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Master memory

Domain : Mathematics and Computer Science
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Theme

On semilinear elliptic equations involving stummel classes

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Contents

Dedication

To my wonderful family,

On this momentous occasion of my graduation, I am overwhelmed with profound gratitude for the unwavering support, boundless love, and endless encouragement you have bestowed upon me throughout my university journey. Today is not just a celebration of my individual achievements but a testament to the remarkable strength and unity that defines our family, I dedicate my graduation to all of you.

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I extend my respect and gratitude to the members of the Jury, Dr. Saadi Abderrachid and Dr. Tallab Abdelhamid, for graciously agreeing to examine and evaluate my work. Their insightful remarks and suggestions will undoubtedly contribute to enhancing the quality of this memory.

My family, especially my parents, played an indispensable role in the completion of this work. Their unwavering support and encouragement have been invaluable.

Last but not least, I would like to express my gratitude to all my teachers and everyone who has directly or indirectly contributed to the realization of this work.

Résumé

Dans ce travail, nous étudions l'existence et la régularité des solutions du problème suivant:

$$\begin{cases} -div(M(x)\nabla v) + g(v) = f, \\ v \in W_0^{1,2}(\Omega). \end{cases}$$

Dans ce cas, nous supposons que $M(x)$ est une nxn matrice, borne, et $g : \mathbb{R} \to \mathbb{R}$ est croissate Lipschitzienne,

Les donnes f appartenant á $\tilde{S}_\alpha(\Omega)$, où $\alpha = 1or2$.

Comme les classes de Stummel ont des propriétés d'inclusion avec d'autres espaces fonctionnels et ont des applications á la régularité de la solution des équations aux dérivées partielles elliptiques, notre étude est donc basé sur l'utilisant du lemme de Stampacchia et une intégration pondérée d'une fonction dans les classes de Stummel où le poids est dans des espaces Sobolev Ã support compact

mots-clés: Classes de Stummel ,Espaces de Morrey , Espaces de Sobolev, Espaces de Hilbert, Existence, Unicite, Regularité.

ملخص

في هذا العمل ندرس وجود و إنتظام حلول المسائل مثل :

$$\begin{cases} -div(M(x)\nabla v) + g(v) = f, \\ v \in W_0^{1,2}(\Omega). \end{cases}$$

حيث Ω جزء مفتوح من $\mathbb{R}^n$ مع $N \geqslant 3$
و $g:\mathbb{R} \to \mathbb{R}$ هي دالة *Lipschitz* متزايدة.

الهدف من هذه العمل هو دراسة الوجود والوحدانية و الإنتظام لحلول مثل هذه المسائل والتي تنطوي على فئات *Stummel* . نظرًا لأن فئات *Stummel* تحتوي على بعض خصائص التضمين مع فضاءات تابعية اخرى وتطبيقات على انتظام حلول بعض المعادلات التفاضلية الجزئية الناقصية ، لذلك فإن دراستنا تعتمد على استخدام توطئة *Stampacchia* والتباين بين فئات *Stummel* التي يكون فيها الوزن في فضاءات *Sobolev* مدعوم بشكل متراص.

الكلمات الرئيسية:

فئات *Stummel* ، فضاءات *Morry* ، فضاءات *Sobolev* ، فضاءات *Hilbert* ، وجود الحل ، وحدانيته و إنتظامه .

Abstract

This memory deals with the study of the existence, uniqueness, and regularity of semilinear elliptic equations, more precisely:

$$\begin{cases} -div(M(x)\nabla v) + g(v) = f, \\ v \in W_0^{1,2}(\Omega). \end{cases}$$

Where Ω is a bounded open subset of $\mathbb{R}^n (n \geqslant 3)$,

In this case, we suppose that $M(x)$ is nxn symmetric matrix, elliptic, bounded, and $g : \mathbb{R} \to \mathbb{R}$ is non decreasing, and Lipschitz.

The datum f is taken belongs to $\tilde{S}_\alpha(\Omega)$, for $\alpha = 1 or 2$.

Since the Stummel classes have some inclusion properties with other function spaces and applications to the regularity of the solution of elliptic partial differential equations, so our studying is based on employing Stampacchia's lemma and a weighted embedding of a function in Stummel classes where the weight is in compactly supported Sobolev spaces.

Key words: Stummel classes, Morrey spaces, Sobolev spaces, Hilbert spaces, Existence, Uniqueness, Regularity.

List of Symbols

In what follows, we will use the following notations.

$\mathbb{R}^n$	Euclidean, n-dimensional space.
x	Vecteur de $\mathbb{R}^n$, $x=(x_1,x_2,...,x_n)$, $x_i \in \mathbb{R}$, $1 \leq i \leq n$.
Ω	Open set in $\mathbb{R}^n$.
$\partial\Omega$	The border of Ω.
$B(x,r)$	Open ball with center x and radius $r>0$.
$O(a,r)$	$=\Omega \cap B(a,r)=\{t \in \Omega : \lvert t-a\rvert < r\}$.
$S_\alpha(\mathbb{R}^n)$	$=\left\{f \in L^1_{loc}(\mathbb{R}^n) : \rho_\alpha f(r) \to 0 \text{ for } r \to 0\right\}$.
$\rho_\alpha f(r)$	$=\sup\limits_{x\in\Omega}\int_{B(x,r)} \frac{\lvert f(y)\rvert}{\lvert x-y\rvert^{n-\alpha}}dy < \infty, \forall r>0.$
$\lVert f\rVert_{L^{1,\gamma}}$	$=\sup\limits_{x\in\Omega, r>0} \frac{1}{r^\gamma}\int_{O(x,r)} \lvert f(y)\rvert dy < \infty.$
$\lVert f\rVert_{L^{p,\gamma}}$	$=\sup\limits_{a\in\Omega, r>0} (\frac{1}{r^\gamma}\int_{O(a,r)} \lvert f(y)\rvert^p dy)^{\frac{1}{p}} < \infty.$
$W^{k,p}(\Omega)$	$=\left\{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega)\ \forall \alpha \in \mathbb{N}^n \text{ such that } \lvert\alpha\rvert \leq j\right\}$.
$W_0^{k,p}(\Omega)$	Sobolev space with 0 on $\partial\Omega$.
$\mathscr{D}(\Omega)$	Space of unlimited differentiable functions on Ω with compact support.
p'	The conjugate exponent of p.
$p^\star$	$=\frac{Np}{N-p}$; Sobolev conjugate.
$C^\infty(\Omega)$	The set of functions in $C^k(\Omega)$ for all k.
$(\cdot,\cdot)$	The scalar product.
∇v	The gradient of v.

Introduction

Functional spaces known as Morrey spaces introduced by Charles B. Morrey in 1938 (see [11]) generalize the idea of Holder spaces. These spaces were initially created by Morrey to investigate the regularity of solutions to elliptic PDEs. Morrey spaces have grown to be an essential tool for mathematical study. However, a class of function space known as Stummel classes is named after Fritz Stummel,(see [17]). They are used to investigate the characteristics of functions and their derivatives, especially in relation to partial differential equations (PDEs), and they are closely related to Morrey spaces and Sobolev spaces. Stummel classes are frequently employed in the study of elliptic and parabolic equations and are crucial for comprehending the regularity and boundedness of solutions to PDEs. This memory is devoted to studying the existence and regularity of the solutions ofproblems where Ω is a bounded open of $\mathbb{R}^n$:

$$\begin{cases} -div(M(x)\nabla v) + g(v) = f, \\ v \in W_0^{1,2}(\Omega). \end{cases} \tag{1}$$

Here we suppose that $M(x)$ is nxn symmetric matrix with the following assmptions:
1. $M(x)$ is supposed elliptic, that is, there exists $\nu > 0$ such that

$$M(x)\xi.\xi \geqslant \nu|\xi|^2 \quad for \quad all \quad \xi \in \mathbb{R}^n.$$

2. $M(x)$ is assumed bounded, that is, there exists $K > 0$ such that

$$|M(x)| \leqslant K, \quad for \quad all \quad x \in \Omega.$$

3. The function $g : \mathbb{R} \to \mathbb{R}$ is non decreasing and Lipschitz, that is, there exists $K_0 > 0$ such that

$$|g(s) - g(t)| \leqslant K_0|s - t| \quad for \quad all \quad s, t \in \mathbb{R}.$$

4. We discuss two cases: First, the right hand side f belongs to $L^{1,\gamma}(\Omega)$ with $g = 0$. Second, the right hand side f belongs to $\tilde{\mathbf{S}}_\alpha(\Omega)$, for $\alpha = 1$ or 2. Our studying based on using a weighted embedding of a function in Stummel classes involving Sobolev spaces and Stampacchia 's lemma. Basically, the Stummel classes have some inclusion properties with other function spaces and applications to the regularity of the solution of elliptic partial differential equations.

The first chapter deals with some facts on Stummel classes, Morrey spaces, Sobolev spaces, Hilbert spaces, some of their properties, and further details on Stummel classes and Morrey spaces as well as the Lax-Milgram Lemma and Stampacchia's Lemma. These spaces also allows us to study the existence and regularity for the problem in Chapters 2 – 3. We refer to [3], [2], [5], [6], [7], [13], [15], [18], and [21] for the theory of these spaces.

In the second chapter of the memory we study the existence and uniqueness of the solution of the following Dirichlet problem:

$$\begin{cases} -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial v}{\partial x_i}\right) = f, \\ v \in W_0^{1,2}(\Omega). \end{cases}$$

where the datum f is a function in Morrey spaces $L^{1,\gamma}(\Omega)$ with $n-2 < \gamma < n$. Our invastigation requires using the weighted estimation in Morrey spaces and applying the Lax-Milgram Lemma. Moreover, we give more details about the method of finding the solutions of problems. We recall that this case of problems have been treated in [19].

In Chapter 3, we discuss the question of existence and uniqueness of the solutions of problems associated to (1) and with data f belonging to $\tilde{\mathbf{S}}_{\alpha}(\Omega)$, for $\alpha = 1$ or 2. Stampacchia's lemma allowed us to apply a weighted estimation for a function in Stummel class with its weight in $W^{1,2}(\Omega)$, which we used to study the existence and uniqueness of the equation of Problem (1). We highlight that all the results in Chapter 3 can be found in [20]. Furthermore, some similar works that employ a variety of techniques can be accessed by looking at [1], [10], and [12].

Tanking everything into account, we have studied the existence and uniqueness of weak solution to Dirichlet problem (1) in the case when $g = 0$ which can be found by assuming that the data belongs to some Morrey spaces. This can be demonstrated by utilizing the Lax-Milgram lemma, a functional analytic method, in conjunction with the weighted embedding in Morrey spaces, where the weight is in Soblev space. Moreover, There is only one solution to problem (1). This is demonstrated by Stampacchia's lemma and a weighted embedding of a function in Stummel classes, where the weight is in compactly supported Sobolev spaces.

Chapter 1

Mathemaical Background

This chapter reviews some of the basics of Hilbert spaces, Sobolev spaces, Morrey spaces, and Stummel classes. Remembering some of their further characteristics as well as more details on Morrey spaces and Stummel classes.

1.1 Lebesgue and Sobolev Spaces

Lebesgue Spaces

Definition 1.1.1. *[2] Assume that* $1 \leq p < \infty$, *we set*

$$L^p(\Omega) = \{v : \Omega \to \mathbb{R}; \quad v \text{ is measurable and } \quad |v|^p \in L^1(\Omega)\}$$

with

$$\|v\|_{L^p} = \|v\|_p = \left(\int_\Omega |v(x)|^p dx\right)^{\frac{1}{p}}$$

It si easy to check that $\| \|_P$ *is a norm.*

Definition 1.1.2. *[21] : We set*

$L^\infty(\Omega) = \{v : \Omega \to \mathbb{R}; \quad v \text{ is measurable and there is a constant } \quad C \text{ such that} c|v| \leqslant C \quad a.e. On \quad \Omega\}$

with

$$\|v\|_{L^p} = \|v\|_p = inf\{C; \quad |v(x)| \leqslant c \quad \text{a.e.On} \quad \Omega\}$$

Definition 1.1.3. *[12] : Suppose that* $1 \leq p < \infty$. *Then*

(i) $L^p_{loc}(\Omega) = \{v : \quad v \in L^p(K) \quad$ *for every compact subset* $\quad K$ *of* $\quad \Omega\}$,

(ii) v *is locally integrable in* Ω *if* $v \in L^1_{loc}(\Omega)$.

(iii) Let v *and* w *be locally integrable functions defined in* Ω.
We define v*as the weak derivative of* u *with respect to* α *if, for every* $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega v D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega w \varphi dx.$$

and we say that $D^\alpha v = w$ *in the weak sense.*

(iiii) Let v *and* w *be in* $L^p_{loc}(\Omega)$.
We define v *as the strong derivative of* v *with respect to* α *if, for every compact subset* K *of* Ω, *there exists a sequence* $\{\varphi_i\}$ *in* $C^{|\alpha|}(K)$ *such that* $\varphi_i \to v$ *in* $L^p(K)$ *and* $D^\alpha \varphi_i \to w$ *in* $L^p(K)$.

The spaces $W^{m,p}(\Omega)$

Definition 1.1.4. *([2], [7], [6], [15]). The sobolev spaces is defined as*

$$W^{m,p}(\Omega) = \left\{ v \in L^p(\Omega), D^\alpha v \in L^p(\Omega), \forall \alpha \in \mathbb{N}^n \text{ such that: } |\alpha| \leq m \right\}$$

which endowed with the norm

$$\| v\|_{W^{m,p}(\Omega)} = \| v\|_{L^p(\Omega)} + \sum_{|\alpha|\leq m} \left\| D^\alpha v \right\|_{L^p(\Omega)}$$

The spaces $W_0^{m,p}(\Omega)$

Definition 1.1.5. *[6] : We have*

$$W_0^{m,p}(\Omega) = \{ \text{ completion of } C_0^\infty(\Omega) \text{ with respect to the norm } || \cdot ||_{W^{m,p}(\Omega)} \}$$

Proposition 1.1.1. *[2] Assume that $\Omega \subset \mathbb{R}^N$ is an open set. Then, the following statemente hold :*

1. *For each* $1 \leq p \leq \infty, W^{1,p}(\Omega)$ *is a Banach space.*
2. *For each* $1 < p < \infty, W^{1,p}(\Omega)$ *is refelexive.*
3. *For each* $1 \leq p < \infty, W^{1,p}(\Omega)$ *is a separable.*

It is easy to see that $E \hookrightarrow F$ is equivalent to the identity mapping from E into F being continuous.

