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AN ELLIPTIC EQUATION IN A DOMAIN WITH PERTURBED BOUNDARY

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بسم الله الرحمن الرحيم

رب أوزعني أن أشكر نعمتك التي أنعمت علي وعلى والدي

وأن أعمل صالحا ترضىه وأدخلني برحمتك

في عبادك الصالحين

DEDICACE

I would like to dedicate this work to
My parents the first love of my life
My big brother and sister my second parents
My other sisters and brothers
The rest of my family
Who loved me for there support
Who hated me because they make me strong.

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Contents

Introduction

Many questions in mathematical physics can be reduced to elliptic differential equations of second order combined with some boundary value conditions. The simplest example of elliptic PDE in a two-dimension domain Ω is the Laplace equation

$$\Delta u = u_{xx} + u_{yy} = 0, \ \ (x, y) \in \Omega \subset \mathbb{R}^2.$$

The solution of the Laplace equation in a bounded domain requires the specification of certain conditions that the solution must satisfy at the boundary of the domain $\partial\Omega$. For instance, such an equation describes the temperature distribution of a planar body in steady-state heat flow. It may also describe other physical phenomena at equilibrium. If the domain Ω is the unit disk, it is more convenient to write the Laplace equation in polar coordinates, which is given by

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, , \ \ 0 \leq r \leq 1, \ \ 0 \leq \theta < 2\pi.$$

In this work, we are interested in solving the Laplace equation when the domain is subject to a small perturbation depending on some parameter $\varepsilon > 0$, i.e.

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, , \ \ 0 \leq r \leq 1 + \varepsilon f(\theta), \ \ 0 \leq \theta < 2\pi$$

where f is some $2\pi-$periodic function. When f is not reduced to a constant, the solution of Laplace's equation can no longer be expressed by a series formula (obtained by separation of variable) or by Laplace's integral formulas. To overcome this difficulty, we use an asymptotic method by assuming that the solution of the perturbed problem can be written as

$$u(r, \theta, \varepsilon) = u_0(r, \theta) + \varepsilon u_1(r, \theta) + O(\varepsilon^2),$$

where u_0 and u_1 are independent of the perturbation parameter ε and both solve the Laplace equation with some boundary conditions, see [6]. To compute u_0 and u_1, we propose three approaches:

- Two analytical methods: separation of variables or written by Laplace's formulas.
- One numerical method: finite differences in polar coordinates.

This work is divided into four chapters. The first one recalls some basic definitions and proprieties used in the sequel such as obtaining the solution of Laplace's equation in the unit disk by separation of variable or Poisson's integral formula. Some basic definitions of perturbation and asymptotic methods are also recalled. In the second chapter, we consider Laplace's equation on a perturbed circular disk and look for an asymptotic solution using the two analytical methods stated above. In the third chapter, we combine the asymptotic approach with finite differences method in polar coordinates, which are less known then finite differences in Cartesian coordinates, see [9–12]. This mixed approach allows us to solve Laplace's equation numerically. In the last chapter, we include numerical results for some examples of the Laplace equation with different boundary conditions and different perturbations of the boundary.

Chapter 1

PRELIMINARIES

In this chapter we have three part. First part, will talk about same basic definition and propriety of functional spaces, then, we consider Fourier series of a periodic function. Second part, we define Laplace equation in the unit disk and solve it with tow methods: separation of variable and Poisson formula. Last part, we give same basic definition about perturbation and asymptotic methods. In this chapter we use the following reference [1–5].

1.1 Functional spaces

1.1.1 $\mathbb{L}^p$ spaces

Definition 1.1 *Let* $p \in \mathbb{R}$ *with* $1 < p < \infty$*; we define* $\mathbb{L}^p$ *space:*

$$\mathbb{L}^p(\Omega) = \{f : \Omega \to \mathbb{R}; \quad f \quad is \quad measurable \quad and \quad |f|^p \in \mathbb{L}^1(\Omega)\} \tag{1.1}$$

with

$$\|f\|_{\mathbb{L}^p} = \|f\|_p = \left[\int_\Omega |f(x)|^p dx\right]^{\frac{1}{p}} \tag{1.2}$$

is a norm.

Definition 1.2 *We set*

$$\mathbb{L}^\infty(\Omega) = \{f : \Omega \to \mathbb{R}; \quad f \quad is \quad measurable \quad and \quad \exists C \quad such \quad that \quad |f(x)| \leq C \quad on \quad \Omega\}$$

with

$$\|f\|_{\mathbb{L}^\infty} = \|f\|_\infty = \inf\{|f(x)| \leq C\}$$

is a norm.

Notation 1.1 *Let* $1 \leq p \leq \infty$*; we denote by* q *the conjugate exponent, such that:*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Theorem 1.1 $\mathbb{L}^p$ *is a:*

- *Normed vector space and* $\|f\|_p$ *his norm for any* p *such that* $1 \leq p \leq \infty$.
- *Banach space for any* p, $1 \leq p \leq \infty$.
- *Reflexive space for any* p *such that* $1 < p < \infty$.
- *Separable space for any* p *such that* $1 \leq p < \infty$*, if we assume that* Ω *is a separable measure space.*

Example 1.1 *we set*

$$\mathbb{L}^2(\Omega) = \{f : \Omega \to \mathbb{R}; \quad f \quad is \quad measurable \quad and \quad |f|^2 \in \mathbb{L}^1(\Omega)\}$$

with

$$\|f\|_{\mathbb{L}^2} = \|f\|_2 = \left[\int_\Omega |f(x)|^2 dx\right]^{\frac{1}{2}}$$

is a Banach, Reflexive and Separable space, moreover is a Hilbert space equipped with scalar product :

$$(f, g) = \int_\Omega fg \, dx.$$

1.1.2 $\mathbb{H}^m$ spaces

Definition 1.3 *Let $m \in \mathbb{N}$. A function $f \in \mathbb{L}^2(\Omega)$ belongs to the Sobolev space of order m, denoted $\mathbb{H}^m(\Omega)$, if all the partial derivatives of f up to the order m, in the distributional sense, belong to $\mathbb{L}^2(\Omega)$. By convention, we note $\mathbb{H}^0(\Omega) = \mathbb{L}^2(\Omega)$.*

Theorem 1.2 *The spaces $\mathbb{H}^m(\Omega)$, $m \geq 0$ endowed with the following inner product:*

$$\langle u, v\rangle = \sum_{|\alpha|\leq m} \int_\Omega (\partial^\alpha u(x)\partial^\alpha v(x))dx. \tag{1.3}$$

are Hilbert spaces. with the associated norm:

$$\|u\| = \left(\sum_{|\alpha|\leq m} \|\partial^\alpha u(x)\|^2_{\mathbb{L}^2(\Omega)}\right)^{\frac{1}{2}}. \tag{1.4}$$

Example 1.2 *When $m = 2$ and $\Omega \subset \mathbb{R}$, we set:*

$$\mathbb{H}^m(\Omega) = \mathbb{H}^2(\Omega) = \left\{f : \Omega \to \mathbb{R};\ \textit{such that } f \textit{ and } f' \textit{ belong to } L^2(\Omega)\right\}.$$

equipped with the norm:

$$\|f\| = \|f'(x)\|_{\mathbb{L}^2(\Omega)}.$$

1.2 Fourier series of periodic functions

Suppose that $f(\theta)$ is a function defined on the real line such that $f(\theta + 2\pi) = f(\theta)$ for all θ. Such functions are said to be periodic with period 2π, or $2\pi-$periodic for short. We shall assume that f is integrable on every bounded interval $[-j; j]$, this will be the case if f is bounded and is continuous except perhaps at finitely many points in each bounded interval. f can be expanded in a Fourier series

$$f(\theta) = \frac{1}{2}a_0 + \sum_1^\infty (a_n \cos(n\theta) + b_n \sin(n\theta)) \tag{1.5}$$

With

$$a_0 = \frac{1}{\pi}\int_{-j}^{j} f(\theta)d\theta \tag{1.6}$$

$$a_n = \frac{1}{2\pi}\int_{-j}^{j} \cos(n\theta)f(\theta)d\theta \qquad (n \geq 1) \tag{1.7}$$

$$b_n = \frac{1}{2\pi}\int_{-j}^{j} \sin(n\theta)f(\theta)d\theta \qquad (n \geq 1) \tag{1.8}$$

Here $\frac{1}{2}a_0$ is the coefficient of the constant function $1 = \cos(0\theta)$.
There is no b_0 because $\sin(0\theta) = 0$.
The real numbers a_n and b_n are called the Fourier coefficients of f.

Lemma 1.1 *$f(\theta)$ is a function defined on the interval $[-j;j]$:*

- *if f is even,*

$$a_n = \frac{2}{\pi}\int_{-j}^{j}\cos(n\theta)f(\theta)d\theta \ \textit{and}\ b_n = 0. \qquad (n \geq 1)$$

- *if f is odd,*

$$b_n = \frac{2}{\pi}\int_{-j}^{j}\sin(n\theta)f(\theta)d\theta \ \textit{and}\ a_n = 0. \qquad (n \geq 1)$$

Whether the Fourier series of a $2\pi-$periodic function the constant term in the series is a_0

Definition 1.4 *A function f is said to be piecewise continuous on an interval $[-j;j]$ if it is continuous at all but finitely many points in $[-j;j]$, where it has jump discontinuities, that is at any discontinuity point x, the function has distinct right-hand side and left-hand side (finite) limits $f(x+)$ and $f(x-)$.*

If both f and f' are continuous on $[-j;j]$, then f is called smooth on $[-j;j]$.

If at least one of f, f' is piecewise continuous on $[-j;j]$, then f is said to be piecewise smooth on $[-j;j]$.

Theorem 1.3 *If f is $2\pi-$periodic, continuous, and piecewise smooth, then the Fourier series of f converges to f absolutely and uniformly on $\mathbb{R}$.*

Lemma 1.2 *If $f \in \mathbb{L}^2([j;j])$ than we get the classical result:*

$$\lim_{n\longrightarrow+\infty}\int_{-j}^{j}|f(\theta) - \Sigma_1^{\infty}(a_n\cos(n\theta) + b_n\sin(n\theta))|^2 d\theta = 0$$

1.3 Laplace equation on a disc

Let's define the two dimensional Laplace equation for a rectangle governed by the equation

$$\Delta u = u_{xx} + u_{yy} = 0$$

with boundary conditions specified on the boundary of a rectangle. However, if the domain of the solution $u(x,y)$ is a circular disk, it is useful to study the two dimensional Laplace equation in polar coordinates. It is well known that the polar coordinates (r,θ) of any point are related to its Cartesian coordinates (x,y) by the familiar formulas:

$$x = r\cos(\theta), \quad y = r\sin(\theta) \tag{1.9}$$

Let us formulate the steady problem in polar coordinates r, θ, where

$$x^2 + y^2 = r^2 \quad \textit{and} \quad \frac{y}{x} = \tan(\theta) \tag{1.10}$$

Consider a function u such that $u = u(r;\theta)$, where $r = r(x;y)$ and $\theta = \theta(x;y)$. That is $u = u(r(x;y);\theta(x;y))$ then:

$$\begin{aligned} u_x &= u_r r_x + u_\theta \theta_x \\ u_{xx} &= u_r r_{xx} + (u_r)_x r_x + u_\theta \theta_{xx} + (u_\theta)_x \theta_x \end{aligned}$$

hence

$$\begin{aligned} u_{xx} &= u_r r_{xx} + (u_{rr} r_x + u_{r\theta}\theta_x) r_x + u_\theta \theta_{xx} + (u_{\theta r} r_x + u_{\theta\theta}\theta_x)\theta_x \\ &= u_r r_{xx} + u_{rr} r_x^2 + 2u_{r\theta} r_x \theta_x + u_\theta \theta_{xx} + u_{\theta\theta}\theta\theta_x^2 \end{aligned}$$

A similar computation using y instead of x also gives:

$$u_{yy} = u_r r_{yy} + u_{rr} r_y^2 + 2u_{r\theta} r_y \theta_y + u_\theta \theta_{yy} + u_{\theta\theta}\theta_y^2$$

If we add these expressions and collect like terms we get:

$$\Delta u = u_r(r_{xx} + r_{yy}) + u_{rr}(r_x^2 + r_y^2) + 2u_{r\theta}(r_x\theta_x + r_y\theta_y) + u_\theta(\theta_{xx} + \theta_{yy}) + u_{\theta\theta}(\theta_x^2 + \theta_y^2) \quad (1.11)$$

Differenting $x^2 + y^2 = r^2$, with respect to x :

$$\begin{aligned} 2x &= 2rr_x \Rightarrow r_x = \frac{x}{r} \\ r_{xx} &= \frac{r - xr_x}{r^2} = \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3} \end{aligned}$$

and with respect to y:

$$\begin{aligned} 2y &= 2rr_y \Rightarrow r_y = \frac{y}{r} \\ r_{yy} &= \frac{r - yr_y}{r^2} = \frac{r^2 - y^2}{r^3} = \frac{x^2}{r^3} \end{aligned}$$

Now differentiate $tan(\theta) = \frac{y}{x}$, with respect to x:

$$\begin{aligned} sec^2(\theta)\theta_x &= -\frac{y}{x^2} \Rightarrow \theta_x = -\frac{ycos^2(\theta)}{x^2} = -\frac{y}{r^2} \\ \theta_{xx} &= \frac{2y}{r^3} r_x = \frac{2yx}{r^4} \end{aligned}$$

and with respect to y:

$$\begin{aligned} sec^2(\theta)\theta_y &= \frac{1}{x} \Rightarrow \theta_y = \frac{\cos^2(\theta)}{x} = \frac{x}{r^2} \\ \theta_{yy} &= -\frac{2x}{r^3} r_y = \frac{2yx}{r^4} \end{aligned}$$

Together we get:

$$\begin{aligned} r_{xx} + r_{yy} &= \frac{y^2 + x^2}{r^3} = \frac{1}{r} \text{ and } r_x^2 + r_y^2 = \frac{x^2 + y^2}{r^2} = 1 \\ \theta_{xx} + \theta_{yy} &= \frac{2xy}{r^4} + \frac{-2xy}{r^4} = 0. \end{aligned}$$

In addition, we have

$$\begin{aligned} 2x+2y &= \frac{y^2+x^2}{r^4} = \frac{1}{r^2} \\ r_x\theta_x + r_y\theta_y &= \frac{-2xy}{r^3} + \frac{2xy}{r^3} = 0 \end{aligned}$$

Finally, going back to (1.11), the Laplace operator in polar coordinates is given by

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}. \tag{1.12}$$

In this part, we will study Laplace's equation for a circular disc of radius 1 where the top and the bottom faces of the disc are insulated. The boundary condition at the circular edge is specified. The phenomenon that the temperature reaches a steady state inside the disc is governed by the Laplace's equation in polar coordinates ,and expressed by the boundary value problem

$$PDE \ : \ \Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}, \ \ 0 \le r \le 1, \ \ 0 \le \theta \le 2\pi. \tag{1.13}$$

$$BC \ : \ \alpha u(1,\theta) + \beta \frac{\partial u}{\partial r}(1,\theta) = g(\theta). \tag{1.14}$$

We will require that g is $2\pi-$periodic, and $g \in \mathbb{H}^2(\Omega)$ where Ω is the unit disk.

