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On the classical character of Sheffer-Meixner 2-orthogonal polynomials type

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Abstract. In this paper we study the classical character of 2-OPS whose generating function satisfies a first order differential equation. Our results give some information about the classical character of the 2-Sheffer-Meixner type polynomials

1 Introduction

Let $\{P_n(x)\}_{n\geq 0}$ be a sequence of monic polynomials and consider their generating function (GF in short) in the following form

$$G(x,t)=\sum_{n\geq 0}\lambda_n P_n(x)t^n. \tag{1.1}$$

A specific form of GF, enables us to determine all the generated families of polynomials and, rather to build explicitly their expressions. On the other hand, when the sequence of polynomials is orthogonal, then the recurrence relation provides further informations to find out many analytic and algebraic properties. Besides the explicit formula, recurrence relations settle on whether the moment problem is determinate, look over the asymptotic behavior, make up the measure of the orthogonality and also it look as tools in further applications.

Many families of polynomials as well as orthogonal polynomials (OP in short) have been found out as a solution of some specific form of GF.

We quote for instance the Sheffer-Meixner OP [22, 15], which consist of five distinct families, i.e., Hermite, Laguerre, Charlier, Meixner and Meixner-Pollaczek polynomials, that generated by the GF of the form $G(x,t)=\exp(xu(t)+v(t))$.

The Boas-Buck type GF, i.e., $G(x,t)=B(t)h(x\psi(t))$, is more general and contains a large class of OP which looks still far to be listed in details. In particular, the case $\psi(t)=t$ is called Brenke type because of his work. Brenke was the first one who tired to determine all OP of Boas-Buck type with

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$\psi(t) = t$. Later, Chihara with two papers in 1968 and 1971, claimed that the Brenke type are classified in four families. No final classification has been done after Chihara's work till 2013 where all the OP of Brenke type have been determined explicitly by a group of Japanese researcher [3].

In the case of the d-orthogonality, Ben Cheikh with his collaborators, has made some d-analogue of Brenke as well as of Boas-Buch type and also some particular classes. In fact, in [8] the authors proved that the only d-OP generated by Boas-Buck type GF with $B(t) = e^t$ and $\psi(t) = t$ are the d-analogue of Laguerre OP. In [7] the authors showed that the d-symmetric d-OP of Brenke type are only two families determined explicitly. Further, the explicit expressions of the corresponding d-components provide two new classes of d-OP of Brenke type.

The GF also could be provides a strong characterization for the orthogonality, see for example [2] for the usual orthogonality. Mainly for d-orthogonality, the quasi-monomiality seems to be powerful tool for checking whether a sequence of polynomials is d-orthogonal or not [10, thm. 2.4]. Further, a new characterization of the d-symmetric d-OP has been given in terms of GF [6, thm. 3.1]. We mention another result of the family of d-OP generated by $G(x,t) = u(t)\,[v(t) - xt]^{-1}$, where $u(t)$ and $v(t)$ are two polynomials of degree $d+1$ in t [5], which allow the construction of the d-analogue of Chebyshev polynomials from the above GF instead of the problem stated by Douak and Maroni in their constructions of the two kind of the d-analogue.

Maroni and Van Iseghem considered in [21], a more general problem that includes the Sheffer class which we call the Maroni-Iseghem type (M-I type in short). This problem deals with the interaction between the coefficients in the three term recurrence relation for OP and the GF above satisfying a first-order linear homogeneous differential equation of the form

$$A(x,t)G'_t(x,t) = H(x,t)G(x,t)$$

with

$$A(x,t) = \sum_{n\geq 0} a_n(x)t^n \quad \text{and} \qquad H(x,t) = \sum_{n\geq 0} d_n(x)t^n \tag{1.2}$$

and where $a_n(x)$ and $d_n(x)$ are polynomials in x.

When we consider the usual orthogonality, i.e., if we suppose that the sequence of polynomials $\{P_n\}_n$ satisfies a three term recurrence relation, then A and H in (1.2), may reduce to

$$A(x,t) = a_0 + a_1(x)t + a_2(x)t^2 \quad \text{and} \qquad H(x,t) = d_0 + d_1(x)t.$$

This is the case treated by Maroni and Iseghem. The authors compute the coefficients in the three term recurrence relation explicitly and then look for the semiclassical character of the obtained polynomials by considering the structure relation. In this line of thought, the authors obtain the Charlier polynomials, the Hermite polynomials, the Meixner polynomials, the Laguerre polynomials, the Tricomi-Carlitz polynomials, the Gegenbauer polynomials, and the Pollaczek polynomials.

In this paper, first we study the problem (1.2) by supposing that the sequence of polynomials generated by the above GF (1.1) satisfying a four term recurrence relation. And then we compute the coefficients in the recurrence relation. In this case, since we have the explicit form of the GF, then we try to study the classical character of the corresponding 2-OPS as well. Obviously, this consideration gives a generalization of the 2-orthogonal polynomials of Sheffer-Meixner type studied by Boukhemis and Maroni [11]. They obtained nine families four of them are classical [13].

Certainly some of the obtained polynomials are of Sheffer-Meixner type. Furthermore, we obtain the 2-analogue of the Hermite polynomials, the Charlier polynomials and some type of Laguerre polynomials which correspond to the family (D) in [11, p. 85, Table 1]. We also give the explicit form of the measures of the 2-orthogonality for this last family since it's not done yet. This means that now we have the explicit expression of the measures for all the four classical families of 2-Sheffer-Meixner type. The measures of two of them are given by Boukhemis [13] and a third one by Ben Cheikh and Zeghouani [9, theorem 4.1]. We also obtain a 2-analogue of Tricomi-Carlitz as well as a 2-analogue of Meixner-Pollaczek polynomials, and for this family we have computed the explicit form of the polynomials. Another interesting result is that we have obtained the family (G) in [11, p. 85, Table 1]. This family contains the Chebyshev polynomials as well as the polynomials studied in [12] as particular case. We also prove that if this family is classical, then it reduces to the polynomials studied by Boukhemis [12]. This gives us an interesting way of motivating since it add up an information on the classical character of the Sheffer-Meixner 2-OPS [11].

In what follows, for a sequence of monic polynomials $\{P_n\}_{n\geq 0}$, we denote its dual sequence by $\{u_n\}_{n\geq 0}$, which is defined as follows $\langle u_n, P_m\rangle := \delta_{n,m}$, $n,m\geq 0$.

Let us denote the Hahn operator by $\Delta_w \pi(x) := \frac{\pi(x+w)-\pi(x)}{w}$, for any polynomial π, and set $Q_n(x) := (n+1)^{-1}\Delta_w P_{n+1}(x)$, $n\geq 0$.

Definition 1.1. [20] A sequence $\{P_n\}_{n\geq 0}$ is d-OPS with respect to $U=(u_0,\dots,u_{d-1})^T$ if it fulfills

$$\begin{aligned} &\langle u_r, x^m P_n(x)\rangle = 0, \quad n\geq md+r+1, \quad m\geq 0,\\ &\langle u_r, x^m P_{md+r}(x)\rangle \neq 0, \quad m\geq 0, \end{aligned} \tag{1.3}$$

for each $0\leq r\leq d-1$.

An important characterization of the d-orthogonality is that the orthogonality conditions (1.3) are equivalent to the fact that the sequence $\{P_n\}_{n\geq 0}$ satisfies a $(d+2)$ term linear recurrence relation written in the following form

$$P_{m+d+1}(x) = (x-\beta_{m+d})P_{m+d}(x) - \sum_{\nu=0}^{d-1}\gamma_{m+d-\nu}^{d-1-\nu}P_{m+d-1-\nu}(x), \ \ m\geq 0, \tag{1.4}$$

with the initial data

$$\begin{aligned} &P_0(x)=1, \ \ P_1(x)=x-\beta_0,\\ &P_m(x) = (x-\beta_{m-1})P_{m-1}(x) - \sum_{\nu=0}^{m-2}\gamma_{m-1-\nu}^{d-1-\nu}P_{m-2-\nu}(x), \ \ 2\leq m\leq d, \end{aligned} \tag{1.5}$$

and the regularity conditions $\gamma_{m+1}^{0}\neq 0, \ \ m\geq 0$.