Proof. We sketch the proof for each property:

1. Banach space property:
   - $W^{1,p}(\Omega)$ is complete with respect to the norm

$$\|u\|_{1,p} = \left(\|u\|_p^p + \|\nabla u\|_p^p\right)^{1/p} \quad \text{for } 1 \leq p < \infty$$

$$\|u\|_{1,\infty} = \max\{\|u\|_\infty, \|\nabla u\|_\infty\} \quad \text{for } p = \infty$$

   - The completeness follows from the completeness of L^p spaces and the fact that L^p-convergence preserves weak derivatives.

2. Reflexivity for $1 < p < \infty$:
   - $W^{1,p}(\Omega)$ is isomorphic to a closed subspace of $L^p(\Omega)^{N+1}$ via the mapping $u \mapsto (u, \partial_1 u, \ldots, \partial_N u)$.
   - Since $L^p(\Omega)$ is reflexive for $1 < p < \infty$, and closed subspaces of reflexive spaces are reflexive, the result follows.

3. Separability for $1 \leq p < \infty$:

   - For $1 \leq p < \infty$, $L^p(\Omega)$ is separable.
   - The same embedding as above shows $W^{1,p}(\Omega)$ is separable as a subspace of a separable space.

The continuity of the identity mapping $E \hookrightarrow F$ is indeed equivalent to the embedding $E \hookrightarrow F$, as it precisely means that $\|u\|_F \leq C\|u\|_E$ for some constant $C > 0$. □

Remark 1.1.1. *These properties are fundamental in the analysis of partial differential equations:*

- *The Banach space structure allows us to apply functional analysis tools*
- *Reflexivity is crucial for weak compactness arguments*
- *Separability is important for approximation techniques*

Definition 1.1.6. *[21] : Suppose that E and F are Banach spaces. If $E \subset F$, we say that A is continuously imbedded in F (in symbols, this is written $E \hookrightarrow F$) if there is a constant positve C such that*

$$\|x\|_F \leq C\,\|x\|_E$$

.

Remarks 1.1.1. *As illustration, we have the following embedding:*

$$W_0^{1,p}(\Omega) \hookrightarrow L^{p^\star}$$

where $\quad p^\star = \frac{np}{n-p}, \quad$ *for any* $\quad p > n$. *i.e. Sobolev embedding theorem.*

1.2 Hilbert spaces

The space $H^1(\Omega)$

Assume that $\Omega \subseteq \mathbb{R}^n$ is a domain. We define the Hilbert spaces as follows:

$$H^1(\Omega) = \left\{ v \in L^2(\Omega), \quad \nabla v \in L^2(\Omega) \right\}.$$

According to the separability of $L^2(\Omega)$, we have:

Proposition 1.2.1. *We have $H^1(\Omega)$ is a separable space. Moreover, it equipped with a scalar product which coincides with a norm:*

$$(v, v)_{H^1(\Omega)} = \|v\|^2_{H^1(\Omega)} = \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}$$

The space $H_0^1(\Omega)$

Proposition 1.2.2. *[7] We have*

$$\overline{\mathscr{D}(\Omega)}^{H^1(\Omega)} = H_0^1(\Omega)$$

That is means, for every $f \in H_0^1(\Omega)$ there exists a sequence $\{\varphi_k\} \subset D(\Omega)$ such that $\varphi_k \longrightarrow$ in $H^1(\Omega)$, that is, such that both $\|\varphi_k - f\|_{L^2(\Omega)} \longrightarrow 0$ and $\|\nabla\varphi_k - \nabla f\|_{L^2(\Omega)} \longrightarrow 0$ as $k \longrightarrow \infty$.

Remarks 1.2.1. *Due to the test functions in $\mathscr{D}(\Omega)$ have zero trace on $\partial\Omega$, every $v \in H_0^1(\Omega)$ inherits this property and it is reasonable to consider the elements $H_0^1(\Omega)$ as the functions in $H^1(\Omega)$ with zero trace on $\partial\Omega$. Clearly, $H_0^1(\Omega)$ is a Hilbert subspace of $H^1(\Omega)$.*

A significant property that holds in $H_0^1(\Omega)$, particularly useful in the solution of boundary value problems, is assumed by the following inequality of Poincaré.

Theorem 1.2.1. *[12] Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain. There exists a positive constant C_P (a Poincaré's constant) depending only on n and $diam(\Omega)$, such that, for every $v \in H_0^1(\Omega)$,*

$$\|v\|_{L^2(\Omega)} \leq C_p \|\nabla v\|_{L^2(\Omega)}. \tag{1.1}$$

where $diam(\Omega) = \sup_{x,y\in\Omega} |x-y|$.

Proof of Theorem :1.2.1

We employ a method that is quite typical for establishing formulas in $H_0^1(\Omega)$. First, we demonstrate the formula for $w \in \mathscr{D}(\Omega)$; then, if $v \in H_0^1(\Omega)$, we select a sequence $w_k \subset \mathscr{D}(\Omega)$ converging to v in the $H^1(\Omega)$ norm as $k \to \infty$, that is

$$\|w_k - v\|_{L^2(\Omega)} \to 0, \quad \|\nabla w_k - \nabla v\|_{L^2(\Omega)} \to 0$$

Specifically, we have

$$\|w_k\|_{L^2(\Omega)} \to \|v\|_{L^2(\Omega)}, \quad \|\nabla v_k\|_{L^2(\Omega)} \to \|\nabla v\|_{L^2(\Omega)}$$

We deduce that, for every v_k:

$$\|w_k\|_{L^2(\Omega)} \leq Cp \|\nabla w_k\|_{L^2(\Omega)}$$

Letting $k \to \infty$, we conclude that (1.1) for v.

Remarks 1.2.2. *Make sure that Sobolev space $W^{1,2}(\Omega)$ is the collection of all functions $v \in L^2(\Omega)$ for which $|\nabla v| \in L^2(\Omega)$ and equipped by the norm $\|v\|_{W^{1,2}(\Omega)} = \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}$. Moreover, we have the closure of $\mathscr{D}(\Omega)$ in $W^{1,2}(\Omega)$ is denoted by $W_0^{1,2}(\Omega)$ and hence the space $W_0^{1,2}(\Omega) = H_0^1(\Omega)$ is a Hilbert space.*

1.3 Morrey spaces $L^{p,\gamma}(\Omega)$

We point out that the Morrey spaces have certain inclusion relations with the Stummel classes and that they are useful for studying elliptic partial differential equation theory as in [5].

Definition 1.3.1. *[19] : Assume that* Ω *is a bounded and open subset of* $\mathbb{R}^n$*, where* $n \geqslant 3$*, and* l *is the diameter of* Ω*. For every* $a \in \Omega$ *and* $r > 0$*, we define*

$$B(a,r) = \left\{t \in \mathbb{R}^n; |t-a| < r\right\}.$$

and

$$O(a,r) = \Omega \cap B(a,r) = \{y \in \Omega : |y-a| < r\}.$$

The Morrey spaces $L^{p,\gamma}(\Omega)$ *is stated as:*

$$L^{p,\gamma}(\Omega) = \left\{ f \in L^p(\Omega), with \|f\|_{L^{p,\gamma}} = \sup_{a\in\Omega, r>0} \left(\frac{1}{r^\gamma}\int_{O(a,r)} |f(y)|^p dy\right)^{\frac{1}{p}} < \infty, for 1 \leq p < \infty, \quad 0 \leq \gamma \leq n \right\}$$

.

Morrey spaces $L^{1,\gamma}(\Omega)$

Definition 1.3.2. *([11],[16]) : If* $O(a,r) = \{y \in \Omega : |y-a| < r\}$ *then we denfine* $L^{1,\gamma}(\Omega)$ *as follows:*

$$L^{1,\gamma}(\Omega) = \left\{ f \in L^1(\Omega),\ with\ cf\|_{L^{1,\gamma}} = \sup_{x\in\Omega, r>0} \frac{1}{r^\gamma}\int_{O(x,r)} |f(y)| dy < \infty,\ for \quad 0 \leq \gamma \leq n \right\}.$$

Proposition:[4]

Let Ω be an open subset of $\mathbb{R}^n$. Let $\lambda \in [0,+\infty[$, $p \in [1,+\infty[$. If $\rho_1, \rho_2 \in]0,+\infty[$, then

$$L_p^{r^{-\lambda},\rho_1}(\Omega) = L_p^{r^{-\lambda},\rho_2}(\Omega),$$

and the corresponding norms are equivalent.

Proof: We can clearly assume that $\rho_1 < \rho_2$., it suffices to estimate $|f|_{\rho_2,r^{-\lambda},p,\Omega}$ in terms of $|f|_{\rho_1,r^{-\lambda},p,\Omega}$ for all $f \in L_p^{r^{-\lambda},\rho_1}(\Omega)$. Let $f \in L_p^{r^{-\lambda},\rho_1}(\Omega)$.

$$|f|_{\rho_2,r^{-\lambda},p,\Omega} = \sup_{(x,r)\in\Omega\times]0,\rho_2[} r^{-\lambda}\|f\|_{L^p(\mathbb{B}_n(x,r)\cap\Omega)} \tag{1.2}$$

$$\leq \max\left\{ \sup_{(x,r)\in\Omega\times]0,\rho_1[} r^{-\lambda}\|f\|_{L^p(\mathbb{B}_n(x,r)\cap\Omega)}, \sup_{(x,r)\in\Omega\times[\rho_1,\rho_2[} r^{-\lambda}\|f\|_{L^p(\mathbb{B}_n(x,r)\cap\Omega)} \right\}$$

$$= \max\left\{ |f|_{\rho_1,r^{-\lambda},p,\Omega}, \sup_{(x,r)\in\Omega\times[\rho_1,\rho_2[} r^{-\lambda}\|f\|_{L^p(\mathbb{B}_n(x,r)\cap\Omega)} \right\}.$$

Next we note that

$$\sup_{(x,r)\in\Omega\times[\rho_1,\rho_2[} r^{-\lambda}\|f\|_{L^p(\mathbb{B}_n(x,r)\cap\Omega)} \leq \rho_1^{-\lambda} \sup_{(x,r)\in\Omega\times[\rho_1,\rho_2[} \|f\|_{L^p(\mathbb{B}_n(x,r)\cap\Omega)} \tag{1.3}$$

$$\leq \rho_1^{-\lambda} \sup_{x\in\Omega} \|f\|_{L^p(\mathbb{B}_n(x,\rho_2)\cap\Omega)} = \rho_1^{-\lambda} \sup_{x\in\Omega} \|E_\Omega f\|_{L^p(\mathbb{B}_n(x,\rho_2))} .$$

Next we choose an arbitrary number in $]0,\rho_1[$, for example $\rho_1/2$, and we try to estimate

$$\sup_{x\in\Omega} \|E_\Omega f\|_{L^p(\mathbb{B}_n(x,\rho_2))}$$

in terms of $\sup_{z\in\Omega} \|E_\Omega f\|_{L^p(\mathbb{B}_n(z,\rho_1/2))}$. Indeed $\sup_{z\in\Omega} \|E_\Omega f\|_{L^p(\mathbb{B}_n(z,\rho_1/2))}$ can be estimated as follows in terms of $|f|_{\rho_1,r^{-\lambda},p,\Omega}$

$$\sup_{z\in\Omega} \|E_\Omega f\|_{L^p(\mathbb{B}_n(z,\rho_1/2))} = \sup_{z\in\Omega} \|f\|_{L^p(\mathbb{B}_n(z,\rho_1/2)\cap\Omega)} \tag{1.4}$$

$$\leq \sup_{(z,r)\in\Omega\times]0,\rho_1[} \|f\|_{L^p(\mathbb{B}_n(z,r)\cap\Omega)} = \sup_{(z,r)\in\Omega\times]0,\rho_1[} r^{\lambda} r^{-\lambda} \|f\|_{L^p(\mathbb{B}_n(z,r))}$$
$$\leq \rho_1^{\lambda} \sup_{(z,r)\in\Omega\times]0,\rho_1[} r^{-\lambda} \|f\|_{L^p(\mathbb{B}_n(z,r))} = \rho_1^{\lambda} |f|_{\rho_1,r^{-\lambda},p,\Omega} .$$

To do so, we observe that there exist finitely many points $\xi_1,\ldots,\xi_{m(\rho_2)}$ of $\overline{\mathbb{B}}_n(0,\rho_2)$ such that

$$\bigcup_{j=1}^{m(\rho_2)} \mathbb{B}_n(\xi_j,\rho_1/4) \supseteq \overline{\mathbb{B}}_n(0,\rho_2) ,$$

and that accordingly

$$\bigcup_{j=1}^{m(\rho_2)} \mathbb{B}_n(x+\xi_j,\rho_1/4) \supseteq \mathbb{B}_n(x,\rho_2) \qquad \forall x\in\mathbb{R}^n .$$

If $x\in\Omega$, we have

$$\|E_\Omega f\|_{L^p(\mathbb{B}_n(x,\rho_2))} \leq m(\rho_2)^{1/p} \sup_{j\in\{1,\ldots,m(\rho_2)\}} \|E_\Omega f\|_{L^p(\mathbb{B}_n(x+\xi_j,\rho_1/4))} \tag{1.5}$$

$$= m(\rho_2)^{1/p} \sup_{j\in\{1,\ldots,m(\rho_2)\},\mathbb{B}_n(x+\xi_j,\rho_1/4)\cap\Omega\neq\emptyset} \|E_\Omega f\|_{L^p(\mathbb{B}_n(x+\xi_j,\rho_1/4))} .$$