- when $\beta = 0$ we get Dirichlet boundary condition .
- when $\alpha = 0$ we get Newman boundary condition .
- when $\alpha \neq 0 \ \ and \ \ \beta \neq 0$ we get Robin boundary condition .
- α and β are arbitrary constants and must be the same sign.

1.4 Solving Laplace equation by separation of variables

1.4.1 Separation of variable method

The method of separation of variable is a suitable technique for determining solutions to linear PDEs, usually with constant coefficients, when the domain is bounded in at least one of the independent variables. We illustrate this procedure for 1-D wave equation and 2-D heat equation for Dirichlet, Newman and Robin boundary conditions, though the idea is equally applicable for diffusion equation and Laplace's equation, and other constant coefficient equations. The idea of separation of variable is first to seek simple solution to the PDE in the form of a product, each term in the product depending on only one independent variable. Solutions are then constrained by boundary conditions. This results in a countably infinite set of solutions. A linear superposition of such solution is also a solution, because of the linearity of the problem. As we shall find later, such linear superposition is capable of describing all reasonable initial conditions. Separation of variable is not restricted to rectangular geometry. The method of solution provides solution to either Laplace, heat or wave equation and other related equations in circular (in $2-D$) and spherical geometries (in $3-D$). We illustrate this for Laplace's equation in a circle with Robin boundary condition.

1.4.2 Solving Laplace equation with separation of variable method

Consider the following 2-D problem:

$$PDE \quad : \quad \Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}, \;\; 0 \leq r \leq 1, \;\; 0 \leq \theta \leq 2\pi. \tag{1.15}$$

$$BC \quad : \quad \alpha u(1,\theta) + \beta\frac{\partial u}{\partial r}(1,\theta) = g(\theta). \tag{1.16}$$

We seek a separation of variable solution to (PDE) in the form:

$$u(r,\theta) = R(r)\Theta(\theta) \tag{1.17}$$

deriving term by term we get:

$$u_{rr} = R''(r)\Theta(\theta) \;\text{ and }\; u_{\theta\theta} = R(r)\Theta''(\theta)$$

Substituting term by term gives:

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0 \tag{1.18}$$

Separating the variables:

$$\frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} \tag{1.19}$$

Now, since left and right sides of (PDE) are functions of r and θ respectively, it follows that each side is a constant . Therefore, $\Theta(\theta)$, $R(r)$ satisfy:

$$\frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = k \tag{1.20}$$

Thus we get two ODEs which are given by:

$$\begin{cases} r^2R''(r) + rR'(r) - kR(r) = 0, \\ \Theta''(\theta) + k\Theta(\theta) = 0. \end{cases} \tag{1.21}$$

Now, since the solution $u(r,\theta)$ sought is uni-valued in the disk of radius 1, it follows that :

$$u(r,\theta+2\pi) = u(r,\theta).$$

Hence, we must have:

$$\Theta(\theta+2\pi) = \Theta(\theta).$$

To obtain a periodic non-trivial solution we muste take $k \geq 0$. We suppose that $k = m^2$,where m is an arbitrary constant.

$$\begin{cases} r^2R''(r) + rR'(r) - m^2R(r) = 0 \\ \Theta''(\theta) + m^2\Theta(\theta) = 0 \end{cases} \Longrightarrow \begin{cases} R(r) = A_m r^{-m} + B_m r^m \\ \Theta(\theta) = C_m \cos m\theta + D_m \sin m\theta \end{cases} \tag{1.22}$$

Since the solution $u(r, \theta)$ we are seeking is a classical solution without any singularities for $r \leq 1$, $R(r)$ must be well-behaved in particular at $r = 0$, The solution has to be finite as r approach to 0 then:

$$A_m = 0 \ \ and \ \ B_m = 1 \Longrightarrow R(r) = r^m$$

the solution is $2\pi-$periodic in θ then:

$$\Theta(0) = \Theta(2\pi) \Longrightarrow \cos 0 = 1 = \cos 2\pi m.$$

Thus $m = n, n \in \mathbb{N}$ and

$$\Theta(\theta) = C_n \cos(n\theta) + D_n \sin(n\theta)$$

It follows that

$$u(r, \theta) = R(r)\Theta(\theta) = \sum_{n=0}^{+\infty} r^n (C_n \cos(n\theta) + D_n \sin(n\theta))$$

when

$$k = 0 \Longrightarrow n = 0 \Longrightarrow B_m D_m = C_0$$

then:

$$u(r, \theta) = C_0 + \sum_{n=1}^{+\infty} r^n (C_n \cos(n\theta) + D_n \sin(n\theta)) \tag{1.23}$$

which is usually written in a more convenient equivalent form by:

$$u(r, \theta) = \frac{C_0}{2} + \sum_{n=1}^{+\infty} r^n (C_n \cos(n\theta) + D_n \sin(n\theta)) \tag{1.24}$$

To determine the constants C_n and D_n we use the boundary condition:

$$\alpha u(1, \theta) + \beta u_r(1, \theta) = g(\theta) \tag{1.25}$$

where:

$$u(1, \theta) = \frac{C_0}{2} + \sum_{n=1}^{+\infty} (C_n \cos(n\theta) + D_n \sin(n\theta))$$

and

$$u_r(1, \theta) = \sum_{n=1}^{+\infty} n(C_n \cos(n\theta) + D_n \sin(n\theta))$$

gives:

$$\begin{aligned} g(\theta) &= \alpha\left(\frac{C_0}{2}+\sum_{n=1}^{+\infty}(C_n\cos(n\theta)+D_n\sin(n\theta))\right)+\beta\left(\sum_{n=1}^{+\infty}n(C_n\cos(n\theta)+D_n\sin(n\theta))\right) \\ &= \alpha\frac{C_0}{2}+\sum_{n=1}^{+\infty}(\alpha+\beta n)(C_n\cos(n\theta)+D_n\sin(n\theta)). \end{aligned}$$

Clearly $(\alpha+\beta n)C_n \ \ and \ \ (\alpha+\beta n)D_n$ are Fourier coefficients, and therefore can be determined by:

$$(\alpha+\beta n)C_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos(n\theta)d\theta; \quad n=0,1,2,3,\cdots, \tag{1.26}$$

$$(\alpha+\beta n)D_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin(n\theta)d\theta; \quad n=1,2,3,\cdots. \tag{1.27}$$

1.4.3 A remak on existance and uniqueness of solutions

As is well known, the solution of the Dirichlet problem $(\beta = 0)$ is unique, and the Newman problem $(\alpha = 0)$ is unique to within an arbitrary constant. The series solution confirms the results for these two special cases.

In fact,with $(\beta = 0)$ defines each of the $C_n, \ \ n = 0, 1, ...$ and $D_n, \ \ n = 1, 2, ...$ uniquely. However, if $(\alpha = 0)$, we see that for $n = 0$ that C_0 is arbitrary. Thus, for any solution u of the Newman problem $u - C_0/2$ is also a solution for any constant C_n.

For the general boundary-value problem where $(\alpha \neq 0, \beta \neq 0)$, we see that solutions are unique if α and β have the same sign. This is a special case of the more general uniqueness result for Laplace equation in an arbitrary domain with the Robin boundary condition .

One can show that solution is unique if α and β have the same sign for all points on the boundary. However, if α and β have different signs, solutions may not exist, and even if they exist they may not be unique. To show this, it suffices to restrict attention to the case where α and β are non zero constants with opposite signs, choosing

$$\left|\frac{\alpha}{\beta}\right| = m, \ \ \text{where} \ \ m \in \mathbb{N}$$

Consider first the case where $g(\theta)$ is orthogonal to both $\cos(m\theta)$ and $\sin(m\theta)$, i.e.,the Fourier series of $g_i(\theta)$ does not contain $\cos(m\theta)$ and $\sin(m\theta)$. Then the Fourier coefficients both vanish for $n = m$. In this case, a solution exists but is not unique since $C_n, \ \ n = 0, 1, \cdots$ and $D_n, \ \ n = 1, 2, \cdots$ are arbitrary. If $g_i(\theta)$ is not orthogonal to either $\cos(m\theta)$ and $\sin(m\theta)$ or both, the Fourier coefficients will be nonzero, whereas both left-hand sides the Fourier coefficients are zero. Therefore, either $C_n, \ \ n = 0, 1, \cdots$ and $D_n, \ \ n = 1, 2, \cdots$ or both will be undefined, and a solution will not exist.

1.5 Poisson formula for the solution of Laplace equation

1.5.1 Poisson formula

We consider the Dirichlet problem (where $\alpha = 1, \beta = 0$) in the unit disk and we find closed form to the solution named Poisson formula it express any harmonic function inside a circle in term of it's boundary condition values.

1.5.2 Solution of Laplace equation with Poisson formula

we have fond that the solution of Laplace equation on the unit disk withe Dirichlet boundary condition is given by:

$$u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{+\infty} r^n (A_n \cos(n\theta) + B_n \sin(n\theta)) \tag{1.28}$$

where

$$\begin{aligned} A_n &= \frac{1}{\pi} \int_0^{2\pi} g(\theta) \cos(n\theta) d\theta; \quad n = 0, 1, 2, 3, \cdots && (1.29) \\ B_n &= \frac{1}{\pi} \int_0^{2\pi} g(\theta) \sin(n\theta) d\theta; \quad n = 1, 2, 3, \cdots . && (1.30) \end{aligned}$$

Now we will rewrite this solution in term of a single integral by subscribing A_n and B_n into the serie form solution we get:

$$\begin{aligned} u(r,\theta) &= \frac{A_0}{2} + \sum_{n=1}^{+\infty} r^n (A_n \cos(n\theta) + B_n \sin(n\theta)) \\ &= \frac{1}{2\pi} \int_0^{2\pi} g(\phi) d\phi + \sum_{n=1}^{+\infty} r^n \left(\frac{1}{\pi} \int_0^{2\pi} g(\phi) \cos(n\phi) d\phi \right) \cos(n\theta) \\ &\quad + r^n \left(\frac{1}{\pi} \int_0^{2\pi} g(\phi) \sin(n\phi) d\phi \right) \sin(n\theta) \end{aligned}$$

If we simplify this equation we get:

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} g(\phi) \left[1 + 2 \sum_{n=1}^{+\infty} r^n (\cos(n\phi) \cos(n\theta) + \sin(n\phi) \sin(n\theta)) \right] d\phi. \tag{1.31}$$

We know that:

$$\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) = \cos(\alpha - \beta)$$

then the equation becames

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} g(\phi) \left[1 + 2 \sum_{n=1}^{+\infty} r^n (\cos(n(\theta - \phi))) \right] d\phi \tag{1.32}$$

and it's known that the exponential form of cossinus is writing as:

$$\cos(x) = \frac{\exp(ix) + \exp(-ix)}{2}$$

then :

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} g(\phi)\left[1 + 2\sum_{n=1}^{+\infty} r^n \left(\frac{\exp(n(\theta-\phi)) + \exp(-n(\theta-\phi))}{2}\right)\right] d\phi$$

when we simplify the equation we get:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} g(\phi)\left[1 + \sum_{n=1}^{+\infty}(r\exp(\theta-\phi))^n + \sum_{n=1}^{+\infty}(r\exp(-(\theta-\phi)))^n\right] d\phi \qquad (1.33)$$

the sum of a geometric serie is given by:

$$\sum_{n=1}^{+\infty} x^n = \frac{x}{1-x}$$

then the equation becames

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} g(\phi)\left[1 + \left(\frac{r\exp(\theta-\phi)}{1-\exp(\theta-\phi)}\right) + \left(\frac{r\exp(-(\theta-\phi))}{1-r\exp(-(\theta-\phi))}\right)\right] d\phi \qquad (1.34)$$

when we united the denominator then simplify the equation we get:

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} g(\phi)\left[\frac{1-r^2}{1+r^2-r\exp(\theta-\phi)-r\exp(-(\theta-\phi))}\right] d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi} g(\phi)\left[\frac{1-r^2}{1+r^2-2r\left(\frac{\exp(\theta-\phi)+\exp(-(\theta-\phi))}{2}\right)}\right] d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi} g(\phi)\left[\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}\right] d\phi \end{aligned}$$

Finally, we obtain the Poisson formula

$$u(r,\theta) = \frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{g(\phi)}{1+r^2-2r\cos(\theta-\phi)} d\phi. \qquad (1.35)$$

1.5.3 A remak on existance and uniqueness of solutions

Theorem 1.4 *The solution of Laplace equation is unique in* Ω . *Moreover* $u \in \mathbb{C}^2(\Omega) \cap \mathbb{C}(\bar{\Omega})$.

