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ALGEBRA 3

كمرجع للدروس لطلبة **السنة الثانية ليسانس رياضيات.**

وهذا بعد الاطلاع على التقارير الإيجابية للأستاذ الخبير المكلف بالمطبوعة.

رئيس اللجنة العلمية

رئيس اللجنة العلمية
لقسم الرياضيات
مرزوقي عبد الكريم

سلمت هذه الشهادة للمعني لاستعمالها في حدود ما يسمح به القانون.

Mohamed Boudiaf University – M'sila
Faculty of Mathematics and Computer Science
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ALGEBRA COURSES

MIDOUNE Noureddine

Algebra Courses ALG3
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Contents

Preface

This handout is the result of years of teaching the ALG3 module, designed for second-year university students.

It contains four chapters, each with examples. The first three chapters cover key concepts such as eigenvalues, eigenvectors, the characteristic polynomial, diagonalization, and trigonalization of matrices. The final chapter focuses on various applications of these concepts.

I hope that both students and teachers find this handout useful and valuable in their learning and teaching.

ALG 3

Chapter 1

Diagonalization of matrices

1.1 Definitions

Let E be an $n-$dimensional space vector over a field K, where $K = \mathbb{R}$ or $\mathbb{C}$.
$dimE = n$, B a basis of E. Let $f : E \longrightarrow E$ a linear application (endomorphism of E), A the square matrix $(n \times n)$ associated with f: $A = \mathbb{M}_B(f) = (a_{ij})$.

Definition 1. Characteristic Polynomial of a Matrix

If A is an $n \times n$ matrix, the **characteristic polynomial** $P(\lambda)$ of A is defined by:

$$P(\lambda) = det(A - \lambda I_n)$$

Definition 2. Eigenvalues and Eigenvectors

If A is $n \times n$ matrix, a number λ is called an eigenvalue of A if there is $V \in E$ such that:

$$AV = \lambda V$$

In this case, V is called an eigenvector of A corresponding to the eigenvalue λ.

Example. If $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ and $V = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ then $AV = \begin{bmatrix} 20 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 5 \\ 1 \end{bmatrix} = 4V$
So $\lambda = 4$ is an eigenvalue of A with corresponding eigenvector V.

Theorem. Let A be an $n \times n$ matrix.

1. The eigenvalues λ of A are the roots of the characteristic polynomial $P(\lambda)$ of A.

$$P(\lambda) = 0$$

2. The $\lambda-$ eigenvectors X are the nonzero solutions to the homogeneous system

$$(A - \lambda I)X = 0$$

Definition 3.

Let A be $n \times n$ matrix and λ an eigenvalue of the matrix A. The set

$$E(\lambda) = \{V \in E, AV = \lambda V\}$$

is called the **eigenspace** of A associated to the eigenvalue λ in which $E(\lambda)$ is vector sub-space of E. Its dimension ($dimE(\lambda)$) is called the the geometric multiplicity of λ.

Definition 4. Similarity and Diagonalization

If A, B are two $n \times n$ matrices, then they are **similar** if and only if there exists an invertible matrix P such that:

$$A = P^{-1}BP$$

Definition 5. Trace of a matrix

If $A = (a_{ij})$ is an $n \times n$ matrix, then the trace of A is

$$trace(A) = tr(A) = \sum_{i=1}^{n} a_{ij}$$

Lemma. Properties of a trace For $n \times n$ matrices A and B, and any $k \in \mathbb{R}$,

1. $tr(A + B) = tr(A) + tr(B)$
2. $tr(kA) = k.tr(A)$
3. $tr(AB) = tr(BA)$

Theorem. Properties of similar matrices If A and B are $n \times n$ matrices and A, B are similar, then

1. $det(A) = det(B)$
2. $rank(A) = rank(B)$
3. $tr(A) = tr(B)$

4. $P_A(\lambda) = P_B(\lambda)$

5. A and B have the same eigenvalues.

Proof. 1. We have $B = P^{-1}AP$, then $det(B) = det(P^{-1}AP) = det(A)$

2. $P_B(\lambda) = det(B - \lambda I_n) = det(P^{-1}AP - P^{-1}\lambda P) = det[P^{-1}(A - \lambda I_n)P] = det(P^{-1}) \times det(A - \lambda I_n) \times det(P)$

□

Definition 6. Digonalizable

Let A be an $n \times n$ matrix. Then A is said to be **diagonalizable** if there exists an invetible matrix P such that

$$P^{-1}AP = D$$

where D is a diagonal matrix.

Proposition. Let λ_1 and λ_2 be two distinct eigenvalues ($\lambda_1 \neq \lambda_2$) of A, then

$$E(\lambda_1) \cap E(\lambda_2) = \{0\}$$

Proof. If $V \in E(\lambda_1) \cap E(\lambda_2)$, then $AV = \lambda_1 V = \lambda_2 V$ i.e. $(\lambda_1 - \lambda_2)V = 0$.
Since $\lambda_1 \neq \lambda_2$, then we have $V = 0$ □

Definition 7. Diagonalization

A square $n \times n$ matrix A is **diagonalizable** if A is similar to a diagonal matrix, i.e.

$$A = PDP^{-1}$$

for a diagonal matrix D and an invertible matrix P.

Proposition. Let A be an $n \times n$ matrix. We suppose that $P(\lambda)$ have k distinct roots $\lambda_1, \lambda_2, ..., \lambda_k$. If $E = E(\lambda_1) \oplus E(\lambda_2) \oplus ... \oplus E(\lambda_k)$, then A is diagonalizable.

