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Theme

Numerical study of incompressible flow problem

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Thanks

First and foremost, I would like to thank Almighty God for the will, health, courage and patience he has given me all these long years.
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I would like to thank all thosewho have helped me in any way to carry out this work.

Thanks
for everything

Signing sessions

After an academic career that brought with it joys and difficulties ....

Here I am today, picking its fruits and dedicating them to the soul of my dear father, may God bless him and grant him peace......

To my dear mother, under whose feet God placed paradise, my dear mother, with whose prayers I am certain that I will succeed...

To my wife....

To the three pleasures of my liver........

To my sisters and brothers, each in his name...

To my companions...to those asking...

To the minute details of my life....

I thank God for granting me success in this moment and reaching the verge of graduation thanks to my teachers. May God bless them, provide them with health, and guide their steps....

DOUMI LOUCIF

Résumé

An approximate method is presented to solve the problem of steady free-surface flow of an ideal fluid over u semi-infinite triangle in the bottom of un open channel. Schwartz-Christoffel transformation is used to map the region of flow, in the complex potential-plane, onto the upper half-plane. The Hilbert transformation as well as the perturbation technique are used as a basis for the approximate solution of the problem for large Froude number and small variation of triangle angle. General equations, in integral form, for any order of approximation are obtained. Solution up to first-order approximation is discussed and illustrated .
Keywords :free-surface flow problems, Schwartz-Christoffel transformation, Hilbert transformation, perturbation technique, nonlinear integral equations.

Une méthode approximative est présentée pour résoudre le problème de l'écoulement stationnaire à surface libre d'un fluide idéal sur un triangle semi-infini au fond d'un canal ouvert.La Transformation de Schwartz-Christoffel est utilisé pour transformer domaine d'écoulement, dans le plan de potentiel complexe, sur le demi-plan supérieur. La transformation de Hilbert ainsi que la technique de perturbation sont utilisées comme base pour la solution approximative du problème pour un grand nombre de Froude et un petit variation l'angle de la rampe. On obtient des équations générales, sous forme intégrale, pour tout ordre d'approximation. La solution jusqu'à l'approximation du premier ordre est discutée et illustrée.
Mots-Clés : problèmes d'écoulement à surface libre, transformation de Schwartz-Christoffel, transformation de Hilbert, technique de perturbation, équations intégrales non linéaires.

ملخص : تم تقديم طريقة تقريبية لحل مشكلة التدفق المستقر للسطح الحر لسائل مثالي فوق المثلث في
الجزء السفلي من القناة المفتوحة ، يستخدم تحويل شوارتز كريستوفل لتحويل منطقة التدفق
في المستو المعقد المحتمل إلى نصف مستو علوي.
يتم استخدام تحويل هيلبرت و كذلك تقنية الاضطراب كأساس للحل التقريبي للمشكل من أجل عدد
فرود كبير و زاوية المثلث صغيرة ، يتم الحصول على معدلات تكاملية
لأي درجة تقريب ، تمت ممناقشة و توضيح الحل حتى التقريب من الدرجة الأولى.
الكلمات الدالة : مشاكل التدفق ذو السطح الحر ، تحويل شوارتز كريستوفل تحويل هيلبرت
تقنية الاضطراب ، المعادلات التكاملية غير الخطية

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Notations

$\overrightarrow{S}$: the position vector of a given particle

$\overrightarrow{s_{0}}$: initial position

t: time

(X, Y, Z): Cartesian coordinates

$\overrightarrow{U}$: the speed vector

(u_x, u_y, u_z): speed field

φ: speed potential

ψ: current function

$f(z) = \varphi(x, y) + i\psi(x, y)$....the complex potential function

$z = x + iy$: the complex variable associated with the function

p: fluid pressure

ρ: fluid density

g: gravity

$m(t)$: the mass

S: the surface

$\overrightarrow{n}$: the unit normal vector to a surface element of S

H: the Hilbert transformation

F: the Froude number

rot: Rotational

$\overrightarrow{grad}$ gradient

$\frac{d}{dt}, \frac{D}{Dt}$material derivative

$\frac{\partial}{\partial x}$derivative with respect to a x

Δ la placien

w: the logarithmic hodograph variable

$f'(t)$: the mapping function

L' :ramp length

α : angle of inclination

β : angle of inclination

General introduction

Free-surface flows of the over- an obstraction are present in many industrial and urban reservoir applications. Given its practical importance, this type of flow is the subject of a large number of theoretical, experimental and numerical studies.

In particular, we deal with plane flows in the incompressible and irrotational cases. To find a numerical solution to the flow problem, where the effect of gravity is taken into account and surface tensions are neglected, we use the Schwarz-Christoffel transformation Hilbert transform and perturbation technique to solve the problem approximatively.

This memory is divided into three chapters. After this brief introduction, the first chapter deals with fundamental notions of fluid-mechanical properties, such as motion, streamline function and velocity potential.

In the second chapter, we treat the problem in the case of the surface tension effet is negliceled adopt a method of conformal transformations which reduces pro- blem only on the free surface. We first use the Schwarz-Christoffel transformation technique, the perturbation technique and the Hilbert method to obtain the shape of the free surface.

In the third chapter, we present the of the free surfaces in curves showing the effect of the angles α and β , of height ε .

Finally, we present the general conclusion of our work.

CHAPTER 1

Preliminaries of
POTENTIAL OUTFLOW

This chapter introduces the basic concepts of fluid mechanics: fluid properties, fundamental equations of fluid motion for a potential flow,
two-dimensional and irrotational incompressible and non-viscous fluids

1.1 Movement descriptions

Fluid motion can be described using two distinct methods. We can choose to follow the fluid particles as they move (Lagrange's method), or we can fix a point in space and observe the velocity field of all the fluid particles at a given instant (Euler's method).

1.1.1 Lagrangian method (description by trajectories)

In the Lagrangian description, we describe the movement by the trajectories of particles of determined identities.
Let $\overrightarrow{S}\left(\overrightarrow{S}_0, t\right)$ be the current position vector of a given particle where $\overrightarrow{S_0}$ is the vector of the initial position at initial time t_0 . In Cartesian coordinates : $\overrightarrow{S} = x\,\overrightarrow{i} + y\,\overrightarrow{j} + z\,\overrightarrow{k}$ which is also written :

$$\begin{cases} x = x\left(x_0, y_0, z_0, t\right), \\ y = y\left(x_0, y_0, z_0, t\right), \\ z = z\left(x_0, y_0, z_0, t\right). \end{cases} \tag{1.1}$$

where x_0, y_0, z_0 are the coordinates of the initial position vector of the current espace position vector $\overrightarrow{S}$ at time t of the particle.The velocity vector denoted by $\overrightarrow{u}\,(u_x, u_y, u_z)$in $\overrightarrow{S_0}$can be calculated by :

$$u_x = \left(\frac{dx}{dt}\right)_{x_0,y_0,z_0};\, u_y = \left(\frac{dy}{dt}\right)_{x_0,y_0,z_0};\, u_z = \left(\frac{dz}{dt}\right)_{x_0,y_0,z_0}. \tag{1.2}$$

The advantages of Lagrangian description are :
- ♦ The trajectory of each fluid particle is known, and its history can be traced.
- ♦Conservation of mass is satisfied.

1.1.2 Eulerian method (description by velocity field)

It consists of establishing at a given time t all the speeds associated with each point of the domain occupied by the fluid. The mathematical representation of the Eulerian method is written for speed:$\overrightarrow{u}\left(\overrightarrow{S}, t\right)$where$\overrightarrow{u} = u_x\,\overrightarrow{i} + u_y\,\overrightarrow{j} + u_z\,\overrightarrow{k}$
Moreover, the components of the velocity field are expressed as :
:

$$\begin{cases} u_x = u_x\left(x, y, z, t\right), \\ u_y = u_y\left(x, y, z, t\right), \\ u_z = u_z\left(x, y, z, t\right) \end{cases} \tag{1.3}$$

1.2 Definitions

1.2.1 Incompressible fluid

A fluid is said to be incompressible if its volume does not vary as a function of external pressure. Liquids can be considered as incompressible fluids (water, oil, etc.).