1.6 Perturbation and Asymptotic Methods.

Owing to the complexity of the PDEs involved in some mathematical models, it is not always possible to find an exact solution to an initial/boundary value problem. The next best thing in such situations is to compute an approximate solution instead. This is the idea behind the method of asymptotic expansion, which is applicable to problems that contain a small positive parameter and relies on the expansion of the solution in a series of powers of the parameter. If the series converges, then the technique is called a perturbation method; when the series diverges but is asymptotic, we have an asymptotic method.

1.6.1 Asymptotic Series

In order to state what an asymptotic series is, we need a mechanism to compare the "magnitude" of functions.

Definition 1.5 *Let f and g be two functions of a real variable x. We say that f is of order g near $x = a$ and write:*

$$f(x) = O(g(x)) \quad as \ x \longrightarrow a$$

If

$$\left|\frac{f(x)}{g(x)}\right| \ is \ bounded \ as \ x \longrightarrow a$$

We also write:

$$f(x) = o(g(x)) \quad as \ x \longrightarrow a$$

If

$$\frac{f(x)}{g(x)} \longrightarrow 0 \ as \ x \longrightarrow a$$

Definition 1.6 *A function $f(x,\varepsilon)$, where $0 < \varepsilon << 1$ is a small parameter, is said to have the asymptotic (power) series:*

$$f(x,\varepsilon) = \sum_{n=0}^{\infty} f_n(x)\varepsilon^n \ as \ \varepsilon \longrightarrow 0^+ \tag{1.36}$$

if, for any positive integer N:

$$f(x,\varepsilon) = \sum_{n=0}^{N-1} f_n(x)\varepsilon^n + O(\varepsilon^N) \ as \ \varepsilon \longrightarrow 0^+ \tag{1.37}$$

uniformly for x in some interval.

Definition 1.7 *Consider an initial boundary value problem that depends (smoothly) on a small parameter $\varepsilon > 0$. The problem obtained by setting $\varepsilon = 0$ in the equation and data functions is called the reduced (unperturbed) problem. If the reduced problem is of the same type and order as the given one and both have unique solutions, then the given problem is called a regular perturbation problem; otherwise, it is called a singular perturbation problem.*

Definition 1.8 *If the solution u of a perturbation problem has the asymptotic series:*

$$u(x,\varepsilon) = \sum_{n=0}^{N-1} u_n(x)\varepsilon^n \tag{1.38}$$

then $u - u_0$ is called a perturbation of the solution of the reduced problem.

1.6.2 Regular Perturbation Problems

Perturbation theory is particularly useful in analyzing differential equations. A small parameter ε can be added to the equation so that when $\varepsilon = 0$, the differential equation is simplified to the point of being almost trivial to solve. We can then find the perturbation series solution to the perturbed equation, and finally replace $\varepsilon = 1$ to reproduce the original equation. We define a regular perturbation problem as one whose perturbation series is a power series in ε having a non vanishing radius of convergence. A basic feature of all regular perturbation problems is that the exact solution for small but nonzero ε smoothly approaches the unperturbed or zeroth-order solution as $\varepsilon \to 0$.

1.6.3 Singular Perturbation Problems

We define a singular perturbation problem as one whose perturbation series either does not take the form of a power series or, if it does, the power series has a vanishing radius of convergence. In singular perturbation theory there is sometimes no solution to the unperturbed problem (the exact solution as a function of ε may cease to exist when $\varepsilon = 0$); when a solution to the unperturbed problem does exist, its qualitative features are distinctly different from those of the exact solution for arbitrarily small but nonzero ε. In either case, the exact solution for $\varepsilon = 0$ is fundamentally different in character from the ”neighboring” solutions obtained in the limit $\varepsilon \to 0$. If there is no such abrupt change in character, then we would have to classify the problem as a regular perturbation problem.

When dealing with a singular perturbation problem, one must take care to distinguish between the zeroth-order solution (the leading term in the perturbation series) and the solution of the unperturbed problem, since the latter may not even exist. There is no difference between these two in a regular perturbation theory, but in a singular perturbation theory the zeroth-order solution may depend on ε and may exist only for nonzero ε.

Chapter 2

ASYMPTOTIC SOLUTION FOR THE LAPLACE EQUATION ON AN PERTURBED DOMAIN

In this chapter we have tow part. First part, we are going to define Laplace equation in the perturbed domain which is a perturbed disk and we will use same mathematical tricks to deal with the perturbation which is based on two Taylor series and asymptotic series. Second part, we are going to solve our problem with tow analytic methods with separation of variable and Poisson formula . In this chapter we used the following reference [3–8].

2.1 Problem setting

We consider Laplace equation with a Robin boundary condition:

$$PDE \quad : \quad \Delta u = 0, \text{ in } \Omega. \tag{2.1}$$

$$BC \quad : \quad \alpha u + \beta \frac{\partial u}{\partial n} = g, \text{ on } \partial\Omega. \tag{2.2}$$

where Ω is given by:

$$\Omega = \{(r,\theta);\ 0 \le \theta \le 2\pi,\ 0 \le r \le 1 + \varepsilon f(\theta)\}, \tag{2.3}$$

i.e. Ω is simply a perturbation of a circle of radius 1, and

- f is smooth on $0 < \theta < 2\pi$, moreover f is $2\pi-$periodic function.
- ε is a small positive constant.
- We will require that g is $2\pi-$periodic, and belongs to $\mathbb{H}^2(\Omega)$.
- α and β are arbitrary constants and must be the same sign, moreover:
  - when $\beta = 0$ we get Dirichlet boundary condition .
  - when $\alpha = 0$ we get Newman boundary condition .
  - when $\alpha \neq 0$ and $\beta \neq 0$ we get Robin boundary condition .

Since the domain of the solution $u(x, y)$ is perturbation of a circular disk, it is useful to study the two dimensional Laplace equation in polar coordinates and expressed by the boundary value problem:

$$PDE \quad : \quad \Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \text{ in } \Omega, \tag{2.4}$$

$$BC \quad : \quad \alpha u(1+\varepsilon f(\theta),\theta,\varepsilon) + \beta \frac{\partial u}{\partial r}(1+\varepsilon f(\theta),\theta,\varepsilon) = g(\theta), \text{ on } \partial\Omega. \tag{2.5}$$

We assume the regular expansion:

$$u(r,\theta,\varepsilon) = u_0(r,\theta) + \varepsilon u_1(r,\theta) + O(\varepsilon^2) \tag{2.6}$$

Mathematically, we are asserting that the solution to the perturbed problem is simply a perturbation of the solution to the unperturbed problem. If we substitute this expansion into Laplace equation, it is easily seen that each $u_i(r;\theta)$ must satisfy

$$\Delta u_i(r;\theta) = 0, \quad i = 0, 1. \tag{2.7}$$

Now we substitute the expansion into our boundary value problem. The equation will becames

$$\Delta u = \frac{\partial^2 u_0}{\partial r^2} + \varepsilon \frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r}\left(\frac{\partial u_0}{\partial r} + \varepsilon \frac{\partial u_1}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 u_0}{\partial \theta^2} + \varepsilon \frac{\partial^2 u_1}{\partial \theta^2}\right), \text{ in } \Omega, \tag{2.8}$$

and the boundary condition:

$$\alpha u_0(1+\varepsilon f(\theta),\theta,\varepsilon)+\varepsilon\alpha u_1(1+\varepsilon f(\theta),\theta,\varepsilon)+\beta\frac{\partial u_0}{\partial r}(1+\varepsilon f(\theta),\theta,\varepsilon)+ \\ \varepsilon\beta\frac{\partial u_1}{\partial r}(1+\varepsilon f(\theta),\theta,\varepsilon)+O(\varepsilon^2)=g(\theta),\ \text{ on }\ \partial\Omega. \quad (2.9)$$

When we expand this boundary condition about $\varepsilon=0$; we find

$$u(1+\varepsilon f(\theta),\theta)=u(1,\theta)+\varepsilon f(\theta)\frac{\partial u}{\partial r}(1,\theta)+O(\varepsilon^2) \quad (2.10)$$

and

$$\frac{\partial u}{\partial r}(1+\varepsilon f(\theta),\theta)=\frac{\partial u}{\partial r}(1,\theta)+\varepsilon f(\theta)\frac{\partial^2 u}{\partial r^2}(1,\theta)+O(\varepsilon^2) \quad (2.11)$$

We will finely get:

$$\alpha u_0(1,\theta)+\varepsilon\alpha u_1(1,\theta)+\varepsilon\alpha f(\theta)\frac{\partial u_0}{\partial r}(1,\theta)+\beta\frac{\partial u_0}{\partial r}(1,\theta)+\varepsilon\beta\frac{\partial u_1}{\partial r}(1,\theta) \\ +\varepsilon\beta f(\theta)\frac{\partial^2 u_0}{\partial r^2}(1,\theta)+\varepsilon\beta\frac{\partial u_1}{\partial r}(1,\theta)+O(\varepsilon^2)=g(\theta),\ \text{ on }\ \partial\Omega. \quad (2.12)$$

Matching the powers of ε, we obtain the following boundary value problems:

$$O(\varepsilon^0):\ \begin{cases} \dfrac{\partial^2 u_0}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u_0}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u_0}{\partial\theta^2},\ \ in\ \ \Omega \\ \alpha u_0(1,\theta)+\beta\dfrac{\partial u_0}{\partial r}(1,\theta)=g_0(\theta),\ \ on\ \ \partial\Omega \end{cases} \quad (2.13)$$

and

$$O(\varepsilon^1):\ \begin{cases} \dfrac{\partial^2 u_1}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u_1}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u_1}{\partial\theta^2},\ \ in\ \ \Omega. \\ \alpha u_1(1,\theta)+\beta\dfrac{\partial u_1}{\partial r}(1,\theta)=g_1(\theta),\ \ on\ \ \partial\Omega. \end{cases} \quad (2.14)$$

where

$$g_0(\theta)=g(\theta)\ \ and\ \ g_1(\theta)=-f(\theta)\left(\alpha\frac{\partial u_0}{\partial r}(1,\theta)+\beta\frac{\partial^2 u_0}{\partial r^2}(1,\theta)\right). \quad (2.15)$$

Once the solution of unperturbed problem for u_0 is known, g_1 is available and the solution of perturbed problem for u_1 will be known as well.

Now, since the solution $u_i(r,\theta)$ sought is uni valued in the disk of radius 1, it follows that :

$$u_i(r,\theta+2\pi)=u_i(r,\theta),\quad i=0,1. \quad (2.16)$$

The solution $u_i(r,\theta)$ we are seeking is a classical solution without any singularities for $r\leq 1$,so $u_i(r,\theta)$ must be well-behaved in particular at $r=0$.

The original problem reduces to a sequence of formally identical problems for each of the u_i, each

of these satisfies Laplace equation in the interior of the unit disc with the Robin boundary condition where $g_i(\theta)$ is known in terms of the previously computed solutions for u_i.
In the next of this chapter we will solve our problem with two different methods Separation of variable and Poisson formula depending on asymptotic analysis .

2.2 Asymptotic solution by separation of variables

2.2.1 Solving Laplace equation with separation of variable method

We seek a separation of variable solution to (PDE) in the form:

$$u_i(r,\theta) = R(r)\Theta(\theta), \quad i = 0, 1. \tag{2.17}$$

Deriving and substituting term by term gives:

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0 \tag{2.18}$$

Separating the variables, we get

$$\frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} \tag{2.19}$$

Now, since left and right sides of (PDE) are functions of r and θ respectively, it follows that each side is some constant . Therefore, $\Theta(\theta)$ and $R(r)$ satisfy:

$$\frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = k \tag{2.20}$$

Thus we get two ODEs which are given by:

$$\begin{cases} r^2R''(r) + rR'(r) - kR(r) = 0, & 0 \leq r \leq 1, \\ \Theta''(\theta) + k\Theta(\theta) = 0, & 0 \leq \theta \leq 2\pi, \end{cases} \tag{2.21}$$

Now, since the solution $u_i(r,\theta)$ sought is uni-valued in the disk of radius 1, it follows that :

$$u_i(r, \theta + 2\pi) = u_i(r,\theta) \tag{2.22}$$

Hence, we must have:

$$\Theta(\theta + 2\pi) = \Theta(\theta). \tag{2.23}$$

since the solution $u_i(r,\theta)$ we are seeking is a classical solution without any singularities for $r \leq 1$, $R(r)$ must be well-behaved in particular at $r = 0$.

To obtain periodic solution, we assume that $k > 0$ and arguing as in the first chapter 1.4.2, we obtain

$$u_i(r,\theta) = \frac{A_0^i}{2} + \sum_{n=1}^{+\infty} r^n(A_n^i\cos(n\theta) + B_n^i\sin(n\theta)), \quad i = 0, 1. \tag{2.24}$$

To determine the constants A_n^i and B_n^i we use the boundary condition:

$$\alpha u_i(1,\theta) + \beta \frac{\partial u_i}{\partial r}(1,\theta) = g_i(\theta) \tag{2.25}$$

where:

$$\begin{aligned} u_i(1,\theta) &= \frac{A_0^i}{2} + \sum_{n=1}^{+\infty} (A_n^i \cos(n\theta) + B_n^i \sin(n\theta)) \quad (2.26) \\ \frac{\partial u_i}{\partial r}(1,\theta) &= \sum_{n=1}^{+\infty} n(A_n^i \cos(n\theta) + B_n^i \sin(n\theta)) \quad (2.27) \end{aligned}$$

This gives:

$$g_i(\theta) = \alpha \left(\frac{A_0^i}{2} + \sum_{n=1}^{+\infty} (A_n^i \cos(n\theta) + B_n^i \sin(n\theta)) \right) + \beta \left(\sum_{n=1}^{+\infty} n(A_n^i \cos(n\theta) + B_n^i \sin(n\theta)) \right)$$

$$g_i(\theta) = \alpha \frac{A_0^i}{2} + \sum_{n=1}^{+\infty} (\alpha + \beta n) A_n^i \cos(n\theta) + (\alpha + \beta n) B_n^i \sin(n\theta) \tag{2.28}$$

Noting that $(\alpha + \beta n)A_n^i$ and $(\alpha + \beta n)B_n^i$ are the Fourier coefficients, and therefore can be determined by:

$$(\alpha + \beta n)A_n^i = \frac{1}{\pi} \int_0^{2\pi} g_i(\theta) \cos(n\theta) d\theta; \quad n = 0, 1, 2, \cdots \tag{2.29}$$

and

$$(\alpha + \beta n)B_n^i = \frac{1}{\pi} \int_0^{2\pi} g_i(\theta) \sin(n\theta) d\theta; \quad n = 1, 2, 3, \cdots . \tag{2.30}$$

for $i = 0, 1$.