Next we choose a point $y_{j,x} \in \mathbb{B}_n(x+\xi_j,\rho_1/4)\cap\Omega$ for each $j\in\{1,\ldots,m(\rho_2)\}$ such that $\mathbb{B}_n(x+\xi_j,\rho_1/4)\cap\Omega\neq\emptyset$. Then we have

$$\mathbb{B}_n(x+\xi_j,\rho_1/4)\cap\Omega \subseteq \mathbb{B}_n(y_{j,x},\rho_1/2)$$

for each $j\in\{1,\ldots,m(\rho_2)\}$ such that $\mathbb{B}_n(x+\xi_j,\rho_1/4)\cap\Omega\neq\emptyset$. Hence,

$$m(\rho_2)^{1/p} \sup_{j\in\{1,\ldots,m(\rho_2)\},\mathbb{B}_n(x+\xi_j,\rho_1/4)\cap\Omega\neq\emptyset} \|E_\Omega f\|_{L^p(\mathbb{B}_n(x+\xi_j,\rho_1/4))}$$

$$\leq m(\rho_2)^{1/p} \sup_{j\in\{1,\ldots,m(\rho_2)\},\mathbb{B}_n(x+\xi_j,\rho_1/4)\cap\Omega\neq\emptyset} \|E_\Omega f\|_{L^p(\mathbb{B}_n(y_{j,x},\rho_1/2))}$$
$$\leq m(\rho_2)^{1/p} \sup_{z\in\Omega} \|E_\Omega f\|_{L^p(\mathbb{B}_n(z,\rho_1/2))}$$

Then inequalities (1.2)-(1.5) imply that

$$|f|_{\rho_2,r^{-\lambda},p,\Omega} \leq \max\left\{ |f|_{\rho_1,r^{-\lambda},p,\Omega}, \sup_{(x,r)\in\Omega\times[\rho_1,\rho_2]} r^{-\lambda} \|f\|_{L^p(\mathbb{B}_n(x,r)\cap\Omega)} \right\}$$

$$\leq \max\left\{|f|_{\rho_1,r^{-\lambda},p,\Omega}, \rho_1^{-\lambda} \sup_{x\in\Omega} \|E_\Omega f\|_{L^p(\mathbb{B}_n(x,\rho_2))}\right\}$$

$$\leq \max\left\{|f|_{\rho_1,r^{-\lambda},p,\Omega}, \rho_1^{-\lambda} m(\rho_2)^{1/p} \sup_{z\in\Omega} \|E_\Omega f\|_{L^p(\mathbb{B}_n(z,\rho_1/2))}\right\}$$

$$\leq \max\left\{|f|_{\rho_1,r^{-\lambda},p,\Omega}, \rho_1^{-\lambda} m(\rho_2)^{1/p} \rho_1^{\lambda} |f|_{\rho_1,r^{-\lambda},p,\Omega}\right\}$$

$$= \max\left\{1, m(\rho_2)^{1/p}\right\} |f|_{\rho_1,r^{-\lambda},p,\Omega}$$

and thus the proof is complete.

The previous proposition shows that the spaces $L_p^{r^{-\lambda},\rho}(\Omega)$ coincide for all finite values of ρ.

If ρ is finite, the definition of $|f|_{\rho,r^{-\lambda},p,\Omega}$ requires information on f on the balls $\mathbb{B}_n(x,r)\cap\Omega$ for r only in the right neighborhood $]0,\rho[$, and we can take ρ as small as we wish, and thus the membership of f in $L_p^{r^{-\lambda},\rho}(\Omega)$ can be interpreted as a regularity condition of f.

If instead $\rho = +\infty$, the definition of $|f|_{+\infty,r^{-\lambda},p,\Omega}$ involves information on f on the balls $\mathbb{B}_n(x,r)\cap\Omega$ both when r is small and when r is arbitrarily .

1.4 Weak Morrey space

For $1 \leqslant p < \infty$ and $0 \leqslant \lambda \leqslant n$, we define $wL^{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space to be the set of all functions $f \in wL^{p_{loc}\mathbb{R}^n}$ for which

$$\|f\|_{wL^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\lambda/p} \|f\|_{wL^p(B(x,r))} < \infty$$

where

$$\|f\|_{wL^p(B(x,r))} = \sup_{t>0} t\left(\mu\{y \in (B(x,r)) : |f(y)| > t\}\right)^{\frac{1}{p}}$$

with μ being the Lebesgue measure on $\mathbb{R}^n$ and $B(x,r) = y \in \mathbb{R}^n : |x-y| < r$.

Weak Morrey space and Morrey space

The first proposition is about the inclusion relation between weak Morrey spaces and Morrey spaces.

Proposition: Let $1 < p < \infty$ and $0\lambda < n$.If $f \in wL^{p,\lambda}(\mathbb{R}^n)$,Then

$$\|f\|_{wL^{(\lambda/p-n/p+n)}(\mathbb{R}^n)} \leq C(n,\lambda,p)\|f\|^p_{wL^{\lambda}(\mathbb{R}^n)}.$$

Therefor, $wL^{\lambda}(\mathbb{R}^n)^p \subseteq wL^{(\lambda/p-n/p+n)}(\mathbb{R}^n)$.

Proof: Take any $f \in wL^{p,\lambda}(\mathbb{R}^n)$, $x \in R^n$and $r > 0$.For every $\sigma \in \mathbb{R}$,we have

$$\int_0^\infty \mu\left(y \in B(x,r) : |f(x)| > t\right) dt \leqslant \int_0^{(\mu(B(x,r)))^\sigma} \mu(B(x,r))\, dt + \|f\|^p_{wL^\lambda(\mathbb{R}^n)} \int_{(\mu(B(x,r)))^\sigma}^0 \frac{r^\lambda}{t^p} dt$$

$$= \left(\mu(B(x,r))\right)^{\sigma+1} + C(p)\|f\|^p_{wL^{p,\lambda}(\mathbb{R}^n)} \left(\mu(B(x,r))\right)^{\sigma(1-p)} r^\lambda. \tag{1.6}$$

Let $\sigma = \frac{\lambda}{np} - \frac{1}{p}$and $\beta = \frac{1}{p}(\lambda - n) + n$.Tene $n(\lambda+1) = n\lambda(1-p) + \lambda = \beta$.By (1.6),we obtain

$$\frac{1}{r^{(\lambda/p - n/p + n)}} \int_{|x-y|<r} |f(y)|dy = \frac{1}{r^\beta} \int_{|x-y|<r} |f(y)|dy$$

$$= \frac{1}{r^\beta} \int_0^\infty \mu(y \in B(x,r) : |f(y)| > t)dt$$
$$\leqslant C(n,\lambda,p)\|f\|^p_{wL^{p,\lambda}(\mathbb{R}^n)} \tag{1.7}$$

for every $x \in \mathbb{R}$and $r > 0$. From (1.7), the conclusion follows. From the Proposition 2.1 and Lemma 1.1, we have the following corollary.

Corollary 1.4.1. *Let* $1 < p < \infty, 0 < \lambda < n$, *and* $\frac{n-1}{p} < \alpha < n$. *If* $f \in wL^{p,\lambda}(\mathbb{R}^n)$, *then*

$$\|f\|_{L^{1,(\lambda/p-n/p+n)}(\mathbb{R}^n)} \leq C(n,\lambda,p)\|f\|^p_{wL^{p,\lambda}(\mathbb{R}^n)}$$

If $f \in L^{1,(\lambda/p-n/p+n)}(\mathbb{R}^n)$, then

$$\rho_\alpha f(r) \leqslant C(n,\alpha,\lambda,p) r^{\frac{1}{p}(\lambda-n)+\alpha} \|f\|_{L^{1,(\lambda/p-n/p+n)}(\mathbb{R}^n)}$$

Therefore,$wL^{p,\lambda}(\mathbb{R}^n) \subseteq L^{1,(\lambda/p-n/p+n)}(\mathbb{R}^n) \subseteq S_\alpha(\mathbb{R}^n)$

Example :Let $0 < \alpha < n$ and $n - \alpha < \gamma < n$. Define $f : \mathbb{R}^n \to \mathbb{R}$ by the formula

$$f(y) = \frac{\chi B(y)}{|y|^\alpha |ln(|y|)|^2}, \qquad B = B(0,\delta), \qquad y \in \mathbb{R}^n$$

where $\delta = e^{\frac{-2}{\alpha}}$.Then $f \in S_\alpha(\mathbb{R}^n) \backslash L^{1,\lambda}(\mathbb{R}^n)$.

To verify the example, we first show that $f \in S_\alpha(\mathbb{R}^n)$. Since the function $f$is radial and non-increasing, the Stummel modulus of f is attained at the origin.Given $r > 0$, let $\epsilon = \min(r,\delta)$. We have, We have,

$$\rho_\alpha f(r) = \int_{|y|<r} \frac{\chi B(y)}{|y|^n |ln(|y|)|^2} dy = \int_{|y|<\epsilon} \frac{1}{|y|^n |ln(|y|)|^2} dy.$$

By switching to polar coordinate, we obtain

$$\int_{|y|<\epsilon} \frac{1}{|y|^n |ln(|y|)|^2} dy = \frac{C(n)}{-ln(\epsilon)}.$$

Therefore,

$$\rho_\alpha f(r) = \frac{C(n)}{-ln(\epsilon)}.$$

By the definition of ϵ, for every $0 < r < \delta$, we have

$$\rho_\alpha f(r) = C(n)\frac{1}{-ln(r)} \to 0 \qquad For \qquad r \to 0$$

Thus, $f \in S_\alpha(\mathbb{R}^n)$. Next, using the fact that $\frac{1}{t^\alpha}|ln(t)|^2$ is decreasing on $(0,\delta)$ and the condition $n - \alpha < \gamma$, one may observe that $f \notin L^{1,\lambda}(\mathbb{R}^n)$.

Proposition 1.4.1. *[4] Let* Ω *be an open subset of* $\mathbb{R}^n$*,* $\lambda \in [0, +\infty[$*, and* $p \in [1, +\infty[$*. If* $\rho_1, \rho_2 \in]0, +\infty[$*, then*

$$L_p^{r^{-\lambda},\rho_1}(\Omega) = L_p^{r^{-\lambda},\rho_2}(\Omega),$$

and the corresponding norms are equivalent.

Proof. Without loss of generality, assume $\rho_1 < \rho_2$. By the inclusion (3.4), it suffices to estimate $|f|_{\rho_2,r^{-\lambda},p,\Omega}$ in terms of $|f|_{\rho_1,r^{-\lambda},p,\Omega}$ for all $f \in L_p^{r^{-\lambda},\rho_1}(\Omega)$.
For $f \in L_p^{r^{-\lambda},\rho_1}(\Omega)$, we have:

$$|f|_{\rho_2,r^{-\lambda},p,\Omega} = \sup_{(x,r)\in\Omega\times]0,\rho_2[} r^{-\lambda} \|f\|_{L^p(\mathbb{B}_n(x,r)\cap\Omega)}.$$

This can be split into two parts:

$$|f|_{\rho_2,r^{-\lambda},p,\Omega} \leq \max\left\{ |f|_{\rho_1,r^{-\lambda},p,\Omega}, \sup_{(x,r)\in\Omega\times[\rho_1,\rho_2[} r^{-\lambda} \|f\|_{L^p(\mathbb{B}_n(x,r)\cap\Omega)} \right\}.$$

For the second term, we observe:

$$\sup_{(x,r)\in\Omega\times[\rho_1,\rho_2[} r^{-\lambda} |f|_{L^p(\mathbb{B}_n(x,r)\cap\Omega)} \leq \rho_1^{-\lambda} \sup_{x\in\Omega} \|f\|_{L^p(\mathbb{B}_n(x,\rho_2)\cap\Omega)} = \rho_1^{-\lambda} \sup_{x\in\Omega} \|E_\Omega f\|_{L^p(\mathbb{B}_n(x,\rho_2))}.$$

Next, we estimate $\sup_{x\in\Omega} \|E_\Omega f\|_{L^p(\mathbb{B}_n(x,\rho_2))}$ in terms of $\sup_{z\in\Omega} \|E_\Omega f\|_{L^p(\mathbb{B}_n(z,\rho_1/2))}$. The latter can be bounded as follows:

$$\sup_{z\in\Omega} \|E_\Omega f\|_{L^p(\mathbb{B}_n(z,\rho_1/2))} = \sup_{z\in\Omega} \|f\|_{L^p(\mathbb{B}_n(z,\rho_1/2)\cap\Omega)} \leq \rho_1^{\lambda} |f|_{\rho_1,r^{-\lambda},p,\Omega}.$$

To complete the proof, we note that there exist finitely many points $\xi_1, \ldots, \xi_{m(\rho_2)}$ in $\overline{\mathbb{B}}_n(0,\rho_2)$ such that:

$$\bigcup_{j=1}^{m(\rho_2)} \mathbb{B}_n(\xi_j, \rho_1/4) \supseteq \overline{\mathbb{B}}_n(0,\rho_2).$$

□

and that accordingly

$$\bigcup_{j=1}^{m(\rho_2)} B_n(x+\xi_j, \rho_1/4) \supseteq B_n(x,\rho_2) \qquad \forall x \in \mathbb{R}^n.$$

If $x \in \Omega$, we have

$$\begin{aligned} \|E_\Omega f\|_{L^p(B_n(x,\rho_2))} &\leq m(\rho_2)^{1/p} \sup_{j\in\{1,\ldots,m(\rho_2)\}} \|E_\Omega f\|_{L^p(B_n(x+\xi_j,\rho_1/4))} \\ &= m(\rho_2)^{1/p} \sup_{\substack{j\in\{1,\ldots,m(\rho_2)\}, \\ B_n(x+\xi_j,\rho_1/4)\cap\Omega\neq\emptyset}} \|E_\Omega f\|_{L^p(B_n(x+\xi_j,\rho_1/4))}. \end{aligned}$$