1.2.2 streamlines

These are the lines which, at each point of the flow, are tangent to the velocity vector at that point. From this definition, we deduce the differential equation that models stream lines

$$\frac{dx_1}{u_1(x_1, x_2, x_3, t)} = \frac{dx_2}{u_2(x_1, x_2, x_3, t)} = \frac{dx_3}{u_3(x_1, x_2, x_3, t)}. \tag{1.4}$$

where t is a fixed time value..

1.3 Some flow properties

1.3.1 Definitions

A flow is said to be :

1) Irrotational if the rotational velocity vector is zero, i.e. $rot\overrightarrow{u} = 0$ where u represent flow velocity .

2) Incompressible si $\overrightarrow{\bigtriangledown} \cdot \overrightarrow{u} = 0$ où $div(\overrightarrow{u}) = 0$.

1.3.2 Speed potential

If a $\overrightarrow{u}$ ovelocity field is irrotational ,we can define a φ scalar function such that

$$\overrightarrow{u} = \overrightarrow{\bigtriangledown}\varphi$$

The symbol φ espace represents the velocity potential. In the Cartesian reference frame and considering a plane flow, we can therefore write that :

$$\begin{cases} u_x = \frac{\partial\varphi}{\partial x}, \\ u_y = \frac{\partial\varphi}{\partial y} \end{cases} \tag{1.5}$$

If, moreover, the fluid is incompressible :

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \Longrightarrow \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \Longrightarrow \triangle\varphi = 0.$$

1.3.3 stream function

If the flow is in a plane domain, then the velocity vector is verified for any point in this domain, at time t we have :

$$\begin{aligned} div(\overrightarrow{u}) &= 0, \\ \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} &= 0. \end{aligned}$$

This implies that the differential form $u_x dx + u_y dy$ is, t fixed,espace the total differential of a certain function ψ :

$$\exists \psi, \; d\,\psi = u_x dx + u_y dy.$$

This means :

$$\begin{cases} u_x = \frac{\partial \, \psi}{\partial y}, \\ u_y = -\frac{\partial \, \psi}{\partial x} \end{cases} \tag{1.6}$$

where ψ is the current function

Moreover, the property of irrotational flow for a plane flow leads to :

$$\overrightarrow{\bigtriangledown} \wedge \overrightarrow{u} = \overrightarrow{0} \Longrightarrow \begin{vmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{vmatrix} \wedge \begin{vmatrix} u_x = \frac{\partial \, \psi}{\partial x} \\ u_y = -\frac{\partial \, \psi}{\partial y} \end{vmatrix} = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = 0,$$

$\Longrightarrow \triangle \psi = 0,$ espace ψ also verifies Laplace's equation.

1.4 Notions of complex potential and complex velocity

The function $f(z) = \varphi\,(x, y) + i\psi\,(x, y)$ is called complex potentiel , where $z = x + iy$ is the complex variable associated with the complex potential function $f(z)$(φ and ψ represent the potential and current functions respectively).

The function f has the following properties:

1-$f(z)$ is a uniform function, i.e. one value of z corresponds to a single value of f

2-$f(z)$ is analytic, its derivative must be defined everywhere, i.e.

$$\begin{aligned} \frac{df}{dz} &= \frac{\partial \varphi}{\partial x} + i\frac{\partial \psi}{\partial x} = -i\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial y}, \\ &\Longrightarrow \begin{cases} \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} = u_x. \\ \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x} = u_y. \end{cases} \end{aligned}$$

This system of equations constitutes the Cauchy-Riemann relations.

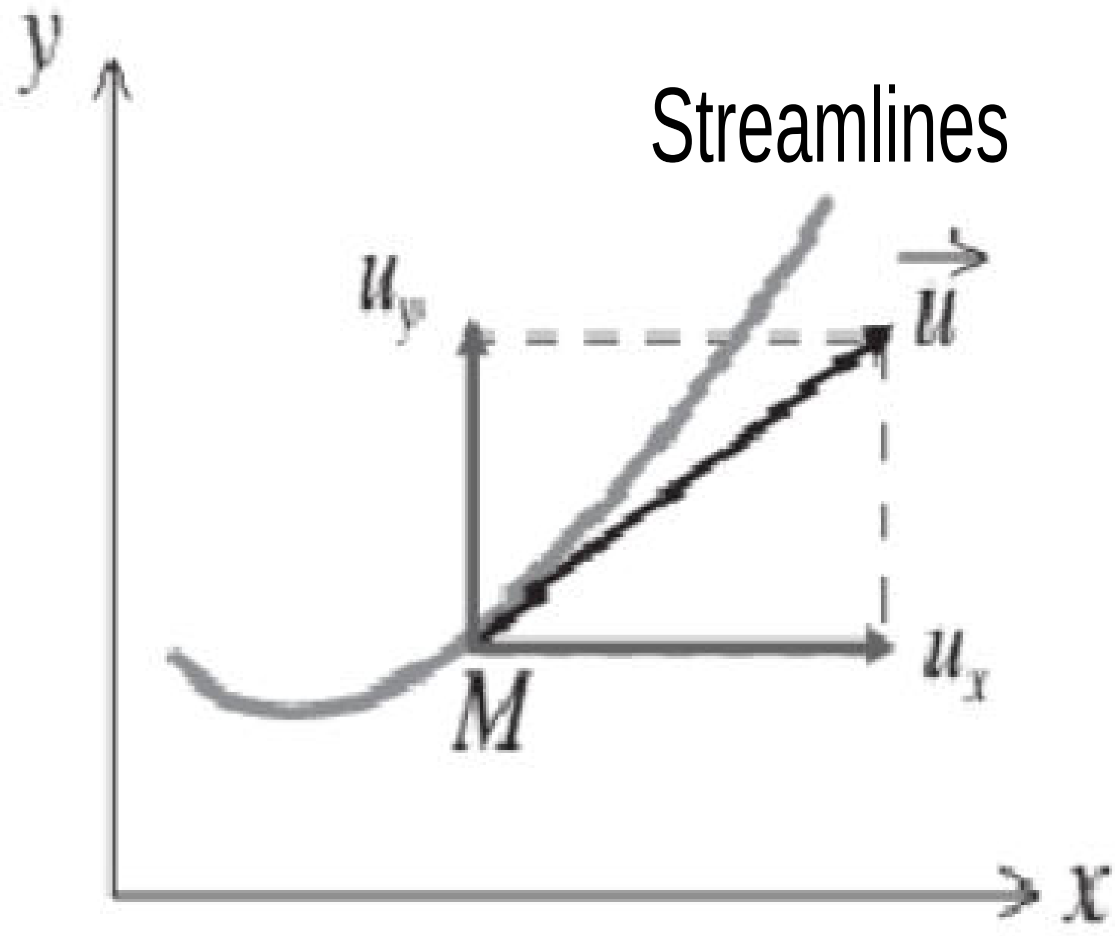

FIGURE 1.1 – Rotational movement of a volume of fluid without deformation

For $f(z)$ to be analytic, φ and ψmust verify the Cauchy-Riemann relations. The function $f(z)$is called the complex velocity potential.

We have seen that for a flow to be described by a current function ψ and a velocity potential φ, these two functions must satisfy the Laplace equation:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0,$$

and

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0.$$

1.5 Bernoulli equation

Bernoulli's theorem is an application of the conservation of energy to the case of moving fluids. In the case of an incompressible fluid, Bernoulli's equation is :

$$\frac{1}{2}q^2+\frac{p}{\rho}+yg=cont.$$

such as $q=\sqrt{u_x^2+v_x^2}$, p fluid pressure, ρ fluid density,$and\, g$ gravity.

1.6 Mass conservation equation

Let's consider a fuid occupying a volumev_0 of density $\rho\ (x,t)$ and boundary a closed surface s.The mass quantity $m$of fuid contained in this volume is equal to :

$$m(t)=\int_{v_0}\rho\,(x,t)\,dv,$$

The variation of the mass m contained in the volume v_0 is given by :

$$\frac{dm\,(t)}{dt}=\frac{d}{dt}\int_{v_0}\rho\,(x,t)\,dv=\int_{v_0}\frac{\partial\rho}{\partial t}(x,t)dv. \tag{1.7}$$

On the other hand, the variation in mass is equal to the mass flux passing through the surface$S$

Let $\overrightarrow{n}$ be the unit normal vector to a surface element of S , $\overrightarrow{V}$ the velocity vector then the mass flow is given by :

$$\int_s\rho V.\overrightarrow{n}\,ds. \tag{1.8}$$

By identifying the two expressions 1.7 and 1.8 we obtain :

$$\frac{dm\,(t)}{dt}=\frac{d}{dt}\int_{v_0}\rho\,(x,t)\,dv=-\int_s\rho V.\overrightarrow{n}\,ds.$$

According to the divergence theorem (Green-Ostogradsky) :

$$\int_s \rho\overrightarrow{V}.\overrightarrow{n}ds \ = \int_{v0} div \ \rho\overrightarrow{V}dv.$$

The result is

$$\int_{v0} \left(\frac{\partial\rho}{\partial t} + div \ \rho\overrightarrow{V}\right) dv = 0.$$

$$\frac{\partial\rho}{\partial t} + div \ \rho\overrightarrow{V} = 0.$$

This equation is known"continuity equation".