Definition 1.2. A d-OPS $\{P_n\}_{n\geq 0}$ is called Δ_w -classical if $\{Q_n\}_{n\geq 0}$ is also a d-OPS.

Now suppose that $\{f_n\}$ is 2-OPS satisfying the following recurrence relation

$$f_{n+3}(x) = (A_{n+2}x+B_{n+2})f_{n+2}(x) + \gamma_{n+2}\, f_{n+1}(x) + \delta_{n+1}\, f_n(x).$$

It is evident that, for each integer $p\geq 1$, we have

$$f_{n+2+p}(x) = T_p^{(1)}(x) f_{n+2}(x) + T_p^{(2)}(x)\ f_{n+1}(x) + T_p^{(3)}(x) f_n(x), \tag{1.6}$$

where $T_p^{(i)}(x)$ are polynomials on x of degree p and $p-1$ for $i=1$ and $i=2,3$ respectively.

Note that, we can deduce the explicit form of that polynomials and, furthermore, we can prove that they are 2-OPS! This is a generalization of Al-Salam's characterization of the orthogonality [1].

On the other hand, the expansion (1.6) means that there is no relation of the form appear in the right hand side of (1.6), between three consecutive polynomials of a 2-OPS. That is, if we suppose that the right hand side of (1.6) is identically zero, then we must have $T_p^{(i)}(x)\equiv 0$ for $i=1,2,3$, otherwise the polynomials f_n satisfy a three term recurrence relation which is in contradiction with the fact that $\{f_n\}$ is a 2-OPS.

2 Main results (Generalization of Sheffer-Meixner class)

Let $\{P_n\}_{n\geq 0}$ be a 2-OPS with respect to $\mathcal{U}=(u_0,u_1)^T$, and $a_n(x)$ and $d_n(x)$ two polynomials of any degree. Next, we think over the 2-OPS of M-I type. Suppose that the GF (1.1) satisfies the following first order differential equation

$$A(x,t)\,G_t'(x,t)=H(x,t)\,G(x,t) \tag{2.1}$$

where

$$\begin{aligned} G(x,t) &= \textstyle\sum_{n\geq 0}\lambda_n P_n(x)\,t^n, \quad \lambda_0=1\\ A(x,t) &= \textstyle\sum_{n\geq 0} a_n(x)\,t^n,\\ H(x,t) &= \textstyle\sum_{n\geq 0} d_n(x)\,t^n, \end{aligned} \tag{2.2}$$

and

$$\begin{cases} P_{n+3}(x)=(x-\beta_{n+2})\,P_{n+2}(x)-\gamma_{n+2}P_{n+1}(x)-\delta_{n+1}P_n(x), & n\geq 0\\ P_2(x)=(x-\beta_1)\,P_1(x)-\gamma_1, \quad P_1(x)=x-\beta_0, \quad P_0(x)=1. \end{cases}$$

The differential equation (2.1) is equivalent to the following relations

$$\begin{aligned} a_0\lambda_1 P_1(x) &= d_0(x)\\ a_0(n+3)\,\lambda_{n+3}P_{n+3} &= (d_0-(n+2)\,a_1)\,\lambda_{n+2}P_{n+2}\\ &\quad +(d_1-(n+1)\,a_2)\,\lambda_{n+1}P_{n+1}\\ &\quad +\sum_{k=0}^{n}(d_{n+2-k}-ka_{n+3-k})\,\lambda_k P_k, \;\; n\geq 0. \end{aligned} \tag{2.3}$$

Remark that if a_0 is zero, then $d_0=0$, which means that the equation can be simplified by t and a_1, d_1 take the place of a_0, d_0. Hence, in the following, we suppose $a_0\neq 0$.

The problem now is restricted to the case

$$a_n(x)=0,\;\; n\geq 4 \;\text{ and }\; d_n(x)=0,\;\; n\geq 3, \tag{2.4}$$

that is the relation (2.3) is a linear recurrence of order 3. Now the problem is: find all 2-orthogonal monic families $\{P_n\}_{n\geq 0}$ such that

$$\begin{aligned} a_0\lambda_1 P_1(x) &= d_0(x)\\ a_0(n+3)\,\lambda_{n+3}P_{n+3} &= (d_0-(n+2)\,a_1)\,\lambda_{n+2}P_{n+2}\\ &\quad +(d_1-(n+1)\,a_2)\,\lambda_{n+1}P_{n+1}\\ &\quad +(d_2-na_3)\,\lambda_n P_n, \;\; n\geq 0 \end{aligned} \tag{2.5}$$

and

$$\begin{cases} P_{n+3}(x)=(x-\beta_{n+2})\,P_{n+2}(x)-\gamma_{n+2}P_{n+1}(x)-\delta_{n+1}P_n(x), & n\geq 0\\ P_2(x)=(x-\beta_1)\,P_1(x)-\gamma_1, \quad P_1(x)=x-\beta_0, \quad P_0(x)=1. \end{cases} \tag{2.6}$$

In this case, from (2.5) and (2.6), it follows

$$\begin{aligned} &[(d_0-(n+2)\,a_1)\,\lambda_{n+2}-a_0(n+3)\,\lambda_{n+3}(x-\beta_{n+2})]\,P_{n+2}\\ &=-[(d_1-(n+1)\,a_2)\,\lambda_{n+1}+a_0(n+3)\,\lambda_{n+3}\gamma_{n+2}]\,P_{n+1}\\ &-\left[(d_2-na_3)\,\lambda_n+a_0(n+3)\,\lambda_{n+3}\gamma_{n+1}^0\right]P_n, \quad n\geq 0. \end{aligned}$$

Now the expansion (1.6), shows that we have necessarily

$$\begin{aligned} a_0(n+3)\lambda_{n+3}(x-\beta_{n+2}) &= (d_0-(n+2)a_1)\lambda_{n+2} \\ a_0(n+3)\lambda_{n+3}\gamma_{n+2} &= (d_1-(n+1)a_2)\lambda_{n+1} \\ a_0(n+3)\lambda_{n+3}\delta_{n+1} &= (d_2-na_3)\lambda_n. \end{aligned} \tag{2.7}$$

Hence, a_0 divides d_0, a_1, d_1, a_2, d_2, a_3, and then the relations (2.5) can be divided by a_0. The last relations prove that $(d_0-(n+2)a_1)$ is exactly of degree one and $(d_1-(n+1)a_2)$ and (d_2-na_3) are a non zero constants for all $n\geq 0$. Hence we have the following 2-analogue of the Maroni and Van Iseghem's results [21]

Theorem 2.1. The 2-OPS $\{P_n\}_{n\geq 0}$, whose generating function $G(x,t)=\sum_{n\geq 0}\lambda_n P_n(x)t^n$ and satisfies a linear differential equation

$$\begin{aligned} A(x,t)\tfrac{\partial G}{\partial t} &= H(x,t)G(x,t) \\ A(x,t) &= a_0(x)+a_1(x)t+a_2(x)t^2+a_3(x)t^3, \\ H(x,t) &= d_0(x)+d_1(x)t+d_2(x)t^2, \end{aligned} \tag{2.8}$$

with $a_i(x)=\sum a_i^k x^k$ and $d_i(x)=\sum d_i^k x^k$, are characterized by the coefficients β_n, γ_n, δ_n of their recurrence relation

$$\begin{aligned} \beta_n &= -\tfrac{d_0^0-na_1^0}{d_0^1-na_1^1}, \\ \gamma_{n+2} &= -\tfrac{(n+2)\left(d_1^0-(n+1)a_2^0\right)}{\left(d_0^1-(n+2)a_1^1\right)\left(d_0^1-(n+1)a_1^1\right)}, \qquad n\geq 0. \\ \delta_{n+1} &= -\tfrac{(n+1)(n+2)\left(d_2^0-na_3^0\right)}{\left(d_0^1-(n+2)a_1^1\right)\left(d_0^1-(n+1)a_1^1\right)\left(d_0^1-na_1^1\right)}, \end{aligned} \tag{2.9}$$

