PEOPLE’S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

Mohamed Boudiaf university of M’sila
Faculty of Mathematics and computer sciences
Department of Mathematics

Master memory

Field : Mathematics and computer sciences
Branch : Mathematics
Option : Functional analysis

Theme

Some Properties of Mixed Morrey Spaces and Applications

Presented by :
MERZAKA ALI SAOICHA

In front of the jury composed of :

Mr. Abdelkader Gasmi	Prof.,	University of M’sila	**President**.
Mr. Aissa DJERIOU	M.C.A,	University of M’sila	**Supervisor.**
Mr. Seghiri Fakhreddine	M.C.A,	University of M’sila	**Examiner.**

University year 2023/2024

Contents

Dedication

*Avant tout, Praise be to **Allah***

*the Lord of the Worlds, and peace and blessings be upon the most honorable and final messenger of the Prophet **Muhammad**, upon whom be the best prayer and the greatest peace, and after..*

*Avant tout, At the outset, I must first thank **Allah***

the Almighty, who enabled me to complete this memorandum and that the researcher may benefit from it. Avant tout, I would also like to express my gratitude and appreciation to the supervisor of my memorandum,

*Mr. **DJERIOU Aissa** for his valuable advice, guidance and support for my work.*

*memory. All gratitude to the president of the jury, **Mr. Abdelkader Gasmi** and the teacher of the examiners **Mr. Fakhreddine Seghiri** for having devoted their time and efforts to read and review my work and all those who supported me from near or far in the realization of this work.*

Thank you

List of Symbols

In what follows, we will use the following notations.

$\mathbb{R}^n$	Euclidean, n-dimensional space.
$\mathbb{Z}$	is the set of all integers.
$\mathbb{C}$	The set of complex numbers.
$\mathcal{S}(\mathbb{R}^n)$	Schwartz space.
$\mathcal{S}'(\mathbb{R}^n)$	Dual of $\mathcal{S}(\mathbb{R}^n)$, space of tempered distributions.
x	Vecteur de $\mathbb{R}^n$, $x = (x_1, x_2, ..., x_n)$, $x_i \in \mathbb{R}$, $1 \leq i \leq n$.
p'	is the conjugate exponent of p where $\frac{1}{p} + \frac{1}{p'} = 1$.
$\vec{p}$	Will denote n-tuples of the numbers in $(0, \infty]$ $(n \geq 1)$, $\vec{p} = (p_1, ..., p_n)$.
$r\vec{p}$	Will denote n-tuples of the numbers in $(0, \infty]$ $(n \geq 1)$, $r\vec{p} = (rp_1, ..., rp_n)$.
$\vec{p'} = (p'_1, ..., p'_n)$	Denotes the conjugate exponent of $\vec{p}$, where p'_j is the conjugate exponent of p_j.
$\frac{1}{\vec{p}}$	Will denote the vector $(\frac{1}{p_1}, ..., \frac{1}{p_n})$.
$d\mu$	or dx Lebesgue measure N-dimensional.
Ω	Denotes the open set in $\mathbb{R}^n$.
a.e.	Almost everywhere.
C^∞	Smooth functions.
C_0^∞	Smooth functions with compact support.
χ_Q	Denotes the characteristic function of a measurable set $Q \subset \mathbb{R}^n$ is defined by $\chi_Q = \begin{cases} 1 & if \quad x \in Q, \\ 0 & if \quad x \notin Q. \end{cases}$
supp f	supp $f = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}$.
$L^0(\Omega)$	The space of all measurable functions from $\Omega \subseteq \mathbb{R}^n$ to $\mathbb{R}$ or $\mathbb{C}$.
$L^1_{loc}(\mathbb{R}^n)$	Is the space of measurable functions on $\mathbb{R}^n$, in tegrable on any compact of $\mathbb{R}^n$.
$L^0_{loc}(\mathbb{R}^n)$	denotes the set of measureable functions on $\mathbb{R}^n$integrable on any compact of $\mathbb{R}^n$.
$B(x, r)$	The ball with center x and radius r , defined by $B(x, r) = \{y \in \mathbb{R}^n : \lvert y - x \rvert < r\}$.
$\hat{f} = \mathcal{F}(f)$	Denotes the Fourier transform.
$f * g$	Denotes the convolution of f and g .

Introduction

Classical Morrey spaces were introduced by Morrey [11], who considered an estimate of the difference of functions in 1938 in order to investigate the local behavior of solutions to second order elliptic partial differential equations, where his focus is on Morrey spaces $\mathcal{M}^p_q(\mathbb{R}^n)$ which are generalizations of Lebesgue spaces $\mathcal{M}^p_p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

In 1961, Benedek and Panzone [3] are introduit the mixed Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$, with $\vec{p} \in (0, \infty]^n$, this spaces as a natural generalization of the classical Lebesgue space $L^p(\mathbb{R}^n)$) via replacing the constant exponent p by an exponent vector $\vec{p}$.

Nogayama [13, 14] studied mixed Morrey spaces $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$, which encompass mixed Lebesgue spaces $L^{\vec{q}}(\mathbb{R}^n)$. In his work, he established the boundedness of the maximal operator, the fractional integral operator, and singular integral operators.

In this memory, we based on the works of Benedek [3] and Nogayama [13], we will delve into the fundamental concepts of Mixed Lebesgue spaces and Mixed Morrey spaces, exploring their definitions, key properties, and applications in various areas of mathematics and its allied fields.The memory is divided into three chapters.

The first chapter begins by defining some basic function spaces such as Lebesgue $L^p(\mathbb{R}^n)$ spaces and Morrey spaces $\mathcal{M}^p_q(\mathbb{R}^n)$ with the aim that we can accommodate as many function spaces as possible, which we will cover in the next two chapters.

In chapter 2 we introduce the definitions and properties of mixed Lebesgue spaces.

Finally, we present the mixed Morrey spaces $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ with $\vec{q} \in (0, \infty]^n$, $p \in (0, \infty]$ and their applications to the boundedness of operators. To this end, we first recall the notion and some examples of mixed Morrey spaces , then we present some basic properties about these spaces as well as the Hardy-Littlewood maximal operator.

CHAPTER 1

PRELIMINARIES

1.1 Some basic function spaces

In this chapter, we discuss some basic preliminary concepts that are directly related to the research topic, including some functional spaces, Lebesgue spaces $L^q(\mathbb{R}^n)$ and Morrey spaces $\mathcal{M}^p_q(\mathbb{R}^n)$.

1.1.1 The spaces of p- summing sequences

We recall in this subsection by the definition of p- summing sequences spaces.

Definition 1.1.1. *Let $0 < p \leq \infty$. Let $\ell^p(\mathbb{N})$ be the space of real-valued or complex-valued sequences $\{f_j\}_{j\geq 0}$ such that,*

$$\|\{f_j\}_{j\geq 0}\|_{\ell^p(\mathbb{N})} = \left(\sum_{j=0}^{\infty} |f_j|^p\right)^{\frac{1}{p}} < \infty,\ if\ 0 < p < \infty$$

and

$$\|\{f_j\}_{j\geq 0}\|_{\ell^\infty(\mathbb{N})} = \sup_{j\in\mathbb{Z}} |f_j| < \infty.$$

Remark 1.1.2.

1. *These spaces $\ell^p(\mathbb{N})$ are Banach spaces for $p \geq 1$.*

2. *If $0 \leq p \leq q \leq \infty$, then we have $\ell^p \hookrightarrow \ell^q$.*

1.1.2 Lebesgue spaces

Definition 1.1.3. *Let $0 < p \leq \infty$ and $\Omega \subset \mathbb{R}^n$. Let*

$$L^p(\Omega) = \{f : \Omega \to \mathbb{C},\ \ \textit{such that} f \in L^0(\Omega) \textit{ and } \|f|L^p(\Omega)\| < \infty\}.$$

With,

$$\|f|\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}}, \qquad if \quad 0 < p < \infty$$

and

$$\sup_{x\in\Omega} ess|f(x)|, \qquad if \quad p = \infty.$$

If $\Omega = \mathbb{R}^n$, we pose $L^p(\mathbb{R}^n) = L^p$. The spaces $L^p(\Omega)$ are Banach spaces for $p \geq 1$.

1.1.3 The locally integrable Lebesgue space

Definition 1.1.4. *Let $0 < p \leq \infty$. The locally integrable Lebesgue space is defined by,*

$$L^p_{loc}(\mathbb{R}^n) = \{f measurable : f\chi_K \in L^p(\mathbb{R}^n), \textit{ for all compact subsets } K \subset \mathbb{R}^n\}.$$

1.1.4 The weighted Lebesgue space

Definition 1.1.5. *We say that ω is a weight function, if is a function locally intégrable function on $\mathbb{R}^n$.*

Definition 1.1.6. *Let $0 < p \leq \infty$, $\Omega \subseteq \mathbb{R}^n$ and ω be a weight function, we pose*

$$L^p(\mathbb{R}^n,\omega) = \{f : \Omega \longrightarrow \mathbb{C}, such\ that\ f is\ measurable\ and\ \|f|L^p(\Omega,\omega)\| < \infty\}.$$

With,

$$\|f\|_{L^p(\Omega,\omega)} = \|f\|_{p.w} = \begin{cases} \left(\int_\Omega |f(x)|^p w(x)dx\right)^{\frac{1}{p}}, & if \;\; 0 < p < \infty \\ \sup_{x\in\Omega} ess\omega(x)|f(x)|, & if \;\; p = \infty. \end{cases}$$

If $\Omega = \mathbb{R}^n$, we pose $L^p(\mathbb{R}^n,\omega) = L^p(\omega)$.