1.7 Hilbert transformation

Definition 1.1. Let s be a function defined on $\mathbb{R}$, The Hilbert transform is the function $\hat{s}$ defined by :

$$\hat{s}\,(t) = H\,\{s\}\,(t) = vp\,\{(h\star s)\,(t)\} = vp\left\{\int_{-\infty}^{+\infty} s(t)h(t-\tau)d\tau\right\} = \frac{1}{\pi}vp\left\{\int_{-\infty}^{+\infty}\frac{s(t)}{t-\tau}d\tau\right\}.$$

where H the Hilbert transformation and where :

$$h(t) = \frac{1}{\pi t}.$$

and :

$$vp\left\{\int_{-\infty}^{+\infty} s(t)h(t-\tau)\right\} = \lim_{\epsilon\to 0}\left\{\int_{-\infty}^{t-\epsilon} s(t)h(t-\tau)d\tau + \int_{t+\epsilon}^{+\infty} s(t)h(t-\tau)d\tau\right\}.$$

vpbeing the abbreviation for Cauchy principal value, we can show that for any real $p>1$, $H$is a bounded operator of the space $L^p\ (\mathbb{R})$in itself..

1.8 Schawarz-Christoffel Tronsformation

The Schwarz-Christoffel tronsformation is a conformal tronsformation widely used to solve flow problems. This tronsformation transforms the interior of a polygon in the complex plane into the upper half-plane of the complex variable λ.

Let $a_1, a_2, ..., a_n$be the points corresponding respectively to $\lambda_1, \lambda_2, ..., \lambda_n$of the real axis of the $plan\lambda$. We define the schwarz-christoffel trosformtion, which represents the interior of the upper half-plane polygon, by the formula :

$$\Omega = K\int(\lambda-\lambda_1)^{\frac{\alpha_1}{\pi}-1}(\lambda-\lambda_2)^{\frac{\alpha_2}{\pi}-1}(\lambda-\lambda_3)^{\frac{\alpha_3}{\pi}-1}......(\lambda-\lambda_n)^{\frac{\alpha_n}{\pi}-1}d\lambda + K.$$

where K and $M$are complex constants , $\lambda_1, \lambda_2, ...and\lambda_n$are real numbers and$\alpha_1, \alpha_2, ...$ and $\alpha_n$the interior angles of the polygon .

1.9 Dimensional analysis

Before solving a problem, we need to rewrite the equations governing the phenomenon in non-dimensional variables. To do this, we introduce a some notions and theorems for moving from a physical equation in dimensional variables to an equation whose va- riables are physically dimensionless.We use Vaschy-Buckingham's π theorem,which shows how to make a physical equation dimensionless. The use of non-dimensional variables reduces thenumber of parameters that determine the solution of a problem. If a physical phenomenon depends on n dimensional variables, we can make these variables dimensionless by reducing them $n-k$, with $(k = 1, ...4)$. The four variables universally are length L ,the mass m ,the temperature T and time t.

CHAPTER 2

Numerical Resolution of the two dimensional flow problem

In this chapter, we focus on the numerical solution of the two-dimensional flow problem underconsideration, by taking into account the forces of gravity, and neglecting the surface tension, using Hilbert's method and the perturbation technique.

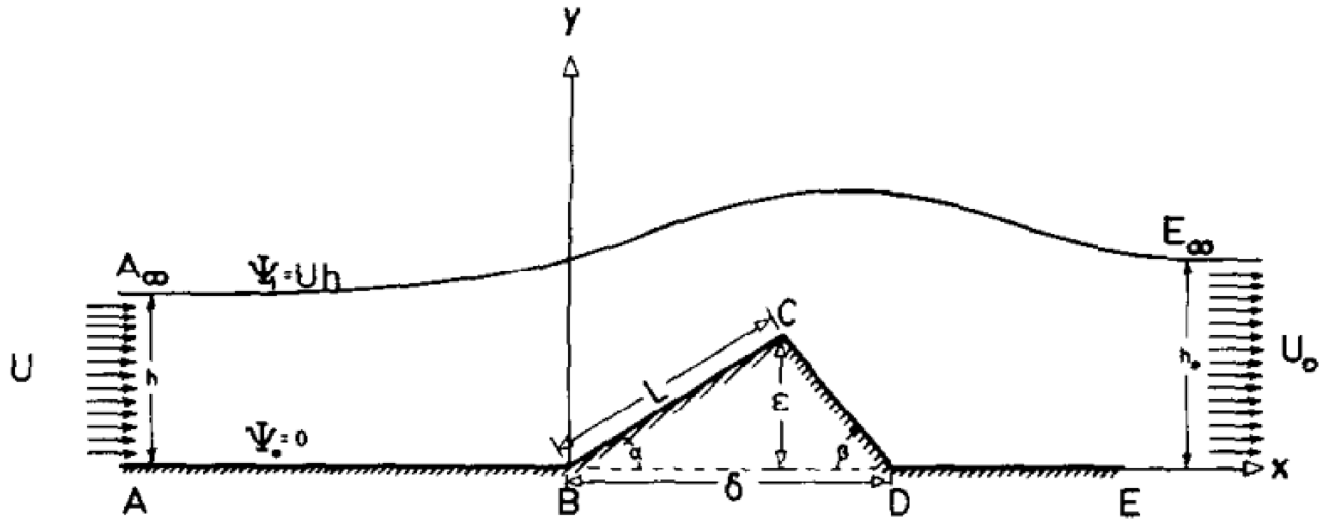

FIGURE 2.1 – z-plane

2.1 **Formulating** the problem

In this study, we consider a problem of two-dimensional irrotational motion, i.e, the velocity of this flow is derived from a potential $\varphi(x, y)$ satisfying
Laplace's equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0. \tag{2.1}$$

The fluid is considered non-viscous and incompressible, subject to the action of gravitational force and with a free upper surface.
At the origin we consider the flow to be uniform with velocity U_1 . in the positive direction of the apsidal axis x and depth $h_1$the bottom of the flow consists of a horizontal wall AB, triangle $BCD$with two angles of inclination α and β,and a horizontal wall DE. Where the bottom extends from $-\infty$ (point A)à$+\infty$ (point E) .The domain is illustrated in figure 1.

For convenience, we choose point Bas the origin in the $zplane$, the x-axisis coincident with wall AB and the y-axis is perponducular, passing through the pointB.The potential function $f(z) = \varphi(x, y) + i\psi(x, y)$ is introduced as an analytical function of z in the flow region, with complex velocity.

