DEMOCRATIC AND POPULAR REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH
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Faculty of Mathematics and Computer Sciences
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Master memory

Field : Mathematics and Computer Sciences
Branch : Mathematics
Option : Algebra and Discrete Mathematics

Theme

Automorphisms group of some trivectors in dimension ≤ 8

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University year 2023/2024

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Introduction

Let V be a finite demensional vector space of dimension n over a commutative field K. The classification of trivectors is the study of the action of the general linear group GL(V) on the vector space $\wedge^3 V$. Since $\wedge^3 V^* \cong (\wedge^3 V)^*$, we refer interchangeablyto alternating trilinear forms or trivectors.

To classifiy trivectors, one often uses invariants,such as the outomorphism groups of a trivector (ω) denoted as $\omega, Aut(\omega)$, two trivectors ω_1 and ω_2 are equivalent if and only their automorphism groups $Aut\omega_1$ and $Aut\omega_2$ are equivalent. Several authors have studied alternating trilinear froms.

For$n \leq 7$, the classification is completely determind by several authors, such as A.M. Cohen and A.G. Hilminck.

For $n = 8$, this classification is determind in cases where $K = \mathbb{R}, \mathbb{C}, K$ (algebraically closed)$\mathbb{F}_q$ of characteristics different from 2 and 3 and $\mathbb{F}_q$.

In the first chapter, we present generalities on the notions of exterior product, splittability, invariants, automorphism group, commutant, and radical polynomial .

In the second chapter, we review the essential results already known abut the classification of trivestors (alternating triliniear forms) in a space of dimension ≤ 8.

In the last chapter we present a known result that gives the automorphism group of certain rank 8 trivectors.

Chapitre 1

Preliminaries

I got the following chapter from : refrences number 5 and 6

1.1 Tensor product

Definition 1.1 *: Let E be a vector space over a commutative field k.*

There exist a vector space over k, denoted $E \otimes E$, read as E tensor E and a bilinear form

$$\varphi_1 : E \times E \longrightarrow E \otimes E$$

$$(u, v) \longrightarrow \varphi_1(u, v)$$

.

Such that, for any bilinear form

$$\varphi_2 : E \times E \longrightarrow E \otimes E$$

$$(u, v) \longrightarrow \varphi_2(u, v)$$

There exists a unique linear map $\varphi : E \otimes E \longrightarrow K$ such that

$\varphi_2 = \varphi \circ \varphi_1$

$$E \times E \xrightarrow{\forall\varphi_2} K$$

$$\exists\varphi_2 \downarrow \quad \nearrow_{\exists!\varphi}$$

$$E \otimes E$$

The set of bilinear maps from $E \times E$ to K, is identified with the set of linear maps from $E \otimes E$ to k.

$$\varphi(E, E; K) \cong (E \otimes E; K)$$

$$\varphi_2 \longrightarrow \varphi$$

In addition, this property characterizes $E \otimes E$ we define $T^3(E) = E \otimes E \otimes E = \otimes^3 E$.

1.2 Exterior product

Definition 1.2 *We denote $\wedge^3 E$ as the quotient of $T^3(E)$ by the vector subspace generated by elements $x_1 \otimes x_2 \otimes x_3$ where $x_i = x_j$ for 2 indices $i \neq j$. we call $\wedge^3 E$ the 3rd exterior power of E.*

We denote $x_1 \wedge x_2 \wedge x_3 = x_1 \otimes x_2 \otimes x_3$ which is rad as x_1 exterior x_2 exterior x_3.

1. The outer power $\wedge^3 E$ can be defined similarly the tensor product .

$$\omega : E \times E \times E \xrightarrow{\forall \omega 1} k$$

$$(x_1, x_2, x_3) \longrightarrow \omega(x_1, x_2, x_3)$$

$$\omega_2 \downarrow \quad \nearrow_{\exists ! \omega}$$

$$\wedge^3 \omega$$

$$\omega_1 = \omega \circ \omega$$

$$\omega(x_1, x_2, x_3) = x_1 \wedge x_2 \wedge x_3$$

$$\omega(x_1 \wedge x_2 \wedge x_3)$$

2.

$$(x + y) \wedge (x + y) = 0$$

$$= x \wedge x + y \wedge y + x \wedge y + y \wedge x$$

$$x \wedge y = -y \wedge x.$$

1.2.1 Support and rank

The support of ω denoted S_ω is the smallest subspace F of E

such that $\omega \in \wedge^3 F$; the dimension of S_ω is called the rank of ω denoted $\mathrm{rk}(\omega)$.

1.2.2 Radical

Let $\omega \in \wedge^3 E^*$ be a trilinear form, the radical of ω is the set :

$$\text{Rad }(\omega) = \{x \in E/\omega(x,y,z) = 0, \forall y, z \in E\}$$

If Rad $\omega = \{0\}$, we say that ω is not-degenerate or of maximum rank.

1.2.3 Decomposable vector

A non-zero trivector ω is called decomposable if there exist x,y,z in E such that

$\omega = x \wedge y \wedge z$(exterior product).

A trivector is the sum of decomposable trivectors, and minimal number of

trivector needed is an interesting invariant, known as the length.

Notation 1.3 *: often written as x_1, x_2, x_3 or instead of $x_1 \wedge x_2 \wedge x_3$.*

1.2.4 Divisible vector

Let ω be a non-zero trivector, ω is a divisible trivector if there existed an

$x \in E - \{0_E\}$ and $u \in \wedge^2 E_2$ such that $E = Kx \oplus E_2$ and $\omega = x \wedge u$.

1.2.5 Group The action on a set

Definition 1.4 *:The action of the linear group GL(E) on the set of alternating trilinear*

forms $Alt_3(E)$, is done as follows :

for $f \in GL(E)$ and $\omega : E \times E \times E \longrightarrow K$ an alternating trilinear

form, we have $f.\omega(x,y,z) = \omega(f(x), f(y), f(z))$ satisfying the following conditions :

for all $f_1 f_2 \in GL(E)$ an alternating trilinear form

$1.(f_1 \circ f_2) = f_1.(f_2.\omega)$

$2.Id_E.\omega = \omega.$

Definition 1.5 *:The action of the linear group GL(E) on the vector space* $\wedge^3 E$*, is defined follows :for all* $f \in GL(E), \omega \in \wedge^3 E, f.\omega = (\wedge^3 f)(\omega)$ *where* $\wedge^3 f$ *is an endomorphism of* $\wedge^3 E$*, defined by :*$\wedge^3 f(x \wedge y \wedge z) = f(x) \wedge f(y) \wedge f(z)$.

According to isomorphism $\wedge^3 E^* \cong (\wedge^3 E)^*$*, both definitions are employed.*

1.2.6 Alternating trilinear forms

The vector space $\wedge^3 E$ can be defined in another way using alternating trilinear forms for any vector space E over a commutative field K, the set $Alt_3(E)$ of alternating trilinear forms

$h : E \times E \times E \longrightarrow K$ is itself a K vector space for the usual term-wese operations

Definition 1.6 *:A trilinear form*

$$\omega : E \times E \times E \longrightarrow \quad K$$

$$(x, y, z) \quad \longrightarrow \omega(x, y, z)$$

Is termed alternating if $\omega(x, y, z) = 0$ *whenver* $x_i = x_j$ *;*

for a pair of indices $i \neq j$ *for each linear application*

$$f : \quad E \quad \longrightarrow E$$

$$\wedge^3 f : \quad \omega^3 E \longrightarrow \omega^3 E \quad \omega^3 E$$

$$x_1 \wedge x_2 \wedge x_3 \longrightarrow \wedge^3 f(x_1 \wedge x_2 \wedge x_3)$$

$$= f(x_1) \wedge f(x_2) \wedge f(x_3)$$

1.2.7 Stable parts

Lemma 1.7 *: Let E be a vector space over the field K and consider the alternate bilinear form definde by :*

$\omega^x(y, z) = \omega(x, y, z)$*(*$\omega$ *an alternating trilinear form).*

Then,the set $R_i = \{x \in E/rk\omega^x = 2_i\}$ $(0 \leq 2_i \leq n)$*is stable under* $Aut(\omega)$*, meaning* $f(R_i(\omega)) \subset R_i(\omega)$.

The set $R_i(\omega) = x \in E/\varpi(x)$ *of type* ω_i *is a stable subset for* $Aut(\omega)$.

Proof. :We use stable parts to determine automorphisms . □

Lemma 1.8 *: Let E be a vector space over field K, of finite dimension, let V_1 and V_2 be two subspaces of E such that dim V_1=dim V_2.*

If f is an endomorphism of E that leaves the union of V_1 and V_2 stable, that is $f(V_1 \cup V_2) \subset V_1 \cup V_2$, then either $[f(V_1) \subset V_1$ and $f(V_2) \subset V_2]$ or $[f(V_1) \subset V_2$ and $f(V_2) \subset V_1]$.