Remark 1.1.7. *The spaces $L^p(\Omega,\omega)$ are Banach spaces for $1 \leq p \leq \infty$.*

1.1.5 The $\mathbf{L^{q,\Omega}}$ spaces

Definition 1.1.8. *If Ω is a compact set of $\mathbb{R}^n$, we write*

$$L^{q,\Omega} = \{f \in \mathcal{S}'(\mathbb{R}^n) : \operatorname{supp} \mathcal{F}f \subset \Omega, \quad \|f\|_q < \infty\}.$$

1.1.6 The $\ell_q(L^p)$ vector-valued spaces

Definition 1.1.9. *Let $0 < p \leq \infty$, $0 < q < \infty$, then $\ell_q(L^p)$ is the space of sequences $\{f_k\}_k \subset \mathcal{S}'(\mathbb{R}^n)$ such that,*

$$\|\{f_k\}_k\|_{\ell_q(L^p)} = \left(\sum_{k=0}^{\infty} \|f_k\|_p^q\right)^{\frac{1}{q}} < \infty.$$

Proposition 1.1.10. ***(Hölder inequality)***

Let $f \in L^q(\mathbb{R}^n)$ and $g \in L^{q'}(\mathbb{R}^n)$, with $1 \leq q, q' \leq \infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then $f \cdot g \in L^1(\mathbb{R}^n)$ and

$$\|f \cdot g\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^q(\mathbb{R}^n)} \|g\|_{L^{q'}(\mathbb{R}^n)}.$$

Proposition 1.1.11. ***(Minkowski inequality)***

Let $f, g \in L^q(\mathbb{R}^n)$ and $1 \leq q \leq \infty$. Then $f + g \in L^q(\mathbb{R}^n)$ and

$$\|f + g\|_{L^q(\mathbb{R}^n)} \leq \|f\|_{L^q(\mathbb{R}^n)} + \|g\|_{L^q(\mathbb{R}^n)}.$$

Theorem 1.1.12. ***(Generalized Minkowski inequality)***

Let $0 < \gamma \leq \beta \leq \infty$ and $\{f_j\}_{j=0}^{\infty} \subset L^{\gamma}_{loc}$, then we have

$$\left(\sum_{j=1}^{\infty} \left(\int_{\mathbb{R}^n} |f_j(x)|^{\gamma} dy\right)^{\frac{\beta}{\gamma}} dx\right)^{\frac{1}{\beta}} \leq c \left(\int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} |f_j(x)|^{\beta} dx\right)^{\frac{\gamma}{\beta}} dy\right)^{\frac{1}{\gamma}}.$$

1.1.6.1 The Morrey spaces

Definition 1.1.13. ***(Morrey's space $L^{p,\lambda}(\Omega)$)***

Let $1 < p < \infty$, $0 < \lambda < n$ and f be a real measurable function defined in $\Omega \subseteq \mathbb{R}^n$. Morrey space is denoted by $L^{p,\lambda}(\Omega)$ is the set of functions $f \in L^p(\Omega)$ such that:

$$\|f\|^p_{L^{p,\lambda}(\Omega)} := \sup_{x \in \Omega, r > 0} r^{-\lambda} \int_{\Omega \cap B_r(x)} |f(y)|^p dy.$$

If $\Omega = \mathbb{R}^n$, we pose $L^{p,\lambda}(\Omega) = L^{p,\lambda}(\mathbb{R}^n)$.

Remark 1.1.14.

1. *The spaces $L^{p,\lambda}(\Omega)$ becomes a Banach space.*

2. *If* $\lambda = 0$ *and* $\lambda = n$, *then* $L^{p,0}(\Omega) = L^p(\Omega)$ *and* $L^{p,n}(\Omega) = L^{\infty}(\Omega)$.

To describe various properties of functions in $L^{p,\lambda}(\mathbb{R}^n)$, it is sometimes convenient to use the notation $\mathcal{M}^p_q(\mathbb{R}^n)$.

Definition 1.1.15. *([16] ($\mathcal{M}^p_q(\mathbb{R}^n)$). Let* $0 < q \leq p \leq \infty$. *For an* $L^q_{loc}(\mathbb{R}^n)$*-function f, its (classical) Morrey norm is defined by*

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |f(y)|^q dy\right)^{\frac{1}{q}} \tag{1.1}$$

If $0 < q \leq p = \infty$, *then the Lebesgue differentiation theorem shows that* $\mathcal{M}^p_q(\mathbb{R}^n) = L^{\infty}(\mathbb{R}^n)$, *so we exclude this case.*

By (1.1.13) and (1.1), we have:

$$L^{p,\lambda}(\mathbb{R}^n) = \mathcal{M}^p_q(\mathbb{R}^n), if \lambda = n(1 - q/p) \text{ or equivalently } p = \frac{qn}{n-\lambda}.$$

Theorem 1.1.16. *For* $0 < p < \infty$, $\mathcal{M}^p_p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ *with coincidence of norms.*

Proof. Let $f \in L^0(\mathbb{R}^n)$. We write down the definition of the norm $\|f\|_{\mathcal{M}^p_p}$,

$$\|f\|_{\mathcal{M}^p_p} \equiv \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |B(x,r)|^{\frac{1}{p}-\frac{1}{p}} \|f\|_{L^p(B(x,r))} = \sup_{(x,r)\in\mathbb{R}^{n+1}_+} \|f\|_{L^p(B(x,r))}.$$

Then, we have

$$\|f\|_{\mathcal{M}^p_p} \leq \left(\int_{B(x,r)} |f(y)|^p dy\right)^{\frac{1}{p}} = \|f\|_{L^p} \tag{1.2}$$

Meanwhile, by the monotone convergence theorem, for any $x \in \mathbb{R}^n$, we have

$$\|f\|_{L^p} = \lim_{m\to\infty} \|f\|_{L^p(B(x,m))} \leq \sup_{r>0} \|f\|_{L^p(B(x,r))} = \|f\|_{\mathcal{M}^p_p} \tag{1.3}$$

Combining (1.2) and (1.3), we obtain the desired result. □

Example 1.1.17. *Let* $0 < q \leq p < \infty$ *and* B *be an open ball. Then*

$$\|\chi_B\|_{\mathcal{M}^p_q} = \|\chi_B\|_{L^p} \tag{1.4}$$

In fact, it is easy to see that

$$\|\chi_B\|_{\mathcal{M}^p_q} \leq \|\chi_B\|_{L^p} \tag{1.5}$$

from Theorem 1.1.16 *and Hölder's inequality. (See also Theorem* 1.1.20 *to follow) If we write out the norm* $\|\chi_B\|_{\mathcal{M}^p_q}$ *in full, then*

$$\|\chi_B\|_{\mathcal{M}^p_q} = \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |\chi_B(y)|^q dy\right)^{\frac{1}{q}}.$$

We can calculate and evaluate the integral precisely. The result is:

$$\|\chi_B\|_{\mathcal{M}^p_q} = \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} |B(x,r)\cap B|^{\frac{1}{q}} \geq |B|^{\frac{1}{p}} = \|\chi_B\|_{L^p} \tag{1.6}$$

Combining (1.5) *and* (1.6), *we obtain* (1.4).

Proposition 1.1.18. *Let* $0 < q \leq p < \infty$.

(1) *The Morrey space* $\mathcal{M}^p_q(\mathbb{R}^n)$ *does not have* $L^\infty_c(\mathbb{R}^n)$ *as dense subspace.*

(2) *The Morrey space* $\mathcal{M}^p_q(\mathbb{R}^n)$ *is not separable.*

Theorem 1.1.19. *([16]) For all* $0 < q \leq p < \infty$, *we have a scaling law:*

$$\|f(t\cdot)\|_{\mathcal{M}^p_q} = t^{-\frac{n}{p}} \|f\|_{\mathcal{M}^p_q}$$

for all $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ *and* $t > 0$.

Proof. By the change of variables $z = ty$ and as, $t^n|B(x,r)| = |B(tx,tr)|$ we have:

$$\begin{aligned} \|f(t\cdot)\|_{\mathcal{M}^p_q(\mathbb{R}^n)} &= \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |f(ty)|^q dy\right)^{\frac{1}{q}} \\ &= \sup_{(x,r)\in\mathbb{R}^{n+1}_+} t^{-\frac{n}{p}} |B(tx,tr)|^{\frac{1}{p}} \left(\frac{1}{t^{-n}|B(tx,tr)|}\int_{B(tx,tr)} |f(z)|^q t^{-n} dz\right)^{\frac{1}{q}} \\ &= t^{-\frac{n}{p}} \|f\|_{\mathcal{M}^p_q(\mathbb{R}^n)}. \end{aligned}$$

□

Theorem 1.1.20. *([16]) For all* $f \in L^0(\mathbb{R}^n)$ *and* $0 < q_1 \leq q_0 \leq p < \infty$,

$$\|f\|_{\mathcal{M}^p_{q_1}} \leq \|f\|_{\mathcal{M}^p_{q_0}}.$$

In particular,

$$\mathcal{M}^p_{q_0}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q_1}(\mathbb{R}^n).$$

Proof. Using (1.1), we write out the norms in full:

$$\|f\|_{\mathcal{M}^p_{q_1}} \equiv \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |B(x,r)|^{\frac{1}{p}-\frac{1}{q_1}} \left(\int_{B(x,r)} |f(y)|^{q_1} dy\right)^{\frac{1}{q_1}} \tag{1.7}$$

$$\|f\|_{\mathcal{M}^p_{q_0}} \equiv \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |B(x,r)|^{\frac{1}{p}-\frac{1}{q_0}} \left(\int_{B(x,r)} |f(y)|^{q_0} dy\right)^{\frac{1}{q_0}} \tag{1.8}$$

By Hölder's inequality (for probability measures), we have

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|^{q_1} dy\right)^{\frac{1}{q_1}} \leq \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|^{q_0} dy\right)^{\frac{1}{q_0}} \tag{1.9}$$

Thus, by inserting inequality (1.9) into (1.7) and (1.8), we obtain the desired result. ☐

1.1.7 A_p Weights spaces

By a weight we mean a measurable function which satysfy $0 < w(x) < \infty$ for almoste all $x \in \mathbb{R}^n$.

Definition 1.1.21. *([8]) Let $1 < p < \infty$ and w be a weigh. Then, w is said to be an A_p weight if*

$$[w]_{A_p} = \sup_{Q\in\mathcal{Q}} \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}dx\right)^{p-1} < \infty.$$

A weight w is said to be an A_1 weight if,

$$[w]_{A_1} = \sup_{Q\in\mathcal{Q}} \left(\frac{1}{|Q|}\int_Q w(x)dx\right) ass\sup_{x\in Q} w(x)^{-1} < \infty.$$

1.1.8 Hardy-Litlewood maximal function

Let us begin with giving the definition of the Hardy-Litlewood maximal function, which plays a very important role in harmonic analysis.

Definition 1.1.22. ***([18], Hardy-Litlewood maximal function)***

Let $f \in L^0(\mathbb{R}^n)$. Then for any $x \in \mathbb{R}^n$, the Hardy-Litlewood maximal function $\mathcal{M}f$ is defined by:

$$\mathcal{M}f(x) = \sup_{Q\in\mathcal{Q}} \frac{\chi_Q(x)}{|Q|}\int_Q |f(y)|dy.$$

Where $\mathcal{Q}$ denotes the set of all cubes in $\mathbb{R}^n$.

Definition 1.1.23. *Let $0 < t < \infty$. We define the powered Hardy–Littlewood maximal operator $\mathcal{M}_{(t)}$ by,*

$$\mathcal{M}_t f(x) = (\mathcal{M}[|f|^t](x))^{\frac{1}{t}}.$$

Theorem 1.1.24. *Let $0 < p < \infty$, then*

$$\|\mathcal{M}f\|_p \leq \|f\|_p$$

for $f \in L^p(\mathbb{R}^n)$.

