.$$

- If $\eta > 1$, we have $\gamma_\eta < -1$ and we obtain this time

$$\omega_n = ni\pi - \frac{1}{2}\ln\left|\gamma_\eta\right|, \quad n \in \mathbb{Z}.$$

Thus, if $\eta \geq 0$ and $\eta \neq 1$, we always have $\ln\left|\gamma_\eta\right| \geq 0$ and ω_n given by (4.10). Due to the superposition principal, it follows that f can be written as

$$f(\xi) = \sum_{n\in\mathbb{Z}} c_n e^{\omega_n \varphi(\xi)}, \qquad c_n \in \mathbb{C},$$

where c_n are complex coefficients to be determined later. Since $f(\xi) = -g(\xi)$, then D'Alembert's formula for the solution gives the series

$$u(x,t) = \sum_{n\in\mathbb{Z}} c_n \left(e^{\omega_n\varphi(t+x)} - e^{\omega_n\varphi(t-x)}\right), \quad \text{for } 0 < x < \ell(t) \text{ and } t \geq 0. \tag{4.22}$$

If $C \neq 0$ in (4.21), then we can check that

$$f(\xi) = \frac{C\gamma_\eta}{1+\gamma_\eta} + \sum_{n\in\mathbb{Z}} c_n e^{\omega_n\varphi(\xi)},$$

solves (4.21). However, this will not affect the solution of (DWP$_2$) since $f(\xi) = -g(\xi)$ and thus the constant parts of f and g will be canceled in the expression (4.22).

- *Computing the coefficients c_n:* We extend $u(\cdot,t)$ to an odd function $\tilde{u}(\cdot,t)$ on the interval $(-\ell(t),\ell(t))$. This ensures in particular that the boundary condition at $x = 0$ is satisfied for $t \geq 0$. It follows also that $\tilde{u}_t(\cdot,t)$ and $\tilde{u}_x(\cdot,t)$ are respectively an odd and an even function. Going back to (4.22), we infer that

$$\tilde{u}_x + \tilde{u}_t = 2\varphi'(t+x)\sum_{n\in\mathbb{Z}} \omega_n c_n e^{\omega_n\varphi(t+x)}, \quad \text{for } x \in (-\ell(t),\ell(t)) \text{ and } t \geq 0.$$

Using the definition of ω_n, we get

$$\tilde{u}_x + \tilde{u}_t = \begin{cases} 2e^{\frac{1}{2}\left(i\pi - \ln\gamma_\eta\right)\varphi(t+x)}\varphi'(t+x)\sum_{n\in\mathbb{Z}} \omega_n c_n e^{ni\pi\varphi(t+x)}, & \text{if } 0 \leq \eta < 1, \\ 2e^{-\frac{1}{2}\ln\left|\gamma_\eta\right|\varphi(t+x)}\varphi'(t+x)\sum_{n\in\mathbb{Z}} \omega_n c_n e^{ni\pi\varphi(t+x)}, & \text{if } 1 < \eta < +\infty, \end{cases} \tag{4.23}$$

which implies that

$$\sum_{n\in\mathbb{Z}} \omega_n c_n e^{ni\pi\varphi(t+x)} = \begin{cases} \frac{1}{2\varphi'(t+x)} e^{\frac{1}{2}\left(-i\pi + \ln\gamma_\eta\right)\varphi(t+x)}\left(\tilde{u}_x + \tilde{u}_t\right), & \text{if } 0 \leq \eta < 1, \\ \frac{1}{2\varphi'(t+x)} e^{\frac{1}{2}\ln\left|\gamma_\eta\right|\varphi(t+x)}\left(\tilde{u}_x + \tilde{u}_t\right), & \text{if } 1 < \eta < +\infty. \end{cases} \tag{4.24}$$

Taking into account that $\left\{e^{ni\pi\varphi(t+x)}/\sqrt{2}\right\}_{n\in\mathbb{Z}}$ is an orthonormal basis of the weighted space $L^2\left(-\ell(t),\ell(t),\varphi'(t+x)\,dx\right)$ (see Appendix A), we deduce that

$$\omega_n c_n = \begin{cases} \frac{1}{4}\int_{-\ell(t)}^{\ell(t)} e^{-\frac{1}{2}\left(i\pi - \ln\gamma_\eta\right)\varphi(t+x)}\left(\tilde{u}_x + \tilde{u}_t\right)e^{-ni\pi\varphi(t+x)}dx, & \text{if } 0 \leq \eta < 1, \\ \frac{1}{4}\int_{-\ell(t)}^{\ell(t)} e^{\frac{1}{2}\ln\left|\gamma_\eta\right|\varphi(t+x)}\left(\tilde{u}_x + \tilde{u}_t\right)e^{-ni\pi\varphi(t+x)}dx, & \text{if } 1 < \eta < +\infty, \end{cases}$$

for $n \in \mathbb{Z}$. Whether $0 \leq \eta < 1$ or $1 < \eta < +\infty$, we have always

$$c_n = \frac{1}{4\omega_n} \int_{-\ell(t)}^{\ell(t)} \left(\tilde{u}_x + \tilde{u}_t\right) e^{-\omega_n \varphi(t+x)} dx, \quad \text{for } n \in \mathbb{Z}. \tag{4.25}$$

Taking $t = 0$, we obtain (4.15) as claimed.

- *Regularity of the solution:* As a consequence of Parseval's equality, we get

$$\sum_{n\in\mathbb{Z}} \left|\omega_n c_n\right|^2 = \begin{cases} \dfrac{1}{8} \displaystyle\int_{-\ell(t)}^{\ell(t)} \left|e^{-\frac{1}{2}\left(i\pi - \ln \gamma_\eta\right)\varphi(t+x)}\right|^2 \left(\tilde{u}_x + \tilde{u}_t\right)^2 \dfrac{dx}{\varphi'(t+x)}, & \text{if } 0 \leq \eta < 1 \\ \dfrac{1}{8} \displaystyle\int_{-\ell(t)}^{\ell(t)} \left|e^{\frac{1}{2}\ln\left|\gamma_\eta\right|\varphi(t+x)}\right|^2 \left(\tilde{u}_x + \tilde{u}_t\right)^2 \dfrac{dx}{\varphi'(t+x)}, & \text{if } 1 < \eta < +\infty. \end{cases}$$

The two identities can be unified in one formula written as

$$\sum_{n\in\mathbb{Z}} \left|\omega_n c_n\right|^2 = \frac{1}{8} \int_{-\ell(t)}^{\ell(t)} \left(\tilde{u}_x + \tilde{u}_t\right)^2 e^{\ln\left|\gamma_\eta\right|\varphi(t+x)} \frac{dx}{\varphi'(t+x)}, \quad \text{for } \eta > 0, \eta \neq 1. \tag{4.26}$$

Due to (4.12) and (4.13), we have for $t = 0$

$$\frac{1}{\sqrt{\varphi'(x)}} e^{\frac{1}{2}\ln\left|\gamma_\eta\right|\varphi(t+x)} \left(\tilde{u}_x^0 + \tilde{u}^1\right) \in L^2\left(-L, L\right).$$

This means that (4.16) holds as claimed.

Due to the continuity and differentiability of $\varphi\left(t+x\right)$ and the exponential function, and since $\left|\omega_n\right| = O\left(n\right)$ for large values of n, the regularity result (4.14) follows from the convergences of the series of the solution (4.22) and its derivatives. ■

In the following, we are mainly interested in the asymptotic behavior in time of the energy of the solution, defined as

$$E_\ell\left(t\right) := \frac{1}{2} \int_0^{\ell(t)} u_t^2\left(x,t\right) + u_x^2\left(x,t\right) dx, \quad \text{for } t \geq 0. \tag{4.27}$$

4.4 THE UNDAMPED CASE

In this section, we show some results for the undamped case, i.e. when $\eta = 0$ in Problem (DWP2). One way to know whether the energy is increasing or decreasing, is to compute $E_\ell'\left(t\right)$. Thus, using Leibniz's rule for differentiation under the integral sign, we get

$$E_\ell'\left(t\right) = \frac{1}{2}\ell'(t)\left[u_x^2\left(\ell(t),t\right) + u_t^2\left(\ell(t),t\right)\right] + \int_0^{\ell(t)} u_t u_{tt} + u_x u_{tx} dx.$$

Since $u_{tt} = u_{xx}$, then $u_t u_{tt} + u_x u_{tx} = \left(u_t u_x\right)_x$ and it comes that

$$E_\ell'\left(t\right) = \frac{1}{2}\ell'(t)\left[u_x^2\left(\ell(t),t\right) + u_t^2\left(\ell(t),t\right)\right] + u_t u_x\left(\ell(t),t\right) \tag{4.28}$$

(Recall that $u_t\left(0,t\right) = 0$). Then, we have the following result.

Lemma 4.1 *Under the assumptions* (4.1) *and* (4.13)*, the energy of solution of Problem* (DWP$_2$)*, with* $\eta = 0$*, is nondecreasing if the interval is shrinking and nonincreasing if the interval is expanding.*

Proof. The boundary condition at $x = \ell(t)$, with $\eta = 0$, reads $u_x(\ell(t), t) = -u_t(\ell(t), t)$, for $t > 0$. Substituting this in (4.28), we get

$$E_\ell'(t) = -\frac{\ell'(t)}{2}\left(1 - |\ell'(t)|^2\right) u_t^2(\ell(t), t), \quad \text{for } t > 0. \tag{4.29}$$

The lemma follows since $1 - |\ell'(t)|^2 > 0$. ■

The next theorem show that the asymptotic behavior of $E_\ell(t)$ is dictated by φ'.