Proof. Since $f(V_1 \cup V_2) \subset V_1 \cup V_2$, we obtain :

$$\begin{cases} f(V_1) \cap (V_1 \cup V_2) = f(V1), \\ f(V_2) \cap (V_1 \cup V_2) = f(V_2). \end{cases} \Longrightarrow \begin{cases} f(V_1) \cap (V_1 \cap V_2) = f(V1), \\ f(V_2) \cap (V_1 \cap V_2) = f(V_2). \end{cases}$$

$$\Longrightarrow \begin{cases} (f(V_1) \cap V_1) \cup (f(V_1) \cap V_2) = f(V1) & I \\ (f(V_2) \cap V_1) \cup (f(V_2) \cap V_2) = f(V_2) & II \end{cases}$$

Now, the union of two vector subspaces is a vector subspaces if and only if one is included in the other, as follows :

$$\begin{cases} (f(V_1) \cap V_1) \subset (f(V_1) \cap (V_2) or ((f(V1) \cap V_2) \subset (f(V_1) \cap V_1)), \\ \text{and} \\ (f(V_2) \cap V_1) \subset (f(V_2) \cap (V_2) or ((f(V_2) \cap V_2) \subset (f(V_2) \cap V_1)). \end{cases}$$

Remplcing in I and II, we obtain :

$$\begin{cases} (f(V_1) \cap V_1) = (f(V_1)) or ((f(V_1) \cap V_2) = f(V_1)), \\ \text{and} \\ (f(V_2) \cap V_1)(f(V_2) or ((f(V_2) \cap V2) = f(V_2)). \end{cases}$$

This implies that :

$$\begin{cases} (f(V_1) \subset V_1) or (f(V_1) \subset V_2), \\ \text{and} \\ (f(V_2) \subset V1)(f(V_2) or (f(V_2) \subset V_2). \end{cases}$$

We have four cases depicted :

$$\begin{cases} (f(V_1) \subset V_1) \; and \; f(V_1) \subset V_2) or (f(V_1)V_2) \; and \; f(V_2) \subset V_1) \\ \text{and} \\ (f(V_1) \subset V_1) \; and \; f(V_2) \subset V_1) or (f(V1) \subset V_2) \; and \; f(V_2) \subset V_2) \end{cases}$$

The last two cases are impossible because, for example :

If $f(V_1) \subset V_1$ and $f(V_2) \subset V_1$ since dim $f(V_1) = dim V_2$ and $dim f(V_2) = dim V_2$.

Thus $f(V_1) = V_1$ and $f(V_2) = V_1 = V_2$(since f is injective), which is absurd because : $v_1 \neq V_2$. $\square$

1.2.8 Scindable elements

Let E_1 and E_2 be two supplementary subspaces of E, $\wedge^3 E$ is identifies as :

$$\otimes_{k=0}^{k=3}(\wedge^k E_1 \otimes \wedge^{3-k} E_2).$$

An element $\omega \in \wedge^3$E is said to be scindable, if there exists a decomposition

$E = E_1 \oplus E_2$ such that :

$\omega \in E_1 \otimes \wedge^2 E_2$ concidered as a direct factor of $\wedge^3 E$.

If $dimE_1 = r$, we say that ω is r-splittable. Splittabilty is a generalization of divisibility. Indeed ω is divisible if and only if ω is 1-splittable, a property that does not depend on the base field because it is equivalent to saying that the mapping :

$$E \longrightarrow \wedge^4 E$$

is not injective

$$x \longrightarrow x\omega$$

Let ω a be an r-scindable element and $\{e_1........e_r\}$be a basis of E_1, $\sum_{i=1}^{r} e_i u_i$ where$u_i \in \wedge^2 E_2$. The u_i are uniquely determined by the basis $e_1, ..., e_r$ of E_1. Then ω is determined by the vector subspace F of $\wedge^2 E_2$ generated by the u_i, indeed if we change the base in E_1, and if the new basis f_j is given by :

$$e_i = \sum_{j=1}^{r} a_{ij} f_j,$$

$$\omega = \sum_1^r e_i u_i = \sum_{j=1}^{r} f_j (\sum_{j=1}^{r} a_{ij} u_i) = \sum_{j=1}^{r} f_j u_j,$$

the v_j are obtained from the u_i by the countragradient change of base from the one that goes fromthe basis f_g to the base e_i this can also be seen using natural isomorphism between $E_1 \otimes \wedge^2 E_2$ and Hom$(E_1^*, \wedge^2 E_2)$ if φ is the element of Hom$(E_1^*, \wedge^2 E_2)$ canonically associated with ω F is nothing other than $\varphi(E_1^*)$. the same trivector can be scindable for several values of the integer r as shown by the example :

$\omega_{7,3} = e_1 + e_2 + e_3 + e_3 + e_4 + e_5 + e_5 e_6 e_7$ which is 2 and 3-scindable :

$\omega_{7,3} = e_3(e_1 e_2 + e_4 e_5) + (e_5 e_6) e_7 = e_1(e_2 e_3) + e_4(e_5 e_3) + e_5 e_6) e_7.$

1.2.9 Exact sequence

Definition 1.9 *: Let* $G' \xrightarrow{f} G \xrightarrow{g} G''$ *be a sequence of group homomorphism.*

We say that this sequence is exact if

$$Im\ f = Ker\ g.$$

Example 1.10 *If H is a distinguished subgroup of G, the sequennce*

$$H \xrightarrow{j} G \xrightarrow{\varphi} G/H$$

Is exact (j being the injection and φ *is the canonical projection)*

Notation 1.11 *Stating that, the sequel*

$$1 \longrightarrow G' \xrightarrow{f} G \xrightarrow{g} G" \longrightarrow 1$$

Is exact, means that f is injective, $Imf = Kerg$ *and g is surjective.*

1.2.10 Invariant and trivectors :the $Aut(\omega)$ invariant

The group of automorphisms of ω, $Aut(\omega)$ is the stabilizer of ω in the action of GL(E) that is, the subgroup of GL(E) consisting of the automorphism of E that leave ω invariant $Aut(\omega) = \{f/f \in GL(E) and \wedge^3 f(\omega) = \omega\} = \{f/f \in GL(E) and f.\omega = \omega\}$. The orbit of ω GL(E) is then in bijection with the set of left classes $GL(E)Aut/(\omega)$.

1.2.11 Commutant

The second invariant was itntroduced by B.Kahn. if $\omega \in \wedge^3 V^*$, the commutant of ω denoted by $C(\omega)$, is defined as the set of endomorphisms $f : V \longrightarrow V$ such that :

$$\omega(f(v_1), v_2, v_3) = \omega(v_1, f(v_2), v_3) = \omega(v_1, v_2, f(v_3)), \text{ for all } v_1, v_2, v_3 \in V.$$

Example 1.12 *.*
1) Let V be a K-vector space of dimension 5,$\{e_1, e_2, e_3, e_4, e_5\}$ *a basis of V, and let* $\omega \in \wedge^3 V$ *with* $\omega_5 = e_1e_2e_3 + e_1e_4e_5$

$$C(\omega_5) = \{f : V \longrightarrow V, \omega(f(e_i), e_j, e_k) = \omega(e_i, f(e_j), e_k) = \omega(e_i, e_j, f(e_k)), 1 \leq i; j; k \leq 5\},$$

si $f \in C(\omega_5)$ *then the matrix associated with* f *is of the form :*

$$\begin{array}{cc} & \begin{array}{ccccc} f(e_1) & f(e_2) & f(e_3) & f(e_4) & f(e_5) \end{array} \\ M(f) = & \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \lambda_1 & \mu_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \lambda_2 & \mu_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \lambda_3 & \mu_3 \\ \alpha_4 & \beta_4 & \gamma_4 & \lambda_4 & \mu_4 \\ \alpha_5 & \beta_5 & \gamma_5 & \lambda_5 & \mu_5 \end{pmatrix} \begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \end{array}$$

We have

$$\begin{cases} \omega(f(e_1), e_2, e_3) = \omega(e_1, f(e_2), e_3) = \omega(e_1, e_2, f(e_3)) \\ and \\ \omega(f(e_1); e_4; e_5) = \omega(e_1, f(e_4), e_5) = \omega(e_1, e_4, f(e_5)). \end{cases}$$

As

$$\begin{cases} \omega(e_1, e_2, e_3) = \omega(e_1, e_4, e_5) = 1 \\ and \\ \omega(e_i, e_j, e_k) = 0. \end{cases}$$

Then

$$\omega(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5, e_2, e_3) = \omega(e_1, \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3 + \beta_4 e_4 + \beta_5 e_5, e_3)$$

$$= \omega(e_1, e_2, \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3 + \gamma_4 e_4 + \gamma_5 e_5)$$

Let's Calculate

$*\omega(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5, e_2, e_3) = \alpha_1 \omega(e_1, e_2, e_3) + \alpha_2 \omega(e_2, e_2, e_3) + \alpha_3 \omega(e_{3,2}, e_3)$

$+ \alpha_4 \omega(e_4, e_2, e_3) + \alpha_5 \omega(e_5, e_2, e_3)$

$= \alpha_1 + 0 + 0 + 0$

$*\omega(e_1\beta_1 + e_2\beta_2 + e_3\beta_3 + e_4\beta_4 + e_5\beta_5, e_2, e_3) = \beta_1 \omega(e_1, e_2, e_3) + \beta_2 \omega(e_2, e_2, e_3) + \beta_3 \omega(e_{3,2}, e_3)$

$+ \beta_4 \omega(e_4, e_2, e_3) + \beta_5 \omega(e_5, e_2, e_3)$

$= 0 + \beta_2 + 0 + 0 + 0$

$*\omega(\gamma_1 e_1+\gamma_2 e_2+\gamma_3 e_3+\gamma_4 e_4+\gamma_5 e_5, e_2, e_3) = \gamma_1\omega(e_1, e_2, e_3)+\gamma_2\omega(e_2, e_2, e_3)+\gamma_3\omega(e_{3,2}, e_3)$