Once the solution of unperturbed problem for u_0 is known, g_1 is available and the solution of perturbed problem of order $O(\varepsilon)$ for u_1 will be known as well. To be precise, g_1 is given by

$$\begin{aligned} g_1(\theta) &= -f(\theta) \left(\alpha \frac{\partial u_0}{\partial r}(1,\theta) + \beta \frac{\partial^2 u_0}{\partial r^2}(1,\theta) \right), \\ &= -f(\theta) \left(\sum_{n=1}^{+\infty} \alpha n(A_n^0 \cos(n\theta) + B_n^0 \sin(n\theta)) + \beta n(n-1)(A_n^0 \cos(n\theta) + B_n^0 \sin(n\theta)) \right) \end{aligned}$$

i.e.

$$g_1(\theta) = -f(\theta) \sum_{n=1}^{+\infty} (\alpha n + \beta n(n-1))(A_n^0 \cos(n\theta) + B_n^0 \sin(n\theta)). \tag{2.31}$$

2.3 Asymptotic solution by Poisson's formula

In this section we consider the Dirichlet problem (where $\alpha = 1, \ \beta = 0$) in the unit disk. We assume again the regular expansion:

$$u(r,\theta,\varepsilon) = u_0(r,\theta) + \varepsilon u_1(r,\theta) + O(\varepsilon^2) \tag{2.32}$$

where

$$O(\varepsilon^0): \begin{cases} \dfrac{\partial^2 u_0}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u_0}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u_0}{\partial \theta^2}, & in \ \ \Omega \\ u_0(1,\theta) = g_0(\theta), & on \ \ \partial\Omega \end{cases} \tag{2.33}$$

and

$$O(\varepsilon): \begin{cases} \dfrac{\partial^2 u_1}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u_1}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u_1}{\partial \theta^2}, & in \ \ \Omega. \\ u_1(1,\theta) = g_1(\theta), & on \ \ \partial\Omega. \end{cases} \tag{2.34}$$

and we find closed form to the solution named Poisson formula it express any harmonic function inside a circle in term of it's boundary condition values.

Arguing as in the first chapter 1.5.2, we can show that the solution of the two problems are given by Poisson's formulas

$$u_i(r,\theta) = \frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{g_i(\phi)}{1+r^2-2r\cos(\theta-\phi)}d\phi, \quad i = 0, 1. \tag{2.35}$$

Recalling that

$$g_1(\theta) = -f(\theta)\left(\alpha\frac{\partial u_0}{\partial r}(1,\theta)\right),$$

i.e.

$$g_1(\theta) = -f(\theta)\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{g_0(\phi)}{\cos(\theta-\phi)-1}d\phi\right). \tag{2.36}$$

Once the solution of unperturbed problem for u_0 is known, g_1 is available and the solution of perturbed problem for u_1 will be known as well.

Chapter 3

FINITE DIFFERENCE FOR LAPLACE EQUATION

In this chapter we will introduce finite difference method the most common numerical methods for solution of partial deferential equations. Then we will solve Laplace equation in the two cases the unperturbed and the perturbed case. .In this chapter we have use the following reference [9–12].

3.1 Finite difference method

Many engineering applications involve boundary value problems that require solving elliptic partial differential equations (PDEs). The discretization of such boundary-value problems leads to linear system $Au = B$, where u is the set of unknowns corresponding to the unknown variables in the PDE and B is the set of discrete values of the known function in the PDE.

Finite deference methods are common numerical methods for solution of partial deferential equations, witch involve discretization of the spatial domain, the deferential equation, and boundary conditions, and a subsequent solution of a large system of linear equations for the approximate solution values in the nodes of the numerical mesh. Simplest discretizations of second-order deferential operators commonly have first or second order accuracy. The resulting large system of linear equations involves a sparse matrix and are solved by iterative methods (Jacobi, Gauss-Seigel, etc.) or Gaussian elimination/LU decomposition,which have been significantly optimized for sparse matrices.

3.1.1 Finite difference approximations

finite deference approximations to partial derivatives are based on Taylor series expansions of a function of one or more variables. Recall that the Taylor series expansion for a function of one variable is given by:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + ... + \frac{h^n}{n!}f^{(n)}(x). \tag{3.1}$$

For a function of more than one independent variable we have the derivatives replaced by partial derivatives. We give here the case of 2 independent variables:

$$f(x+h,y+k) = f(x,y)+hf_x(x,y)+kf_y(x,y)+\frac{h^2}{2!}f_{xx}(x,y)+\frac{hk}{2!}f_{yx}(x,y)+\frac{k^2}{2!}f_{yy}(x,y)+\cdots . \tag{3.2}$$

The remainder can be written in the form:

$$\frac{1}{n!}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^{(n)} f(x,y).$$

Depending on Taylor series expansions of a function we can define the finite deference approximations to partial derivative of a function such that:

- difference approximations to first derivatives:
  - forward deference approximation:

$$\frac{\partial f}{\partial x} = \frac{f_{i+1} - f_i}{\Delta x}. \tag{3.3}$$

- backward deference approximation:

$$\frac{\partial f}{\partial x}=\frac{f_i-f_{i-1}}{\Delta x}. \tag{3.4}$$

- centered deference approximation:

$$\frac{\partial f}{\partial x}=\frac{f_{i+1}-f_{i-1}}{2\Delta x}. \tag{3.5}$$

- difference approximations to second derivatives:

$$\frac{\partial^2 f}{\partial x^2}=\frac{f_{i+1}-2f_i+f_{i-1}}{(\Delta x)^2}. \tag{3.6}$$

3.2 Solving Laplace equation with finite difference method on circular domain

We choose a grid which the grid points are half-integered in radial direction and integered in azimuthal direction,that is:

$$r_i = (i-\frac{1}{2})\triangle r,\ \ \triangle r=\frac{2}{2N+1},\ \ i=1,2,.......N+1, \tag{3.7}$$

$$\theta_J = (j-1)\triangle\theta,\ \ \triangle\theta=\frac{2\pi}{M},\ \ J=1,2,........M+1 \tag{3.8}$$

Let the discrete values be denoted by:

$$u_{i,j}=u(r_i,\theta_J)\ \text{ and }\ g_j=g(\theta_J) \tag{3.9}$$

The value at any grid point requires the knowledge of the solution at the five points below. We describe this by the following computational molecule:

By the choice of the radial mesh width, the boundary values are defined on the grid points.

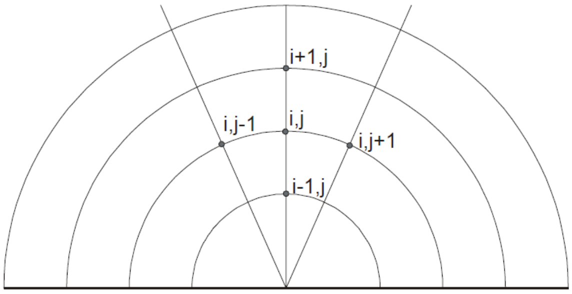

Figure 3.1: five points discretization for laplace equation

Using the centered difference method to discretize equation and for $i=1,2,.......N$ and $J=1,2,........M$ we have:

$$\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{(\triangle r)^2}+\frac{1}{r_i}\frac{u_{i+1,j}-u_{i-1,j}}{2(\triangle r)}+\frac{1}{(r_i)^2}\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{(\triangle\theta)^2} \tag{3.10}$$

Among the above representations, the numerical boundary values are given by

$$u_{N+1;j} = g_j, \;\; u_{i;0} = u_{i;M}, \text{ and } u_{i;1} = u_{i;M+1} \tag{3.11}$$

since the solution is 2π periodic in θ. At $i = 1$, we immediately observe that $r_{1/2} = 0$, so the coefficient of $u_{0,j}$ is zero. This implies that the scheme does not need any extrapolation for the inner numerical boundary value $u_{0,j}$ so that there is no pole condition needed. It is also easy to check that the matrix of linear. Let us order the unknown $u_{i,j}$ by first grouping the same ray then moving counterclockwise to caver the whole domain. thus, the unknown vector v is defined by:

$$v = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ \vdots \\ u_M \end{pmatrix}, u_J = \begin{pmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ \vdots \\ \vdots \\ u_{N,J} \end{pmatrix} \tag{3.12}$$

the remain problem is to solve a large sparse linear system $Av = b$ where A can be written as

$$A = \begin{pmatrix} T-2D & D & & & & D \\ D & T-2D & D & & & \\ & D & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & T-2D & D \\ D & & & & D & T-2D \end{pmatrix} \tag{3.13}$$

where D is the diagonal matrix

$$D = \begin{pmatrix} \beta_1 & & 0 \\ & \ddots & \\ 0 & & \beta_N \end{pmatrix}, \text{ with } \beta_i = \frac{1}{(i-\frac{1}{2})^2(\triangle\theta)^2}, 1 \leq i \leq N \tag{3.14}$$

and

$$T = \begin{pmatrix} -2 & 1+\lambda_1 & & & & 0 \\ 1-\lambda_2 & -2 & 1+\lambda_2 & & & \\ & 1-\lambda_3 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1-\lambda_{N-1} & -2 & 1+\lambda_{N-1} \\ 0 & & & & 1-\lambda_N & -2 \end{pmatrix} \tag{3.15}$$

with

$$\lambda_I = \frac{1}{2(i-\frac{1}{2})}, 1 \leq i \leq N.$$

The unknown b is defined by

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{pmatrix}, b_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -(1+\lambda_N)g_j \end{pmatrix} \tag{3.16}$$

Remark 3.1 *For the Neumann or Robin problem, we still use the same grid but with different choice of* $\triangle r = \frac{1}{N}$ *. With the choice of this mesh width, the discrete values of u are defined midway between boundary so that the first derivative can be centered on the grid points.That is, at r = 1:*

$$\alpha u + \beta \frac{\partial u}{\partial r} = \alpha \frac{u_{N+1,j} - u_{N,j}}{2} + \beta \frac{u_{N+1,j} - u_{N,j}}{\Delta r} \tag{3.17}$$

3.3 Using finite differences method to compute the asymptotic solution

Assuming that the solution u of the perturbed problem

$$PDE \quad : \quad \Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \ \text{ in } \ \Omega, \tag{3.18}$$

$$BC \quad : \quad \alpha u(1+\varepsilon f(\theta), \theta, \varepsilon) + \beta \frac{\partial u}{\partial r}(1+\varepsilon f(\theta), \theta, \varepsilon) = g(\theta), \ \text{ on } \ \partial\Omega. \tag{3.19}$$

can be expressed by the regular expansion:

$$u(r,\theta,\varepsilon) = u_0(r,\theta) + \epsilon u_1(r,\theta) + O(\varepsilon^2) \tag{3.20}$$

we can now apply the above finit difference method to compute both u_0 and u_1. Inparticular,

- when computing u_0, we take $g_j = g^0(\theta_J)$ in the last row of b_j given by (3.16).
- once u_0 is known, g_1 is available by the formula

  $$g^1(\theta) = -f(\theta)\alpha \frac{\partial u_0}{\partial r}(1,\theta), \tag{3.21}$$

  so we take

  $$g_j^1 = -f_j(u_{M+1,j}^0 - u_{M,j}^0)/h \tag{3.22}$$

  and when computing u_1, we take $g_j = g^1(\theta_J)$ in the last row of b_j given by (3.16).

Chapter 4

SOME NUMERICAL RESULTS

In this chapter we will solve the boundary value problem of Laplace equation where we will give the unknown $g(\theta)$ and $f(\theta)$. We will solve the Laplace equation with three methods separation of variable, Poisson formula and finite difference.Then,we will make some changes and some numerical result will be given.

We consider Laplace's equation in polar coordinates, expressed by the boundary value problem

$$\begin{aligned} PDE &: \quad \Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \ \text{ in } \Omega, && (4.1) \\ BC &: \quad u(1+\varepsilon f(\theta), \theta, \varepsilon) = sin(2\theta), \ \text{ on } \partial\Omega. && (4.2) \end{aligned}$$

Where Ω is given by:

$$\Omega = \{(r,\theta);\ 0 \le \theta \le 2\pi,\ 0 \le r \le 1 + \epsilon f(\theta)\}, \tag{4.3}$$

Now, we solve Laplace equation with two analytic methods: separation of variable and Poisson formula. Then, we use a numerical method, which is finite difference. We give two examples of function f in boundary perturbation with two choices for ε in each one.