Theorem 4.2 *Under the assumptions* (4.1), (4.12) *and* (4.13)*, the solution of Problem* (DWP$_2$) *satisfies*

$$\int_0^{\ell(t)} \left(\frac{1}{\varphi'(t+x)} + \frac{1}{\varphi'(t-x)}\right)(u_x^2 + u_t^2) + 2\left(\frac{1}{\varphi'(t+x)} - \frac{1}{\varphi'(t-x)}\right) u_x u_t \, dx = 4\mathcal{S}_0, \quad \textit{for } t \geq 0, \tag{4.30}$$

where $\mathcal{S}_0 := \frac{\pi^2}{2}\sum_{n\in\mathbb{Z}} |(2n+1)\, c_n|^2$. *Moreover, it holds that*

$$\mathcal{S}_0 m(t) \leq E_\ell(t) \leq \mathcal{S}_0 M(t), \quad \textit{for } t \geq 0, \tag{4.31}$$

where

$$m(t) := \min_{x\in[0,\ell(t)]} \{\varphi'(t-x), \varphi'(t+x)\} \ \textit{and}\ M(t) := \max_{x\in[0,\ell(t)]} \{\varphi'(t-x), \varphi'(t+x)\}. \tag{4.32}$$

Proof. If $\eta = 0$, then $\gamma_\eta = 1$ and $\omega_n = (2n+1)\, i\pi/2$. The identity (4.26) becomes

$$\frac{1}{8}\int_{-\ell(t)}^{\ell(t)} (\tilde{u}_x + \tilde{u}_t)^2 \frac{dx}{\varphi'(t+x)} = \frac{\pi^2}{4}\sum_{n\in\mathbb{Z}} |(2n+1)\, c_n|^2 = \frac{1}{2}\mathcal{S}_0, \quad \text{for } t \geq 0. \tag{4.33}$$

Since $\tilde{u}_x$ is an even function of x and that $\tilde{u}_t$ is an odd one, then changing x by $-x$ in the last formula, we also obtain

$$\frac{1}{8}\int_{-\ell(t)}^{\ell(t)} (\tilde{u}_x - \tilde{u}_t)^2 \frac{dx}{\varphi'(t-x)} = \frac{1}{2}\mathcal{S}_0, \quad \text{for } t \geq 0. \tag{4.34}$$

Taking the sum of (4.33) and (4.34), it comes that

$$\int_{-\ell(t)}^{\ell(t)} (\tilde{u}_x + \tilde{u}_t)^2 \frac{dx}{\varphi'(t+x)} + \int_{-\ell(t)}^{\ell(t)} (\tilde{u}_x - \tilde{u}_t)^2 \frac{dx}{\varphi'(t-x)} = 8\mathcal{S}_0.$$

Expanding $(u_x \pm u_t)^2$ and collecting similar terms, we get

$$\int_{-\ell(t)}^{\ell(t)} \left(\frac{1}{\varphi'(t+x)} + \frac{1}{\varphi'(t-x)}\right)(\tilde{u}_x^2 + \tilde{u}_t^2) + 2\left(\frac{1}{\varphi'(t+x)} - \frac{1}{\varphi'(t-x)}\right) \tilde{u}_x \tilde{u}_t \, dx = 8\mathcal{S}_0, \quad \text{for } t \geq 0. \tag{4.35}$$

As the function under the integral sign is even, then (4.30) follows.

Next, we use the algebraic inequality $\pm 2u_x u_t \leq u_t^2 + u_x^2$ to obtain

$$\int_0^{\ell(t)} \left(\frac{1}{\varphi'(t+x)} + \frac{1}{\varphi'(t-x)} - \left| \frac{1}{\varphi'(t+x)} - \frac{1}{\varphi'(t-x)} \right| \right) \left(u_x^2 + u_t^2\right)\, dx \leq 4\mathcal{S}_0$$
$$\leq \int_0^{\ell(t)} \left(\frac{1}{\varphi'(t+x)} + \frac{1}{\varphi'(t-x)} + \left| \frac{1}{\varphi'(t+x)} - \frac{1}{\varphi'(t-x)} \right| \right) \left(u_x^2 + u_t^2\right)\, dx,$$

for $t \geq 0$. Recalling that

$$(a+b) - |a-b| = 2\min\{a,b\} \quad \text{and} \quad (a+b) + |a-b| = 2\max\{a,b\}, \tag{4.36}$$

for $a, b \in \mathbb{R}$, then

$$\int_0^{\ell(t)} \min\left\{ \frac{1}{\varphi'(t+x)}, \frac{1}{\varphi'(t-x)} \right\} \left(u_x^2 + u_t^2\right)\, dx \leq 2\mathcal{S}_0$$
$$\leq \int_0^{\ell(t)} \max\left\{ \frac{1}{\varphi'(t+x)}, \frac{1}{\varphi'(t-x)} \right\} \left(u_x^2 + u_t^2\right)\, dx,$$

for $t \geq 0$. Recalling that $E_\ell(t)$ is defined by (4.27), we deduce that

$$\min_{x\in[0,\ell(t)]} \left\{ \frac{1}{\varphi'(t+x)}, \frac{1}{\varphi'(t-x)} \right\} E_\ell(t) \leq \mathcal{S}_0 \leq \max_{x\in[0,\ell(t)]} \left\{ \frac{1}{\varphi'(t+x)}, \frac{1}{\varphi'(t-x)} \right\} E_\ell(t),$$

hence

$$\min_{x\in[0,\ell(t)]} \{\varphi'(t+x), \varphi'(t-x)\} \mathcal{S}_0 \leq E_\ell(t) \leq \max_{x\in[0,\ell(t)]} \{\varphi'(t+x), \varphi'(t-x)\} \mathcal{S}_0,$$

which is (4.31). ■

A second proof for (4.31). First, we denote

$$\begin{aligned}
\tilde{A} &= 2\sum_{n\in\mathbb{Z}} c_n \omega_n \varphi'(t+x) e^{2\omega_n \varphi(t+x)}, \\
\tilde{B} &= 2\sum_{n\in\mathbb{Z}} c_n \omega_n \varphi'(t-x) e^{2\omega_n \varphi(t-x)}, \\
a &= \frac{1}{\varphi'(t+x)}, \quad b = \frac{1}{\varphi'(t-x)}.
\end{aligned}$$

From (4.30), we take

$$\underbrace{\int_0^{\ell(t)} \left(\frac{1}{\varphi'(t+x)} + \frac{1}{\varphi'(t-x)} \right) \left(u_x^2 + u_t^2\right) dx}_{I_1} + 2\underbrace{\int_0^{\ell(t)} \left(\frac{1}{\varphi'(t+x)} - \frac{1}{\varphi'(t-x)} \right) u_x u_t dx}_{I_2}. \tag{4.37}$$

On one hand, by using parallelogram identitie, we obtain

$$\begin{aligned}
I_1 &= \int_0^{\ell(t)} \left(\frac{1}{\varphi'(t+x)} + \frac{1}{\varphi'(t-x)} \right) \left(u_x^2 + u_t^2\right) dx, \\
&= \int_0^{\ell(t)} (a+b) \left\{ \left(\tilde{A} + \tilde{B}\right)^2 + \left(\tilde{A} - \tilde{B}\right)^2 \right\}, \\
&= 2\int_0^{\ell(t)} (a+b) \left\{ \tilde{A}^2 + \tilde{B}^2 \right\}.
\end{aligned} \tag{4.38}$$

On the other hand, we have

$$I_2 = \int_0^{\ell(t)} 2\left(\frac{1}{\varphi'(t+x)} - \frac{1}{\varphi'(t-x)}\right) u_x u_t \, dx = \int_0^{\ell(t)} 2\,(a-b)\left\{\tilde{A}^2 - B^2\right\}. \tag{4.39}$$

Taking the sum of (4.38) and (4.39), it comes that

$$\begin{aligned} I_1 + I_2 &= 2\int_0^{\ell(t)} (a+b)\left\{\tilde{A}^2 + \tilde{B}^2\right\} + 2\int_0^{\ell(t)} (a-b)\left\{\tilde{A}^2 - \tilde{B}^2\right\}, \\ &= 4\int_0^{\ell(t)} \left(a\tilde{A}^2 + b\tilde{B}^2\right), \end{aligned}$$

i.e.,

$$\begin{aligned} \mathcal{S}_0 = \int_0^{\ell(t)} \frac{1}{\varphi'(t+x)} &\left(2\sum_{n\in\mathbb{Z}} c_n\omega_n\varphi'(t+x)e^{2\omega_n\varphi(t+x)}\right)^2 \\ &+ \frac{1}{\varphi'(t-x)}\left(2\sum_{n\in\mathbb{Z}} c_n\omega_n\varphi'(t-x)e^{2\omega_n\varphi(t-x)}\right)^2 dx. \end{aligned}$$

Considering the change of variable $t \pm x = z$ and by combining the two integrals, it comes that

$$\mathcal{S}_0 = \int_{t-\ell(t)}^{t+\ell(t)} \frac{1}{\varphi'(z)}\left(2\sum_{n\in\mathbb{Z}} c_n\omega_n\varphi'(z)e^{2\omega_n\varphi(z)}\right)^2 dz. \tag{4.40}$$

Next, recalling that $E_\ell(t)$ is defined by (4.27), using parallelogram identitie and change of variable, we obtain

$$\begin{aligned} E_\ell(t) &= \int_0^{\ell(t)} \left(2\sum_{n\in\mathbb{Z}} c_n\omega_n\varphi'(t+x)e^{2\omega_n\varphi(t+x)}\right)^2 + \left(2\sum_{n\in\mathbb{Z}} c_n\omega_n\varphi'(t-x)e^{2\omega_n\varphi(t-x)}\right)^2 dx, \\ &= \int_{t-\ell(t)}^{t+\ell(t)} \left(2\sum_{n\in\mathbb{Z}} c_n\omega_n\varphi'(z)e^{2\omega_n\varphi(z)}\right)^2 dz. \end{aligned} \tag{4.41}$$

Comparing (4.40), (4.41) and using the definition (4.32), we can easily have (4.31). ■

Remark 4.1 *An estimation analogue to* (4.31) *was obtained in* [19] *for the case of homogeneous Dirichlet boundary conditions at both ends.*

If φ' is monotone, then we can tie the asymptotic behavior of $E_\ell(t)$ directly with $\ell'(t)$ and refine the result of Lemma 4.1.