All the possible cases are obtained with the following notations and conditions

$$\begin{aligned} &a_0(x)=1,\ \ a_1(x)=a_1^0+a_1^1x,\ \ a_2(x)=a_2^0,\ \ a_3(x)=a_3^0, \\ &d_0(x)=d_0^0+d_0^1x,\ \ d_1(x)=d_1^0,\ \ d_2(x)=d_2^0, \\ &d_0^1-na_1^1\neq 0,\ n\geq 0 \ \text{ and } \ d_2^0-na_3^0\neq 0,\ \ n\geq 1. \end{aligned} \tag{2.10}$$

Conversely, if the coefficients β_n, γ_n, δ_n are in the previous form, then $G(x,t)$ satisfies the differential equation (2.1).

Comparing the series A and H, taking account the coefficients (2.10), with that of Sheffer-Meixner [11, p.84], show that we have replaced the constant θ_1 in the case of 2-Sheffer-Meixner by $a_1(x)$, i.e., a function of the variable x, in the case of M-I type.

Now we determine the coefficients in the four term recurrence relation. First off, we begin with the coefficient λ_n in the GF. The recurrence relation for λ_n follows from (2.7) by equating the coefficients of x

$$\frac{\lambda_{n+2}}{(n+3)\lambda_{n+3}}\left(d_0^1-(n+2)a_1^1\right)=1.$$

As the same resonance presented in [21], we distinguish two cases according to the fact that either a_1^1 is zero or not

$$\begin{aligned} &a_1^1=0,\quad \lambda_n=(n)^{-1}d_0^1\lambda_{n-1}=(n!)^{-1}\left(d_0^1\right)^n, \\ &a_1^1\neq 0,\quad \lambda_{n+3}=\tfrac{\left(d_0^1-(n+2)a_1^1\right)}{(n+3)}\lambda_{n+2}=\left(-a_1^1\right)\tfrac{(n+2+\rho)}{(n+3)}\lambda_{n+2}, \end{aligned}$$

that is, if $a_1^1\neq 0$, then

$$\lambda_n = \left(-a_1^1\right) \frac{(n+\rho-1)}{n} \lambda_{n-1} = \left(-a_1^1\right)^n \frac{(\rho)_n}{n!},$$

with $\rho = \left(-d_0^1/a_1^1\right)$ and $(\rho)_n$ is the Pochhammer symbol. On the other hand, by a dilation on the variable t in the function $G : (t \to \alpha t)$, with

$$\begin{aligned} \alpha &= d_0^1, \quad \text{if} \quad a_1^1 = 0, \\ \alpha &= -a_1^1, \quad \text{if} \quad a_1^1 \neq 0, \end{aligned}$$

two canonical forms appear for λ_n

$$\begin{aligned} a_1^1 &= 0, \quad \lambda_n = (n!)^{-1}, \\ a_1^1 &\neq 0, \quad \lambda_n = (n!)^{-1} (\rho)_n . \end{aligned}$$

The coefficients of the recurrence can be now determined explicitly from (2.9). Furthermore, eight genetic cases are obtained as follows. For $n \geq 0$, we have

(i) If $a_1^1 = 0$, we have four cases

$$\begin{aligned} &\textbf{A)}\ a_2^0 = 0,\ a_3^0 = 0, \\ &\textbf{B)}\ a_2^0 = 0,\ a_3^0 \neq 0, \\ &\textbf{C)}\ a_2^0 \neq 0,\ a_3^0 = 0, \\ &\textbf{D)}\ a_2^0 \neq 0,\ a_3^0 \neq 0. \end{aligned}$$

(ii) If $a_1^1 \neq 0$, also we have four cases

$$\begin{aligned} &\textbf{E)}\ a_2^0 = 0,\ a_3^0 = 0, \\ &\textbf{F)}\ a_2^0 = 0,\ a_3^0 \neq 0, \\ &\textbf{G)}\ a_2^0 \neq 0,\ a_3^0 = 0, \\ &\textbf{H)}\ a_2^0 \neq 0,\ a_3^0 \neq 0. \end{aligned}$$

2.1 Determination of A and H

Now we give the explicit form of the coefficients of the linear recurrence as well as of A and H in order to calculate the expression of the corresponding GF and then study the classical character. Note that we often rely on the GF to study the classical character.

Case A). $a_1^1 = a_2^0 = a_3^0 = 0$. In this case, since $\lambda_n = (n!)^{-1}$ and $d_0^1 = 1$, then from (2.9), one gets

$$\left.\begin{aligned} \beta_n &= -d_0^0 + n a_1^0 := \beta_0 + n\tau, \\ \gamma_{n+1} &= -(n+1)\, d_1^0 := \gamma(n+1), \\ \delta_{n+1} &= -(n+1)(n+2)\, d_2^0 := \delta(n+1)(n+2), \end{aligned}\right. \qquad n \geq 0.$$

On the other hand, the functions $A(x,t)$ and $H(x,t)$ are in the following form

$$\begin{aligned} A(x,t) &= 1 + \tau t, \\ H(x,t) &= x - \beta_0 - \gamma t - \delta t^2. \end{aligned}$$

Case B). $a_1^1 = a_2^0 = 0$ and $a_3^0 \neq 0$. In this case, $\lambda_n = (n!)^{-1}$, $d_0^1 = 1$ and

$$\begin{aligned}\beta_n &= \beta_0 + n\tau,\\ \gamma_{n+1} &= \gamma(n+1),\\ \delta_{n+1} &= -(n+1)(n+2)\left(d_2^0 - na_3^0\right) := a(n+1)(n+2)(n+\eta+1),\end{aligned}$$

with $a = a_3^0$ and $d_2^0 = -a(\eta+1)$. Hence, the functions $A(x,t)$ and $H(x,t)$ are

$$\begin{aligned}A(x,t) &= 1 + \tau t + at^3,\\ H(x,t) &= x - \beta_0 - \gamma t - a(\eta+1)t^2.\end{aligned}$$

Case C). $a_1^1 = a_3^0 = 0$, $a_2^0 \neq 0$. In this case, $\lambda_n = (n!)^{-1}$, $d_0^1 = 1$ and

$$\begin{aligned}\beta_n &= \beta_0 + n\tau,\\ \gamma_{n+1} &= -(n+1)\left(d_1^0 - na_2^0\right) := b(n+1)(n+\mu+1),\\ \delta_{n+1} &= \delta(n+1)(n+2),\end{aligned}$$

with $b = a_2^0$ and $d_1^0 = -b(\mu+1)$. Hence

$$\begin{aligned}A(x,t) &= 1 + \tau t + bt^2,\\ H(x,t) &= x - \beta_0 - b(\mu+1)t - \delta t^2.\end{aligned}$$