Proof. For $i = 1, 2, ..., k$, we choose the basis B_i of $E(\lambda_i)$. The basis $B' = \cup_{i=1}^{i=k} B_i$ of E consists of the eigenvectors of A associated with the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$, then the matrix $D = \mathbb{M}_{B'}(f)$ is diagonal. □

Examples. Find the characteristic polynomial, eigenvalues and eigenvectors of the matrices:

1. $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 4 & -5 \\ 0 & 2 & -2 \end{bmatrix}$

Solution.

1. $P(\lambda) = (\lambda - 4)(\lambda + 2)$
   $\lambda_1 = -2$ and $\lambda_2 = 4$
   $V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

2. $P(\lambda) = -\lambda(\lambda - 1)(\lambda - 2)$
   $\lambda_1 = 0, \lambda_2 = 4$ and $\lambda_3 = 2$
   $V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $V_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $V_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

1.2 Sufficient condition for a matrix to be diagonalizable

Proposition. An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Proof. We have $P(\lambda) = (-1^n)(\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_n)$, where $\lambda_1, \lambda_2, ..., \lambda_n$ n distinct eigenvalues of A and $V_1, V_2, ..., V_n$ the n eigenvectors associated with λ_i.
$AV_1 = \lambda_1 V_1$
$AV_2 = \lambda_2 V_2$
$\vdots$
$AV_n = \lambda_n V_n$
We can prove that $B' = (V_1, V_2, ..., V_n)$ is a basis of E by induction:
We prove that the set $(V_1, V_2, V_3, ..., V_{k+1})$ is linearly independent of E.

$$\alpha_1 V_1 + \alpha_2 V_2 + ... + \alpha_k V_k + \alpha_{k+1} V_{k+1} = 0 \tag{1.1}$$

We have $A(\alpha_1 V_1 + \alpha_2 V_2 + ... + \alpha_k V_k + \alpha_{k+1} V_{k+1}) = 0$, then
$\alpha_1 AV_1 + \alpha_2 AV_2 + ... + \alpha_k AV_k + \alpha_{k+1} AV_{k+1} = 0$

$$\alpha_1 \lambda_1 V_1 + \alpha_2 \lambda_2 V_2 + ... + \alpha_k \lambda_k V_k + \alpha_{k+1} \lambda_{k+1} V_{k+1} \tag{1.2}$$

From $(2) - \lambda_{k+1}(1)$:
$(\lambda_1 - \lambda_{k+1})\alpha_1 V_1 + (\lambda_2 - \lambda_{k+1})\alpha_2 V_2 + ... + (\lambda_k - \lambda_{k+1})\alpha_k V_k = 0$

Since the set $(V_1, V_2, ..., V_k)$ is linearly independent of E by induction hypothesis, then $(\lambda_1 - \lambda_{k+1})\alpha_1 = (\lambda_2 - \lambda_{k+1})\alpha_2 = ... = (\lambda_k - \lambda_{k+1})\alpha_k = 0$ (because λ_k are distinct).
Therefore $\alpha_1 = \alpha_2 = ... = \alpha_k = 0$
By (1) we have $\alpha_{k+1}V_{k+1} = 0$, then $\alpha_{k+1} = 0$ □

1.3 Necessary and sufficient condition for diagonalizability

Proposition 1. Let A be an $n \times n$ matrix, then

$$dim(E(\lambda_1)) \leq m_1$$

where λ_1 is an eigenvalue of A multiplicity m_1.

Proof. Let $(e_1, e_2, ..., e_r)$ the basis of $E(\lambda_1)$, then we can find the basis $B = (e_1, e_2, ..., e_r, e_{r+1}, ..., e_n)$ of E.
The matrix A is similar of the matrix A' of the form

$$A' = \left(\begin{array}{cccc|c} \lambda_1 & & & & \\ & \lambda_1 & & & A_1 \\ & & \ddots & & \\ & & & \lambda_1 & \\ \hline & & 0 & & A_2 \end{array}\right)$$

$$P(\lambda) = det(A - \lambda I_n) = \left[\begin{array}{cccc|c} \lambda_1 - \lambda & & & & \\ & \lambda_1 - \lambda & & & A_1 \\ & & \ddots & & \\ & & & \lambda_1 - \lambda & \\ \hline & & 0 & & A_2 - \lambda I_{n-r} \end{array}\right]$$

$$= (\lambda_1 - \lambda)^r det(A_2 - \lambda I_{n-r})$$

Then $m \geq r$, where $r = dimE(\lambda_1)$ □

Proposition 2. Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if:

1. $P(\lambda)$ is factored.

