REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE
MINISTERE DE L'ENSEIGNEMENT SUPERIEUR ET DE LA RECHERCHE SCIENTIFIQUE
Université Mohamed Boudiaf de M'sila
Faculté de Mathématiques et de l'Informatique
Département de Mathématiques

N^o d'ordre :.........................

Thèse

Présentée pour l'obtention du diplôme
de Doctorat 3ème cycle LMD

Spécialité
Mathématiques

Option
EDPs et Optimisation

Par

KAOUTHER BOUCHAMA

Thème

Méthodes numériques pour la résolution des équations différentielles fractionnaires

Soutenue le 22/02/2022 devant le jury composé de :

BENHAMIDOUCHE Noureddine	Professeur	Université de M'sila	Président
MERZOUGUI Abdelkrim	Professeu	Université de M'sila	Rapporteur
ARIOUA Yacine	M.C.A	Université de M'sila	Co-Rapporteur
GAGUI Bachir	M.C.A	Université de M'sila	Examinateur
MERIKHI Bachir	Professeur	Université de Sétif 1	Examinateur
BENDJEDOU Ahmed	Professeur	Université de Sétif 1	Examinateur

Année universitaire : 2021/2022

Thanks

I cannot begin and finish my work without thanking the greatest and the most powerful " **Allah**."

At the end of this modest work, I would like to extend my sincere thanks to the people who have helped me and who have contributed to the preparation of this brief.

I would like to express my heartfelt thanks to my supervisor Professor MERZOUGUI ABDELKRIM and Professor ARIOUA YACINE for their patience, their competence, and their precious help during the realization of this work and their remarks and their suggestions which allowed me to finalize this thesis.

I express all my gratitude, my most respectful thanks to all the teachers of the Department of Mathematics who contributed to my training, without forgetting the pedagogical managers of the Department who have always been attentive and very available throughout along the two years of my information.

Furthermore, my sincere thanks also go to the Jury members who will be kind enough to read and judge this modest work. My special thanks to Professor BENHAMIDOUCHE NOUREDDINE whokindly who has agreed to chair this jury.

I wish to express my sincere thanks to Professors GUAGUI BACHIR, MERIKHI. BACHIR and BENDJEDOU AHMED who kindly accepted to be jury members of my thesis.

I also extend my sincere thanks and gratitude to my dear father and mother for their help to me, my husband, for their contribution, their support and patience. Finally I extend my sincere thanks to all my loved ones, my friends and to all those who have contributed from near and far to the accomplishment of this modest work by help, sympathy and moral support.

Dédicaces

Je dédie cette thèse :

Ma Mère

Mon Père

Mon Mari

Mes enfants :

Ahmed

Ilyes

Table of contents

List of tables

List of figures

Introduction

Fractional calculus (FC) is a mathematical analysis branch investigating the properties of derivatives and integrals of non-integer orders (called fractional derivatives and integrals, briefly differ-integrals). It dates back to the end of the 17th century and continuous to the present days. The number of publications and scientific meetings in the recent period which are dedicated to it reflects the importance of the problems that this notion has raised, both theoretical and applied. One can say that it has become a discipline in its own. Specialists agree that the beginning of its history can be traced back to the end of 1695, when Leibniz, in a letter to L'Hospital, wondered of a possible theory of the non-whole derivation of a of the non-integer derivation of a function. In his reply, l'Hospital wondered about the meaning one could give to the derivative of order 1/2. Indeed, 1/2 is at equal distance from order 0 which is supposed to designate continuity and order 1 which is supposed to designate classical derivability. Leibniz's answer contained the following sentence : "...this would lead to a paradox from which, one day, one could draw useful consequences". It was not until the 1990s that the first "useful consequences" appeared. The first works are due to Liouville between 1832 and 1837. Independently, Riemann proposed an approach that turned out to be essentially that of Liouville. Since then, this theory has been called the Riemann-Liouville theory. Later, other theories have been developed and become more diversified. The reader interested in the subject of FC is referred to the books by Samko et al. [42] which provides a comprehensive study of the subject, Miller and Ross [34] 1993, Podlubny 1999 [39], Kilbas et al. 2006 [29], Diethelm 2010 [16]. The first applications of FC started to appear in the 1990's. In particular in control engineering and fractal geometry. Engineers found in the fractional derivative a convenient tool to propose models that describe physical phenomena in a more accurate way.

The classical fractional calculus is based on several definitions for the operators of integration and differentiation of arbitrary order [30]. Among the various definitions of fractional differentiation, the Riemann–Liouville fractional integral plays a major role in FC introduced in $1837-1847$. The Caputo fractional derivative has also been defined via a modified Riemann–Liouville fractional integral, the Riesz and Grunwald–Letnikov are just a few [29]- [42]. In 1891, J.Hadamard [21] introduced a new fractional derivative called the Hadamard operator. Then Butzer et al. investigated the properties of the Hadamard fractional integral and the derivative in [14]- [28]- [42]. Pooseh *et al.* derived expansion formulas for the Hadamard operators in terms of integer order derivatives. On the other hand, it is natural to look for and study generalized fractional operators, for which the known ones are particular cases. Recently, Jarad *et al* in [24] defined the generalization of the Hadamard fractional derivatives and presented some properties of such derivatives. This new generalization is now know as the Caputo–Hadamard fractional derivative. In 2011, Udita Katugampola [25] introduced the Katugampola operator, which generalizes the familiar Riemann–Liouville and the Hadamard fractional derivatives to a single form. He studied their basic properties such as expansion formulas, variational calculus applications, control theoretical applications, convexity and integral inequalities. A recent generalization introduced by Ricardo *et al* in [1]. The authors define the generalization of derivatives and presented the properties of such derivatives. This new generalization is now know as the Caputo–Katugampola fractional derivatives, which generalizes the concept of Caputo and Caputo–Hadamard fractional derivatives.

Fractional differential equations and partial fractional equations have been used in the study of models of many phenomena in various fields of science and engineering, for instance bioengineering [32], [48], chaos theory [37], viscoelasticity [33], control system engineering [35], fractional signal processing techniques [44] and many others areas (see e.g. [41]). Recent investigations have shown that sometimes physical systems can be modeled more accurately using fractional derivative formulations [22]. Fractional diffusion equations represent extensions of basic equations of mathematical physics, many researchers focused on the their solutions. Some analytical methods have been proposed to provide better solutions to some of fractional differential equations or partial fractional equations, such as the Laplace transform method, the Fourier transform method [29], but these solutions are expressed in terms of special functions which are even inaccessible for some of fractional nonlinear equations and they are usually difficult for numerical evaluation. Recently, considerable research has been devoted to the study of numerical methods lead to a rapid increase development of numerical methods for fractional differential and partial equations. Many methods have been presented ([9], [13], [18], [43], [47], [49]) to overcome above mentioned problems. Among them yields only an approximate solution which can be derived using perturbation method, adomian decomposition method, generalized differential transform method or finite difference method associated with Current, who created the mathematical foundations in the 1940 and apply it to various fields of physics. The terminology "finite element" was introduced by Clough in 1960 and the finite volume method and the finite difference method are an important class of numerical methods for solving fractional differential equations, which became very popular and a large number of schemes has been published very recently. Consequently it becomes important to understand how do they compare in terms of accuracy, convergence and the stability. Several authors have done much work on this topic ([12], [15], [20], [27], [36], [38], [40], [45], [46]).

In this thesis, we use the finite difference method (FDM) to solve numerically some linear fractional differential equations and linear fractional partial differential equations, in particular the time fractional diffusion-wave equation (TFDWE) and the space-time fractional diffusion equation (STFDE), the fractional derivatives are described in Caputo–Hadamard and Caputo-Katugampola sense.

This thesis is organized as follows :

The first chapter, entitled **Basic Fractional Calculus**, will be devoted to elementary fractional calculus. We present some preliminaries concerning the basic tools of fractional calculus, such as the gamma and the Beta functions which play an important role in the theory of fractional differential equations. Then we, give the definitions and some properties of fractional integrals and fractional derivatives of various types namely the Riemann–Liouville, Caputo and Hadamard, as well as the generalized Katugampola fractional integrals and derivatives. Then, we present some preliminaries of finite difference method.

In **the second chapter**, finite difference methods to approximate for the fractional derivatives are discussed in detail. It can be derived from the discretization of the domain and from the partial or mean derivative approximation of Taylor expansions. The discretization of the domain of the problem approaches the continuous domain by a finite number of sub domains, in which the numerical values of unknowns is a determined quantity. Then we investigate the approximation of Riemann–Liouville derivatives, Caputo derivatives, Caputo–Hadamard derivatives and Caputo–Katugampola derivatives. The natural generalization of the above methods for the Caputo–Riesz–Katugampola derivatives are also introduced. These discretized schemes are useful for the following chapters

In **the third chapter**, the finite difference methods for fractional differential and partial differential equations are presented involving Caputo–Hadamard derivative. Firstly, we started by solving a linear fractional differential equation with dependence on the Caputo-Hadamard derivative given by

$$ {}^{CH}\mathcal{D}_{a^+}^{\alpha} u\left(t\right) + c(t)u(t) = f(t),\ 0 < a \leq t \leq b < \infty, \tag{1}$$

with an initial condition

$$ u(a) = u_0, \tag{2}$$

where ${}^{CH}\mathcal{D}^{\alpha}$ denotes the Caputo-Hadamard fractional derivative operator of order α between zero and one [4]- [24].

Secondly, we discuss the numerical solution of time fractional diffusion-wave equation (TFDWE) given by

$$ {}^{CH}\partial_t^{\alpha} u\left(x,t\right) = \frac{\partial^2 u\left(x,t\right)}{\partial x^2} + f(x,t),\ \ (x,t) \in \left]x_0, L\right[\times \left]t_0, T\right[\ \ (t_0 > 0), \tag{3}$$

with Dirichlet-Neumann initial conditions

$$u\left(x,t_0\right)=\varphi_1(x),\ \frac{\partial u}{\partial t}\left(x,t_0\right)=\varphi_2(x),\ \ x\in\left[x_0,L\right], \tag{4}$$

and Dirichlet-Neumann boundary conditions

$$u\left(x_0,t\right)=\psi_1(t),\ \frac{\partial u}{\partial x}\left(L,t\right)=\psi_2(t),\ \ t\in\left[t_0,T\right]. \tag{5}$$

where ${}^{CH}\partial_t^{\alpha}$ denotes the Caputo–Hadamard time–fractional derivative operator of order $1<\alpha\leq 2$, $\varphi_1,\ \varphi_2,\ \psi_1,\ \psi_2$ are continuous functions and $f(x,t)$ is the source term. The stability, convergence and error estimate of this method is also carefully studied. Also, we present three examples to explain the application of our main results.