Case D). $a_1^1 = 0$, $a_2^0 \neq 0$, $a_3^0 \neq 0$. We have $\lambda_n = (n!)^{-1}$, $d_0^1 = 1$ and Hence

$$\begin{aligned}\beta_n &= \beta_0 + n\tau,\\ \gamma_{n+1} &= -(n+1)\left(d_1^0 - na_2^0\right) := b(n+1)(n+\mu),\\ \delta_{n+1} &= -(n+1)(n+2)\left(d_2^0 - na_3^0\right) := a(n+1)(n+2)(n+\eta+1),\end{aligned}$$

with $b = a_2^0$, $d_1^0 = -b(\mu+1)$, $a = a_3^0$ and $d_2^0 = -a(\eta+1)$. In this case, we have

$$\begin{aligned}A(x,t) &= 1 + \tau t + bt^2 + at^3,\\ H(x,t) &= (x - \beta_0) - b(\mu+1)t - a(\eta+1)t^2.\end{aligned}$$

Case E). $a_1^1 \neq 0$, $a_2^0 = a_3^0 = 0$. In this case, we have $\lambda_n = \lambda_n = (n!)^{-1}(\rho)_n$, i.e. $\lambda_1 = \rho = d_0^1$ and $d_0^0 = -\rho\beta_0$. Hence

$$\begin{aligned}\beta_n &= \tfrac{-d_0^0 + na_1^0}{d_0^1 - na_1^1} = \tfrac{\rho\beta_0 + \tau n}{\rho - na_1^1} := \tfrac{\rho\beta_0 + \tau n}{n+\rho},\\ \gamma_{n+2} &= -\tfrac{(n+2)d_1^0}{(n+\rho+2)(n+\rho+1)} := \tfrac{\gamma(n+2)}{(n+\rho+2)(n+\rho+1)},\\ \delta_{n+1} &= -\tfrac{(n+1)(n+2)d_2^0}{(n+\rho+2)(n+\rho+1)(n+\rho)} := \tfrac{\delta(n+1)(n+2)}{(n+\rho+2)(n+\rho+1)(n+\rho)}.\end{aligned}$$

Then,

$$\begin{aligned}A(x,t) &= 1 + (\tau - x)t,\\ H(x,t) &= \rho(x - \beta_0) - \gamma t - \delta t^2.\end{aligned}$$

Case F). $a_1^1 \neq 0$, $a_2^0 = 0$, $a_3^0 \neq 0$. In this case, we have $\lambda_n = \lambda_n = (n!)^{-1}(\rho)_n$ and

$$\begin{aligned}\beta_n &= \tfrac{\rho\beta_0+\tau n}{n+\rho},\\ \gamma_{n+1} &= \tfrac{\gamma(n+1)}{(n+\rho+1)(n+\rho)},\\ \delta_{n+1} &:= \tfrac{a(n+1)(n+2)(n+\eta+1)}{(n+\rho+2)(n+\rho+1)(n+\rho)},\end{aligned}$$

with $a=-a_3^0$ and $d_2^0=-a(\eta+1)$. Then,

$$\begin{aligned}A(x,t) &= 1+(\tau-x)t+at^3,\\ H(x,t) &= \rho(x-\beta_0)-\gamma t-a(\eta+1)t^2.\end{aligned}$$

Case G). $a_1^1\neq 0$, $a_2^0\neq 0$, $a_3^0=0$. We have $\lambda_n=\lambda_n=(n!)^{-1}(\rho)_n$ and

$$\begin{aligned}\beta_n &= \tfrac{\rho\beta_0+\tau n}{n+\rho},\\ \gamma_{n+1} &= -\tfrac{(n+1)\left(d_1^0-na_2^0\right)}{(n+\rho+1)(n+\rho)} := \tfrac{b(n+1)(n+\mu)}{(n+\rho+1)(n+\rho)},\\ \delta_{n+1} &:= \tfrac{\delta(n+1)(n+2)}{(n+\rho+2)(n+\rho+1)(n+\rho)},\end{aligned}$$

with $b=a_2^0$ and $d_1^0=-b(\mu+1)$ and

$$\begin{aligned}A(x,t) &= 1+(\tau-x)t+bt^2,\\ H(x,t) &= \rho(x-\beta_0)-b(\mu+1)t-\delta t^2.\end{aligned}$$

Case H). $a_1^1\neq 0$, $a_2^0\neq 0$, $a_3^0\neq 0$. We have $\lambda_n=\lambda_n=(n!)^{-1}(\rho)_n$ and

$$\begin{aligned}\beta_n &= \tfrac{\rho\beta_0+\tau n}{n+\rho},\\ \gamma_{n+1} &= -\tfrac{(n+1)\left(d_1^0-na_2^0\right)}{(n+\rho+1)(n+\rho)} := \tfrac{b(n+1)(n+\mu)}{(n+\rho+1)(n+\rho)},\\ \delta_{n+1} &:= \tfrac{a(n+1)(n+2)(n+\eta+1)}{(n+\rho+2)(n+\rho+1)(n+\rho)},\end{aligned}$$

with $b=a_2^0$, $d_1^0=-b(\mu+1)$, $a=-a_3^0$ and $d_2^0=-a(\eta+1)$. In this case, we have

$$\begin{aligned}A(x,t) &= 1+(\tau-x)t+bt^2+at^3,\\ H(x,t) &= \rho(x-\beta_0)-b(\mu+1)t-a(\eta+1)t^2.\end{aligned}$$

3 Study of the classical character

According to the Hahn's property, now we try to study the classical (when $w=1$) as well as the Δ_w-classical character for the all possible form of the GF.

Case A). The GF satisfies the following first order differential equation

$$(1+\tau t)\,G_t'(x,t)=\left(x-\beta_0-\gamma t-\delta t^2\right)G(x,t).$$

First suppose that $\tau=0$. In this case the GF is as follows

$$G(x,t)=\exp\left\{(x-\beta_0)t-\frac{\gamma}{2}t^2-\frac{\delta}{3}t^3\right\}=\sum\nolimits_{n\geq 0}\frac{1}{n!}P_n(x)t^n.$$

In this case, we have $G'_x(x,t) = tG(x,t)$, i.e.,

$$P'_{n+1}(x) = (n+1)P_n(x),$$

that is, $\{P_n\}_{n\geq 0}$ is classical Appell sequence. By translation, we can take $\beta_0 = 0$, which corresponds to Douak one's [18], and then it is 2-OPS with respect to $\mathcal{U} = (u_0, -u'_0)^T$ [18].

Second, suppose that $\tau \neq 0$. The GF takes the following form

$$G(x,t) = (1+\tau t)^{(1/\tau)\left[(x-\beta_0)+(\gamma/\tau)-\left(\delta/\tau^2\right)\right]} \times \exp\left\{-\left[(\gamma/\tau)+\left(\delta/\tau^2\right)\right]t - (\delta/2\tau)t^2\right\}. \tag{3.1}$$

This GF is of Sheffer Meixner type [11, 13, 23]. In addition, the 2-OPS are called Charlier 2-OPS of type I, studied in [9] who have showed that this is the only family which is at the same time d-OPS and Δ_τ-Appell. Since $\Delta_\tau G(x,t) = \tau t G(x,t)$, i.e., $\Delta_\tau P_{n+1} = (n+1)P_n$ which means that the sequence of derivative is also 2-OPS and hence it's classical according to Hahn property. Meanwhile, the dual sequence is built in the same work.