4.1 Example 1: Boundary perturbation with $f(\theta) = sin(9\theta)$

For $\epsilon = 0.1$

Solution of Laplace equation with separation of variable 1

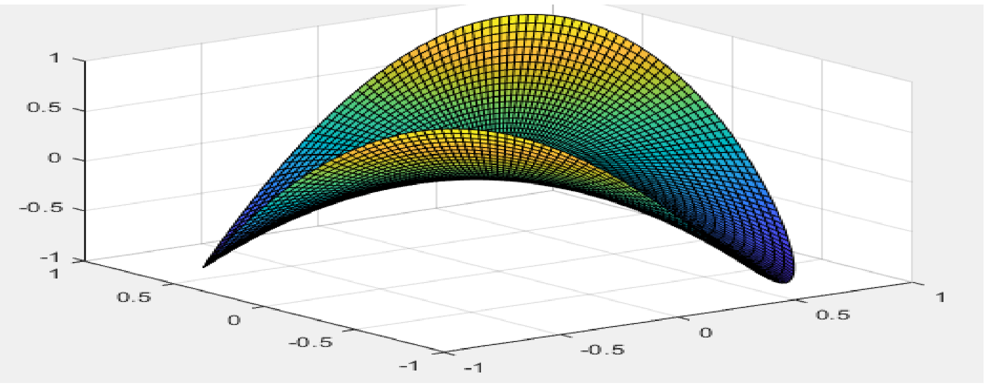

Figure 4.1: Regular solution u_0

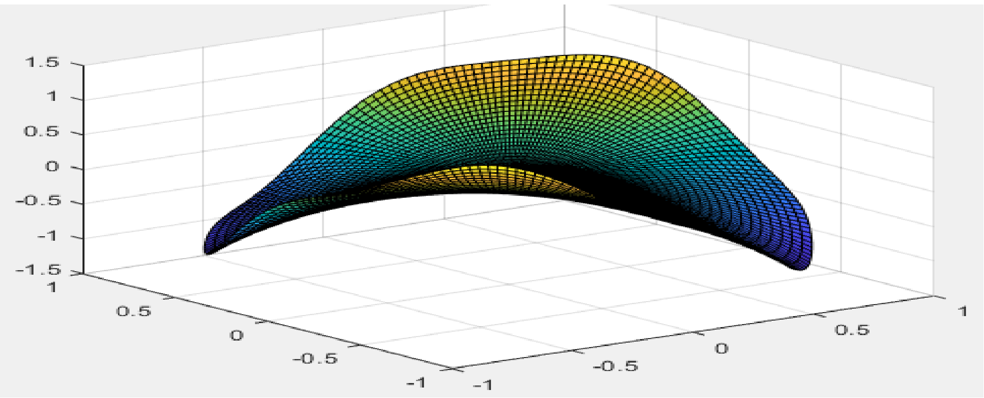

Figure 4.2: Perturbed solution solution $u_0 + \varepsilon u_1$

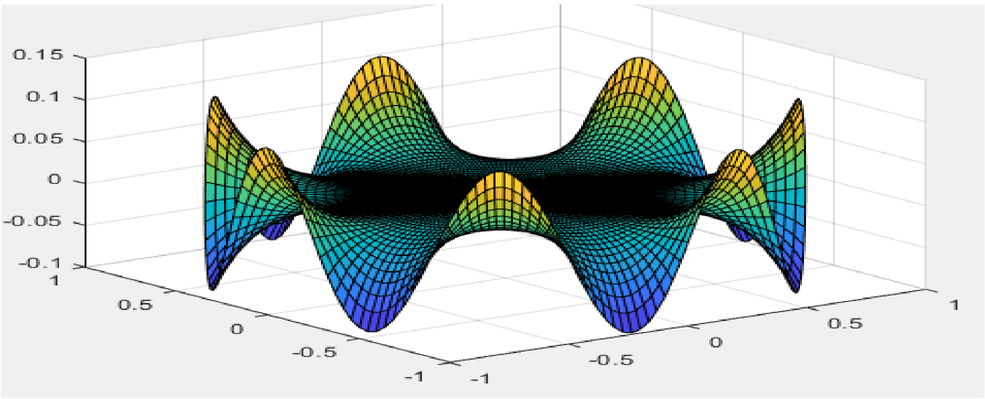

Figure 4.3: Perturbation value εu_1

Solution of Laplace equation with Poisson formula 2

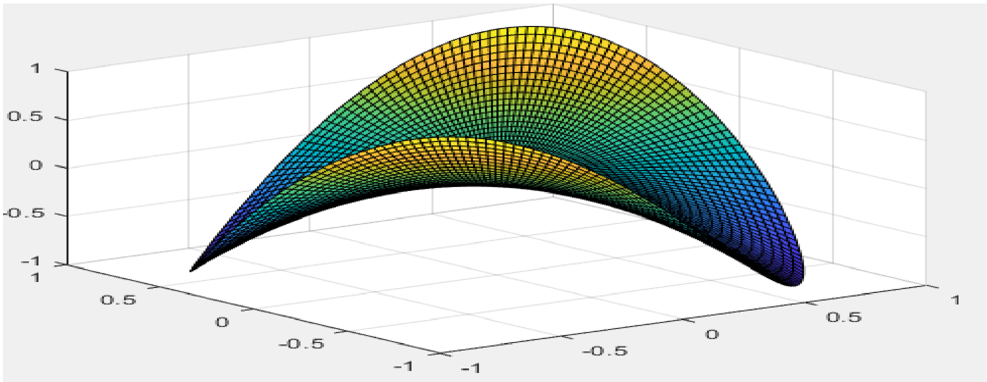

Figure 4.4: Regular solution u_0

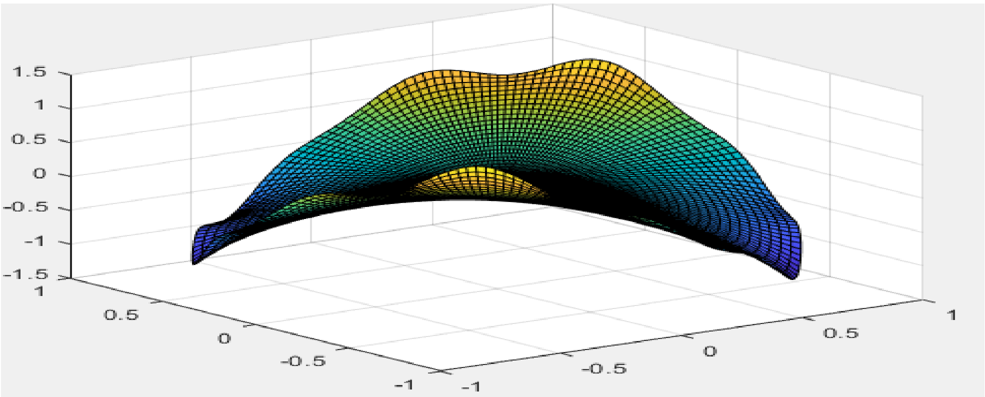

Figure 4.5: Perturbed solution solution $u_0 + \varepsilon u_1$

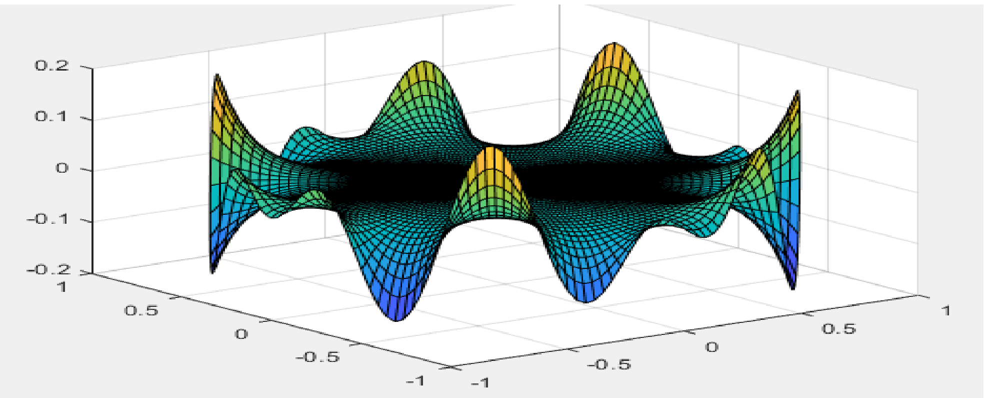

Figure 4.6: Perturbation value εu_1

Solution of Laplace equation with finite difference 3

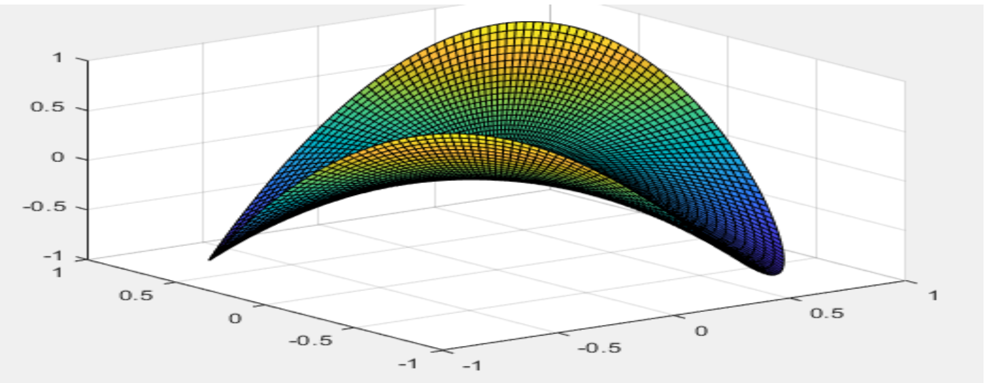

Figure 4.7: Regular solution u_0

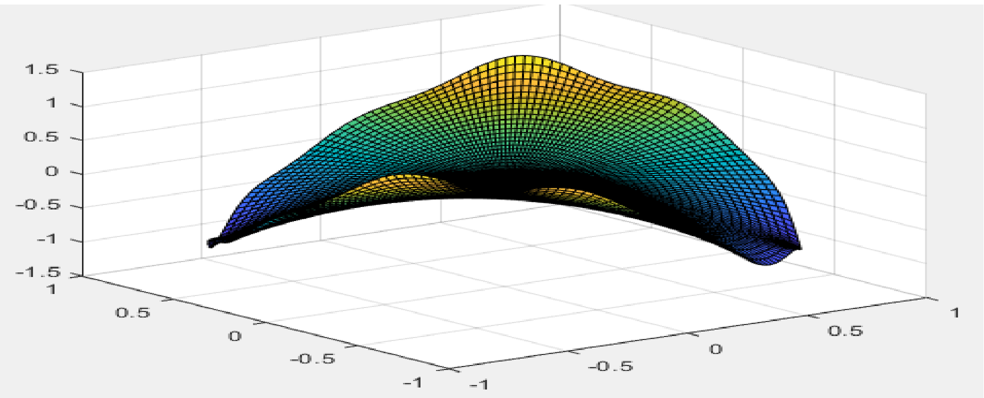

Figure 4.8: Perturbed solution solution $u_0 + \varepsilon u_1$

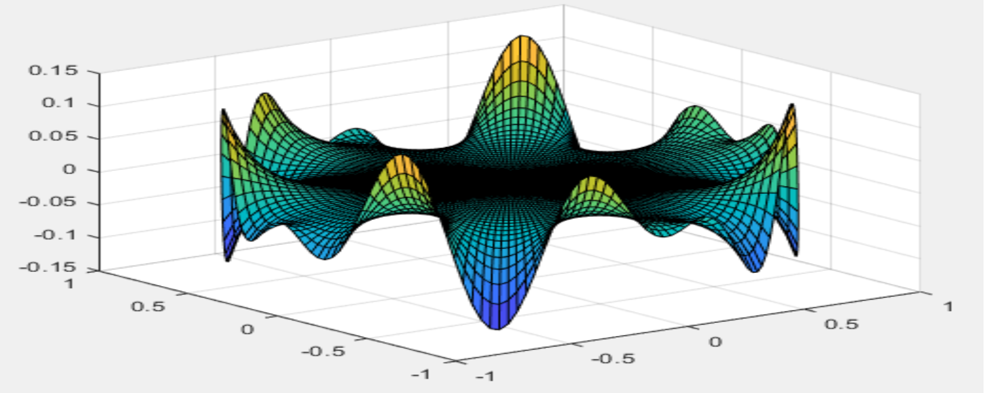

Figure 4.9: Perturbation value εu_1

For $\epsilon = 0.01$

Solution of Laplace equation with separation of variable

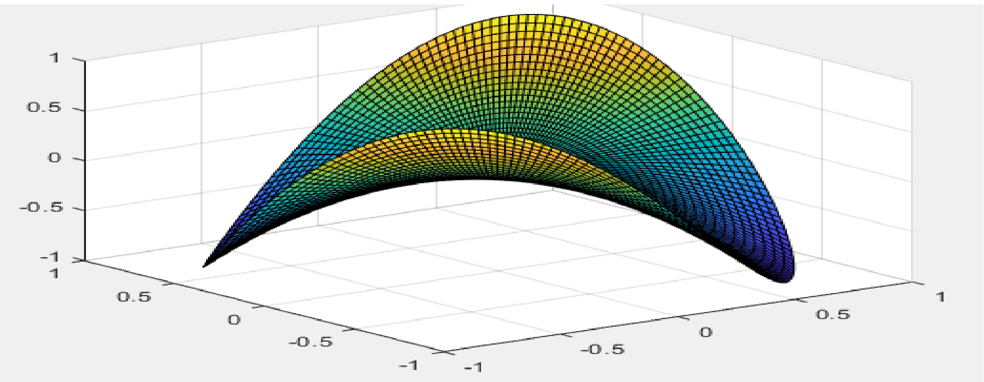

Figure 4.10: Regular solution u_0

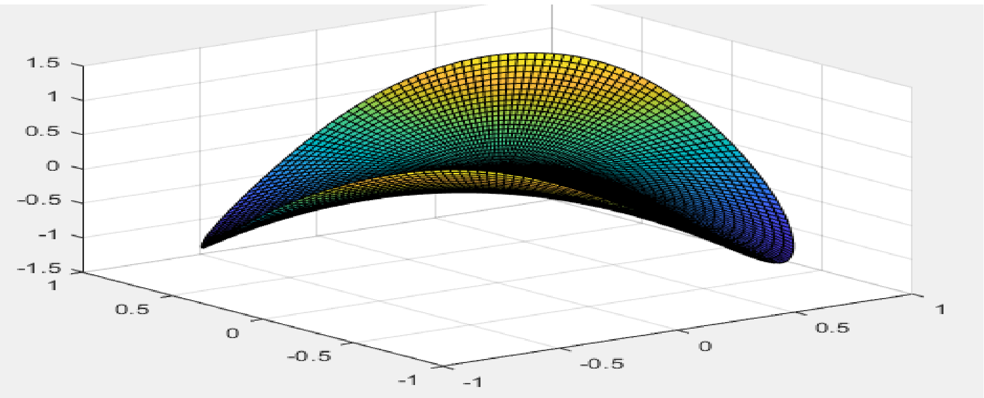

Figure 4.11: Perturbed solution solution $u_0 + \varepsilon u_1$

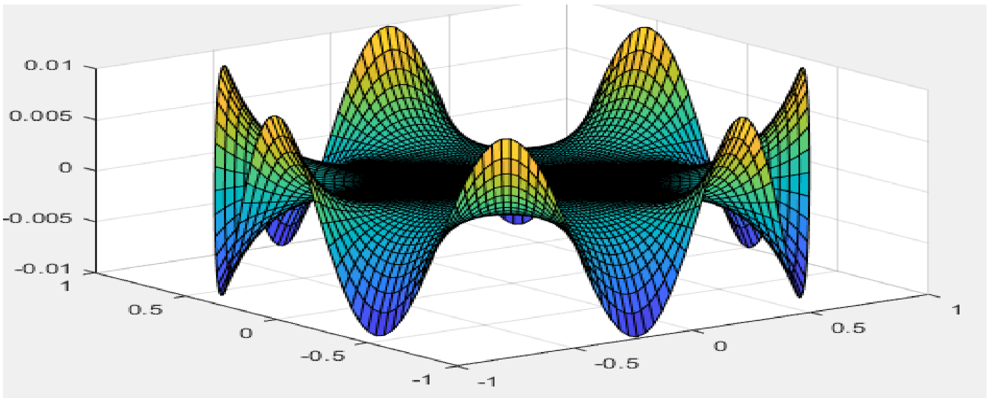

Figure 4.12: Perturbation value εu_1

Solution of Laplace equation with Poisson formula

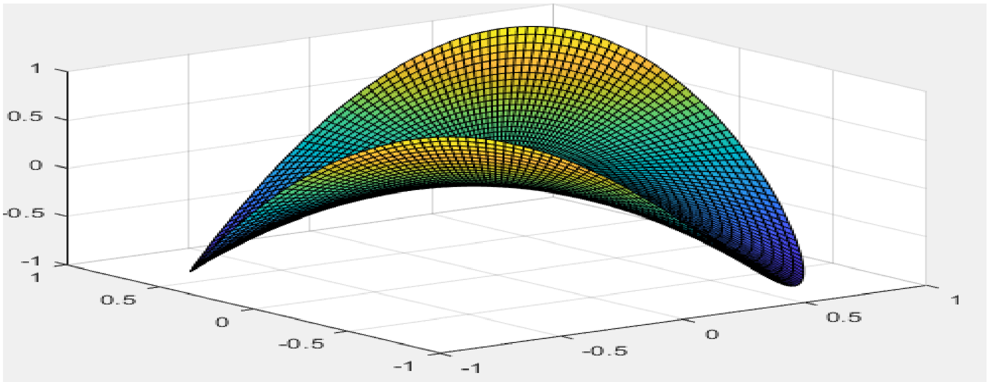

Figure 4.13: Regular solution u_0

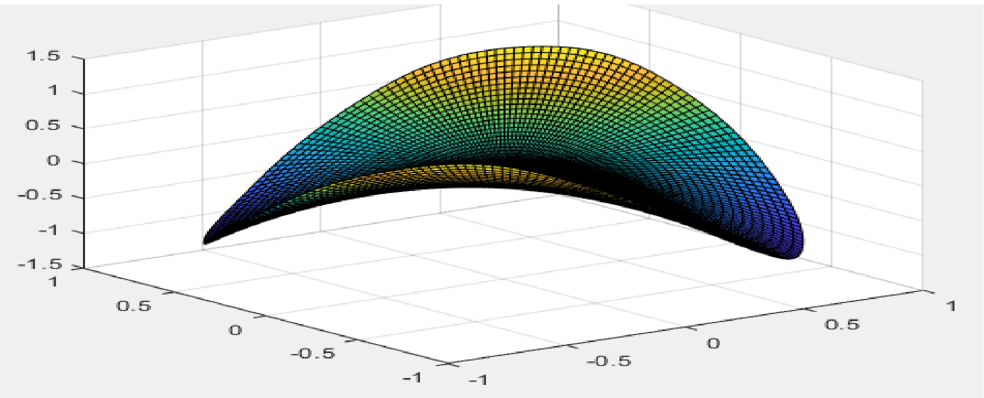

Figure 4.14: Perturbed solution solution $u_0 + \varepsilon u_1$

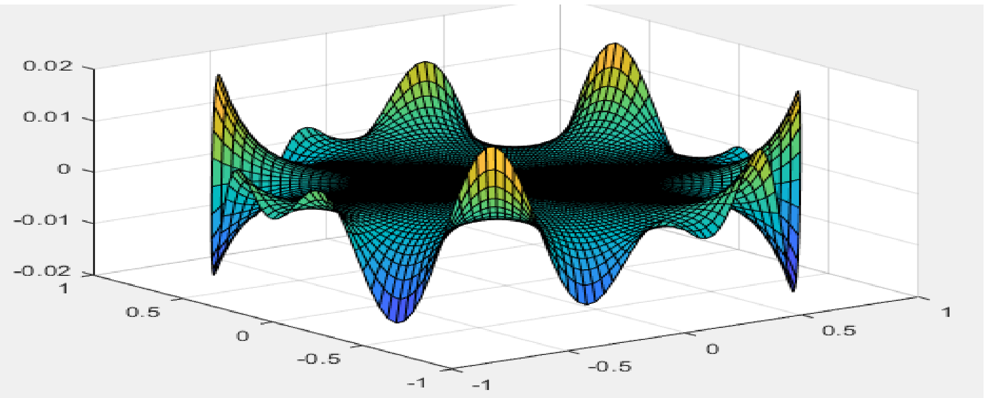

Figure 4.15: Perturbation value εu_1

Solution of Laplace equation with finite difference

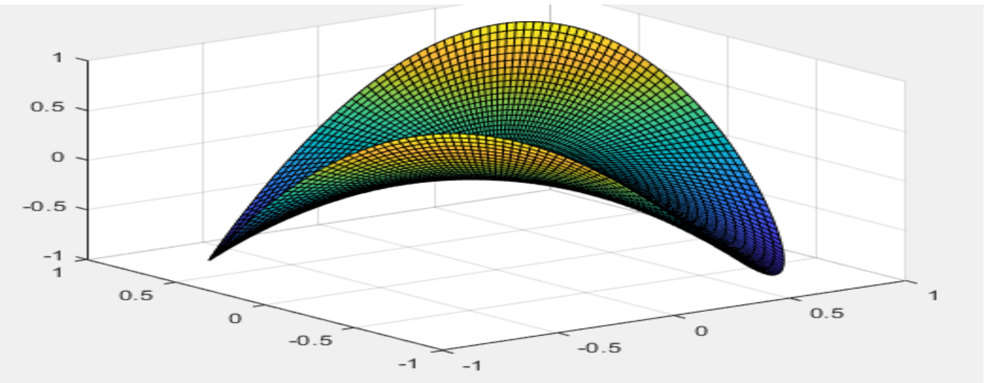

Figure 4.16: Regular solution u_0

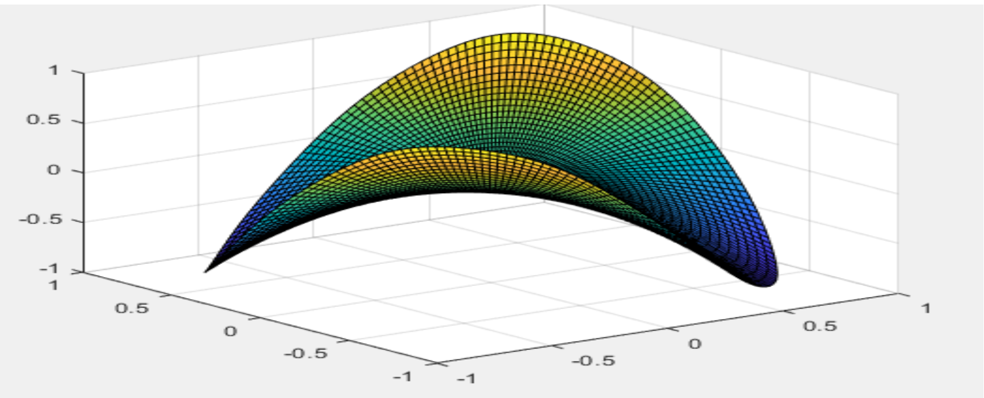

Figure 4.17: Perturbed solution solution $u_0 + \varepsilon u_1$

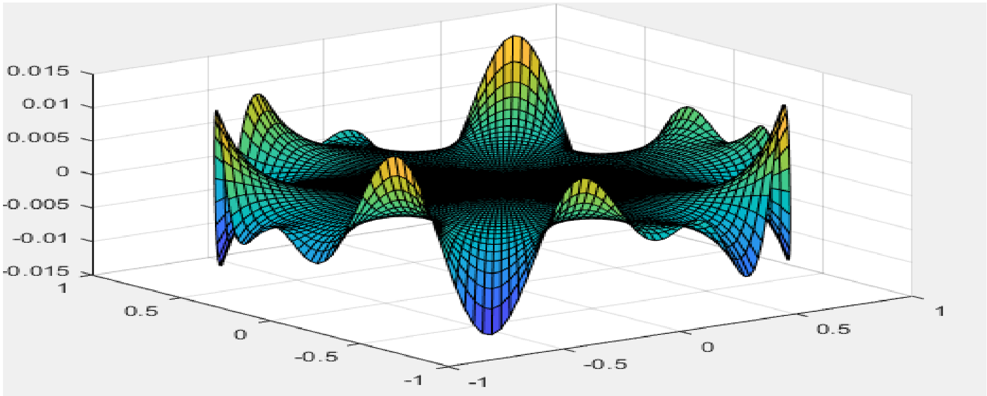

Figure 4.18: Perturbation value εu_1

4.2 Example 2: Boundary perturbation with $f(\theta) = sin(3\theta) + cos(5\theta)$

For $\epsilon = 0.1$**:**

Solution of Laplace equation with separation of variable

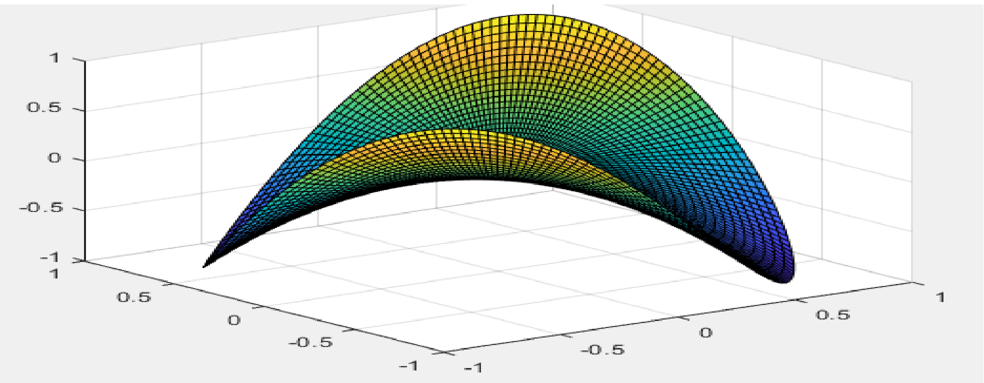

Figure 4.19: Regular solution u_0

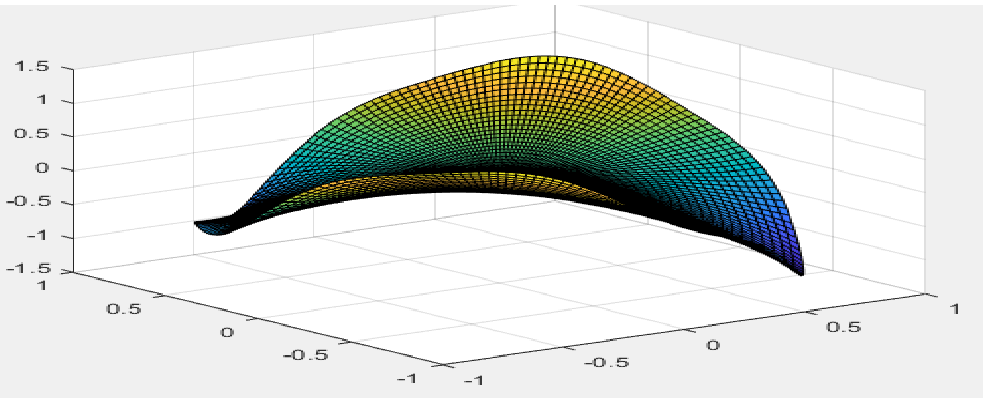

Figure 4.20: Perturbed solution solution $u_0 + \varepsilon u_1$

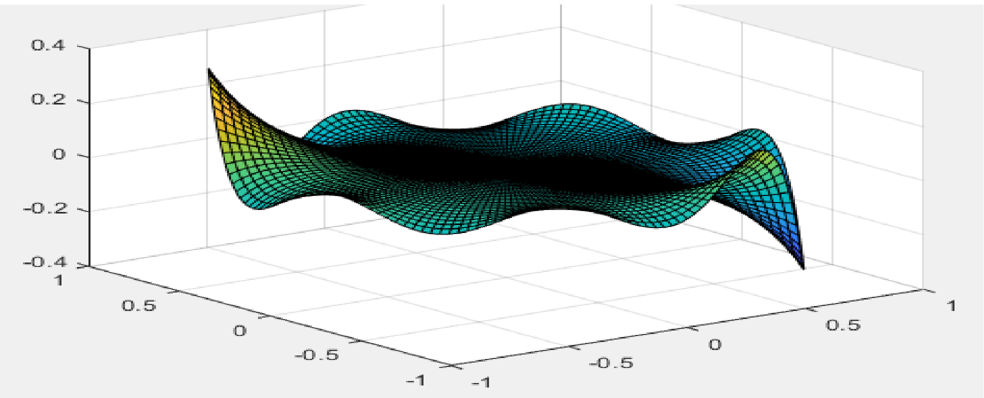

Figure 4.21: Perturbation value εu_1

Solution of Laplace equation with Poisson formula

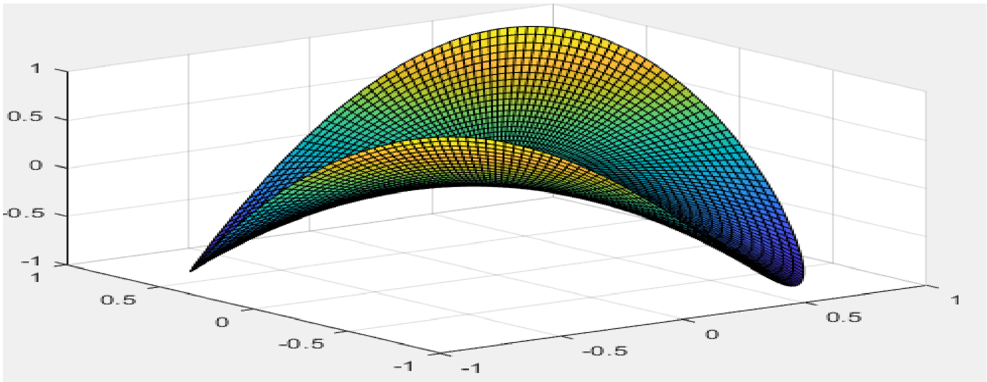

Figure 4.22: Regular solution u_0

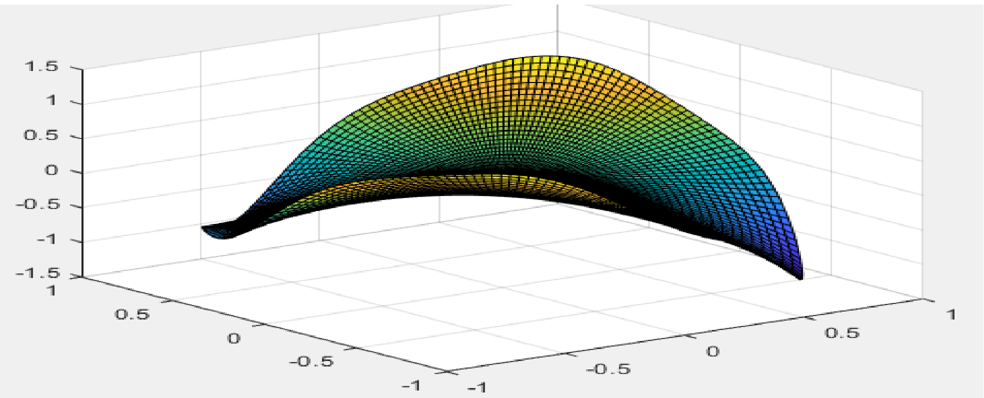

Figure 4.23: Perturbed solution solution $u_0 + \varepsilon u_1$

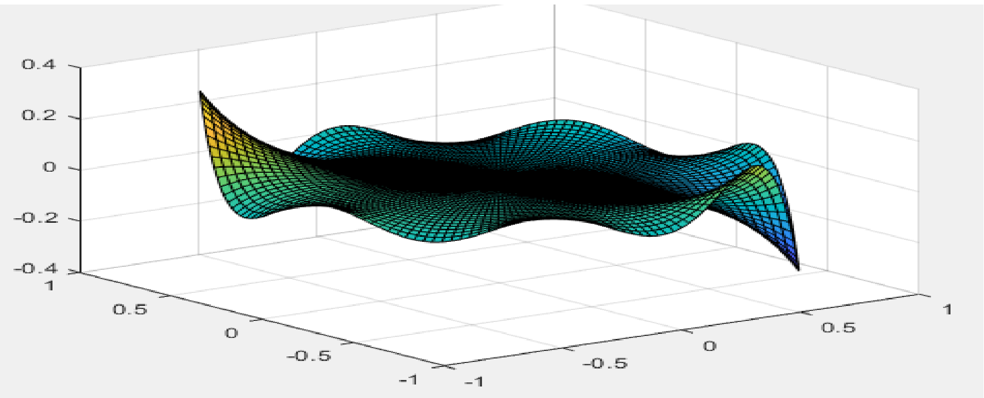