Corollary 4.1 *Under the assumption of Theorem* 4.2, *assume that*

$$\varphi' \text{ is monotone for } t \in [t_0, t_1],\ 0 \le t_0 < t_1. \tag{4.42}$$

- If $-1 < \ell'(t) \le 0$ on $[t_0, t_1]$, then $\varphi'$and $E_\ell(t)$ are nondecreasing and

$$\mathcal{S}_0\varphi'(t-\ell(t)) \le E_\ell(t) \le \mathcal{S}_0\varphi'(t+\ell(t)), \quad \text{for } t \in [t_0, t_1]. \tag{4.43}$$

- If $0 \leq \ell'(t) < 1$ on $[t_0, t_1]$, then $\varphi'$and $E_\ell(t)$ are nonincreasing and
$$\mathcal{S}_0\varphi'(t+\ell(t)) \leq E_\ell(t) \leq \mathcal{S}_0\varphi'(t-\ell(t)), \quad \text{for } t \in [t_0, t_1]. \tag{4.44}$$

Proof. Taking the derivative of the identity (4.11) we get
$$\frac{1+\ell'(t)}{1-\ell'(t)}\varphi'(t+\ell(t)) = \varphi'(t-\ell(t)). \tag{4.45}$$
Considering the variation of the function $s \mapsto (1+s)/(1-s)$ on the interval $(-1,1)$, it follows that
$$0 < \frac{1+\ell'(t)}{1-\ell'(t)} \leq 1 \text{ if } -1 < \ell'(t) \leq 0 \quad \text{and} \quad 1 \leq \frac{1+\ell'(t)}{1-\ell'(t)} \quad \text{if } 0 \leq \ell'(t) < 1.$$
Hence
$$\text{if } \ -1 < \ell'(t) \leq 0, \text{ then } \ \varphi'(t-\ell(t)) \leq \varphi'(t+\ell(t)), \tag{4.46}$$
$$\text{if } \ 0 \leq \ell'(t) < 1, \text{ then } \ \varphi'(t-\ell(t)) \geq \varphi'(t+\ell(t)). \tag{4.47}$$
Of course, if φ' is monotone and $-1 < \ell'(t) \leq 0$ then (4.46) means that φ' is necessarily nondecreasing. Then $\varphi'(t \pm \ell(t))$ is also nondecreasing since $t \pm \ell(t)$ is an increasing function due to (4.1). By consequence $m(t) = \varphi'(t-\ell(t)), M(t) = \varphi'(t+\ell(t))$ and (4.44) follows from (4.31).

A similar argument can be made when $0 \leq \ell'(t) < 1$ and the corollary is proved. ■

Remark 4.2 *The assumption (4.42) is satisfied in all the examples of the last section.*

Recall that $\mathcal{S}_0$, defined in Theorem 4.2, can also be computed using by setting $t = 0$ in the left hand side of (4.30). If one needs to compare $E_\ell(t)$ with $E_\ell(t_0)$ for $0 \leq t_0 < t$, then we have the next result.

Corollary 4.2 *Under the assumption of Theorem 4.2, we have*
$$\frac{m(t)}{M(t_0)}E_\ell(t_0) \leq E_\ell(t) \leq \frac{M(t)}{m(t_0)}E_\ell(t_0), \quad \textit{for } 0 \leq t_0 < t. \tag{4.48}$$
Moreover, if φ satisfies (4.42), then:

- If $-1 < \ell'(t) \leq 0$ on $[t_0, t_1]$, then $\varphi'$and $E_\ell(t)$ are nondecreasing and
$$\frac{\varphi'(t-\ell(t))}{\varphi'(t_0+\ell(t_0))}E_\ell(t_0) \leq E_\ell(t) \leq \frac{\varphi'(t+\ell(t))}{\varphi'(t_0-\ell(t_0))}E_\ell(t_0), \quad \text{for } t \in [t_0, t_1]. \tag{4.49}$$
- If $0 \leq \ell'(t) < 1$ on $[t_0, t_1]$, then $\varphi'$and $E_\ell(t)$ are nonincreasing and
$$\frac{\varphi'(t+\ell(t))}{\varphi'(t_0-\ell(t_0))}E_\ell(t_0) \leq E_\ell(t) \leq \frac{\varphi'(t-\ell(t))}{\varphi'(t_0+\ell(t_0))}E_\ell(t_0), \quad \text{for } t \in [t_0, t_1]. \tag{4.50}$$

Proof. Since (4.31) holds also for $t = t_0$, then the corollary follows from the inequalities
$$\frac{E_\ell(t)}{M(t)} \leq \mathcal{S}_0 \leq \frac{E_\ell(t_0)}{m(t_0)} \quad \text{and} \quad \frac{E_\ell(t_0)}{M(t_0)} \leq \mathcal{S}_0 \leq \frac{E_\ell(t)}{m(t)}.$$
■

4.5 THE DAMPED CASE

In this section, we investigate the case with boundary damping, i.e.

$$\eta > 0 \text{ with } \eta \neq 1$$

in Problem (DWP$_2$). Recall that we still have (4.28), i.e.

$$E_\ell'(t) = \frac{1}{2}\ell'(t)\left[u_x^2(\ell(t),t) + u_t^2(\ell(t),t)\right] + u_t u_x(\ell(t),t), \quad \text{for } t \geq 0.$$

and the boundary condition at $x = \ell(t)$ is given by

$$\left(1 + \eta\ell'(t)\right) u_x(\ell(t),t) + \left(\eta + \ell'(t)\right) u_t(\ell(t),t) = 0, \quad \text{for } t \geq 0$$

Lemma 4.2 *Under the assumptions* (4.1) *and* (4.13), *the energy of the solution of Problem* (DWP$_2$), *with $\eta > 0$ and $\eta \neq 1$, satisfies:*

- *If the interval is shrinking, i.e. $-1 < \ell'(t) < 0$ on $[t_1, t_2]$ where $0 \leq t_1 < t_2$, then*
  - *$E_\ell(t)$ is nonincreasing if the damping factor η satisfies*

    $$\eta_1(t) < \eta < \eta_2(t), \quad \textit{for } t \in [t_1, t_2],$$

    where

    $$\eta_1(t) := \frac{1 - \sqrt{1 - |\ell'(t)|^2}}{|\ell'(t)|} \quad \textit{and} \quad \eta_2(t) := \frac{1 + \sqrt{1 - |\ell'(t)|^2}}{|\ell'(t)|}. \tag{4.51}$$

    (Note that $0 < \eta_1(t) < 1 < \eta_2(t)$).
  - *$E_\ell(t)$ is constant if $\ell'(t) = -2\eta/\left(\eta^2 + 1\right)$ on $[t_1, t_2]$.*
  - *$E_\ell(t)$ is nondecreasing if $\eta \in \left]0, \eta_1(t)\right[\cup \left]\eta_2(t), +\infty\right[$ on $[t_1, t_2]$.*
- *If the interval is expanding or time-independent, i.e. $0 \leq \ell'(t) < 1$ on $[t_1, t_2]$, then $E_\ell(t)$ is nonincreasing for every $\eta > 0$.*

Proof. Let us discuss the sign of $E_\ell'(t)$ for different values of $\ell'(t)$.

- If the interval is shrinking $-1 < \ell'(t) < 0$ on $[t_1, t_2]$, then we have the following cases:

  - If $\ell'(t) = -1/\eta$, then the boundary condition at $x = \ell(t)$ reads $u_t(\ell(t),t) = 0$ and thus

  $$E_\ell'(t) = -\frac{1}{2\eta}u_x^2(\ell(t),t) \leq 0, \quad \text{for } t \in [t_1, t_2].$$

- Assume that $\ell'(t) \neq -1/\eta$, then after some computation we can rewrite (4.28) as

$$E_\ell'(t) = -\frac{1}{2}\left(\ell'(t)\eta^2 + 2\eta + \ell'(t)\right)\frac{1-|\ell'(t)|^2}{(1+\eta\ell'(t))^2}u_t^2(\ell(t),t), \quad \text{for } t \in [t_1,t_2]. \tag{4.52}$$

The sign of $E_\ell'(t)$ is opposite to the sign of the polynomial

$$P_\ell(\eta) = \ell'(t)\eta^2 + 2\eta + \ell'(t).$$

Due to (4.1), the polynomial $P_\ell(\eta)$ has a discriminant $\Delta = 4(1-|\ell'(t)|^2) > 0$. Thus $P_\ell(\eta)$ has two real positive roots given by (4.51), see Figure 4.1.

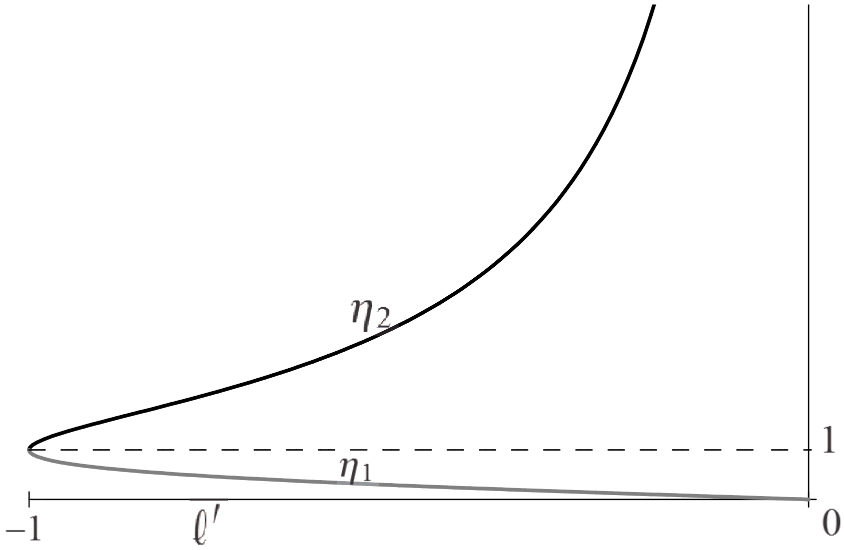

Figure 4.1: Variation of η_1 and η_2 in function of ℓ' when $-1 < \ell' < 0$.

It follows from (4.52) that:

- If $\eta_1(t) < \eta < \eta_2(t)$ on $[t_1,t_2]$, then $P_\ell(\eta) > 0$ and by consequence $E_\ell'(t) \leq 0$.