Case B). The GF verifies

$$\left(1+\tau t+at^3\right)G'_t(x,t) = \left(x-\beta_0-\gamma t - a(\eta+1)t^2\right)G(x,t).$$

Since $A(x,t)$ is a polynomial of degree 3 on t, then

$$1+\tau t+at^3 = (1-\sigma_1 t)(1-\sigma_2 t)(1-\sigma_3 t).$$

Remark that we have necessarily $\sigma_1 \neq \sigma_2 \neq \sigma_3 \neq 0$. Otherwise, we get $a = 0$ or $\tau = 0$. This means that we have three simple zeros. In this case, one can write

$$\frac{1}{(1-\sigma_1 t)(1-\sigma_2 t)(1-\sigma_3 t)} = \frac{\alpha_1}{1-\sigma_1 t} + \frac{\alpha_2}{1-\sigma_2 t} + \frac{\alpha_3}{1-\sigma_3 t},$$

with

$$\alpha_i = \frac{\sigma_i^2}{(\sigma_i-\sigma_j)(\sigma_i-\sigma_k)}, \quad i,j,k = 1,2,3 \text{ and } i \neq j \neq k. \tag{3.2}$$

Put $e_i = -(\alpha_i/\sigma_i)$ in (3.2), and using the fact that $e_1+e_2+e_2 = 0$, and after some calculations we find the explicit form of the corresponding GF

$$\begin{aligned} G(x,t) &= (1-\sigma_1 t)^{e_1\left[(x-\beta_0)-(\gamma/\sigma_1)-\left(a(\eta+1)/\sigma_1^2\right)\right]} \\ &\times (1-\sigma_2 t)^{e_2\left[(x-\beta_0)-(\gamma/\sigma_2)-\left(a(\eta+1)/\sigma_2^2\right)\right]} \\ &\times (1-\sigma_3 t)^{e_3\left[(x-\beta_0)-(\gamma/\sigma_3)-\left(a(\eta+1)/\sigma_3^2\right)\right]}. \end{aligned} \tag{3.3}$$

The present GF is also of Sheffer-Meixner type and it corresponds to the case (G) in [11, p. 85, Table 1]. On the other hand, since it is analogue of the case **D**, we prefer to discuss the case **D** as well as the case **H** below, because it's a more general.

Case C). The differential equation is in the following form

$$\left(1+\tau t+bt^2\right)G'_t(x,t) = \left(x-\beta_0-b(\mu+1)t-\delta t^2\right)G(x,t).$$

Case C1). Write $(1-\sigma_1 t)(1-\sigma_2 t) = 1+\tau t+bt^2$, with $\sigma_1 \neq \sigma_2$ and

$$\frac{1}{1+\tau t+bt^2}=\frac{A_1}{1-\sigma_1 t}+\frac{A_2}{1-\sigma_2 t}.$$

Then after some calculations we get

$$\begin{aligned}G(x,t)&=(1-\sigma_1 t)^{\frac{1}{\sigma_2-\sigma_1}\left[(x-\beta_0)-(b(\mu+1)/\sigma_1)-\left(\delta/\sigma_1^2\right)\right]}\\&\times(1-\sigma_2 t)^{\frac{1}{\sigma_1-\sigma_2}\left[(x-\beta_0)-(b(\mu+1)/\sigma_2)-\left(\delta/\sigma_2^2\right)\right]}\\&\times\exp\left\{-(\delta/b)\,t\right\}.\end{aligned}\tag{3.4}$$

The family of polynomials generated by this GF is a classical discrete 2-OPS. Indeed, we can easily obtain a structure relation as follows

$$\begin{aligned}nP_{n-1}(x)&=\Delta_{(\sigma_1-\sigma_2)}P_n(x)-n\sigma_1\Delta_{(\sigma_1-\sigma_2)}P_{n-1}(x)\\nP_{n-1}(x)&=\Delta_{(\sigma_2-\sigma_1)}P_n(x)-n\sigma_2\Delta_{(\sigma_2-\sigma_1)}P_{n-1}(x).\end{aligned}\tag{3.5}$$

Now, if we set for example, $w=\sigma_1-\sigma_2$, and apply the operator Δ_w to the four term recurrence relation satisfied by $\{P_n\}$, we get

$$\Delta_w P_{n+1}=(x-\beta_n+w)\,\Delta_w P_n+P_n-\gamma_n\Delta_w P_{n-1}-\delta_{n-1}\Delta_w P_{n-2},$$

and hence, by the first structure relation in (3.5) we obtain the recurrence of the derivative sequence in four terms, i.e.,

$$\frac{n}{n+1}\Delta_w P_{n+1}=(x-\beta_n+w-\sigma_2)\,\Delta_w P_n+\gamma_n\Delta_w P_{n-1}+\delta_{n-1}\Delta_w P_{n-2}.$$

According to Hahn's property, this family is classical discrete 2-OPS.

Remark that this family corresponds to the case (D) in [11, p. 85, Table 1]. We can then determine the dual sequence by using the Pearson equation [17] as well as by using the quasi-monomiality principle [10]. We prefer here the last idea in which the dual sequence can be done via, we have kept the notation in [10]

$$\langle u_r,f\rangle=\frac{\sigma^r}{r!A(\sigma)}f(0)=\sigma^r\exp\{\delta\sigma/b\}f(0)\ \text{ for }\ r=0,1.\tag{3.6}$$

Let us denote by $\eta:=\delta/b$ and

$$\begin{aligned}\alpha\ &:=\beta_0+(b\,(\mu+1)\,/\sigma_1)+\left(\delta/\sigma_1^2\right)\\\beta\ &:=\beta_0+(b\,(\mu+1)\,/\sigma_2)+\left(\delta/\sigma_2^2\right).\end{aligned}$$

For the sake of simplicity, we suppose here that $\alpha=\beta=0$. Now, this GF maybe written in the form

$$G(x,t)=\exp\{-\eta t\}\exp\left\{\frac{x}{\sigma_2-\sigma_1}\ln\left(\frac{1-\sigma_2 t}{1-\sigma_1 t}\right)\right\},$$

hence according to [4, 10], the operator σ is defined by

$$\sigma:=C^*(D)=\frac{e^{wD}-1}{\sigma_1 e^{wD}-\sigma_2}=\frac{\Delta_w}{1+\sigma_1\Delta_w}.$$

First, by the binomial theorem we have

$$\langle u_0, f\rangle = \sum_{n,k=0}^{\infty} \frac{\eta^n (n)_k (-\sigma_1)^k}{n!k!} \Delta_w^{n+k} f(0).$$

A short manipulation of power series [24, p. 100-102], justified by the expansion

$$\Delta_w^n f(0) = \left(\frac{-1}{w}\right)^n \sum_{j=0}^{n} \binom{n}{j} (-1)^j f(wj), \tag{3.7}$$

shows that

$$\begin{aligned}\langle u_0, f\rangle = &\sum_{n,k,j=0}^{\infty} \tfrac{\eta^{n+j+1}(n+j+1)_k(-\sigma_1)^k}{(n+j+1)!k!} \left(\tfrac{-1}{w}\right)^{n+j+k+1} \tbinom{n+j+k+1}{j} (-1)^j f(wj) \\ &+ \sum_{n,k,j=0}^{\infty} \tfrac{\eta^n(n)_{k+j}(-\sigma_1)^{k+j}}{n!(k+j)!} \left(\tfrac{-1}{w}\right)^{n+j+k} \tbinom{n+k+j}{n+j} (-1)^{n+j} f(w(n+j)).\end{aligned}$$

Since $(n+k+j+1)! = (n+j+2)_k (n+j+1)!$ and $(n)_{k+j} = (n)_j (n+j)_k$ we get finally

$$\begin{aligned}\langle u_0, f\rangle = &\sum_{n,j=0}^{\infty} \tfrac{(-1)^j(-\eta/w)^{n+j+1}}{n!j!} \, {}_2F_1\left(\left.\begin{matrix} n+j+1,\, n+j+2 \\ n+2 \end{matrix}\right| \tfrac{\sigma_1}{w}\right) f(wj) \\ &+ \sum_{n,j=0}^{\infty} \tfrac{(-1)^j(\eta/w)^n(\sigma_1/w)^j(n)_j}{n!j!} \, {}_2F_1\left(\left.\begin{matrix} n+j,\, n+j+1 \\ j+1 \end{matrix}\right| \tfrac{\sigma_1}{w}\right) f(w(n+j)).\end{aligned}$$