2. For each eigenvalue λ_i of A, $dim(E(\lambda_i)$ is equal to the multiplicity of λ_i i.e.
$$dimE(\lambda_i) = m_i, i = 1, ..., k$$

Proof. By induction, the sub-spaces $E(\lambda_i)$, $i = 1, ..., j$, verify

$$E = E(\lambda_1) \oplus E(\lambda_2) \oplus ... \oplus E(\lambda_k)$$

for $j = 1, ..., k$
Denote $S_j = E(\lambda_1) \oplus E(\lambda_2) \oplus ... \oplus E(\lambda_j)$
It is sufficient to demonstrate that $S_j \cap E(\lambda_{j+1}) = \{0\}$
Let $V \in S_j \cap E(\lambda_{j+1})$, then

$$\begin{cases} V = V_1 + V_2 + ... + V_j \\ \text{and} \\ AV = \lambda_{j+1} V \end{cases} \tag{1.3}$$

For (3), we have $AV = AV_1 + AV_2 + ... + AV_j$, then

$$\lambda_{j+1} V = \lambda_1 V_1 + \lambda_2 V_2 + ... + \lambda_j V_j \tag{1.4}$$

For $(4) - \lambda_{j+1}(3)$, we have

$$0 = (\lambda_1 - \lambda_{j+1})V_1 + (\lambda_2 - \lambda_{j+1})V_2 + ... + (\lambda_j - \lambda_{j+1})V_j$$

Using induction hypothesis, we get $V_1 = V_2 = ... = V_j = 0$
Since $\sum_{i=1}^{n} dimE(\lambda_i) = \sum_{i=1}^{n} m_i = n$, we see that $E = \oplus_{i=1}^{k} E(\lambda_i)$. Then A is diagonalizable and we write:

$$D = \begin{bmatrix} \lambda_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \lambda_1 & & & & & & \\ & & & \lambda_2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \lambda_2 & & & \\ & & & & & & \ddots & & \\ & & & & & & & \lambda_k & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \lambda_k \end{bmatrix}$$

□

Examples.

1. $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$
   $P(\lambda) = -\lambda(\lambda - 1)^2$
   $P(\lambda) = 0 \Rightarrow \begin{cases} \lambda_1 = 0, m_1 = 1 \\ \lambda_2 = 1, m_2 = 2 \end{cases}$
   $E(\lambda_1) = E(0) =< V_1 >$, where $V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $dimE(\lambda_1) = 1 = m_1$
   $E(\lambda_2) = E(1) =< V_2, V_3 >$, where $V_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, $V_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $dimE(\lambda_2) = 1 = m_2 = 2$.
   Then the matrix A is diagonalizable.
2. $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -7 \\ 1 & 3 & -4 \end{bmatrix}$
   $P(\lambda) = -\lambda(\lambda - 1)^2$
   $P(\lambda) = 0 \Rightarrow \begin{cases} \lambda_1 = 0, m_1 = 1 \\ \lambda_2 = 1, m_2 = 2 \end{cases}$
   $E(\lambda_1) = E(0) =< V_1 >$, where $V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $dimE(\lambda_1) = 1 = m_1$
   $E(\lambda_2) = E(1) =< V_2 >$, where $V_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $dimE(\lambda_2) = 1 \neq m_2 = 2$
   Then the matrix A isn't diagonalizable.

Chapter 2

Triangulability of matrices

2.1 Example

Consider the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -7 \\ 1 & 3 & -4 \end{bmatrix}$, then

$P(\lambda) = -\lambda(\lambda - 1)^2$

$P(\lambda) = 0 \Rightarrow \begin{cases} \lambda_1 = 0, m_1 = 1 \\ \lambda_2 = 1, m_2 = 2 \end{cases}$

$E(\lambda_1) = E(0) =< V_1 >$, where $V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $dimE(\lambda_1) = 1 = m_1$

$E(\lambda_2) = E(1) =< V_2 >$, where $V_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $dimE(\lambda_2) = 1 \neq m_2 = 2$

Then the matrix A isn't diagonalizable.

What to do if matrix A is not diagonalizable?
Therefore, we use triangulation:

2.2 Proposition

Let $f : E \to F$ a linear map and A the matrix of f, we suppose the characteristic polynomial $P(\lambda)$ of f (or A) is factored in $K[\lambda]$. Then f (or A) is triangulable.

Proof. By induction over $dimE$: the result is true for the space of dimension 1. Suppose they are true for spaces of dimension $\leq n-1$ and let E be a space of dimension n.
Let $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_n)$ in $K[\lambda]$, ($K = \mathbb{R}$ or $\mathbb{C}$).
We suppose that the eigenvalues λ_i are not necessarily distinct. We denote V_1, an eigenvector associated with λ_1 (i.e $f(V_1) = \lambda_1 V_1$).
By the incomplete basis theorem, there exists a basis B' of E where
$B' = (V_1, e_2, e_3, ..., e_n)$ then the matrix A' has the form

$$A' = M_{B'}(f) = \begin{bmatrix} \lambda_1 & a_{12} & . & . & . & a_{1n} \\ 0 & a_{22} & & & & . \\ 0 & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ 0 & a_{n2} & . & . & . & a_{nn} \end{bmatrix}$$

The family $B_1 = (e_2, ..., e_n)$ is a basis of the subspace $F = < e_2, ..., e_n >$ of E. We denote $g : F \to F$, the linear map such that the associated matrix is

$$A_1 = \begin{bmatrix} a_{12} & . & . & . & a_{1n} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ a_{n2} & . & . & . & a_{nn} \end{bmatrix} = M_{B_1}(g)$$

Then $P(\lambda) = (\lambda_1 - \lambda) \times det(A_1 - \lambda I_{n-1})$
i.e. $P(\lambda)$ is factored and since $dimF = n - 1$, by induction hypothesis, there exists a basis $B_2 = (V_2, ..., V_n)$ of F such that $M_{B_2}(g)$ is upper triangular. We get

$$M_{B'=(V_1,V_2,...,V_n)}(f) = \begin{bmatrix} \boxed{\lambda_1} & a_{12} & . & . & . & a_{1n} \\ & \lambda_2 & . & . & . & . \\ & & & . & . & . \\ & & & & . & . \\ & & & & & \lambda_n \end{bmatrix}$$

□

Remark 1.
1/ If A is triangulable, the diagonal of the matrix $T = M_{B'}(f)$ are the eigenvalues of A.
2/ All matrix of $A \in M_n(\mathbb{C})$ is triangulable.

