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Naasira Bentoumi

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Weak Solutions for elliptic equations with lower-order terms and L^1 data

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Mr. Mokhtari Abdelhak	M.C.A. Univ of M'sila	president
Mr. Rabah Mecheter	M.C.A. Univ of M'sila	Rapporteur
Mr. Dahmane Bouafia	M.C.A. Univ of M'sila	Examiner

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Dedication

I dedicate this modest work :

-To my mom and dad,

-To my brothers and sisters,

-To all my family,

-To my best friend,

-To all friends and all my department family,

-To all my adorable ones that i have known during all my life . . .

Bentoumi Nassira

Contents

Notation

We introduce the necessary notations and definition which are used in the sequel.

Ω	open subset of $\mathbb{R}^N$.
$\mathbb{R}^N$	Euclidean space of dimension N, where N is a nonzero natural number.
$\partial\Omega = \Gamma$	boundary of Ω.
X'	topological dual of X.
x	vector in $\mathbb{R}^N$, $x = (x_1, x_2, ..., x_N)$, $x_i \in \mathbb{R}$, $1 \leq i \leq N$.
dx	Lebesgue measure in N-dimensional space.
$B(x_0, r)$	open ball of radius r centered at $x0$.
$\lvert . \rvert$	Hilbert norm.
$\mathcal{H}$	Hilbert space.
$\langle ., . \rangle$	duality bracket between X and its dual space.
$(., .)$	scalar product.
$\int_\Omega f(x)\, dx$	integral of f in Ω with respect to the Lebesgue measure.
supp u	Support of the function u.
div u	divergence of the vector u, $divu = \frac{\partial u}{dx_1} + \frac{\partial u}{dx_2} + ... + \frac{\partial u}{dx_N}$.
$\Delta u \ = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$	Laplacian of u.
$L^p(\Omega)$	Lebesgue space.
$L^p(\Omega)$	$= \{u : \Omega \to \mathbb{R}$ measurable and $\left(\int_\Omega \lvert u(x)\rvert^p dx\right)^{1/p} < +\infty$ such that $1 \leq p < \infty\}$.
$L^\infty(\Omega)$	$= \{u : \Omega \to \mathbb{R}$ measurable, $\exists M > 0 \mid \lvert u(x)\rvert \leq M$ a.e.$\}$.
q	Hölder conjugate of p : $q = \frac{p}{p-1}$ if $p > 1$ and $q = \infty$ if $p = 1$.
$L^q(\Omega) \subset L^p(\Omega), \quad \forall\, 1 \leq p \leq q < \infty$.	
$\lvert u\rvert_p$	$= \left[\int_\Omega \lvert u(x)\rvert^p dx\right]^{1/P} = \lvert u\rvert_{L^p}$.
$\lVert u\rVert_{L^\infty}$	$= \inf\{C : \lvert u(x)\rvert \leq C$ a.e. on $\Omega\}$.
$W^{1,p}(\Omega)$	$= \left\{u \in L^p(\Omega) \mid \nabla u \in (L^p(\Omega))^N\right\}$.
$W_0^{1,p}(\Omega)$	$= \left\{u \in W^{1,p}(\Omega), \quad \text{with} \quad u = 0 \quad \text{on} \quad \partial\Omega\right\}$.
$W^{1,2}(\Omega)$	$= H^1(\Omega)$.

$W_0^{1,2}(\Omega)$	$= H_0^1(\Omega)$.
$C_c(\Omega)$	space of continuous functions with compact.
$\mathcal{D}(\Omega)$	space of infinitely differentiable functions on Ω with compact support in Ω.
a.e.	almost everywhere.

Introduction

This thesis focuses on the existence of *weak solutions* for a class of *linear elliptic boundary value problems* involving *lower-order terms* and right-hand sides in $L^1(\Omega)$. The general form of the problem is:

$$(P)\begin{cases} -\operatorname{div}(M(x)\nabla u) + a(x)u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open domain, $M(x)$ is a bounded elliptic matrix, $a(x) \in L^1(\Omega)$ is a non-negative function, and $f \in L^1(\Omega)$ is the source term.

We first illustrate the main difficulties that may arise when studying of the problem (P).

- The irregularity of the problem (P) in the case where $f \in L^1(\Omega)$ i.e. the second member of (P) does not belong to space $H^{-1}(\Omega)$.

To solve this issue, we are going to approximate the problem (P), next, we will prove some uniform estimates on the sequence of approximate solutions, and we shall finally pass to the limit in the approximate problems to establish the existence of a weak solution for the problem (P).

In [1], David Arcoya et al. studied the existence of solutions for the problem (P) where $f \in L^1(\Omega)$. In this work, we follow the same ideas and method used in [1] to study a similar elliptic problem. We use the same approach to prove the existence of a solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

We are going to prove the existence of the weak solution of the problem (P). To do this, we approximate the problem (P) by a sequence of approximate problems (P_n) given in $L^{\infty}(\Omega)$ whose existence of the solution approximate is guaranteed (See [6]). Then we will prove some estimates uniform on the sequence of solutions of these problems (P_n) and their partial derivatives. Using the linearity of the operator with respect to the gradient and the boundedness and continuity of the map $u \mapsto \nabla u$, we can pass to the limit and obtain a solution.

Thesis Plan

- **The first chapter** is dedicated to giving some basic definitions and results with functional analysis tools essential to the achievement of the objectives for the study of the (P) problem. For example we recall functional spaces (Lebesgue, Sobolev) .

- **In the second chapter** we recall some definitions on the operators (Bounded, hime-continuous, Monotone and coercive). We also present the method of monotonic (pseudo-monotonic) operators in the general framework to prove the existence of a solution for the equation :

  $$Au = -\mathrm{div}(M(x)\nabla u) + a(x)u = f.$$

  Prove the existence and the regularity of the solution we use the Theorem 2.1.

- **In the third chapter**, we will study existence of solutions for the problem

  $$(P) \qquad \begin{cases} -\mathrm{div}(M(x)\nabla u) + a(x)u &= f \quad \Omega; \\ u &= 0 \quad \partial\Omega, \end{cases}$$

  with $M(x)$ such that (3.1) holds, and the functions $a(x)$ in $L^1(\Omega)$ and $f(x)$ in $L^1(\Omega)$ are such that there exists $Q > 0$ such that

  $$|f(x)| \leq Qa(x)$$

  For this equation we will prove existence of a solution $u \in H_0^1(\Omega) \cap L^{\infty}(\Omega)$.

Chapter 1

Preliminaries

In this chapter, we will introduce the basic concepts and analytical tools from Sobolev space. We will also clarify the main symbols used in it, which will contribute to a better understanding of the topic. **For more information on these concepts, we refer to** ([4],[2],[7])

1.1 Reminders and some definitions

The symbol Ω will always stand for an bounded open subset of $\mathbb{R}^N$. Its boundary will be denoted by Γ or $\partial\Omega$ and its closure by $\overline{\Omega}$.

The space $\mathbb{R}^N$ is endowed with the Lebesgue measure dx, and all integrals are to be considered in the sense of Lebesgue.