Proof. See for example [17, p. 21] and [19, p. 55-57] or [21, p. 89]. $\square$

1.2 Interpolation spaces

Theorem 1.2.1. ***([4], Riesz-Thorin theorem)***

Let (X, μ), (Y, ν) be two measured spaces and $p_0, p_1, q_0, q_1 \in [1, \infty]$ with $p_0 \neq p_1, q_0 \neq q_1$ and a linear map T is continuous from $L^{p_0}(X, \mu)$ into $L^{q_0}(Y, \nu)$ and from $L^{p_1}(X, \mu)$ into $L^{q_1}(Y, \nu)$ such that, for any simple function f :

$$\|Tf\|_{q_i} \leq C_i \|f\|_{p_i} \quad (i = 0, 1),$$

then for $0 < \theta < 1$ it is continuous from $[L^{p_0}, L^{p_1}]_\theta = L^p$ into $[L^{q_0}, L^{q_1}]_\theta = L^q$, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

moreover

$$\|Tf\|_q \leq C_0^{1-\theta} C_1^{\theta} \|f\|_p.$$

Theorem 1.2.2. *([10]) Let $1 < p_0 \leq q_0 < \infty$ and $1 < p_1 \leq q_1 < \infty$. Let $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Then:*

(1) *We have $[\mathcal{M}^{p_0}_{q_0}(\mathbb{R}^n), \mathcal{M}^{p_1}_{q_1}(\mathbb{R}^n)]_\theta \subset \mathcal{M}^p_q(\mathbb{R}^n)$ (complex interpolation).*

(2) *We have $[\mathcal{M}^{p_0}_{q_0}(\mathbb{R}^n), \mathcal{M}^{p_1}_{q_1}(\mathbb{R}^n)]_\theta = \mathcal{M}^p_q(\mathbb{R}^n)$ (with equivalence of norms) if and only if $\frac{p_0}{q_0} = \frac{p_1}{q_1}$.*

(3) *We have $[\mathcal{M}^{p_0}_{q_0}(\mathbb{R}^n), \mathcal{M}^{p_1}_{q_1}(\mathbb{R}^n)]_{\theta.p} \subset \mathcal{M}^p_q(\mathbb{R}^n)$ ((real interpolation).*

(4) *We have $\mathcal{M}^p_q(\mathbb{R}^n) \subset [\mathcal{M}^{p_0}_{q_0}(\mathbb{R}^n), \mathcal{M}^{p_1}_{q_1}(\mathbb{R}^n)]_{\theta.\infty}$ (continuous embedding) if and only if $\frac{p_0}{q_0} = \frac{p_1}{q_1}$.*

(5) *$[\mathcal{M}^{p_0}_{q_0}(\mathbb{R}^n), \mathcal{M}^{p_1}_{q_1}(\mathbb{R}^n)]_{\theta.\infty} \subset \mathcal{M}^p_q(\mathbb{R}^n)$ if and only if $p_0 = q_0$.*

CHAPTER 2

SOME BASIC PROPERTIES OF MIXED LEBESGUE SPACES AND APPLICATIONS

Mixed Lebesgue spaces are a generalization of $L^p(\mathbb{R}^n)$ spaces that occur naturally when considering functions that depend on quantities with different properties, such as space and time. We first present mixed Lebesgue versions of several classical results and other vector-valued inequalities in mixed-Lebesgue spaces.

2.1 The mixed Lebesgue spaces and their properties

In this section, we discuss some properties and examples of mixed Lebesgue spaces, we recall the definition of mixed Lebesgue spaces.

2.1.1 Definiton of mixed Lebesgue spaces

Definition 2.1.1. *([3], Mixed Lebesgue space)*

Let $\vec{p} = (p_1, ..., p_n) \in (0, \infty]^n$. The mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that their quasi-norms

$$\|f\|_{\vec{p}} \;=\; \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, ..., x_n)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{p_3}{p_2}} \cdots dx_n \right)^{\frac{1}{p_n}} < \infty,$$

with the standard modification when $p_j = \infty$, for some $1 \leq j \leq n$.

Remark 2.1.2.

1. *The quasi-norm $\|\cdot\|_{\vec{p}}$, is a norm when $\min(p_1, ..., p_n) \geq 1$ and turns $(L^{\vec{p}}(\mathbb{R}^n), \|\cdot\|_{\vec{p}})$ into a Banach space.*

2. *If $\vec{p} = (p, ..., p)$, then $L^{\vec{p}}(\mathbb{R}^n)$ coincides with the classical Lebesgue space L^p.*

3. *If $\vec{p} = (p_1, ..., p_n) \in (0, \infty]^n$, then the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ is complete.*

4. *For $\vec{p} \in [1, \infty]^n$ we define the conjugate $\vec{p}\,' := (p_1', ..., p_n') \in [1, \infty]^n$ by requiring that $1/p_j + 1/p_j' = 1$ for every $j \in \{1, ..., n\}$. Note that $L^{\vec{p}\,'}$ is the dual of $L^{\vec{p}}$ for every $\vec{p} \in [1, \infty]^n$); (see, e.g.,[3]).*

5. *We will denote by $\|f\|_{(p_1,...,p_n)}$ the measurable function on $\mathbb{R}^{n-j}$ that results from successively applying the L^{p_1}-norm in the x_1 variable, up to the L^{p_j}-norm in the x_j variable. Of course $\|\|f\|_{p_1}\|_{p_2} = \|f\|_{p_1 p_2}$, etc., and finally $\|f\|_{(p_1,...,p_n)} = \|f\|_{\vec{p}}$.*

Next, we consider the examples of $L^{\vec{p}}(\mathbb{R}^n)$.

Example 2.1.3. *Let $Q = I_1 \times ... \times I_n$ be a cube, where each I_j is an intervale of equal length, for $0 < \vec{p} \leq \infty$,*

$$\|\chi_Q\|_{\vec{p}} = |Q|^{\frac{1}{n}(\frac{1}{p_1}+\cdots+\frac{1}{p_n})}.$$

Hence, $\chi_Q(x) = \prod_{j=1}^{n} \chi_{I_j}(x_j)$, we have

$$\|\chi_Q\|_{\vec{p}} = \prod_{j=1}^{n} \|\chi_{I_j}\|_{p_j} = \prod_{j=1}^{n} \left(\int_{I_j} dx_j \right)^{\frac{1}{p_j}} = \prod_{j=1}^{n} |I_j|^{\frac{1}{p_j}}.$$

Notice that since Q is a cube, $|I_j| = l(Q) = |Q|^{\frac{1}{n}}$. This

$$\|\chi_Q\|_{\vec{p}} = \prod_{j=1}^{n} |I_j|^{\frac{1}{p_j}} = |Q|^{\frac{1}{n}(\frac{1}{p_1}+\cdots+\frac{1}{p_n})}.$$

2.1.2 Properties of mixed Lebesgue spaces

In this subsection we present some properties of mixed norm Lebesgue spaces. For additional properties of $L^{\vec{p}}$, see for example [1, 2, 3].

Proposition 2.1.4. *([13], Fatou's property for $L^{\vec{p}}(\mathbb{R}^n)$).*

Let $0 < \vec{p} \leq \infty$, $\{f_j\}_{j=1}^{\infty}$ be a sequence of non-negative measurable functions on $\mathbb{R}^n$.

Then,

$$\left\| \lim_{J \longrightarrow \infty} f_j \right\|_{\vec{p}} \leq \lim_{J \longrightarrow \infty} \|f_j\|_{\vec{p}}. \tag{2.1}$$

Theorem 2.1.5. *([12]) The following inclusions hold:*

$$\mathcal{S} \hookrightarrow L^{\vec{p}}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'.$$

Furthermore, they are dense and continuous for $\vec{p} \in [1, \infty)^n$.

Proof. see [12, theorem. 3]. ☐

Proposition 2.1.6. *([13]) Let $0 < \vec{p} \leq \infty$, the mixed Lebesgue norm has the dilation relation; for all $f \in L^{\vec{p}}(\mathbb{R})$ and $t > 0$,*

$$\|f(t\cdot)\|_{\vec{p}} = t^{-\sum_{j=1}^{n} \frac{1}{p_j}} \|f\|_{\vec{p}}.$$

Proof. see [13, pro. 2.5]. ☐

Proposition 2.1.7. ***([9], The triangle inequality)***

Let $\vec{p} = (p_1, ..., p_n) \in [1, \infty]^n$ and $f, g \in L^{\vec{p}}$, then it holds,

$$\|f + g\|_{\vec{p}}^r \leq \|f\|_{\vec{p}}^r + \|g\|_{\vec{p}}^r \tag{2.2}$$

for every $0 < r < \min(1, p_1, ..., p_n)$.

Proof. In fact $\int |f + g|^p d\mu \leq \int |f|^p d\mu + \int |g|^p d\mu$ hold for arbitrary measures if $0 < p < 1$, so $\|f + g\|_p \leq (\|f\|_p^\nu + \|f\|_p^\nu)^{\frac{1}{\nu}}$ for all $\nu \in]0, \min(1, p)]$. Using this, it is easy to see that $\||\cdot\||_p^r$ is sub additive for $r \leq \min(1, p, q)$. For $r \in]0, \min(1, p_1, ..., p_n)]$ a repeated use of this yields (2.2). ☐

The mixed Hölder inequality is the following estimate.

Theorem 2.1.8. ***([6], Hölder Inequality)***

Let $f \in L^{\vec{p}}(\mathbb{R}^n)$ and $g \in L^{\vec{q}}(\mathbb{R}^n)$, with $1 \leq \vec{p}, \vec{q} \leq \infty$ and define $\vec{r}$ so that $\frac{1}{\vec{p}} + \frac{1}{\vec{q}} = \frac{1}{\vec{r}}$ then $f \cdot g \in L^{\vec{r}}(\mathbb{R}^n)$ and:

$$\|f \cdot g\|_{L^{\vec{r}}(\mathbb{R}^n)} \leq \|f\|_{L^{\vec{p}}(\mathbb{R}^n)} \|g\|_{L^{\vec{q}}(\mathbb{R}^n)}.$$

Proof. By successive applications of Hölder's inequality we obtain the result. ☐

We will also need an adapted version of the Hausdorff-Young inequality.

Theorem 2.1.9. *([6]) The mixed Hausdorff-Young's Theorem asserts that if* $s = (s_1, ..., s_n)$ *with* $1 \leq s_n \leq s_{n-1} \leq ... \leq s_1 \leq 2$*, then for every* $f \in L^{\vec{s}}$

$$\|\hat{f}\|_{\vec{s'}} \leq \|f\|_{\vec{s}}.$$

Theorem 2.1.10. *([6]) The mixed convolution Young's inequality asserts that if* $\vec{p} \in [1, \infty]^n$, $f \in L^{\vec{p}}$*, and* $g \in L^1$*, then* $f * g \in L^{\vec{p}}$ *and*

$$\|f * g\|_{\vec{p}} \leq \|g\|_1 \|f\|_{\vec{p}}.$$

A mixed Nikol'skij's inequality can be found in [9, Proposition 4].