Corollary.
$tr(A) = \sum_i \lambda_i$
$det(A) = \prod_i \lambda_i$

Remark 2.
We can triangulate the matrix A of Example 1.

We consider the basis B' of E where $\begin{cases} V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = e_1 + e_2 + e_3 \\ V_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = e_1 + 3e_2 + 2e_3 \\ V_3 = e_1 \end{cases}$

Because $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 2 & 0 \end{vmatrix} = 2 - 3 = -1 \neq 0$

And $\begin{cases} e_1 = V_3 \\ e_2 = -2V_1 + V_2 + V_3 \\ e_3 = 3V_1 - V_2 - 2V_3 \end{cases}$

Then $T = M_{B'}(f) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = P^{-1}AP$

Where $\begin{cases} f(V_1) = \lambda_1 V_1 = 0 \\ f(V_2) = \lambda_2 V_2 = V_2 \\ f(V_3) = f(e_1) = e_1 + 2e_2 + e_3 = -V_1 + V_2 + V_3 \end{cases}$

Finally, $T = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is the upper triangular matrix,

$P = (V_1V_2V_3) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 2 & 0 \end{bmatrix}$ and $P^{-1} = (e_1e_2e_3) = \begin{bmatrix} 0 & -2 & 3 \\ 0 & 1 & -1 \\ 1 & 1 & -2 \end{bmatrix}$

2.3 Annihilating polynomials

Let E a vector space over K and $R \in K[\lambda]$
$R(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_2\lambda_2 + a_1\lambda^1 + a_0\lambda^0$
If $f \in End_K(E)$, we denote $R(f)$, the linear map of E defined by
$R(f) = a_nf^n + a_{n-1}f^{n-1} + ... + a_2f^2 + a_1f^1 + a_0id$
or $R(A)$ the matrix

$R(A) = a_2 A^n + a_{n-1} A^{n-1} + ... + a_2 A^2 + a_1 A^1 + a_0 I_n$
Where $f^k = \underbrace{f \circ f \circ ... \circ f}_{\text{k times}}$

Remark.
We have $P(f) \circ Q(f) = Q(f) \circ P(f)$.

Definition.

Let $f \in End_K(E)$, the polynomial $R \in K[\lambda]$ is called annihilating polynomial of f (or A), if

$$R(f) = 0$$

or

$$R(A) = 0$$

.

2.4 Cayley-Hamilton theorem

Let $f \in End_k(E)$ and $P(\lambda)$ the characteristic polynomial of f (or A). Then

$$P(f) = 0$$

(or $P(A) = 0$). i.e $P(\lambda)$ annihilates f (or A).

Proof. We suppose $K = \mathbb{C}$, in this case f (or A) is triangulable.
Let $B' = (V_1, V_2, .., V_n)$, a basis of E such that

$$M_{B'}(f) = \begin{pmatrix} \lambda_1 & a_{12} & . & . & . & a_{1n} \\ & \lambda_2 & a_{23} & . & . & a_{2n} \\ & & . & & & . \\ & & & & . & . \\ & & & & & \lambda_n \end{pmatrix} = T \text{ is an upper triangular matrix}$$

We have $f(V_1) = \lambda_1 V_1 \Rightarrow (\lambda_1 id - f)(V_1) = 0$ and
$P(\lambda) = det(T - \lambda I_n) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)...(\lambda_n - \lambda)$
Then $P(f) = (\lambda_1 id - f) \circ ... \circ (\lambda_n id - f)$ and
$P(f)(V_1) = (\lambda_2 id - f) \circ ... \circ (\lambda_n id - f) \circ (\lambda_1 id - f)(V_1) = 0$. Therefore, $P(f)(V_1) = 0$
$P(f)(V_2) = (\lambda_3 id - f) \circ ... \circ (\lambda_n id - f) \circ (\lambda_1 id - f) \circ (\lambda_2 id - f)(V_2) = (\lambda_3 id - f) \circ ... \circ (\lambda_n id - f) 0 (\lambda_1 id - f)(-a_{12} V_1) = 0$. Therefore, $P(f)(V_2) = 0$

We can similarly show that $P(f)(V_3) = 0$
By induction, we find $P(f)(V_i) = 0, \forall i = 1, ..., n$. Finally, $P(f) = 0$.

$\square$

Example.
$A = \begin{bmatrix} 4 & 1 & -1 \\ -6 & -1 & 2 \\ 6 & 1 & 1 \end{bmatrix}$
$P(\lambda) = det(A - \lambda I_3) = (2-\lambda)(1-\lambda)^2 = -\lambda^3 + 4\lambda^2 - 5\lambda + 2$
Since $det(A) = P(0) = 2 \neq 0$, A is invertible.
By the Cayley-Hamilton theorem, we have $P(A) = 0$
i.e $-A^3 + 4A^2 - 5A + 2I_3 = 0$. Then $-A^3 + 4A^2 - 5A = -2I_3 \Rightarrow$
$A[-A^2 + 4A - 5I_3] = -2I_3 \Rightarrow A[\frac{1}{2}A^2 - 2A + \frac{5}{2}I_3] = I_3$
Therefore,

$$A^{-1} = \frac{1}{2}A^2 - 2A + \frac{5}{2}I_3$$

Proposition 1.