Next, the finite difference methods for fractional partial differential equations involving Caputo–Katugampola derivative are presented in **chapter 4**. We will discuss firstly the approximate solution of time fractional diffusion-wave equation using the Caputo–Katugampola time-fractional derivative in one spatial dimension, which is given by the following equation

$${}^{C}\partial_t^{\alpha,\rho}u\left(x,t\right)=\frac{\partial^2 u\left(x,t\right)}{\partial x^2}+f(x,t),\ 0\leq x\leq L,\ t_0\leq t\leq T, \tag{6}$$

with Dirichlet-Neumann initial and boundary conditions

$$u\left(x,t_0\right)=\varphi_1(x),\ \frac{\partial u}{\partial t}\left(x,t_0\right)=\varphi_2(x),\ \ 0\leq x\leq L \tag{7}$$

$$u\left(0,t\right)=\psi_1(t),\ \frac{\partial u}{\partial x}\left(L,t\right)=\psi_2(t),\ \ t_0\leq t\leq T \tag{8}$$

where $T>0$, $L>0$, $\rho>1$, $1<\alpha\leq 2$ and ${}^{C}\partial_t^{\alpha,\rho}u\left(x,t\right)$ is the fractional order Caputo–Katugampola sense derivative. $\varphi_1,\ \varphi_2,\ \psi_1,\ \psi_2$ are continuous functions and $f(x,t)$ is the source term. Next, we give the approximate solution for fractional diffusion equation in time-space in one spatial dimension, where the first order derivative in time is the Caputo-Katugampola derivative of order $\alpha,\ 0<\alpha\leq 1$ and the second-order space derivative is a Riesz–Caputo–Katugampola derivative of order $\beta,\ 1\leq\beta\leq 2$, which is given by the following equation

$${}^{C}\partial_t^{\alpha,\rho}u\left(x,t\right)=\frac{\partial^{\beta,\rho}u\left(x,t\right)}{\partial\left|x\right|^{\beta}}+f(x,t),\ \ (x,t)\in\left]x_0,L\right[\times\left]t_0,T\right[, \tag{9}$$

with conditions

$$\begin{aligned}&u\left(x,t_0\right)=u_0(x),\ x\in\left[x_0,L\right],\\&\frac{\partial u}{\partial x}\left(x_0,t\right)=\psi(t),u\left(x_0,t\right)=\phi(t),\ u\left(L,t\right)=\varphi(t),\ \ t\in\left[t_0,T\right].\end{aligned}$$

where ${}^{C}\partial_t^{\alpha,\rho}$, $\dfrac{\partial^{\beta,\rho}}{\partial\left|x\right|^{\beta}}$ denotes the Caputo–Katugampola fractional derivative and Riesz–Caputo–Katugampola fractional derivative of order α and β respectively, with $\rho > 1$, $t_0, x_0 > 0$ and $f(x,t)$ is the source term and $u_0(x)$, ψ, ϕ, φ are continuous functions. The stability, convergence and error estimates are studied. Many numerical examples are also displayed, which support the theoretical analysis. Most parts of results presented in this thesis have already been published or submitted for publication in peer-reviewed international journals. Results included in chapter 3, have been accepted in [Palestine Journal of Mathematics] and those included in chapter 4, have been published in "Journal of Numerical Algebra, Control and Optimization".

List of abbreviations and symbols

Symboles	Meaning	Pages
${}^{C}\mathcal{D}^{\alpha,\rho}_{a^+}u$	Left-sided Caputo–Katugampola fractional derivative	25
${}^{\rho}\mathcal{D}^{\alpha}_{a+}u$	Left-sided Katugampola fractional derivative	23
${}^{\rho}\mathcal{D}^{\alpha}_{b-}u$	Right-sided Katugampola fractional derivative	23
${}^{\rho}\mathcal{I}^{\alpha}_{a+}u$	Left-sided Katugampola fractional integral	22
${}^{\rho}\mathcal{I}^{\alpha}_{b-}u$	Right-sided Katugampola fractional integral	22
${}^{C}\mathcal{D}^{\alpha,\rho}_{b^-}u$	Right-sided Caputo–Katugampola fractional derivative	25
${}^{RC}_{a}\mathcal{D}^{\alpha,\rho}_{b}u$	Riesz–Caputo–Katugampola fractional derivative	27
${}^{RC}_{a}\mathcal{D}^{\alpha}_{b}u$	Riesz–Caputo fractional derivative	27
${}^{R}_{a}\mathcal{D}^{\alpha}_{b}u$	Riesz fractional derivative	26
$\delta = tD(D = d/dt)$	The δ-derivative	16
$\dfrac{\partial^{\beta,\rho}}{\partial \lvert x\rvert^{\beta}}$	Riesz–Caputo–Katugampola fractional derivative of order β	79
$\Gamma(z)$	Euler gamma function	9
$\mathcal{D}^{\alpha}_{a+}u$	Left-sided Riemann–Liouville fractional derivative of order α	13
$\mathcal{D}^{\alpha}_{b-}u$	Right-sided Riemann–Liouville fractional derivative of order α	13
$\mathcal{I}^{\alpha}_{a+}u$	Left-sided Riemann-Liouville fractional integral of order α	11
$\mathcal{I}^{\alpha}_{b-}u$	Right-sided Riemann-Liouville fractional integral of order α	12
$\mathcal{M}\left[u\left(t\right)\right]$	The Mellin transform of a function u	9
$\Re\left(\alpha\right)$	Real part of complex α	9
$\operatorname{ess\,sup}_{t\in\Omega}\lvert u(t)\rvert$	The essential maximum of the function $\lvert u(t)\rvert$	8
${}^{CH}\mathcal{D}^{\alpha}_{a+}u$	Left-sided Caputo–Hadamard fractional derivative of order α	19
${}^{CH}\mathcal{D}^{\alpha}_{b-}u$	Right-sided Caputo–Hadamard fractional derivative of order α	19

CHAPTER 1

PRELIMINARIES

This chapter is preliminary in nature and in its content, in which we give definitions and certain properties of some fractional integrals and fractional derivatives of different types. We also give the definition and properties of some special functions that will be used in this thisis. For additional details on the content of this chapter, reader can refer to the works of [1], Y. Arioua and al. [4], Albadarneha and al. [9], Diethelm [16], Gambo and al. [17], Fortin [19], Jarad and al [24], Katugampola [25] and [26], Kilbas and al. [29], LI [31], Podlubny [39], Samko and al. [42].

In general, the results presented in this chapter will be taken into account for the appropriate functions. precise details can be found in the references cited above
First, let $\Omega = [a,b](-\infty \leq a < b \leq \infty)$ be a finite or infinite interval of the real axis $\mathbb{R}$. We denote by $L^p(a,b)(1 \leq p < \infty)$ the set of Lebesgue measurable functions $u(t)$ on Ω for which $\int_\Omega |u| < \infty$, we set

$$\|u\|_{L_p(\Omega)} = \left\{ \int_\Omega |u(t)|^p dt \right\}^{1/p}$$

and

$$\|u\|_{L_\infty(\Omega)} = \operatorname*{ess\,sup}_{t\in\Omega} |u(t)|$$

where esssup $|u(t)|$ is an essential maximum of the function $|u(t)|$.

1.1 Basic fractional calculus

In this section, we present the most commonly used fractional integration operators and definitions of fractional derivatives and give the most important properties of these concepts. We begin by giving some definitions on special functions and functional spaces.

1.1.1 Special functions of fractional calculus

In the following, we present the Euler Gamma function, the Beta functions ; these functions play a very important role in the theory of fractional calculus. For more details refer to ([29])

Euler Gama function

One of the basic function of the fractional calculus is Euler gamma function $\Gamma(z)$, the simplest interpretation of the gamma function is the genrealization of the notion of factorial, for real numbers. which generalizes the factorial $n!$ and allows n to take also non-integer and even complex values.

Definition 1.1 (Euler gamma function [29])**.** The Euler gamma is defined on $\mathbb{C}$, by the improper integral

$$\Gamma\left(\alpha\right)=\int_{0}^{\infty}s^{\alpha-1}e^{-s}ds,\left(\Re\left(\alpha\right)>0\right) \tag{1.1}$$

which converges on the complex half-planes $\Re\left(\alpha\right)>0$. The Euler gamma function is the Mellin transform of the exponential function $\Gamma\left(\alpha\right)=\mathcal{M}\left[e^{-s}\right]\left(\alpha\right)$, where the Mellin transform is defined for a function $u(t)$ of a real variable $t\in\mathbb{R}_{+}$ by : $\mathcal{M}\left[u\left(t\right)\right]\left(s\right)=\int_{0}^{\infty}t^{s-1}u\left(t\right)dt,\Re\left(s\right)>0,$

Properties of Euler gamma function : We give some properties of the Euler gamma function :

1. $\Gamma\left(1\right)=1,\Gamma\left(0^{+}\right)=+\infty.$
2. The Euler gamma function $\Gamma\left(\alpha\right)$ is a monotonous and strictly decreasing for $0<\alpha\leq 1$.
3. The Euler gamma function is a monotonous and strictly increasing function for $\alpha\geq 2$, so it is convex for $\alpha\in\left]0,+\infty\right[$, with point of minimum equal to $\Gamma\left(\dfrac{3}{2}\right)=\dfrac{\sqrt{\pi}}{2}$.
4. By integrating by part, we obtain

$$\Gamma\left(\alpha+1\right)=\alpha\Gamma\left(\alpha\right),\Re\left(\alpha\right)>0, \tag{1.2}$$

   this is one of the basic properties of the Euler gamma function and we have

$$\Gamma\left(n+1\right)=n\Gamma\left(n\right)=n!,\forall n\in\mathbb{N},$$

   this is know by the connection between the Euler gamma function and the factorial. This is provided by the formula (*1.2*) and by the factorial that $\Gamma\left(1\right)=1$.
5. Among the properties of the gamma function

$$\frac{1}{\Gamma\left(-m\right)}=0,\ \text{for } m=0,1,2,...$$

   because $\Gamma\left(\alpha\right)$ is infinite for all the negative integer values of α.

Beta function

Also, known as integral Euler of the first type, it shares the form that typically resembles the fractional integral derivative of many functions.