$+\gamma_4\omega(e_4, e_2, e_3)+\gamma_5\omega(e_5, e_2, e_3)$

$=\gamma_3$
$\Rightarrow \alpha_1=\beta_2=\gamma_3$

of

$$\omega(f(e_1), e_4, e_5) = \omega(e_1, f(e_4), e_5) = \omega(e_1, e_4, f(e_5)),$$

that is

$\omega(\alpha_1 e_1+\alpha_2 e_2+\alpha_3 e_3+\alpha_4 e_4+\alpha_5 e_5, e_4, e_5) = \omega(e_1, \lambda_1 e_1+\lambda_2 e_2+\lambda_3 e_3+\lambda_4 e_4+\lambda_5 e_5, e_5)$

$= \omega(e_1, e4, \mu_1 e_1+\mu_2 e_2+\mu_3 e_3+\mu_4 e_4+\mu_5 e_5)$

so

$\omega(\alpha_1 e_1+\alpha_2 e_2+\alpha_3 e_3+\alpha_4 e_4+\alpha_5 e_5, e_4, e_5) = \alpha_1\omega(e_1, e_4, e_5) = \alpha_1$

$\omega(e_1, \lambda_1 e_1+\lambda_2 e_2+\lambda_3 e_3+\lambda_4 e_4+\lambda_5 e_5, e_5) = \lambda_4\omega(e_1, e_4, e_5) = \lambda_4$

$\omega(e_1, e_4, \mu_1 e_1+\mu_2 e_2+\mu_3 e_3+\mu_4 e_4+\mu_5 e_5) = \mu_5\omega(e_1, e_4, e_5) = \mu_5$

$$\alpha_1=\beta_2=\lambda_3=\lambda_4=\mu_5.$$

Similarly

$\beta_1=\gamma_1=\lambda_1=\mu_1=\alpha_2=\gamma_2=\lambda_2=\mu_2=\alpha_3=\beta_3=\lambda_3=\mu_3=\alpha_4=\beta_4=\gamma_4=$

$\mu_4=\alpha_5=\beta_5=\gamma_5=\lambda_5=0.$
Finally, the matrix associated with f will take the forme :

$$MB(f) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & 0 & 0 & \alpha_1 \end{pmatrix} = \alpha_1 I_5$$
Hence $C(\omega_5) = K$.

2)*Let* $\omega_{6,2}$ *the trivector* $e_1e_2e_4+e_2e_3e_5+e_1e_3e_6$, *let's calculate* $C(\omega_{6,2})$ *:*

We have

$$\begin{cases} \omega(e_1, e_2, e_4) = \omega(e_2, e_3, e_5) = \omega(e_1, e_3, e_6) = 1 \\ \omega(e_i, e_j, e_k) = 0. \end{cases}$$

If $f \in C(\omega_{6,2})$ *then*

$$\begin{cases} \omega(f(e_1), e_2, e_4) = (e1; f(e2); e4) = \omega(e_1, e_2, f(e_4)) \\ and \\ \omega(f(e_2), e_3, e_5) = \omega(e_2, f(e_3), e_5) = \omega(e_2, e_3, f(e_5)) \\ and \\ \omega(f(e_1), e_3, e_6) = \omega(e_1, f(e_3), e_6) = \omega(e_1, e_3, f(e_6)). \end{cases}$$

The matrix associated with f in the basis $B = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ *is of the forme :*

$$MB(f) = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \lambda_1 & \mu_1 & \sigma_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \lambda_2 & \mu_1 & \sigma_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \lambda_3 & \mu_1 & \sigma_3 \\ \alpha_4 & \beta_4 & \gamma_4 & \lambda_4 & \mu_1 & \sigma_4 \\ \alpha_5 & \beta_5 & \gamma_5 & \lambda_5 & \mu_1 & \sigma_5 \\ \alpha_6 & \beta_6 & \gamma_6 & \lambda_6 & \mu_1 & \sigma_6 \end{pmatrix}$$

by direct calculation , we find :

$$\omega(f(e_1), e_2, e4_{)} = \omega(e_1, f(e_2), e_4) = \omega(e_1, e_2, f(e_4)) = \alpha_1 = \lambda_4 = \beta_2$$

$$\omega(f(e_2), e_3, e_5) = \omega(e_2, f(e_3), e_5) = \omega(e_2, e3, f(e_5)) = \beta_2 = \gamma_3 = \mu_5$$

$$\omega(f(e_1), e_3, e_6) = \omega(e_1, f(e_3), e_6) = \omega(e_1, e_3, f(e_6)) = \alpha_1 = \gamma_3 = \sigma_6$$

and

$$\alpha_1 = \beta_2 = \gamma_3 = \lambda_4 = \mu_5 = \sigma_6 = \lambda, \lambda \in K.$$

We also have :

$$\omega(f(e_1), e_2, e_3) = \omega(e_1, f(e_2), e_3) = \omega(e_1, e, 2, f(e_3))$$

$$\Rightarrow \alpha_5 = -\beta_6 = \gamma_4 = \beta, \beta \in K$$

$$\omega(f(e_i), e_j, e_k) = \omega(e_i, f(e_j), e_k) = \omega(e_i, e_j, f(e_k)) = 0$$

$\Rightarrow \alpha_2 = \alpha_3 = \alpha_4 = \alpha_6 = \beta_1 = \beta_3 = \beta_4 = \beta_5 = \gamma_1 = \gamma_2 = \gamma_5 = \gamma_6$

$= \lambda_1 = \lambda_2 = \lambda_3 = \lambda_5 = \lambda_6 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_6 = \sigma_1 = \sigma_2$

$= \sigma_3 = \sigma_4 = \sigma_5 = 0.$

Therefor, the matrixe MB(f) $= \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ *where* $A = \lambda I_3, D = \lambda I_3$ *with* $C = \begin{pmatrix} 0 & 0 & \beta \\ \beta & 0 & 0 \\ 0 & -\beta & 0 \end{pmatrix}$
where $\alpha, \beta \in K$,

let's denot by $\epsilon : V \longrightarrow V$ *linear application defined by* $\epsilon(e_1) = e_5, \epsilon(e_2) = -e_6$,

$\epsilon(e_3) = e_4, \epsilon(e_i) = 0$, *if* $i \in \{4; 5; 6\}$, *thus* $f = \lambda_i d_v + \beta\epsilon, \epsilon^2 = \epsilon \circ \epsilon = 0$ *which proves that* $C(\omega_{6,2}) = K[\epsilon]$,*wher* $\epsilon^2 = 0$.

1.2.12 Radical Polynomial

The third invariant was introducted by Hora , the radical polinomial of ω.

Let V be an n-dimensional vector space over a finite field $\mathbb{F}_q$ and ω be an alternating trininear form .

We fix $v \in V$, and define the radical $Rad_\omega(v)$ of v as :

$$Rad_\omega(v) = \{u \in V, \omega(v, u, z) = 0, \forall z \omega V\}.$$

The rank of v and the codimoension of $Rad_\omega(v)$ in V ,
that is $r_\omega(v) = n - dimRad_\omega(v)$.

we define

$$p(\omega) = \sum_{v \in V} x^{r(v)} y^{n-r(v)}$$

$p(\omega)$ is a homogeneous polynomial of degree n ; and it can be written as :

$$p(\omega) = \sum_{i=0}^{n-1} \alpha_i x^i y^{n-1}$$

where$\sum_{i=0}^{n-1} \alpha_i = q^n$

Example 1.13 . *Let V be a 5-dimensional vector space de over* $\mathbb{F}_2$, *for* $\omega 5 = e_1(e_2e_3 + e_4e_5) = e_1e_2e_3 + e_1e_4e_5$.

Let's Calculate $p(\omega_5) = \sum_{i=0}^{4} \alpha_i x^i y^{5-i}$ *with* $\sum_{i=0}^{4} \alpha_i = 2^5 = 32$.

$dimRad_{\omega_5}(0) = 5,$ so $r_{\omega_5}(0) = 0, \alpha_0 = 1,$

Ther exist 15 vectors :

$e_2, e_3, e_4, e_5, e_2 + e_3, e_2 + e_4, e_2 + e_5, e_3 + e_4, e_3 + e_5, e_4 + e_5, e_2 + e_3 + e_4, e_2 + e_3 + e_5, e_2 + e_4 + e_5, e_3 + e_4 + e_5, e_2 + e_3 + e_4 + e_5$ *where the radical is dimension 3 so* $r_{\omega_5}(v) = 2, \alpha_2 = 15,$

there exist 16 vectors :

$e_1, e_1 + e_2, e_1 + e_3, e_1 + e_4, e_1 + e_5, e_1 + e_2 + e_3, e_1 + e_2 + e_4, e_1 + e_2 + e_5, e_1 + e_3 + e_4, e_1 + e_3 + e_5, e_1 + e_4 + e_5, e_1 + e_2 + e_3 + e_4, e_1 + e_2 + e_3 + e_5, e_1 + e_2 + e_4 + e_5, e_1 + e_3 + e_4 + e_5, e_1 + e_2 + e_3 + e_4 + e_5$

where the radical is dimension 1 so $r_{\omega_5}(v) = 4, \alpha_4 = 16.$

Hence, the radical polynomail of ω_5 is :

$$p(\omega_5) = y^5 + 15x^2y^3 + 16x^4y, \text{ with } 1 + 15 + 16 = 32 = 2^5.$$

Chapitre 2

Classification of trivectors and automorphisms groups

I got the following chapter from : refrence number 7

2.1 Classification of trivectors

Let K be an algebraically closed field,and E a K-vector space.