$$\frac{df}{dz} = u(x, y) - iv(x, y) = qe^{-i\theta}. \tag{2.2}$$

If we take the unit of length $h_1$and the unit of speed U_1we can write the one-dimensional variables :

$$z' = \frac{z}{h_1}, q' = \frac{q}{u_1}, f' = \frac{f}{\psi_1}. \tag{2.3}$$

Where

$$\psi_1 = u_1 h_1.$$

LThe condition on the free surface where the pressure is uniform is given by Bernoulli's equation:

we have :

$$\begin{aligned}
&\frac{1}{2}q^2 + yg = \frac{1}{2}u^2 + hg, \\
\Rightarrow &\frac{1}{2}q^2 + g\left(y - h\right) = \frac{1}{2}u^2, \\
\Rightarrow &\frac{1}{2}\left(\frac{q}{u}\right)^2 + \frac{g}{u^2}\left(y'h - h\right) = \frac{1}{2}, \\
\Rightarrow &q'^2 + \frac{2gh}{u^2}\left(y' - 1\right) = 1,
\end{aligned}$$

then :

$$q'^2 + \frac{2}{F^2}\left(y' - 2\right) = 1 \tag{2.4}$$

Such as :

$$\begin{aligned}
y' &= \frac{y}{h}, \\
q' &= \frac{q}{u},
\end{aligned}$$

$q = \sqrt{u_x^2 + u_y^2}$ ..$F$the Froude number defined by :

$$F = \frac{u_1}{\sqrt{gh_1}}. \tag{2.5}$$

The height of the triangle noted by ϵ, where :

$$\epsilon = \frac{L\ \sin\alpha}{h_1} = L'\sin\alpha. \tag{2.6}$$

Using the unitary variables (2.3) equation (2.2) in adimentional form gives t:

$$\xi = \frac{df'}{dz'} = q'e^{-i\theta}. \tag{2.7}$$

We
have

$$\omega = \ln\ \xi = \ln\ q' - i\theta. \tag{2.8}$$

$$\xi = e^{\omega},$$

as

$$\xi = \frac{df'}{dz'} = q'e^{-i\theta},$$

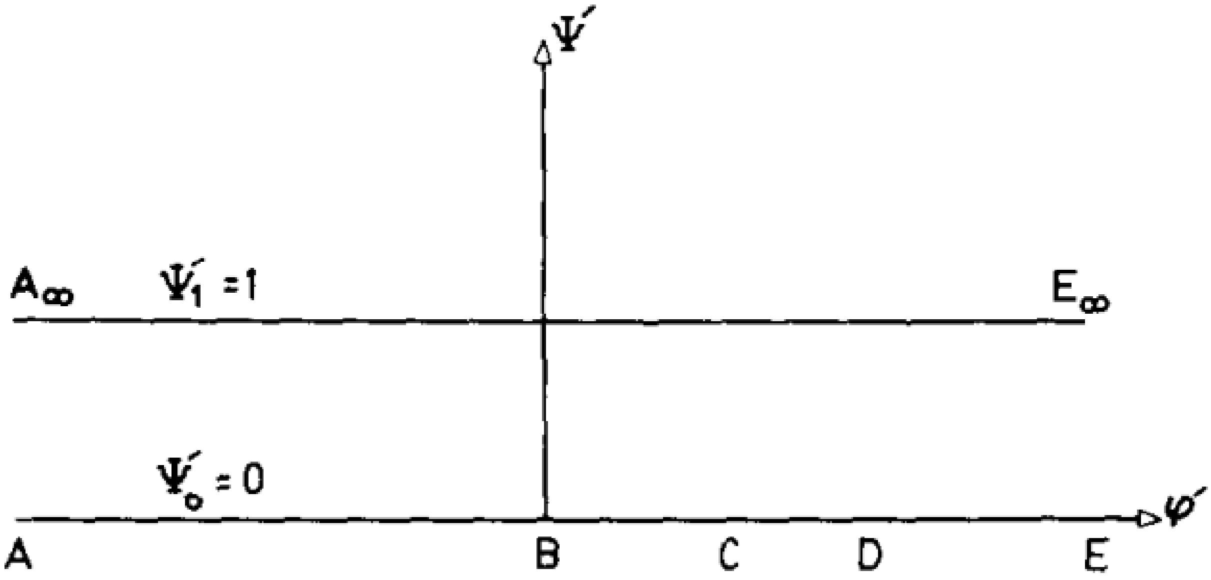

FIGURE 2.2 – f-plane

then

$$dz' = \frac{1}{\xi} df',$$
$$z' = \int \frac{1}{e^f} df',$$

so

$$z' = \int e^{-f} df'. \tag{2.9}$$

Where $\omega$is called the logarithmic hodograph variable. Using the transformation of

Schwartz-Christoffel,we transform the flow region of the f'plane onto the upper half of a t auxilary plane, so that the following points correspond.(see figures 2.2 and 2.3)

point	f'	t
B	0	0
E, E_∞	$+\infty$	1
A, A_∞	$-\infty$	$+\infty$

LThe transformatiuon used is given by :

$$f'(t) = -\frac{1}{\pi} \ln(1-t)\,; 0 \leq \arg(1-t) \leq \pi. \tag{2.10}$$

To express ω as a function of the single variable t, we introduce Hilbert's method for a mixed boundary value problem in the upper half-plane,
$Q(t)$is an analytical function defined in the upper half-plane of the $tplane$,and let's assume that $Im\,(Q(t))$ satisfies the Höldercondition in the limit, $Im\,(t) = 0$,of the plane $tplane$.

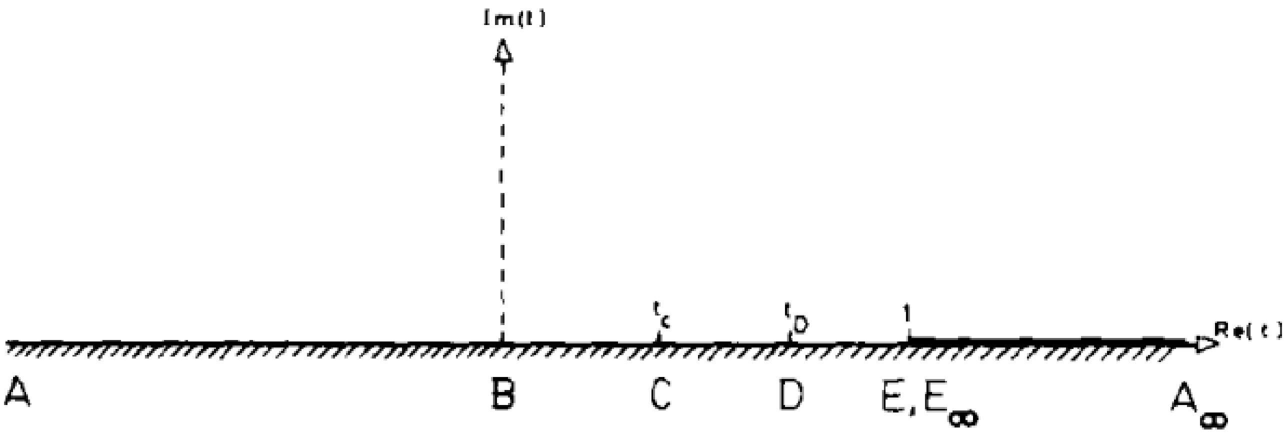

FIGURE 2.3 – t-plane

If $Im\left(Q(t)\right)$is known on the boundary, $Q(t)$ is given by the Poisson integral formula:

$$Q(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{Im\left(q\left(v\right)\right)}{v-t}dv + \sum_{j=0}^{\infty}A_j t^j. \tag{2.11}$$

WhereA_j are constant realms, and f ddenotes a singular integral in Cauchy's serenity.Now we try to relate the function $\omega\left(t\right)$to the function $Q\left(t\right)$. From (2.11),we find that $Q(t)$is expressed in terms of its imaginary parts along the real axis, the limit of the t plane.

Thus, we need to examine the value of $\omega\left(t\right)$along the t plane boundary,and we find that we have $\omega = \ln q' e^{-i\theta}$then

$$Im\left(\omega\left(t\right)\right) = -\theta\left(t\right), \tag{2.12}$$

and

$$\begin{aligned} q'^2 + \frac{2}{F^2}\left(y'-1\right) &= 1, \\ \Rightarrow q'^2 &= 1 - \frac{2}{F^2}\left(y'-1\right), \\ \Rightarrow q' &= \sqrt{1-\frac{2}{F^2}\left(y'-1\right)}, \\ \Rightarrow \ln q' &= \frac{1}{2}\ln(1-\frac{2}{F^2}\eta'(t)). \end{aligned}$$

So we find

$$\begin{aligned} Im\left(\omega\left(t\right)\right) &= -\theta\left(t\right). \\ Re\left(\omega\left(t\right)\right) &= \frac{1}{2}\ln(1-\frac{2}{F^2}\eta'(t)). \end{aligned} \tag{2.13}$$

Where

$$\theta(t) = \begin{cases} 0, & t<0, \\ \alpha, & 0<t<t_c, \\ -\beta, & t_c<t<t_D, \\ 0, & t_D<t<1, \\ \theta(t), & t>1. \end{cases} \qquad (2.14)$$