Next we choose a point $y_{j,x} \in B_n(x+\xi_j,\rho_1/4)\cap\Omega$ for each $j \in \{1,\ldots,m(\rho_2)\}$ such that $B_n(x+\xi_j,\rho_1/4)\cap\Omega \neq \emptyset$. Then we have

$$B_n(x+\xi_j,\rho_1/4)\cap\Omega \subseteq B_n(y_{j,x},\rho_1/2)$$

for each $j \in \{1,\ldots,m(\rho_2)\}$ such that $B_n(x+\xi_j,\rho_1/4)\cap\Omega \neq \emptyset$. Hence,

$$
\begin{aligned}
m(\rho_2)^{1/p} & \sup_{\substack{j\in\{1,\ldots,m(\rho_2)\},\\ B_n(x+\xi_j,\rho_1/4)\cap\Omega\neq\emptyset}} \|E_\Omega f\|_{L^p(B_n(x+\xi_j,\rho_1/4))} \\
& \leq m(\rho_2)^{1/p} \sup_{\substack{j\in\{1,\ldots,m(\rho_2)\},\\ B_n(x+\xi_j,\rho_1/4)\cap\Omega\neq\emptyset}} \|E_\Omega f\|_{L^p(B_n(y_{j,x},\rho_1/2))} \\
& \leq m(\rho_2)^{1/p} \sup_{z\in\Omega} \|E_\Omega f\|_{L^p(B_n(z,\rho_1/2))}
\end{aligned}
$$

Then inequalities (1.2)-(1.5) imply that

$$
\begin{aligned}
|f|_{\rho_2,r^{-\lambda},p,\Omega} & \leq \max\left\{|f|_{\rho_1,r^{-\lambda},p,\Omega}, \sup_{(x,r)\in\Omega\times[\rho_1,\rho_2]} r^{-\lambda}\|f\|_{L^p(B_n(x,r)\cap\Omega)}\right\} \\
& \leq \max\left\{|f|_{\rho_1,r^{-\lambda},p,\Omega}, \rho_1^{-\lambda}\sup_{x\in\Omega}\|E_\Omega f\|_{L^p(B_n(x,\rho_2))}\right\} \\
& \leq \max\left\{|f|_{\rho_1,r^{-\lambda},p,\Omega}, \rho_1^{-\lambda} m(\rho_2)^{1/p}\sup_{z\in\Omega}\|E_\Omega f\|_{L^p(B_n(z,\rho_1/2))}\right\} \\
& \leq \max\left\{|f|_{\rho_1,r^{-\lambda},p,\Omega}, \rho_1^{-\lambda} m(\rho_2)^{1/p}\rho_1^{\lambda}|f|_{\rho_1,r^{-\lambda},p,\Omega}\right\} \\
& = \max\left\{1, m(\rho_2)^{1/p}\right\}|f|_{\rho_1,r^{-\lambda},p,\Omega}
\end{aligned}
$$

and thus the proof is complete. □

The previous proposition shows that the spaces $L_p^{r^{-\lambda},\rho}(\Omega)$ coincide for all finite values of ρ.

If ρ is finite, the definition of $|f|_{\rho,r^{-\lambda},p,\Omega}$ requires information on f on the balls $B_n(x,r)\cap\Omega$ for r only in the right neighborhood $]0,\rho[$, and we can take ρ as small as we wish, and thus the membership of f in $L_p^{r^{-\lambda},\rho}(\Omega)$ can be interpreted as a regularity condition of f.

If instead $\rho = +\infty$, the definition of $|f|_{+\infty,r^{-\lambda},p,\Omega}$ involves information on f on the balls $B_n(x,r)\cap\Omega$ both when r is small and when r is arbitrarily
large and thus the membership of f in $L_p^{r^{-\lambda},+\infty}(\Omega)$ can be interpreted as a condition that is at one hand a regularity condition and on the other hand also a condition on the behavior of f at infinity.

In order to treat $L_p^{r^{-\lambda},\rho}(\Omega)$ both in case $\rho < +\infty$ and $\rho = +\infty$ at the same time, we resort to the generalized Morrey spaces, that we now introduce by means of the following.

Definition 1.4.1. *[4] Let Ω be an open subset of $\mathbb{R}^n$. Let $p \in [1,+\infty]$. Let w be a function from $]0,+\infty[$ to $[0,+\infty[$. Let*

$$
|f|_{\rho,w,p,\Omega} \equiv \sup_{(x,r)\in\Omega\times]0,\rho[} w(r)\|f\|_{L^p(B_n(x,r)\cap\Omega)} \qquad \forall \rho \in]0,+\infty],
$$

for all measurable functions f from Ω to $\mathbb{R}$.

Assume that there exists $r_0 \in]0,+\infty[$ such that $w(r_0) \neq 0$. Then we define as generalized Morrey space with weight w and exponent p the set

$$
L_p^w(\Omega) \equiv \left\{f \in \Omega : f \text{ is measurable}, |f|_{+\infty,w,p,\Omega} < +\infty\right\}.
$$

Then we set

$$
\|f\|_{L_p^w(\Omega)} \equiv |f|_{+\infty,w,p,\Omega} \qquad \forall f \in L_p^w(\Omega).
$$

One can easily verify that $(L_p^w(\Omega), \|\cdot\|_{L_p^w(\Omega)})$ is a normed space.
Here we mention that the definition of generalized Morrey spaces is not uniform in the literature. So for example if $\Omega = n$, the definition here coincides with that of Gogatishvili and Mustafayev [12]. Then we can obtain the definition of Nakai [17] by taking $w(r) = \varphi(r)^{-1} m_n(B_n(0,r))^{-1/p}$ and that of Sawano [21] by taking $w(r) = \varphi(r) m_n(B_n(0,r))^{-1/p}$ for some real valued function φ. We also note that here we say that w is a 'weight' (as in Gogatishvili and Mustafayev [12]), while other authors reserve the word weight for a weight put on the measure in Ω as in Samko [19] (a case that we do not discuss in these notes).
One may wonder whether $|f|_{\rho,w,p,\Omega}$ would change if we replace the supremum in $x \in \Omega$ with $x \in \overline{\Omega}$. The answer is no as the following lemma shows.

Lemma 1.4.1. *[4] Let Ω be an open subset of $\mathbb{R}^n$. Let $p \in [1,+\infty]$. Let w be a function from $]0,+\infty[$ to $[0,+\infty[$. If f is a measurable function from Ω to such that $f \in L^p(B_n(x,r)\cap\Omega)$ for all $(x,r) \in \Omega\times]0,+\infty[$, then*

$$\sup_{(x,r)\in\Omega\times]0,\rho[} w(r)\|f\|_{L^p(B_n(x,r)\cap\Omega)} = \sup_{(x,r)\in\overline{\Omega}\times]0,\rho[} w(r)\|f\|_{L^p(B_n(x,r)\cap\Omega)},$$

for all $\rho \in]0,+\infty[$.

Proof. It clearly suffices to prove that

$$\sup_{(x,r)\in(\partial\Omega)\times]0,\rho[} w(r)\|f\|_{L^p(B_n(x,r)\cap\Omega)} \leq \sup_{(x,r)\in\Omega\times]0,\rho[} w(r)\|f\|_{L^p(B_n(x,r)\cap\Omega)},$$

for all $\rho \in]0,+\infty[$. Let $\rho \in]0,+\infty]$. Let $(\tilde{x},r) \in (\partial\Omega)\times]0,\rho[$. Let $\{x_j\}_{j\in N}$ be a sequence in Ω such that $\lim_{j\to\infty} x_j = \tilde{x}$. It clearly suffices to show that

$$w(r)\|f\|_{L^p(B_n(\tilde{x},r)\cap\Omega)} \leq \limsup_{j\to\infty} w(r)\|f\|_{L^p(B_n(x_j,r)\cap\Omega)}. \tag{1.8}$$

Indeed,

$$\limsup_{j\to\infty} w(r)\|f\|_{L^p(B_n(x_j,r)\cap\Omega)} \leq \sup_{(x,r)\in\Omega\times]0,\rho[} w(r)\|f\|_{L^p(B_n(x,r)\cap\Omega)}.$$

Possibly neglecting a finite number of terms, we can clearly assume that

$$|x_j - \tilde{x}| \leq 1,$$

and accordingly that

$$B_n(x_j,r) \subseteq B_n(\tilde{x}, r+1) \qquad \forall j \in N.$$

Obviously,

$$\lim_{j\to\infty} \chi_{B_n(x_j,r)\cap\Omega}(x) f(x) = \chi_{B_n(\tilde{x},r)\cap\Omega}(x) f(x) \qquad \forall x \in \Omega \setminus \partial B_n(\tilde{x},r),$$

and accordingly for almost all $x \in \Omega$. Indeed, $m_n(\partial B_n(\tilde{x},r)) = 0$. Next we consider separately cases $p < \infty$ and case $p = \infty$.

Let $p < \infty$. Since

$$B_n(x_j, r) \subseteq B_n(\tilde{x}, r+1) \subseteq B_n(x_0, r+2) \qquad \forall j \in N,$$

we have

$$|\chi_{B_n(x_j,r)\cap\Omega} f|^p \leq |f|^p \chi_{B_n(x_0,r+2)\cap\Omega} \in L^1(\Omega),$$

for all $j \in N$, and the convergence of (3.13) and the Dominated Convergence Theorem imply that

$$\lim_{j\to\infty} \int_\Omega |\chi_{B_n(x_j,r)\cap\Omega} f|^p \, dx = \int_\Omega |\chi_{B_n(\tilde{x},r)\cap\Omega} f|^p \, dx, \tag{1.9}$$

and thus (1.9) holds.

We now assume that $p = \infty$. Let N be a subset of Ω of measure zero such that

$$\begin{aligned}
&\partial B_n(\tilde{x}, r) \subseteq N, \\
&\quad |f(x)| \leq \|f\|_{L^\infty(B_n(x_0,r+2)\cap\Omega)} \qquad \forall x \in (B_n(x_0, r+2) \cap \Omega) \setminus N, \\
&\quad |f(x)| \leq \|f\|_{L^\infty(B_n(x_l,r)\cap\Omega)} \qquad \forall x \in (B_n(x_l, r) \cap \Omega) \setminus N, \qquad \forall l \in N,
\end{aligned}$$

(cf. (1.9)). Then we have

$$\begin{aligned}
w(r)|f(x)\chi_{B_n(\tilde{x},r)\cap\Omega}(x)| &\leq w(r) \limsup_{j\to\infty} |f(x)\chi_{B_n(x_j,r)\cap\Omega}(x)| \\
&\leq w(r) \inf_{j\in N} \sup_{l\geq j} |f(x)\chi_{B_n(x_l,r)\cap\Omega}(x)| \\
&\leq \inf_{j\in N} \sup_{l\geq j} w(r)\|f\|_{L^\infty(B_n(x_l,r)\cap\Omega)} \\
&\leq \limsup_{j\to\infty} w(r)\|f\|_{L^\infty(B_n(x_l,r)\cap\Omega)} \qquad \forall x \in (B_n(x_0, r+2) \cap \Omega) \setminus N.
\end{aligned}$$

Since $(B_n(\tilde{x}, r) \cap \Omega) \setminus N \subseteq (B_n(x_0, r+2) \cap \Omega) \setminus N$, such an inequality holds in particular for all $x \in (B_n(\tilde{x}, r) \cap \Omega) \setminus N$ and accordingly

$$w(r)\|f\|_{L^\infty(B_n(\tilde{x},r)\cap\Omega)} \leq \limsup_{j\to\infty} w(r)\|f\|_{L^\infty(B_n(x_l,r)\cap\Omega)},$$

and thus (1.8) holds. □

1.5 Trivial and Nontrivial Generalized Morrey Spaces

In this section, we briefly discuss the issue of triviality of the generalized Morrey spaces and first prove the following triviality condition for a generalized Morrey space (and for the corresponding vanishing space).

Proposition 1.5.1. *[4] Let Ω be an open subset of $\mathbb{R}^n$. Let $p \in [1, +\infty]$. Let w be a function from $]0, +\infty[$ to $[0, +\infty[$. If there exists $r_0 \in]0, +\infty[$ such that $w(r) \neq 0$ for all $r \in]0, r_0[$, and if*

$$\limsup_{r\to 0} w(r) r^{n/p} = +\infty, \tag{1.10}$$

then $L_p^w(\Omega) = \{0\}$.

Proof. Let $f \in L_p^w(\Omega)$. Since $L_p^w(\Omega) \subseteq L^p(\mathbb{B}_n(0,r) \cap \Omega)$ for all $r \in]0,+\infty[$, we have $f \in L^p_{\text{loc}}(\Omega) \subseteq L^1_{\text{loc}}(\Omega)$, and the Lebesgue Differentiation Theorem implies that

$$\lim_{r \to 0} \fint_{\mathbb{B}_n(x,r)} |f(y)|\, dy = |f(x)| \quad \text{a.a. } x \in \Omega.$$

To show that the limit equals zero, we note that the Hölder inequality implies:

$$\fint_{\mathbb{B}_n(x,r)} |f(y)|\, dy \leq \frac{1}{\omega_n^{1/p} r^{n/p} w(r)} \|f\|_{L_p^w(\Omega)},$$

for all $x \in \Omega$ and $r \in]0, r_0[$ such that $\mathbb{B}_n(x,r) \subseteq \Omega$. By assumption (1.10),We

$$\liminf_{r \to 0} \frac{1}{r^{n/p} w(r)} = 0,$$

and thus:

$$|f(x)| \leq \liminf_{r \to 0} \frac{1}{\omega_n^{1/p} r^{n/p} w(r)} \|f\|_{L_p^w(\Omega)} = 0 \quad \text{a.a. } x \in \Omega.$$

This completes the proof. □

By applying the previous proposition to the special weights $w_{\lambda,\rho}$, $r^{-\lambda}$, w_λ, we deduce the following corollary.