2.1.1 Classification of trivectors in dimensions below 6

1. For $dimE = 3$, there is only one orbit of non-zero trivectors :if $\omega \in \wedge^3 E_0$ there exists a base e_1, e_2, e_3 of E such that $\omega = e_1, e_2, e_3$.

2. For $dimE = 4$, all non-zero trivectors are decomposable,so $\wedge^3 E$ has two orbits,in a base (e_i),$1 \leq i \leq 5$
a representative of each orbit is given by :

0

$e_1 e_2 e_3$

3. For $dim = 5$, the isomorphism $\wedge^3 E \cong \wedge^2 E^*$ shows that there are three orbits in $\wedge^3 E$:

Indeed, if a trivector is non-zero and non-decomposable, it is necessarily of maximal rank, thus

divisible by a vector e_1 : $\omega = e_1 u$ where u is a bivector of $rank4$; we can choose any supplement

S_u of Ke_1 in E and there exists a base (e_1),$1 \leq i \leq 5$, of E such that a representative of each

orbit is given by :

0

$e_1e_2e_3$

$e_1(e_2e_3 + e_4e_5)$.

2.1.2 Classification of trivectors in dimensions 6

Let E be a vector space of $dimE = 6$ and let $\omega \in \wedge^3 E$ a trivector of maximum rank $rk(\omega) = dimE = 6$.

There exist a basis (e_i),$1 \leq i \leq 6$, of E such that ω can be written as :

$$\omega_{6,1} = e_1e_2e_3 + e_4e5e_6$$

$$\omega_{6,2} = e_1e_2e_3 + e_2e_3e_5 + e_1e_3e_6.$$

That is there are 2 orbit of maximal ranke.

2.1.3 Classification of trivectors in dimensions 7

There are five orbits of rank 7, two orbits orbits of rank 6, one orbits of rank 5, one orbits of rank 3, and one orbits orbites 0 :

$\omega_3 = e_1e_2e_3, rk(\omega_3) = 3$

$\omega_5 = e_1(e_2e_3 + e_4e_5), rk(\omega_5) = 5$

$\omega_{6,1} = e_1e_2e_3 + e_4e_5e_6$

$\omega_{6,2} = e_1e_2e_4 + e_2e_3e_5 + e_1e_3e_6, rk(\omega_{6,1} = rk(\omega_{6,2} = 6$

$\omega_{7,1} = e_1(e_2e_3 + e_4e_5 + e_6e_7)$

$\omega_{7,2} = \omega_{7,1} + e_2e_4e_6$

$\omega_{7,3} = e_1e_2e_3 + e_3e_4e_5 + e_5e_6e_7$

$\omega_{7,4} = e_1(e_2e_3 + e_4e_5) + e_2e_4e_6 + e_3e_5e_7$

$\omega_{7,5} = \omega_{7,2} + e_3e_5e_7$

$rk(\omega_{7,i}) = 7, i = \overline{1,5}.$

2.1.4 Classification of trivectors in dimensions 8

There are 13 equivalence classes of trivectors of rank 8. In a base (e_i) of E ,$1 \leq i \leq 13$, a representative of each class is given by $\omega_{8,i}$,$1 \leq i \leq 13$ from table 1.

$\omega_{8,i}$	Expression of of an orbital representative
$\omega_{8,1}$	$e_1(e_2e_3 + e_4e_5) + e_6e_7e_8$
$\omega_{8,2}$	$e_1(e_2e_3 + e_4e_5 + e_6e_7) + e_5e_6e_8$
$\omega_{8,3}$	$e_1(e_3e_4 + e_5e_6) + e_2(e_3e_5 + e_7e_8)$
$\omega_{8,4}$	$e_1(e_2e_3 + e_4e_5) + e_6(e_2e_7 + e_4e_8)$
$\omega_{8,5}$	$e_1(e_2e_3 + e_4e_5) + e_6(e_2e_3 + e_7e_8)$
$\omega_{8,6}$	$e_1(e_2e_3 + e_4e_5 + e_6e_7) + e_8(e_4e_3 + e_5e_6)$
$\omega_{8,7}$	$e_1(e_2e_3 + e_4e_6 + e_5e_7) + e_2(e_5e_6 + e_7e_8)$
$\omega_{8,8}$	$e_1(e_2e_8 + e_3e_6 + e_4e_7) + e_6e_7e_8 + e_3e_4e_5$
$\omega_{8,9}$	$e_1\left[e_2(e_3 + e_4) + e_5e_6\right] + e_3e_5e_7 + e_4e_6e_8$
$\omega_{8,10}$	$e_1(e_2e_8 + e_6e_7) + e2e3e5 + e_3e_4e_6 + e_4e_5e_7$
$\omega_{8,11}$	$e_1(e_3e_7 + e_5e_4 + e_8e_2) + e_8(e_4e_3 + e_6e_7) + e_2e_4e_6$
$\omega_{8,12}$	$e_1\left[(e_4 - e_7)(e_3 - e_8) + e_5e_7\right] + e_2(e_3e_4 + e_5e_6) + e_6e_7e_8$
$\omega_{8,13}$	$e_1\left[e_5(e_3 - e_7) + e_8e_4\right] + e_2(e_3e_4 + e_5e_6) + e_6e_7e_8$

2.2 Automorphisms groups of trivectors

2.2.1 automorphism group of Trivector below 6

Property 2.1 *The group of automorphisms $Aut(\omega_3)$,$\omega_3 = e_1e_2e_3$ is the special group $SL_3(E)$.*

$$Aut(\omega_3) \cong SL_3(E).$$

Proof. We have $Aut(\omega_3) = \{f/f \in GL(E)$ and $f.\omega_3 = \omega_3$.

Let $B = \{e_1, e_2, e_3\}$be a basis of E. Since $\omega_3(e_1, e_2, e_3) = 1$, $\omega_3(e_i, e_j, e_k) = 0$ otherwise.

$f \in Aut(\omega_3) \Rightarrow f.\omega_3 = \omega_3.$

$f.e_1e_2e_3 = f(e_1e_2e_3)$

$= e_1e_2e_3$

Let $M_B(f)$ be the matrix of f :

$$M_B(f) = \begin{pmatrix} x_1 & y_2 & z_3 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$$

We have,

$e_1e_2e_3 = f(e_1)f(e_2)f(e_3)$

$e_1e_2e_3 = (x_1e_1 + x_2e_2 + x_3e_3)(y_1e_1 + y_2e_2 + y_3e_3)(y_1e_1 + y_2e_2 + y_3e_3)(z_1e_1 + z_2e_2 + z_3e_3)$

$e_1e_2e_3 = x_1y_2z_3(e_1e_2e_3) + x_1y_3z_2(e_1e_3e_2) + x_2y_1z_3(e_2e_1e_3) + x_2y_3z_1 + (e_2e_3e_1)$

$+ x_3y_1z_2(e_3e_1e_2) + x_3y_2z_1(e_3e_2e_1)$.

$$e_1e_2e_3 = \begin{vmatrix} x_1 & y_2 & z_3 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} e_1e_2e_3$$

This yields :

$det(M_B(f)) = 1$ $Aut(\omega) = \{f \in GL(E), det(M_B(f)) = 1\}$.
So :$Aut(\omega) = SL_3(E)$. $\square$

Property 2.2 *The Automorphism Group* $A = Aut(\omega_5)$, $\omega_5 = e_1(e_2e_3 + e_4e_5)$ *is given by the following exact sequences :*

$$1 \longrightarrow A' \longrightarrow A \longrightarrow k^* \longrightarrow 1$$

$$1 \longrightarrow K^4 \longrightarrow A' \longrightarrow sp_4(k) \longrightarrow 1$$

Proof. Let's Consider the set $E_1 = \{x \in E/x\omega_5 = 0\}$; So $x = \sum_{i=1}^{5} \alpha_i e_i \in E_i$,

$x = \alpha_i e_i$ and $E_i = Vect\,\{e_1\}$.

So,

$x \wedge \omega_5 = 0 \Longrightarrow (\alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_5 e_5) \wedge (e_1 e_2 e_3 + e_1 e_4 e_5 = 0$:

$$\Psi : A \longrightarrow K^*$$

$$f \longrightarrow \Psi(f) = \lambda.f$$

for $\alpha \in K^*$, we consider the linear mapping $f : E \longrightarrow E$

defined by $f(e_1) = \alpha_1, f(e_{2i}) = \alpha^{-1} e_{2i}, f(e_{2i+1}) = e_{2i+1}$, for $i = 1, 2$. Let $f \in A' = Ker\psi$,

as $f.\omega_5, e_1 \wedge^2 f(e_2 e_3 + e_4 e_5) = e_1(e_2 e_3 + e_4 e_5)$, in other words $\wedge^2 f(e_2 e_3 + e_4 e_5) = e_2 e_3 +$

$e_4 e_5 + x e_1$ with $x \in E_2 = Vect\{e_2, e_3, e_5, e_5\}$.