Figure 4.24: Perturbation value εu_1

Solution of Laplace equation with finite difference

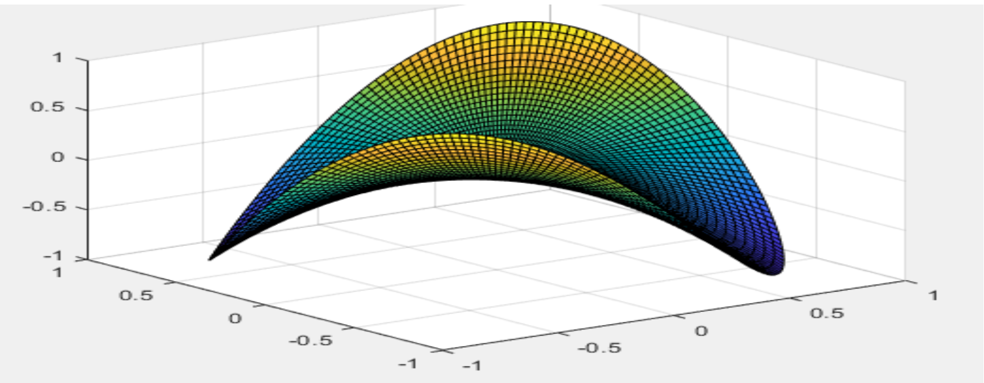

Figure 4.25: Regular solution u_0

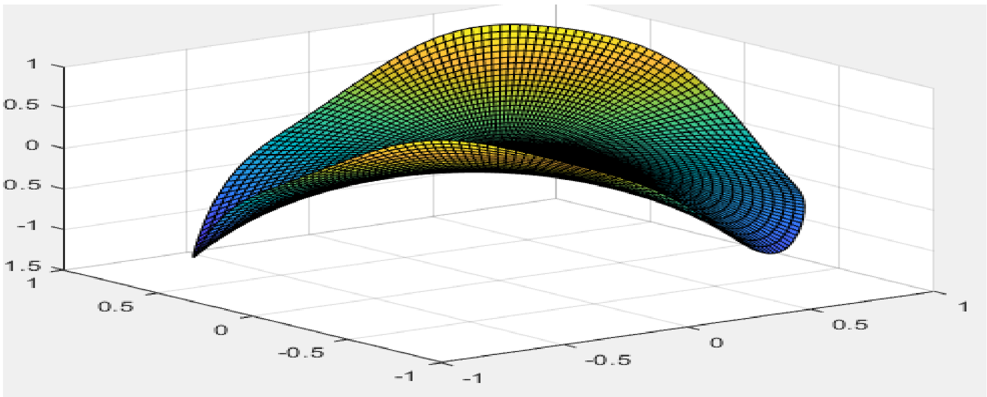

Figure 4.26: Perturbed solution solution $u_0 + \varepsilon u_1$

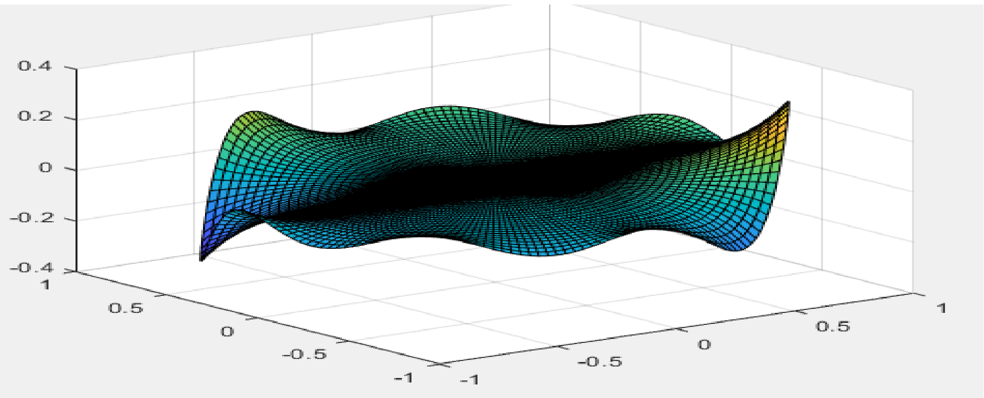

Figure 4.27: Perturbation value εu_1

For $\epsilon = 0.01$

Solution of Laplace equation with separation of variable

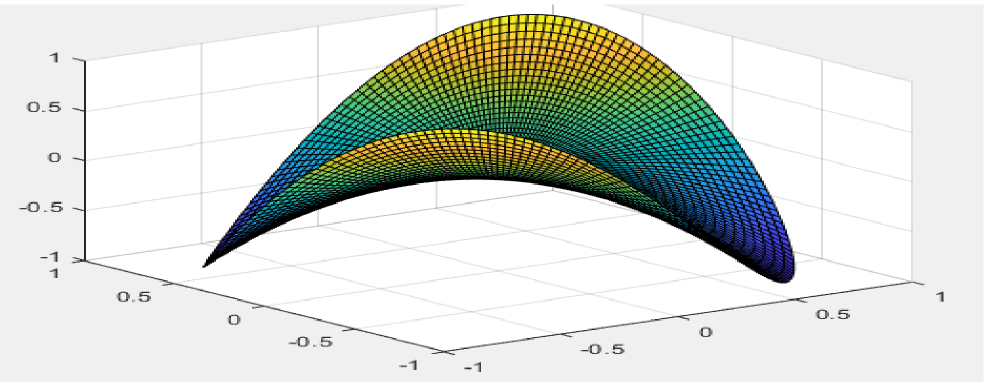

Figure 4.28: Regular solution u_0

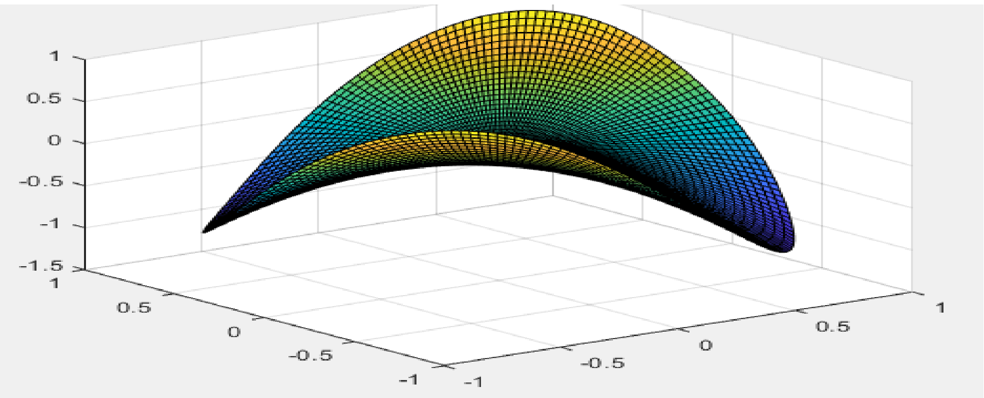

Figure 4.29: Perturbed solution solution $u_0 + \varepsilon u_1$

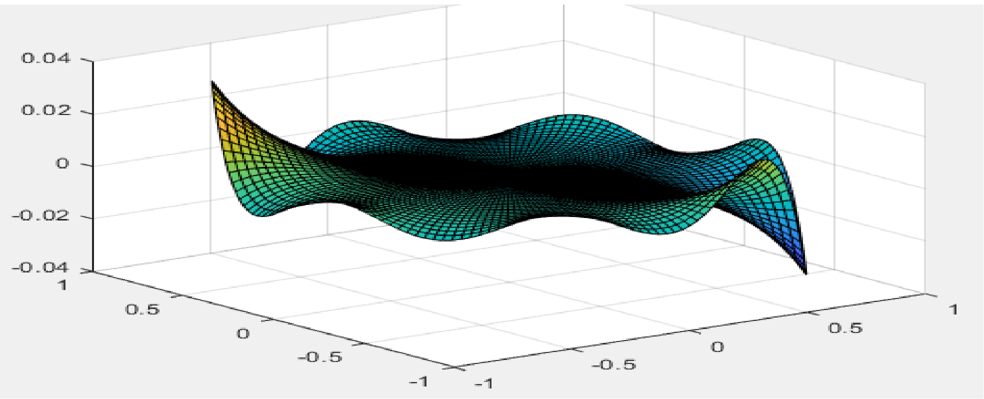

Figure 4.30: Perturbation value εu_1

Solution of Laplace equation with Poisson formula

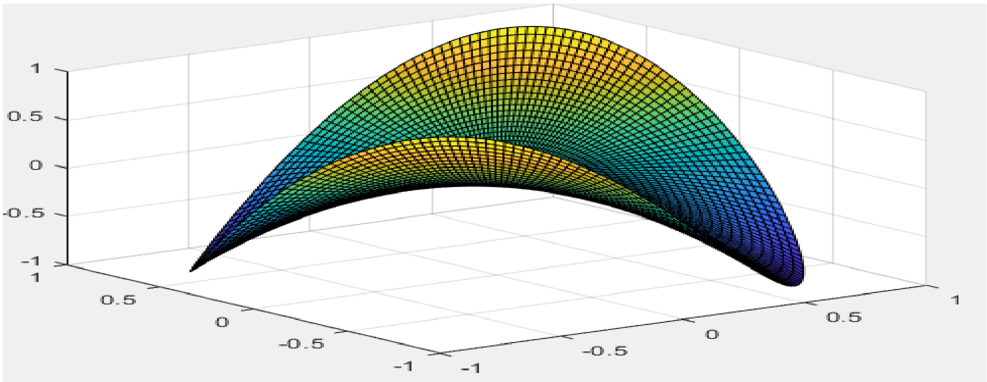

Figure 4.31: Regular solution u_0

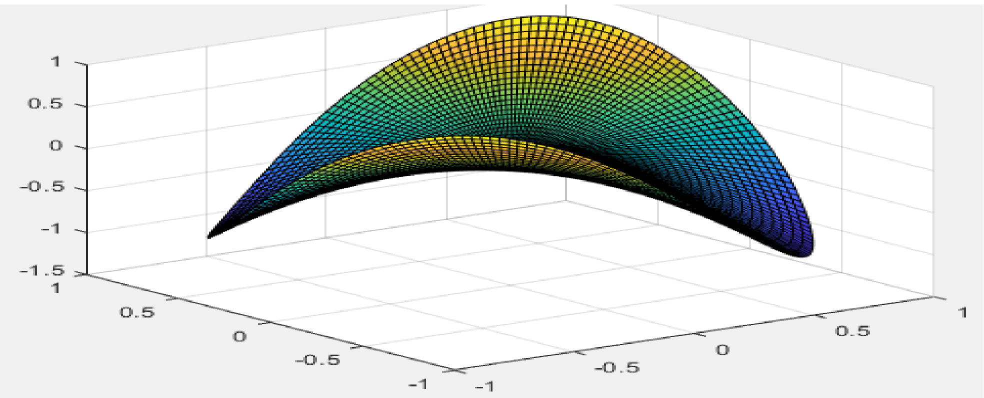

Figure 4.32: Perturbed solution solution $u_0 + \varepsilon u_1$

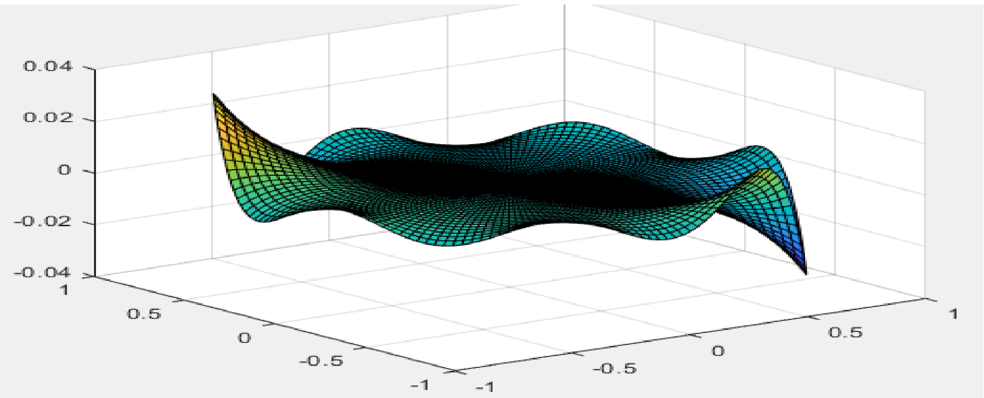

Figure 4.33: Perturbation value εu_1

Solution of Laplace equation with finite difference

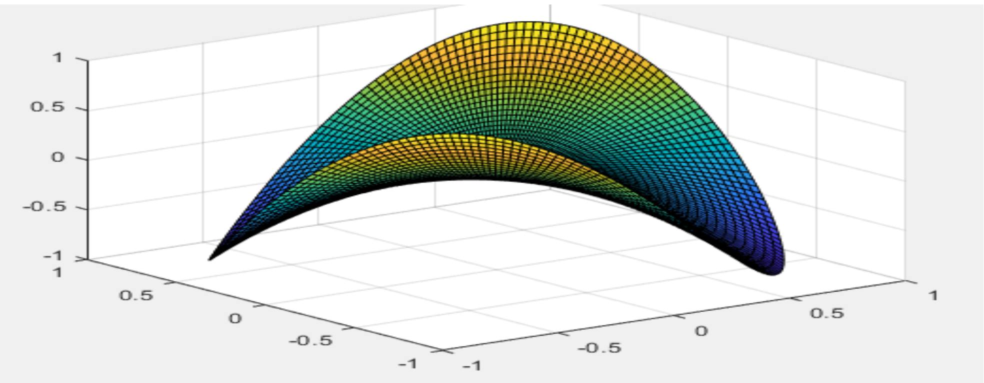

Figure 4.34: Regular solution u_0

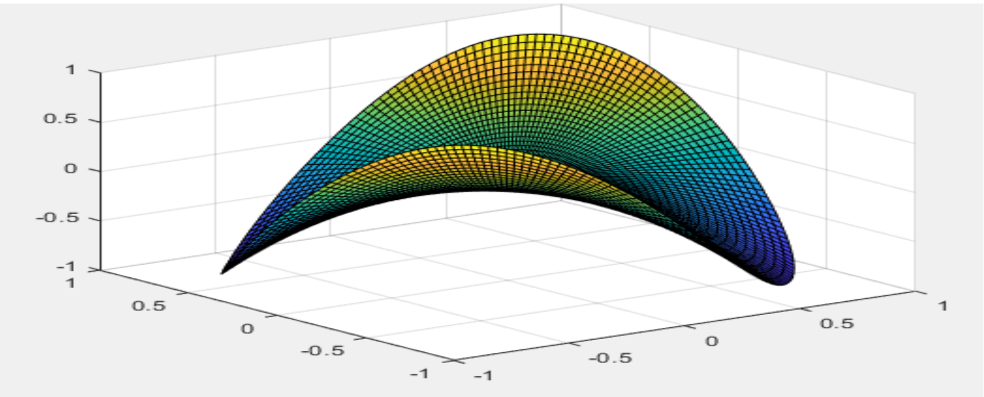

Figure 4.35: Perturbed solution solution $u_0 + \varepsilon u_1$

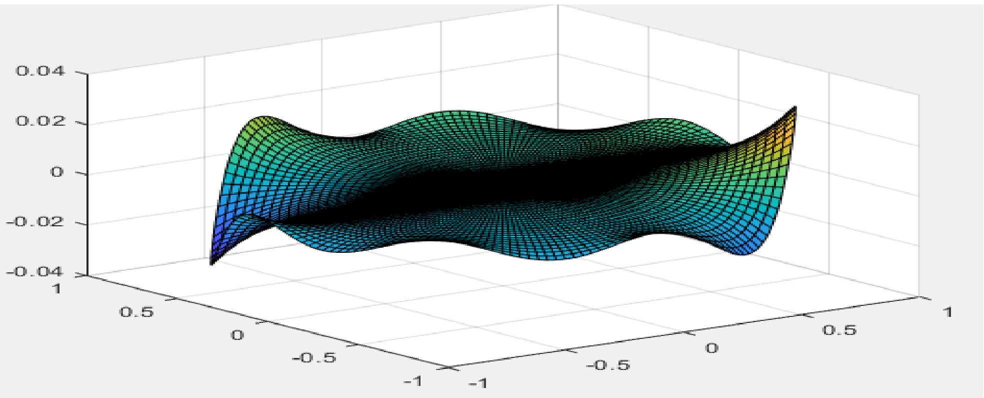

Figure 4.36: Perturbation value εu_1

ALGORITHM 1: Separation of variable method for Laplace equation .

```
Input  : M– Discritization number for R
         N – Discritization number for theta
         h – Discritization step for R
         k – Discritization step for theta
         g0– Boundary condition
         f – Perturbation function
         e – Perturbation value epsilon
         X – Length of the series
Output: u0– Regular solution
        u –Perturbed solution
        u-u0 – perturbation value
r ← [0 : h : M]                                              /* Discritization for R */
th← [0 : k : N]                                              /* Discritization for theta */
$A0 = (1/2pi)int(g0dth)$                                     /* Calculating the Fourier coefficient A0 */
for i ← M+1 do
    for j ← N+1 do
        for n ← X+1 do
            $A0_n = int(g0cos(nth(j)))$                      /* Calculating the Fourier coefficient $A0_n$ */
            $B0_n = int(g0sin(nth(j)))$                      /* Calculating the Fourier coefficient $B0_n$ */
            $F0 = r(i)^n(A0_n(n)cos(nth(j)) + B0_n(n)sin(nth(j)))$   /* the sum of the serie */
        end
        $u0 = A0 + sum(F0)$
    end
end
$g1 = -f(a(diff)(u0, r)(1, th) + b$(diff)((diff)(u0,r)(1,th)))
A1=(1/2pi)int(g0dth)                                         /* Calculating the Fourier coefficient A1 */
for i ← M+1 do
    for j ← N+1 do
        for n ← X+1 do