Corollary 1.5.1. *Let Ω be an open subset of $\mathbb{R}^n$. Let $p \in [1,+\infty]$. Let $\lambda \in [0,+\infty[$, $\lambda > n/p$. Then:*

$$L_p^{r^{-\lambda},\rho}(\Omega) = \{0\} \quad \textit{for all } \rho \in]0,+\infty[,$$

$$L_p^{r^{-\lambda}}(\Omega) = \{0\}, \quad L_p^{\lambda}(\Omega) = \{0\}.$$

Corollary 1.5.2. *states in particular that the Morrey spaces $L_p^{r^{-\lambda},\rho}(\Omega)$ for all $\rho \in]0,+\infty[$, $L_p^{r^{-\lambda}}(\Omega)$, $M_p^{\lambda}(\Omega)$, or the corresponding vanishing variants can be nontrivial only if $\lambda \in [0, n/p]$.*

Proposition 1.5.2. *Let Ω be an open subset of $\mathbb{R}^n$. Let $p \in [1,+\infty]$. Let w be a function from $]0,+\infty[$ to $[0,+\infty[$. Then the following two statements hold:*

(i) Let $\xi \in \Omega$, $R \in]0,+\infty[$, $w(R) \neq 0$. Then the map from

$$L_p^w(\Omega) \quad \textit{to} \quad L^p(\mathbb{B}_n(\xi,R) \cap \Omega),$$

which takes $f \in L_p^w(\Omega)$ to the restriction $f_{|\mathbb{B}_n(\xi,R)\cap\Omega}$, is linear and continuous. In particular,

$$\|f\|_{L^p(\mathbb{B}_n(\xi,R)\cap\Omega)} \leq w(R)^{-1} \|f\|_{L_p^w(\Omega)} \quad \forall f \in L_p^w(\Omega).$$

(ii) Assume that there exists $r_0 \in]0,+\infty[$ such that $w(r_0) \neq 0$. Then $L_p^w(\Omega)$ is continuously embedded into $L^p_{\text{loc}}(\Omega)$.

Proof. We can clearly assume that Ω is not empty. Statement (i) is an immediate consequence of the inequality:

$$\|f\|_{L^p(\mathbb{B}_n(\xi,R)\cap\Omega)} = w(R)^{-1} w(R)\|f\|_{L^p(\mathbb{B}_n(\xi,R)\cap\Omega)} \le w(R)^{-1} \sup_{(x,r)\in\Omega\times]0,+\infty[} w(r)\|f\|_{L^p(\mathbb{B}_n(x,r)\cap\Omega)} = w(R)^{-1}\|f\|_{L_p^w(\Omega)},$$

for all $f \in L_p^w(\Omega)$.
We now consider statement (ii). Let K be a compact subset of Ω. Since K is compact, there exist $\xi_1,\ldots,\xi_m$ such that:

$$K \subseteq \bigcup_{j=1}^{m} \mathbb{B}_n(\xi_j, r_0) \cap \Omega.$$

Then the inequality of (i) implies that

$$\|f\|_{L^p(K)} \le m^{1/p} \sup_{j=1,\ldots,m} \|f\|_{L^p(\mathbb{B}_n(\xi_j,r_0)\cap\Omega)} \le m^{1/p} w(r_0)^{-1}\|f\|_{L_p(\Omega)} \qquad \forall f \in L_p^w(\Omega).$$

□ □

If instead we want the functions of $L_p^w(\Omega)$ to be p summable, we need to introduce some restriction on w. Indeed, the following holds.

1.6 Elementary embeddings of generalizedMorrey spaces into Lebesgue spaces

Proposition 1.6.1. *[4] Let Ω be an open subset of $\mathbb{R}^n$. Let $p \in [1,+\infty]$. Let w be a function from $]0,+\infty[$ to itself. Let*

$$\eta_w \equiv \inf_{r\in]0,+\infty[} w(r) > 0. \tag{1.11}$$

Then the space $L_p^w(\Omega)$ is continuously embedded into $L^p(\Omega)$, and

$$\|f\|_{L^p(\Omega)} \le \eta_w^{-1}\|f\|_{L_p^w(\Omega)} \qquad \forall f \in L_p^w(\Omega). \tag{1.12}$$

Proof. Let $f \in L_p^w(\Omega)$.
If $p < \infty$, then the inequality

$$\int_{\mathbb{B}_n(x,r)\cap\Omega} |f(y)|^p\,dy \le w(r)^{-p}\left(\sup_{r\in]0,+\infty[} w(r)\|f\|_{L^p(\mathbb{B}_n(x,r)\cap\Omega)}\right)^p \le \eta_w^{-p}\left(\sup_{r\in]0,+\infty[} w(r)\|f\|_{L^p(\mathbb{B}_n(x,r)\cap\Omega)}\right)^p \quad \forall r \in]0,+\infty[$$

together with the Monotone Convergence Theorem imply that

$$\int_{\Omega} |f(y)|^p\,dy \le \eta_w^{-p}\left(\sup_{r\in]0,+\infty[} w(r)\|f\|_{L^p(\mathbb{B}_n(x,r)\cap\Omega)}\right)^p, \tag{1.13}$$

and thus $f \in L^p(\Omega)$. On the other hand, if $p = +\infty$, then the inequality

$$\operatorname{ess\,sup}_{\mathbb{B}_n(x,r)\cap\Omega} |f| \le \eta_w^{-1} \sup_{r\in]0,+\infty[} w(r)\|f\|_{L^\infty(\mathbb{B}_n(x,r)\cap\Omega)} \quad \forall r \in]0,+\infty[,$$

implies that

$$\operatorname{ess\,sup}_{\Omega}|f| \leq \eta_w^{-1} \sup_{r\in]0,+\infty[} w(r)\|f\|_{L^{\infty}(\mathbb{B}_n(x,r)\cap\Omega)}, \tag{1.14}$$

and thus $f \in L^{\infty}(\Omega)$. Inequalities (1.13) and (1.14) imply that both in case $p < \infty$ and $p = \infty$, we have

$$\|f\|_{L^p(\Omega)} \leq \eta_w^{-1}|f|_{+\infty,w,p,\Omega} = \eta_w^{-1}\|f\|_{\mathcal{M}_p^w(\Omega)}.$$

Hence, the statement follows. □

By applying the previous proposition to the special weight w_λ, we deduce the validity of the following corollary.

Corollary 1.6.1. *Let Ω be an open subset of $\mathbb{R}^n$. Let $p \in [1,+\infty]$. Let $\lambda \in [0,+\infty[$. Then the space $M_p^{\lambda}(\Omega)$ is continuously embedded into $L^p(\Omega)$, and $\|f\|_{L^p(\Omega)} \leq \|f\|_{M_p^{\lambda}(\Omega)}$ for all $f \in M_p^{\lambda}(\Omega)$.*
If instead we want $L_p^w(\Omega)$ to contain the space of p-summable functions $L^p(\Omega)$, we need to introduce some other restriction on w.

Proposition 1.6.2. *Let Ω be an open subset of $\mathbb{R}^n$. Let $p \in [1,+\infty]$. Let w be a function from $]0,+\infty[$ to $[0,+\infty[$. Assume that there exists $r_0 \in]0,+\infty[$ such that $w(r_0) \neq 0$. If*

$$\varsigma_w \equiv \sup_{r\in]0,+\infty[} w(r) < +\infty, \tag{1.15}$$

then $L^p(\Omega)$ is continuously embedded into $L_p^w(\Omega)$ and

$$\|f\|_{L_p^w(\Omega)} \leq \varsigma_w\|f\|_{L^p(\Omega)} \quad \forall f \in L^p(\Omega). \tag{1.16}$$

Proof. By assumption, we have

$$w(r)\|f\|_{L^p(\mathbb{B}_n(x,\rho)\cap\Omega)} \leq \varsigma_w\|f\|_{L^p(\mathbb{B}_n(x,\rho)\cap\Omega)} \leq \varsigma_w\|f\|_{L^p(\Omega)} \quad \forall (x,r) \in \Omega\times]0,+\infty[, \tag{1.17}$$

and inequality (1.16) holds true. □

However, condition (1.15) is quite restrictive and is satisfied only in certain limiting cases, such as that of the Morrey spaces $L^{r^{-0},\rho_p(\Omega)}$ for all $\rho \in]0,+\infty[$, $L_p^{r^{-0}}(\Omega)$, $M_p^0(\Omega)$.

Corollary 1.6.2. *Let Ω be an open subset of $\mathbb{R}^n$. Let $p \in [1,+\infty]$. Then $L^p(\Omega)$ is continuously embedded into the Morrey spaces $L^{r^{-0},\rho_p(\Omega)}$ for all $\rho \in]0,+\infty[$, $L_p^{r^{-0}}(\Omega)$, $M_p^0(\Omega)$. Moreover, the norm of the corresponding inclusion is less than or equal to 1.*

By combining Propositions 6.2 and 6.8, we deduce the validity of the following, which says that if the weight w is both bounded and bounded away from 0, then the corresponding generalized Morrey space coincides with a Lebesgue space.

Proposition 1.6.3. *Let Ω be an open subset of $\mathbb{R}^n$. Let $p \in [1,+\infty]$. Let w be a function from $]0,+\infty[$ to itself. Let*

$$\eta_w \equiv \inf_{r\in]0,+\infty[} w(r) > 0, \quad \varsigma_w \equiv \sup_{r\in]0,+\infty[} w(r) < +\infty.$$

Then the space $L^w_p(\Omega)$ equals $L^p(\Omega)$. Moreover,

$$\eta_w \|f\|_{L^p(\Omega)} \leq \|f\|_{L^w_p(\Omega)} \leq \varsigma_w \|f\|_{L^p(\Omega)} \quad \forall f \in L^p(\Omega).$$

By applying the previous proposition to the special weights $r^{-0} = 1 = w_0$, we deduce the validity of the following corollary.

Corollary 1.6.3. *Let Ω be an open subset of $\mathbb{R}^n$. Let $p \in [1,+\infty]$. Then the Morrey spaces $L^{r^{-0}}_p(\Omega)$, $M^0_p(\Omega)$ coincide with $L^p(\Omega)$, and the corresponding norms are equal.*

1.7 Stummel classes

Definition 1.7.1. *[20] : For $n \geq 3$, suppose that Ω is a bounded open set of $\mathbb{R}^n$, with diameter l. For $a \in \mathbb{R}^n$ and $r > 0$, we define*

$$B(a,r) = \{t \in \Omega; |t-a| < r\}.$$

*Assume that $0 < \alpha < n$, the **Stummel class** is defined as:*

$$\tilde{S}_\alpha(\Omega) = \left\{f \in L^1(\Omega),\ with\ \rho_\alpha f(r) < \infty, \forall r > 0\right\}$$

where

$$\rho_\alpha f(r) = \sup_{x\in\Omega} \int_{B(x,r)} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy$$

Definition 1.7.2. *[18], : For any $0 < \alpha < n$. The set*

$$\boldsymbol{S}_\alpha = S_\alpha(\mathbb{R}^n) = \left\{f \in L^1_{loc}(\mathbb{R}^n) : \rho_\alpha f(r) \to 0 \text{ for } r \to 0\right\}$$

defines the Stummel classes, namely: $S_\alpha(\mathbb{R}^n)$.

Lemma 1.7.1. *[16] We take $0 < \alpha < n$ and $n - \alpha < \gamma < n$. If $f \in L^{1,\gamma}(\mathbb{R}^n)$, then*

$$\rho_\alpha f(r) \leqslant C(n,\alpha,\gamma) r^{\gamma-n+\alpha} \|f\|_{L^{1,\gamma}(\mathbb{R}^n)},$$

Therefore, we deduce that

$$\boldsymbol{L}^{1,\gamma}(\mathbb{R}^n) \hookrightarrow \boldsymbol{S}_\alpha(\mathbb{R}^n)$$

Remarks 1.7.1. *We note that the inclusions*

$$L^{1,\gamma}(\Omega) \hookrightarrow \tilde{S}_2(\Omega) \hookrightarrow \tilde{S}_\alpha(\Omega)$$

hold provided that $n - 2 < \gamma < n$, (see [18]).

1.8 The generalized Stummel classes

For $1 \leqslant p < \infty$ and a measurable function $\psi : (0,\infty) \to (0,\infty)$, the generalized Morrey space $L^{p,\psi} = L^{p,\psi}(\mathbb{R})^n$ is the collection of all functions $f \in L^p(\mathbb{R})^n$ for which

$$\|f\|_{L^{p,\psi}} = \sup_{x\in\mathbb{R}^n, r>0} \frac{|(B(x,r)|^{\frac{-1}{p}}}{\psi(r)} \|f\|_{L^p(B(x,r))} < \infty$$

where $|(B(x,r)|$ denotes the Lebesgue measure of $B(x,r)$. Observe that, for $\psi(t) = t^{\frac{\lambda-n}{p}}$ $(0 \leqslant \lambda \leqslant n)$, we have $L^{p,\psi} = L^{p,\lambda}$.

we assume that $\psi : (0,\infty) \to (0,\infty)$ is a measurable function, we consider the following conditions on ψ:

$$\int_0^1 \frac{\psi(t)}{t} dt < \infty; \tag{1.18}$$

$$\frac{1}{A_1} \leqslant \frac{\psi(s)}{\psi(r)} \leqslant A_2 \quad for \quad 1 \leqslant \frac{S}{r} \leqslant 2; \tag{1.19}$$

$$\frac{\psi(r)}{r^n} \leqslant A_2 \frac{\psi(s)}{s^n} \quad for \quad s \leqslant r, \tag{1.20}$$

Where $A_i > 0, i = 1,2$, are independent of $r,s > 0$. The condition (1.19) is known as thedoubling condition on ψ. In some cases, we can weaken the doubling condition by the right doubling condition:

$$\frac{\psi(s)}{\psi(r)} \leqslant A_3 \quad for \quad 1 \leqslant \frac{S}{r} \leqslant 2, \tag{1.21}$$

whereA_3 is independent of $r,s > 0$.

Definition 1.8.1. *For $1 \leqslant p < \infty$, we define the generalized Stummel p-class $S_{p,\Psi} = S_{p,\Psi}(\mathbb{R}^n)$ by*

$$S_{p,\Psi} = \{f \in L^p_{loc}(\mathbb{R}^n) : \rho_{p,\Psi} f(r) \to 0 \qquad for \qquad r \to 0\}$$

where

$$\rho_{p,\Psi} f(r) = \sup_{x\in\mathbb{R}^n} \left(\int_{|x-y|<r} \frac{|f(y)|^p \Psi|x-y|}{|x-y|^n} dy \right)^{\frac{1}{p}}, \qquad r > 0.$$

We call $\rho_{p,\Psi} f(r)$ the Stummel p-modulus of f . Observe that the Stummel p-modulusis nondecreasing on $(0,\infty)$. For $p = 1$, we have $S_{1,\Psi} = S_\Psi$ the generalized Stummel class introduced in [6]. For $\Psi(t) = t^\alpha (0 < \alpha < n)$, we write Cinstead of $S_{p,\Psi}$ and $\rho_{p,\alpha}$ instead of $\rho_{p,\alpha}$. Observe that $S_{1,\alpha} = S_\alpha$ the Stummel class introduced in [4, 13].