Let $S(\lambda)$ a annihilating polynomial of f $[S(f) = 0]$.
All eigenvalue λ_1 of f (of A) is a root of $S(\lambda)[S(\lambda_1) = 0]$.

Proof. If λ_1 is a $V.P, f(V) = \lambda_1 V$
or $S(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0$
$S(f) = a_n f^n + a_{n-1}f^{n-1} + ... + a_1 f + a_0 id = 0$
Therefore $a_n f^n(V) + a_{n-1}f^{n-1}(V) + ... + a_1 \underbrace{f(V)}_{\lambda V} + a_0 id(V) = 0$
$\Rightarrow a_n\lambda^n V + a_{n-1}\lambda^{n-1}V + ... + a_1\lambda V + a_0 V = 0$
$\underbrace{(a_n\lambda_1^n + a_{n-1}\lambda_1^{n-1} + ... + a_1\lambda_1 + a_0)}_{S(\lambda_1)} V = 0.$
Consequently, $[V \neq 0] \Rightarrow S(\lambda_1) = 0$
i.e λ is a root of $S(\lambda)$. $\square$

Proposition 2.

Let $f \in End(E)$ and $P(\lambda)$ the characteristic polynomial of f i.e

$$P(\lambda) = (-1)^n(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2}...(\lambda - \lambda_p)^{m_p}$$

If f is diagonalizable, then the polynomial $Q(\lambda) = (\lambda - \lambda_1)...(\lambda - \lambda_p)$ annihilates $f[Q(f) = 0]$.

Proof. If f is diagonalizable, there exists a basis $B' = (V_1, V_2, ..., V_n)$ formed of eigenvectors.
Let $\lambda_1, \lambda_2, ..., \lambda_p$ be the eigenvalues of A. For all $V_i \in B' i = \overline{1, n}$, there exists $\lambda_i \quad 1 \leq j \leq p$, such that $f(V_1) = \lambda_j V_i$
i.e $(f - \lambda_j id)(V_i = 0)$
$Q(f) = (f - \lambda_1 id) \circ (f - \lambda_2 id) \circ ... \circ (f - \lambda_p id)$
$Q(f)(V_i) = [(f - \lambda_1 id) \circ (f - \lambda_2 id) \circ ... \circ (f - \lambda_p id)](V_i) = (f - \lambda_1 id) \circ ... \circ \underbrace{(f - \lambda_j id)(V_i)}_{0} = 0$ □

2.5 Minimal polynomial

Definition.

We call the **minimal polynomial** of f (or of A) denoted $Q(f)$ (or $Q(A)$), the normalized annihilating polynomial of f (or of A) of the smallest degree.

$$Q(f) = 0 \text{ or } Q(A) = 0$$

Remark. If $S(\lambda)$ is a multiple of $Q(\lambda)$, then

$$S(\lambda) = Q(\lambda) \times T(\lambda)$$
$$S(f) = Q(f) \circ T(f) = 0$$

i.e $S(\lambda)$ is an annihilating polynomial.

Proposition 1. The annihilating polynomials of f are the polynomials of the type:

$$S(\lambda) = Q(\lambda) \times T(\lambda)$$

Then $S(\lambda) = Q(\lambda) \times T(\lambda) + R(\lambda)$
$S(f) = R(f) = 0 \;\; R(f) = 0$
i.e R is annihilating and since $d^\circ R(\lambda) < d^\circ Q(\lambda)$. This contradicts the hypothesis that $Q(\lambda)$ is a minimal polynomial. Then $R(\lambda) = 0$.
Remark.
$Q(\lambda)/P(\lambda)$ or $P(\lambda) = Q(\lambda) \times T(\lambda])$

Proposition 2. The roots of $Q(\lambda)$ are exactly the roots of $P(\lambda)$, i.e the eigenvalues but with a different multiplicity
If

$$P(\lambda) = (-1)^n(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2}...(\lambda - \lambda_p)^{m_p} \qquad \lambda_i \neq \lambda_j$$

Then

$$Q(\lambda) = (\lambda - \lambda_1)^{\alpha_1}(\lambda - \lambda_2)^{\alpha_2}...(\lambda - \lambda_p)^{\alpha_p}$$

with $1 \leq \alpha_i \leq m_i, \;\; i = 1, ..., p$

Proof. We know that $P(\lambda) = Q(\lambda)T(\lambda)$, then if λ is a root of $Q(\lambda)$, then it is a root of $P(\lambda)$.
Conversely, let λ a root of $P(\lambda)$ i.e λ is an eigenvalue of A, then λ is a root of $Q(\lambda)$ because $Q(\lambda)$ annihilates A. □

Theorem 1. The minimal polynomial and characteristic polynomial of f (or A) share the same roots, except for multiplicities.

Examples.

- $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$
  We have $P(\lambda) = -(\lambda + 1)(\lambda + 2)(\lambda - 3)$, then $Q(\lambda) = (\lambda + 1)(\lambda + 2)(\lambda - 3)$
- $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$
  We have $P(\lambda) = -(\lambda - 1)(\lambda + 2)^2$, then there exists two possibilities:

$$Q(\lambda) = (\lambda - 1)(\lambda + 2)$$
$$Q(\lambda) = (\lambda - 1)(\lambda + 2)^2$$

Theorem 2. An endomorphism f (or A) is diagonalizable if and only if the minimal polynomial of f (or A) is factored and has all its simple roots.
i.e

$$Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_m)$$

Examples.