- If $\eta = \eta_1(t)$ or $\eta = \eta_2(t)$ on $[t_1,t_2]$, then $P_\ell(\eta) = E_\ell'(t) = 0$, i.e. the energy is constant. Observe that

$$\frac{-1 \pm \sqrt{1-|\ell'(t)|^2}}{\ell'(t)} = \eta \quad \text{implies that} \quad \ell'(t) = \frac{-2\eta}{\eta^2+1}.$$

- If $\eta \in]0,\eta_1(t)[\cup]\eta_2(t),+\infty[$ on $[t_1,t_2]$, then $P_\ell(\eta) < 0$ and we have $E_\ell'(t) \geq 0$.

- If the interval is independent of time, i.e. $\ell'(t) = 0$ on $[t_1,t_2]$, then $u_x(\ell(t),t) = -\eta u_t(\ell(t),t)$ and thus

$$E_\ell'(t) = -\eta u_t^2(\ell(t),t) \leq 0, \quad \text{for } t \in [t_1,t_2].$$

- If the interval is expanding, i.e. $0 < \ell'(t) < 1$ on $[t_1,t_2]$, then $P_\ell(\eta) > 0$ for every $\eta > 0$, hence $E_\ell'(t) \leq 0$ for $t \in [t_1,t_2]$.

■

Remark 4.3 *If we do not exclude the case $\ell'(t) \to 1$ as $t \to +\infty$, then a boundary damping may fail to improve the decay of the energy even in expanding domains, see the last example in Section 4.6.*

Let us now estimate $E_\ell(t)$ using φ' and $\exp\left(-\ln\left|\gamma_\eta\right|\varphi\right)$.

Theorem 4.3 *Under the assumptions (4.1), (4.12) and (4.13), the solution of Problem (DWP₂) satisfies*

$$\int_0^{\ell(t)} \left(\frac{e^{\ln|\gamma_\eta|\varphi(t+x)}}{\varphi'(t+x)} + \frac{e^{\ln|\gamma_\eta|\varphi(t-x)}}{\varphi'(t-x)}\right)(u_x^2+u_t^2) + 2\left(\frac{e^{\ln|\gamma_\eta|\varphi(t+x)}}{\varphi'(t+x)} - \frac{e^{\ln|\gamma_\eta|\varphi(t-x)}}{\varphi'(t-x)}\right)u_x u_t dx = 4\mathcal{S}_\eta, \quad (4.53)$$

for $t \geq 0$, where $\mathcal{S}_\eta := 2\sum_{n\in\mathbb{Z}}|\omega_n c_n|^2$. Moreover, it holds that

$$\mathcal{S}_\eta \tilde{m}(t) \leq E_\ell(t) \leq \mathcal{S}_\eta \tilde{M}(t), \quad \text{for } t \geq 0, \quad (4.54)$$

where

$$\begin{aligned}\tilde{m}(t) &:= \min_{x\in[0,\ell(t)]}\left\{\varphi'(t-x)e^{-\ln|\gamma_\eta|\varphi(t-x)}, \varphi'(t+x)e^{-\ln|\gamma_\eta|\varphi(t+x)}\right\},\\ \tilde{M}(t) &:= \max_{x\in[0,\ell(t)]}\left\{\varphi'(t-x)e^{-\ln|\gamma_\eta|\varphi(t-x)}, \varphi'(t+x)e^{-\ln|\gamma_\eta|\varphi(t+x)}\right\}.\end{aligned}$$

Proof. We argue as in the proof of Theorem 4.2. First, since $\tilde{u}_x$ is an even function of x and that $\tilde{u}_t$ is an odd one, then the identity (4.26) yields

$$\int_{-\ell(t)}^{\ell(t)} e^{\ln|\gamma_\eta|\varphi(t\pm x)}(\tilde{u}_x \pm \tilde{u}_t)^2 \frac{dx}{\varphi'(t\pm x)} = 4\mathcal{S}_\eta, \quad \text{for } t \geq 0. \quad (4.55)$$

Summing up, we get

$$\int_{-\ell(t)}^{\ell(t)} e^{\ln|\gamma_\eta|\varphi(t+x)}(\tilde{u}_x + \tilde{u}_t)^2 \frac{dx}{\varphi'(t+x)} + \int_{-\ell(t)}^{\ell(t)} e^{\ln|\gamma_\eta|\varphi(t-x)}(\tilde{u}_x - \tilde{u}_t)^2 \frac{dx}{\varphi'(t-x)} = 8\mathcal{S}_\eta.$$

Expanding squares, we obtain

$$\int_{-\ell(t)}^{\ell(t)} \left(\frac{e^{\ln|\gamma_\eta|\varphi(t+x)}}{\varphi'(t+x)} + \frac{e^{\ln|\gamma_\eta|\varphi(t-x)}}{\varphi'(t-x)}\right)(\tilde{u}_x^2+\tilde{u}_t^2) + 2\left(\frac{e^{\ln|\gamma_\eta|\varphi(t+x)}}{\varphi'(t+x)} - \frac{e^{\ln|\gamma_\eta|\varphi(t-x)}}{\varphi'(t-x)}\right)\tilde{u}_x\tilde{u}_t\, dx = 8\mathcal{S}_\eta, \quad \text{for } t \geq 0. \quad (4.56)$$

As the function under the integral sign is even, then (4.53) follows.

For $0 \leq x \leq \ell(t)$ and $t \geq 0$, let us denote

$$A(x,t) = \frac{e^{\ln|\gamma_\eta|\varphi(t-x)}}{\varphi'(t-x)} \text{ and } B(x,t) = \frac{e^{\ln|\gamma_\eta|\varphi(t+x)}}{\varphi'(t+x)}$$

and rewrite (4.53) as

$$\int_0^{\ell(t)} (A+B)\left(u_t^2+u_x^2\right)dx + 2(A-B)\,u_t u_x dx = 4\mathcal{S}_\eta.$$

Using the algebraic inequality

$$-\left|A-B\right|\left(u_t^2+u_x^2\right) \leq 2(A-B)\,u_t u_x \leq \left|A-B\right|\left(u_t^2+u_x^2\right),$$

we get

$$\int_0^{\ell(t)} \left(A+B-\left|A-B\right|\right)\left(u_t^2+u_x^2\right)dx \leq 4\mathcal{S}_\eta \leq \int_0^{\ell(t)} \left(A+B+\left|A-B\right|\right)\left(u_t^2+u_x^2\right)dx.$$

Thanks to (4.36), the precedent estimation yields

$$\int_0^{\ell(t)} \min\left\{\frac{e^{\ln\left|\gamma_\eta\right|\varphi(t-x)}}{\varphi'(t-x)}, \frac{e^{\ln\left|\gamma_\eta\right|\varphi(t+x)}}{\varphi'(t+x)}\right\}\left(u_t^2+u_x^2\right)dx \leq 2\mathcal{S}_\eta$$
$$\leq \int_0^{\ell(t)} \max\left\{\frac{e^{\ln\left|\gamma_\eta\right|\varphi(t-x)}}{\varphi'(t-x)}, \frac{e^{\ln\left|\gamma_\eta\right|\varphi(t+x)}}{\varphi'(t+x)}\right\}\left(u_t^2+u_x^2\right)dx,$$

for $t \geq 0$. By consequence

$$\min_{x\in[0,\ell(t)]}\left\{\frac{e^{\ln\left|\gamma_\eta\right|\varphi(t-x)}}{\varphi'(t-x)}, \frac{e^{\ln\left|\gamma_\eta\right|\varphi(t+x)}}{\varphi'(t+x)}\right\} E_\ell(t) \leq \mathcal{S}_\eta$$
$$\leq \max_{x\in[0,\ell(t)]}\left\{\frac{e^{\ln\left|\gamma_\eta\right|\varphi(t-x)}}{\varphi'(t-x)}, \frac{e^{\ln\left|\gamma_\eta\right|\varphi(t+x)}}{\varphi'(t+x)}\right\} E_\ell(t)$$

and (4.54) follows as claimed. ■

Since $\ln\left|\gamma_\eta\right| \geq 0$ for $\eta \geq 0$, and φ is an increasing function, then we have the following immediate corollary.

Corollary 4.3 *Under the assumptions of Theorem 4.3, it holds that*

$$\mathcal{S}_\eta m(t)\, e^{-\ln\left|\gamma_\eta\right|\varphi(t+\ell(t))} \leq E_\ell(t) \leq \mathcal{S}_\eta M(t)\, e^{-\ln\left|\gamma_\eta\right|\varphi(t-\ell(t))}, \quad \textit{for } t \geq 0, \tag{4.57}$$

where $m(t)$ and $M(t)$ are given by (4.32).

If φ' is monotone on $[t_0, t_1]$, then (4.57) can be replaced by more explicit estimation.

Corollary 4.4 *Under the assumptions of Theorem 4.3, assume that φ satisfies (4.42) on $[t_0, t_1]$, then:*

- If $-1 < \ell'(t) \leq 0$ on $[t_0, t_1]$, then φ' is nondecreasing and $E_\ell(t)$ satisfies

$$\mathcal{S}_\eta \varphi'(t-\ell(t)) e^{-\ln\left|\gamma_\eta\right|\varphi(t+\ell(t))} \leq E_\ell(t)$$
$$\leq \mathcal{S}_\eta \varphi'(t+\ell(t))\, e^{-\ln\left|\gamma_\eta\right|\varphi(t-\ell(t))}, \quad \text{for } t \in [t_0, t_1]. \tag{4.58}$$

- If $0 < \ell'(t) < 1$ on $[t_0, t_1]$, then $\varphi'$and $E_\ell(t)$ are nonincreasing and

$$\mathcal{S}_\eta \varphi'(t+\ell(t)) e^{-\ln|\gamma_\eta|\varphi(t+\ell(t))} \le E_\ell(t) \le \mathcal{S}_\eta \varphi'(t-\ell(t)) e^{-\ln|\gamma_\eta|\varphi(t-\ell(t))}, \quad \text{for } t \in [t_0, t_1]. \tag{4.59}$$

Proof. It suffices to argue as in the proof of Corollary 4.1. ■

Remark 4.4 *If the interval is shrinking, there is competition between the nondecreasing φ' and $e^{-\ln|\gamma_\eta|\varphi}$ in Estimation (4.58). The behavior of $E_\ell(t)$ depends on how close the value of the damping η to 1, as stated in Lemma 4.2.*

To compare $E_\ell(t)$ with the energy $E_\ell(t_0)$ for $0 \le t_0 < t$, we have the following result.