The following two theorems confirm that $\rho_{p,\alpha} f$ is continuous (hencemeasurable) and satisfies the doubling condition.

Theorem 1.8.1. *If $f \in S_{p,\Psi}$, then $\rho_{p,\alpha} f$ is continuous on* $(0,\infty)$.

Proof: Let r_k be a sequence in $(0,\infty)$ with $r_k \to r \in (0,\infty)$ and $x \in \mathbb{R}^n$,Choose r_* such that $r, r_k \leqslant r_*$ for every $k \in \mathbb{N}$.Next, for every $k \in \mathbb{N}$, define

$$g_k(y) = \frac{|f(y)|^p \Psi |x-y|}{|x-y|^n} \chi_{B(x,r_k)}(y)dy$$

and

$$g_k(y) = \frac{|f(y)|^p \Psi |x-y|}{|x-y|^n} \chi_{B(x,r)}(y)dy,$$

for$y \in B(x,r_*)$. We see that g_k is a sequence of nonnegativemeasurable functions on $B(x,r_*)$, and $g_k \to g$ almost everywhere on $B(x,r_*)$. By the Dominated Convergence Theorem we obtain

$$\int_{|y-x|<r_*} g_k(y)dy \to \int_{|y-x|<r_*} g(y)dy.$$

Therefore

$$\left(\int_{|y-x|<r_*} \frac{|f(y)|^p \Psi |x-y|}{|x-y|^n} dy\right)^{\frac{1}{p}} \to \left(\int_{|y-x|<r} \frac{|f(y)|^p \Psi |x-y|}{|x-y|^n} dy\right)^{\frac{1}{p}}. \tag{1.22}$$

Let e be any positive real number. By (1.22), there exists $k_0 \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ with $k \geqslant k_0$we have

$$\left(\int_{|y-x|<r} \frac{|f(y)|^p \Psi |x-y|}{|x-y|^n} dy\right)^{\frac{1}{p}} - \epsilon < \left(\int_{|y-x|<r_k} \frac{|f(y)|^p \Psi |x-y|}{|x-y|^n} dy\right)^{\frac{1}{p}}$$
$$< \left(\int_{|y-x|<r} \frac{|f(y)|^p \Psi |x-y|}{|x-y|^n} dy\right)^{\frac{1}{p}} + \epsilon.$$

Since $x \in \mathbb{R}^n$ is arbitrary, we conclude that

$$C - \epsilon \leqslant \rho_{p,\Psi} f(r_k) \leqslant \rho_{p,\Psi} f(r) + \epsilon.$$

Thus, we have proved that $\rho_{p,\Psi} f(r_k) \to \rho_{p,\Psi} f(r)$ for any sequence r_k in $(0,\infty)$ with $r_k \to r \in (0,\infty)$. This means that hp,Y f is continuous on $(0,\infty)$.

Theorem 1.8.2. *Let ψ satisfy the condition (1.20). If $f \in S_{p,\psi}$, then $\rho_{p,\Psi} f$ satisfies the doubling condition.*

Proof: Let $x \in \mathbb{R}_n$ and $r > 0$. Choose$m = m(n) \in \mathbb{N}$ and $x_1, ..., x_m \in B(x,r)$ such that

$$B(x,r) \subseteq \bigcup_{i=1}^{m} B(x_i, r/2).$$

Note that

$$\left(\int_{|y-x|<r} \frac{|f(y)|^p \Psi |x-y|}{|x-y|^n} dy\right)^{\frac{1}{p}} \leqslant \sum_{i=1}^{m} \left(\int_{|y-x_i|<r/2} \frac{|f(y)|^p \Psi |x-y|}{|x-y|^n} dy\right)^{\frac{1}{p}}$$
$$= \sum_{i=1}^{m} I_i \tag{1.23}$$

For $i = 1, ..., m$, we have

$$I_i = \left(\int_{|y-x_i|<r/2} \frac{|f(y)|^p \Psi |x-y|}{|x-y|^n} dy\right)^{\frac{1}{p}}$$

$$\leqslant \left(\int_{|y-x|>|y-x_i|,|y-x_i|<r/2} \frac{|f(y)|^p \Psi|x-y|}{|x-y|^n} dy\right)^{\frac{1}{p}} + \left(\int_{|y-x|\leqslant|y-x_i|<r/2} \frac{|f(y)|^p \Psi|x-y|}{|x-y|^n} dy\right)^{\frac{1}{p}}$$

$$= A_i + B_i \tag{1.24}$$

By the condition (1.20) on ψ, we obtain

$$\begin{aligned} A_i &= \left(\int_{|y-x|>|y-x_i|,|y-x_i|<r/2} \frac{|f(y)|^p \Psi|x-y|}{|x-y|^n} dy\right)^{\frac{1}{p}} \\ &\leqslant c(p) \left(\int_{|y-x_i|<r/2} \frac{|f(y)|^p \Psi|x-y|}{|x-y|^n} dy\right)^{\frac{1}{p}} \\ &\leqslant c(p) \left(\int_{|y-x|>|y-x_i|,|y-x_i|<r/2} \frac{|f(y)|^p \Psi|x-y|}{|x-y|^n} dy\right)^{\frac{1}{p}} \\ &\leqslant c(p) \rho_{p,\Psi} f(r/2). \end{aligned}$$

From (1.23)and(1.24), we get

$$\begin{aligned} \left(\int_{|y-x_i|<r} \frac{|f(y)|^p \Psi|x-y|}{|x-y|^n} dy\right)^{\frac{1}{p}} &\leqslant m(n)\big(c(p)+1\big)\rho_{p,\Psi} f(r/2). \\ &= c\big(n,p\big)\rho_{p,\Psi} f(r/2). \end{aligned} \tag{1.25}$$

Since the inequality (1.25) holds for all $x \in \mathbb{R}^n$, we obtain

$$\rho_{p,\Psi} f(r) \leqslant c\big(n,p\big)\rho_{p,\Psi} f(r/2).$$

According to the fact that $\rho_{p,\Psi} f(r)$ is nondecreasing, we conclude that $\rho_{p,\Psi} f(r)$ satisfies the doubling condition.

1.9 Some basic Formulas

Theorem 1.9.1. *(**Holder's inequality**).*
Assume that $f \in L^P$ and $g \in L^{p'}$ with $1 \leqslant p < \infty$. then $fg \in L^1(\Omega)$ and

$$\int_\Omega |fg| \leqslant \|f\|_p \|g\|_{p'}$$

Proof. We prove this important inequality in several steps:

1. First recall that for $a, b \geq 0$ and $p > 1$,

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. This follows from the concavity of the logarithm.

2. Without loss of generality, assume $\|f\|_p = \|g\|_{p'} = 1$, since otherwise we can consider $\tilde{f} = f/\|f\|_p$ and $\tilde{g} = g/\|g\|_{p'}$.

3. For each $x \in \Omega$, apply Young's inequality to $|f(x)|$ and $|g(x)|$:

$$|f(x)g(x)| \le \frac{|f(x)|^p}{p} + \frac{|g(x)|^{p'}}{p'}$$

4. Integrate both sides over Ω:

$$\int_\Omega |fg|\,dx \le \frac{1}{p}\int_\Omega |f|^p\,dx + \frac{1}{p'}\int_\Omega |g|^{p'}\,dx = \frac{1}{p} + \frac{1}{p'} = 1$$

which proves the normalized case.

5. For arbitrary $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$, the result follows by multiplying both sides by $\|f\|_p\|g\|_{p'}$.

□

Remark 1.9.1. *Key observations about Hölder's inequality:*

- *The inequality is sharp - equality holds when* $|f|^p$ *and* $|g|^{p'}$ *are proportional*
- *When* $p = p' = 2$*, this reduces to the Cauchy-Schwarz inequality*
- *The conjugate exponent* p' *satisfies* $\frac{1}{p} + \frac{1}{p'} = 1$
- *The inequality remains valid when* $p = 1$ *and* $p' = \infty$ *with the appropriate interpretation*

Theorem 1.9.2. ***(Poincaré's inequality)***
If $v \in W_0^{1,2}(\Omega)$*, then there exists a positive constant* $C = C(l)$ *such that*

$$\int_\Omega |v|^2 \leqslant C\int_\Omega |\nabla v|^2.$$

Proof. We provide a detailed proof with the following steps:

1. First consider $v \in C_c^\infty(\Omega)$, the space of smooth functions with compact support in Ω, which is dense in $W_0^{1,2}(\Omega)$.

2. For any $x = (x_1, \ldots, x_n) \in \Omega \subset \mathbb{R}^n$, since v has compact support, we can write:

$$v(x) = \int_{-\infty}^{x_1} \frac{\partial v}{\partial x_1}(t, x_2, \ldots, x_n)\,dt$$

3.

$$\begin{aligned}
|v(x)|^2 &= \left|\int_{-\infty}^{x_1} \frac{\partial v}{\partial x_1}(t, x_2, \ldots, x_n)\,dt\right|^2 \\
&\le \left(\int_{-\infty}^{x_1} 1^2\,dt\right)\left(\int_{-\infty}^{x_1} \left|\frac{\partial v}{\partial x_1}(t, x_2, \ldots, x_n)\right|^2 dt\right) \\
&\le \operatorname{diam}(\Omega)\int_{-\infty}^{\infty} \left|\frac{\partial v}{\partial x_1}(t, x_2, \ldots, x_n)\right|^2 dt
\end{aligned}$$

4.

$$\int_\Omega |v(x)|^2 dx \le \operatorname{diam}(\Omega) \int_\Omega \int_{-\infty}^{\infty} \left| \frac{\partial v}{\partial x_1}(t, x_2, \ldots, x_n) \right|^2 dt dx$$
$$\le \operatorname{diam}(\Omega)^2 \int_\Omega \left| \frac{\partial v}{\partial x_1}(x) \right|^2 dx$$
$$\le \operatorname{diam}(\Omega)^2 \int_\Omega |\nabla v(x)|^2 dx$$

5. For any $v \in W_0^{1,2}(\Omega)$, take a sequence $\{v_n\} \subset C_c^\infty(\Omega)$ converging to v in $W^{1,2}(\Omega)$. Applying the inequality to v_n and taking the limit $n \to \infty$ gives the result.

The constant C depends on the diameter of Ω, with $C = \operatorname{diam}(\Omega)^2$ for convex domains. For more general domains, the constant depends on the Poincaré constant of the domain. □

Remark 1.9.2. *The Poincaré inequality has several important features:*

- *The inequality shows that the L^2 norm of a function is controlled by the L^2 norm of its gradient for functions with zero boundary conditions*
- *The proof demonstrates how the constant depends on the domain size*
- *The inequality fails for functions that don't vanish on the boundary or for unbounded domains*
- *This is a special case of more general Poincaré inequalities that hold for $W_0^{1,p}(\Omega)$ spaces*

Theorem 1.9.3. *(**Sub Representation Formula**)[20]*
If $v \in W_0^{1,2}(\Omega)$, then there exists a positive constant $C = C(n)$ such that

$$|v(x)| \leqslant C \int_\Omega \frac{|\nabla v(y)|}{|x-y|^{n-1}} dy$$

for a.e $x \in \Omega$.

Theorem 1.9.4. *[19]*
Assume that $n-2 < \gamma < n$. If $v \in L^{1,\gamma}(\Omega)$ then there exists a constant $C = C(n, \gamma, l) > 0$ such that

$$\int_\Omega \frac{|v(x)|}{|z-x|^{n-1}} dx \le C \|v\|_{L^{1,\gamma}(\Omega)}$$

for every $z \in \Omega$.

Theorem 1.9.5. *(**Tonelli's Theorem**) Assume that $E \subset \mathbb{R}^n$ and $E \subset \mathbb{R}^n$ are measurable sets and $f : E \times F \to \mathbb{R}$ is a nonnegative measurable function. Then, we have the followings:*

- *For a.e. $x \in E$, $f(x, .)$ is measurable in F.*

- *For a.e.* $y \in F, f(., y)$ *is measurable in E.*
- *The function*
$$g(x) = \int_F f(x, y)dy, \quad (resp. \quad h(y) = \int_E f(x, y)dx)$$
is measurable in E (resp. F).
- *We have the following equality of the integrals(in* $\overline{\mathbb{R}}$*)*
$$\int_{E\times F} f(x, y)dxdy = \int_E \left(\int_F f(x, y)dy\right)dx = \int_F \left(\int_E f(x, y)dx\right)dy$$

1.10 The Lemmas of Lax-Milgram and Stampacchia

Definition 1.10.1. *Let H be a Hilbert space with norm* $\|\cdot\|$ *and* $A : H \times H \to \mathbb{R}$ *be a bilinear mapping. The mapping A is called continuous if there exists a constant* $C_1 > 0$ *such that*

$$|A(v, w)| \leqslant C_1\|v\|\|w\|,$$

for all $v, w \in H$*,*
and called coercive if there exists a constant $C_2 > 0$ *such that*

$$|A(v, v)| \geqslant C_2\|v\|^2,$$

for all $v \in H$*.*

Definition 1.10.2. *Assume that H is a Hilbert space with norm* $\|\cdot\|$ *and* H^{-1} *is the set of all liner functional onH.*
The mapping $A : H \times H \to \mathbb{R}$ *is called continuous and linear in the second variable if* $v, \varphi, \omega \in H$ *and for every sequence* $\{\varphi_n\}_n$ *such that*

$$\lim_{n\to\infty} \|\varphi_n - \varphi\| = 0, \textit{then} \lim_{n\to\infty} |A(v, \varphi_n) - A(v, \varphi)| = 0$$

and

$$A(v, \varphi + \omega) = A(v, \varphi) + A(v, \omega).$$

Meanwhile, $F \in H^{-1}$ *is called bounded linear functional if there exists a constant* $C > 0$ *such that*

$$|F(v)| \leq C\|v\|$$

for all $v \in H$

Lemma 1.10.1. *(**Lax-Milgram's Lemma**), [2]*
Let $A : H \times H \to \mathbb{R}$ *be a continuous and coercive bilinear mapping. Then, for every bounded linear functional* $F : H \times H \to \mathbb{R}$*, there exists a unique element* $v \in H$ *such that*

$$A(v, w) = F(w)$$

for every $v \in H$

Lemma 1.10.2. *(**Stampacchia's Lemma**), ([1],[14])*

Let H be a Hilbert space and $A : H \times H \to \mathbb{R}$ be a continuous and linear in the second variable, and there exists two constants $K_1 > 0$ and $K_2 > 0$ such that :

(1) $|A(v_1,\varphi) - A(v_2,\varphi)| \leq K_1 \| v_1 - v_2 \| \| \varphi \|$, $\forall v_1, v_2, \varphi \in H$,

(2) $A(v_1, v_1 - v_2) - A(v_2, v_1 - v_2) \geqslant K_2 \| v_1 - v_2 \|^2$, $\forall v_1, v_2 \in H$, then, for every bounded linear functional $f \in H^{-1}$, there exists a unique $v \in H$ such that

$$F(w) = A(v, w)$$

for every $w \in H$.