If $u : \Omega \to \mathbb{R}$ is differentiable, the gradient of u at $x = (x_1, ..., x_N)$ is the vector :

$$\nabla u(x) = \Big(\frac{\partial u}{\partial x_1}(x), ..., \frac{\partial u}{\partial x_N}(x)\Big).$$

Definition 1.1 (**Dual Space**). *Whenever X is a Banach space, we denote by X' its dual, namely the space of continuous linear functionals from X to $\mathbb{R}$. We recall that X' is a Banach space endowed with the norm:*

$$\|A\|_{X'} = \sup_{u\in X, \|u\|=1} |A(u)| = \sup_{u\in X, \|u\|\leq 1} |A(u)| = \sup_{u\in X, \|u\|\neq 0} \frac{|A(u)|}{\|u\|}.$$

Remark 1.1. *We denote by $W^{-1,p'}(\Omega)$ the dual space of $W_0^{1,p}(\Omega)$ (with $1 \leq p < \infty$). And By $H^{-1}(\Omega)$the dual space of $H_0^1(\Omega)$ we identify $L^2(\Omega)$ and its dual, we have the diagram:*

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$$

Definition 1.2 (**Separable Spaces**). . *We say that a Banach space E is Separable if there exists a countable and dense subset* $\mathcal{D} \subset E$

Definition 1.3 (**Reflexive Spaces**). . *Let E be a Banach space and let i be the canonical injection i :E→E" is surjective ,We say that E is reflexive if* $i(E) = E''$.

Proposition 1.1. *From[4]. Let E a separable metric spaces and let F a subset of E. Then F is separable*

Corollary 1.1. *Let E be a Banach space, and* E' *is dual,*
then (E is reflexive and separable)⇔(E′ is reflexive and separable)

1.2 Functional spaces

- $C^k(\Omega)$, for $k = 1, 2, \ldots$, is the space of functions $u : \Omega \to \mathbb{R}$ that are k times differentiable in Ω and whose k-th derivatives are continuous in Ω.
- $C^\infty(\Omega)$ is the space of functions $u : \Omega \to \mathbb{R}$ that are infinitely many times differentiable in Ω.
- $C_0^\infty(\Omega)$ is the subspace of $C^\infty(\Omega)$ consisting of functions with compact support in Ω; the support of a (continuous) function $u : \Omega \to \mathbb{R}$, denoted by $\operatorname{spt} u$, is the closure (in $\mathbb{R}^N$) of the set

$$\{x \in \Omega \mid u(x) \neq 0\}.$$

Likewise, $C_0^k(\Omega)$ is the subset of $C^k(\Omega)$ containing only functions with compact support.

1.2.1 Lebesgue spaces

Definition 1.4. *Let* Ω *be an open bounded set of* $\mathbb{R}^N$ *and let* $p \in \mathbb{R}$ *with* $1 \leq p < \infty$ *, we set*

$$L^p(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \quad \textit{is measurable and} \quad \int_\Omega \mid u(x) \mid^p < \infty \Big\}.$$

with the norm

$$\|u\|_{L^p(\Omega)} = \|u\|_p = \left(\int_\Omega |u(x)|^p \right)^{\frac{1}{p}}, \quad p \geq 1.$$

if p=2:

$$\|u\|_{L^2(\Omega)} = \|u\|_2 = \Big(\int_\Omega |u(x)|^2 \Big)^{\frac{1}{2}}$$

The space $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{2,\Omega} = \int_\Omega uvdx.$$

Definition 1.5. *we set*

$$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R} \ measurable, \exists C > 0, | \ |u(x)| \leq C \quad a.e. \quad on \quad \Omega\},$$

with the norm

$$\|u\|_{L^\infty(\Omega)} = \|u\|_\infty = inf \quad \{C > 0 :| \ u(x) \ |\leq C \quad a.e. \quad in \quad \Omega\}.$$

Proposition 1.2. *(**Density** [7]) Let Ω be an open subset of $\mathbb{R}^n$ and let p be a real number such that $1 < p < +\infty$. Then, $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$.*

1.2.2 Some Inequalities

Remark 1.2. *Let $1 \leq p \leq \infty$,be denoted by p' the conjugate exponent of p :*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Theorem 1.1 (**Hölder's inequality** [4]). *Let $1 \leq p \leq \infty$ and p' the conjugate exponent of p. Assume that $u \in L^p(\Omega)$, and $v \in L^{p'}(\Omega)$, then $u \cdot v \in L^1(\Omega)$ and:*

$$\| \ u \cdot v \ \|_{L^1(\Omega)} \leq \| \ u \ \|_{L^p(\Omega)} \| \ v \ \|_{L^{p'}(\Omega)} .$$

Remark 1.3. *The case p=2 is known as **the Cauchy-Schwarz inequality***

$$\int_\Omega uvdx = \Big(\int_\Omega |u(x)|^2 dx \Big)^{\frac{1}{2}} \Big(\int_\Omega |v(x)|^2 dx \Big)^{\frac{1}{2}}$$

Theorem 1.2 (**Young's inequality**[4]). *Let $1 < p < \infty$, then we have:*

$$ab \leq \frac{1}{p}a^p + \frac{1}{p'}b^{p'}, \quad \forall a \geq 0, \quad \forall b \geq 0 \tag{1.1}$$

Theorem 1.3 (**Green's formulas**[2]). *Let $\Omega \subset \mathbb{R}^N$ be open, bounded and smooth. Let $u \in C^2(\Omega)$ and $v \in C^1(\Omega)$ Then,*

$$\int_{\Omega} (\Delta u) v \quad dx = \int_{\partial\Omega} \frac{\partial u}{\partial \mathcal{V}} v \quad d\sigma - \int_{\Omega} \nabla u \cdot \nabla v \quad dx$$

Where $\mathcal{V}$ =$\mathcal{V}(x)$ is the outward normal to $\partial\Omega$ at x, $\frac{\partial u}{\partial \mathcal{V}}(x) = \nabla u(x) \cdot \mathcal{V}(x)$ and σ is the surface measure on $\partial\Omega$

1.2.3 Convergence theorems

Theorem 1.4 (Lebesgue's Dominated Convergence Theorem [2]). *Let $\Omega \subset \mathbb{R}^N$ be open and (u_n) be a sequence of functions in $L^1(\Omega)$ such that :*

- *$u_n(x) \to u(x)$ a.e in Ω as $n \to \infty$.*
- *there exists $g \in L^1$ such that for all n :*

$$|u_n(x)| \leq g(x) \quad a.e \ in \quad \Omega$$

Then $u \in L^1$and $u_n \to u$ in $L^1(\Omega)$ norm, that is :

$$\|u_n - u\|_{L^p(\Omega)} \to 0 \quad as \quad n \to \infty.$$

Theorem 1.5 (Lebesgue's Dominated Convergence Inverse Theorem [2]). *Let $\Omega \subset \mathbb{R}^N$ be open and (u_n) be a sequence of functions in $L^p(\Omega)$ and $p \in [1, \infty)$ such that $u_n \to u$ in $L^p(\Omega)$ as $n \to \infty$. Then there exist a subsequence u_{nk} and a function $h \in L^p(\Omega)$ such that*