Theorem 2.1.11. *([9]) Let* $0 < p_j \leq r_j \leq \infty$ *for* $j = 1, ..., n$*. There exists a constant* $c > 0$ *such that for every* $f \in \mathcal{S}'(\mathbb{R}^n) \cap L^{\vec{p}}$ *with spectrum in a compacte rectangle given by* $|\xi_k| \leq R_k$ *for* $k = 1, ..., n$*, i.e.*

$$\operatorname{supp} \mathcal{F}f \subset [-R_1, R_1] \times \cdots \times [-R_n, R_n]$$

it also holds that $f \in L^{\vec{r}}$ *and*

$$\|f\|_{\vec{r}} \leq c \left(\prod_{j=1}^{n} R_j^{\frac{1}{p_j} - \frac{1}{r_j}} \right) \|f\|_{\vec{p}}.$$

Theorem 2.1.12. *([12], **Interpolation inequality**)*

If $\vec{p}, \vec{q}, \vec{r} \in [1, \infty]^n$ *are such that* $\lambda \in (0,1)$ *exists for which it holds that,*

$$\frac{1}{q_i} = \frac{\lambda}{p_i} + \frac{1-\lambda}{r_i} \qquad i \in \{1, ..., n\},$$

then, $L^{\vec{p}}(\mathbb{R}^n) \cap L^{\vec{r}}(\mathbb{R}^n) \subseteq L^{\vec{q}}(\mathbb{R}^n)$*, and for any* $f \in L^{\vec{p}}(\mathbb{R}^n) \cap L^{\vec{r}}(\mathbb{R}^n)$*, we have the inequality*

$$\|f\|_{\vec{q}} \leq \|f\|_{\vec{p}}^{\lambda} \|f\|_{\vec{r}}^{1-\lambda}.$$

Proof. The proof is obtained by induction on the dimension n, where the claim for $n = 1$ is the classical interpolation inequality.

Suppose that the claim is valid for $n - 1$ and split $x = (\bar{x}, x_n) = (x_1, ..., x_{n-1}; x_n)$, and similarly for $\vec{p}, \vec{q}$, and $\vec{r}$.

By the induction hypothesis, we have

$$\|f(\cdot, x_n)\|_{\bar{q}} \leq \|f(\cdot, x_n)\|_{\bar{p}}^{\lambda} \|f(\cdot, x_n)\|_{\bar{r}}^{(1-\lambda)} \qquad (ae \;\; x_n).$$

Clearly, for $p_n = r_n = q_n = \infty$, we can take the supremum and obtain the result.

Next, we assume that $p_n, r_n < \infty$. By hypothesis, we have

$$\|f\|_{\vec{q}}^{q_n} = \int_{\mathbb{R}} \|f(\cdot, x_n)\|_{\vec{q}}^{q_n} dx_n \leq \int_{\mathbb{R}} \|f(\cdot, x_n)\|_{\vec{p}}^{\lambda_{q_n}} \|f(\cdot, x_n)\|_{\vec{r}}^{(1-\lambda)_{q_n}} dx_n \tag{2.3}$$

We continue the estimate using Hölder's inequality with exponents $\frac{p_n}{\lambda_{q_n}}$ and $\frac{r_n}{(1-\lambda)_{q_n}}$ to obtain:

$$\|f\|_{\vec{q}}^{q_n} \leq \left(\int_{\mathbb{R}} \|f(\cdot, x_n)\|_{\vec{p}}^{p_n} dx_n\right)^{\frac{\lambda_{q_n}}{p_n}} \left(\int_{\mathbb{R}} \|f(\cdot, x_n)\|_{\vec{r}}^{r_n} dx_n\right)^{\frac{(1-\lambda)_{q_n}}{r_n}} = \|f\|_{\vec{p}}^{\lambda_{q_n}} \|f\|_{\vec{r}}^{(1-\lambda)_{q_n}}.$$

In other cases, without loss of generality, we can assume that $p_n < r_n = \infty$. In this case, $\lambda q_n = p_n$ and $\|f(\bar{x}, \cdot)\|_{\bar{r}} \leq \|f\|_{\vec{r}}$, and the estimate (2.3) can be continued as follows:

$$\|f\|_{\vec{q}}^{q_n} \leq \|f\|_{\vec{r}}^{(1-\lambda)q_n} \int_{\mathbb{R}} \|f(\cdot, x_n)\|_{\vec{p}}^{p_n} dx_n = \|f\|_{\vec{p}}^{\lambda q_n} \|f\|_{\vec{r}}^{(1-\lambda)q_n}.$$

We only state the (complex) interpolation theorem [3, Theorem 2, page 316] , as follows. □

Theorem 2.1.13. *([3], **Mixed Riesz-Thorin**)*

Let T be a linear operator defined on the space of simple functions on $\mathbb{R}^n$, which takes values in the space of measurable functions on $\mathbb{R}^n$, and take $\vec{p_0}, \vec{p_1}, \vec{q_0}, \vec{q_1} \in [1, \infty]^n$. If for any f in the domain of T we have (for some $M_0, M_1 > 0$)

$$\|Tf\|_{\vec{q_0}} \leq M_0 \|f\|_{\vec{p_0}},$$

$$\|Tf\|_{\vec{q_1}} \leq M_1 \|f\|_{\vec{p_1}},$$

then for any $t \in (0, 1)$, we also have the estimate,

$$\|Tf\|_{\vec{q_t}} \leq M_0^{1-t} M_1^t \|f\|_{\vec{p_t}}.$$

Where $1/\vec{p_t} = (1-t)/\vec{p_0} + t/\vec{p_1}$ and $1/\vec{q_t} = (1-t)/\vec{q_0} + t/\vec{q_1}$. Furthermore, for $\vec{p_t} \in [1, \infty)^n$, operator T can be extended by continuity to a bounded operator from $L^{\vec{p_t}}(\mathbb{R}^n)$ to $L^{\vec{q_t}}(\mathbb{R}^n)$ in a unique manner.

2.2 Boundedness of some operators in mixed Lebesgue spaces

In this section, we investigate the boundedness of some operators in $L^{\vec{p}}(\mathbb{R}^n)$.

2.2.1 Boundedness of the Hardy-Littlewood maximal operator

In this subsection, we investigate the boundedness of the iterated maximal operator in $L^{\vec{p}}(\mathbb{R}^n)$.

Definition 2.2.1. *Let* $1 \leq j \leq n$. *We define for* $f \in L^1_{loc}(\mathbb{R}^n)$,

$$\mathcal{M}_j f(x) = \sup_{I_j \in I^j_x} \frac{1}{|I_j|} \int_{I_j} |f(x_1, ..., y_j, ..., x_n)| dy_j,\ x = (x_1, ..., x_n) \in \mathbb{R}^n,$$

where I^j_x *is the set of all intervals* I_j *in* $\mathbb{R}_{x_j}$ *containing* x_j.

Definition 2.2.2. *Let* $1 \leq j \leq n$. *We will use extensively the following iterated maximal operator: for* $f \in L^1_{loc}(\mathbb{R}^n)$ *and* $\vec{t} \in (0, \infty)^n$ *we set,*

$$\mathcal{M}_{\vec{t}} f(x) := \left(M_n \left(\cdots M_2 (M_1 |f|^{t_1})^{\frac{t_2}{t_1}} \cdots \right)^{\frac{t_n}{t_{n-1}}} \right)^{\frac{1}{t_n}} (x),\ x \in \mathbb{R}^n.$$

The operator $\mathcal{M}_{(t,...,t)}$ *will be briefly denoted by* $\mathcal{M}_t$ *for every* $t > 0$.

Theorem 2.2.3. *([20]) Let* $0 < \vec{p} < \infty$. *If* $0 < t < \min(p_1, ..., p_n)$, *then*

$$\|\mathcal{M}_t f\|_{\vec{p}} \lesssim \|f\|_{\vec{p}} \tag{2.4}$$

for $f \in L^{\vec{p}}(\mathbb{R}^n)$.

In order to prove the Theorem 2.2.3, we need to recall the following Lemma due to Bagby in 1975 (see [2]).

Lemma 2.2.4. *([2]) Let* $1 < q_i < \infty (i = 1, ..., m)$ *and* $1 < p < \infty$. *Let* $(\Omega_i, \mu_i)(i = 1, ...m)$ *be* σ*-finite measure spaces, and let* $\Omega = \Omega_1 \times \cdots \times \Omega_m$. *For* $f \in L^0(\mathbb{R}^n \times \Omega)$,

$$\int_{\mathbb{R}^n} \|Mf(x, \cdot)\|^p_{(q_1,...,q_m)} dx \lesssim \int_{\mathbb{R}^n} \|f(x, \cdot)\|^p_{(q_1,...,q_m)} dx.$$

Now, we show the proof of theorem 2.2.3.

Proof. Since

$$\|\mathcal{M}_t f\|_{\vec{p}} = \|(M_n \cdots M_1[|f|^t])^{\frac{1}{t}}\|_{\vec{p}} = \|M_n \cdots M_1[|f|^t]\|^{\frac{1}{t}}_{(\frac{p_1}{t},...,\frac{p_n}{t})}.$$

We have only to check (2.4) for $t = 1$ and $1 < \vec{p} < \infty$.

Let $t = 1$. Then the conclution can be written as

$$\|\mathcal{M}_1 f\|_{\vec{p}} = \|M_n \cdots M_1 f\|_{\vec{p}} \lesssim \|f\|_{\vec{p}}.$$

We use induction on n. Let $n = 1$. Then, the result follows by the classical case of the boundedness of the Hardy-Littlewood maximal operator.

Suppose that the result holds for $n-1$, that is, for $h \in L^0(\mathbb{R}^{n-1})$ and $1 < (q_1, ..., q_{n-1}) < \infty$,

$$\|M_{n-1} \cdots M_1 h\|_{(q_1,...,q_{n-1})} \lesssim \|h\|_{(q_1,...,q_{n-1})}.$$

By Lemma 2.2.4

$$\|M_n f\|_{\vec{p}} = \big\| \big[\|M_n f\|_{(p_1,...,p_{n-1})} \big] \big\|_{(p_n)} \lesssim \big\| \big[\|f\|_{(p_1,...,p_{n-1})} \big] \big\|_{(p_n)} = \|f\|_{\vec{p}}.$$

This, by induction assumption, we obtain

$$\begin{aligned}
\|M_n M_{n-1} \cdots M_1 f\|_{\vec{p}} &= \|M_n [M_{n-1} \cdots M_1 f]\|_{\vec{p}} \\
&\lesssim \|M_{n-1} \cdots M_1 f\|_{\vec{p}} \\
&= \big\| \|[M_{n-1} \cdots M_1 f]\|_{(p_1,...,p_{n-1})} \big\|_{p_n} \\
&\lesssim \big\| \|f\|_{(p_1,...,p_{n-1})} \big\|_{p_n} \\
&\lesssim \|f\|_{\vec{p}}.
\end{aligned}$$

□

Remark 2.2.5.