Definition 1.2 (Beta function [29])**.** For a positive values of the two parameters $p, q \in \mathbb{C}$, the Beta function is defined on $\mathbb{C} \times \mathbb{C}$ by the following integral

$$B(p,q) = \int_0^1 s^{p-1} (1-s)^{q-1} \, ds, \left(\Re(p) > 0 \text{ and } \Re(q) > 0\right), \tag{1.3}$$

Properties of Beta function : For all $p, q \in \mathbb{C}$ with $(\Re(p) > 0$ and $\Re(q) > 0)$

1. $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, (Connection between the Euler gamma function and beta function).
2. The Beta function is symmetric, i.e.,

$$B(p,q) = B(q,p).$$

1.1.2 Riemann-Liouville fractional integrals

The Riemann-Liouville's fractional integral plays a major role in fractional calculus [42], we will follow Riemanns approach to propose a first definition of fractional integral.

Fractional integration

The definition of fractional integrals is to start with an iterated n-tuple integral and show that it can be expressed as a single integral involving the n parameter, and then we replace the n integer with the real positive α. let u be a measurable continuous function on $[a,b]$ in $\mathbb{R}$, where $[a,b]$ is a finite closed interval of the real axis $\mathbb{R} = (-\infty, \infty)$, we pose

$$\mathcal{I}_{a^+}^1 u(t) = \int_a^t u(s)\, ds, \tag{1.4}$$

$\mathcal{I}_{a^+}^1$ is the primitive of u, for a primitive second of u and according to the theorem of Fubini, we have

$$\begin{aligned} \mathcal{I}_{a^+}^2 u(t) &= \mathcal{I}_{a^+}^1 \circ \mathcal{I}_{a^+}^1 u(t) = \int_a^t \left(\int_a^s u(\tau)\, d\tau \right) ds = \int_a^t u(\tau) \left(\int_\tau^t ds \right) d\tau \\ &= \int_a^t (t-\tau)\, u(\tau)\, d\tau. \end{aligned}$$

The Riemann–Liouville approach is based on the Cauchy formula (1.4) by repeating n times, we get the following relation

$$\mathcal{I}_{a^+}^{n} u(t) = \int_a^t \int_a^{t_1} \cdots \int_a^{t_{n-1}} u(t_n)\, dt_n dt_{n-1} \ldots dt_1 = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} u(s)\, ds. \tag{1.5}$$

Now it is clear how to get an integral of arbitrary order. We simply generalize the Cauchy formula (1.4), the integer n is substituted by a positive real number α and using the Euler gamma function, we get the following definition

Definition 1.3 ([29]). Let $u \in L^1([a,b], \mathbb{R})$ and $\alpha > 0$. Then, for any $t \in [a,b]$, the integrals

$$\mathcal{I}_{a^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s)\, ds, \quad t \in [a,b], \tag{1.6}$$

is called the left-sided Riemann–Liouville fractional integral of order $\alpha > 0$. The extension to the real axis $\mathbb{R}$ and $\mathbb{R}^+$ known by fractional integrals on the real line are noted $\mathcal{I}_{+}^{\alpha}$ and $\mathcal{I}_{0^+}^{\alpha}$ (respectively) and given by the following

$$\mathcal{I}_{+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t (t-s)^{\alpha-1} u(s)\, ds, \quad t \in \mathbb{R}, \tag{1.7}$$

and

$$\mathcal{I}_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds, \quad t > 0, \tag{1.8}$$

Similarly, if we go back to the starting relationship (1.4) for a function $u : [a,b] \to \mathbb{R}$, we can notice that the integral

$$\mathcal{I}_{b^-}^{1} u(t) = \int_t^b u(s)\, ds,$$

is also a primitive of u, which this time involves the values to the right of u, from (1.5) we could define in the same way the right-sided integral of order n of u by

$$\forall t \in [a,b], \; \mathcal{I}_{b^-}^{n} u(t) = \frac{1}{(n-1)!} \int_t^b (s-t)^{n-1} u(s)\, ds,$$

by replacing the integer n by a positive real number α, we obtain the the right-sided Riemann-Liouville fractional integral defined by :

Definition 1.4 ([29]). The right-sided Riemann–Liouville fractional integral of order $\alpha > 0$ of function $u \in L^1([a,b], \mathbb{R})$ is given by :

$$\mathcal{I}_{b^-}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} u(s)\, ds, \; t \in [a,b], \tag{1.9}$$

The extension on $[a, +\infty)$ and $\mathbb{R}$ is noted $\mathcal{I}_{-}^{\alpha}$

$$\mathcal{I}_{-}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{+\infty} (s-t)^{\alpha-1} u(s)\, ds, \quad t \in \mathbb{R}. \tag{1.10}$$

Properties of fractional integrals

The fractional integral $\mathcal{I}^{\alpha}$ of arbitrary real order $\alpha > 0$ defined by (1.6) and (1.9), has the following important properties (see [29])

1. For any $t \in (a, b)$, $\alpha > 0$ and $\beta > 0$, we have :

$$\begin{aligned} \left(\mathcal{I}_{a^+}^{\alpha} (x-a)^{\beta-1}\right)(t) &= \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (t-a)^{\beta+\alpha-1}, \\ \left(\mathcal{I}_{b^-}^{\alpha} (b-x)^{\beta-1}\right)(t) &= \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (b-t)^{\beta+\alpha-1}. \end{aligned}$$

2. Suppose $u \in L^1([a, b])$, then for any $t \in (a, b)$,the fractional integrals $\mathcal{I}_{a^+}^{\alpha}$ and $\mathcal{I}_{b^-}^{\alpha}$ are well defined.
3. By convention we pose : $\mathcal{I}^0 u(t) = u(t)$.
4. If the function $u = e^{\lambda t}$, $\lambda > 0$, we have

$$\left(\mathcal{I}^{\alpha}\left(e^{\lambda x}\right)\right)(t) = \lambda^{-\alpha} e^{\lambda t}.$$

5. The Riemann-Liouville fractional integral of order $\alpha\,(\alpha > 0)$ of a constant function $u(t) = C$ is given by

$$\mathcal{I}_{a}^{\alpha}(C) = \frac{C}{\Gamma(\alpha+1)} (t-a)^{\alpha}, a \in \mathbb{R}, C \in \mathbb{R}.$$

6. Semigroup property : for $u \in L^1([a, b])$

$$\mathcal{I}_{a+}^{\alpha} \mathcal{I}_{a+}^{\beta} u = \mathcal{I}_{a+}^{\alpha+\beta} u, \quad \mathcal{I}_{b-}^{\alpha} \mathcal{I}_{b-}^{\beta} u = \mathcal{I}_{b-}^{\alpha+\beta} u, \quad \alpha > 0, \quad \beta > 0.$$

7. $\mathcal{I}_{a^+}^{\alpha} u(t) = 0$ implies that $u = 0$ almost everywhere.
8. Fractional integration by parts formula : Let $u \in L^p([a, b])$ and $v \in L^q([a, b])$ either with $\alpha \geq 1$, $p = q = 1$, or with $0 < \alpha < 1$, $\dfrac{1}{p} + \dfrac{1}{q} \leq 1 + \alpha$, $p, q > 1$. Then,

$$\int_{a}^{b} u(s)\, \mathcal{I}_{a^+}^{\alpha} v(s)\, ds = \int_{a}^{b} \mathcal{I}_{a^+}^{\alpha} u(s)\, v(s)\, ds.$$

1.1.3 Riemann-Liouville fractional derivatives

The most frequently used definitions of the fractional calculus involves the Riemann-Liouville fractional derivative. Then to define a fractional derivative there is no formula for the nth derivative analogous to (1.5) so, we need to generalize the derivatives through a fractional integral. For $\alpha > 0$, we denote $\lceil \alpha \rceil$ the integer part of α, $\lceil \alpha \rceil$ is the unique integer satisfying

$$\lceil \alpha \rceil \leq \alpha < \lceil \alpha \rceil + 1.$$

Let $u : [a, b] \to \mathbb{R}$. From the classic relationship $\frac{d}{dt} = \frac{d^2}{dt^2} \circ \mathcal{I}^1_{a^+}$ we can define a fractional derivative of order $0 \leq \alpha < 1$ by

$$\frac{d^\alpha}{dt^\alpha} = \frac{d}{dt} \circ \mathcal{I}^{1-\alpha}_{a^+}.$$

More generally, if $\alpha > 0$ and if $n = \lceil \alpha \rceil + 1$, we can put

$$\frac{d^\alpha}{dt^\alpha} = \left(\frac{d}{dt}\right)^n \circ \mathcal{I}^{n-\alpha}_{a^+}, \tag{1.11}$$

we get exactly the left-sided Riemann–Liouville derivative given by the following definition

Definition 1.5 ([29]). Let $\alpha > 0$ and $n = \lceil \alpha \rceil + 1$. The left-sided Riemann–Liouville fractional derivative of order α of a continuous function $u : [a, b] \to \mathbb{R}$ is given by

$$\mathcal{D}^\alpha_{a^+} u(t) = \left(\frac{d}{dt}\right)^n \circ \mathcal{I}^{n-\alpha}_{a^+} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} u(s)\, ds. \tag{1.12}$$

Moreover, the right integral was associated with $\left(-\frac{d}{dt}\right)$. The previous reasoning therefore leads to the following definition :

Definition 1.6 ([29]). Let $\alpha > 0$ and $n = \lceil \alpha \rceil + 1$.The right-sided Riemann–Liouville fractional derivative of order α of a continuous function $u : [a, b] \to \mathbb{R}$ is given by

$$\mathcal{D}^\alpha_{b^-} u(t) = \left(-\frac{d}{dt}\right)^n \circ \mathcal{I}^{n-\alpha}_{a^+} u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_t^b (s-t)^{n-\alpha-1} u(s)\, ds. \tag{1.13}$$

If $u : \mathbb{R} \to \mathbb{R}$ the previous definitions are directly generalized and are called Liouville derivatives.