- *$u_{nk}(x) \to u(x)$ a.e in Ω as $k \to \infty$*
- *for all k , $|u_{nk}(x)| \leq h(x)$ a.e in Ω .*

Theorem 1.6 (Fatou's Lemma [4]). *Let (u_n) be a sequence of function on of $L^1(\Omega)$ such that*

- *(1) for evry n ,$u_n(x) \geq 0$*
- *(2) $\sup\limits_n \int_\Omega u_n(x) \leq \infty$*

for every $x \in \Omega$ we put $u(x) = \liminf\limits_{n\to\infty} u_n(x)dx$. then $f \in L^1(\Omega)$ and

$$\int_\Omega u(x)dx \leq \liminf_{n\to\infty} \int_\Omega u_n(x)dx.$$

Theorem 1.7. *[3] Let $1 < p < +\infty$, and let $\{u_n\}$ be a bounded sequence in $W_0^{1,p}(\Omega)$. Then, there exists a subsequence u_{n_k} and a function $u \in W_0^{1,p}(\Omega)$ such that:*

$$\nabla u_{n_k} \rightharpoonup \nabla u \quad \textit{weakly in } L^p(\Omega).$$

Definition 1.6 (**Strong Convergence**). *In a Banach space X with dual X'.*

We write $u_n \to u$ when the sequence $(u_n)_{n\in N}$ converges strongly to u, that is, in the strong topology of X, which means that:

$$\|u_n - u\|_X \to 0 \quad as \quad n \to \infty$$

Definition 1.7 (**Weak Convergence**). *In a Banach space X with dual X', and $\langle .,. \rangle$ the duality bracket XX'.*

We write $u_n \rightharpoonup u$ if (u_n) converges weakly to u, i.e. in the weak topology of X, which means that:

$$\langle f, u_n \rangle \longrightarrow \langle f, u \rangle, \quad as \quad n \longrightarrow +\infty \quad \forall f \in X'.$$

1.3 Sobolev Spaces

In this section, we will study Sobolev spaces, which are normed vector spaces well suited for solving many problems involving partial differential equations.

Let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

Definition 1.8. *The Sobolev space* $W^{1,p}(\Omega)$ *, is defined by:*

$$W^{1,p}(\Omega) = \Big\{ u \in L^p(\Omega), \nabla u \in L^p(\Omega) \Big\}.$$

We denote

$$H^1(\Omega) = W^{1,2}(\Omega)$$

Remark 1.4. *The space* $W^{1,p}(\Omega)$ *is equipped with the norm*

$$\| \ u \ \|_{W^{1,p}(\Omega)} = \|u\|_{L^p} + \|\nabla u\|_{L^p} \tag{1.2}$$

The space H^1 *is a Hilbert space equipped with the scalar product*

$$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)}$$

that is

$$(u, v)_{H^1(\Omega)} = \int_\Omega u(x) \cdot v(x) \quad dx + \int_\Omega \nabla u(x) \cdot \nabla v(x) \quad dx \quad ;$$

the associated norm:

$$\|u\|_{H^1(\Omega)} = (\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)})^{1/2}$$

is a norm equivalent to that of $W^{1,2}(\Omega)$ *.*

Remark 1.5. *It is clear that if* $u \in C^1(\Omega) \cap L^p(\Omega)$ *and if* $\nabla u \in L^P(\Omega)$ *(here* ∇u *is the usual derivative of* u*), then* $u \in W^{1,p}(\Omega)$*. Moreover, the usual derivative of* u *coincides with the derivative of* u *in the sense of* $W^{1,p}(\Omega)$*. In particular, if* Ω *is bounded, then* $C^1(\overline{\Omega}) \subset W^{1,p}(\Omega)$ *for all* $1 \leq p \leq \infty$

Proposition 1.3. *From [4].*

1. $W^{1,p}(\Omega)$ *is a Banach space for*$1 \leq p \leq \infty$

2. $W^{1,p}(\Omega)$ *is reflexive for* $1 < p < \infty$

3. $W^{1,p}(\Omega)$ *is separable for* $1 \leq p < \infty$ *.*

In particular $H^1(\Omega)$ *is a separable Hillbert space .*

Definition 1.9. *Let* $1 \leq p < \infty$*;* $W_0^{1,p}(\Omega)$ *designates the closure of* $C_c^1(\Omega)$ *in* $W^{1,p}(\Omega)$ *we denote :*

$$H_0^1(\Omega) = W_0^{1,2}(\Omega)$$

The space $W_0^{1,p}(\Omega)$ *equipped with the norm induced by* $W^{1,p}(\Omega)$ *is a separable Banach space; it is reflexive if* $1 < p < \infty$. $H_0^1(\Omega)$ *is a Hilbert space for the scalar product of* $H^1(\Omega)$*.*

Definition 1.10. *Let* $1 \leq p < \infty$, $W_0^{1,p}(\Omega)$ *denoted the closure of* $\mathcal{D}(\Omega)$ *in* $W^{1,p}(\Omega)$ *and*

$$W_0^{1,p}(\Omega) = \Big\{ u \in W^{1,p}(\Omega), \quad with \quad u = 0 \quad on \quad \partial\Omega \Big\}.$$

With the norm:

$$\| u \|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_{L^p}$$

if p=2 the norm :

$$\| u \|_{H_0^1(\Omega)} = \|\nabla u\|_{L^2}$$

Lemma 1.1. *From[4]. Let* $u \in W^{1,p}(\Omega)$ *and* $1 \leq p < \infty$ *With compact support of* u *included in* Ω *then* $u \in W_0^{1,p}(\Omega)$

Proposition 1.4 (**Poincare's inequality**[4])**.** *Assume that* Ω *is a bounded open set. Then there exists a constant* C *(depending on* Ω *and* p*) such that:*

$$\| u \|_{L^p(\Omega)} \leq C \| \nabla u \|_{L^p(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega), \quad (1 \leq p < \infty). \tag{1.3}$$

Corollary 1.2. *In particular, the expression* $\| \nabla u \|_{L^p(\Omega)}$ *is a norm on* $W_0^{1,p}(\Omega)$ *that is equivalent to the norm* $\| u \|_{W^{1,p}(\Omega)}$*; on* $H_0^1(\Omega)$ *the expression* $\| \nabla u \|_{L^2(\Omega)}$ *that is equivalent to the norm* $\| u \|_{H^1(\Omega)}$*.*

1.3.1 Sobolev embeddings

Theorem 1.8 (**Continuous embedding**)**.** *[4] Assume that* Ω *is bounded open subset of* $\mathbb{R}^N$ *with* $N \geq 1$ *and* $1 \leq p < \infty$*.*

- $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ $,\forall q \in [1, p^*[$ *if* $p < N$. *Where* $p^* = \frac{Np}{N-p}$
- $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ $,\forall q \in [1, \infty[$ *if* $p = N$.
- $W_0^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$. *,if* $p > N$.