And

$$\eta'(t) = y'(t) - 1.$$

This means that we know the Real or Imaginary part of $\omega(t)$ along the boundary of the $tplane$.then we need to construct an auxilliary function $H(t)$that makes known the Imaginary part of the quotient $Q(t) = \omega(t)/H(t)$ at the boundary of the tplane. the general form of $H(t)$ is :

$$H(t) = \begin{cases} -\sqrt{(1-t)}, & t<1, \\ -i\sqrt{t-1}, & t>1. \end{cases} \qquad (2.15)$$

chanson [20]has shown that the final solution is independent of the particular choice of $H(t)$.we have :

$$Q(t) = \frac{\omega(t)}{H(t)} = \begin{cases} \frac{\ln q' - i\theta}{-\sqrt{(1-t)}}, & t<1, \\ \frac{\ln q' - i\theta}{-i\sqrt{(t-1)}}, & t>1. \end{cases} \qquad (2.16)$$

using (2.12) and (2.15) ,we obtain :

$$Im(Q(t)) = \begin{cases} \frac{\theta(t)}{\sqrt{1-t}}, & t<1, \\ \frac{\ln\left(1-\frac{2}{F^2}\eta'(t)\right)}{2\sqrt{(t-1)}}, & t>1. \end{cases} \qquad (2.17)$$

next ,we examine the condition rn upstream ,As we approach that is point $A_\infty thatis$ $t \to +\infty, H(t) \simeq -i\sqrt{t}$ et $\omega(t) \simeq \ln q'(t) = 0$. Danc ,$Q(t) \simeq \omega(t)/\left(-i\sqrt{t}\right) = 0$, and from , $A_j = 0$ for $j = 0, 1, ..., n$.
Thus , (2.11)takes the form:

$$Q(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{Im(q(v))}{v-t} dv. \qquad (2.18)$$

Using (2.6),we obtain :

$$Q(t) = \frac{}{H(t)} = \begin{cases} \frac{-\ln q'(t)}{\sqrt{1-t}} + i\frac{\theta(t)}{\sqrt{1-t}}; & t<1, \\ \frac{i\ln q'(t)}{\sqrt{t-1}} + \frac{\theta(t)}{\sqrt{t-1}}; & t>1, \end{cases} \\ = R(t) + iS(t). \qquad (2.19)$$

The form (2.18) is equivalent to :

$$\begin{array}{c} R(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{s(v)}{v-t}dv, \\ S\left(t\right) = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{R(v)}{v-t}dv, \end{array} \tag{2.20}$$

using (2.20),we obtain a set of integral equations, and for our work we only need the following equations:

$$\begin{aligned} \theta(t) &= \frac{\sqrt{t-1}}{\pi}\int_{1}^{+\infty}\frac{\ln q(s)}{(s-t)\sqrt{s-1}}ds + \frac{2\alpha}{\pi}\tan^{-1}\left(\frac{r\sqrt{t-1}}{t-r}\right) \\ &- \frac{2\beta}{\pi}\tan^{-1}\left(\frac{m\sqrt{t-1}}{t-1+n}\right), \qquad t>1, \end{aligned} \tag{2.21}$$

$$\begin{aligned} \ln(q_j(t)) &= \frac{\sqrt{1-t}}{\pi}\int_{1}^{+\infty}\frac{\ln(q(s))}{(s-t)\sqrt{s-1}}ds - \alpha\frac{\sqrt{1-t}}{\pi}\int_{0}^{t_C}\frac{ds}{(s-t)\sqrt{1-s}} \\ &+ \beta\frac{\sqrt{1-t}}{\pi}\int_{t_C}^{t_D}\frac{ds}{(s-t)\sqrt{1-s}}, \end{aligned} \tag{2.22}$$

where

$$\begin{cases} r = 1-\sqrt{1-t_c}, \\ m = \sqrt{1-t_C}-\sqrt{1-t_D}, \\ n = \sqrt{\left(1-t_C\right)\left(1-t_D\right)}, \end{cases}$$

and $j = 1,2,3,4$, $q_1(t)$ defined for $t<0$, $q_2(t)$ for $0<t<t_C$, $q_3(t)$ for $t_C<t<t_D$ and $q_4(t)$ for $t_D<t<1$.
The coordinates (x', y') of a point on the free surface can be obtained using (2.8) and(2.9) as follows:

we have

$$z' = \int e^{-\omega}dw',$$

such that

$$\omega = \ln q' - i\theta,$$

then

$$z' = \int e^{-\ln q' + i\theta}dw',$$

using (2.10), we obtain :

$$z'(t) = \frac{1}{\pi}\int\frac{e^{i\theta(v)}}{q'(v)}\times\frac{1}{1-v}dv + const.$$

So

$$z'(t) = \frac{1}{\pi}\int_{\infty}^{t}\frac{e^{i\theta(v)}}{q'(v)\,(1-v)}dv + (x'_{\infty}+i)$$

separating the real and imaginary parts :

$$x'(t) = x'_{\infty} + \frac{1}{\pi}\int_{\infty}^{t}\frac{\cos\theta(v)}{(1-v)\,q'(v)}dv. \tag{2.23}$$

$$y'(t) = 1 + \frac{1}{\pi}\int_{\infty}^{t}\frac{\sin\theta(v)}{(1-v)\,q'(v)}dv \tag{2.24}$$

from (2.24) we can find the length of BC :

$$L' = \frac{1}{\pi}\int_{0}^{t_c}\frac{1}{(1-v)\,q'_2(v)}dv. \tag{2.25}$$

consequently,the system of equations (2.4) (2.21),(2.22),(2.23)and(2.24)(2.21),(2.22),(2.23) and(2.24) fully describes our problem.

2.2 Approximate equations

For a higher Froud number Fand for a small value of the angle of inclination α, in which case the change in $\theta(t)$will be very small, we can approximate $\sin\theta(t)$ by $\theta(t)$ and $\cos\theta(t)$ par 1. After that, we proceed to the prime number drop and the equation system takes the form :

$$q(t) \simeq 1 - \frac{1}{F^2}\eta(t), \qquad t>1, \tag{2.26}$$

$$\eta(t) \simeq -\frac{1}{\pi}\int_{t}^{\infty}\frac{}{1-v}\left(1+\frac{1}{F^2}\eta(v)\right)dv, \quad t>1, \tag{2.27}$$

$$\theta(t) \simeq -\frac{\sqrt{t-1}}{\pi F^2}\int_{1}^{\infty}\frac{\eta(v)}{(v-t)\sqrt{v-1}}dv + \frac{2\alpha}{\pi}\tan^{-1}\left(\frac{r\sqrt{t-1}}{t-r}\right)$$
$$-\frac{2\beta}{\pi}\tan^{-1}\left(\frac{m\sqrt{t-1}}{t-1+n}\right), \qquad t>1, \tag{2.28}$$

$$x(t) \simeq x_{\infty} - \frac{1}{\pi}\int_{t}^{\infty}\frac{1}{1-v}\left(1+\frac{1}{F^2}\eta(v)\right)dv, \quad t>1, \tag{2.29}$$

2.3 DISRUPTION TECHNOLOGY

Expand $q(t), \eta(t), \theta(t)$ and $x(t)$in terms of small parameters α and β,

$$
\begin{aligned}
q(t) &= q_0(t) + \alpha q_{\alpha,1}(t) + \alpha^2 q_{\alpha,2}(t) + ... + \beta q_{\beta,1}(t) + \beta^2 q_{\beta,2}(t) + ..., && (2.30)\\
\eta(t) &= \eta_0(t) + \alpha \eta_{\alpha,1}(t) + \alpha^2 \eta_{\alpha,2}(t) + ... + \beta \eta_{\beta,1}(t) + \beta^2 \eta_{\beta,2}(t) + ..., && (2.31)\\
\theta(t) &= \theta_0(t) + \alpha \theta_{\alpha,1}(t) + \alpha^2 \theta_{\alpha,2}(t) + ... + \beta \theta_{\beta,1}(t) + \beta^2 \theta_{\beta,2}(t) + ..., && (2.32)\\
x(t) &= x_0(t) + \alpha x_{\alpha,1}(t) + \alpha^2 x_{\alpha,2}(t) + ... + \beta x_{\beta,1}(t) + \beta^2 x_{\beta,2}(t) + ... && (2.33)
\end{aligned}
$$

using the expansions $(2.30)-(2.33)$ in equations $(2.22)-(2.26)$and assimilating similar power terms α and β, we obtain the :