            $A1_n = int(g1cos(nth(j)))$                      /* Calculating the Fourier coefficient $A1_n$ */
            $B1_n = int(g1sin(nth(j)))$                      /* Calculating the Fourier coefficient $B1_n$ */
            $F1 = r(i)^n(A1_n(n)cos(nth(j)) + B1_n(n)sin(nth(j)))$   /* the sum of the serie */
        end
        $u1 = A1 + sum(F1)$
    end
end
u=u0+eu1                                                     /* the general solution */
draw u0
draw u
draw u-u0
```

ALGORITHM 2: Poisson formula method for Laplace equation .

```
Input  : M– Discritization number for R
         N – Discritization number for theta
         h – Discritization step for R
         k – Discritization step for theta
         g0– Boundary condition
         f – Perturbation function
         e – Perturbation value epsilon
Output: u0– Regular solution
         u –Perturbed solution
         (u-u0)– perturbation value
r ← [0 : h : M]                                    /* Discritization for R */
th← [0 : k : N]                                    /* Discritization for theta */
for i ← M+1 do
    for j ← N+1 do
        $u0 = int((1 - (r(i)^2))/(2pi))g0/(1 + (r(i)^2) - 2r(i)cos(th(j) - th))$
    end
end
g1=-f(adiff(u0,r)(1,th)+bdiff(diff(u0,r)(1,th)))
for i ← M+1 do
    for j ← N+1 do
        $u1 = int((1 - (r(i)^2))/(2pi))g1/(1 + (r(i)^2) - 2r(i)cos(th(j) - th))$
    end
end
u=u0+eu1                                           /* the general solution */
draw u0
draw u
draw u-u0
```

ALGORITHM 3: Finite difference method for Laplace equation .

```
Input  : M– Discritization number for R
         N – Discritization number for theta
         h – Discritization step for R
         k – Discritization step for theta
         g0– Boundary condition
         f – Perturbation function
         e – Perturbation value epsilon
Output: U0– Regular solution
         L –Perturbed solution
         (L-U0)– perturbation value
r ← [0 : h : M]                                              /* Discritization for R */
th← [0 : k : N]                                          /* Discritization for theta */
beta← zeros(1,M)                                                /* Main diagonal  */
th← zeros(1,M)                                                /* Second diagonal */
$n = N * M$;                                      /* the dimonssion of the matrix A */
for $i \leftarrow M$ do
                                      /* planing diagonals for matrices D and T */
    $beta(i) = 1/((i - 0.5)^2 k^2)$
    $lamd(i) = 1/(2(i - 0.5))$
end
D=diag(beta)                  /* T=infereur diagonale + diagonale +supereur diagonale */
T=diag(1-lamd(2:end),-1)-2*diag(ones(1,M))+diag(1+lamd(1:end-1),1)
                                          /* the matrix A have five diagonals  */