Definition 1.7. [[29]] Let $\alpha > 0$ and $n = \lceil \alpha \rceil + 1$.The left-sided Liouville fractional derivative on the real line is given by

$$\mathcal{D}^\alpha_{+} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_{-\infty}^t (t-s)^{n-\alpha-1} u(s)\, ds.$$

Moreover, the right-integral was associated with $\left(-\dfrac{d}{dt}\right)$. The preceding reasoning therefore leads to the following definition :

Definition 1.8 ([29]). Let $\alpha > 0$ and $n = \lceil \alpha \rceil + 1.$ The right-sided Liouville fractional derivative on the real line is given by

$$\mathcal{D}_{-}^{\alpha} u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_t^{+\infty} (s-t)^{n-\alpha-1} u(s)\, ds.$$

Properties of fractional derivatives

The fractional derivative $D_{a^+}^{\alpha}$ has the following properties (see [29])

1. In general the fractional derivative of Riemann–Liouville of a constant function is neither zero nor constant, we have

$$\mathcal{D}^{\alpha}(C) = \frac{C}{\Gamma(1-\alpha)} (t-a)^{-\alpha}$$

2. Let $0 < \alpha < 1$, we have
   (a) For any $u \in L^1([a,b])$, we that $D_{a^+}^{\alpha} \mathcal{I}_{a^+}^{\alpha} u = u.$
   (b) The latter can be generalized. In fact, if the function $I_{a^+}^{1-\alpha} u$ is absolutely continuous on $[a,b]$, then

$$\mathcal{I}_{a^+}^{\alpha} D_{a^+}^{\alpha} u(t) = u(t) - \frac{\mathcal{I}_{a^+}^{1-\alpha} u(a)}{\Gamma(\alpha)} (t-a)^{\alpha-1},\ t \in (a,b),$$

   where $\mathcal{I}_{a^+}^{1-\alpha} u(a) = \lim_{s \to a^+} \left(\mathcal{I}_{a^+}^{1-\alpha} u\right)(s)$, which is in general non zero.
3. For any $0 \leqslant n-1 < \alpha < n$ and $\beta > -1,$ the Riemann–Liouville fractional derivative of the function $u(t)$ such that : $u(t) = (t-a)^{\beta}$ is given by :

$$\mathcal{D}_{a^+}^{\alpha} (t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (t-a)^{\beta-\alpha}. \tag{1.14}$$

1.1.4 Caputo-type fractional derivatives

The definition of Caputo fractional derivative is based on the inversion of the compositions in the right sided of (1.11), it also seems reasonable to define the fractional derivative called the Caputo derivative, wich is given by :

Definition 1.9 ([29]). Let $\alpha > 0$ and $n = \lceil \alpha \rceil + 1$. The left-sided Caputo fractional derivative of order α of a function $u \in C^n([a,b], \mathbb{R}),$ is given by

$$^{C}\mathcal{D}_{a^+}^{\alpha} u(t) = \mathcal{I}_{a^+}^{n-\alpha} \circ \left(\frac{d}{dt}\right)^n u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} \left(\frac{d}{ds}\right)^n u(s)\, ds, \tag{1.15}$$

The right-sided Caputo fractional derivative of order α of a function $u \in C^n([a,b],\mathbb{R})$, is given by

$$^{C}\mathcal{D}_{b-}^{\alpha}u(t) = \mathcal{I}_{b-}^{n-\alpha} \circ \left(-\frac{d}{dt}\right)^{n} u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} \left(\frac{d}{ds}\right)^{n} u(s)\, ds, \tag{1.16}$$

If $u \in C^n(\mathbb{R},\mathbb{R})$, the Caputo fractional derivatives can be also defined in the real line.

Definition 1.10 ([29]). Let $\alpha > 0$ and $n = \lceil \alpha \rceil + 1$. The left-sided Caputo fractional derivative of order α of a function $u \in C^n(\mathbb{R},\mathbb{R})$, is given by

$$^{C}\mathcal{D}_{+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^{t} (t-s)^{n-\alpha-1} \left(\frac{d}{ds}\right)^{n} u(s)\, ds.$$

The right-sided Caputo fractional derivative of order α of a function $u \in C^n(\mathbb{R},\mathbb{R})$, is given by

$$^{C}\mathcal{D}_{-}^{\alpha}u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_{t}^{+\infty} (s-t)^{n-\alpha-1} \left(\frac{d}{ds}\right)^{n} u(s)\, ds.$$

Properties of Caputo fractional derivatives

1. The Caputo fractional derivative of a constant function is zero.
2. Let $\alpha \in \mathbb{R}^+$, $n \in \mathbb{N}$, such that $n = \lceil \alpha \rceil + 1$. If $u \in AC^n([a,b])$, then

$$\lim_{\alpha \to n^-} {}^{C}\mathcal{D}_{a+}^{\alpha}u(t) = u^{(n)}(t),$$
$$\lim_{\alpha \to n^-} {}^{C}\mathcal{D}_{b-}^{\alpha}u(t) = (-1)^n u^{(n)}(t).$$

almost everywhere

3. If $\alpha \notin \mathbb{N}$ and $u(t)$ is a function, for which the Caputo fractional derivatives $^{C}\mathcal{D}_{a+}^{\alpha}u(t)$ and $^{C}\mathcal{D}_{b-}^{\alpha}u(t)$ of order $\alpha > 0$ exist together with the Riemann-Liouville fractional derivatives $\mathcal{D}_{a+}^{\alpha}u(t)$ and $\mathcal{D}_{b-}^{\alpha}u(t)$, then we have

$$^{C}\mathcal{D}_{a+}^{\alpha}u(t) = \mathcal{D}_{a+}^{\alpha}u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha}, \quad n = \lceil \alpha \rceil + 1, \tag{1.17}$$

and

$$^{C}\mathcal{D}_{b-}^{\alpha}u(t) = \mathcal{D}_{b-}^{\alpha}u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(b)}{\Gamma(k-\alpha+1)} (b-t)^{k-\alpha}, \quad n = \lceil \alpha \rceil + 1. \tag{1.18}$$

The Caputo fractional derivatives $^{C}\mathcal{D}_{a+}^{\alpha}u(t)$ and $^{C}\mathcal{D}_{b-}^{\alpha}u(t)$, coincide with the Riemann-Liouville fractional derivatives $\mathcal{D}_{a+}^{\alpha}u(t)$ and $\mathcal{D}_{b-}^{\alpha}u(t)$, in the following cases :

$$^{C}\mathcal{D}_{a+}^{\alpha}u(t) = \mathcal{D}_{a+}^{\alpha}u(t), \qquad \text{if } u(a) = u'(a) = \cdots = u^{(n-1)}(a) = 0.$$

and

$$^{C}\mathcal{D}_{b^-}^{\alpha}u\left(t\right)=\mathcal{D}_{b^-}^{\alpha}u\left(t\right), \qquad \text{if } u\left(b\right)=u'\left(b\right)=\cdots=u^{(n-1)}\left(b\right)=0.$$

4. For α and β, such that : $0\leqslant n-1<\alpha<n$ and $\beta>n-1$, the Caputo fractional derivative of the functions $u\left(t\right)$ and $v\left(t\right)$ where, $u\left(t\right)=\left(t-a\right)^{\beta}$ and $v\left(t\right)=\left(b-t\right)^{\beta}$, are given by :

$$^{C}\mathcal{D}_{a^+}^{\alpha}u\left(t\right)=\frac{\Gamma\left(\beta+1\right)}{\Gamma\left(\beta-\alpha+1\right)}\left(t-a\right)^{\beta-\alpha}.$$
$$^{C}\mathcal{D}_{a^+}^{\alpha}v\left(t\right)=\frac{\Gamma\left(\beta+1\right)}{\Gamma\left(\beta-\alpha+1\right)}\left(b-t\right)^{\beta-\alpha}.$$

5. The Caputo fractional derivative of the function $u\left(t\right)$ such that : $u\left(t\right)=t^{\beta}$, is given by :

$$^{C}\mathcal{D}_{a^+}^{\alpha}u\left(t\right)=\begin{cases}\dfrac{\Gamma\left(\beta+1\right)}{\Gamma\left(\beta-\alpha+1\right)}t^{\beta-\alpha}, & \beta>\lceil\alpha\rceil,\\ 0, & \beta\leqslant\lceil\alpha\rceil.\end{cases}.$$

1.1.5 Hadamard fractional integrals and fractional derivatives

In 1891, J.Hadamard introduced new types of fractional operators. The Hadamard approach to the fractional integral was based on the generalization of the nth integral

$$\int_a^t\frac{d\tau_1}{\tau_1}\int_a^{\tau_1}\frac{d\tau_2}{\tau_2}\cdots\int_a^{\tau_{n-1}}\frac{d\tau_n}{\tau_n}u\left(\tau_n\right)=\frac{1}{(n-1)!}\int_a^t\left(\log\frac{t}{\tau}\right)^{n-1}u\left(\tau\right)\frac{d\tau}{\tau}, \tag{1.19}$$

by replacing the integer n by a positive real number α, we obtain the following definition

Definition 1.11 (Hadamard fractional integral)**.** (see [29]) Let $(a,b)(0\leqq a<b\leqq\infty)$ be a finite or infinite interval of the half-axis $\mathbb{R}^+$, $u:(a,b)\to\mathbb{R}$ and let $\Re(\alpha)>0$. We consider the left-sided and right-sided integrals of fractional order $\alpha\in\mathbb{C}(\Re(\alpha)>0)$ defined by

$$\left(^{H}\mathcal{I}_{a^+}^{\alpha}u\right)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{u(s)ds}{s},\ (a<t<b), \tag{1.20}$$

and

$$\left(^{H}\mathcal{I}_{b^-}^{\alpha}u\right)(t)=\frac{1}{\Gamma(\alpha)}\int_t^b\left(\log\frac{s}{t}\right)^{\alpha-1}\frac{u(s)ds}{s},\ (a<t<b), \tag{1.21}$$

Definition 1.12 (Hadamard fractional derivative)**.** (see [29]).
The Let $\delta=tD(D=d/dt)$ be the δ-derivative, The left- and right-sided Hadamard fractional

derivative of order $\alpha \in \mathbb{C}(\Re(\alpha) \geqq 0)$ on (a,b) are defined by

$$\begin{aligned} {}^{H}\mathcal{D}_{a^+}^{\alpha} u(t) &= \delta^n \left({}^{H}\mathcal{I}_{a^+}^{n-\alpha} u\right)(t) \\ &= \left(t\frac{d}{dt}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log\frac{t}{s}\right)^{n-\alpha-1} \frac{u(s)\,ds}{s}, \quad (a<t<b), \end{aligned} \tag{1.22}$$

and

$$\begin{aligned} {}^{H}\mathcal{D}_{b^-}^{\alpha} u(t) &= (-\delta)^n \left({}^{H}\mathcal{I}_{b^-}^{n-\alpha} u\right)(t) \\ &= \left(-t\frac{d}{dt}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_t^b \left(\log\frac{s}{t}\right)^{n-\alpha-1} \frac{u(s)\,ds}{s}, \quad (a<t<b). \end{aligned} \tag{1.23}$$

where, $n = \lceil \Re(\alpha) \rceil + 1$ with $\lceil \Re(\alpha) \rceil$ denotes the integer part of the number α.