1. *Jessen, Marcinkiewicz and Zygmund showed the boundedness of the iterated maximal operator in the classical L^p spaces in 1935, (see [7]).*

2. *Bagby showed the boundedness of the Hardy-Littlewood maximal operator for the functions taking values in mixed Lebesgue spaces in 1975, (see [2]).*

Example 2.2.6. *Let R be a set of all rectangles in $\mathbb{R}^n$. By M_R, denote the strong maximal operator which is generated by a rectangle R: for $f \in L^0(\mathbb{R}^n)$,*

$$M_R f(x) = \sup_{R \in \mathcal{R}} \frac{\chi_R(x)}{|R|} \int_R |f(y)| dy.$$

Then, the followings follow (see [7]):

$$M_R f(x) \leq M_n \cdots M_1 f(x) = \mathcal{M}_1 f(x)$$

and

$$M_R f(x) \leq M_1 \cdots M_n f(x),$$

and so on. Thus, the iterated maximal operator cancontroll the strong maximal operator. On the other hand, the relation between $M_1 \cdots M_n$ and $M_n \cdots M_1$ is not comparable. To see this, we give the following example. For the sake of simplicity, let $n = 2$. Let $f(x,y) = \chi_\Delta(x,y)$, where

$$\Delta = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq x\}.$$

First, we calculate $M_1 f$ and $M_2 f$:

$$M_1 f(x,y) \;=\; \begin{cases} 0 & (y \leq 0, 1 \leq y), \\ 1 & (0 \leq y \leq 1, y \leq x), \\ \frac{1-y}{1-x} & (0 \leq y \leq 1, x \leq y), \\ \frac{1-y}{x-y} & (0 \leq y \leq 1, 1 \leq x). \end{cases}$$

and

$$M_2 f(x,y) = \begin{cases} 0 & (y \leq 0, 1 \leq x), \\ \frac{x}{x-y} & (0 \leq x \leq 1, y \leq 0), \\ 1 & (0 \leq x \leq 1, 0 \leq y \leq x), \\ \frac{x}{y} & (0 \leq x \leq 1, x \leq y). \end{cases}$$

Next, we calculate $M_2 M_1 f$ and $M_1 M_2 f$. In particular, we consider two cases. For $0 \leq x \leq 1, \quad y \geq 1$, we get

$$M_2 M_1 f(x,y) = \frac{-x^2 - y^2 + 2y}{2y(1-x)}, \quad M_1 M_2 f(x,y) = \frac{x+1}{2y}$$

For $x \geq 1, 0 \leq y \leq 1$, we have

$$M_2 M_1 f(x,y) = \frac{1}{y}\left(y + (x-1)\log\frac{x-1}{x}\right), \qquad M_1 M_2 f(x,y) = \frac{x - \sqrt{x^2 - 1}}{y}$$

Thus, we obtain

$$M_2M_1f \leq M_1M_2f \quad (0 \leq x \leq 1, y \geq 1),$$

while,

$$M_2M_1f \geq M_1M_2f \quad (x \geq 1, 0 \leq y \leq 1).$$

Theorem 2.2.7. ***([13] , Dual inequality of stein type for*** $L^{\vec{p}}$ ***)***

Let f *be a measurable function on* $\mathbb{R}^n$ *and* $w_j (j = 1, ..., n)$ *be a non-negative measurable function on* $\mathbb{R}$. *Then, for* $1 \leq \vec{p} < \infty$, *if* $0 < t < \min(p_1, ..., p_n)$ *and* $w_j^t \in A_{p_j}$,

$$\left\| \mathcal{M}_t f \cdot \prod_{j=1}^{n} (w_j)^{\frac{1}{p_j}} \right\|_{\vec{p}} \lesssim \left\| f \cdot \prod_{j=1}^{n} (M_j w_j)^{\frac{1}{p_j}} \right\|_{\vec{p}} .$$

Theorem 2.2.8. ***([6], Maximal operators)***

Maximal operators will be an essential tool in the proof of some of our results. If $\vec{p} = (p_1, ..., p_n) \in (0, \infty)^n$ *and* $\vec{r} = (r_1, ..., r_n) \in (0, \infty)^n$ *with*

$r_j \leq \min(p_1, ..., p_n)$ *for every* $j = 1, ..., n$, *then*

$$\|\mathcal{M}_{\vec{r}}(f)\|_{\vec{p}} \leq c\|f\|_{\vec{p}}.$$

Theorem 2.2.9. *Let* $0 < \vec{p} < \infty$, *then*

$$\|Mf\|_{\vec{p}} \leq \|f\|_{\vec{p}}$$

for $f \in L^{\vec{p}}(\mathbb{R}^n)$.

Proof. Putting $\vec{r} = (1, ..., 1)$ in Theorem 2.2.8 we find the result. □

2.2.2 Boundedness of the Fefffferman-Stein vector-valued maximal inequality in mixed Lebesgue spaces

Definition 2.2.10. ***(spaces*** $\ell_q(L^{\vec{p}})$***)***

Let $0 < \vec{p} \leq \infty$, $0 < q < \infty$, *then* $\ell_q(L^{\vec{p}})$ *is the space of sequences* $\{f_k\}_k \subset \mathcal{S}'(\mathbb{R}^n)$ *such that,*

$$\|\{f_k\}_k\|_{\ell_q(L^{\vec{p}})} = \left(\sum_{k=0}^{\infty} \|f_k(x)\|_{\vec{p}}^q \right)^{\frac{1}{q}} < \infty.$$

We can also extend the Fefffferman-Stein vector-valued maximal inequality for mixed spaces.

Theorem 2.2.11. *Let* $0 < \vec{p} < \infty$, $1 < u \leq \infty$, *satisfy* $\frac{n}{p} \leq \sum_{j=1}^{n} \frac{1}{q_j}$.

Then, for every sequence $\{f_j\}_{j=1}^{\infty} \subset L^0(\mathbb{R}^n)$,

$$\left\| \left(\sum_{j=1}^{\infty} [M f_j]^u \right)^{\frac{1}{u}} \right\|_{\vec{p}} \lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^u \right)^{\frac{1}{u}} \right\|_{\vec{p}}.$$

Theorem 2.2.12. *([13]) Let* $0 < \vec{p} < \infty$, $0 < u \leq \infty$ *and* $0 < t < \min(p_1, ..., p_n, u)$*. Then, for every sequence* $\{f_j\}_{j=1}^{\infty} \subset L^0(\mathbb{R}^n)$,

$$\left\| \left(\sum_{j=1}^{\infty} [\mathcal{M}_t f_j]^u \right)^{\frac{1}{u}} \right\|_{\vec{p}} \lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^u \right)^{\frac{1}{u}} \right\|_{\vec{p}}.$$

2.2.3 Boundedness of the singular integral operators

Definition 2.2.13. *([13]) Let* T *be a singular integral operators with a kernel* $k(x, y)$ *which satisfies the following condition*

(1) *There existe a conctant* $C > 0$ *such that* $|k(x, y)| \leq \dfrac{C}{|x - y|^n}$.

(2) *There existe* $\epsilon > 0$ *and* $C > 0$ *such that*

$$|k(x, y) - k(z, y)| + |k(y, x) - k(y, z)| \leq C \frac{|x - z|^{\epsilon}}{|x - y|^{n+\epsilon}}$$

if $|x - y| \geq 2|x - z|$, *with* $x \neq y$.

(3) *If* $f \in L_c^{\infty}(\mathbb{R}^n)$, *the set of all compactly supported* L^{∞}*-functions, then*

$$Tf(x) = \int_{\mathbb{R}^n} k(x, y) f(y) dy \qquad (x \notin \operatorname{supp} f).$$

Keeping in mind that T extend to a bounded linear operator on $L^{\vec{q}}(\mathbb{R}^n)$, we prove the follawing theorem.

Theorem 2.2.14. *([13]) Let* $1 < \vec{q} < \infty$ *if* $f \in L^{\vec{q}}(\mathbb{R}^n)$, *then:*

$$\|Tf\|_{\vec{q}} \leq \|f\|_{\vec{q}}.$$

Proof. Put $\vec{q} = \theta \vec{r}$, where $\theta > 1$ and $\vec{r} > 1$. Then, using the $L^{\vec{r}}(\mathbb{R}^n) - L^{\vec{r}'}(\mathbb{R}^n)$ duality argument, for $g \in L^{\vec{r}'}(\mathbb{R}^n)$, we have

$$\|Tf\|_{\vec{q}} = \left\| |Tf|^{\theta} \right\|_{\vec{r}}^{\frac{1}{\theta}} = \left(\int_{\mathbb{R}^n} |Tf(x)|^{\theta} g(x) dx \right)^{\frac{1}{\theta}}.$$

Since $g(x) \le M[|g|^{\frac{1}{\eta}}](x)^\eta$ and $M[|g|^{\frac{1}{\eta}}]^\eta \in A_1$ for $\eta > 1$ we get

$$\|Tf\|_{\vec{q}} \le \left(\int_{\mathbb{R}^n} |Tf(x)|^\theta M[|g|^{\frac{1}{\eta}}](x)^\eta dx\right)^{\frac{1}{\theta}} \lesssim \left(\int_{\mathbb{R}^n} |f(x)|^\theta M[|g|^{\frac{1}{\eta}}](x)^\eta dx\right)^{\frac{1}{\theta}}.$$

By Hölder's inequality and the boundedness of the Hardy-Littlewood maximal operator.

$$\|Tf\|_{\vec{q}} \lesssim \left\||f|^\theta\right\|_{\vec{r}}^{\frac{1}{\theta}} \left\|\left(M[|g|^{\frac{1}{\eta}}]\right)^\eta\right\|_{\vec{r'}} \lesssim \||f|\|_{\theta\vec{r}} \left\||g|^{\frac{1}{\eta}}\right\|_{\eta\vec{r'}} = \|f\|_{\vec{q}}\|g\|_{\vec{r'}}.$$

□

CHAPTER 3

SOME BASIC PROPERTIES OF MIXED MORREY SPACES AND APPLICATIONS

The aim of this chapter is to recall the definition of Morrey mixed spaces along with some of their main characteristics these properties extend the classical ones. We investigate the boundedness in these spaces of the iterated maximal operator, the fractional integtral operator and singular integral operator.

3.1 Mixed Morrey spaces

In this section, we discuss some properties and examples of mixed Morrey spaces.

3.1.1 Definiton of mixed Morrey spaces

We recall in this subsection by the definition of mixed Morrey spaces.

Definition 3.1.1. *([13]) Let* $0 < \vec{q} \leq \infty, 0 < p \leq \infty$ *satisfy* $\sum_{j=1}^{n} \frac{1}{q_j} \geq \frac{n}{p}$*, is the set of function* $f \in L^0(\mathbb{R}^n)$*. Then define the mixed Morrey norm:* $\|\cdot\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}$ *by*

$$\|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} = \sup\left\{ |Q|^{\frac{1}{p} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{q_j}\right)} \|f\chi_Q\|_{\vec{q}} : Q \textit{ is a cube in } \mathbb{R}^n \right\} < \infty.$$

Let us give some examples.