Now for the linear functional u_1, we have

$$\begin{aligned}\langle u_1, f\rangle &= \sigma \exp\{\eta\sigma\} f(0) \\ &= \sum_{n,k=0}^{\infty} \tfrac{\eta^n(-\sigma_1)^k(n+1)_k}{n!k!} \Delta_w^{n+k+1} f(0).\end{aligned}$$

Following the same method for u_0 we get the following representation

$$\begin{aligned}\langle u_1, f\rangle = &\tfrac{1}{\eta} \sum_{n,j=0}^{\infty} \tfrac{(-1)^j(-\eta/w)^{n+j+1}(n+j+1)}{(n+1)!j!} f(wj) \\ &\times {}_2F_1\left(\left.\begin{matrix} n+j+1,\, n+j+2 \\ n+2 \end{matrix}\right| \tfrac{\sigma_1}{w}\right) \\ &+\tfrac{1}{\sigma_1} \sum_{n,j=0}^{\infty} \tfrac{(-1)^j(\eta/w)^n(\sigma_1/w)^{j+1}(n+1)_j}{n!j!} f(w(n+j+1)) \\ &\times {}_2F_1\left(\left.\begin{matrix} n+j+1,\, n+j+2 \\ j+1 \end{matrix}\right| \tfrac{\sigma_1}{w}\right).\end{aligned}$$

Case C2). If $\sigma_1 = \sigma_2$, then the GF becomes

$$\begin{aligned}G(x,t) = &(1-\sigma t)^{-(\mu+1)-\left(2\delta/\sigma^3\right)} \\ &\times \exp\left\{\tfrac{t}{(1-\sigma t)} \left[x - \beta_0 - \sigma(\mu+1) - \left(\delta/\sigma^2\right)\right] - \left(\delta/\sigma^2\right) t\right\}.\end{aligned}$$

This family is of Laguerre type and contain the 2-Laguerre polynomials of Douak [19] as well as that given in [10], as particular case and it is again classical 2-OPS. Indeed, put for simplicity

$$\omega = (\mu+1) + \left(2\delta/\sigma^3\right) \text{ and } \theta = \beta_0 + \sigma(\mu+1) + \left(\delta/\sigma^2\right),$$

and take the first partial derivative of $G(x,t)$ with respect to x and t, we get

$$(1-\sigma t)\,G'_x(x,t) = tG(x,t),$$
$$t(1-\sigma t)\,G'_t(x,t) = \left[x-\theta-\omega(1-\sigma t)-\left(\delta/\sigma^2\right)(1-\sigma t)^2\right]G'_x(x,t).$$

From these equalities, we obtain by using (1.1) with $\lambda_n = \frac{1}{n!}$, the following structure relation

$$nP_{n-1}(x) = P'_n(x) - \sigma n P'_{n-1}(x),$$

as well as the four term recurrence relation of the derivative sequence

$$\frac{n}{n+1}P'_{n+1} = \left[x+\sigma(2n-1)-\theta-\omega-\left(\delta/\sigma^2\right)\right]P'_n$$
$$+n\sigma\left[\omega+2\left(\delta/\sigma^2\right)-\sigma(n-1)\right]P'_{n-1} - \delta n(n-1)\,\delta P'_{n-2},$$

which proves that $\{P_n\}$ is classical 2-OPS. Furthermore, the matrix polynomials in the Pearson equation is analog to the Douak one [19]. Then we can give the integral representation in terms of some special functions like Bessel and Whittaker functions.

Case D). The differential equation is in the following form

$$\left(1+\tau t+bt^2+at^3\right)G'_t(x,t) = \left[(x-\beta_0)-b(\mu+1)t-a(\eta+1)t^2\right]G(x,t).$$

As in the second case, write $1+\tau t+bt^2+at^3 = (1-\sigma_1 t)(1-\sigma_2 t)(1-\sigma_3 t)$ and suppose that we have

Case D1). Three simple zero which are all different from zero, we obtain the following GF

$$\begin{aligned} G(x,t) &= (1-\sigma_1 t)^{e_1\left[(x-\beta_0)-(b(\mu+1)/\sigma_1)-\left(a(\eta+1)/\sigma_1^2\right)\right]} \\ &\times(1-\sigma_2 t)^{e_2\left[(x-\beta_0)-(b(\mu+1)/\sigma_2)-\left(a(\eta+1)/\sigma_2^2\right)\right]} \\ &\times(1-\sigma_3 t)^{e_3\left[(x-\beta_0)-(b(\mu+1)/\sigma_3)-\left(a(\eta+1)/\sigma_3^2\right)\right]}, \end{aligned} \tag{3.8}$$

which is similar to (3.3).

First, remark that if we suppose that $e_1 = e_2$ (remember that we have $e_1+e_2 = -e_3$), and set $w = 1/e_1$, we can have

$$w(1-\sigma_3 t)^2\Delta_w G(x,t) = \left[(-\sigma_1-\sigma_2+2\sigma_3)t+\left(\sigma_1\sigma_2-\sigma_3^2\right)t^2\right]G(x,t), \tag{3.9}$$

in this case, the sequence of polynomials generated by (3.8) which satisfies (3.9) is classical 2-OPS if and only if $\sigma_1\sigma_2-\sigma_3^2 = 0$. But this choice of parameters also makes the coefficient of t zero, which is impossible. Otherwise, we obtain the following structure relation

$$\alpha nP_{n-1}+\beta n(n-1)P_{n-2} = w\Delta_w P_n - 2\sigma wn\Delta_w P_{n-1} + w\sigma_3^2 n(n-1)\Delta_w P_{n-2}, \tag{3.10}$$

with $\alpha = 2\sigma_3-\sigma_1-\sigma_2$ and $\beta = \sigma_1\sigma_2-\sigma_3^2$.

Now by applying the operator Δ_w to the recurrence of $P_n(x)$, and replace then P_{n-1} and P_{n-2} from the resulting recurrence in (3.10) we obtain

$$\begin{aligned} \left(\tfrac{\alpha n-w}{n}\right)\Delta_w P_n &= \left[\alpha(x-\beta_{n-1})-\beta(n-1)-2w\sigma_3\right]\Delta_w P_{n-1}+ \\ &+\left[\beta(n-1)(x-\beta_{n-2})+w(n-1)\sigma_3^2-\alpha\gamma^1_{n-1}\right]\Delta_w P_{n-2} \\ &-\left[\alpha\gamma^0_{n-2}+(n-1)\beta\gamma^1_{n-2}\right]\Delta_w P_{n-3}-\beta(n-1)\gamma^0_{n-3}\Delta_w P_{n-4}. \end{aligned} \tag{3.11}$$

Hence, $\{\Delta_w P_n\}_n$ verifies a recurrence relation in five terms! and then $\{P_n\}_n$ is not classical according to the Hahn property.

A more general situation of that GF appears in the case **H** (last case) in which we prove that the only classical families generated by these type of GF are those studied in [12].