Properties of Hadamard fractional operators

Here, we give some interesting prthoperties of the modified derivatives are necessary in order to formulate some important outcomes, we mainly restrict our attention to the left-sided operators (1.20), (1.22) (of course, the right-sided Hadamard operators (1.21), (1.23) possesses similar properties.)

Property 1.1. (see [29]). Let $\Re(\alpha) > 0, \Re(\beta) > 0, (0 < a < b < \infty)$ and $1 \leqq p \leqq \infty$.

1. Index property : for $u \in L^p(a,b)$,

$${}^{H}\mathcal{I}_{a^+}^{\alpha H}\mathcal{I}_{a^+}^{\beta} u = {}^{H}\mathcal{I}_{a^+}^{\alpha+\beta} u \quad \text{and} \quad {}^{H}\mathcal{I}_{b^-}^{\alpha H}\mathcal{I}_{b^-}^{\beta} u = {}^{H}\mathcal{I}_{b^-}^{\alpha+\beta} u.$$

2. Composition property : for $\Re(\alpha) > \Re(\beta) > 0$ and $u \in L^p(a,b)$

$${}^{H}\mathcal{D}_{a^+}^{\beta H}\mathcal{I}_{a^+}^{\alpha} u = {}^{H}\mathcal{I}_{a^+}^{\alpha-\beta} u \quad \text{and} \quad {}^{H}\mathcal{D}_{b^-}^{\beta H}\mathcal{I}_{b^-}^{\alpha} u = {}^{H}\mathcal{I}_{b^-}^{\alpha-\beta} u\,,$$

in particular, if $\beta = m \in N$, then

$${}^{H}\mathcal{D}_{a^+}^{m H}\mathcal{I}_{a^+}^{\alpha} u = {}^{H}\mathcal{I}_{a^+}^{\alpha-m} u \quad \text{and} \quad {}^{H}\mathcal{D}_{b^-}^{m H}\mathcal{I}_{b^-}^{\alpha} u = {}^{H}\mathcal{I}_{b^-}^{\alpha-m} u\,,$$

Theorem 1.1. *(see [29]). Let $\Re(\alpha) > 0$, $n = [\Re(\alpha)]$ and $0 < a < b < \infty$. Also let $\left({}^{H}\mathcal{I}_{a^+}^{n-\alpha} u\right)(t)$ be the Hadamard fractional integral of the form (1.20). If $u(t) \in L(a,b)$ and $\left({}^{H}\mathcal{I}_{a^+}^{n-\alpha} u\right)(t) \in AC_{\delta}^{n}[a,b]$, then*

$$\left({}^{H}\mathcal{I}_{a^+}^{\alpha H}\mathcal{D}_{a^+}^{\alpha} u\right)(t) = u(t) - \sum_{k=1}^{n} \frac{\left(\delta^{n-k}\left({}^{H}\mathcal{I}_{a^+}^{n-\alpha} u\right)\right)(a)}{\Gamma(\alpha-k+1)} \left(\log\frac{t}{a}\right)^{\alpha-k},$$

in particular, if $\alpha = n \in N$, and $u(t) \in AC_{\delta}^{n}[a,b]$ then

$$\left({}^{H}\mathcal{I}_{a^+}^{n}{}^{H}\mathcal{D}_{a^+}^{n}u\right)(t) = u(t) - \sum_{k=0}^{n-1}\frac{\left(\delta^k u\right)(a)}{k!}\left(\log\frac{t}{a}\right)^k.$$

Lemma 1.1. *(see [29]). Let $\alpha \in \mathbb{C}$ and be such $\Re(\alpha) > 0$, if $0 < a < b < \infty$, then for $u \in L^p(a,b)$*

$$\left({}^{H}\mathcal{I}_{a^+}^{\alpha}{}^{H}\mathcal{D}_{a^+}^{\alpha}u\right)(t) = u(t),$$

and

$$\left({}^{H}\mathcal{I}_{b^-}^{\alpha}{}^{H}\mathcal{D}_{b^-}^{\alpha}u\right)(t) = u(t).$$

Proposition 1.1. *(see [29]). If $\Re(\alpha) > 0$ and $\Re(\beta) > 0$, $0 < a < b < \infty$, then*

1. $\left({}^{H}\mathcal{I}_{a^+}^{\alpha}\left(\log\frac{x}{a}\right)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\log\frac{t}{a}\right)^{\beta+\alpha-1},$
2. $\left({}^{H}\mathcal{D}_{a^+}^{\alpha}\left(\log\frac{x}{a}\right)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{t}{a}\right)^{\beta-\alpha-1},$
3. $\left({}^{H}\mathcal{I}_{b^-}^{\alpha}\left(\log\frac{b}{x}\right)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\log\frac{b}{t}\right)^{\beta+\alpha-1},$
4. $\left({}^{H}\mathcal{D}_{b^-}^{\alpha}\left(\log\frac{b}{x}\right)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{b}{t}\right)^{\beta-\alpha-1}.$

In particular, if $\beta = 1$ and $\Re(\alpha) \geqq 0$, then the Hadamard fractional derivatives of a constant, in general, are not equal to zero :

$$\left({}^{H}\mathcal{D}_{a^+}^{\alpha}1\right)(t) = \frac{1}{\Gamma(1-\alpha)}\left(\log\frac{t}{a}\right)^{-\alpha} \text{ and } \left({}^{H}\mathcal{D}_{b^-}^{\alpha}1\right)(t) = \frac{1}{\Gamma(1-\alpha)}\left(\log\frac{b}{t}\right)^{-\alpha},$$

when $0 < \Re(\alpha) < 1$. On the other hand, for $j = [\Re(\alpha)] + 1$,

$$\left({}^{H}\mathcal{D}_{a^+}^{\alpha}\left(\log\frac{x}{a}\right)^{\alpha-j}\right)(t) = 0 \text{ and } \left({}^{H}\mathcal{D}_{b^-}^{\alpha}\left(\log\frac{b}{x}\right)^{\alpha-j}\right)(t) = 0.$$

Proof. For $\Re(\alpha) > 0$, and $\Re(\beta) > 0$, we have

1.
$$\left({}^{H}\mathcal{I}_{a^+}^{\alpha}\left(\log\frac{x}{a}\right)^{\beta-1}\right)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\log\frac{s}{a}\right)^{\beta-1}\frac{ds}{s}.$$

letting $\Pi = \dfrac{\log\left(\frac{s}{a}\right)}{\log\left(\frac{t}{a}\right)}$, we obtain

$$\left({}^{H}\mathcal{I}_{a^+}^{\alpha}\left(\log\frac{x}{a}\right)^{\beta-1}\right)(t) = \frac{1}{\Gamma(\alpha)}\left(\log\frac{t}{a}\right)^{\beta+\alpha-1}\int_0^1 (1-\Pi)^{\alpha-1}\,\Pi^{\beta-1}d\Pi,$$

by the definition of the beta function, we obtain

$$\left({}^{H}\mathcal{I}_{a^+}^{\alpha}\left(\log\frac{x}{a}\right)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\log\frac{t}{a}\right)^{\beta+\alpha-1}.$$

2. From proposition 1.1(1) and 1.22, we obtain

$$\begin{aligned}\left({}^{H}\mathcal{D}_{a^+}^{\alpha}\left(\log\frac{x}{a}\right)^{\beta-1}\right)(t) &= \delta^n\left({}^{H}\mathcal{I}_{a^+}^{n-\alpha}\left(\log\frac{x}{a}\right)^{\beta-1}\right)(t)\\ &= \left(t\frac{d}{dt}\right)^n\frac{\Gamma(\beta)}{\Gamma(n+\beta-\alpha)}\left(\log\frac{t}{a}\right)^{\beta-\alpha+n-1},\end{aligned}$$

   furthermore, $\Gamma(n+\beta-\alpha) = (n-1+\beta-\alpha)(n-2+\beta-\alpha)\ldots(\beta-\alpha)\Gamma(\beta-\alpha)$, it follows that

$$\left({}^{H}\mathcal{D}_{a^+}^{\alpha}\left(\log\frac{x}{a}\right)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{t}{a}\right)^{\beta-\alpha-1},$$

3. Similarly, for all $\Re(\alpha) > 0$ and $\Re(\beta) > 0$, we can proof (2) and (3) of the proposition.

□

Caputo type of Hadamard fractional derivative

The Caputo-Hadamard fractional derivative is given by the following definition :

Definition 1.13 (Caputo–Hadamard fractional derivative)**.** (see [24]). The left-sided Caputo-type modification of left- Hadamard fractional derivatives of order α, $(\Re(\alpha) \geq 0)$ of a functionis $u(t) \in AC_{\delta}^{n}[a,b]$, where $0 < a < b < \infty$ is given by

$${}^{CH}\mathcal{D}_{a^+}^{\alpha}u(t) = {}^{H}\mathcal{D}_{a^+}^{\alpha}\left[u(t) - \sum_{k=0}^{n-1}\frac{\delta^k u(a)}{k!}\left(\log\frac{t}{a}\right)^k\right](t),$$

with $n = \lceil\Re(\alpha)\rceil + 1$ and the right-sided Caputo-type modification of left- Hadamard integral defined by

$${}^{CH}\mathcal{D}_{b^-}^{\alpha}u(t) = {}^{H}\mathcal{D}_{b^-}^{\alpha}\left[u(t) - \sum_{k=0}^{n-1}\frac{(-1)^k\delta^k u(b)}{k!}\left(\log\frac{b}{t}\right)^k\right](t).$$

Theorem 1.2. *(see [24]). Let* $\Re(\alpha) \geq 0$ *and* $n = \lceil \Re(\alpha) \rceil + 1$. *If* $u(t) \in AC_\delta^n[a,b]$, *where* $0 < a < b < \infty$. *Then* ${}^{CH}\mathcal{D}_{a+}^{\alpha} u(t)$ *and* ${}^{CH}\mathcal{D}_{b-}^{\alpha} u(t)$ *exist everywhere on* $[a,b]$ *and*

1. *If* $\alpha \notin \mathbb{N}$, ${}^{CH}\mathcal{D}_{a+}^{\alpha} u(t)$ *can be represented by*

$$
\begin{aligned}
{}^{CH}\mathcal{D}_{a+}^{\alpha} u(t) &= {}^{H}\mathcal{I}_{a+}^{n-\alpha}\, \delta^n u(t) \\
&= \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \delta^n u(s) \frac{ds}{s},
\end{aligned}
\tag{1.24}
$$

and ${}^{CH}\mathcal{D}_{b-}^{\alpha} u(t)$ *can be represented by*

$$
\begin{aligned}
{}^{CH}\mathcal{D}_{b-}^{\alpha} u(t) &= {}^{H}\mathcal{I}_{b-}^{n-\alpha}\, \delta^n u(t) \\
&= \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \left(\log \frac{s}{t}\right)^{n-\alpha-1} \delta^n u(s) \frac{ds}{s},
\end{aligned}
\tag{1.25}
$$

2. *If* $\alpha \in \mathbb{N}$, *then*

$$
{}^{CH}\mathcal{D}_{a+}^{\alpha} u(t) = \delta^n u(t), \quad {}^{CH}\mathcal{D}_{b-}^{\alpha} u(t) = (-1)^n \delta^n u(t),
$$

in particular

$$
{}^{CH}\mathcal{D}_{a+}^{0} u(t) = {}^{CH}\mathcal{D}_{b-}^{0} u(t) = u(t).
$$

The Caputo-modifications of the left and right Hadamard fractional derivatives have the same basic properties.