Theorem 1.9 (**Compact embedding (Rellich-Kondrachon)**[4])**.** *Assume that* Ω *is bounded domain of class* C^1 *we have :*

- $W_0^{1,p}(\Omega) \hookrightarrow_c L^q(\Omega)$ $,\forall q \in [1, p^*[$ *if* $p < N$. *Where* $p^* = \frac{Np}{N-p}$
- $W_0^{1,p}(\Omega) \hookrightarrow_c L^q(\Omega)$ $,\forall q \in [1, \infty[$ *if* $p = N$.
- $W_0^{1,p}(\Omega) \subset C(\overline{\Omega})$ *,if* $p > N$.

Remark 1.6.

1. *If* Ω *is not bounded,the embedding* $W^{1,p}(\Omega) \subset L^p(\Omega)$ *is not compact in general.*
2. *The injection* $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$*is never compact even if* Ω *is bounded and regular*

CHAPTER 2

Elliptic equations with regular data

In this chapter, we study the existence of weak solutions for linear elliptic boundary value problems with regular data. The focus is on establishing existence results under classical assumptions, where the source term f and the coefficients of the equation are sufficiently regular, typically belonging to $L^2(\Omega)$ or $L^\infty(\Omega)$. To this end, we introduce several classes of operators such as bounded, monotone, hemicontinuous, coercive, and pseudo-monotone operators. These properties are fundamental in the framework of functional analysis and play a crucial role in proving the existence of solutions via monotonicity methods.

2.1 Properties of Operators

2.1.1 The p-Laplacian

The p-Laplacian is the second-order nonlinear differential operator defined as

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right).$$

Here, $p > 1$, and of course, when $p = 2$, the p-Laplacian reduces to the usual Laplacian.

$$\Delta_p u = \Delta u$$

2.1.2 Monotone Operators

Definition 2.1. *Let X is a reflexive and separable Banach space, and A is an application from $X \to X'$ (generally non-linear). We say that:*

(i) *A is monotone :*

$$\forall u, v \in X, \quad \langle Au - Av, u - v\rangle \geq 0.$$

(ii) *A is strictly monotone if moreover :*

$$\forall u, v \in X, \quad \langle Au - Av, u - v\rangle > 0, \quad u \neq v$$

Example 2.1. *We define $Au = -\Delta u$. The operator A maps $H_0^1(\Omega)$ into its dual $H^{-1}(\Omega)$. It is monotone :*

for all $u, v \in H_0^1(\Omega)$ we have:

$$\begin{aligned}\langle Au - Av, u - v\rangle &= \int_\Omega \nabla(u-v).\nabla(u-v)\\ &= \|u-v\|^2_{H_0^1(\Omega)}\\ &\geq 0.\end{aligned}$$

2.1.3 Hemicontnuous Operators

Definition 2.2. *Let X is a reflexive and separable Banach space, and A is an application from $X \to X'$. We say that:*

A is hemicontinuous if for all $u, v \in X$, the application $t \mapsto \langle A(u+tv), v\rangle$ is continuous from $\mathbb{R}$ to $\mathbb{R}$

Example 2.2. *The operator $A : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ defined by*

$Au = -\Delta u$ is hemicontinuous. Indeed, let $u, v \in H_0^1(\Omega)$ and $t \in \mathbb{R}$:

$$\begin{aligned}\langle A(u+tv), v\rangle &= \int_\Omega \nabla(u+tv).\nabla v\\ &= \int_\Omega \nabla u.\nabla v + t\int_\Omega \nabla v \nabla v\\ &= a + bt.\end{aligned}$$

This shows that $t \longrightarrow \langle A(u+tv), v\rangle$ is continuous.

2.1.4 Coercive Operator

Definition 2.3. *An operator $A : X \to X^{'}$, we say that A is coercive if*

$$\lim_{\|v\|_X \to +\infty} \frac{\langle Av, v\rangle}{\|v\|_X} = +\infty.$$

Example 2.3. *The operator* $A : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ *defined by :*

$Av = -\mathrm{div}(\nabla v) = -\Delta v.$

$$\frac{\langle Av, v\rangle}{\|v\|_{H_0^1}} = \frac{\int_\Omega |\nabla v|^2}{\|v\|_{H_0^1}} \longrightarrow +\infty \quad \text{as} \quad \|v\|_{H_0^1} \to +\infty$$

2.1.5 Bounded Operator

Definition 2.4. *Let* V *and* W *be two Banach spaces and let* $A : V \to W$ *be an operator. We say that* A *is bounded if it maps every bounded set in* V *to a bounded set in* W*; i.e*

$$\forall \rho > 0, \quad \exists C_\rho > 0 : \quad A(B_V(0,\rho)) \subset B_W(0, C_\rho).$$

where $B_V(0,\rho)$ *designates the open ball in* V *with center* 0 *and radius* $\rho > 0$ *and* $B_W(0, C_\rho)$ *designates the open ball in* W *with center* 0 *radius* $C_\rho > 0$ *.*

Example 2.4. *The operator* $Au = -\triangle_p$ *is bounded from* $W_0^{1,p}(\Omega)$ *in* $W^{-1,p'}(\Omega)$

From the expression of the norm in a dual space, let $\rho > 0$*, for* $u \in B_V(0,\rho)$*, we can write:*

$$\| Au \|_{V'} = \sup_{\substack{\varphi \in V \\ \|\varphi\| \leq 1}} |\langle Au, \varphi\rangle| = \sup_{\substack{\varphi \in V \\ \|\varphi\| \leq 1}} \left| \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla \varphi \right|.$$

So,

$$\begin{aligned} \left| \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla \varphi \right| &\leq \int_\Omega |\nabla u|^{p-1} . |\nabla \varphi| \\ &\leq \left(\int_\Omega |\nabla u||^p \right)^{\frac{1}{p'}} \left(\int_\Omega |\nabla \varphi|^p \right)^{\frac{1}{p}} \\ &= \| u \|_V^{p-1} \| \varphi \|_V \\ &\leq \rho^{p-1} \end{aligned}$$

Hence $\| Au \|_{V'} \leq \rho^{p-1}$*. this shows that* $A(B_V(0,\rho)) \subset B_{V'}(0, \rho^{p-1})$

2.1.6 Pseudo-monotone Operators

Definition 2.5. *Let* X *be a separable and reflexive Banach space. An operator* $A : X \to X'$ *is pseudo-monotone if:*

(i) A is bounded.

(ii) whenever $u_n \rightharpoonup u$ in X weak and if $\limsup_{n\to+\infty}\langle A(u_n), u_n - v\rangle \leq 0$ then

$$\liminf_{n\to+\infty}\langle A(u_n), u_n - v\rangle \geq \langle A(u), u - v\rangle, \quad \forall v \in X. \tag{2.1}$$

where $\langle . \rangle$ refers to the duality product between X' and X.