Example 3.1.2. *In the classical case, it is known that* $f(x) = |x|^{-\frac{n}{p}} \in \mathcal{M}^p_q(\mathbb{R}^n)$ *if* $q < p$*. Let* $\vec{q} = (q_1, ..., q_n)$*. Using the above embedding, we have*

$$\mathcal{M}^p_{\widetilde{q}}(\mathbb{R}^n) = \mathcal{M}^p_{\underbrace{(\widetilde{q}, ..., \widetilde{q})}_{n\, times}}(\mathbb{R}^n) \subset \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n),$$

Where $\widetilde{q} = \max(q_1, ..., q_n)$ *this , if* $\max(q_1, ..., q_n) = \widetilde{q} < p$*,*

$$f(x) = |x|^{-\frac{n}{p}} \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n.$$

Remark 3.1.3. *In Example* 3.1.2*, the condition*

$$\max(q_1, ..., q_n) = \widetilde{q} < p \tag{3.1}$$

is a sufficient condition but is not a necessary condition for $f(x) = |x|^{-\frac{n}{p}} \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n$*.*

In fact, let $\vec{s} = (s_1, \underbrace{\infty, ..., \infty}_{(n-1)\, times})$ *and* $s_1 < \frac{q_1}{n}$*. Then, by Proposition* 3.1.9*,*

$$\begin{aligned}
\|f\|_{\mathcal{M}^p_{\vec{s}}(\mathbb{R}^n)} &= \sup_{Q=Q(x,r)} |Q(x,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)} \|f\chi_{Q(x,r)}\|_{\vec{s}} \\
&= \sup_{r>0} |Q(0,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)} \|f\chi_{Q(0,r)}\|_{\vec{s}} \\
&= |Q(0,1)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)} \|f\chi_{Q(0,1)}\|_{\vec{s}} \\
&= |Q(0,1)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)} \left\| \left(\int_{-1}^{1} |x|^{-\frac{n}{p}s_1} dx_1 \right)^{\frac{1}{s_1}} \chi_{[-1,1]^{n-1}} \right\|_{\underbrace{(\infty, ..., \infty)}_{(n-1)\, times}}.
\end{aligned}$$

Since $s_1 < \frac{q_1}{n}$, $\|f\|_{\mathcal{M}^p_{\vec{s}}(\mathbb{R}^n)} < \infty$ *and* $f \in \mathcal{M}^p_{\vec{s}}(\mathbb{R}^n)$*. But* $\vec{s}$ *does not satisfy* (3.1).

Example 3.1.4. *Let* $0 < \vec{q} \leq \infty$ *and assume that* $q_j < p_j$ *if* $p_j < \infty$ *and that* $q_j \leq \infty$ *if* $p_j = \infty$ $(j = 1, ..., n)$*. Let*

$$\sum_{j=1}^n \frac{1}{p_j} = \frac{n}{p}. \tag{3.2}$$

Then,

$$f(x) = \prod_{j=1}^n |x|^{-\frac{1}{p_j}} \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n).$$

In fact , letting $Q = I_1 \times ... \times I_n$*, we obtain*

$$\begin{aligned}
\|f\chi_Q\|_{\vec{q}} &= \left(\int_{I_n} \cdots \left(\int_{I_2} \left(\int_{I_1} \prod_{j=1}^n |x|^{-\frac{q_1}{p_j}} dx_1 \right)^{\frac{q_2}{q_1}} dx_2 \right)^{\frac{q_3}{q_2}} \cdots dx_n \right)^{\frac{1}{q_n}} \\
&= \prod_{j=1}^n \left(\int_{I_j} |x_j|^{-\frac{q_j}{p_j}} dx_j \right)^{\frac{1}{q_j}}.
\end{aligned}$$

To estimate, this integrale, letting $\ell(Q) = r$*, we have*

$$\int_{I_j} |x_j|^{-\frac{q_j}{p_j}} dx_j \leq \int_{-\frac{r}{2}}^{\frac{r}{2}} |x_j|^{-\frac{q_j}{p_j}} dx_j = 2\int_0^{\frac{r}{2}} |x_j|^{-\frac{q_j}{p_j}} dx_j \lesssim r^{1-\frac{q_j}{p_j}}$$

Thus,

$$\|f\chi_Q\|_{\vec{q}} \lesssim \prod_{j=1}^{n} \left(r^{1-\frac{q_j}{p_j}} \right)^{\frac{1}{q_j}} = \prod_{j=1}^{n} r^{\frac{1}{q_j}-\frac{1}{p_j}} = r^{\sum_{j=1}^{n} \frac{1}{q_j} - \sum_{j=1}^{n} \frac{1}{p_j}}.$$

Since $\sum_{j=1}^{n} \frac{1}{pj} = \frac{n}{p}$ *,*

$$r^{\frac{n}{p} - \sum_{j=1}^{n} \frac{1}{q_j}} \|f\chi_Q\|_{\vec{q}} \lesssim 1.$$

Taking supremum over all the cubes, we obtain

$$\|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \lesssim 1.$$

That is,

$$f(x) = \prod_{j=1}^{n} |x|^{-\frac{1}{p_j}} \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n).$$

Remark 3.1.5. *In Example* 3.1.4 *the condition* (3.2) *is a necessary and sufficient condition for,* $f(x) = \prod_{j=1}^{n} |x|^{-\frac{1}{p_j}}$ *to be a member in* $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$.

In fact, let $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ *and* $f \neq 0$*. Using the Proposition* 3.1.9*, we have*

$$\|f(t.)\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} = t^{-\frac{n}{p}} \|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \quad (t > 0). \tag{3.3}$$

On the other hand, since $f(tx) = t^{-\sum_{j=1}^{n} \frac{1}{p_j}} f(x)$ *,*

$$\|f(t.)\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} = t^{-\sum_{j=1}^{n} \frac{1}{p_j}} \|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}. \tag{3.4}$$

By (3.3) *and* (3.4)*, for all* $t > 0$*,*

$$t^{-\sum_{j=1}^{n} \frac{1}{p_j}} = t^{-\frac{n}{p}}.$$

This, we obtain (3.2).

Theorem 3.1.6. *([12]) For* $1 \leq p < \infty$*,* $\vec{q} \in [1, \infty)^n$*, if for some* $i \in \{1, ..., n\}$*, it holds that* $p < q_i$*, then*

$$\mathcal{M}^p_{\vec{q}} = \{0\}.$$

3.1.2 Some properties of mixed Morrey spaces

Firste, we give some properties of the mixed Morrey spaces.

Proposition 3.1.7. *([13]) Let* $1 \leq \vec{q} \leq \infty$ *and* $0 < p \leq \infty$*. The mixed Morrey space* $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ *is also a Banach space.*

Proof. Although the proof is easy, we give the proof for the sake of completeness. First , we will check the triangle inquality . For $f, g \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$,

$$\begin{aligned}\|f+g\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} &= \sup_Q |Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)} \|(f+g)\chi_Q\|_{\vec{q}} \\ &\leq \sup_Q |Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)} \left(\|f\chi_Q\|_{\vec{q}} + \|g\chi_Q\|_{\vec{q}}\right) \\ &\leq \|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} + \|g\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}.\end{aligned}$$

The positivity and the homogeneity are both clear. This $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ is anormed space. It remains to check the completeness.

Let $\{f_j\}_{j=1}^\infty \subset \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ and $\sum_{j=1}^\infty \|f_j\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} < \infty$. Then,

$$\left\|\sum_{j=1}^J |f_j|\right\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \leq \sum_{j=1}^J \|f_j\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \leq \sum_{j=1}^\infty \|f_j\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} < \infty.$$

By Proposition 2.1.4.

$$\begin{aligned}\left\|\sum_{j=1}^\infty |f_j|\right\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} &= \left\|\lim_{J\longrightarrow\infty} \sum_{j=1}^J |f_j|\right\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \\ &\leq \lim_{J\longrightarrow\infty} \left\|\sum_{j=1}^J |f_j|\right\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \\ &\leq \lim_{J\longrightarrow\infty} \sum_{j=1}^J \|f_j\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \\ &= \sum_{j=1}^\infty \|f_j\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} < \infty.\end{aligned}$$

Thus, for almoste everywhere $x \in \mathbb{R}^n$. $\sum_{j=1}^\infty |f_j(x)| < \infty$ There fore, there exists a function g such that the limit:

$$g(x) = \lim_{J\longrightarrow\infty} \sum_{j=1}^J f_j(x),$$

exists for almoste everywhere $x \in \mathbb{R}^n$.

If $\sum_{j=1}^{\infty} |f_j(x)| = \infty$, then it will be understood that $g(x) = 0$. Again, by Proposition 2.1.4, for $m > 1$

$$\begin{aligned}
\left\| g - \sum_{j=1}^{m-1} f_j \right\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} &= \left\| \sum_{j=m}^{\infty} f_j \right\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \\
&= \left\| \lim_{J \longrightarrow \infty} \sum_{j=m}^{J} f_j \right\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \\
&\leq \lim_{J \longrightarrow \infty} \left\| \sum_{j=m}^{J} f_j \right\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \\
&\leq \lim_{J \longrightarrow \infty} \sum_{j=m}^{J} \|f_j\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \\
&= \sum_{j=m}^{\infty} \|f_j\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}.
\end{aligned}$$

Letting $m \longrightarrow \infty$, we obtain

$$g = \sum_{j=1}^{\infty} f_j$$

in $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$. $\square$

Proposition 3.1.8. *([13]) Let $\vec{q} \in (0;\infty]^n$ and $f \in L^0(\mathbb{R}^n)$.*

(i) If for each $q_i = q$, then $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n) = \mathcal{M}^p_q(\mathbb{R}^n)$

(ii) If $p = n/(\frac{1}{q_1} + ... + \frac{1}{q_n})$, then $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n) = L^{\vec{q}}(\mathbb{R}^n)$.

Proof. (i) If for each $q_i = q$, then by Remark 2.1.2

$$|Q|^{\frac{1}{p} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{q_j}\right)} \|f\chi_Q\|_{\vec{q}} = |Q|^{\frac{1}{p} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{q}\right)} \|f\chi_Q\|_{\vec{q}} = |Q|^{\frac{1}{p} - \frac{1}{q}} \|f\chi_Q\|_{\vec{q}}.$$

This taking the supremum over the all cubes in $\mathbb{R}^n$, we obtain:

$$\|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} = \|f\|_{\mathcal{M}^p_q(\mathbb{R}^n)}$$

and

$$\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n) = \mathcal{M}^p_q(\mathbb{R}^n).$$

With coincidence of norme.

(ii) In particular. Let

$$p = \frac{n}{\frac{1}{q_1} + \dots + \frac{1}{q_n}}.$$

Then, since

$$\begin{aligned}
\|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} &= \sup\left\{ |Q|^{\frac{1}{p} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{q_j}\right)} \|f\chi_Q\|_{\vec{q}} : Q \textit{ is a cube in } \ \mathbb{R}^n \right\}. \\
&= \sup\left\{ \|f\chi_Q\|_{\vec{q}} : Q \textit{ is a cube in } \ \mathbb{R}^n \right\} \\
&= \|f\|_{\vec{q}},
\end{aligned}$$

we obtain $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n) = L^{\vec{q}}(\mathbb{R}^n)$.