2.3.1 zero order approximation

$$
\begin{aligned}
q_0(t) &= 1 - \frac{1}{F^2}\eta_0(t), \ \ t > 1, && (2.34)\\
\eta_0(t) &= -\frac{1}{\pi}\int_t^{\infty} \frac{}{1-v}\left(1 + \frac{1}{F^2}\eta_0(v)\right) dv, && (2.35)\\
\theta_0(t) &= -\frac{\sqrt{t-1}}{\pi F^2}\int_1^{\infty} \frac{\eta_0(v)}{(v-t)\sqrt{v-t}} dv, \ t > 1, && (2.36)\\
x_0(t) &= x_{\infty} - \frac{1}{\pi}\int_t^{\infty} \frac{1}{1-v}\left(1 + \frac{1}{F^2}\eta_0(v)\right) dv, \ \ t > 1, && (2.37)
\end{aligned}
$$

This zero-order approximation corresponds to the flat-bottom flow case .solution
solution deorder zero approximation is: dimensionless variables $x' = \frac{x}{h_1}, y' = \frac{}{h_1}, z' = \frac{z}{h_1}$

$$q_0(t) = 1 - \frac{1}{F^2}\eta_0(t),$$

such that

$$
\begin{aligned}
\eta_0(t) &= -\frac{1}{\pi}\int_t^{\infty} \frac{}{1-v}\left(1 + \frac{1}{F^2}\eta_0(v)\right) dv,\\
&= y' - 1,\\
&= \frac{y}{h_1} - 1.
\end{aligned}
$$

The flow is flat, so $y = h_1$. So :

$$\eta_0(t) = 0; \implies q_0(t) = 1. \tag{2.38}$$

And as :

$$\eta_0(t) = 0 \tag{2.39}$$

then

$$\theta_0(t) = 0 \tag{2.40}$$

and

$$x_0(t) = x_\infty - \frac{1}{\pi}\int_t^\infty \frac{dv}{1-v}.$$

write

$$x_\infty = \frac{1}{\pi}\int_0^\infty \frac{dv}{1-v}.$$

Therefore

$$x_0 = -\frac{1}{\pi}\ln(t-1). \tag{2.41}$$

2.3.2 first-order approximation

$$\begin{cases} q_{1,\alpha}(t) \approx -\frac{1}{F^2}\eta_{1,\alpha}(t), \\ q_{1,\beta}(t) \approx -\frac{1}{F^2}\eta_{1,\beta}(t)). \end{cases} \tag{2.42}$$

$$\begin{cases} \eta_{1,\alpha}(t) \approx & -\frac{1}{\pi}\int_t^\infty \frac{\theta_{1,\alpha}(v)}{1-v}dv, \\ \eta_{1,\beta}(t) \approx & -\frac{1}{\pi}\int_t^\infty \frac{\theta_{1,\beta}(v)}{1-v}dv. \end{cases} \tag{2.43}$$

$$\begin{cases} \theta_{1,\alpha}(t) \approx & -\frac{\sqrt{t-1}}{\pi F^2}\int_1^\infty \frac{\eta_{1,\alpha}(v)}{(v-t)\sqrt{v-t}}dv + \frac{2}{\pi}\tan^{-1}\left(\frac{r\sqrt{t-1}}{t-r}\right), \\ \theta_{1,\beta}(t) \approx & -\frac{\sqrt{t-1}}{\pi F^2}\int_1^\infty \frac{\eta_{1,\beta}(v)}{(v-t)\sqrt{v-t}}dv - \frac{2}{\pi}\tan^{-1}\left(\frac{m\sqrt{t-1}}{t-1+n}\right), \end{cases} \tag{2.44}$$

and

$$\begin{cases} x_{1,\alpha}(t) \approx & -\frac{1}{\pi F^2}\int_t^\infty \frac{\eta_{1,\alpha}(v)}{1-v}dv \\ x_{1,\beta}(t) \approx & -\frac{1}{\pi F^2}\int_t^\infty \frac{\eta_{1,\beta}(v)}{1-v}dv \end{cases} \tag{2.45}$$

2.4 Problem solution

Solution of first-order approximations: from (2.44), for the very large Froud number F,we can neglect the first term with respect to the second and obtain :

$$\begin{cases} \theta_{1,\alpha}(t) \approx & \frac{2}{\pi}\tan^{-1}\left(\frac{r\sqrt{t-1}}{t-r}\right), \\ \theta_{1,\beta}(t) \approx & -\frac{2}{\pi}\tan^{-1}\left(\frac{m\sqrt{t-1}}{t-1+n}\right) \quad , t>1, \end{cases} \tag{2.46}$$

Replacing (2.43)and completing the intergration,we find:

$$\begin{cases} \eta_{1,\alpha}(t) \approx & \frac{4r}{\pi^2\sqrt{1-r}}\tan^{-1}\left(\sqrt{\frac{1-r}{t-1}}\right), \\ \eta_{1,\beta}(t) \approx & -\frac{4m}{\pi^2\sqrt{1-n}}\tan^{-1}\left(\sqrt{\frac{1-n}{t-1}}\right) \quad , t>1, \end{cases} \tag{2.47}$$

Consequently,relation (2.42)becomes of the form :

$$\begin{cases} q_{1,\alpha}(t) \approx & -\frac{4r}{\pi^2F^2\sqrt{1-r}}\tan^{-1}\left(\sqrt{\frac{1-r}{t-1}}\right), \\ q_{1,\beta}(t) \approx & \frac{4m}{\pi^2F^2\sqrt{1-n}}\tan^{-1}\left(\sqrt{\frac{1-n}{t-1}}\right) \quad , t>1, \end{cases} \tag{2.48}$$

and

$$\begin{cases} x_{1,\alpha}(t) \approx & \frac{8r}{\pi^3 F \sqrt[2]{t-1}}, \\ q_{1,\beta}(t) \approx & -\frac{8m}{\pi^3 F \sqrt[2]{t-1}} \end{cases}, \; t>1, \tag{2.49}$$

Consequently,the system solution (2.26) − (2.29)becomes by :

$$q(t) \simeq 1-\alpha\frac{4r}{\pi^2F^2\sqrt{1-r}}\tan^{-1}\left(\sqrt{\frac{1-r}{t-1}}\right)+\beta\frac{4m}{\pi^2F^2\sqrt{1-n}}\tan^{-1}\left(\sqrt{\frac{1-n}{t-1}}\right), \tag{2.50}$$

$$y(t) \approx 1+\alpha\frac{4r}{\pi^2\sqrt{1-r}}\tan^{-1}\left(\sqrt{\frac{1-r}{t-1}}\right)-\beta\frac{4r}{\pi^2\sqrt{1-r}}\tan^{-1}\left(\sqrt{\frac{1-r}{t-1}}\right), \tag{2.51}$$

$$\theta(t) \simeq \frac{2\alpha}{\pi}\tan^{-1}\left(\frac{r\sqrt{t-1}}{t-r}\right)-\frac{2\beta}{\pi}\tan^{-1}\left(\frac{m\sqrt{t-1}}{t-n}\right) \tag{2.52}$$

$$x(t) \simeq -\frac{1}{\pi}\ln(t-1)+\alpha\frac{8r}{\pi^3F^2\sqrt{t-1}}+-\frac{1}{\pi}\ln(t-1)+\beta\frac{8m}{\pi^3F^2\sqrt{t-1}}. \tag{2.53}$$

CHAPTER 3

Numerical Results

in this chapter, we give the results in curves showing the effect of angle α,height ε
.