- $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$
  We saw that $Q(\lambda) = (\lambda + 2)(\lambda - 1)$, then A is diagonalizable.

- $A = \begin{bmatrix} 3 & 2 & -2 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$
  We have $P(\lambda) = -(\lambda - 1)^3$, then $Q(\lambda) = \lambda - 1$ or $(\lambda - 1)^2$ or $(\lambda - 1)^3$
  If $Q(\lambda) = \lambda - 1$, $Q(A) = 0$ or $Q(A) = A - I_3 = \begin{bmatrix} 2 & 2 & -2 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \neq 0$
  then A is not diagonalizable.

- $A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$
  We have $P(\lambda) = -(\lambda - 1)(\lambda - 2)^2$, then $Q(\lambda) = (\lambda - 1)(\lambda - 2)$ or $Q(\lambda) = (\lambda - 1)(\lambda - 2)^2$
  If $Q(\lambda) = (\lambda - 1)(\lambda - 2)$, then
  $Q(A) = (A - I_3)(A - 2I_3) = \begin{bmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & . & . \\ . & . & . \\ . & . & . \end{bmatrix} \neq 0$
  Then A is not diagonalizable.

2.6 Kernel decomposition theorem

1. We suppose there exists $P \in K[\lambda]$ of the form $P = S \times T$ with $S, T \in K[\lambda]$ relatively prime, such that $P(f) = 0$. Then $E = kerS(f) \oplus kerT(f)$.

2. We suppose there exists $P \in K[\lambda]$ of the form $P = P_1 \times P_2 \times ... \times P_k$ with $P_1, P_2, ..., P_k \in K[\lambda]$ relatively prime pairwise, such that $P(f) = 0$. Then, $E = kerP_1(f) \oplus ... \oplus kerP_k(f)$.

Proof.

1. We prove that $kerS(f) \cap kerT(f) = \{0\}$
   Let $v \in kerS(f) \cap kerT(f)$
   $S(f)(v) = 0$ and $T(f)(v) = 0$
   Or $P(\lambda) = S(\lambda) \times T(\lambda) \Rightarrow P(f) = S(f) \circ T(f)$, since $S(\lambda) \wedge T(\lambda) = 1$.
   Using Besout theorem, $\exists S_1(\lambda), T_1(\lambda)$ such that $S_1(\lambda) \times S(\lambda) + T_1(\lambda) \times T(\lambda) = 1$
   Therefore, $S_1(f) \circ S(f) + T_1(f) \circ T(f) = id$ and
   $v = id(v) = S_1(f)[\underbrace{S(f)(v)}_{0}] + T_1(f)[\underbrace{T(f)(v)}_{0}]$. Then $v = 0$.
   Let $v \in E$
   $v = id(v) = \underbrace{S_1(f) \circ S(f)(v)}_{V_2 \in kerT(f)} + \underbrace{T_1(f) \circ T(f)(v)}_{V_1 \in kerS(f)}$
   $v_1 \in kerS(f)$
   i.e $S(f)(v_1) = S(f)[T_1(f) \circ T(f)(v)] = T_1(f) \circ \underbrace{S(f) \circ T(f)}_{P(f)=0}(v) = 0.$
   Similarly for v_2, we obtain $v_2 \in kerT(f)$
   i.e $v = v_1 + v_2$

2. We obtain the result of 2. by induction. □

Proposition. An endomorphism f (or A) is diagonalizable if and only if the minimal polynomial of f (or A) is factored and has all its simple roots.

Proof. If f is diagonalizable $\Rightarrow Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_p)$
If $Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_p)$
$Q(\lambda) = P_1 \times P_2 \times ...P_p$ with $P_i = \lambda - \lambda_i \in K[\lambda], i = 1, ..., p$ relatively prime pairwise, such that $Q(f) = 0$
Then $E = kerP_1(f) \oplus ... \oplus kerP_p(f) = ker(f - \lambda_1 id) \oplus ... \oplus ker(f - \lambda_p id) = E(\lambda_1) \oplus E(\lambda_2) \oplus ... \oplus E_{\lambda_p}$.
i.e E is the direct sum of the eigenspace $E(\lambda_i), i = 1, ..., p$. Then f (or A) is diagonalizable. □

2.7 Applications

- **Compute the power of the matrix**
  Let A be an $n \times n$ matrix.

Method 1. Using the formula $A = PDP^{-1}$
We suppose A is diagonalizable, then $D = P^{-1}AP$, i.e $A = PDP^{-1}$, then

$$A^k = (PDP^{-1})(PDP^{-1})...(PDP^{-1}) = PD^kP^{-1}$$

Or $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$, $D^k = \begin{bmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}$

and it's easy to compute A^k using the following formula $A^k = P \begin{bmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} P^{-1}$

$$A^n = PD^nP^{-1}$$

Method 2. Using the minimal polynomial $Q(\lambda)$
$\lambda^n = Q(\lambda) \times S(\lambda) + R(\lambda)$. Then $A^n = R(A)$

Example.
Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$
We have $P(\lambda) = (\lambda - 2)(\lambda - 3)$
$E(\lambda_1) = E(2) =< V_1 >$, where $V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
$E(\lambda_2) = E(3) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
Therefore, $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} -2 & 1 \\ -1 & -1 \end{bmatrix}$
We obtain $A^k = \begin{bmatrix} 2^{k+1} - 3^k & 2^{k+1} - 2.3^k \\ -2^k + 3^k & -2^k + 2.3^k \end{bmatrix}$