Corollary 4.5 *Under the assumptions of Theorem 4.3, we have*

$$\frac{m(t) e^{-\ln|\gamma_\eta|\varphi(t+\ell(t))}}{M(t_0) e^{-\ln|\gamma_\eta|\varphi(t_0-\ell(t_0))}} E_\ell(t_0) \le E_\ell(t) \le \frac{M(t) e^{-\ln|\gamma_\eta|\varphi(t-\ell(t))}}{m(t_0) e^{-\ln|\gamma_\eta|\varphi(t_0+\ell(t_0))}} E_\ell(t_0), \text{ for } 0 \le t_0 < t. \tag{4.60}$$

Moreover, if φ' is monotone on $[t_0, t_1]$, then:

- If $-1 < \ell'(t) \le 0$ on $[t_0, t_1]$, then φ' is nondecreasing and $E_\ell(t)$ satisfies

$$\frac{\varphi'(t-\ell(t)) e^{-\ln|\gamma_\eta|\varphi(t+\ell(t))}}{\varphi'(t_0+\ell(t_0)) e^{-\ln|\gamma_\eta|\varphi(t_0-\ell(t_0))}} E_\ell(t_0) \le E_\ell(t) \le \frac{\varphi'(t+\ell(t)) e^{-\ln|\gamma_\eta|\varphi(t-\ell(t))}}{\varphi'(t_0-\ell(t_0)) e^{-\ln|\gamma_\eta|\varphi(t_0+\ell(t_0))}} E_\ell(t_0), \quad \text{for } t \in [t_0, t_1]. \tag{4.61}$$

- If $0 < \ell'(t) < 1$ on $[t_0, t_1]$, then $\varphi'$and $E_\ell(t)$ are nonincreasing and

$$\frac{\varphi'(t+\ell(t)) e^{-\ln|\gamma_\eta|\varphi(t+\ell(t))}}{\varphi'(t_0-\ell(t_0)) e^{-\ln|\gamma_\eta|\varphi(t_0-\ell(t_0))}} E_\ell(t_0) \le E_\ell(t) \le \frac{\varphi'(t-\ell(t)) e^{-\ln|\gamma_\eta|\varphi(t-\ell(t))}}{\varphi'(t_0+\ell(t_0)) e^{-\ln|\gamma_\eta|\varphi(t_0+\ell(t_0))}} E_\ell(t_0), \quad \text{for } t \in [t_0, t_1]. \tag{4.62}$$

Proof. Since (4.54) holds also for $t = t_0$, then the corollary follows by combining the inequalities

$$\frac{e^{\ln|\gamma_\eta|\varphi(t-\ell(t))}}{M(t)} E_\ell(t) \le \mathcal{S}_\eta \le \frac{e^{\ln|\gamma_\eta|\varphi(t_0+\ell(t_0))}}{m(t_0)} E_\ell(t_0)$$

and

$$\frac{e^{\ln|\gamma_\eta|\varphi(t_0-\ell(t_0))}}{M(t_0)} E_\ell(t_0) \le \mathcal{S}_\eta \le \frac{e^{\ln|\gamma_\eta|\varphi(t+\ell(t))}}{m(t)} E_\ell(t).$$

■

Remark 4.5 *For the special case* $\eta = 1$, *the boundary condition at* $x = \ell(t)$ *reads*

$$u_x(\ell(t), t) + u_t(\ell(t), t) = 0.$$

This is a transparent condition, i.e. there is no reflections of waves from the moving endpoint and consequently all the initial disturbances leave the interval $(0, \ell(t))$ *at most after a time*[1]

$$T_\ell := \beta^{-1}(L),$$

where $\beta(t) := t - \ell(t)$, *see Figure 4.2. Hence, whether the interval is expanding or shrinking, the linear velocity feedback* $-u_t(\ell'(t), t)$ *steers the solution to the zero state in the finite time* T_ℓ. *See for instance [18, 19], and for the particular case* $\ell(t) = L$ *see [12, 49]. In the remaining of this paper, we will assume that* $\eta \geq 0$ *and* $\eta \neq 1$.

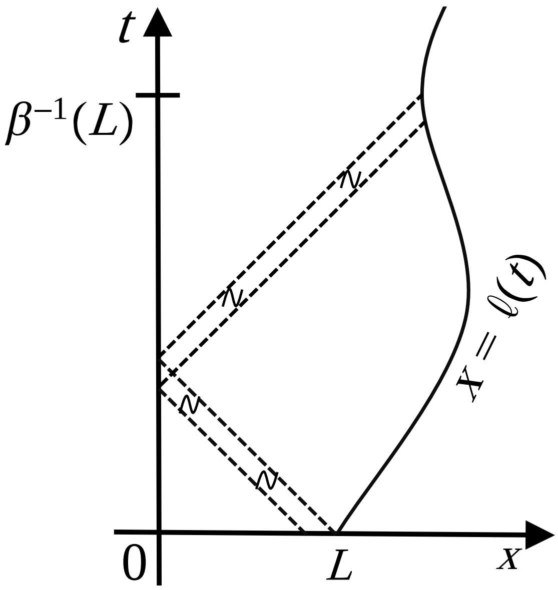

Figure 4.2: An initial disturbance will leave the interval $(0, \ell(t))$ at most after time T_ℓ.

4.6 ENERGY BEHAVIOR FOR SOME EXAMPLES OF LENGTH VARIATION LAWS

The functional equation (4.11) is difficult to solve in a direct way, i.e., for a given $\ell(t)$, we search for a function φ satisfying (4.11). However, the inverse Problem is much easier to solve, i.e. for a given φ we solve (4.11) to obtain $\ell(t)$. A family of solutions obtained by the latter approach can be found in [52].

---

[1] For the uniform variation of length, i.e. $\ell(t) = L + \mathbf{v}t$, then $T_{L+\mathbf{v}t} = 2L/(1 - \mathbf{v})$, see for instance [18, 19],

In the following, we give some examples of $\ell(t)$ and $\varphi(t)$ with the estimations (4.61) and (4.62) written for each case. The obtained results shows that the variation of the length of the string has an important impact on the asymptotic behaviors of $E_\ell(t)$.

4.6.1 *Strings with constant length*

For $\ell(t) = L > 0$, we have $\varphi(t) = a + t/L$, for $t \geq 0$ where $a \in \mathbb{R}$. We can always assume that $a = 0$ since it will give a common factor $e^{w_n a}$ in (4.9) that can be absorbed by the coefficient c_n.

- If $\eta = 0$, then $m(t) = M(t) = 1/L$ and by (4.48) the energy $E_L(t)$ is constant, i.e.

$$E_L(t) = E_L(0), \quad \text{for } t \geq 0.$$

- If $\eta > 0$ with $\eta \neq 1$, then due to (4.61) the energy $E_L(t)$ is decaying and satisfies

$$\frac{1}{\gamma_\eta^2} E_L(0)\, e^{-\frac{1}{L}\ln|\gamma_\eta| t} \leq E_L(t) \leq \gamma_\eta^2 E_L(0)\, e^{-\frac{1}{L}\ln|\gamma_\eta| t}. \tag{4.63}$$

The constants in this estimate are sharp. To see this, first observe that the solution given by the series formulas (4.9) satisfies $u(x, t+2L) = -u(x,t)/\gamma_\eta$. Hence, the energy satisfies

$$E_L(t+2L) = E_L(t)/\gamma_\eta^2, \quad \text{for } t \geq 0. \tag{4.64}$$

Then, consider an initial disturbance of the string supported on a small interval of length $\delta > 0$. Let T_1 be the time of the first reflection from the boundary $x = L$. Necessarily, $T_1 < 2L$, see Figure 4.3 (Right).

On one hand, assume that the initial disturbance travels to the right, see Figure 4.3 (left). Then, we have

$$\begin{aligned} E_L(t) &= E_L(0), \quad \text{for } 0 \leq t < T_1, \\ E_L(t) &= E_L(2L) = E_L(0)/\gamma_\eta^2, \quad \text{for } T_1 + \delta\sqrt{2} < t < T_1 + 2L. \end{aligned}$$

The last assertion is a consequence of (4.64) and the fact that the energy remains constant if there is no reflection from the boundary $x = L$ where the dash-pot is placed. In particular, we have

$$\frac{1}{\gamma_\eta^2} E_L(0)\, e^{-\frac{1}{L}\ln|\gamma_\eta| t} \leq E_L(t) = \frac{1}{\gamma_\eta^2} E_L(0), \quad \text{for } T_1 + \delta\sqrt{2} < t < T_1 + 2L, \tag{4.65}$$

since $e^{-\frac{1}{L}\ln|\gamma_\eta| t} < 1$. As $\delta \to 0, T_1 \to 0$ and then $t \to 0$, the left hand side of (4.65) tends to $E_L(0)/\gamma_\eta^2$. This show that $1/\gamma_\eta^2$ is the best constant for the lower bound in (4.63).

On the other hand, when the initial disturbance travels to the left, see Figure 4.3 (Right), we have

$$E_L(0) = E_L(t) \leq \gamma_\eta^2 E_L(0)\, e^{-\frac{1}{L}\ln|\gamma_\eta| t}, \text{ for } 0 \leq t < T_1 < 2L,$$

since $\gamma_\eta^2 e^{-\frac{1}{L}\ln|\gamma_\eta| t} = \left|\gamma_\eta\right|^{2-\frac{t}{L}} > 1$, for $t < 2L$. As $\delta \to 0, T_1 \to 2L$ and then $t \to 2L$, the right hand side tends to $E_L(0)$. This show the sharpness of the constant γ_η^2 for the upper bound in (4.63).