Proposition 2.1. *If $A : X \to X^{'}$ is bounded, hemicontinuous and monotonic, then A is pseudo-monotone.*

Proof. a) Let $\{u_n\}$ a sequence weakly converging to u in X. Suppose that

$$\lim_{n\to\infty}\sup\langle Au_n, u_n - u\rangle \leq 0$$

If A is monotone, we have

$$\lim_{n\to\infty} \quad \langle Au_n, u_n - u\rangle \quad \to \quad 0 \tag{2.2}$$

Indeed, the monotonicity of A and the weak convergence of $\{u_n\}$ towards u implies that

$$\langle Au_n, u_n - u\rangle \geq \langle Au, u_n - u\rangle \to 0 \quad pour \quad n \to \infty$$

And so

$$0 \geq \lim_{n\to\infty}\sup\langle Au_n, u_n - u\rangle \geq \lim_{n\to\infty} inf\langle Au_n, u_n - u\rangle \geq \lim_{n\to\infty}\sup\langle Au, u_n - u\rangle = 0$$

Hence (2.2)

b) for $v \in X$ and $t \in]0, 1[$, let $w = (1 - t)u + tv$. We have $\langle Au_n - Aw, u_n - w\rangle$ so that:

$$t\langle Au_n, u - v\rangle \geq -\langle Au_n, u_n - u\rangle + \langle Aw, u_n - u\rangle - t\langle Aw, v - u\rangle.$$

From which, thanks to à (2.2):

$$t\lim_{n\to\infty} inf\langle Au_n, u - v\rangle \geq -t\langle Aw, v - u\rangle,$$

from which, dividing by t and taking into account (2.2):

$$\lim_{n\to\infty} inf\langle Au_n, u_n - v\rangle \;\geq\; \langle Aw, u-v\rangle \tag{2.3}$$

$$w = (1-t)u + tv \quad \forall t \in]0,1[$$

By making t tend towards 0 in (2.3), and using hemicontinuity, we deduce

$$\lim_{n\to\infty} inf\langle Au_n, u_n - v\rangle \geq \langle Au, u-v\rangle, \quad \forall v \in X$$

Which means that A is pseudo-monotonic. □

2.2 Application

We consider the following problem :

$$(P_0) \qquad \begin{cases} -\text{div}\,(M(x)\nabla u) + a(x)u = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Where Ω be a bounded open set in $\mathbb{R}^n$, where M is a bounded elliptic matrix which satisfies, for some positive constants α, β, for every $\xi \in \mathbb{R}$

$$M(x)\xi\xi \geq \alpha|\xi|^2, \qquad |M(x)| \leq \beta. \tag{2.4}$$

and $0 \leq a(x) \in L^\infty$ and $f \in L^\infty(\Omega)$.

We define the following operator:

$$A(u) = -\,\text{div}(M(x)\nabla u) + a(x)u, \tag{2.5}$$

2.3 Monotonic operator theorems

2.3.1 General result

Let X be a reflexive and separable Banach space, and $A : X \to X'$ be an operator.

Theorem 2.1 (Existence theorem). *Suppose that* $A : X \longrightarrow X'$ *is an operator :*

- *bounded*
- *hemicontinuous*
- *coercive*
- *monotone*

Let $f \in X'$. Then, there exists a function $u \in X$ and only one such that:

$$Au = f$$

For the proof this theorem, we refer the reader to the book [6]

Corollary 2.1. *If A is a coercive pseudomonotone operator, then*

$$\forall f \in X', \quad \exists u \in X \quad \textit{such that} \quad Au = f.$$

Lemma 2.1. *The operator A is pseudo-monotone from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$. Moreover, A is coercive in the following sense.*

$$\frac{\langle Au, u\rangle}{\|u\|_{H_0^1(\Omega)}} \to \infty \quad as \quad \|u\|_{H_0^1(\Omega)} \to \infty$$

Proof. .

(1) Proof of Coercivity:

We compute the inner product of the operator A with u in the space $H_0^1(\Omega)$:

$$\langle A(u), u\rangle = \int_\Omega A(u)u\,dx = \int_\Omega M(x)\nabla u \cdot \nabla u\,dx + \int_\Omega a(x)u^2\,dx.$$

Since $M(x)$ satisfies the ellipticity condition (2.4),

$$M(x)\nabla u \cdot \nabla u \geq \alpha|\nabla u|^2, \quad \forall x \in \Omega.$$

Thus, we obtain the following estimate:

$$\int_\Omega A(u)u\,dx \geq \alpha \int_\Omega |\nabla u|^2\,dx + \int_\Omega a(x)u^2\,dx \geq \alpha \int_\Omega |\nabla u|^2\,dx = \alpha\|u\|^2_{H_0^1(\Omega)}.$$

which implies

$$\lim_{\|u\|_{H_0^1(\Omega)}\to+\infty} \frac{\langle Au, u\rangle}{\|u\|_{H_0^1(\Omega)}} \longrightarrow +\infty.$$

as $\|u\|_{H_0^1(\Omega)} \longrightarrow +\infty$ since $\alpha > 0$.

(2) Proof of Pseudo-Monotonicity:

(i) A **is bounded.** Indeed, let u be a bounded function in $H_0^1(\Omega)$ that is : let $\rho > 0$, for $u \in B_{H_0^1(\Omega)}(0,\rho)$, and for all $v \in H_0^1(\Omega)$, we have

$$\| Au \|_{H_0^1(\Omega)} = \sup_{\substack{v\in H_0^1(\Omega)\\ \|v\|\leq 1}} |\langle Au, v\rangle|.$$

then, we have

$$\langle Au, v\rangle = \int_\Omega M(x)\nabla u.\nabla v dx + \int_\Omega a(x)u.v dx$$

So,

$$|\langle Au, v\rangle| \leq \int_\Omega |M(x)||\nabla u|.|\nabla v| dx + \int_\Omega a(x)|u|.|v| dx$$

The fact that $a \in L^\infty(\Omega)$, we have

$$|\langle Au, v\rangle| \leq \beta \int_\Omega |\nabla u|.|\nabla v| dx + \operatorname{ess\,sup}_{x\in\Omega} a(x) \int_\Omega |u|.|v| dx$$

Using the cauchy Schwarz inequality, we obtain

$$|\langle Au, v\rangle| \leq \beta \|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)} + C\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}, \quad C = \sup_{x\in\Omega} a(x).$$

Since $\|u\|_{L^2(\Omega)} \leq \|u\|_{H_0^1(\Omega)}$ and $\|v\|_{H_0^1(\Omega)} \leq 1$, we have

$$|\langle Au, v\rangle| \leq (\beta + C)\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)} \leq (\beta + C)\rho.$$

Hence $\| Au \|_{H_0^1(\Omega)} \leq (\beta + C)\rho$. This shows that

$$A(B_{H_0^1(\Omega)}(0,\rho)) \subset B_{H_0^1(\Omega)}(0,(\beta + C)\rho).$$

(ii) If $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, as $n \to +\infty$, and for any $v \in H_0^1(\Omega)$

$$\begin{aligned} 0 &\geq \limsup_{n\to+\infty} \langle Au_n, u_n - v\rangle \\ &= \limsup_{n\to+\infty} \left(\int_\Omega M(x)\nabla u_n \nabla(u_n - v) dx + \int_\Omega a(x)u.(u_n - v) dx \right). \end{aligned}$$