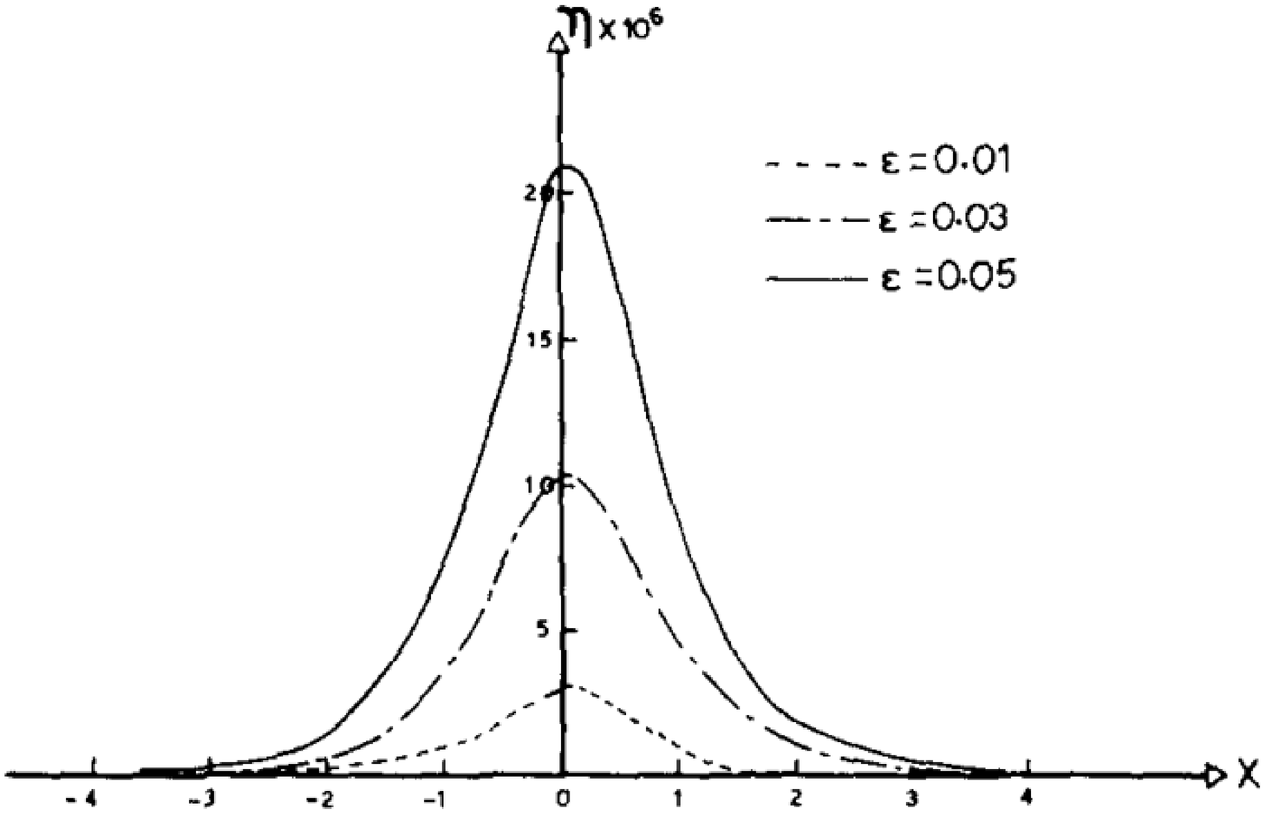

FIGURE 3.1 – Effect of triangular hump height Eon the free surface profile for $\alpha = \beta, \delta = 0,1187, F^2 = 10,0$

Approximate solutions for flows with a large Froude number are found over a wide range of triangular hump heights. Results showing free surface elevation over a range of triangular hump heights $\epsilon : 0,01 - 0,05$, for$(\alpha = \beta, \delta = 0,1187, F^2 = 10,0$ and over a range of

inclination angle $\beta, 0,1027$ rad to $0,3$ for $\alpha = 0,3$, $F^2 = 10$ and ϵranges from $0,010,02$are given in Figures 3.1 and 3.2. It is clear that for $(\alpha = \beta$,the free surface profile is almost symmetrical with respect to point C,as expected. The effect of bottom shape on free surface elevation for $\epsilon = 0,005$ and$F^2 = 10$fixed is given in Figure3.3.3.3. These results demonstrate the monotonic rise in free surface elevation with increases in ϵ andβ.

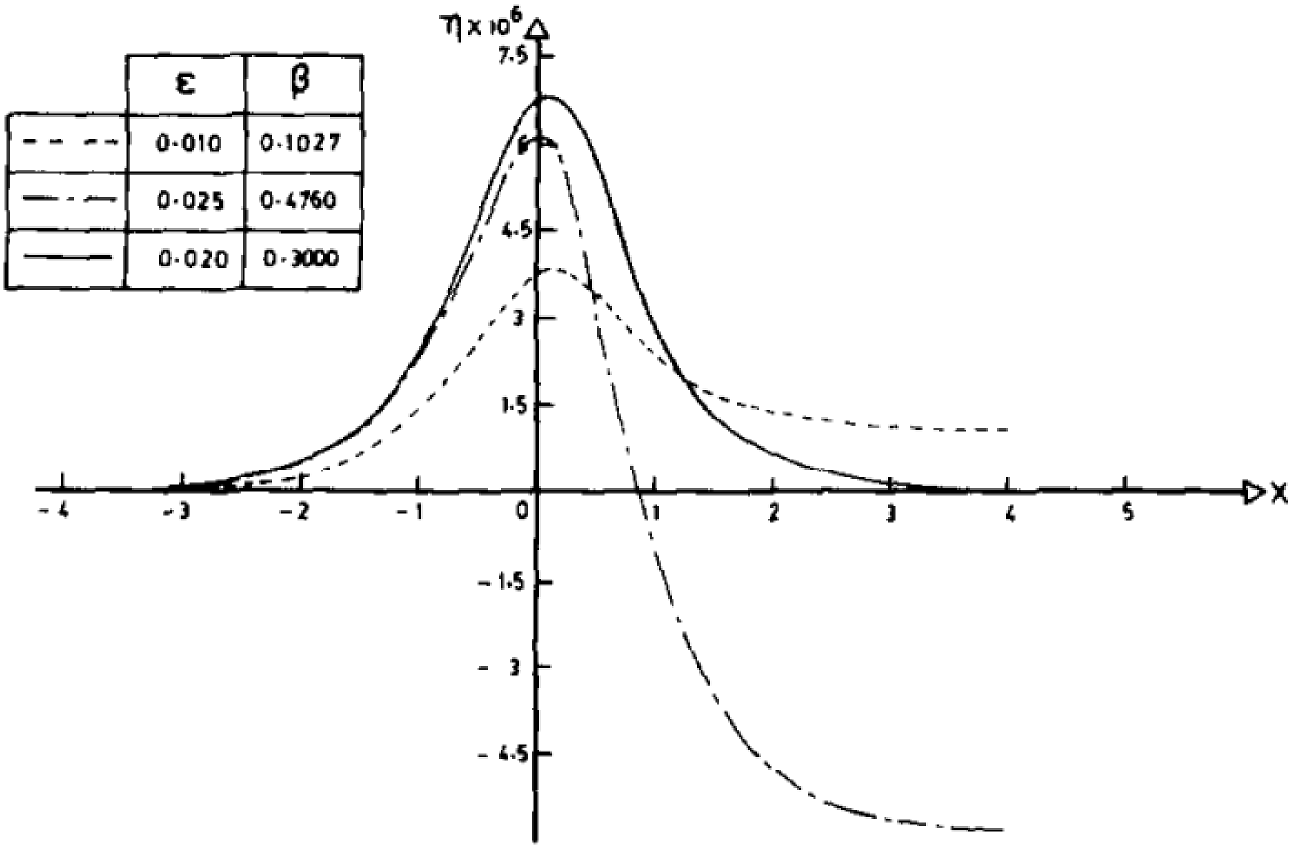

FIGURE 3.2 – Effect of bottom shape on free surface profile for $\alpha = 0,3$rad, $F^2 = 10$ and ϵ