- **Solving a system of recurrence relations**
  Let's illustrate this with an example. This involves determining two sequences (u_n), v_n such that:

$$(1) \begin{cases} u_{n+1} = u_n - v_n \\ v_{n+1} = 2u_n + 4v_n \end{cases} \quad \text{and such that} \begin{cases} u_0 = 2 \\ v_0 = 1 \end{cases}$$

We put $X_n = \begin{bmatrix} u_n \\ v_n \end{bmatrix}$. We can write the system (1):

$$X_{n+1} = AX_n \text{ with } A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

Hence, by induction

$$X_n = A^n X_0 \text{ with } X_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We have $\begin{bmatrix} u_n \\ v_n \end{bmatrix} = A^k = \begin{bmatrix} 2^{k+1} - 3^k & 2^{k+1} - 2.3^k \\ -2^k + 3^k & -2^k + 2.3^k \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Finally,

$$\begin{cases} u_n = 3.2^{n+1} - 4.3^n \\ v_n = -3.2^n + 4.3^n \end{cases}$$

- **Solving a first-order linear differential system**

Let the system $X' = AX$, where $X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, $X' = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$

Example.

(I) $\begin{cases} x_1' = x_1 + 2x_2 + -3x_3 \\ x_2' = x_1 + 4x_2 - 5x_3 \\ x_3' = 2x_2 - 2x_3 \end{cases}$

Chapter 3

Nilpotent and exponential matrix

3.1 Nilpotent Matrix

Definition.

A nilpotent matrix is a square matrix, there exists an integer m such that

$$N^m = 0$$

.
The integer m is called the nilpotency index. It is the smallest integer such that $N^m = 0$.

Examples.

(a) $A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$
The matrix is nilpotent because by squaring matrix A we get the zero matrix as a result:

$$A^2 = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(b) $B = \begin{bmatrix} 1 & -2 & 1 \\ 3 & 0 & 3 \\ -1 & 2 & -1 \end{bmatrix}$
Although when raising the matrix to 2 we do not obtain the null matrix:

$$B^2 = \begin{bmatrix} 1 & -2 & 1 \\ 3 & 0 & 3 \\ -1 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 1 \\ 3 & 0 & 3 \\ -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} -6 & 0 & -6 \\ 0 & 0 & 0 \\ 6 & 0 & 6 \end{bmatrix}$$

When calculating the cube of the matrix we do not get the matrix with all the elements equal to zero:

$$\begin{bmatrix} -6 & 0 & -6 \\ 0 & 0 & 0 \\ 6 & 0 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 1 \\ 3 & 0 & 3 \\ -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So matrix B is a nilpotent matrix, and since the null matrix is obtained with the third power, its nilpotency index is 3.

3.2 Exponential of a matrix

Definition.

If A is a constant $n \times n$ matrix, the matrix exponential e^{At} is given by:

$$e^{At} = I + At + A^2\frac{t^2}{2!} + ... + A^n\frac{t^n}{n!} + ...,$$

where the right-hand side indicates the $n \times n$ matrix whose elements are power series with coefficients given by the entries in the matrices.

Example. The exponential is easiest to compute when A is diagonal. For the matrix $A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$, we calculate

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, A^3 = \begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix}, ..., A^n = \begin{bmatrix} (-1)^n & 0 \\ 0 & 2^n \end{bmatrix}$$

Then we get

$$e^{At} = \sum_{n=0}^{\infty} A^n \frac{t^n}{n!} = \begin{bmatrix} \sum_{n=0}^{\infty}(-1)^n\frac{t^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} 2^n\frac{t^n}{n!} \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}$$

Remark. In general, if A is an $n \times n$ matrix with entries $\lambda_1, \lambda_2, ...\lambda_n$, then e^{At} is the diagonal matrix with entries $e^{\lambda_1 t}, e^{\lambda_2 t}, ..., e^{\lambda_n t}$ on the main diagonal.

Theorem 1. Let A and B be $n \times n$ constant matrices, and $r, s, t \in \mathbb{R}$. Then

(a) $e^{A0} = e^0 = I$

(b) $e^{A(t+s)} = e^{At}e^{As}$

(c) $(e^{At})^{-1} = e^{-At}$

(d) $e^{(A+B)^t} = e^{At}e^{Bt}$ if $AB = BA$

(e) $e^{rIt} = e^{rt}I$

Theorem 2. If A is an $n \times n$ constant matrix, then the columns of the matrix exponential e^{At} form of a fundamental solution set for the system $x'(t) = Ax(t)$. Therefore, e^{At} is a fundamental matrix for the system, and a general solution is $x(t) = ce^{At}$.