The compact embedding (see Theorem 1.8), yields that $u_n \longrightarrow u$ in $L^2(\Omega)$ for a subsequence still denoted as (u_n). Moreover, we assume that $u_n \longrightarrow u$ a.e. in Ω, and using the fact that $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\Omega)$, we get

$$M(x)\nabla u_n \rightharpoonup M(x)\nabla u \quad \text{weakly in} \quad L^2(\Omega), \tag{2.6}$$

and

$$a(x)u.(u_n - u)dx \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty$$

and we have that

$$\int_\Omega (M(x)\nabla u_n - M(x)\nabla u)\, \nabla(u_n - u)dx \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty \tag{2.7}$$

Now, let us prove that

$$\liminf_{n\to+\infty} \langle Au_n, u_n - v\rangle \geq \langle Au, u - v\rangle,\ \forall v \in H_0^1(\Omega).$$

Because that

$$M(x)\nabla u_n \rightharpoonup M(x)\nabla u \quad \text{weakly in} \quad L^2(\Omega),$$

we have

$$\lim_{n\to+\infty} \int_\Omega M(x)\nabla u_n \nabla v dx = \int_\Omega M(x)\nabla u \nabla v dx,\ \forall v \in H_0^1(\Omega). \tag{2.8}$$

By virtue of Fatou's lemma, we get

$$\liminf_{n\to+\infty} \int_\Omega M(x)\nabla u_n \nabla u_n dx \geq \int_\Omega M(x)\nabla u \nabla u dx \tag{2.9}$$

Finally, combining (2.8) and (2.9), we obtain

$$\begin{aligned}\liminf_{n\to+\infty} \langle Au_n, u_n - v\rangle &= \liminf_{n\to+\infty} \int_\Omega M(x)\nabla u_n \nabla(u_n - v)dx \\ &\geq \int_\Omega M(x)\nabla u \nabla(u - v)dx = \langle Au, u - v\rangle.\end{aligned}$$

This completes the proof that A is pseudo-monotone. Then, according to the corollary 2.1, there exists at least one solution u in $H_0^1(\Omega)$ to Problem (P_0).

□

Chapter 3

Weak Solutions for elliptic equations with L^1 data

In this chapter, we study the existence of weak solutions for a class of elliptic boundary value problems involving an elliptic operator with minimal assumptions on the coefficients. Specifically, we consider a lower-order term and a right-hand side belonging to $L^1(\Omega)$, which does not necessarily lie in the dual space of the solution space.

3.1 Application to the problem (P)

We are interested in problems of the type

$$(P) \quad \begin{cases} -\mathrm{div}\,(M(x)\nabla u) + a(x)u = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$, where M is a bounded elliptic matrix which satisfies, for some positive constants α, β, for every $\xi \in \mathbb{R}$

$$M(x)\xi\xi \geq \alpha|\xi|^2, \qquad |M(x)| \leq \beta. \tag{3.1}$$

and $0 \leq a(x) \in L^1$. Even if $f(x)$ only belongs to $L^1(\Omega)$, the assumption

$$f(x), a(x) \in L^1(\Omega) \tag{3.2}$$

$$\text{there exists} \quad Q > 0 \quad \text{such that} \quad |f(x)| \leq Q.a(x). \tag{3.3}$$

implies the existence of a weak solution u belonging to $H_0^1(\Omega)$ and to $L^\infty(\Omega)$.

3.1.1 Weak solutions

Definition 3.1. *We say that* $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *is a weak solution of problem (p) if and only if:*

$$\int_\Omega M(x)\nabla u \cdot \nabla v \, dx + \int_\Omega a(x)u.v \, dx = \int_\Omega f(x)v \, dx \quad \forall v \in H_0^1(\Omega) \cap L^\infty(\Omega). \tag{3.4}$$

Now, The result proved in this chapter is the following theorem

Theorem 3.1. *Assume* (3.1)*,* (3.2)*) and* (3.3)*.Then there exists a unique solution* $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *of problem (P) in the sense of Definition 3.1.*

The proof of the Theorem 3.1 needs several steps: First, we approximate the problem (P) with sequence of problems (P_n) having smooth solutions (u_n). Then, after deriving uniform estimates on u_n , we pass to the limit

3.1.2 Approximation of (P)

Consider the following approximated problem

$$(P_n) \quad \begin{cases} -\operatorname{div}(M(x)\nabla u_n) + a_n(x)u_n = f_n(x), & \text{in } \Omega, \\ u_n = 0, & \text{on } \partial\Omega. \end{cases}$$

i.e., satisfying

$$\int_\Omega M(x)\nabla u_n \cdot \nabla v \, dx + \int_\Omega a_n(x)u_n.v \, dx = \int_\Omega f_n(x)v \, dx \tag{3.5}$$

for every $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$, with f_n, a_n be a sequence of bounded functions defined in Ω defined by

$$a_n(x) = \frac{a(x)}{1 + \frac{Qa(x)}{n}}, \quad f_n(x) = \frac{f(x)}{1 + \frac{|f(x)|}{n}}. \tag{3.6}$$

Note that the definition of a_n and f_n in (3.6) and since $\psi(s) = s(1 + \frac{s}{n})^{-1}$ is increasing and satisfies $\psi(s) \leq s$ for all $s \geq 0.$, we obtain:

(1) L^1**-norm estimates:** Since

$$|f_n(x)| = \frac{|f(x)|}{1 + \frac{|f(x)|}{n}} = \psi(|f(x)|) \leq |f(x)|, \quad \forall x \in \Omega.$$

And also we have:

$$|a_n(x)| \leq |a(x)|, \quad \forall x \in \Omega.$$

We get

$$\|f_n\|_{L^1(\Omega)} = \int_\Omega |f_n(x)|\,dx \leq \int_\Omega |f(x)|\,dx = \|f\|_{L^1(\Omega)}.$$

And also,

$$\|a_n\|_{L^1(\Omega)} \leq \|a\|_{L^1(\Omega)}.$$

(2) **Pointwise bounds:** Using the expression:

$$\psi(s) = \frac{s}{1+\frac{s}{n}} = \frac{ns}{n+s},$$

we observe that:

$$\frac{ns}{n+s} \leq n \quad \text{for all } s \geq 0.$$

Therefore,

$$|f_n(x)| = \psi(|f(x)|) \leq n, \quad \forall x \in \Omega.$$

And also we have:

$$|a_n(x)| \leq n, \quad \forall x \in \Omega.$$

(3) **Estimate: $|f_n(x)| \leq Qa_n(x)$:**

From assumption (3.3), and the function $\psi(s) = \frac{s}{1+\frac{s}{n}}$ being increasing in $s \geq 0$, we obtain:

$$|f_n(x)| = \frac{f(x)}{1+\frac{|f(x)|}{n}} \leq \frac{Qa(x)}{1+\frac{Qa(x)}{n}} = Qa_n(x). \tag{3.7}$$

We Conclude:

$$\begin{cases} \|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}, & \|a_n\|_{L^1(\Omega)} \leq \|a\|_{L^1(\Omega)}, \\ |f_n| \leq n, & |a_n| \leq n. \end{cases}$$

and

$$|f_n(x)| \leq Qa_n(x).$$

The operator $A_n : H_0^1(\Omega) \to H^{-1}(\Omega)$ defined by

$$\langle A_n u, v\rangle = \int_\Omega M(x)\nabla u_n \cdot \nabla v \, dx + \int_\Omega a_n(x) u_n . v \, dx, \quad \forall v \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

is coercive due to the fact (3.1) and is pseudo-monotone on $H_0^1(\Omega)$ Thus, the existence of the approximate solution u_n is proved in **Lemma 2.1** (see chapter 2) .