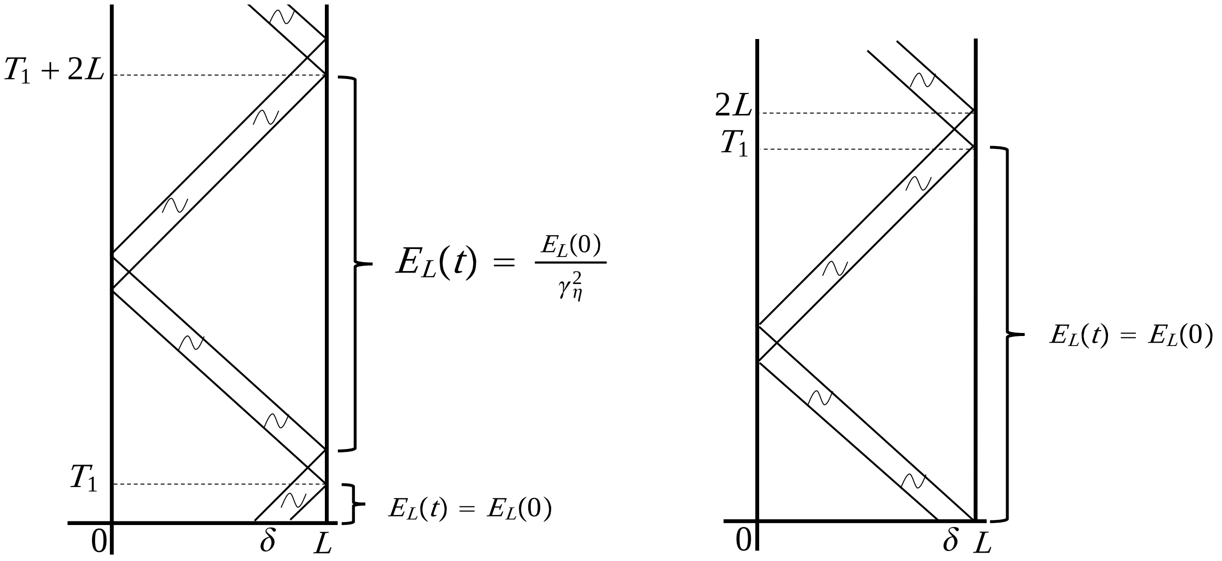

Figure 4.3: Energy and propagation of an initial disturbance with a small support.

4.6.2 *Strings with uniform variation of length*

For $\ell(t) = L + \mathbf{v}t$, where $L > 0$ and $0 < |\mathbf{v}| < 1$, we have $\varphi(t) = 2\ln(L + \mathbf{v}t) / \ln\gamma_{\mathbf{v}}$. Here we take the time

$0 \leq t < L/|\mathbf{v}|$, if $-1 < \mathbf{v} < 0$, i.e. when the interval is shrinking.

$t \geq 0$, if $0 < \mathbf{v} < 1$, i.e. when the interval is expanding.

- For the undamped case, the energy is nondecreasing if $-1 < \mathbf{v} < 0$, and

$$\frac{\gamma_{\mathbf{v}} L}{L + \mathbf{v}t} E_\ell(0) \leq E_\ell(t) \leq \frac{L}{\gamma_{\mathbf{v}}(L + \mathbf{v}t)} E_\ell(0), \quad \text{for } 0 \leq t < L/|\mathbf{v}|. \tag{4.66}$$

  If $0 < \mathbf{v} < 1$, the energy is nonincreasing and

$$\frac{L}{\gamma_{\mathbf{v}}(L + \mathbf{v}t)} E_\ell(0) \leq E_\ell(t) \leq \frac{\gamma_{\mathbf{v}} L}{L + \mathbf{v}t} E_\ell(0), \quad \text{for } t \geq 0. \tag{4.67}$$

The latter result is known in [43, 47] for the Dirichlet boundary conditions case (which is equivalent to tacking $\eta = +\infty$ in the boundary condition at $x = \ell(t)$). Observe that if $\mathbf{v} \to 0$, hence $\gamma_{\mathbf{v}} \to 1$, we get $E_\ell(t) = E_\ell(0)$ as in the time-independent case.

- For the damped case, $\eta > 0$ with $\eta \neq 1$, the energy $E_\ell(t)$ is *still nondecreasing* if $-1 < \mathbf{v} < 0$, and

  $$\frac{\gamma_{\mathbf{v}}}{\gamma_\eta^2}\left(\frac{L}{L+\mathbf{v}t}\right)^{1+2\frac{\ln|\gamma_\eta|}{\ln\gamma_{\mathbf{v}}}} E_\ell(0) \leq E_\ell(t) \leq \frac{\gamma_\eta^2}{\gamma_{\mathbf{v}}}\left(\frac{L}{L+\mathbf{v}t}\right)^{1+2\frac{\ln|\gamma_\eta|}{\ln\gamma_{\mathbf{v}}}} E_\ell(0), \quad \text{for } 0 \leq t < \frac{L}{|\mathbf{v}|}. \quad (4.68)$$

  If $0 \leq \mathbf{v} < 1$, the energy is nonincreasing and

  $$\frac{1}{\gamma_{\mathbf{v}}\gamma_\eta^2}\left(\frac{L}{L+\mathbf{v}t}\right)^{1+2\frac{\ln|\gamma_\eta|}{\ln\gamma_{\mathbf{v}}}} E_\ell(0) \leq E_\ell(t) \leq \gamma_{\mathbf{v}}\gamma_\eta^2\left(\frac{L}{L+\mathbf{v}t}\right)^{1+2\frac{\ln|\gamma_\eta|}{\ln\gamma_{\mathbf{v}}}} E_\ell(0), \quad \text{for } t \geq 0. \quad (4.69)$$

4.6.3 *Strings with nonlinear variation of length I*

For $\ell(t) = 1/(1+at)$, $a > 0$, we have $\varphi(t) = t(at+2)/2$. The assumption (4.1) is satisfied for $0 < a < 1$ and the interval is shrinking with

$$-1 < \ell'(t) = \frac{-a}{(1+at)^2} < 0, \text{ for } t \geq 0.$$

- For the undamped case, the energy is nondecreasing and

  $$\frac{(1+at)^2 - a}{(1+a)(1+at)} E_\ell(0) \leq E_\ell(t) \leq \frac{(1+at)^2 + a}{(1-a)(1+at)} E_\ell(0), \quad \text{for } t \geq 0. \quad (4.70)$$

- For the damped case, $\eta > 0$ with $\eta \neq 1$, the energy $E_\ell(t)$ satisfies

  $$C_1 \frac{(1+at)^2 - a}{(1+at)} e^{-\ln|\gamma_\eta|\varphi(t+\ell(t))} E_\ell(0) \leq E_\ell(t) \leq C_2 \frac{(1+at)^2 + a}{(1+at)} e^{-\ln|\gamma_\eta|\varphi(t-\ell(t))} E_\ell(0), \quad (4.71)$$

  for $t \geq 0$, where $C_1 := \left|\gamma_\eta\right|^{1-\frac{a}{2}}/(1+a)$, $C_2 := \left|\gamma_\eta\right|^{1+\frac{a}{2}}/(1-a)$ and

  $$\varphi(t \pm \ell(t)) = \frac{(1+at)^2 + 1 + at \pm a}{2(1+at)^2}\left(at^2 + t \pm 1\right).$$

Since $\varphi(t-\ell(t)) > 0$, for $t > 1$, the right hand side of (4.71) decays exponentially for large value of t and we have

$$E_\ell(t) = O\left(t\, e^{-\frac{a}{2}\ln\left|\gamma_\eta\right| t^2}\right), \quad \text{as } t \to +\infty.$$

Observe that (4.71) does not mean necessarily that $E_\ell(t)$ is (monotone) nonincreasing. However, since $\ell'(t) \to 0$ as $t \to \infty$, then $\eta_1(t) \to 0$ and $\eta_2(t) \to +\infty$, (see Figure 4.1). By consequence, for every $\eta > 0$, we have $\eta_1(t) < \eta < \eta_2(t)$ for $t \geq t_1$, where t_1 is large enough and thus the energy is nonincreasing for $t \geq t_1$.

4.6.4 *Strings with nonlinear variation of length II*

For $\ell(t) = \sqrt{1+at}$, $a > 0$, we have $\varphi(t) = \sqrt{4at+a^2+4}/a$. The interval is expanding and the assumption (4.1) is satisfied for $0 < a < 2$ and

$$0 < \ell'(t) = \frac{a}{2\sqrt{1+at}} < 1, \quad \text{for } t \geq 0.$$

- For the undamped case, the energy is nonincreasing and

$$\frac{2-a}{\sqrt{4a\left(t+\sqrt{1+at}\right)+a^2+4}} E_\ell(0) \leq E_\ell(t) \leq \frac{2+a}{\sqrt{4a\left(t-\sqrt{1+at}\right)+a^2+4}} E_\ell(0), \quad \text{for } t \geq 0.$$

- For the damped case, the energy is decaying

$$C_3 \frac{\exp\left(-\frac{\ln\left|\gamma_\eta\right|}{a}\sqrt{4a\left(t+\sqrt{1+at}\right)+a^2+4}\right)}{\sqrt{4a\left(t+\sqrt{1+at}\right)+a^2+4}} E_\ell(0) \leq E_\ell(t) \leq C_4 \frac{\exp\left(-\frac{\ln\left|\gamma_\eta\right|}{a}\sqrt{4a\left(t-\sqrt{1+at}\right)+a^2+4}\right)}{\sqrt{4a\left(t-\sqrt{1+at}\right)+a^2+4}} E_\ell(0), \quad \text{for } t \geq 0,$$

where $C_3 = (2-a)\left|\gamma_\eta\right|^{-1+2/a}$ and $C_4 = (2+a)\left|\gamma_\eta\right|^{1+2/a}$. Here also, the energy is decaying exponentially,

$$E_\ell(t) = O\left(\frac{1}{\sqrt{t}} e^{-\frac{2}{\sqrt{a}}\ln\left|\gamma_\eta\right|\sqrt{t}}\right), \quad \text{as } t \to +\infty.$$