3.3 Exponential of a nilpotent matrix

If A is nilpotent of index m, i.e $A^m = 0$, then

$$e^{At} = I + At + ... + A^{m-1}\frac{t^{k-1}}{(k-1)!}$$

Example. Find the fundamental matrix e^{At} for the system $x' = Ax$, where

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{bmatrix}$$

Solution. We find the polynomial of A

$$p(r) = |A - rI| = \begin{vmatrix} 2-r & 1 & 1 \\ 1 & 2-r & 1 \\ -2 & -2 & -1-r \end{vmatrix} = -(r-1)^3$$

Therefore, $r = 1$ is the only eigenvalue of A, so $(A - I)^3 = 0$ and

$$e^{At} = e^t e^{(A-I)t} e^t \{I + (A-I)t + (A-I)^2 \frac{t^2}{2}\}.....(1)$$

We calculate

$$A - I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix} \text{ and } (A - I)^2 = 0$$

Substitution into (1) gives us

$$e^{At} = e^t I + te^t (A - I) = \begin{bmatrix} e^t + te^t & te^t & te^t \\ te^t & e^t + te^t & te^t \\ -2te^t & -2te^t & e^t - 2te^t \end{bmatrix}$$

Chapter 4

Exercises and mock exam

Exercise 1. In each case, find the characteristic polynomial, eigenvalues, eigenvectors, and (if possible) an invertible matrix P such that $P^{-1}AP$ is diagonal.

a. $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, **b.** $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$, **c.** $A = \begin{bmatrix} 7 & 0 & -4 \\ 0 & 5 & 0 \\ 5 & 0 & -2 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 6 \\ 1 & -1 & 5 \end{bmatrix}$, **e.** $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{bmatrix}$, **f.** $A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}$

g. $A = \begin{bmatrix} 3 & 1 & 1 \\ -4 & -2 & -5 \\ 2 & 2 & 5 \end{bmatrix}$, **h.** $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$

Exercise 2. In each case of Ex1, compute A^n (if possible).

Exercise 3. If $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ verify that A and B are diagonalizable, but AB is not.

Exercise 4. Diagonalize the following matrices:

- $A = \begin{bmatrix} 2 & 3 & -3 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{bmatrix}$

- $A = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix}$

Exercise 5. Find the triangular matrix T similar to A where

- $A = \begin{bmatrix} 4 & 3 & 4 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$
- $A = \begin{bmatrix} 3 & -4 & 0 & 2 \\ 4 & -5 & -2 & 4 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 2 & -1 \end{bmatrix}$

Exercise 6. Using the Hamilton-Cayley theorem, find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

Exercise 7. We consider the matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & m-1 \\ 1 & 0 & m \end{bmatrix}$, $m \in \mathbb{R}$

1. Find the characteristic polynomial of A.
2. Triangulate the matrix A.

Exercise 8. Let the matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

1. Find the minimal polynomial of A.
2. Find A^n.

Exercise 9.

1. Find the sequences (x_n) and (y_n) verifying the system of recurrence relations:
   $(I)\begin{cases} x_{n+1} = 5x_n - 3y_n \\ y_{n+1} = 6x_n - 6y_n \end{cases}$

2. Solve the differential system:
   $(I) = \begin{cases} x' = 5x - 3y \\ y' = 6x - 6y \end{cases}$

Exercise 10. Let A be a 3×3 matrix, such that $A = \begin{bmatrix} 1 & 5 & -2 \\ 1 & 2 & -1 \\ 3 & 6 & -3 \end{bmatrix}$

1. Verify that A is nilpotent.
2. Using exponential formula, solve the differential system $X' = \frac{dX}{dt} = AX$.

Exercise 11. Let A be a 4×4 matrix such that: $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

1. Find the characteristic subspaces of A.
2. Find the new basis B' such that $A = PDP^{-1}$.

Exercise 12. Find the Jordan Form of the following matrices:

1. $A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ $A = \begin{bmatrix} 11 & 4 \\ -4 & 3 \end{bmatrix}$
2. $A = \begin{bmatrix} 5 & -9 & -4 \\ 6 & -11 & -5 \\ -7 & 13 & 6 \end{bmatrix}$ $A = \begin{bmatrix} 3 & 0 & -1 \\ -2 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}$

Mock exam 2024/2025

Exercise 1.

We consider the following matrix: $A = \begin{bmatrix} 3 & 1 & 1 \\ -4 & -2 & -5 \\ 2 & 2 & 5 \end{bmatrix}$

1. Show that $P(\lambda) = (\lambda - 1)(2 - \lambda)(\lambda - 3)$.
2. Diagonalise A.
3. Find A^n.
4. Solve the system of recurrence $X_{n+1} = AX_n$.

Exercise 2.

We consider the following matrix: $B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{bmatrix}$

1. Show that $P(\lambda) = (1 - \lambda)^3$.
2. Find the minimal polynomial $Q(\lambda)$, what can we deduce?
3. Find J, the Jordan form of B (find the matrices P and P^{-1} such that $J = P^{-1}AP$).
4. We put $N = B - I_3$:
   - Verify that N is nilpotent and that $e^{tB} = e^t e^{tN}$.
   - Solve the differential system $X' = BX$.
5. Show that:
   - If $B = PJP^{-1}$, then $e^{Bt} = Pe^{Jt}P^{-1}$.
   - Solve the system $X' = BX$ again.

Exercise 3.

Prove by induction that $J^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$, where $J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Bibliography

[1] Robert A. Beezer, *A First Course in Linear Algebra*, University of Puget Sound, 2023.

[2] Jean-Pierre Escofier, *Toute l'algèbre de la Licence*, 6th edition, Dunod, 2023, Paris.

[3] Sheldon Axler, *Linear Algebra Done Right*, 3rd edition, Springer, 2023, ISBN 978-3319110790.

[4] David C. Lay, Steven R. Lay, and Judi J. McDonald, *Linear Algebra and Its Applications*, 5th edition, Pearson, 2023, ISBN 978-0321982384.

[5] Serge Lang, *Linear Algebra*, 3rd edition, Springer, 2023, ISBN 978-1441921291.