□

Proposition 3.1.9. *([13]) Let* $0 < \vec{q} \le \infty$ *and* $0 < p \le \infty$. *For* $f \in L^0(\mathbb{R}^n)$ *and* $t > 0$, *then*

$$\|f(t.)\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} = t^{-\frac{n}{p}} \|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}. \tag{3.5}$$

Proof. To see (3.5), using (2.1), we obtain

$$\begin{aligned}
\|f(t.)\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} &= \sup_{Q=Q(x,r)} |Q(x,r)|^{\frac{1}{p} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{q_j}\right)} \|f(t.)\chi_{Q(x,r)}\|_{\vec{q}} \\
&= \sup_{Q=Q(x,r)} |Q(x,r)|^{\frac{1}{p} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{q_j}\right)} t^{-\sum_{j=1}^{n} \frac{1}{q_j}} \|f\chi_{Q(tx,tr)}\|_{\vec{q}} \\
&= \sup_{Q=Q(x,r)} |Q(tx,tr)|^{\frac{1}{p} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{q_j}\right)} t^{-\frac{n}{p}} \|f\chi_{Q(tx,tr)}\|_{\vec{q}} \\
&= t^{-\frac{n}{p}} \|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}.
\end{aligned}$$

□

Proposition 3.1.10. *([13]) Let* $0 < \vec{q} < \vec{r} \le \infty$, $0 < p < \infty$, *and assume* $\frac{1}{r_1} + \dots + \frac{1}{r_n} \ge \frac{n}{p}$. *Then*

$$\mathcal{M}^p_{\vec{r}}(\mathbb{R}^n) \subset \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n).$$

Proof. To get this inclusion, it suffices to show that for all $f \in L^0(\mathbb{R}^n)$ and all cubes Q,

$$|Q|^{\frac{1}{p} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{q_j}\right)} \|f\chi_Q\|_{\vec{q}} \le |Q|^{\frac{1}{p} - \frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{r_j}\right)} \|f\chi_Q\|_{\vec{r}}. \tag{3.6}$$

Once we can show (3.6), taking the supremum over the all cubes in $\mathbb{R}^n$, we have

$$\|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \leq \|f\|_{\mathcal{M}^p_{\vec{r}}(\mathbb{R}^n)},$$

this implies that

$$\mathcal{M}^p_{\vec{r}}(\mathbb{R}^n) \subset \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n).$$

So, we shall show (3.6). Note that we can write $Q = I_1 \times ... \times I_n$, where each I_j is an interval of aqual length. Using Hölder's inequality, we have

$$\begin{aligned}
\|f\chi_Q\|_{\vec{q}} &= \left(\int_{I_n} \cdots \left(\int_{I_2}\left(\int_{I_1}|f(x)|^{q_1}dx_1\right)^{\frac{q_2}{q_1}}dx_2\right)^{\frac{q_3}{q_2}} \cdots dx_n\right)^{\frac{1}{q_n}} \\
&\leq \left(\int_{I_n} \cdots \left(\int_{I_2}\left[\left(\int_{I_1}|f(x)|^{q_1\frac{r_1}{q_1}}dx_1\right)^{\frac{q_1}{r_1}}\left(\int_{I_1}dx_1\right)^{1-\frac{q_1}{r_1}}\right]^{\frac{q_2}{q_1}}dx_2\right)^{\frac{q_3}{q_2}} \cdots dx_n\right)^{\frac{1}{q_n}} \\
&= \left(\int_{I_n} \cdots \left(\int_{I_2}\|f\chi_{I_1\times\mathbb{R}^{n-1}}\|_{(r_1)}(x_2,...,x_n)^{q_2}|I_1|^{\frac{q_2}{q_1}-\frac{q_2}{r_1}}dx_2\right)^{\frac{q_3}{q_2}} \cdots dx_n\right)^{\frac{1}{q_n}}.
\end{aligned}$$

Since $|I_1| = \ell(Q)$,

$$\begin{aligned}
\|f\chi_Q\|_{\vec{q}} &\leq \left(\int_{I_n} \cdots \left(\int_{I_2}\|f\chi_{I_1\times\mathbb{R}^{n-1}}\|_{(r_1)}(x_2,...,x_n)^{q_2}\ell(Q)^{\frac{q_2}{q_1}-\frac{q_2}{r_1}}dx_2\right)^{\frac{q_3}{q_2}} \cdots dx_n\right)^{\frac{1}{q_n}} \\
&= \ell(Q)^{\frac{1}{q_1}-\frac{1}{r_1}}\left(\int_{I_n} \cdots \left(\int_{I_2}\|f\chi_{I_1\times\mathbb{R}^{n-1}}\|_{(r_1)}(x_2,...,x_n)^{q_2}dx_2\right)^{\frac{q_3}{q_2}} \cdots dx_n\right)^{\frac{1}{q_n}}.
\end{aligned}$$

Ilterating this procedure, we get

$$\begin{aligned}
\|f\chi_Q\|_{\vec{q}} &\leq \ell(Q)^{\left(\sum_{j=1}^{n-1}\frac{1}{q_j}\right)-\left(\sum_{j=1}^{n-1}\frac{1}{r_j}\right)}\left(\int_{I_n}\|f\chi_{I_1\times\ldots\times I_{n-1}\times\mathbb{R}}\|_{(r_1,\ldots,r_{n-1})}(x_n)^{q_n}dx_n\right)^{\frac{1}{q_n}} \\
&\leq \ell(Q)^{\left(\sum_{j=1}^{n}\frac{1}{q_j}\right)-\left(\sum_{j=1}^{n}\frac{1}{r_j}\right)}\|f\chi_Q\|_{\vec{r}}.
\end{aligned}$$

This, we obtain $f \in L^0(\mathbb{R}^n)$ and all cubes Q,

$$|Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{q_j}\right)}\|f\chi_Q\|_{\vec{q}} \leq |Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{r_j}\right)}\|f\chi_Q\|_{\vec{r}}.$$

$\square$

Example 3.1.11. *Let Q be a cube and $0 < \vec{q} \leq \infty$. Then,*

$$\|\chi_Q\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} = |Q|^{\frac{1}{p}}.$$

To check this, put $\sum_{j=1}^n \frac{1}{q_j} = \overline{q}$. First, using Example 2.1.3 we get

$$\|\chi_Q\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} = \sup_{R\in\mathcal{Q}} |R|^{\frac{1}{p}-\frac{\overline{q}}{n}} \|\chi_Q\chi_R\|_{\vec{q}} \geqslant |Q|^{\frac{1}{p}-\frac{\overline{q}}{n}} \|\chi_Q\|_{\vec{q}} = |Q|^{\frac{1}{p}-\frac{\overline{q}}{n}} |Q|^{\frac{\overline{q}}{n}} = |Q|^{\frac{1}{p}}.$$

On the other hand, by Proposition 3.1.10,

$$\|\chi_Q\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \leq \|\chi_Q\|_{\mathcal{M}^p_{\max(q_1,\ldots,q_n)}(\mathbb{R}^n)} = |Q|^{\frac{1}{p}}.$$

Combining the above two inequalities, we obtain

$$\|\chi_Q\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} = |Q|^{\frac{1}{p}}.$$

3.2 Boundedness of some operators in mixed Morry spaces

In this section, we investigate the boundedness of the some operators in $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$.

3.2.1 Boundedness of the Hardy-Littlewood maximal operator

Now, we consider the boundedness of the maximal operator in both classical and mixed Morrey spaces. The following Lemma is crucial for demonstrating the boundedness of the Hardy-Littlewood maximal operator in these spaces.

Lemma 3.2.1 ([15], Lemma 4.2)**.** *For all measurable functions f and cubes Q, we have*

$$M[\chi_{\mathbb{R}^n\setminus 5Q} f](y) \lesssim \sup_{Q\subset R\in\mathcal{Q}} \frac{1}{|R|}\int_R |f(x)|dx \qquad (y \in Q).$$

First, we prove the boundedness of the Hardy-Littlewood maximal operator in mixed Morrey spaces.

Theorem 3.2.2. *([13]) Let $1 < \vec{q} < \infty$ and $1 < p \leq \infty$ satisfy $\frac{n}{p} \leq \sum_{j=1}^n \frac{1}{q_j}$. Then,*

$$\|Mf\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}$$

for all $f \in L^0(\mathbb{R}^n)$.

Proof. It suffices to verify that, for any cube $Q = Q(x, r)$,

$$|Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{q_j}\right)}\|(Mf)\chi_Q\|_{\vec{q}} \lesssim \|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}.$$

Now, we decompose

$$|f(y)| = \chi_{Q(x,5r)}(y)|f(y)| + \chi_{Q(x,5r)^c}(y)|f(y)| \equiv f_1(y) + f_2(y) \quad (y \in \mathbb{R}^n).$$

Using the subadditivity of M, we obtain

$$Mf(y) \leq Mf_1(y) + Mf_2(y) \quad (y \in \mathbb{R}^n).$$

First, the boundedness of M on the mixed Lebesgue space $L^{\vec{q}}(\mathbb{R}^n)$ yields,

$$\begin{aligned}
|Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{q_j}\right)}\|(Mf)\chi_Q\|_{\vec{q}} &\leq |Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{q_j}\right)}\|(Mf_1)\|_{\vec{q}} \\
&\lesssim |Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{q_j}\right)}\|f_1\|_{\vec{q}} \\
&= |Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{q_j}\right)}\|f\chi_{Q(x,5r)}\|_{\vec{q}} \\
&= |Q(x,5r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{q_j}\right)}\|f\chi_{Q(x,5r)}\|_{\vec{q}} \\
&\leq \|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}.
\end{aligned}$$

Second, by Proposition 3.2.1, we get

$$Mf_2(y) = M[\chi_{\mathbb{R}^n\setminus 5Q}f](y) \lesssim \sup_{Q\subset R\in\mathcal{Q}}\frac{1}{|R|}\int_R |f(x)|dx \qquad (y \in Q).$$

Thus, we see that

$$\begin{aligned}
&|Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{q_j}\right)}\|(Mf_2)\chi_Q\|_{\vec{q}} \\
&\lesssim \sup_{Q\subset R\in\mathcal{Q}}|Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{q_j}\right)}\left\|\frac{1}{|R|}\int_R |f(x)|dx \times \chi_Q\right\|_{\vec{q}}. \qquad (3.7)
\end{aligned}$$

Thanks to Example 2.1.3, we have

$$\begin{aligned}
(3.7) &= \sup_{Q\subset R\in\mathcal{Q}}|Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{q_j}\right)}\frac{1}{|R|}\int_R |f(x)|dx \times \|\chi_Q\|_{\vec{q}} \\
&= \sup_{Q\subset R\in\mathcal{Q}}|Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{q_j}\right)}\frac{1}{|R|}\int_R |f(x)|dx \times |Q|^{\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{q_j}\right)} \\
&\leq \sup_{R\in\mathcal{Q}}|R|^{\frac{1}{p}-1}\int_R |f(x)|dx.
\end{aligned}$$

By Proposition 3.1.10, taking into account

$$\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{\underbrace{(1, ..., 1)}_{n\ times}}(\mathbb{R}^n) = \mathcal{M}^p_1(\mathbb{R}^n)$$

with embedding constant 1, we get

$$|Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)} \|(Mf_2)\chi_Q\|_{\vec{q}} \leq \|f\|_{\mathcal{M}^p_1(\mathbb{R}^n)} \leq \|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}.$$

Thus, taking the supremum over all the cubes, we obtain

$$\|Mf_2\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}.$$

Hence, the result holds.

Remark 3.2.3. *The boundedness of the Hardy-Littlewood maximal operator in classical Morrey spaces is showed by Chiarenza and Frasca in* 1987 *(see [5]).*

Theorem 3.2.4. *([13]) Let* $0 < \vec{q} \leq \infty$ *and* $0 < p < \infty$ *satisfy*

$$\frac{n}{p} \leq \sum_{j=1}^n \frac{1}{q_j}, \quad \frac{n-1}{n} p < \max(q_1, ..., q_n).$$

If $0 < t < \min(q_1, ..., q_n, p)$*, then*

$$\|\mathcal{M}_t f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}$$

for all $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$.