4.6.5 *Strings with nonlinear variation of length III*

If $\varphi(t) = e^{at+b}$ with $a > 0, b \in \mathbb{R}$, then $\ell(t) = \operatorname{arcsinh}\left(e^{-at-b}\right)/a$. The interval is shrinking and Assumption (4.1) is satisfied since

$$-1 < \ell'(t) = \frac{-1}{\sqrt{1+e^{2(at+b)}}} < 0, \text{ for } t \geq 0.$$

- For the undamped case, the energy of the solution satisfies

  $$\frac{\exp\left(at - \operatorname{arcsinh}\left(e^{-at-b}\right)\right)}{\exp\left(\operatorname{arcsinh}\left(e^{-b}\right)\right)} E_\ell(0) \leq E_\ell(t) \leq \frac{\exp\left(at + \operatorname{arcsinh}\left(e^{-at-b}\right)\right)}{\exp\left(-\operatorname{arcsinh}\left(e^{-b}\right)\right)} E_\ell(0),$$

  for $t \geq 0$. Since $\operatorname{arcsinh}\left(e^{-at-b}\right) \to 0$, as $t \to +\infty$, the energy is nondecreasing and satisfies

  $$E_\ell(t) = O\left(e^{at}\right), \text{ as } t \to +\infty.$$

- For the damped case, the energy $E_\ell(t)$ satisfies

  $$\begin{aligned} C_5 \exp\left(at - \operatorname{arcsinh}\left(e^{-at-b}\right) - \ln\left|\gamma_\eta\right| e^{\left(at+b+\operatorname{arcsinh}\left(e^{-at-b}\right)\right)}\right) E_\ell(0) \leq E_\ell(t) \\ \leq C_6 \exp\left(at + \operatorname{arcsinh}\left(e^{-at-b}\right) - \ln\left|\gamma_\eta\right| e^{\left(at+b-\operatorname{arcsinh}\left(e^{-at-b}\right)\right)}\right) E_\ell(0), \end{aligned}$$

  for $t \geq 0$, where

  $$\begin{aligned} C_5 &= \exp\left(-\operatorname{arcsinh}\left(e^{-b}\right)\right) \exp\left(\ln\left|\gamma_\eta\right| e^{-\operatorname{arcsinh}\left(e^{-b}\right)+b}\right), \\ C_6 &= \exp\left(\operatorname{arcsinh}\left(e^{-b}\right)\right) \exp\left(\ln\left|\gamma_\eta\right| e^{\operatorname{arcsinh}\left(e^{-b}\right)+b}\right). \end{aligned}$$

  Since $\operatorname{arcsinh}\left(e^{-at-b}\right) \to 0$, as $t \to +\infty$, then

  $$at \pm \operatorname{arcsinh}\left(e^{-at-b}\right) - \ln\left|\gamma_\eta\right| e^{\left(at+b+\operatorname{arcsinh}\left(e^{-at-b}\right)\right)} = O\left(at - \ln\left|\gamma_\eta\right| e^{at+b}\right)$$

  and the energy is nondecreasing and satisfies

  $$E_\ell(t) = O\left(\exp\left(at - \ln\left|\gamma_\eta\right| e^{at+b}\right)\right), \quad \text{as } t \to +\infty.$$

  The damping is *very effective* in this case.

4.6.6 *Strings with nonlinear variation of length IV*

If $\varphi(t) = 1/(1+at)$ and $a > 0$, then $\ell(t) = \left(1 + \sqrt{1+(1+at)^2}\right)/a$. The interval is expanding since

$$\frac{1}{\sqrt{2}} \leq \ell'(t) = \frac{1+at}{\sqrt{1+(1+at)^2}} < 1, \text{ for } t \geq 0.$$

However, the assumption (4.1) is not (strictly) satisfied since $\ell'(t) \to 1$, as $t \to +\infty$.

- For the undamped case, the energy of the solution is nonincreasing and

$$\frac{2}{\left(\sqrt{1+(1+at)^2}+at+2\right)^2}E_\ell(0) \le E_\ell(t) \le \left(\frac{\sqrt{2}+2}{\sqrt{1+(1+at)^2}-at}\right)^2 E_\ell(0), \text{ for } t \ge 0.$$

  One can check that

$$0 < \frac{1}{\sqrt{1+(1+at)^2}+at+2} \le \frac{1}{\sqrt{1+(1+at)^2}-at} \le 1, \text{ for } t \ge 0. \tag{4.72}$$

  By consequence,

$$\frac{K_1}{t^2}E_\ell(0) \le E_\ell(t) \le K_2 E_\ell(0), \text{ as } t \to +\infty,$$

  for some positive constants K_1 and K_2.

- For the damped case, the energy $E_\ell(t)$ is nonincreasing and

$$C_7 \frac{\exp\left(-2\ln\left|\gamma_\eta\right| / \left(\sqrt{1+(1+at)^2}+at+2\right)\right)}{\left(\sqrt{1+(1+at)^2}+at+2\right)^2} E_\ell(0) \le E_\ell(t)$$
$$\le C_8 \frac{\exp\left(2\ln\left|\gamma_\eta\right| / \left(\sqrt{1+(1+at)^2}-at\right)\right)}{\left(\sqrt{1+(1+at)^2}-at\right)^2} E_\ell(0), \text{ for } t \ge 0, \tag{4.73}$$

  where $C_7 = 2e^{-\ln|\gamma_\eta|\sqrt{2}}$ and $C_8 = \left(\sqrt{2}+2\right)^2 e^{2\ln|\gamma_\eta|/(2+\sqrt{2})}$.

  Due to (4.72), the functions inside the exponentials in (4.73) are bounded. It follows that we still have

$$\frac{K_3}{t^2}E_\ell(0) \le E_\ell(t) \le K_4 E_\ell(0), \text{ as } t \to +\infty,$$

  for some positive constants K_3 and K_4.

In this example, the boundary damping *failed to improve* the energy estimates obtained in the undamped case. This can be explained by Figure 4.4 which shows that every initial disturbance of the string may have one reflection from the moving boundary, where the damper is placed, but it fails to catch up again with this boundary. See also [2] where the similar case $\ell(t) = \sqrt{1+t^2}$, with Dirichlet boundary conditions, was investigated.

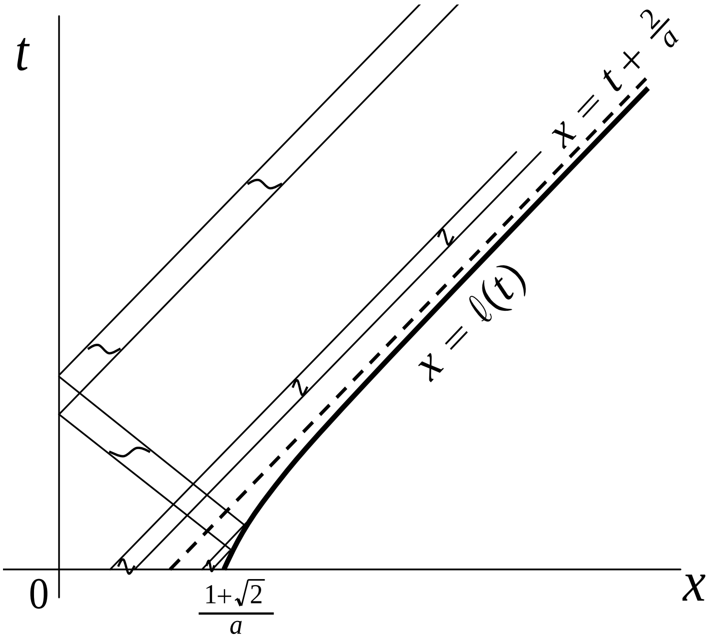

Figure 4.4: Propagation of initial disturbances of a string when $\ell\left(t\right) = \frac{1}{a}\left(1 + \sqrt{1 + \left(1 + at\right)^2}\right)$. Observe that $x = t + 2/a$ is an asymptote for $\ell(t)$.

A

APPENDICES

The appendix of the thesis contains definitions, equalities, inequalities, and notation. Textbooks on functional analysis and partial differential equations (PDEs) provide proofs and more complex results, as shown in the standard literature. [37, 46].

A.1 NOTATION

We will define some terms and concepts before we begin to make the thesis easier to understand. We will start by introducing the general concepts, and then provide more detailed definitions of the specific terms. This will help readers follow our argument and understand the key concepts we are discussing.

- $:=$ equal by definition.
- $\gamma_{\mathbf{v}} := \frac{1+\mathbf{v}}{1-\mathbf{v}}$.
- $L_1 := \frac{1-\mathbf{v}}{1+\mathbf{v}} L, \quad L_2 := \frac{2}{1-\mathbf{v}} L$.
- $T_{\mathbf{v}} := \frac{2L}{1-\mathbf{v}^2}, \quad \tilde{T}_{\mathbf{v}} := \frac{L}{1-\mathbf{v}}$.
- $\mathbf{I}_t := (\mathbf{v}t, L+\mathbf{v}t)$, for $t \in (0,T)$.
- $\alpha(t) := t + \ell(t), \quad \beta(t) := t - \ell(t)$.
- $E' :=$ dual of space E.
- $\|.\|_E :=$ norm of space E, $\langle ., . \rangle_E :=$ duality product of a space E, E'.
- $x.y := \sum_{i=1}^n x_i y_i$, for $x, y \in \mathbb{R}^n$, $|x| := \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$, for $x \in \mathbb{R}^n$.
- $f(x) = O(g(x))$ when $x \to x_0$ ($x_0 \in \mathbb{R}$), if there exists a constant $C > 0$ such that $f(x) \leq Cg(x)$ for all x in a neighborhood of x_0 .