Lemma 1.2. *(see [17]). Let* $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $0 < a < b < \infty$, *then*

2. *Let* $u(t) \in L^p[a,b]$, *such that* $\Re(\alpha) > \Re(\beta) > 0$, *then*

$$
{}^{CH}\mathcal{D}_{a^+}^{\alpha H}\mathcal{I}_{a^+}^{\beta} u(t) = {}^{H}\mathcal{I}_{a^+}^{\beta-\alpha} u(t) \ \text{ and } \ {}^{CH}\mathcal{D}_{b^-}^{\alpha H}\mathcal{I}_{b^-}^{\beta} u(t) = {}^{H}\mathcal{I}_{b^-}^{\beta-\alpha} u(t),
$$

2. *Let* $u(t) \in AC_\delta^n[a,b]$ *or* C_δ^n *and* $\alpha \in \mathbb{C}$, *then*

$$
{}^{H}\mathcal{I}_{a^+}^{\alpha}\left({}^{CH}\mathcal{D}_{a+}^{\alpha} u\right)(t) = u(t) - \sum_{k=0}^{n-1} \frac{\delta^k u(a)}{k!} \left(\log \frac{t}{a}\right)^k,
$$

and

$$
{}^{H}\mathcal{I}_{b-}^{\alpha}\left({}^{CH}\mathcal{D}_{b-}^{\alpha} u\right)(t) = u(t) - \sum_{k=0}^{n-1} \frac{\delta^k u(b)}{k!} \left(\log \frac{b}{t}\right)^k.
$$

Theorem 1.3 (Semigroup property for Caputo–Hadamard derivatives)**.** *(see [17])*
Let $u(t) \in C_\delta^{m+n}[a,b]$ *and* $\alpha \geq 0$, $\beta \geq 0$, *such that* $n-1 < \alpha \leq n$ *and* $m-1 < \beta \leq m$, *then*

$$
{}^{CH}\mathcal{D}_{a^+}^{\alpha CH}\mathcal{D}_{a^+}^{\beta} u(t) = {}^{CH}\mathcal{D}_{a^+}^{\alpha+\beta} u(t).
$$

Proposition 1.2. *(see [24]) Let* $\Re(\alpha) \geq 0$, $n = \lceil \Re(\alpha) \rceil + 1$, $\Re(\beta) > 0$. *Then*

$$\left({}^{CH}\mathcal{D}_{a^+}^{\alpha} \left(\log \frac{x}{a} \right)^{\beta-1} \right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\log \frac{t}{a} \right)^{\beta-\alpha-1}, \quad \Re(\beta) > n, \tag{1.26}$$

$$\left({}^{CH}\mathcal{D}_{b^-}^{\alpha} \left(\log \frac{b}{x} \right)^{\beta-1} \right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\log \frac{b}{t} \right)^{\beta-\alpha-1}, \quad \Re(\beta) > n. \tag{1.27}$$

Let us give in particular

(i) $\left({}^{CH}\mathcal{D}_{a^+}^{\alpha} \left(\log \frac{x}{a} \right)^{k} \right)(t) = 0$ *and* $\left({}^{CH}\mathcal{D}_{b^-}^{\alpha} \left(\log \frac{x}{a} \right)^{k} \right)(t) = 0,\ k = 0, 1, \ldots, n-1,$

(ii) ${}^{CH}\mathcal{D}_{a^+}^{\alpha} 1 = 0,\ {}^{CH}\mathcal{D}_{b^-}^{\alpha} 1 = 0.$

Proof. For $\Re(\alpha) > 0$, and $\Re(\beta) > 0$ and $n = \lceil \Re(\alpha) \rceil + 1$, we have from definition (1.24)

$$\left({}^{CH}\mathcal{D}_{a^+}^{\alpha} \left(\log \frac{x}{a} \right)^{\beta-1} \right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{s} \right)^{n-\alpha-1} \delta^n \left(\log \frac{s}{a} \right)^{\beta-1} \frac{ds}{s}, \tag{1.28}$$

and

$$\delta^n \left(\log \frac{s}{a} \right)^{\beta-1} = (\beta-1)(\beta-2)\ldots(\beta-n) \left(\log \frac{s}{a} \right)^{\beta-n-1}.$$

In particular, if we put $\Pi = \dfrac{\log\left(\frac{s}{a}\right)}{\log\left(\frac{t}{a}\right)}$, then from (1.28), we obtain

$$\begin{aligned} \left({}^{CH}\mathcal{D}_{a^+}^{\alpha} \left(\log \frac{x}{a} \right)^{\beta-1} \right)(t) = & \frac{(\beta-1)(\beta-2)\ldots(\beta-n)}{\Gamma(n-\alpha)} \left(\log \frac{t}{a} \right)^{\beta-\alpha-1} \\ & \times \int_0^1 (1-\Pi)^{n-\alpha-1} \Pi^{\beta-n-1} d\Pi, \end{aligned}$$

by the definition of the beta function, we obtain

$$\begin{aligned} \left({}^{CH}\mathcal{D}_{a^+}^{\alpha} \left(\log \frac{x}{a} \right)^{\beta-1} \right)(t) &= \frac{(\beta-1)(\beta-2)\ldots(\beta-n)\Gamma(\beta-n)}{\Gamma(\beta-\alpha)} \left(\log \frac{t}{a} \right)^{\beta-\alpha-1} \\ &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\log \frac{t}{a} \right)^{\beta-\alpha-1}. \end{aligned} \tag{1.29}$$

If we put $k = \beta - 1$, we obtain from (1.28) :

$$\left({}^{CH}\mathcal{D}_{a^+}^{\alpha} \left(\log \frac{x}{a} \right)^{\beta-1} \right)(t) = \frac{\Gamma(\beta-n)}{\Gamma(\beta-\alpha)} k(k-1)\ldots(k-(n-1)) \left(\log \frac{t}{a} \right)^{\beta-\alpha-1},$$

so, for $k = 0, 1, \ldots, n-1$, we have $\left({}^{CH}\mathcal{D}_{a^+}^{\alpha} \left(\log \frac{x}{a} \right)^{k} \right)(t) = 0.$

Similarly, for all $\Re(\alpha) > 0$ and $\Re(\beta) > 0$, we have (1.27). □

1.1.6 Katugampola fractional integrals and fractional derivatives

Introduced by UDITA Katugampola (2011) [25], which generalizes the Riemann–Liouville and Hadamard fractional integrals into a single form (see Kilbas).The generalized fractional integral is based on the observation that, for $n \in \mathbb{N}$,

$$\int_a^t \tau_1^{\rho-1} d\tau_1 \int_a^{\tau_1} \tau_2^{\rho} d\tau_2 \ldots \int_a^{\tau_{n-1}} \tau_n^{\rho-1} u(\tau_n)\, d\tau_n = \frac{\rho^{1-n}}{(n-1)!} \int_a^t \frac{\tau^{\rho-1} y(\tau)}{(t^\rho - \tau^\rho)^{1-n}} d\tau, \tag{1.30}$$

by replacing the integer n by positive real number α, we obtain the following definition

Definition 1.14 (Katugampola fractional integral)**.** (see [25]). Let $\Omega = [a,b]\,(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$, for $(1 \leq p < \infty)$. The generalized fractional integral ${}^\rho\mathcal{I}_{a^+}^\alpha u(t)$ of order $\alpha \in \mathbb{C}$ $(\Re(\alpha) > 0)$ of $u(t) \in X_c^p(a,b)$ is defined by

$$\left({}^\rho\mathcal{I}_{a^+}^\alpha u\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1} u(\tau)}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau, \tag{1.31}$$

for $t > a$ and $\rho > 0$. This integral is called the left-sided fractional integral. Similarly we can define the right-sided fractional integral ${}^\rho\mathcal{I}_{b^-}^\alpha u$ by

$$\left({}^\rho\mathcal{I}_{b^-}^\alpha u\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b \frac{\tau^{\rho-1} u(\tau)}{(\tau^\rho - t^\rho)^{1-\alpha}} d\tau, \tag{1.32}$$

for $t < b$ and $\Re(\alpha) > 0$.

Definition 1.15 (Katugampola fractional derivative)**.** (see [25]). Let $\alpha \in \mathbb{C}$, $\Re(\alpha) \geq 0$, $n = \lceil \Re(\alpha) \rceil + 1$ and $\rho > 0$. The generalized fractional derivatives for $0 \leq a < t < b \leq \infty$ is defined by

$$\begin{aligned}\left({}^\rho\mathcal{D}_{a^+}^\alpha u\right)(t) &= \left(t^{1-\rho}\frac{d}{dt}\right)^n \left({}^\rho\mathcal{I}_{a^+}^{n-\alpha} u\right)(t) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t \frac{\tau^{(\rho-1)} u(\tau)}{(t^\rho - \tau^\rho)^{\alpha-n+1}} d\tau,\end{aligned} \tag{1.33}$$

respectively

$$\begin{aligned}{}^\rho\mathcal{D}_{b^-}^\alpha u(t) &= \left(-t^{1-\rho}\frac{d}{dt}\right)^n \left({}^\rho\mathcal{I}_{b^-}^{n-\alpha} u\right)(t) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(-t^{1-\rho}\frac{d}{dt}\right)^n \int_t^b \frac{\tau^{(\rho-1)} u(\tau)}{(\tau^\rho - t^\rho)^{\alpha-n+1}} d\tau.\end{aligned} \tag{1.34}$$

Properties of Katugampola fractional operators

Here, we give some lemmas and remarks that present the main properties of the generalized fractional operators, we mainly restrict our attention to the left-sided operators (1.31), (1.33) (of course, the right-sided Katugampola operators (1.32), (1.34) possesses similar properties).