Chapter 2

Elliptic equation involving Morrey spaces

In this chapter, we study the existence and uniqueness of this Dirichlet problem by directly using the Lax-Milgram Lemma and the weighted estimation in Morrey spaces.

2.1 Statement of the results

We consider the following Dirichlet problem

$$\begin{cases} Lv = f, \\ v \in W_0^{1,2}(\Omega). \end{cases} \tag{2.1}$$

Where Ω is a bounded open subset of $\mathbb{R}^n$, $n \geqslant 3$, and L is a divergent elliptic operator defined by :

$$Lv = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial v}{\partial x_i}\right) \tag{2.2}$$

with

$$a_{i,j} \in L^{\infty}(\Omega), i, j = 1, ..., n$$

and there exists $\nu > 0$ such that

$$\nu|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j, \tag{2.3}$$

for every $\xi = (\xi_1, ..., \xi_n)$ and for almost every $x \in \Omega$, the data f is a function in Morrey spaces $L^{1,\gamma}(\Omega)$, and γ satisfies a certain condition $n-2 < \gamma < n$.

Definition 2.1.1. *We say $v \in W_0^{1,2}(\Omega)$ is a weak solution of the Dirichlet problem if :*

$$\int_{\Omega} \sum a_{i,j}(x)\frac{\partial v(x)\partial\varphi(x)}{\partial x_i \partial x_j} = \int_{\Omega} f(x)\varphi(x)dx \tag{2.4}$$

for all $\varphi \in W^{1,2}(\Omega)$.

Remarks 2.1.1. *In this chapter, we will directly study the existence and uniqueness of the weak solution to the Dirichlet problem 2.1, assuming $f \in L^{1,\gamma}(\Omega)$ for $n-2 < \gamma < n$. The Lax-Millgram lemma, a functional analysis technique, is combined with weighted embeddings in Morrey and Sobolev spaces in our method.*

2.2 Method of Research

Now, we associate the operator L with the mapping $A: W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$ defined by the formula

$$A(v,\varphi) = \int_\Omega \sum a_{i,j}(x) \frac{\partial v(x) \partial \varphi(x)}{\partial x_i \partial x_j} dx \tag{2.5}$$

For $n-2 < \gamma < n$ and $f \in L^{1,\gamma}(\Omega)$, we define $F_f : W_0^{1,2}(\Omega) \to \mathbb{R}$ by the formula

$$F_f(\varphi) = \int_\Omega f(x)\varphi(x). \tag{2.6}$$

By the linearity of the weak derivative and integration, it is easy to show that A defined by (2.5), is a bilinear mapping. We notice that, according to (2.4), (2.5), and (2.6), $v \in W_0^{1,2}(\Omega)$ is a weak solution of problem (2.1) if

$$A(v,\varphi) = F_f(\varphi) \tag{2.7}$$

for every $\varphi \in W_0^{1,2}(\Omega)$. Here, we apply the Lax-Millgram lemma as a method of functional analysis tool. Before we proceed, we establish the continuity and coercivity of the bilinear mapping A, as defined by (2.5).

Lemma 2.1:[19]

The mapping A that is specified in equation (2.5) is coercive and continuous.

Proof of Lemma 2.1: We take $v \in W_0^{1,2}(\Omega)$.

First, we establish the property of coercivity. Using (2.3) and Poincarś inequality after that, we obtain

$$A(v,v) = \int_\Omega \sum_{i,j=1} a_{i,j}(x) \frac{\partial v(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j} dx$$

$$\geqslant \nu \int_\Omega |\nabla v|^2$$

$$\geqslant \frac{\nu}{2} \int_\Omega |\nabla v|^2 + C \int_\Omega |v|^2$$

$$\geqslant C \|v\|^2_{W_0^{1,2}(\Omega)}$$

where $C = C(\nu, l)$ is a positive constant.

The continuity property is now established. Assume that $v, \varphi \in W_0^{1,2}(\Omega)$,

Observe that,

$$M = \sum_{i,j=1}^{n} \|a_{i,j}\|_{L^\infty(\Omega)}$$

Based on $a_{i,j} \in L^\infty(\Omega), i, j = 1, ..., n$. Applying Holder's inequality, we obtain

$$|A(v,\varphi)| \leq \sum_{i,j=1}^{n} \|a_{i,j}\|_{L^\infty(\Omega)} \left|\frac{\partial v(x)}{\partial x_i}\right| \left|\frac{\partial \varphi(x)}{\partial x_j}\right| dx$$

$$\leqslant M \int_\Omega |\nabla v(x)||\nabla \varphi(x)| dx$$

$$\leqslant M \left(\int_\Omega |\nabla v(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla \varphi(x)|^2 dx\right)^{\frac{1}{2}}$$

$$\leqslant M \|v\|_{W^{1,2}(\Omega)} \|\varphi\|_{W^{1,2}(\Omega)}.$$

The proof is now complete.

To demonstrate the bounded linear functional nature of the function F_f described by (2.6), we require the following theorem which states about a weighted estimation in **Morrey spaces** where the weight in **Sobolev spaces**.

Theorem 2.2.1. *[19] Suppose that $n-2 \leqslant \gamma \leqslant n$.*
If $f \in L^{1,\gamma}(\Omega)$, then there exists a constant $C = C(n,\gamma,l) > 0$ such that,

$$\int_\Omega |fv| \leq C\|f\|_{L^{1,\gamma}(\Omega)} \|v\|_{W^{1,2}(\Omega)}.$$

for every $v \in W_0^{1,2}(\Omega)$.

Proof of theorem 2.2.1:

We take $v \in W_0^{1,2}(\Omega))$. Using Holder's inequality and the sub representation formula of v, we get

$$\int_\Omega |f(x)v(x)| dx \leqslant C(n) \int_\Omega |f(x)| \left(\int_\Omega \frac{|\nabla v(y)|}{|x-y|^{n-1}} dy\right) dx$$

$$= C(n) \int_\Omega |\nabla v(y)| \left(\int_\Omega \frac{|f(x)|}{|x-y|^{n-1}} dx\right) dy$$

$$\leqslant C(n) \|\nabla v\|_{L^2(\Omega)} \left(\int_\Omega \left(\int_\Omega \frac{|f(x)|}{|x-y|^{n-1}} dx\right)^2 dy\right)^{\frac{1}{2}} \tag{2.8}$$

Observe that

$$\int_\Omega |f(z)| dz \leqslant C(n,\gamma,l) \|f\|_{L^{1,\gamma}(\Omega)} \tag{2.9}$$

We deduce that

$$\int_\Omega\left(\int_\Omega \frac{|f(x)|}{|x-y|^{n-1}}dx\right)^2 dy = \int_\Omega\left(\int_\Omega \frac{|f(z)|}{|z-y|^{n-1}}dz\right)\left(\int_\Omega \frac{|f(x)|}{|x-y|^{n-1}}dx\right)dy$$

$$= \int_\Omega\int_\Omega |f(z)||f(x)|\left(\int_\Omega \frac{1}{|z-y|^{n-1}|x-y|^{n-1}}dy\right)dxdz$$

$$\leqslant C(n)\int_\Omega\int_\Omega |f(z)||f(x)|\frac{1}{|z-x|^{n-2}}dxdz$$

$$= C(n)\int_\Omega |f(z)|\left(\int_\Omega \frac{|f(x)|}{|z-x|^{n-2}}dx\right)dz$$

$$\leqslant C(n,\gamma,l)\|f\|_{L^{1,\gamma}(\Omega)}\int_\Omega |f(z)|dz$$

$$\leqslant C(n,\gamma,l)\|f\|^2_{L^{1,\gamma}(\Omega)} \quad . \tag{2.10}$$

According to Theorem (1.9.4) and (2.9). When (2.8) and (2.10) are combined, we get

$$\int_\Omega |f(x)v(x)|dx \leqslant C(n)\|\nabla v\|_{L^2(\Omega)}\left(\int_\Omega\left(\int_\Omega \frac{|f(x)|}{|x-y|^{n-1}}dx\right)^2)dy\right)^{\frac{1}{2}}$$

$$\leqslant C(n,\gamma,l)\|\nabla v\|_{L^2(\Omega)}\|f\|_{L^{1,\gamma}(\Omega)}$$

$$\leqslant C(n,\gamma,l)\|v\|_{W^{1,2}(\Omega)}\|f\|_{L^{1,\gamma}(\Omega)}.$$

Finally, the proof of this theorem is finished.

We get the following consequence from Theorem (2.2.1).

Corollary 2.1[19]

Suppose that $n-2<\gamma<n$. The mapping $F_f : W_0^{1,2}(\Omega)\to\mathbb{R}^n$ defined by (2.6) is a continuous linear functional.

Proof of Corollary 2.1:

Since integration is linear, F_f is a linear functional on $W_0^{1,2}(\Omega)$.

For every $v\in W_0^{1,2}(\Omega)$, Thanks to theorem (2.2.1), we deduce that

$$\left|F_f(v)\right| \leqslant \int_\Omega |fv| \leqslant C\|f\|_{L^{1,\gamma}(\Omega)}\|v\|_{W^{1,2}(\Omega)} \leqslant C(n,\gamma,l,\|f\|_{L^{1,\gamma}(\Omega)})\|v\|_{W_0^{1,2}(\Omega)}$$

This means F_f is also continuous.

Lemma (2.2), Corollary (2.2), and the Lax-Milgram lemma are combined to provide the following assertion on the existence and uniqueness of the Dirichlet problem's weak solution.

Theorem 2.2.2. *[19] Suppose that $n-2<\gamma<n$. and $f \in L^{1,\gamma}(\Omega)$ in Dirichlet problem (2.1). Then there exists a unique element $v \in W_0^{1,2}(\Omega)$ which is the weak solution of the Dirichlet problem (2.1).*

Proof. We establish the existence and uniqueness through the following steps:

1. Functional framework:
    - Consider the Gelfand triple $W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow W^{-1,2}(\Omega)$
    - The Morrey space condition $f \in L^{1,\gamma}(\Omega)$ with $n-2<\gamma<n$ ensures sufficient regularity
2. Variational formulation:
    - The weak formulation seeks $v \in W_0^{1,2}(\Omega)$ such that

      $$a(v,\phi) = \langle f,\phi\rangle \quad \forall \phi \in W_0^{1,2}(\Omega)$$

      where $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$ is the bilinear form
3. Existence proof:
    - The bilinear form $a(\cdot,\cdot)$ is coercive on $W_0^{1,2}(\Omega)$ by Poincaré's inequality
    - The functional $\phi \mapsto \langle f,\phi\rangle$ is continuous on $W_0^{1,2}(\Omega)$ since

      $$|\langle f,\phi\rangle| \le \|f\|_{L^{1,\gamma}(\Omega)} \|\phi\|_{W_0^{1,2}(\Omega)}$$

    - Apply the Lax-Milgram theorem to obtain existence
4. Uniqueness proof:
    - Suppose v_1, v_2 are two solutions
    - Then $a(v_1 - v_2, \phi) = 0$ for all $\phi \in W_0^{1,2}(\Omega)$
    - Taking $\phi = v_1 - v_2$ and using coercivity gives $\|v_1 - v_2\|_{W_0^{1,2}(\Omega)} = 0$

□

Remark 2.2.1. *Key aspects of this result:*

- *The condition $\gamma > n-2$ ensures the Morrey space is not too singular*
- *The upper bound $\gamma < n$ guarantees f has sufficient decay*
- *The proof combines functional analysis with Morrey space theory*
- *The result extends classical L^2 theory to more singular right-hand sides*

Chapter 3

Elliptic equation involving Stummel classes

This chapter discusses the existence and uniqueness of the problem 1 solution by taking $f \in \tilde{S}_\alpha(\Omega)$ and $g \neq 0$ utilizing Stampacchia's lemma and a weighted estimation for functions in the Stummel class with a weight in $W^{1,2}(\Omega)$. Our approach handles a continuous linear functional and a continuous linear mapping in the second variable.

3.1 Statement of the problem via Stampacchia's Lemma

Now, we take a map $A : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$ by the formula

$$A(v,\varphi) = \int_\Omega M(x)\nabla v.\nabla\varphi + \int_\Omega g(v)\varphi. \tag{3.1}$$

For $f \in \tilde{S}_\alpha(\Omega)$, we also take, $F_f : W_0^{1,2}(\Omega) \to \mathbb{R}$ by the formula

$$F_f(\varphi) = \int_\Omega f\varphi \tag{3.2}$$

We remark that, due to $M(x)$ is bounded, g is Lipschitz, and Holder inequality, for every $v,\varphi \in W_0^{1,2}(\Omega)$, we obtain

$$|A(v,\varphi)| \leq \int_\Omega |M(x)\nabla v.\nabla\varphi| + \int_\Omega |g(v)\varphi|$$

$$\leq C(k,n)\int_\Omega |\nabla v||\nabla\varphi| + \int_\Omega (C|v| + |g(0)|)|\varphi| < \infty$$

With $C = C(k_0)$, so (3.1) is well defined; therefore, one can verify that F_f defined by (3.2) is a linear functional on $W_0^{1,2}(\Omega)$ that comes from the linearity of integration.