As a corollary, we obtain this boundedness of $\mathcal{M}_t$ *in classical Morrey spaces.*

Corollary 3.2.5. *([13]) Let*

$$0 < \frac{n-1}{n} p < q \leq p < \infty.$$

If $0 < t < q$*, then*

$$\|\mathcal{M}_t f\|_{\mathcal{M}^p_q(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{M}^p_q(\mathbb{R}^n)}$$

for all $f \in \mathcal{M}^p_q(\mathbb{R}^n)$.

□

3.2.2 Boundedness of the Feffferman-Stein vector-valued maximal inequality

Next, we show the boundedness of the Feffferman-Stein vector-valued maximal operator on mixed Morrey spaces. To show this, we need auxiliary estimates.

Theorem 3.2.6. *([13]) Let $0 < \vec{q} < \infty$, $1 < u \leq \infty$, and $1 < p \leq \infty$ satisfy $\frac{n}{p} \leq \sum_{j=1}^{n} \frac{1}{q_j}$. Then, for every sequence $\{f_j\}_{j=1}^{\infty} \subset L^0(\mathbb{R}^n)$,*

$$\left\| \left(\sum_{j=1}^{\infty} [Mf_j]^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}$$

Theorem 3.2.7. *([13]) Let $0 < \vec{q} < \infty$ and $1 < p \leq \infty$ satisfy*

$$\frac{n}{p} \leq \sum_{j=1}^{n} \frac{1}{q_j}, \quad \frac{n-1}{n} p < \max(q_1, ..., q_n).$$

If $0 < t < \min(q_1, ..., q_n, u)$, then

$$\left\| \left(\sum_{j=1}^{\infty} [\mathcal{M}_t f_j]^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \lesssim \left\| \left(\sum_{k=1}^{\infty} |f_j|^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}$$

for $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$.

Corollary 3.2.8. *Let*

$$0 < \frac{n-1}{n} p < q \leq p < \infty.$$

If $0 < t < min(q, u)$, then

$$\left\| \left(\sum_{j=1}^{\infty} [\mathcal{M}_t f_j]^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_{q}(\mathbb{R}^n)} \lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_{q}(\mathbb{R}^n)}$$

for $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}^p_{q}(\mathbb{R}^n)$.

3.2.3 Boundedness of the fractional integral operator

We now study the boundedness of the fractional integral operator I_α. Adams proved that this operator is bounded in classical Morry spaces.

Definition 3.2.9. *Let* $0 < \alpha < n$*. Define the fractional integral operator* I_α *of order* α *by,*

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

for $f \in L^1_{loc}(\mathbb{R}^n)$ *as long as the right-hand side makes sence.*

Theorem 3.2.10. *Let* $0 < \alpha < n$*,* $1 < \vec{q}, \vec{s} < \infty$ *and* $0 < p, r < \infty$*. Assume that* $\frac{n}{p} \leq \sum_{j=1}^{n} \frac{1}{q_j}$*, and* $\frac{n}{r} \leq \sum_{j=1}^{n} \frac{1}{s_j}$*. Also, assume that*

$$\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{n}, \frac{\vec{q}}{p} = \frac{\vec{s}}{r}.$$

Then, for $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$*,*

$$\|I_\alpha f\|_{\mathcal{M}^p_{\vec{s}}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}.$$

Proof. See [13, Theorem 1.11]. ☐

3.2.4 Boundedness of the singular integral operators

Finally, we show that the singular integral operators are bounded in mixed Morry spaces. Their boundedness in classical Morry spaces is proved by Chiarenza and Frasca (see [5]).

Theorem 3.2.11. *([13]) Let* $0 < \vec{p} \leq \infty$ *and* $1 < p < \infty$*, for* $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ *satysfy*

$$\frac{n}{p} \leq \sum_{j=1}^{n} \frac{1}{q_j}.$$

Then,

$$\|Tf\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n} \lesssim \|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n}$$

for $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$*.*

Conclusion

We have reached the final letters of this memory, and in the course of this modest work we have encountered new concepts and interesting ideas such as mixed Lebesgue's spaces, mixed Morrey's spaces and the boundedness in these spaces of the Hardy-Litlewood maximal function, ... The goal is to study the mixed Lebesgue's spaces $L^{\vec{q}}(\mathbb{R}^n)$, the mixed Morrey's spaces $\mathcal{M}^{p}_{\vec{q}}(\mathbb{R}^n)$ and their applications, which generalizes mixed Lebesgue spaces $L^{\vec{q}}(\mathbb{R}^n)$, classical Lebesgue spaces and Morrey $\mathcal{M}^{p}_{q}(\mathbb{R}^n)$ spaces, with some of the main properties of these spaces.

This work raises a number of questions that deserve further research, including, generalising these results to generalised mixed Morrey spaces.

Bibliography

[1] N. Antonić and I. Ivec, *On the Hormander-Mihlin theorem for mixed-norm Lebesgue spaces*, J. Math.Anal. Appl., **433**(1)(2016), 176–199.

[2] R.J. Bagby, *An extended inequality for the maximal function*. Proc. Am. Math. Soc., **48**(1975), 419–422

[3] A. Benedek and R. Panzone, *The spaces L^p , With mixed norme*, Duke Math .J., **28**(1961) 301-324.

[4] J. Bergh and J. Löfström, *Interpolation spaces, An introduction*, Springer Science.Berlin, 1976.

[5] F. Chiarenza and M. Frasca, *Morrey spaces and Hardy-Littlewood maximal function*, Rend. Mat., **7**(1987), 273-279.

[6] C. Galatia, G.G. Athanasios, *Mixed-Norm α-Modulation Spaces*, American. Math. Society, **373**(2020), 3323-3356.

[7] B. Jessen, J. Marcinkiewicz and A. Zygmund, *Note on the differentiability of multiple integrals*, Fund. Math., **25**(1935), 217-234.

[8] A. Kufner and L.E. Persson, Weighted Inequalities of Hardy Type, World Scientific, Singapore, 2003.

[9] J. Jon , S. Winfried , *A direct proof of Sobolev embeddings for quasi-homogeneous Lizorkin-Triebel spaces with mixed norms*, J. Function Spaces and Appl., **2**(2007), 183-198.

[10] P. G. Lemariè-Rieusset, *Multipliers and Morrey Spaces*, Publ. Math, **38**(2013), 741-752.

[11] C.B. Morrey, Jr., *On the solutions of quasi-linear elliptic partial differential equations*. Trans. Amer. Math. Soc.,**43**(1)(1938), 126-166.

[12] A. Nenad , I.Ivan , *On the Hörmander-Mihlin theorem for mixed-norm Lebesgue space*, J. Math. Analysis Appl., **433**(2016), 176-199.

[13] T.Nogayama, *Mixed Morrey spaces*, Positivity, **23**(4)(2019), 961-1000.

[14] T. Nogayama, *Boundedness of commutators of fractional integral operators on mixed Morrey spaces*, Integral Transforms and Special Functions, **30(10)**(2019), 790-816.

[15] Y. Sawano, D.I. Hakim and H. Gunawan, *Non-smooth atomic decomposition for generalized Orlicz-Morrey spaces*, Math. Nachr, **288**(14-15)(2015), 1741-1775.

[16] Y. Sawano, G. Di Fazio and D. F. Hakim, *Morrey Spaces : Introduction and Applications to Integral Operators and PDE's, Volumes I & II*, CRC Press, 2020.

[17] T. Runst and W. Sickel, Sobolev spaces of fractional order, Nemytskii operators and non-linear partial differential equations, de Gruyter, Berlin, 1996.

[18] L. Shanzhen, *Sungular Integrals and Related Topics*, Sientific Publishing Co Pte Ltd , Singapare, 2007.

[19] E. M. Stein, Harmonic Analysis, real-variable methods, orthogonality and oscillatory integrals, Press, Princeton New Jersey, 1993.

[20] B. Stöckert, *von Plancherel-Polya-Nikol'skij typ in gewichteten* L_p^{Ω}*- Räumen mit gemischten Norm*, Math. Nach., **86**(1978), 19-32.

[21] H. Triebel, Theory of function spaces II, Birkhäuser, Basel, 1992.

Acknowledgments

Avant tout, My Lord, the night is not pleasant without your thanks .. and the day is not pleasant without your obedience .. and heaven is not pleasant without seeing you

To my angel in life .. To the meaning of love and tenderness' .. To the one whose prayer were the secret of my success and whose tenderness was the balm of my wounds .. To the most precieous beloved, my mother ***Al-yakout****.*

To the one who taught me to give without waiting .. To the owner of the big heart .. My dear father ***Al-Masoud****.*

To the companion of my path and the apple of my eye .... Who shared my joy and sorrow and shared my moments of success and failure .. My dear husband, ***DJamal Hamidi****.. thank you for your continuous support.*

To my source of strength and pride .... To my pride and honor, my brothers ***Rahal*** *and* ***Kamal****.*

To the people with good hearts and sincere intentions .. To those with whom I gain strength.. ***Samira****..* ***Houria****..* ***Laila****, and sister-in-law* ***Nora****.*

To the winds of my life and the balm of my wounds .. ***Abdul Hayy*** *..* ***Ali*** *..* ***Loujain*** *..* ***Aseel*** *..* ***Yasser*** *....* ***Amjad*** *..* ***Ashwaq*** *..* ***Amani*** *..* ***Malak*** *.. Sondos ..* ***Kawthar*** *..* ***Arwa****,* ***Diaa Eldain*** *and those whose presence will illuminate my life,* ***osaid****.*

To the symbol of kindness and love, second father ***Ahmed Hamidi****, my aunt* ***Merzaka****, and my sisters* ***Thalja*** *and* ***Wassila****.*

To my companions on my path and the sisters that my mother did not give birth to .. ***Samira*** *..****Khawla*** *..* ***Asmaa*** *..* ***Ahlam****..* ***Nadjat*** *And* ***Sumaya****.*

To my classmates, thank you so much for your support.

To all those who are in my memory .. and not in my notebook .

ملخص

في هذه المذكرة، قدمنا بعض الخصائص والأمثلة لفضاءات لوبيغ و فضاءات موراي ذات النظائم المختلطة، والتي تعمم فضاءات لوبيغ وموراي الكلاسيكية. لقد استخدمنا هذه العائلة من فضاءات الدوال لدراسة محدودية بعض المؤثرات.

كلمات مفتاحية

فضاءات لوبيغ ذات النظائم المختلطة، فضاءات موراي ذات النظائم المختلطة، الدالة القصوى.

Abstract

In this memory, we present some properties and examples of mixed norm Lebesgue spaces and Morrey spaces, which generalize classical Lebesgue and Morrey spaces. We utilized this family of function spaces to study the boundedness of certain operators.

Key words :

Mixed Lebesgue spaces, mixed Morrey spaces, maximal function.

Résumé

Dans ce mémoire, nous présentons quelques propriétés et exemples d'espaces des espaces de Lebesgue et des espaces de Morrey à normes mixtes, qui généralisent les espaces de Lebesgue et de Morrey classiques. Nous avons utilisé cette famille d'espaces de fonctions pour étudier la continuité de certains opérateurs.

Mot-clés :

Les spaces de Lebesgue mixte, les spaces de Morrey mixte , la fonction maximal.