3.2 Uniform Estimates of Approximate solutions

Lemma 3.1 (Stampacchia)**.** *Let $T : \mathbb{R} \to \mathbb{R}$ be a globally lipschitz function,i.e.*

$$\exists C > 0 \quad \textit{such that} \quad |T(s) - T(t)| \leq C|s - t|, \quad \forall s, t \in \mathbb{R},$$

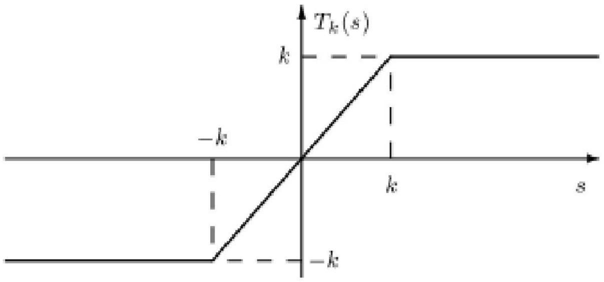
such that $T(0) = 0$. then, $\forall v \in W_0^{1,p}(\Omega)$ with $1 \leq p \leq \infty$ we have :

$$T(v) \in W_0^{1,p}(\Omega) \quad \textit{and} \quad \nabla T(v) = T'(v)\nabla v \quad \textit{in} \quad \mathcal{D}'(\Omega) \quad \textit{and} \quad \textit{a.e.} \quad \textit{in} \quad \Omega$$

Example 3.1. *Let $k > 0$. The truncation at levels $-k$ and k is defined by the function T_k of $\mathbb{R}$ from $\mathbb{R}$ given by*

$$T_k(s) = \begin{cases} k, & \textit{if } s \geq k, \\ s, & \textit{if } |s| < k, \\ -k, & \textit{if } s \leq -k. \end{cases}$$

It can be verified that the function T_k is a globally Lipschitz function, The graph of T_k is:



We can verify that the function T_k is a globally Lipschitz function satisfying $|T_k(s)| \leq k$ and $|T_k(s)| \leq |s|$.

Example 3.2. *Let $k > 0$. We will use the following function defined for $s \in \mathbb{R}$ by*

$$G_k(r) = r - T_k(r) = \begin{cases} r-k, & \text{if } r > k, \\ 0, & \text{if } |r| \leq k, \\ r+k, & \text{if } r < -k. \end{cases}$$

so,

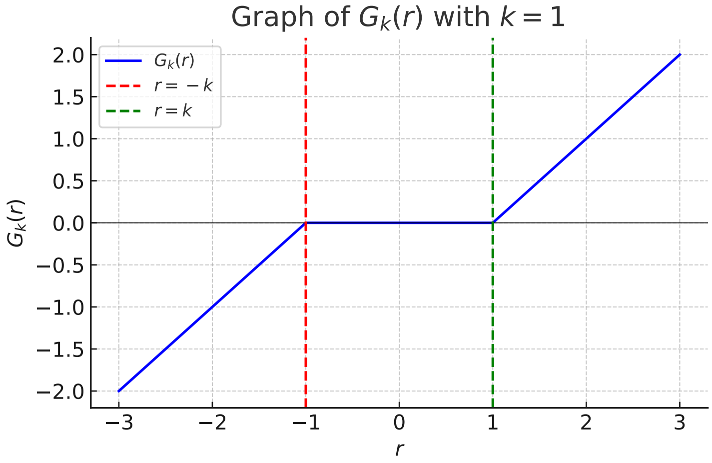


Figure 3.1: Graph of the function $G_k(r) = r - T_k(r)$ for $k = 1$.

$$G_k(u_n) = u_n - T_k(u_n) = \begin{cases} u_n-k, & \text{if } u_n > k, \\ 0, & \text{if } |u_n| \leq k, \\ u_n+k, & \text{if } u_n < -k. \end{cases}$$

and

$$\nabla G_k(u_n) = \begin{cases} \nabla u_n, & \text{if } |u_n| > k, \\ 0, & \text{if } |u_n| \leq k. \end{cases}$$

Lemma 3.2. *Let u_n be the solutions to problems (P_n). Then, there exists a positive constant C such that*

$$\|u_n\|_{L^\infty(\Omega)} \leq C, \quad \|u_n\|_{H_0^1(\Omega)} \leq C, \quad \forall n \in \mathbb{N}.$$

Proof. We choose $G_Q(u_n)$ as test function in (P_n) , we get

$$\int_\Omega M(x)\nabla u_n \nabla(G_Q(u_n))dx + \int_\Omega a_n(x)u_n G_Q(u_n)dx = \int_\Omega f_n(x)G_Q(u_n)dx. \tag{3.8}$$

Since $\nabla u_n.\nabla(G_Q(u_n)) = |\nabla G_Q(u_n)|^2$ and thanks to (3.1), we obtain

$$\alpha \int_\Omega |\nabla G_Q(u_n)|^2 dx + \int_\Omega a_n(x) u_n G_Q(u_n) dx \leq \int_\Omega |f_n||G_Q(u_n)|dx. \tag{3.9}$$

The fact that $u_n G_Q(u_n) > 0$ and using (3.3), we obtain

$$\alpha \int_\Omega |\nabla G_Q(u_n)|^2 dx + \int_\Omega a_n(x)(|u_n| - Q)|G_Q(u_n)|dx \leq 0.$$

Dropping the non negative term, we have

$$\int_\Omega a_n(x)(|u_n| - Q)|G_Q(u_n)|dx \leq 0$$

which implies

$$|u_n| \leq Q. \tag{3.10}$$

This implies that $(u_n)_n$ is bounded in $L^\infty(\Omega)$.

Now, to show that the sequence $\{u_n\}_n$ is bounded in $H_0^1(\Omega)$, inserting u_n in (P_n), we get

$$\alpha \int_\Omega |\nabla u_n|^2 dx + \int_\Omega a_n(x)|u_n|^2 dx \leq \int_\Omega |f_n||u_n|dx,$$

dropping non negative terms $\int_\Omega a_n(x)|u_n|^2 dx$ and using (3.10), we can rewrite the above inequality as follows

$$\alpha \int_\Omega |\nabla u_n|^2 dx \leq Q\|f_n\|_{L^1(\Omega)} \leq Q\|f\|_{L^1(\Omega)}.$$

Since $f \in L^1(\Omega)$, we find

$$\int_\Omega |\nabla u_n|^2 dx \leq C \quad ,C = \frac{Q}{\alpha}\|f\|_{L^1(\Omega)}.$$

Thus,

$$\|u_n\|_{H_0^1(\Omega)} \leq C. \tag{3.11}$$

This implies that $(u_n)_n$ is bounded in $H_0^1(\Omega)$. □

3.3 Passing to the limit

Proposition 3.1. *since $(u_n)_n$ is bounded in the space $H_0^1(\Omega)$, according to Rellich-Kondrachov theorem, we can extract from the sequence (u_n) a sub-sequence, also denote by (u_n) such that*

$$u_n \rightharpoonup u \quad \textit{weakly in} \quad H_0^1(\Omega), \tag{3.12}$$

$$u_n \longrightarrow u \quad \textit{strongly in} \quad L^2(\Omega). \tag{3.13}$$

Therefore

$$u_n \longrightarrow u \quad \textit{a.e in} \quad \Omega, \tag{3.14}$$

For $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ (See (3.5)), we have

$$\int_\Omega M(x)\nabla u_n \nabla v dx + \int_\Omega a_n(x) u_n v dx = \int_\Omega f_n v dx,$$

1)Passage to the limit in $\int_\Omega f_n v dx$

Using that $f_n \to f$ strongly in $L^1(\Omega)$, we have $\forall v \in L^\infty(\Omega)$

$$\lim_{n\to+\infty} \int_\Omega f_n v dx = \int_\Omega f v dx.$$

2)Passage to the limit in $\int_\Omega a_n(x) u_n v dx$.