A=zeros(n) D0=diag(kron(eye(N-1),D)),D1=diag(D0,-M),D2=diag(D0,M)   /* A=inferior corner block +
  infereur diagonale + diagonale + supereur diagonale + uper coner block  */
A=diag(beta,n-M)+D1+kron(eye(N),T-2D)+D2+diag(beta,-n+M)
B=zeros(N*M,1)                  /* The B vector containing the boundary conditions */
for $j \leftarrow N$ do
    B(jM)=-(1+lamd(M))sin(2(j-1)k);                         /* the bondary condition */
end
$u0 = B/A$                                            /* solution of the system Av=B  */
B1=zeros(N*M,1)                /* The B1 vector containing the boundary conditions */
for $j \leftarrow N$ do
    B1(jM)=(1+lamd(M))(f(U0(M+1,j)-U0(M,j))/h;              /* the bondary condition  */
end
$u1 = B1/A$                                           /* solution of the system Av=B  */
u=u0+eu1                                                      /* the general solution */
draw u0
draw u
draw u-u0
```

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ABSTRACT

We consider the Laplace equation, with different boundary conditions, on a planar domain defined as a disc with a small perturbation of its boundary. First, we find an asymptotic solution in the interior of the disc using two analytical methods: separations of variables and Laplace's formula. A third approach consists in combining the asymptotic approach with finite differences in polar coordinates. Some numerical examples are given at the end of this work.

Key words:

Laplace's equation, separation of variable, Poisson's formula, finite difference in polar coordinates, perturbed domains .

RÉSUMÉ

On considère l'équation de Laplace, avec différentes conditions aux limites, sur un domaine planaire défini comme un disque avec une petite perturbation de sa frontière. Tout d'abord, nous trouvons une solution asymptotique à l'intérieur du disque en utilisant deux méthodes analytiques: séparations de variables et formule de Laplace. Un troixième approche consiste à combiner l'approche asymptotique avec des différences dans les coordonnées polaires. Quelques exemples numériques sont donnés au fin de ce travail.

Mots clé:

Équation de Laplace,séparation de variable, formule de Poisson, différences finies on coordinates polar , domaine perturbé.

الملخص

نعتبر معادلة لابلاس بشروط حدية مختلفة و يُعرَفْ المجال المستوي على أنه قرص به اضطراب صغير في حدوده. أولاً ، نجد حلاً مقاربًا في داخل القرص باستخدام اثنين من الطرق التحليلية: فصل المتغيرات وصيغة بواسون. اما الحل الثالث فيجمع بين النهج المقارب و الفروق المحدودة في الإحداثيات القطبية. يتم إعطاء بعض الأمثلة العددية في نهاية هذا العمل

الكلمات المفتاحية:

معادلة لا بلاس، فصل المتغير، صيغة بواسون، الفروق المحدودة في الاحداثيات القطبية ،مجال مضطرب