Let's consider $g : E_2 \longrightarrow E$, $M_B(g) = B$, the linear application whose matrix is

$B : \wedge^2 g(e_2 e_3 + e_4 e_5) = e_2 e_3 + e_4 e_5$ the homomorphism $\varphi : A' \longrightarrow Sp_4(K)$

defined by :$\varphi(f) = B$ is surjective and has a carnel isomorphic to K^4, hence the result. □

2.2.2 automorphisms group of rank 6 trivector

Property 2.3 *The automorphisms group $A = Aut(\omega_{6,1})$ is determined by the following exact sequence :*

$$1 \longrightarrow SL_3(K) \times SL_3(K) \longrightarrow Aut(\omega_{6,1}) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

Proof. $\omega_{6,1} = e_1 e_2 e_3 + e_4 e_5 e_6$

We have $R_i = vect\{e_1, e_2, e_3\} \cup vect\{e_4, e_5, e_6\} = V_1 \cup V_2$ is a stable subset for f,

so if $f \in Aut(\omega_{6,1}, f(R_1) \subset R_1$. Let $V = V_1 \cup V_2$ where $V_1 vect\{e_1, e_2, e_3\}$ and

$V_2 = vect\{e_4, e_5, e_6\}.f(V_1 \cup V_2) \subset (V_1 \cup V_2)$.
Then,

$f(V_1) \subset V_1$ and $f(V_2) \subset V_2$ or $f(V_1) \subset V_2$ and $f(V_2) \subset V_1$.
This allows defining a group homomorphism form $A = Aut(\omega_{6,1})$ into $\mathbb{Z}/2\mathbb{Z}$ by :

$$\varphi : A \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

$$f \longrightarrow \varphi(f)$$

where :

$$\varphi(f) = \begin{cases} 1 \, if \; f(V_1) \subset V_1 \; and \; f(V_2) \subset V_2 \\ -1 \, if \; f(V_1) \subset V_2 \; and \; f(V_2) \subset V_1 \end{cases}$$

Let's calculate Ker $\varphi = \{f/f(v_1) \subset V1$ and $f(V_2) \subset V_2$.

Let $f_1 = f/V_1$ and $f_2 = f/V_2$.

Therefore, the matrix of f is in the form :

$M_B(f) = \begin{pmatrix} A_1 & 0, \\ 0 & A_2. \end{pmatrix}$, $A_1 = M_B(f_1)$ and $A_2 = M_B(f_2)$.

Since $f \in A = Aut(\omega_{6,1})$that is $f\omega_{6,1} = \omega_{6,1}$ thus,

$e_1e_2e_3 + e_4e_5e_6 = detA_1.e_1e_2e_3 + detA_2.e_4e_5e_6$.

This proves that $detA_1 = 1$ and $detA_2 = 1$

that is $A_1, A_2 \in SL_3(K)$

Hence $Ker\varphi = SL_3(K) * SL_3(K)$

φ is surjective because :

$\varphi(id_E) = 1$ and $\varphi(f_0 = -1)$, f_0 is defined as :

$$\begin{cases} f_0(e_1) = e_4 \\ f_0(e_2) = e_5 \\ f_0(e_3) = e_6 \\ f_0(e_4) = e_1 \\ f_0(e_5) = e_2 \\ f_0(e_6) = e_3 \end{cases}$$

Henc, the exactness of the sequence

$$1 \longrightarrow SL_3(K) \; SL_3(K) \longrightarrow A \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

□

Property 2.4 *The automorphism group $A = Aut(\omega_{6,2})$ is determined by the following exact sequence :*

$$1 \longrightarrow K^8 \longrightarrow Aut(\omega_{6,2}) \longrightarrow GL_1(K) \longrightarrow 1.$$

Proof. We have :
$\omega_{6,2} = e_1e_2e_4 + e_2e_3e_5 + e_1e_3e_6$

Let $V_1 = vecte_1, e_2, e_3, V_1$ is a stable subset.

Thus,$f(V_1) \subset V_1$, allowing us to define a group homomorphism from $A_2 = Aut(\omega_{6,2})$ to $GL_3(K)$ as follows :

$$\varphi : A_2 \longrightarrow GL_3(K)$$

$$f \longrightarrow \varphi(f) = f/V_1$$

Therefore, Ker $\varphi = f/f(V_1) = id$.
For f $\in Ker\varphi$.

$$M_B(f) = \begin{pmatrix} 1 & 0 & 0 & x_1 & y_1 & z_1 \\ 0 & 1 & 0 & x_2 & y_2 & z_2 \\ 0 & 0 & 1 & x_3 & y_3 & z_3 \\ 0 & 0 & 0 & x_4 & y_4 & z_4 \\ 0 & 0 & 0 & x_5 & y_5 & z_5 \\ 0 & 0 & 0 & x_6 & y_6 & z_6 \end{pmatrix}$$

Of $f.\omega = \omega$

$$\Longrightarrow \; f(e_1)f(e_2)f(e_4) + f(e_2)f(e_3)f(e_5) + f(e_1)f(e_3)F(e_6) = e_1e_2e_4 + e_2e_3e_5 + e_1e_3e_6.$$

By identification,we find

$e_1e_2e_4 : x_4 = 1, x_5 = x_6 = 0$

$e_2e_3e_5 : y_4 = 1, y_4 = x_6 = 0$

$e_1e_3e_6 : z_4 = 1, z_4 = z_5 = 0$

$e_1e_2e_3 : x_3 + y_1 - z_2 = 0 \longrightarrow z_2 = x_3 + y_1.$

So :

$$M_B(f) = \begin{pmatrix} 1 & 1 & 1 & x_1 & y_1 & z_1 \\ 1 & 1 & 1 & x_2 & y_2 & x_3 + y_1 \\ 1 & 0 & 1 & x_3 & y_3 & z_3 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

The kernel ker ω is a subgroup of $GL_6(K)$ formed by triangular matrices $M_B(f) = \begin{pmatrix} I_3 & 0 \\ 0_3 & I_3 \end{pmatrix}$ where $A \in M_3(K)$ satisfies the condition $z_2 = x_3 + y_1$, thus it is the additive group K^8.

Hence Ker $\psi \cong K^8$

Let's demontrate thatψ is surjective :

Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ be an element of $GL_3(K)$,and let's determine $f \in Aut(\omega_{6,2})$ such that $f/V_1 = A$.

Then, M(f) is of the form :

$$M(f) = \begin{pmatrix} a_1 & b_1 & c_1 & x_1 & y_1 & z_1 \\ a_2 & b_2 & c_2 & x_2 & y_2 & z_2 \\ a_3 & b_3 & c_3 & x_3 & y_3 & z_3 \\ a & b & c & x_4 & y_4 & z_4 \\ 0 & 0 & 0 & x_5 & y_5 & z_5 \\ 0 & 0 & 0 & x_6 & y_6 & z_6 \end{pmatrix}$$

Since $f.\omega = \omega$, then,

$$f(e_1)f(e_2)f(e_4) + f(e_2)f(e_3)f(e_5) + f(e_1)f(e_3)F(e_6) = e_1e_2e_4 + e_2e_3e_5 + e_1e_3e_6.$$

Thus,

$e_1e_2e_4 : (a_1b_1 - a_2b_1)x_4 + (b_2c_3 - b_3c_2)y_4 + (a_1c_3 - a_3c_1)z_4 = 1$

$e_2e_3e_5 : (a_2b_3 - a_3b_2)x_5 + (b_2c_3 - b_3c_2)y_5 + (a_2c_3 - a_3c_2)z_5 = 1$

$e_1e_3e_6 : (a_1b_3 - a_3b_1)x_6 + (b_1c_3 - b_3c_1)y_6 + (a_1c_3 - a_3c_1)z_6 = 1$

$e_1e_3e_4 : (a_1b_3 - a_3b_1)x_4 + (b_1c_3 - b_3c_1)y_4 + (a_1c_3 - a_3c_1)z_4 = 0$

$e_2e_3e_4 : (a_2b_3 - a_3b_2)x_4 + (b_2c_3 - b_3c_2)y_4 + (a_2c_3 - a_3c_2)z_4 = 0.$

Let's set :

$$M_B = \begin{pmatrix} a_1b_3 - a_3b_1 & b_1c_3 - b_3c_1 & a_1c_3 - a_3c_1 \\ a_1b_3 - a_3b_1 & b_1c_3 - b_3c_1 & a_1c_3 - a_3c_1 \\ a_2b_3 - a_3b_2 & b_2c_3 - b_3c_2 & a_2c_3 - a_3c_2 \end{pmatrix}$$

Then,

$$B = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$B = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$B = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{As det B} = \begin{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} & \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} \\ \\ \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} \\ \\ \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} & \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} \end{vmatrix}$$

Let's define :

$$\begin{cases} u_1 = a_1e_1 + a_2e_2 + a_3e_3 \\ u_2 = b_1e_1 + b_2e_2 + b_3e_3 \\ u_3 = c_1e_1 + c_2e_2 + c_3e_3 \end{cases}$$

If $V = vectu_1, u_2, u_3$, then u_1u_2, u_1u_3, u_2u_3 is a base of $\wedge^2 V$.

So det $u_1u_2, u_1u_3, u_2u_3 \neq 0$ that is $detB \neq 0$.