3.2 Main Results

This section is devoted to studying the linearity in the second variable and continuity of the map A which defined by (3.1).

Lemma 3.2.1. *[20] In the second variable, the map A described by* (3.1) *is continuous and linear.*

Proof of Lemma 3.2.1:

We assume that $v, \varphi \in W_0^{1,2}(\Omega)$ and $\{\varphi_n\}_n$ is any sequence in $W_0^{1,2}(\Omega)$ such that

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_{W_0^{1,2}(\Omega)} = 0$$

We note that,

$$\left|\int_\Omega \big(M(x)\nabla v.\nabla\varphi_n - M(x)\nabla v.\nabla\varphi\big)\right| \leqslant \int_\Omega \big|M(x)\nabla v.\nabla(\varphi_n - \varphi)\big| \tag{3.3}$$

The right-hand side of (3.3) is estimated in the following way by applying Holder's inequality and the boundedness of $M(x)$.

$$\int_\Omega \big|M(x)\nabla v.\nabla(\varphi_n-\varphi)\big| \leqslant C(k,n)\int_\Omega |\nabla v|\big|\nabla(\varphi_n-\varphi)\big| \leqslant C(k,n)\|\nabla v\|_{L^2(\Omega)}\|\nabla(\varphi_n-\varphi)\|_{L^2(\Omega)} \tag{3.4}$$

When we combine (3.3) and (3.4), we obtain

$$\left|\int_\Omega (M(x)\nabla v.\nabla\varphi_n - M(x)\nabla v.\nabla\varphi)\right| \leqslant C(k,n)\|\nabla v\|_{L^2(\Omega)}\|\varphi_n-\varphi\|_{L^2(\Omega)} \tag{3.5}$$

Additionally, the fact that $|g(v)| \leqslant C(k_0)|v| + |g(0)| \in L^2(\Omega)$ and g is Lipschitz gives us

$$\begin{aligned}\left|\int_\Omega \big(g(v)\varphi_n - g(v)\varphi\big)\right| &\leqslant \int_\Omega |g(v)||\varphi_n-\varphi| \\ &\leqslant \int_\Omega (C(k_0)|v| + |g(0)|)|\varphi_n-\varphi| \\ &\leqslant \|C(k_0)|v| + |g(0)|\|_{L^2(\Omega)}\|\nabla(\varphi_n-\varphi)\|_{L^2(\Omega)}\end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we conclude that

$$\big|A(v,\varphi_n) - A(v,\varphi)\big| \leqslant \left|\int_\Omega (M(x)\nabla v.\nabla\varphi_n - M(x)\nabla v.\nabla\varphi\right| + \left|\int_\Omega (g(v)\varphi_n - g(v)\varphi)\right|$$

$$\leqslant C(k,n)\|\nabla v\|_{L^2(\Omega)}\|\nabla(\varphi_n-\varphi)\|_{L^2(\Omega)} + \|C(K_0)|v| + |g(0)|\|_{L^2(\Omega)}\|\nabla(\varphi_n-\varphi)\|_{L^2(\Omega)}$$

$$\leqslant C\|\varphi_n-\varphi\|_{L^2(\Omega)}$$

With $C = C(k,n)\|\nabla v\|_{L^2(\Omega)} + \|C(K_0)|v| + |g(0)|\|_{L^2(\Omega)}$. This inequality provides us with

$$\lim_{n\to\infty} |A(v,\varphi_n) - A(v,\varphi)| = 0$$

This indicates that A is continuous in the second variable. Moreover, based on the integration and the linearity property of ∇, the map A is linear in the second variable.

Lemma 3.2.2. *[20] Assume that A is defined by (3.1). Then there exist two constants $K_1 > 0$ and $K_2 > 0$ such that*

- $\left|A(v_1,\varphi) - A(v_2,\varphi)\right| \le K_1 \|v_1 - v_2\|_{W_0^{1,2}(\Omega)} \|\varphi\|_{W_0^{1,2}(\Omega)}, \forall v_1, v_2, \varphi \in W_0^{1,2}(\Omega)$
- $|A(v_1, v_1 - v_2) - A(v_2, u_1 - v_2)| \geqslant k_2 \|v_1 - v_2\|^2_{W_0^{1,2}(\Omega)}, \forall v_1, v_2 \in W_0^{1,2}(\Omega)$

Proof of Lemma 3.2.2:

We take $v_1, v_2, \varphi \in W_0^{1,2}(\Omega)$. Using Holder's inequality and g is Lipschitz and $M(x)$ bounded, produces

$$\left|A(v_1,\varphi) - A(v_2,\varphi)\right| = \left|\int_\Omega M(x)\nabla(v_1 - v_2)\nabla\varphi + \int_\Omega (g(v_1) - g(v_2))\varphi\right|$$

$$\leqslant C\int_\Omega |\nabla(v_1 - v_2)|.|\nabla\varphi| + C\int_\Omega |v_1 - v_2||\varphi| \tag{3.7}$$

$$\leqslant C\|\nabla(v_1 - v_2)\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2\Omega} + c\|v_1 - v_2\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}$$

$$\leqslant c\|v_1 - v_2\|_{W_0^{1,2}(\Omega)}\|\varphi\|_{W_0^{1,2}(\Omega)}$$

where $C = C(k, n, k_0) = K_1 > 0$. Additionally, in view of $M(x)$ is elliptic and g is non-decreasing, we deduce that

$$A(v_1, v_1 - v_2) - A(v_2, v_1 - v_2) =:$$

$$= \int_\Omega M(x)\nabla v_1\nabla(v_1 - v_2) + \int_\Omega g(v_1)(v_1 - v_2) - \int_\Omega M(x)\nabla v_2\nabla(v_1 - v_2) - \int_\Omega g(v_2)(v_1 - v_2) \tag{3.8}$$

$$= \int_\Omega M(x)\nabla v_1\nabla(v_1 - v_2)\nabla(v_1 - v_2) + \int_\Omega g(v_1 - v_2)\nabla(v_1 - v_2)$$

$$\geqslant \nu\int_\Omega |\nabla(v_1 - v_2)|^2$$

$$= \nu\|v_1 - v_2\|^2_{W_0^{1,2}}$$

Assuming $K_2 = \nu$. The proof of the lemma is given by (3.7) and (3.8).

To continue we need the following lemma that help us to study the next theorem.

Lemma 3.2.3. *[20] If $f \in \tilde{S}_1\Omega)$, then there exists a constant $C = C(n) > 0$ such that,*

$$\int_\Omega |f\varphi| \leq C\rho_\alpha f(l) \int_\Omega |\nabla\varphi|$$

for every $\varphi \in W_0^{1,2}(\Omega)$.

We shall now demonstrate a weighted approximation of a function in Stummel classes, with the weight in compactly supported Sobolev spaces $W_0^{1,2}(\Omega)$.

Theorem 3.2.1. *[20] Suppose that $\varphi \in W_0^{1,2}(\Omega)$.*
If $f \in \tilde{S}_1(\Omega)$,then there exists a constant $C = C(n,l) > 0$ such that,

$$\int_\Omega |f\varphi| \leq C\rho_\alpha f(l)\|\varphi\|_{W^{1,2}(\Omega)}$$

If $f \in \tilde{S}_2(\Omega)$ then there exists a constant $C = C(n) > 0$ such that,

$$\int_\Omega |f\varphi| \leqslant C\left(\rho_\alpha f(l))\|f\|_{L^1\Omega}\right)^{\frac{1}{2}}\|\varphi\|_{W^{1,2}(\Omega)}$$

Proof of Theorem 3.2.1 : Suppose that $f \in \tilde{S}_\alpha\Omega)$.
We begin with the case $\underline{\alpha = 1}$.
According to Lemma 3.2.3 and Holder s inequality, there exists a constant $C = C(n,l) > 0$ such that.

$$\int_\Omega |f\varphi| \leq C\rho_f(l)\int_\Omega |\nabla\varphi| \leqslant C\rho_f(l)\|\varphi\|_{W^{1,2}(\Omega)}$$

Next, we take $\underline{\alpha = 2}$. Using the Sub representation of φ, Tonelli's theorem, and Holder s inequality, we obtain that

$$\int_\Omega |f\varphi| = \int_\Omega \left|f(x)\varphi(x)dx\right|$$

$$\leqslant C(n)\int_\Omega \left|f(x)\right|\left(\int_\Omega \frac{|\nabla\varphi(y)|}{|x-y|^{n-1}}dy\right)dx$$

$$\leqslant C(n)\int_\Omega \left|\nabla\varphi(y)\right|\left(\int_\Omega \frac{|f(x)|}{|x-y|^{n-1}}dx\right)dy$$

$$\leqslant C(n)\|\nabla\varphi\|_{L^2\Omega}\left(\int_\Omega\left(\int_\Omega \frac{|f(x)|}{|x-y|^{n-1}}dx\right)^2 dy\right)^{\frac{1}{2}}$$

We see that

$$\int_\Omega \left(\int_\Omega \frac{|f(x)|}{|x-y|^{n-1}}dx\right)^2 dy = \int_\Omega \left(\int_\Omega \frac{|f(z)|}{|z-y|^{n-1}}dz\right)\left(\int_\Omega \frac{|f(x)|}{|x-y|^{n-1}}dx\right)dy$$

$$= \int_\Omega\int_\Omega |f(x)||f(z)|\left(\int \frac{1}{|z-y|^{n-1}|x-y|^{n-1}}dy\right)dxdz$$

$$\leqslant C(n)\int_\Omega\int_\Omega |f(x)||f(z)|\frac{1}{|x-z|^{n-2}}dxdz$$

$$= C(n)\int_\Omega |f(z)|\left(\int_\Omega \frac{|f(x)|}{|x-z|^{n-2}}dx\right)dz$$

$$\leqslant C(n)\rho_f(l)\|f\|_{L^1(\Omega)}.$$

When we sum up the previous two disparities, we get

$$\int_\Omega |f\varphi| \leqslant C(n)\|\nabla\varphi\|_{L^2(\Omega)}\left(\int_\Omega\left(\int_\Omega \frac{|f(x)|}{|x-y|^{n-1}}dx\right)^2 dy\right)^{\frac{1}{2}}$$

$$\leqslant C(n)\|\nabla\varphi\|_{L^2(\Omega)}\left(C(n)\rho_f(l)\|f\|_{L^1(\Omega)}\right)^{\frac{1}{2}}.$$

$$\leqslant C(n)\left(\rho_f(l)\|f\|_{L^1(\Omega)}\right)^{\frac{1}{2}}\|\varphi\|_{W^{1,2}(\Omega)}.$$

The proof is finished.

The following corollary, which indicates that the linear functional defined by (3.2) is bounded (i.e. Continuous), is obtained as a result of Theorem (3.2.1).

Corollary 3.2.1. *[20] The linear functional $F_f : W_0^{1,2}(\Omega) \to \mathbb{R}$ stated in (3.2) is bounded.*

Proof of Corollary 3.2.1:

We take $\varphi \in W_0^{1,2}(\Omega)$. Thanks to Theorem (3.2.1), we find

$$|F_f(\varphi)| = \int_\Omega |f\varphi| \leqslant C\|\varphi\|_{W_0^{1,2}(\Omega)}$$

where $C = C(n, l, \rho_f(l))$ or $C = C(n, l, \rho_f(l), \|f\|_{W_0^{1,2}(\Omega)})$

Regarding to Lemma (3.2.2) and Corollary (3.2.1), we can now apply Stampacchia's lemma to find the existence and uniqueness of solution of the problem (1) which stated as follows:

Theorem 3.2.2. *[20] There exists $u \in W_0^{1,2}(\Omega)$ which is unique solution of the problem 1 in the following sense:*

$$A(v, \varphi) = \int_\Omega M(x)\nabla v.\nabla\varphi + \int_\Omega g(v)\varphi = \int_\Omega f\varphi = F_f(\varphi),$$

for all $\varphi \in W_0^{1,2}(\Omega)$.

Conclusion and Further Prospects

In this memory, we have examined the existence and uniqueness of a weak solution to a semilinear monotone elliptic equation when the lower order term equal zero. This weak solution may be identified by presuming that the data is a member of a certain Morrey space. This can be shown by using the weighted embedding in Morrey spaces, where the weight is in Soblev space, together with the Lax-Milgram lemma, a functional analytic technique. Furthermore, our problem has a unique solution. A weighted embedding of a function in Stummel classes, where the weight is in compactly supported Sobolev spaces, and Stampacchia's lemma serve as examples of this case. This thesis memory contributes to the understanding of the semilinear monotone elliptic problem, providing important theoretical insights and establishing conditions for the existence and regularity of solutions. The results obtained in this study lay the groundwork for further investigations in this field and can serve as a valuable resource for researchers and practitioners working in related areas.

Researcher exploration of the following problems is warranted by the findings of this study. We recommend researching the problem below.

$$\begin{cases} -div(A(x)\nabla u) + F(u) = h, \\ u \in W_0^{1,2}(\Omega). \end{cases}$$

Where **A(x)** is nxn symmetric matrix, elliptic, bounded, $\mathbf{F} : \mathbb{R} \to \mathbb{R}$ is non decreasing and Lipschitz, and the right hand side **h** belongs to the generalized Stummel classes, that is:

$$S_{p,\varphi} = S_{p,\varphi}(\mathbb{R}^n) = \left\{ f \in L^1_{loc}(\mathbb{R}^n) : \rho_{p,\varphi} f(r) \to 0 \text{ for } r \to 0 \right\}$$

Where

$$\rho_{p,\varphi} f(r) = \sup_{x \in \mathbb{R}^n} \left(\int_{|x-y|<r} \frac{|f(y)|^p \varphi(|x-y|)}{|x-y|^n} \right)^{\frac{1}{p}} dy, \forall r > 0$$

Regarding these spaces, the reader can look at [18].

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