Theorem 1.4. *([25], [26]) Let $\alpha, \beta \in \mathbb{C}$ be such that $0 < \Re(\alpha) < 1$ and $0 < \Re(\beta) < 1$: If $0 < a < b < \infty$ and $1 \leq p \leq \infty$, then for any $u, v \in X_c^p(a,b)$, $\rho > 0$ we have :*

(i) Index property :

$$^{\rho}\mathcal{I}_{a^+}^{\alpha\rho}\mathcal{I}_{a^+}^{\beta}u = {}^{\rho}\mathcal{I}_{a^+}^{\alpha+\beta}u \quad \text{and} \; {}^{\rho}\mathcal{D}_{a^+}^{\alpha\rho}\mathcal{D}_{a^+}^{\beta}u = {}^{\rho}\mathcal{D}_{a^+}^{\alpha+\beta}u \,.$$

(ii) Composition property : for $0 < \Re(\alpha) < \Re(\beta) < 1$ and $u \in L^p(a,b)$

$$^{\rho}\mathcal{D}_{a^+}^{\alpha\rho}\mathcal{I}_{a^+}^{\beta}u = {}^{\rho}\mathcal{I}_{a^+}^{\beta-\alpha}u \quad \text{and} \quad {}^{\rho}\mathcal{D}_{b^-}^{\alpha\rho}\mathcal{I}_{b^-}^{\beta}u = {}^{\rho}\mathcal{I}_{b^-}^{\beta-\alpha}u \,.$$

(iii) Linearity property :

$$^{\rho}\mathcal{I}_{a^+}^{\alpha}(u+v) = {}^{\rho}\mathcal{I}_{a^+}^{\alpha}u + {}^{\rho}\mathcal{I}_{a^+}^{\alpha}v \,,$$

and

$$^{\rho}\mathcal{D}_{a^+}^{\alpha}(u+v) = {}^{\rho}\mathcal{D}_{a^+}^{\alpha}u + {}^{\rho}\mathcal{D}_{a^+}^{\alpha}v \,.$$

Theorem 1.5. *([26]). Let $\alpha \in \mathbb{C}, \Re(\alpha) \geq 0$, $n = \lceil \Re(\alpha) \rceil + 1$ and $\rho > 0$, then*

$$\begin{aligned}
\lim_{\rho\to 1}\left({}^{\rho}\mathcal{I}_{a^+}^{\alpha}u\right)(t) &= {}^{RL}\mathcal{I}_{a^+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)\,ds,\\
\lim_{\rho\to 0^+}\left({}^{\rho}\mathcal{I}_{a^+}^{\alpha}u\right)(t) &= {}^{H}\mathcal{I}_{a^+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{u(s)}{s}ds,\\
\lim_{\rho\to 1}\left({}^{\rho}\mathcal{D}_{a^+}^{\alpha}u\right)(t) &= {}^{RL}\mathcal{D}_{a^+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_a^t (t-s)^{n-\alpha-1}u(s)\,ds,\\
\lim_{\rho\to 0^+}\left({}^{\rho}\mathcal{D}_{a^+}^{\alpha}u\right)(t) &= {}^{H}\mathcal{D}_{a^+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_a^t \left(\log\frac{t}{s}\right)^{n-\alpha-1}\frac{u(s)}{s}ds.
\end{aligned}$$

Proposition 1.3. *([25], [26]. For $\alpha, \rho > 0$ and $\nu > -\rho$, we quote*

(i) $$^{\rho}\mathcal{D}_{0^+}^{\alpha}t^{\upsilon} = \frac{\Gamma\left(1+\frac{\upsilon}{\rho}\right)\rho^{\alpha-1}}{\Gamma\left(1+\frac{\upsilon}{\rho}-\alpha\right)}t^{\upsilon-\alpha\rho}.$$

Let us give in particular

$$^{\rho}\mathcal{D}_{0^+}^{\alpha}t^{\rho(\alpha-m)} = 0 \quad \text{for each} \;\; m = 1,2,\ldots,n.$$

(ii)

$$
{}^{\rho}\mathcal{I}_{0^+}^{\alpha} t^{\upsilon} = \frac{\rho^{-\alpha}\Gamma\left(1+\dfrac{\upsilon}{\rho}\right)}{\Gamma\left(1+\dfrac{\upsilon}{\rho}+\alpha\right)} t^{\upsilon+\rho\alpha}, \text{ for all } \nu > -\rho.
$$

Proof. For $\alpha, \rho > 0$ and $\nu > -\rho,$ also from definition (1.33), we find

$$
{}^{\rho}\mathcal{D}_{0^+}^{\alpha} t^{\upsilon} = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_0^t (t^{\rho}-\tau^{\rho})^{n-\alpha-1} \tau^{\upsilon+\rho-1} d\tau,
$$

letting $\Pi = \dfrac{\tau^{\rho}}{t^{\rho}},$ then we get

$$
\begin{aligned}
{}^{\rho}\mathcal{D}_{0^+}^{\alpha} t^{\upsilon} &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^n \\
&\times \int_0^1 t^{\rho(n-\alpha-1)} (1-\Pi)^{n-\alpha-1} \left(t\Pi^{\frac{1}{\rho}}\right)^{\upsilon+\rho-1} \frac{t}{\rho}\Pi^{\frac{1}{\rho}-1} d\Pi \\
&= \frac{\rho^{\alpha-n}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^n t^{\rho(n-\alpha)+\nu} \int_0^1 (1-\Pi)^{n-\alpha-1} \Pi^{\frac{1}{\rho}(\nu+\rho)-1} d\Pi,
\end{aligned}
$$

by the definition of the beta function, we obtain

$$
\begin{aligned}
{}^{\rho}\mathcal{D}_{0^+}^{\alpha} t^{\upsilon} &= \frac{\rho^{\alpha-n}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^n t^{\rho(n-\alpha)+\nu} \beta(n-\alpha, 1+\frac{\nu}{\rho}) \\
&= \frac{\rho^{\alpha-1}\Gamma\left(1+\dfrac{\nu}{\rho}\right)}{\Gamma\left(1+n-\alpha+\dfrac{\nu}{\rho}\right)} \left(n-\alpha+\frac{\nu}{\rho}\right) \cdots \left(1-\alpha+\frac{\nu}{\rho}\right) t^{\upsilon-\alpha\rho},
\end{aligned} \tag{1.35}
$$

furthermore, $\Gamma\left(1+n-\alpha+\dfrac{\nu}{\rho}\right) = \left(n-\alpha+\dfrac{\nu}{\rho}\right)\cdots\left(1-\alpha+\dfrac{\nu}{\rho}\right)\Gamma\left(1-\alpha+\dfrac{\nu}{\rho}\right)$, it follows that

$$
{}^{\rho}\mathcal{D}_{0^+}^{\alpha} t^{\upsilon} = \frac{\rho^{\alpha-1}\Gamma\left(1+\dfrac{\nu}{\rho}\right)}{\Gamma\left(1-\alpha+\dfrac{\nu}{\rho}\right)} t^{\upsilon-\alpha\rho}.
$$

If we put $m = \alpha - \dfrac{\nu}{\rho}$ from (1.35), we obtain

$$
{}^{\rho}\mathcal{D}_{0^+}^{\alpha} t^{\rho(\alpha-m)} = \frac{\rho^{\alpha-1}\Gamma(\alpha-m+1)}{\Gamma(n-m+1)} (n-m)(n-m-1)\ldots(1-m)\, t^{-\rho m},
$$

so, for $m = 1, 2, \ldots, n,$ we have ${}^{\rho}\mathcal{D}_{0^+}^{\alpha} t^{\rho(\alpha-m)} = 0$ for all $\alpha, \rho > 0$. Similarly, for all $\alpha, \rho > 0$, we can proof (ii). □

Caputo type of Katugampola fractional derivatives

The Caputo–Katugampola fractional derivatives are given by the following definition

Definition 1.16 (Caputo–Katugampola fractional derivative)**.** ([1]). Let n be the smallest integer greater than α. Then, the left and right Caputo–Katugampola fractional derivative of order $\alpha > 0$ are defined by

$$ {}^{C}\mathcal{D}_{a^+}^{\alpha,\rho} u\left(t\right) = \frac{\rho^{\alpha-n+1}}{\Gamma\left(n-\alpha\right)} \int_a^t \frac{\tau^{(\rho-1)(1-n)}}{\left(t^{\rho}-\tau^{\rho}\right)^{\alpha-n+1}} u^{(n)}(\tau) d\tau, \tag{1.36} $$

and

$$ {}^{C}\mathcal{D}_{b^-}^{\alpha,\rho} u\left(t\right) = \frac{(-1)^n \rho^{\alpha-n+1}}{\Gamma\left(n-\alpha\right)} \int_t^b \frac{\tau^{(\rho-1)(1-n)}}{\left(\tau^{\rho}-t^{\rho}\right)^{\alpha-n+1}} u^{(n)}(\tau) d\tau, \tag{1.37} $$

In this section we define and give some basic properties of the Caputo type modification of the Katugampola fractional derivative.