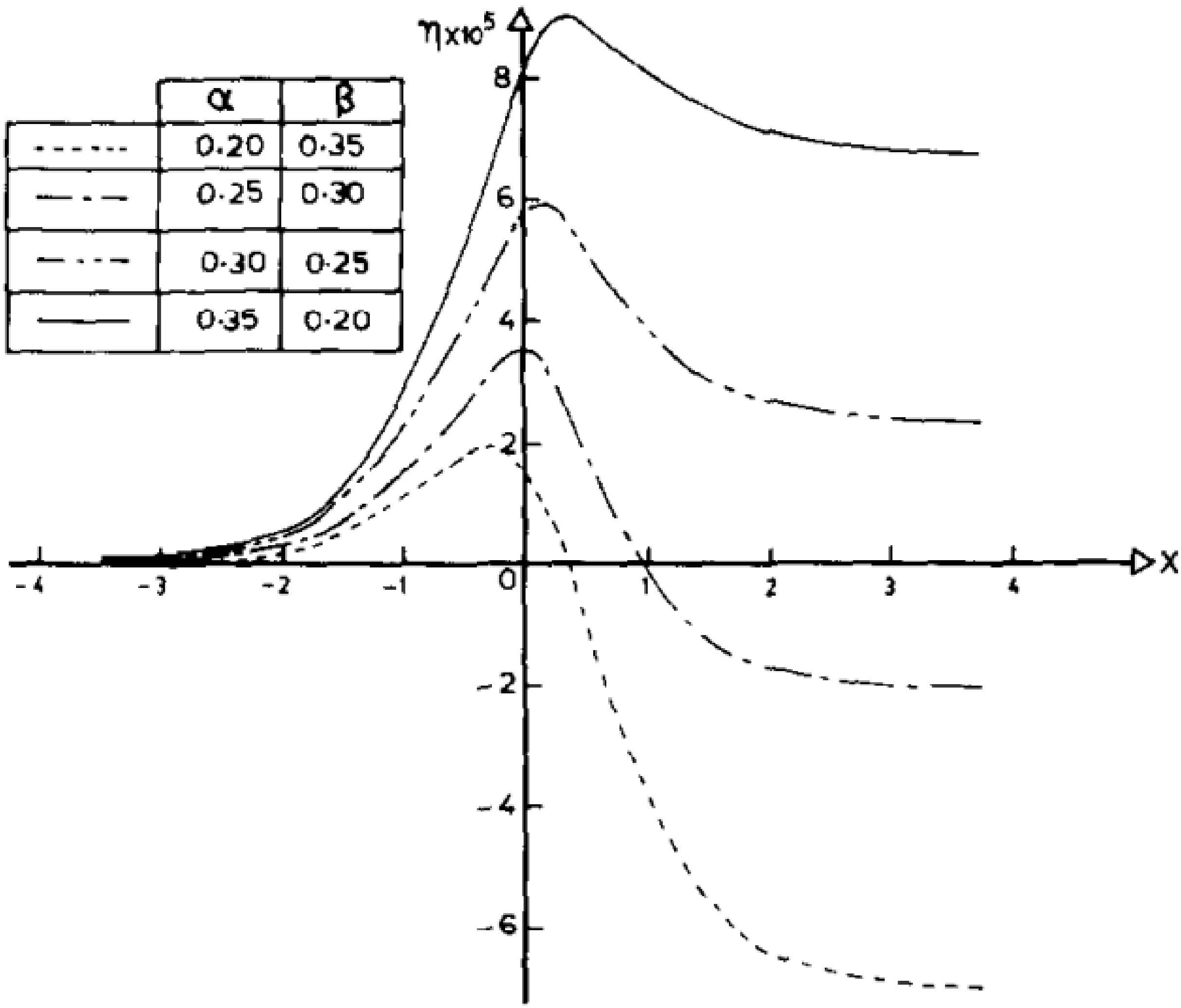

FIGURE 3.3 – Effect of bottom shape on free surface profile for $\epsilon = 0,005$ and $F^2 = 10$

Conclusion

Free-surface flow problems are very difficult and even impossible to solve explicitly, especially if the effects of gravity are included. This is due to the non-linearity of the edge condition of Bernoulli's equation, on a surface of unknown shape. In this thesis, we have studied a two-dimensional incompressible and non-viscous fluid flow problem above the triangular threshold using Hilbert's method and the perturbation technique.

Bibliography

[1] Alfio Quarteroni, Fausto Saleri et Paola Gervasio, Calcul Scientifique Cours, exercices corrigées et illustrations en MATLAB et Octave, Deuxième édition , Springer.

[2] H.Serguine, Numerical resolution of a non-linear problem in a flield at free border due to a jet against a infinite plate.,Thesis of Magister University of M'sila,(2004).

[3] Laiadi abdelkader, effet de la tension de surface sur des écoulements au dessus des obstacles ,thèse magistère de universite de ouargla(2012).

[4] Riadh , Ben Hamoduda Notions de Mécanigue des fluides, cours et exercices corrigés.

[5] Sakir Amiroudine et Jean-Luc Battaglia, Mécanigue des fluides, Dunod,Paris,2011 ISBN 978-2-10-056922-9.

[6] Y. f. Li, J. M. chuang et C. C. Hsiung, Computation Nonlinear 2-D free surface flow using the Hilbert method, Univesity of Nova Scotia Halifax, Canada.

[7] Abd-el-Malek, Mina B and Hanna, Sarwat N. Approximate solution of a flow over a ramp for large Froude number. J. Comput.Appl.Math.28(1989) : 105117.

[8] M.B.Abd-el-Malel,Boudray integral methods and free surface problems,ph.D.dissertation,University of Windsor ,Windsor ,Ontario,canada,1981.

[9]K.M.Agrawal,Non-linear theories in two-dimensional fully and partially cavitatong flow,ph.D.dissertation,University of Windsor ,Windsor ,Ontario,canada,1969.

[10]B.E.Larock and R.L.Street ,A Rieman -Hilbert problem for non-linear fully cavitating flow,J.Ship Research 9 (3) (1965) 170-178.

[11]T.H.Lim,Integral equation methodes for the numerical solution of free surface problems in inviscid flow ,ph.D.dissertation,University of Windsor ,Windsor ,Ontario, canada ,1978.

[12]N.I.Muskhelishvili,Singular Integral Equations(Noordhoff,Gronigen,The Netherlands ,1953).

[13]A.C.Smith and M.B.Abd-el-Malk,Hilbert's methods for numerical solution of flow from a uniform channel over a shelf ,j.Engrg .Math.17(1983) 27-39.

[14]A.C.Smith and T.H.Lim,The steady water wave :A numerical solution using the Riamann-Hilbert methods,Internat.J.Engrg.sci.18 (1980) 139-152.

[15]S.C.Song ,A quasi-linear and linear for non-separated and separated two-dimensional flow about lifting bodies ,Univ.Minnesota,Min.SAF Lab .Tech.paper,B-43,1963.

[16]L.K.Forbes, l.W. Schwartz, Free-surface flow over a semicircular obstruction, *J. Fluid Mech*,**114**(1982),299–314.

[17]A. Gasmi, H. Mekias, The effect of surface tension on the contraction coefficient of a jet, *J. Phys. A : Math. Gen*,**36**(2003), 851–862.

[18] A. Gasmi, H. Mekias, jet from container and flow past a vertical flat plate in a channel with the surface tension effects,*Appl. Math. Sci. (Ruse)*,**53(56)**(2007), 2687 ?2698.

[19] A. Gasmi, A. Amara, Free-surface profile of a jet flow in U-shaped channel without gravity effects,*ASCM (Kyungshang)*,**28(3)**(2018),393–400.

[20]A. Laiadi, A.Merzougui, Free surface flows over a successive obstacles with surface tension and gravity effects, *AIMS Mathematics*,**4(2)**(2019),316–326.

[21]A. Laiadi, A.Merzougui,Numerical solution for a free surface flow over multiple obstacles. *Electron, J. Math. Anal. Appl*,**6(1)**(2018) ,266–275.

[22] A. Mina B, S.N. Hanna,Approximate solution of a flow over a ramp for large Froude number,*J. Comput.Appl.Math*,**28**(1989), 105–117.

[23] A. Mina B, S.N. Hanna and M.K. Kamel, Approximate solution of gravity flow from a uniform channel over triangular bottom for large Froude number, *Appl. Math. Model*,**15(1)**(1991), 25–32.

[24] A. Mina B , S.Z Masoud, Linearized solution of a flow over a ramp, *Appl. Math. Model*,*124*(1988),406–410.

[25]A.Merzougui, A. Laiadi, Free surface flow over a triangular depression, *TWMS J. Appl. Eng. Math*,**4(1)**(2014), 67–73.

[26]H.Sekhri, F. Guechi and H. Mekias, A waveless free surface flow past a submerged triangular obstacle in presence of surface tension, *Electron. J. Differential Equations*,**190**(2016),1–8.

[28]J-M. Vanden-Broeck, J.Marc,Free-surface flow over an obstruction in a channel, *The Physics of fluids*,**30(8)**(1987), 2315–2317.