Hence the existence of $x_4, y_4, z_4, x_5, y_5, z_5, x_6, y_6, z_6$ and $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3$

satisfying the relation :

$x_1(a_2b_3 - a_3b_2) - x_2(a_1b_3 - a_3b_1) + x_3(a_2b_3 - a_3b_2) + y_1(b_2c_3 - b_3c_2) - y_2(b_1c_3 - b_3c_1) +$

$y_3(b_1bc_2 - b_2c_1) + z_1(a_2c_3 - a_3c_2) - z_2(a_1c_3 - a_3c_1) + z_3(a_1c_2 - a_2c_1) = 0.$

Thus it is surjective .

(we can take $x_1 = x_2 = x_3 = y_1 = y_2 = y_3 = z_1 = z_2 = z_3 = 0$).

Hence the result. $\square$

Chapitre 3

Group of outomorphisms of the alternating trilinear form of rank 8

I got the following chapter from : refrence number 3

3.1 The Group of outomorphisms of ω where the invariant $d_1 = 3$

Property 3.1 *The automrphism group $A_1 = Aut(\omega_{8,1})$ is determined by the following exact sequences :*

$$1 \longrightarrow SL_3(K) \longrightarrow A_1 \longrightarrow Aut(\omega_5) \longrightarrow 1$$

$$1 \longrightarrow A_0 \longrightarrow Aut(\omega_5) \longrightarrow K^* \longrightarrow 1$$

$$1 \longrightarrow K_4 \longrightarrow A_0 \longrightarrow Sp_4(K) \longrightarrow 1$$

$$1 \longrightarrow SL_3(K) \longrightarrow A_1 \longrightarrow Aut(\omega_5) \longrightarrow 1$$

$$\text{where } \omega_5 = e_1(e_2e_3 + e_4e_5)$$

Proof. :
Let $\omega = e_1^*(e_2^*e_3^* + e_4^*e_5^*) + e_6^*e_7^*e_8^*$, we determent $\bar{R}_{\leq 1}(\omega) = \bar{R}_0(\omega) \bigcup \bar{R}_1(\omega)$.

Let us therfore assume :

$f^a : f^a(x, y) = \omega(a, x, y)$ with $a = \sum_{i=1}^{8} a_i e_i$

then $\omega^a = a_1e_2^*e_3^* - a_2e_1^*e_3^* + a_3e_1^*e_2^* + a_1e_4^*e_5^* + a_4e_1^*e_5^* + a_5e_1^*e_4^* +$

$a_6e_7^*e_8^* - a_7e_6^*e_8^* + a_8e_6^*e_7^*$ therefore :

$\omega^a = (a_1e_2^* - a_2e_1^*)e_3^* + (a_1e_4^* - a_4e_1^*)e_5^* + (a_6e_7^* - a_7e_6^*)e_8^* +$

$e_1^*(a_3e_2^* - a_5e_4^*) + a_8e_6^*e_7^*$. Let's calculate :$\gamma_2(\omega^a) = \frac{(\omega^a)^2}{2!}$with $\gamma_2(\omega^a)$ we find :

$$a = 0 \text{ and } \begin{cases} a_2a_6 = 0 \\ a_2a_7 = 0 \\ a_2a_8 = 0 \end{cases} \text{ and } \begin{cases} a_3a_6 = 0 \\ a_3a_7 = 0 \\ a_3a_8 = 0 \end{cases} \text{ and } \begin{cases} a_4a_6 = 0 \\ a_4a_7 = 0 \\ a_4a_8 = 0 \end{cases} \text{ and } \begin{cases} a_5a_6 = 0 \\ a_5a_7 = 0 \\ a_5a_8 = 0 \end{cases}$$

$\begin{cases} a_2V = 0 \\ a_3V = 0 \\ a_4V = 0 \\ a_5V = 0 \end{cases}$ where $V = \begin{pmatrix} a_6 \\ a_7 \\ a_8 \end{pmatrix}$ if $V \neq 0$ then $a_2 = a_3 = a_4 = a_5 = 0$ so we have the s.v $< e_6, e_7, e_8 >$,

if $V = 0$ then $a_6 = a_7 = a_8 = 0$ so we have the s.v $< e_2, e_3, e_4, e_5 >$,

finally $\bar{R}_{\leq 1}(\omega) =< e_2, e_3, e_4, e_5 > \bigcup < e_6, e_7, e_8 >= V_1 \bigcup V_2$.

We remark that $V_1 \bigcap V_2 = \{0\}$, so if $f_1 = Aut(\omega)$ we have :

$f(\bar{R}_{\leq 1}(\omega)) \subset \bar{R}_{\leq 1}(\omega)$ from where $\{f(V_1) \subset V_1 \ \ and \ \ f(V_1) \subset V_2\}$ or

$\{f(V_1) \subset V_2 \ \ and \ \ f(V_2) \subset V_1\}$.

this case $\{f(V_1) \subset V_2 \ \ and \ \ f(V_2) \subset V_1\}$ is impossible because that case

$f(e_1), f(e_2), f(e_3), f(e_4), f(e_5) \in< e_6, e_7, e_8, >$ and $f(e_6), f(e_7), f(e_8) \in< e_2, e_3, e_4, e_5 >$

and $f \in A_1$

that is to say $\wedge^3 f(\omega) = \omega$ il suffices to consider the cofficient of $e_1e_2e_3, 1 = 0$,

we fined $1 = 0$,where

$\{f(V_1) \subset V_1 \ \ and \ \ f(V_1) \subset V_2\}$ and the matrix associated with f takes the following from :

$$M_B(f) = \begin{pmatrix} \alpha_1 & 1 & 1 \\ \alpha_2 & & \\ \alpha_3 & & \\ \alpha_4 & A & 0_{4\times 3} \\ \alpha_5 & & \\ \alpha_6 & & \\ \alpha_7 & 0_{3\times 4} & B \\ \alpha_8 & & \end{pmatrix}$$

Let $f \in A_1, \wedge^3 f(\omega)$, entrains :

$\wedge^3 f\left[e_1(e_2e_3 + e_4e_5)\right] + \wedge^3 f(e_6e_7e_8) = e_1(e_2e_3 + e_4e_5) + (e_6e_7e_8)$

so $\wedge^3 f\left[e_1(e_2e_3 + e_4e_5)\right] + \lambda(e_6e_7e_8) = e_1(e_2e_3 + e_4e_5) + (e_6e_7e_8)$

then $e_{678} : \lambda = 1$with $\lambda = detB$ so :$B \in SL_3(K)$

subsequently $\wedge^3 f\left[e_1(e_2e_3 + e_4e_5)\right] = e_1(e_2e_3 + e_4e_5)$ that is to say

$f(e_1) \wedge^3 f(e_2e_3 + e_4e_5) = e_1(e_2e_3 + e_4e_5)$,so :

$f(e_1)e_1(e_2e_3+e_4e_5) = 0$ subsequently $(\alpha_1e_1+...+\alpha_8e_8)e_1e_2e_3+(\alpha_1e_1+...+\alpha_8e_8)e_1e_4e_5 = 0$

they give $\alpha_2 = \alpha_3 = ... = \alpha_8 = 0$,

from wher $f(e_1)\alpha_1e_1$ with $\wedge^3 f\,[e_1(e_2e_3 + e_4e_5)] = e_1(e_2e_3 + e_4e_5)$,

it is deduced that $f\,|< e_1, e_2, e_3, e_4, e_5, >\in Aut(\omega_5)$ this allows defining a group

homomorphism Ψ :

$$A_1 \xrightarrow{\Psi} Aut(\omega_5)$$

$$f \mapsto \Psi(f) = f\,|< e_1, ..., e_5 >$$

Ψ it is obviously surjective , because all you have to do it is take$B = Id_3$ other side :

$\mathrm{Ker}(\Psi) = \{f/f_1 \;\; and \;\; \alpha_1 = 1, A = Id_4\}$,

or $\wedge^3 f(\omega) = \omega$ entrains Ker $(\Psi)_3(K)$,

the exactness of the following sequence, hence :

$$1 \longrightarrow SL_3(K) \xrightarrow{\Psi} A_1 \longrightarrow Aut(\omega_5) \longrightarrow 1(3.1)$$

and as $Aut\omega_5$ checks the exact sequences :

$$1 \longrightarrow A_0 \longrightarrow Aut(\omega_5) \longrightarrow K^* \longrightarrow 1 \tag{3.2}$$

$$1 \longrightarrow K^4 \longrightarrow A_0 \longrightarrow Sp_4(K) \longrightarrow 1(3.3)$$

so $A_1 = Aut(\omega_{8,1})$ is defined as 3.1,3.2and3.1.

II-2-1.group $\mathrm{A}_2 Aut(\omega_{8,2})$ $\square$

Property 3.2 *:*
The automorphism group $A_2 = Aut(\omega_{8,2})$ is determined by the following exact sequences :

$$1 \longrightarrow A_2' \longrightarrow A_2 \longrightarrow K^* \longrightarrow 1$$

$$1 \longrightarrow A_2'' \longrightarrow A_2' \longrightarrow SL_2(K) \longrightarrow 1$$

$$1 \longrightarrow A_2''' \longrightarrow A_2'' \longrightarrow K^* \longrightarrow 1$$

$$1 \longrightarrow K^{16} \longrightarrow A_2''' \longrightarrow SL(K) \longrightarrow 1$$

Proof. :
Let's first determine the stable part scientifically $R_3(\omega) = \{x \in E/\varpi(x)\ is\ of\ rank3\}\,, \omega = \omega_{8,2}$, we remarck that $e_1 \in R_3(\omega)$.