Case D2). Suppose that $\sigma_1 = \sigma_2 = \sigma_3 = \sigma \neq 0$. In this case, the GF reads as

$$\begin{aligned} G(x,t) &= (1-\sigma t)^{-(\eta+1)} \\ &\quad \times \exp\left\{ \tfrac{2t-\sigma t^2}{2(1-\sigma t)^2} \left[x - \beta_0 + 3\sigma(\mu-\eta) + 2\sigma^2(2\eta - 3\mu - 1)t \right] \right\}. \end{aligned} \tag{3.12}$$

This family is a generalization of (E) in [11, p. 85, Table 1]. It reduces also to the generating function [26, (2.8)] with $\sigma = 1$, $\beta_0 = 3(\mu-\eta)$ and $2\eta = 3\mu+1$ where the 2-orthogonality measures are determined using lowering operator. On the other hand, set

$$\omega = \beta_0 - 3\sigma(\mu-\eta) \ \text{ and } \ \theta = 2\sigma^2(2\eta - 3\mu - 1),$$

by taking the first partial derivative of $G(x,t)$ with respect to x, we get

$$2(1-\sigma t)^2 G'_x(x,t) = \left(2t - \sigma t^2\right) G(x,t), \tag{3.13}$$

and by replacing $G(x,t)$ from (1.1) taking account $\lambda_n = \frac{1}{n!}$, we obtain the following structure relation which is of type of the previous one, i.e., (3.10)

$$nP_{n-1} - \frac{\sigma}{2} n(n-1) P_{n-2} = P'_n - 2\sigma n P'_{n-1} + \sigma^2 n(n-1) P'_{n-2}. \tag{3.14}$$

Again, by differentiating the recurrence of $P_n(x)$ with respect to x and replacing the resulting in (3.14) we obtain

$$\begin{aligned} \left(\tfrac{n-1}{n}\right) P'_n &= (x - \beta_{n-1} + (n-5)\sigma/2) P'_{n-1} \\ &\quad - \left[\gamma^1_{n-1} + (x - \beta_{n-1} + 2)(n-1)\sigma/2\right] P'_{n-2} \\ &\quad - \left[\gamma^0_{n-2} - \gamma^1_{n-2}(n-1)\sigma/2\right] P'_{n-3} + \left[\gamma^0_{n-3}(n-1)\sigma/2\right] P'_{n-4}. \end{aligned} \tag{3.15}$$

Hence, $\{P'_n\}_n$ is not 2-OPS and then $\{P_n\}_n$ is not classical according to the Hahn property. The recurrence relation (3.11) as well as (3.15) is of the type presented in the Rainville book's [23, Ch. 14, Ex. 3-4]. Not also that de Bruin [16] has considered further generalization of that recurrence relation in the complex plane and give some informations about the zeros of polynomials satisfying a linear recurrence of k terms where the coefficients are polynomials of degree at most one.

Case D3). Suppose that $1 + \tau t + bt^2 + at^3 = (1-\sigma_1 t)^2 (1-\sigma_2 t)$ with $\sigma_1 \neq \sigma_2 \neq 0$. In this case, by denoting $g_1 = \left(\sigma_2/(\sigma_1-\sigma_2)^2\right)$, one can write

$$\frac{1}{(1-\sigma_1 t)^2(1-\sigma_2 t)} = \frac{\alpha_1}{1-\sigma_1 t} + \frac{\alpha_2}{(1-\sigma_1 t)^2} + \frac{\alpha_3}{1-\sigma_2 t}.$$

Hence, in this case, the GF may be written

$$\begin{aligned} G(x,t) &= (1-\sigma_1 t)^{g_1\left[(x-\beta_0) + (b(\mu+1)/\sigma_1) - \left(a(\eta+1)/\sigma_1^2\right) - (b(\mu+1)(\sigma_1-\sigma_2)/\sigma_1\sigma_2)\right]} \\ &\quad \times (1-\sigma_2 t)^{-g_1\left[(x-\beta_0) - (b(\mu+1)/\sigma_2) - \left(a(\eta+1)/\sigma_2^2\right)\right]} \\ &\quad \times \exp\left\{ \tfrac{1}{(\sigma_1-\sigma_2)(1-\sigma_1 t)} \left[x - \beta_0 - (b(\mu+1)/\sigma_1) - \left(a(\eta+1)/\sigma_1^2\right) \right] \right\} \\ &\quad \times \exp\left\{ \tfrac{a(\eta+1)}{2(\sigma_1-\sigma_2)} \left[4t + t^2\right] \right\}. \end{aligned}$$

We can use Iseghem's approach to give some remarks. Indeed, this GF may be written, by use of the logarithm, in the form of theorem 2.3 in [25], i.e., in the following form

$$G(x,t) = \lambda e^{h(t)} e^{xg(t)}.$$

In this case, if we differentiate with respect to x we get a relation between $G(x,t)$ and its first partial derivative in terms of logarithm. And then, by the idea of Iseghem, we can't extract neither differential recurrence nor a recurrence relations. This means that the sequence of polynomials generated by this family is not classical according to the Hahn property as well.

Case E). We have $\lambda_n = (n!)^{-1}(\rho)_n$ and

$$(1+(\tau-x)t)\,G'_t(x,t) = \left[\rho(x-\beta_0) - \gamma t - \delta t^2\right] G(x,t).$$

After some calculations, we obtain

$$\begin{aligned} G(x,t) &= (1+(\tau-x)t)^{-\rho+\rho(\tau-\beta_0)/(\tau-x)+\gamma/(\tau-x)^2-\delta/(\tau-x)^3} \\ &\quad \times \exp\left\{\left[-\gamma/(\tau-x)+\delta/(\tau-x)^2\right]t - \delta/(2(\tau-x))t^2\right\}. \end{aligned}$$

If we choose, for example $\tau = 0$, $\gamma = 1$ and $\delta = 1$, the GF takes the following form

$$G(x,t) = (1-xt)^{-\rho+\rho(\beta_0/x)-\gamma/x^2+1/x^3} \exp\left\{t/x^2 - t^2/(2x)\right\},$$

and when $\beta_0 = 0$, i.e., $\beta_n = 0$, and $\gamma = 0$, the sequence $\{P_n\}_{n\geq 0}$ will be a 2-symmetric 2-OPS and it is analogue of Tricomi-Carlitz polynomials. For this end, one can bring out a generalized d-analogue of the Tricomi-Carlitz polynomials, i.e., the d-generalized Tricomi-Carlitz are the d-OPS generated by the following GF

$$\begin{aligned} G(x,t) &= (1+(\tau-x)t)^{-\rho+\rho(\tau-\beta_0)/(\tau-x)+\sum_{i=2}^{d+1}(-1)^{i+1}\gamma_i/(\tau-x)^i} \\ &\quad \times \prod_{j=1}^{d} \exp(-t)^j \left\{\sum_{i=j}^{d}(-1)^{i+1}\gamma_i/(\tau-x)^i\right\}, \end{aligned}$$

which corresponds to the d-Tricomi-Carlitz polynomials when $\tau = 0$ (for a d-symmetric sequence).

Case F). This case is analogue to the second one. Then, we can consider the following decomposition

$$1+(\tau-x)t+at^3 = (1-\sigma_1(x)t)(1-\sigma_2(x)t)(1-\sigma_3(x)t).$$

It is of type (3.18) in the case **H** below.

Case G). Is analogue to the third one. Write

$$1+(\tau-x)t+bt^2 = (1-\sigma_1(x)t)(1-\sigma_2(x)t),$$

then after some calculations we get

$$\begin{aligned} G(x,t) &= (1-\sigma_1(x)t)^{\frac{1}{\sigma_1-\sigma_2}\left[\rho(x-\beta_0)-(b(\mu+1)/\sigma_1)-\left(\delta/\sigma_1^2\right)\right]} \\ &\quad \times (1-\sigma_2(x)t)^{\frac{1}{\sigma_2-\sigma_1}\left[\rho(x-\beta_0)-(b(\mu+1)/\sigma_2)-\left(\delta/\sigma_2^2\right)\right]} \\ &\quad \times \exp\{-(\delta/b)t\}. \end{aligned}$$

Which can be written using the logarithm in the following form

$$G(x,t) = \exp\{\varphi_1(x)\ln(1-\sigma_1(x)t)\} \\ \times\exp\{\varphi_2(x)\ln(1-\sigma_2(x)t)\}\exp\{-(\delta/b)t\},$$

where

$$\varphi_1(x) = \tfrac{1}{\sigma_1-\sigma_2}\left[\rho(x-\beta_0)-(b(\mu+1)/\sigma_1)-\left(\delta/\sigma_1^2\right)\right]$$
$$\varphi_2(x) = \tfrac{1}{\sigma_1-\sigma_2}\left[\rho(x-\beta_0)-(b(\mu+1)/\sigma_2)-\left(\delta/\sigma_2^2\right)\right].$$

The partial derivative with respect to x follows

$$G'_x(x,t) = \Big[\varphi'_1(x)\ln(1-\sigma_1(x)t)-\tfrac{\varphi_1(x)\sigma'_1(x)t}{1-\sigma_1(x)t} \\ -\varphi'_2(x)\ln(1-\sigma_2(x)t)+\tfrac{\varphi_2(x)\sigma'_2(x)t}{1-\sigma_2(x)t}\Big]G(x,t).$$

Remark that the sequence $\{P_n\}_{n\geq 0}$ is classical 2-OPS if and only if $\varphi'_1(x)=\varphi'_2(x)=0$. Then, in this case we should have $\delta=0$, which is in contradiction since $\{P_n\}_{n\geq 0}$ cannot be an 2-OPS but rather an OPS which reduces to the case II_2 of [21]. (The computations are analogue of that in the last case below).