A.2 FUNCTION SPACES

We will end this section by introducing some function spaces and the symbols used to represent them

- $C^k(\mathbf{I}_t) \quad := \quad \{w : \mathbf{I}_t \to \mathbb{R} |\ w \text{ is } k \text{ times continuously differentiable}\}, \quad k \in \mathbb{N},$
- $L^2(\mathbf{I}_t) \quad := \quad \left\{w : \mathbf{I}_t \to \mathbb{R} |\ w \text{ is measurable and } \int_{\mathbf{I}_t} |w|^2 \rho ds \ < \infty\right\}.$
- $H_0^1(\mathbf{I}_t) \quad := \quad$ the closure of $\mathcal{D}(\mathbf{I}_t)$ in $H^1(\mathbf{I}_t)$.
- $C^k([0,T]; E) \quad := \quad$ space of k times continuously differentiable functions from an interval $[0,T]$.
- $H^{-1}(\mathbf{I}_t) \quad := \quad$ the dual space of $H^1(\mathbf{I}_t)$.
- $L^2(a,b,\rho ds) \quad := \quad \left\{w : (a,b) \to \mathbb{R} |\ w \text{ is measurable and } \int_a^b |w|^2 \rho ds \ < \infty\right\}.$
- $\mathcal{H}_{L+\mathbf{v}t}(\mathbf{I}_t) \quad := \quad \{w \in H^1(\mathbf{I}_t), w(L+\mathbf{v}t) = 0\}, \quad \text{for } t \geq 0.$
- $V_0(0,\ell(t)) \quad := \quad \{w \in H^1(0,\ell(t)), w(0) = 0\}, \quad \text{for } t \geq 0.$

A.3 SOME USEFUL IDENTITIES AND INEQUALITIES

The following inequalities are often used to derive estimates in Analysis

Parallelogram law

For any two elements u and v of an inner product space, we have

$$\|v+w\|^2 + \|v-w\|^2 = 2\left(\|v\|^2 + \|w\|^2\right).$$

Algebraic equality

Let $|a| = \max\{a, -a\}$, then

$$\max\{a,b\} = \frac{a+b+|a-b|}{2}, \quad \min\{a,b\} = \frac{a+b-|a-b|}{2}.$$

Hölder's inequality

Let $p, q \in [1,\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Cauchy-Schwartz inequality

$$|\langle f, g\rangle| \leq |f|\,|g|\,,\ \forall f,\ g \in \mathbb{R}^n.$$

Integration by parts formula

Let Ω be a réguiler set of class C^1. Let u, v two functions of $C^1\left(\overline{\Omega}\right)$. Then, we have

$$\int\limits_{\Omega} u\left(x\right)\frac{\partial v}{\partial x_i}\left(x\right)dx = \int\limits_{\partial\Omega} w\left(x\right)v\left(x\right).n_i\left(x\right)ds - \int\limits_{\Omega} v\left(x\right)\frac{\partial u}{\partial x_i}\left(x\right)dx,$$

where n_i is the i-th component of the unit exterior normal of Ω.

Leibnitz's formula

Let $a(t)$ and $b(t)$ be twice differentiable functions in some interval and let $f(s,t)$ be both integrable with respect to x over the interval $a(t) \leq x \leq b(t)$ and differentiable with respect to t. Then

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(s,t)ds = \left(\frac{db}{dt}\right)f\left(b(t),t\right) - \left(\frac{da}{dt}\right)f\left(a(t),t\right) + \int_{b(t)}^{a(t)}\frac{\partial f}{\partial t}(s,t)ds.$$

A.4 GENERALIZED FOURIER SERIES IN WEIGHTED HILBERT SPACES

Let $a, b \in \mathbb{R}, b > a$, and consider a positive (weight) function $\rho : (a,b) \to \mathbb{R}$. We denote by $L^2\left(a,b,\rho ds\right)$ the weighted Hilbert space of measurable complex-valued functions on $\mathbb{R}$, endowed by the scalar product

$$\int\limits_a^b h\left(s\right)\overline{g\left(s\right)}\rho\left(s\right)ds$$

and its associated norm. As usual, we drop ρds in the L^2 space notation if ρ is identically one.

Definition A.1 *Let the set of functions $\{\varphi_n\}_{n\in\mathbb{Z}}$ from an orthonormal set over $[a,b]$, so that*

$$\int_a^b \varphi_n\left(s\right)\varphi_m\left(s\right)\rho(s)ds = \begin{cases} 1, & for\ \ m = n, \\ 0, & for\ \ m \neq n. \end{cases}$$

If the set of functions $\{\varphi_n\}_{n\in\mathbb{Z}}$ is a complete orthonormal basis of the space $L^2\left(a,b,\rho ds\right)$, then the generalized Fourier series representation of an integrable function $h \in L^2\left(a,b,\rho ds\right)$ can be written as

$$h\left(s\right) = \sum_{n\in\mathbb{Z}} \mathbf{c}_n\varphi_n\left(s\right),$$

where the generalized Fourier coefficients of $h\left(s\right)$ are given by

$$\mathbf{c}_n = \int_a^b h\left(s\right) \overline{\varphi_n\left(s\right)} \rho(s) ds.$$

In particular, the following Parseval equality for generalized Fourier serie holds[1]

$$\int_a^b \left|h(s)\right|^2 \rho(s) ds = \sum_{n\in\mathbb{Z}} \left|\mathbf{c}_n\right|^2,$$

where $\mathbf{c}_n$ are generalized Fourier coefficients of $h\left(s\right)$, see for instance [37].

Example A.1 *A well-known result of analysis is that the set of functions* $\left\{e^{i\pi nt}/\sqrt{2}\right\}_{n\in\mathbb{Z}}$ *is a complete orthonormal set in the space* $L^2\left(-1,1\right)$.

Example A.2 *Under the assumption* $0 < \mathbf{v} < 1$*, then, the set of functions* $\left\{\sqrt{\frac{1-\mathbf{v}}{2L}} e^{n\pi i(1-\mathbf{v})(t+x)/L}\right\}_{n\in\ \mathbb{Z}}$ *is a complete and orthonormal basis of the space* $L^2\left(\mathbf{v}t, L_2 + \mathbf{v}t\right)$.

Example A.3 *Under the assumption* $0 < \mathbf{v} < 1$*, then, the set of functions* $\left\{\sqrt{\frac{1+\mathbf{v}}{2L}} e^{n\pi i(1+\mathbf{v})(t-x)/L}\right\}_{n\in\ \mathbb{Z}}$ *is a complete and orthonormal basis of the space* $L^2\left(-L_1 + \mathbf{v}t, L + \mathbf{v}t\right)$.

Example A.4 *Under the assumption* $\left|\ell'\left(t\right)\right| < 1$*, then, the set of functions* $\left\{\ e^{i\pi n\varphi(t\pm x)}/\sqrt{2}\right\}_{n\in\mathbb{Z}}$ *is a complete and orthonormal basis of the weighted space* $L^2\left(-\ell\left(t\right), \ell\left(t\right), \varphi'\left(t\pm x\right)dx\right)$*, the proof can be found in [19].*

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[1] $\left\{\varphi_n\right\}_{n\in\mathbb{Z}}$ is a complete orthonormal basis that is important for Parseval's theorem. if the set $\left\{\varphi_n\right\}_{n\in\mathbb{Z}}$ is not a complete orthonormal basis, then satisfy Bessel's inequality $\sum_{n\in\mathbb{Z}} \left|\mathbf{c}_n\right|^2 \leq \int_a^b \left|h(s)\right|^2 \rho(s) ds$.

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العنوان: بعض النتائج حول معادلة الموجة في مجال مرتبط بالزمن

الملخص: في هذه الرسالة، ندرس الاهتزازات الصغيرة لخيط، حيث يتم وصف هذه الاهتزازات بواسطة معادلة موجية ذات بعد واحد مع أطراف متحركة مع الزمن. في الحالة الأولى، نتعامل مع نقطتي نهاية تتحركان في نفس الاتجاه بسرعة ثابتة. يتم التعبير عن الحل بواسطة صيغة سلسلة حيث يتم حساب المعاملات بدلالة المعطيات الأولية. نحصل على كمية محفوظة مكافئة لطاقة الحل. بعد ذلك، نثبت قابلية الملاحظة عند أطراف الخيط في وقت دقيق. في الحالة الثانية، أضفنا مثبطا في أحد طرفيه. نشتق تقديرات دقيقة علويه وسفلية متناقصة أسيا للطاقة مع ثوابت محددة .علاوة على ذلك، تم إعطاء بعض الأمثلة العددية لتوضيح فعالية هذا المثبط .أخيرا، نحن مهتمون بمجال أحد طرفيه ثابت والآخر متحرك. تحصلنا على تقديرات سفلية وعلوية لطاقة الخيط عند وضع مثبط في الطرف المتحرك .

الكلمات المفتاحية: معادلة الموجة، المجالات المرتبطة بالزمن، سلسلة فورييه المعممة، تقديرات الطاقة، قابلية الملاحظة عند الأطراف، التناقص الأسي، قابلية الاستقرار عند الحدود.

Title: Some results on the wave equation in time-dependent domains

Abstract: In this thesis, we study the small vibrations of a string, where the vibrations are described by a 1–d wave equation in a time-dependent interval. First, we deal with two endpoints moving in the same direction at a constant speed. The solution is expressed by a series formula where the coefficients are explicitly computed in the function of the initial data. We obtain a conserved functional equivalent to the energy of the solution. Then, we establish the boundary observability in a sharp time. Then, we added a dashpot damper at one end. We derive precise upper and lower exponentially decaying estimates for the energy with explicit constants. Finally, we are interested in domain one of its sides is fixed while the other one is moving. We establish lower and upper estimates for the energy of the string when a damper is placed at the moving end.

Keywords: Wave equation, time-dependent domains, generalized Fourier series, energy estimates, boundary observability, exponential decay, boundary stabilization.

Titre : Quelques résultats sur l’équation d’onde dans les domaines dépendant du temps

Résumé : Dans cette thèse, nous étudions les petites vibrations d’une corde, où les vibrations sont décrites par une équation d’onde 1–d dans un intervalle dépendant du temps. Dans un premier temps, nous avons affaire à deux extrémités se déplaçant dans la même direction à vitesse constante. La solution est exprimée par une formule de série où les coefficients sont explicitement calculés en fonction des données initiales. On obtient une fonctionnel conservé équivalent à l'énergie de la solution. Ensuite, on étudie l'observabilité à la frontière de la corde. Dans un deuxième temps, nous avons ajouté un amortisseur à une extrémité. Nous dérivons des estimations supérieures et inférieures précises de décroissance exponentielle pour l’énergie avec des constantes explicites. Finalement, nous nous intéressons au domaine dont l’un de ses côtés est fixe tandis que l’autre est en mouvement. Nous établissons des estimations inférieures et supérieures pour l’énergie de la corde lorsqu’un amortisseur est placé à l’extrémité mobile.

Mots-clés : Équation d’onde, domaines dépendant du temps, Fourier généralisé séries, estimations d’énergie, observabilité frontière, décroissance exponentielle, stabilisation frontière.