Let $x = \sum_{i=1}^{8} \alpha_i e_i$ so if $\alpha_i \neq 0$, we have $e_1\lambda_1 x + \lambda_2 e_2 + ...\lambda_8 e_8$

then :$\varpi(x)\lambda_1 x + \lambda_2 e_2 + ... + \lambda_8 e_8)(e_2e_3 + e_4e_5 + e_6e_7) + e_5e_6e_8$,

subsequently :$\varpi(x) = \lambda_2\bar{e_2}\bar{e_4}\bar{e_5}\varpi(x) = \lambda_2\bar{e_2}\bar{e_6}\bar{e_7} + \lambda_3\bar{e_3}\bar{e_4}\bar{e_5} + \lambda_3\bar{e_3}\bar{e_6}\bar{e_7}\lambda_4\bar{e_4}\bar{e_2}\bar{e_3}$

$\lambda_4\bar{e_4}\bar{e_7}\bar{e_6} + \lambda_5\bar{e_5}\bar{e_2}\bar{e_3} + \lambda_5\bar{e_5}\bar{e_6}\bar{e_7} + \lambda_6\bar{e_6}\bar{e_2}\bar{e_3} + \lambda_6\bar{e_6}\bar{e_4}\bar{e_5}$

$+ \lambda_7\bar{e_7}\bar{e_2}\bar{e_3} + \lambda_7\bar{e_7}\bar{e_4}\bar{e_5} + \bar{e_5}\bar{e_6}\bar{e_8}$

$[rank(\varpi(x))] \Leftrightarrow [\varpi(x)$is decomposable$]$

Let the relationships be :

$$\sum_{i\in j-H} \epsilon_{i,j,H}\alpha_{j-\{i\}}\alpha_{H\cup\{i\}} = 0.(*)$$
$$\text{whrer } \epsilon_{i,j,H} = \mp 1$$

the (*) relations are called the grassmann relations :

thus they are necessary and sufficient conditions for $\omega(x)$ to be decomposable.([1] [2])

Let's us assume $J = \{2,4,5,6\}\,; H = \{2,7\}$ so$J - H = \{4,5,6\}$ .Subsequently(*)they give :

$\epsilon_4 a_{256}a_{247} + \epsilon_5 a_{246}a_{257} + \epsilon_6 a_{245}a_{267} = 0,$

so $\epsilon_4(0) + \epsilon_5(0) + \epsilon_6(\lambda_2)(\lambda_2) = 0$ from where $\lambda_2^2 = 0$ that is to say $\lambda_2 = 0$.

Let's us assume $J = \{3,4,5,6\}\,; H = \{3,7\}$ so $J - H = \{4,5,6\}$subsequently(*)they give :

$\epsilon_4 a_{356}a_{347} + \epsilon_5 a_{346}a_{357} + \epsilon_6 a_{345}a_{367} = 0,$

then $\epsilon_4(0) + \epsilon_5(0) + \epsilon_6(\lambda_3)(\lambda_3) = 0$ from where $\lambda_3 = 0$

let's us assume $J = \{2,3,4,6\}\,;H = \{4,7\}$ so $J - H = \{2,3,6\}$subsequently(*)they give :

$\epsilon_2 a_{346}a_{247} + \epsilon_3 a_{246}a_{347} + \epsilon_6 a_{234}a_{467} = 0,$

then $\epsilon_6(0) + (\lambda_4)(\lambda_4) = 0$, from where$\lambda_4 = 0$,

let's us assume $J = \{2,3,5,6\}\,;H = \{5,7\}$ so$J - H = \{2,3,6\}$ subsequently(*)

$\epsilon_2 a_{356}a_{257} + \epsilon_3 a_{256}a_{357} + \epsilon_6 a_{235}a_{567} = 0,$

then $\epsilon_6(0) + \epsilon_5(0) + \epsilon_5 = 0$,from where $\lambda_5 = 0$,

let's us assume :$J = \{2,3,4,6\}\,;H = \{5,6\}$ so , $J - H = \{2,3,4\}$ subsequently(*)

$\epsilon_2 a_{346}a_{256} + \epsilon_3 a_{246}a_{356} + \epsilon_4 a_{236}a_{456} = 0$, then : $\epsilon_4(\lambda_6)(\lambda_6) = 0$from where :$\lambda_6 = 0$,

let's assume : $J = \{2,3,4,7\}\,;H = \{5,7\}$ so $J - H = \{2,3,4\}$

then $\epsilon_2 a_{348}a_{257} + \epsilon_3 a_{247}a_{357} + \epsilon_4 a_{237}a_{457} = 0$,from where

let's us assume :$J = \{2,3,4,8\}\,;H = \{5,8\}$ so $J - H = \{2,3,4\}$,then :

$\epsilon_2(0)a_{348}a_{258} + \epsilon_3(0)a_{248}a_{358} + \epsilon_4(0)a_{238}a_{458} = 0$ from where $\lambda_8 = 0$

let's us assume $J = \{2,3,4,8\}\,;H = \{5,8\}$ so $J - H = \{2,3,5\}$ then :

$\epsilon_2(0)a_{348}a_{258} + \epsilon_3(0)a_{248}a_{358} + \epsilon_4(0)a_{238}a_{458} = 0$ from where $\lambda_8 = 0$

it is deduced thet $\lambda_2 = \lambda_3 = ... = \lambda_8 = 0$, that is to say $e_1 = \lambda_1 x$ where : $x = \alpha_1 e_1$ we obain :

$R_3(\omega) =< e_1 >$
Permutation

$$\downarrow \boxed{\begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 e_7 & e_8 \\ e_1 & e_5 & e_6 & e_8 & e_2 & e_3 e_7 & e_4 \end{array}}$$

$\omega_{8,2}$ becomes $\omega = e_1(e_5e_6 + e_8e_2 + e_3e_7) + e_2e_3e_4$.

As $R_3(\omega) =< e_1 >$ is a stable part so we can define a group homomorphism π :

$$A_2 \xrightarrow{\pi} K^*$$

$$f \mapsto \pi(f) = \lambda \text{ where } f(e_1) = \lambda e_1.$$

π is surjective indeed for $\lambda \in K^*$; there exists $f_0 \in A_2$ defined by :

$f_0(e_1) = \lambda e_1, f_0(e_2) = \lambda^{-1}e_2, f_0(e_3) = e_3, f_0(e_4) = \lambda e_4,$

$f_0(e_5) = e_5, f_0(e_6) = \lambda^{-1}e_6, f_0(e_7) = \lambda^{-1}e_7, f_0(e_8) = e_8$

Hence the accuracy of the sequence $1 \longrightarrow A_2' \longrightarrow A_2 \xrightarrow{\pi} K^* \longrightarrow 1$

where$A_2' = Ker\pi$. Let $f \in A_2'$ so $\omega = e_1(e_5e_6 + e_8e_2 + e_3e_7) + e_2e_3e_4$ so $e_1\omega = e_1e_2e_3e_4$

subsequently $\wedge^4 f(e_1\omega) = f(e_1) \wedge^3 f(\omega) = e_1\omega = \wedge^4 f(e_1e_2e_3e_4)$, that is to say :

$\wedge^4 f(e_1e_2e_3e_4) = e_1e_2e_3e_4$ form where $< e_1e_2e_3e_4 >$ is stable by f.In this case, the matrix of f is the form :

$$M_B(f) = \begin{pmatrix} 1 & a_1 & b_1 & c_1 & x_1 & y_1 & z_1 & t_1 \\ 0 & a_2 & b_2 & c_2 & x_2 & y_2 & z_3 & t_2 \\ 0 & a_3 & b_3 & c_3 & x_3 & y_3 & z_3 & t_3 \\ 0 & a_4 & b_4 & c_4 & x_4 & y_4 & z_4 & t_4 \\ & & & & x_5 & y_5 & z_5 & t_5 \\ & & & & x_6 & x_6 & z_6 & t_6 \\ & 0_{4\times4} & & & x_7 & x_7 & z_7 & t_7 \\ & & & & x_8 & y_8 & z_8 & t_8 \end{pmatrix}$$

the equality $\wedge^3 f(\omega) = \omega$ they give $[(x_1e_1 + ...x_8e_8)(x_1e_1 + ...x_8e_8) + (t_1e_1 + ...t_8e_8)(a_1e_1 + a_2e_2 +$

$(a_1e_1 + ... + a_4e_4)(b_1e_1 + ... + b_4e_4)(c_1e_1 + ... + c_4e_4) = e_1e_8e_6 + e_1e_8e_2 + e_1e_3e_7 + e_2e_3e_4$

so $c_{156} : \begin{vmatrix} x_5 & y_5 \\ x_6 & y_6 \end{vmatrix} = 1$

from where $\begin{pmatrix} x_5 & y_5 \\ x_6 & y_6 \end{pmatrix} \in SL_2(K)$

$$\begin{cases} c_{157} : x_5y_7 - x_7y_5 = 0 \\ c_{167} : x_6y_7 - x_7y_6 = 0 \end{cases} \quad \text{They give} \quad x_7 = y_7 = 0$$

and

$$\begin{cases} c_{158} : x_5y_8 - x_8y_5 = 0 \\ c_{168} : x_6y_8 - x_8y_6 = 0 \end{cases}$$

they give $x_8 = y_8 = 0$

this allows us to define the homomorphism t :

$$A_2' \longrightarrow SL_2(K)$$

$$f \longrightarrow t(f) = \begin{pmatrix} a_5 & b_5 \\ a_6 & b_6 \end{pmatrix}$$

t is surjective for $\begin{pmatrix} x_5 & y_5 \\ x_6 & y_6 \end{pmatrix} \in SL_2(K)$,there exists $f_1 \in A_2$ defined by :