Thanks to (3.10), we have

$$\int_\Omega a_n(x) u_n dx \leq \int_\Omega a_n(x)|u_n| dx \leq Q \int_\Omega a_n(x) dx. \tag{3.15}$$

Since $a(x) \in L^1(\Omega)$ and $a_n \leq a$, the sequence $\{a_n(x)u_n\}_n$ is bounded in $L^1(\Omega)$ and thanks to (3.14), we have that

$$a_n(x)u_n \to a(x)u \quad \text{strongly in} \quad L^1(\Omega). \tag{3.16}$$

So, we have

$$\begin{aligned} \left| \int_\Omega a_n(x)u_n v - \int_\Omega a(x)uv \right| &= \left| \int_\Omega (a_n(x)u_n - a(x)u)v \right| \\ &\leq C\|a_n(x)u_n - a(x)u\|_{L^1(\Omega)} \to 0, \end{aligned}$$

as $n \to +\infty$

ensure that

$$\lim_{n\to+\infty}\int_\Omega a_n(x)u_n v dx = \int_\Omega a(x)uvdx.$$

3)Passage to the limit in $\int_\Omega M(x)\nabla u_n \nabla v dx$.

By (3.11), we have that $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\Omega)$ (see Thoerem 1.7), we obtain

$$M(x)\nabla u_n \rightharpoonup M(x)\nabla u \quad \text{weakly in} \quad L^2(\Omega),$$

So,

$$\lim_{n\to+\infty}\int_\Omega M(x)\nabla u_n \nabla v dx = \int_\Omega M(x)\nabla u \nabla v dx,\ \forall v \in H_0^1(\Omega)\cap L^\infty(\Omega). \tag{3.17}$$

Finally, we pass to the limit in the formulation (3.5), we obtain

$$\int_\Omega M(x)\nabla u \nabla v dx + \int_\Omega a(x)uvdx = \int_\Omega fvdx,$$

for every $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$. Therefore. we have to prove that u is a solution to problem (P). This finishes the proof of theorem 3.1.

Conclusion

In this thesis, we focused on a class of linear elliptic equations involving lower-order terms and right-hand sides in $L^1(\Omega)$.

$$(P) \quad \begin{cases} -\mathrm{div}\,(M(x)\nabla u) + a(x)u = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

We investigated the existence of weak solutions for such equations, which involve lower-order terms and integrable data. The main challenge lies in the low regularity of the source term, which prevents the direct use of classical variational methods.

To overcome this difficulty, we employed an approximation strategy based on truncation techniques. This involved constructing a sequence of auxiliary problems for which existence is guaranteed.

After establishing uniform estimates for the approximate solutions, we used tools from functional analysis such as monotonicity, pseudo-monotonicity, and weak convergence to pass to the limit and prove the existence of a weak solution to the original problem.

These results generalize existing theories to cases with lower regularity data and open the way for future studies on the uniqueness, regularity, and numerical approximation of weak solutions.

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Abstract

In this work, we study the existence of weak solutions for a class of linear elliptic equations with lower-order terms and integrable data. More precisely, we consider problems of the form:

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + a(x)u = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $M(x)$ is an elliptic matrix, $a(x) \in L^1(\Omega)$, and $f \in L^1(\Omega)$. Since the right-hand side lies in $L^1(\Omega)$, standard variational methods are not applicable.

To address this, we construct a sequence of approximate problems whose solutions are well-defined, and establish uniform a priori estimates. Then, using compactness arguments and the theory of pseudo-monotone operators, we prove the existence of a weak solution to the original problem.

Keywords: Elliptic equations, weak solution, integrable data, pseudo-monotone operators, lower-order terms.

Résumé

Dans ce travail, nous étudions l'existence de solutions faibles pour une classe d'équations elliptiques linéaires comportant des termes d'ordre inférieur et un second membre dans $L^1(\Omega)$. Le problème étudié est de la forme :

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + a(x)u = f(x), & \text{dans } \Omega, \\ u = 0, & \text{sur } \partial\Omega, \end{cases}$$

où $\Omega \subset \mathbb{R}^N$ est un domaine borné, $M(x)$ est une matrice elliptique, $a(x) \in L^1(\Omega)$ et $f \in L^1(\Omega)$. En raison de la faible régularité des données, les méthodes variationnelles classiques ne s'appliquent pas.

Nous avons contourné cette difficulté en construisant une suite de problèmes approchés, pour lesquels l'existence est assurée, puis nous avons obtenu des estimations a priori uniformes. Grâce à des arguments de compacité et à la théorie des opérateurs pseudo-monotones, nous établissons l'existence d'une solution faible au problème initial.

Mots-clés : Équations elliptiques, solution faible, données intégrables, opérateurs pseudo-monotones, termes d'ordre inférieur.

ملخص

في هذا العمل، قمنا بدراسة وجود الحلول الضعيفة لفئة من المعادلات الإهليلجية الخطية التي تحتوي على حدود من الرتبة الدنيا وطرف أيمن ينتمي إلى الفضاء $L^1(\Omega)$. تتمثل المعادلة المدروسة في الشكل:

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + a(x)u = f(x), & \text{في } \Omega, \\ u = 0, & \text{على } \partial\Omega, \end{cases}$$

حيث $\Omega \subset \mathbb{R}^N$ مجال مفتوح محدود، $M(x)$ مصفوفة إهليلجية، $a(x) \in L^1(\Omega)$، و $f \in L^1(\Omega)$. ونظراً لضعف انتظام المعطيات، لا يمكن تطبيق الطرق التغايرية الكلاسيكية.

لذلك اعتمدنا على بناء سلسلة من المسائل التقريبية التي نضمن وجود حلول لها، ثم قمنا بإثبات تقديرات موحدة للحلول. باستخدام أدوات التحليل الدالي ونظرية المؤثرات شبه التزايدية، أثبتنا وجود حل ضعيف للمشكلة الأصلية.

الكلمات المفتاحية: معادلات إهليلجية، حل ضعيف، بيانات مندمجة، مؤثرات شبه تزايدية، حدود من الرتبة الدنيا.