On the other hand, by the change of variable $x-\tau=2\sqrt{b}\cos\theta$, i.e., $\sigma_1(x)=\sqrt{b}e^{i\theta}$ and $\sigma_2(x)=\sqrt{b}e^{-i\theta}$, we obtain

$$\varphi_1(x) = \lambda+p\cos\theta+i\phi(\theta) := -\omega+i\phi$$
$$\varphi_2(x) = \lambda+p\cos\theta-i\phi(\theta) := -\omega-i\phi,$$

where $2\lambda=\mu+1$, $p=\delta/b^{3/2}$ and

$$\phi(\theta) = \tfrac{q+(\lambda-\rho)\cos\theta+p\cos^2\theta}{\sin(\theta)},$$
$$q = \left(\rho(\beta_0-\tau)/2\sqrt{b}\right)-p/2.$$

This gives the GF for a 2-analogue of the Meixner-Pollaczek polynomials (i.e. Meixner polynomials of second kind). Further, remark that we have just multiplied the generating function of the Pallaczek polynomials by $\exp(-ct)$ to get the 2-analogue of Meixner-Pollaczek. This means that we can move from the d-orthogonality to the (d+1) orthogonality by just multiplying the first corresponding GF by $\exp(\pi_d)$. This is also the case for the 2-analogue of Laguerre polynomials studied by Douak [19, prop. 3.3].

For the explicit form of the polynomial, we use the Cauchy product of the power series

$$\left(1-\sqrt{b}e^{i\theta}t\right)^{-(\omega-i\phi)} = \textstyle\sum_{n\geq 0}(\omega-i\phi)_n\tfrac{b^{n/2}e^{in\theta}t^n}{n!}$$
$$\left(1-\sqrt{b}e^{-i\theta}t\right)^{-(\omega+i\phi)} = \textstyle\sum_{n\geq 0}(\omega+i\phi)_n\tfrac{b^{n/2}e^{-in\theta}t^n}{n!},$$

and of the exponential to have

$$G(x,t) = \textstyle\sum_{n\geq 0}\sum_{k=0}^{n}\sum_{r=0}^{n-k}(\omega+i\phi)_r(\omega-i\phi)_{n-k-r} \\ \times(-\gamma)^k b^{(n-k)/2}\tfrac{\exp\ (-i(n-k-2r)\theta)}{r!k!(n-k-r)!}t^n.$$

Now from the fact that $(n-k-m)!=\frac{(-1)^m(1)_{n-k}}{(k-n)_m}$ and $(x)_{u-v}=\frac{(x)_u(-1)^v}{(1-x-u)_v}$, we obtain the following

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} (\psi - i\phi)_n b^{(n-k)/2} \gamma^k \frac{e^{i(n-k)\theta}}{n!(1-\psi+i\phi+k-n)_k} \times {}_2F_1 \left(\begin{matrix} k-n,\ \omega+i\phi \\ 1-\omega+i\phi+k-n \end{matrix} ; e^{-2i\theta} \right), \tag{3.16}$$

where ${}_2F_1$ appearing in (3.16) is

$$\sum_{r=0}^{n-k} \frac{(\omega+i\phi)_r (k-n)_r e^{-2ir\theta}}{(1-\omega+i\phi+k-n)_r r!}.$$

A more detailed and general study of the above d-analogue of Meixner-Pollaczek polynomials will be done in forthcoming paper.

Case H). This case is also analogue to **D**. One can consider the following decomposition

$$1 + (\tau - x)t + bt^2 + at^3 = (1 - \sigma_1(x)t)(1 - \sigma_2(x)t)(1 - \sigma_3(x)t). \tag{3.17}$$

And the GF is similar to those in (3.8), i.e.,

$$\begin{aligned} G(x,t) &= (1 - \sigma_1(x)t)^{e_1\left[\rho(x-\beta_0)+(b(\mu+1)/\sigma_1(x))-\left(a(\eta+1)/\sigma_1^2(x)\right)\right]} \\ &\times (1 - \sigma_2(x)t)^{e_2\left[\rho(x-\beta_0)+(b(\mu+1)/\sigma_2(x))-\left(a(\eta+1)/\sigma_2^2(x)\right)\right]} \\ &\times (1 - \sigma_3(x)t)^{e_3\left[\rho(x-\beta_0)+(b(\mu+1)/\sigma_3(x))-\left(a(\eta+1)/\sigma_3^2(x)\right)\right]}. \end{aligned} \tag{3.18}$$

This can be written using the logarithm in the following form

$$G(x,t) = \exp\left\{ \sum_{i=1}^{3} \theta_i(x) \ln(1 - \sigma_i(x)t) \right\},$$

where

$$\theta_i(x) = e_i \left[\rho(x - \beta_0) + (b(\mu+1)/\sigma_i(x)) - \left(a(\eta+1)/\sigma_i^2(x) \right) \right]$$

for $i = 1, 2, 3$. The partial derivative with respect to x follows

$$G'_x(x,t) = \sum_{i=1}^{3} \left[\theta'_i(x) \ln(1 - \sigma_i(x)t) - \frac{\theta_i(x)\sigma'_i(x)t}{1 - \sigma_i(x)t} \right] G(x,t). \tag{3.19}$$

In order that the sequence $\{P_n\}_{n\geq 0}$ is classical 2-OPS, it's necessary and sufficient that we have $\theta'_i(x) = 0$, $i = 1, 2, 3$. After simplifications of these last systems, we get

$$\begin{gathered} \rho(\tau - \beta_0)\sigma_i^4 + b(2\rho + \mu + 1)\sigma_i^3 + a(3\rho - \eta - 1)\sigma_i^2 \\ +b[(2\rho + \mu + 1)x - (2\rho\beta_0 + (\mu + 1)\tau)]\sigma'_i\sigma_i^2 + \\ ba[2(\eta + 1) + 3(\mu + 1)] = 0. \end{gathered}$$

This means that the sequence is classical 2-OPS only in the following cases

$$\begin{aligned} &\tau = \beta_0, \\ &\rho = -\tfrac{1}{2}(\mu + 1) = \tfrac{1}{3}(\eta + 1). \end{aligned}$$

In this case, we obtain the explicit form of θ_i, i.e., $\theta_i(x) = -\rho$ for $i = 1, 2, 3$. Then, the equation (3.19) has the following form

$$\left(1+(\beta_0-x)\,t+bt^2+at^3\right)G'_x(x,t)=\rho t G(x,t)\,,$$

i.e.,

$$G(x,t)=\left(1+(\beta_0-x)\,t+bt^2+at^3\right)^{-\rho}. \tag{3.20}$$

which corresponds to the family analyzed by Boukhemis [12]. This is the unique classical family that we know in the literature which satisfies a third order linear differential equation of type [14]. In addition, any derivative of this family is again classical, i.e., the derivative of order m is again 2-OPS.

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