$f_1(e_1) = e_1, f_1(e_2) = e_2, f_1(e_3) = e_3, f_1(e_4) = e_4,$

$f_1(e_5) = x_5e_5 + x_6e_6, f_1(e_6) = y_5e_5 + y_6e_6, f_1(e_7) = e_7, f_1(e_8) = e_8,$

from this, it is deduced that the following sequence is exact :

$1 \longrightarrow A''_2 \longrightarrow A'_2 \xrightarrow{t} SL_2(K) \longrightarrow 1$

where $A''_2 = Ker(t)$,

set $f \in A"_2 so x_5 = y_6 = 1$ and $x_6 = y_5 = 0$ and as $\wedge^3 f(\omega) = \omega$ we obtain

$e_1\left[(x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + e_5(y_1e_1 + y_2e_2 + y_3e_3 + y_4e_4 + e_6) + (t_1e_1 + ... + t_8e_8)(a_1e_1 + ... + a_4e_4\right.$

so we have

$$c_{234} : \begin{vmatrix} x_2 & y_2 & c_3 \\ x_3 & y_3 & c_3 \\ x_4 & y_4 & c_3 \end{vmatrix} = 1 \quad and(I) \begin{cases} c_{182} : t_8a_2 - b_2z_8 = 1 \\ c_{183} : t_8a_3 - b_3z_8 = 0 \\ c_{184} : t_8a_4 - b_4z_8 = 0 \end{cases} and(II) \begin{cases} c_{137} : t_8a_3 - b_3z_7 = 1 \\ c_{127} : t_8a_2 - b_2z_7 = 0 \\ c_{147} : t_8a_4 - b_4z_7 = 0 \end{cases}$$

System (I) yields $\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} = 0$,as well as system (II), yields $\begin{vmatrix} a_2 & a_2 \\ a_4 & a_4 \end{vmatrix} = 0$

one obtains(III)

$$\begin{cases} a_2b_4 - b_2a_4 = 0 \\ a_3b_4 - b_3a_4 = 0 \end{cases}$$

if : $\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0$ then :

c_{234} : $\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} c_4 - \begin{vmatrix} a_2 & b_2 \\ a_4 & b_4 \end{vmatrix} c_3 + \begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix} c_2 = 1$ that is to say $=1$ (absurd),Hence $\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} \neq 0$

The system (III) yields $a_4 = b_4 = 0$ and c_{234} deviates $\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} .c_4 = 1$. This allows defining a homomorphism η :

$$A''_2 \xrightarrow{\eta} K^*$$

$$f \longrightarrow \eta(f) = c_4$$

η is surjective, indeed, the linear map f_2 defined by : $f_2(e_1) = e_1, f_2(e_2) = e_2, f_2(e_3) = c_4^{-1}e_3, f_2(e_4) = c_4e_4, f_2(e_5) = e_5, f_2(e_6) = e_6,$

$f_2(e_7) = c_4e_7, f_2(e_8) = e_8$is the predecessor of c_4

hence the accuracy of the follwing sequence :

$1 \longrightarrow A_2''' \longrightarrow A_2'' \overset{\eta}{\to} K^* \to 1$ or $A_2''' = Ker(\eta)$,

let $f \in A_2'''$ so $c_4 = 1$ or c_{234} they give $\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} = 1$ that is to say $\begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \in SL_3(K)$;

This allows us to define the homomorphism P :

$$A_2''' \overset{P}{\to} SL_2(K)$$

$$\text{f} \mapsto P(f) = \begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}$$

P is surjective indeed for :

$\begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \in SL_2(K)$ there exists $f_3 \in A_2'''$ defined by : $f_3(e_1) = e_1, f_3(e_2) = a_2e_1 + a_3e_3$,

$f_3(e_3) = b_2e_1 + b_3e_3, f_3(e_4) = e_4, f_3(e_5) = e_5, f_3(e_6) = e_6, f_3(e_7) = a_2e_7 + a_3e_8$,

$f_3(e_8) = b_2e_7 + b_3e_8$.

Let $f \in Ker(P)$ so $a_2 = b_3 = 1$ and $a_3 = b_2 = 0$ as $\wedge^3 f(\omega) = \omega$ then :

$e_1 [(x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + e_5)(y_1e_1 + y_2e_2 + y_3e_3 + y_4e_4 + e_6) + (t_1e_1 + ... + x_8e_8 +) \times$

$\times (a_1e_1 + e_2) + (b_1e_1 + e_3)(g_1e_1 + ... + g_8e_8) + (a_1e_1 + e_2)(b_1e_1 + e_3)(c_1e_1 + c_2e_2 + c_3e_3 + e_4)$

$= e_1e_5e_6 + e_1e_8e_2 + e_1e_3e_7 + e_2e_3e_4$ so $c_{137} : z_7 = 1, c_{127} : t_7 = 0, c_{182} : t_8 = 1, c_{183} : z_8 = 0$,

$c_{145} : y_4 = 0, c_{146} : x_4 = 0, c_{142} : t_4 + b_1 = 0$ then $t_4 = -b_1, c_{134} : z_4 + a_1 = 0$ so $z_4 = -a_1 = 0$ then $z_4 = -a_1$

$c_{125} : y_2 - t_5 = 0$ then $t_5 = -y_5, c_{126} : x_2 - t_6 = 0$ then $t_6 = x_2, c_{135} : -y_3 + z_5 = 0$ then

$z_5 = y_3, c_{136} : x_3 + z_6 = 0$ so $z_6 = -x_3$

$c_{123} : \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} - t_3 - z_2 + c_1 - b_1c_3 - a_1c_2 = 0$ so $t_3 = \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} - z_2 + c_1 - b_1c_3 - a_1c_2$.(**) finally,

the matrix associated with f takes the form :

$$M_B(f) = \begin{pmatrix} 1 & a_1 & b_1 & c_1 & x_1 & y_1 & z_1 & t_1 \\ 0 & 1 & 0 & c_2 & x_2 & y_2 & z_3 & t_2 \\ 0 & 0 & 1 & c_3 & x_3 & y_3 & z_3 & t_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & -a_1 & -b_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & y_3 & -y_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -x_3 & x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

or t_3 is given by (**), one obtains Ker$P \approx K^{16}$ hence the accuracy of the following sequence :

$1 \longrightarrow K^{16} \longrightarrow A_2'' \xrightarrow{P} SL_2(K) \longrightarrow 1$ $\square$

Conclusion

the classification of trivectors over a finite field plays in important role in solving certain problems in coding theory and in CCEGand BMC.The spectrum of codes is closely related to the weights of cod and and the cardinalities of orbits of trivectors.

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Abstract

The study presented in this thesis focuses on the determination of the automorphisms group of some trivectors (or alternating trilinear forms) in dimension ≤ 8. To classify the trivectors, we use algebraic invariants which make it possible to better understand the classification of these forms. For example, we can use the automorphisms group of the trivector ω, $Aut(\omega)$, because two trivectors ω_1 and ω_2 are equivalent if and only if their automorphisms group $Aut(\omega_1)$ and $Aut(\omega_2)$ are the same. In this thesis, we recall the main part of the known results on the classi cation of trivectors, and we determine the group of automorphisms of some trivectors.

Résumé

L'étude présentée dans cette thèse se concentre sur la détermination du groupe des automorphismes de certains trivecteurs (ou formes trilinéaires alternées) de dimension ≤ 8. Pour classer les trivecteurs, nous utilisons des invariants algébriques qui permettent de mieux comprendre la classification de ces formes. Par exemple, nous pouvons utiliser le groupe des automorphismes du trivecteur ω, $Aut(\omega)$, car deux trivecteurs ω_1 et ω_2 sont équivalents si et seulement si leurs groupes d'automorphismes $Aut(\omega_1)$ et $Aut(\omega_2)$ sont les mêmes. Dans cette thèse, nous rappelons la partie principale des résultats connus sur la classification des trivecteurs, et nous déterminons le groupe des automorphismes de certains trivecteurs.

ملخص

الدراسة المقدمة في هذه الرسالة تركز على تحديد مجموعة الاوتومورفيزمات لبعض المتجهات ثلاثية القطبية (او الاشكال التكاملية ثلاثية الخطية المتناوبة) في ابعاد اقل من او تساوي 8 . لتصنيف هذه المتجهات ، نستخدم العناصر الجبرية التي تمكننا من فهم افضل لتصنيف هذه الاشكال. على سبيل المثال ، يمكننا استخدام مجموعات الاوتومورفيزمات للمتجه ثلاثي القطبية ω ، $Aut(\omega)$، لان متجهين ثلاثية القطبية ω_1 وω_2 يعتبران مكافئين اذا وحسب اذا كانت مجموعات اوتومورفيزماتهم $Aut(\omega_1)$ و $Aut(\omega_2)$ متطابقة. في هذه الرسالة ، نستعرض الجزء الرئيسي من النتائج المعروفة في تصنيف المتجهات ثلاثية القطبية ، ونحدد مجموعة الاوتومورفيزمات لبعض هذه المتجهات.