Theorem 1.6. *([1]). The relationship between the Caputo–Katugampola derivative and the Katugampola fractional integral of order α is given by*

(i) Let $u \in C([a, b])$. Then,

$$ {}^{C}\mathcal{D}_{a^+}^{\alpha,\rho} {}^{\rho}\mathcal{I}_{a^+}^{\alpha} u(t) = u(t). $$

(ii) Let $u \in C^1([a, b])$. Then,

$$ {}^{\rho}\mathcal{I}_{a^+}^{\alpha} {}^{C}\mathcal{D}_{a^+}^{\alpha,\rho} u(t) = u(t) - u(a). $$

Lemma 1.3. *([1]). For $\nu > 0$ and $\alpha \in (0, 1)$, define*

$$ u(t) = \left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\upsilon} \text{ and } y(t) = \left(\frac{b^{\rho} - t^{\rho}}{\rho}\right)^{\upsilon}. $$

Then,

$$ {}^{C}\mathcal{D}_{a^+}^{\alpha,\rho} u(t) = \frac{\rho^{\alpha-\upsilon}\Gamma\left(1+\upsilon\right)}{\Gamma\left(1-\alpha+\upsilon\right)} \left(t^{\rho}-a^{\rho}\right)^{\upsilon-\alpha}, \tag{1.38} $$

Performing the change of variables $\Pi = \left(\dfrac{\tau^{\rho} - a^{\rho}}{t^{\rho} - a^{\rho}}\right)$, and by the definition of the beta function, we get

$$
\begin{aligned}
{}^{C}\mathcal{D}_{a^+}^{\alpha,\rho}u(t) &= \frac{\rho^{\alpha-\nu}\nu}{\Gamma(1-\alpha)}\left(t^\rho-a^\rho\right)^{\nu-\alpha}\int_0^1(1-\Pi)^{-\alpha}\,\Pi^{\nu-1}d\Pi \\
&= \frac{\rho^{\alpha-\nu}\nu}{\Gamma(1-\alpha)}\left(t^\rho-a^\rho\right)^{\nu-\alpha}\beta(1-\alpha,\nu) \\
&= \frac{\rho^{\alpha-\upsilon}\Gamma(1+\upsilon)}{\Gamma(1-\alpha+\upsilon)}\left(t^\rho-a^\rho\right)^{\upsilon-\alpha}.
\end{aligned}
$$

Similarly, for all $\alpha,\rho>0,$ we have

$$
{}^{C}\mathcal{D}_{b^-}^{\alpha,\rho}y(t) = \frac{\rho^{\alpha-\upsilon}\Gamma(1+\upsilon)}{\Gamma(1-\alpha+\upsilon)}\left(b^\rho-t^\rho\right)^{\upsilon-\alpha}.
$$

Remark 1.1. ([1]). From definition (1.3) and (1.36)

(i) If we put $\rho=1$, then we obtain the left and right Caputo fractional derivatives (1.15) and (1.16).

(ii) Attending that $\lim\limits_{\rho\to0^+}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)=\ln\left(\frac{t}{\tau}\right)$, we obtain the left and right Caputo-Hadamard fractional derivatives as defined in (1.24) and (1.25).

1.1.7 Riesz fractional derivatives

The Riesz fractional derivatives are given by the following definition

Definition 1.17. (see. [39]) Let $\alpha>0$ and $n=\lceil\alpha\rceil+1$. The Riesz fractional derivative of order α of a continuous function $u:[a,b]\to\mathbb{R}$ is given by

$$
{}_{a}^{R}\mathcal{D}_{b}^{\alpha}u(t) = -\psi_\alpha\left({}_{a}\mathcal{D}_{t}^{\alpha}+(-1)^n\,{}_{t}\mathcal{D}_{b}^{\alpha}\right), \tag{1.39}
$$

where $\psi_\alpha=\left(\frac{1}{2}\right)$ or sec $\left(\frac{\pi\alpha}{2}\right)$ and ${}_{a}\mathcal{D}_{t}^{\alpha}$, ${}_{t}\mathcal{D}_{b}^{\alpha}$ are the left and right side Riemann–Liouville fractional derivatives defined in (1.12) and (1.13) respectively.

Caputo type of Riesz fractional derivatives

In the following, we briefly introduce two interesting definitions of derivatives (Riesz–Caputo and Riesz–Caputo-katugampola), which are essential for the continuation of this work, in order to formulate some important results, we restrict in particular to the left-sided operators (1.15), (1.36) and the right-sided operators (1.16), (1.37).

Definition 1.18. (see. [39]) (Riesz–Caputo fractional derivative). Let $\alpha > 0$, and $n = \lceil \alpha \rceil + 1$.The Riesz–Caputo fractional derivative of order α of a continuous function $u : [a, b] \to \mathbb{R}$ is given by

$$\begin{aligned} {}_{a}^{RC}\mathcal{D}_{b}^{\alpha}u(t) &= \frac{1}{\Gamma(n-\alpha)} \int_{a}^{b} \frac{u^{(n)}(\tau)}{|t-\tau|^{(\alpha-n+1)}} d\tau \\ &= \frac{1}{2}\left({}_{a}^{C}\mathcal{D}_{t}^{\alpha} + (-1)^{n}\, {}_{t}^{C}\mathcal{D}_{b}^{\alpha}\right) u(t), \end{aligned} \tag{1.40}$$

here ${}_{a}^{C}\mathcal{D}_{t}^{\alpha}$ and ${}_{t}^{C}\mathcal{D}_{b}^{\alpha}$ stand for the left and right Caputo derivative defined in (1.15) and (1.16) respectively.

From the Riesz–Caputo fractional derivative (1.40), we define the Riesz–Caputo–Katugampola as follows :

Definition 1.19. (The Riesz–Caputo–Katugampola)(see. [11]). Let $\alpha > 0$, $n = \lceil \alpha \rceil + 1$ and $\rho > 0$. The Riesz–Caputo–Katugampola fractional derivative of order α of a continuous function $u : [a, b] \to \mathbb{R}$ is given by

$$\begin{aligned} {}_{a}^{RC}\mathcal{D}_{b}^{\alpha,\rho}u(t) =& \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_{a}^{b} \frac{s^{(\rho-1)(1-n)}}{|x^{\rho}-s^{\rho}|^{\alpha-n+1}} u^{(n)}(s) ds \\ =& \frac{1}{2}\left({}^{C}\mathcal{D}_{a,t}^{\alpha,\rho} + (-1)^{n}\, {}^{C}\mathcal{D}_{t,b}^{\alpha,\rho}\right) u(t), \end{aligned} \tag{1.41}$$

with ${}^{C}\mathcal{D}_{a,t}^{\alpha,\rho}$ and ${}^{C}\mathcal{D}_{t,b}^{\alpha,\rho}$ are the left and right Caputo-Katugampola fractional derivative defined in (1.36)-(1.37).

1.2 Finite difference method

In numerical analysis, the finite difference method is a common technique for finding approximate solutions of DE and PDE which consists in solving a system of relations (numerical scheme) linking the values of unknown functions at certain points sufficiently close to each other. This method appears to be the simplest to implement because it proceeds in two stages : on the one hand the discretization by finite differences of the operators of derivation / differentiation, on the other hand the convergence of the numerical scheme thus obtained when the distance between the points decreases..

1.2.1 The numerical scheme

It is obtained from the discretization of the domain and the discretization of the equation from the partial or mean derivative approximation of Taylor expansions. The discretization of the domain of the problem approaches the continuous domain by a finite number of sub domains, in which the numerical values of unknowns is a determined quantity. The set of relations for the calculation of these values will be obtained by the discretization of the equation, which approaches the systems continuous.

The numerical differentiation

Usually, Taylor series is used to obtain numerical differentiation methods. Methods using Taylor series are : backward difference scheme, forward difference scheme and central difference scheme to evaluate the derivative [19]. Finite difference formulas are the most common formulas which are used to solve the ordinary and partial differential equations numerically [31] and [19]. The derivatives in these equations can be replaced with suitable finite difference approximatins on a discretized domain. However, Taylor Series can be used to derive some of finite difference approximations. Let h be the discretization step, take for example $[a,b] \subset \mathbb{R}$ and $x \in [a,b]$, then

$$h = \frac{(b-a)}{n}, \text{ where } n \text{ being the number of intervals in } [a,b],$$

we denote by : $x(i) = x_i = a + ih,\ i = 0, 1, ..., n$. we can agree to evaluate the derivative at x. In this case, we use the values $u(x+h)$ and $u(x+2h)$ for the forward difference and the values $u(x+h)$ and $u(x-h)$ for the central difference. With regard to the error term, we only retain its order. Then we summarizes the situation

$$u'(x) = \frac{u(x+h) - u(x)}{h} + O(h), \text{ Forward difference of ordre } 1, \tag{1.42}$$

$$u'(x) = \frac{u(x) - u(x-h)}{h} + O(h), \text{ Backward difference of ordre } 1, \tag{1.43}$$

$$u'(x) = \frac{-u(x+2h) + 4u(x+h) - 3u(x)}{2h} + O(h^2), \text{ Forward difference of ordre } 2, \tag{1.44}$$

$$u'(x) = \frac{u(x+h) - u(x-h)}{2h} + O(h^2), \text{ Central difference of ordre } 2, \tag{1.45}$$

$$u'(x) = \frac{3u(x) - 4u(x-h) + u(x-2h)}{2h} + O(h^2), \text{ Backward difference of ordre } 2. \tag{1.46}$$

Similarly, for higher order derivatives we act more or less in the same way as with the derivatives of order 1

$$u''(x) = \frac{u(x-2h) - 2u(x-h) + u(x)}{h^2} + O(h), \text{ Backward difference of ordre } 1, \tag{1.47}$$

$$u''(x) = \frac{u(x+2h) - 2u(x+h) + u(x)}{h^2} + O(h), \text{ Forward difference of ordre } 1, \tag{1.48}$$

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2), \text{ Central difference of ordre } 2, \tag{1.49}$$

$$u''(x) = \frac{-u(x+2h) + 16u(x+h) - 30u(x) + 16u(x-h) - u(x-2h)}{12h^2} + O(h^4),$$

Central difference of ordre 4.

Then, from [9] and [19] the central difference, forward difference and backward difference formula for the 3^{rd} derivatives are known by

$$u'''(x) = \frac{-u(x-2h) + 2u(x-h) - 2u(x+h) + u(x+2h)}{2h^3} + O(h^2),$$

Central difference of ordre 2,

$$u'''(x) = \frac{u(x+2h) - 4u(x+h) + 6u(x) - 4u(x-h) + u(x-2h)}{h^4} + O(h^4),$$

Central difference of ordre 4,

$$u'''(x) = \frac{-u(x) + 3u(x+h) - 3u(x+2h) + u(x+3h)}{h^3} + O(h),$$

Forward difference of ordre 1,

$$u'''(x) = \frac{u(x) - 3u(x-h) + 3u(x-2h) - u(x-3h)}{h^3} + O(h),$$

Backward difference of ordre 1,

Explicit scheme and implicit scheme

Definition 1.20. The explicit method consists in determining the solution at $t + \Delta t$ according to the value of the function in t, whereas the implicit method consists in determining the solution at $t + \Delta t$ by solving an equation taking into account the value of the function in t and $t + \Delta t$.

Numerical stability, convergence and consistency

Once the discrete scheme has been chosen, it will be necessary to solve it. The process of solving, in view of the equations will be most of the time iterative. We will calculate the values of u step by step. A given value of u will be therefore calculated using the result of calculating other values u. Errors rounded being unavoidable on the machine.

Definition 1.21. A numerical method is stable if these errors do not amplify (too much) in the during the calculation.

Definition 1.22. A numerical method is consistent if the discretization error of the equation tends towards 0 when the discretization step tends towards 0.

Definition 1.23. The convergence of a digital diagram is a global theoretical property ensuring that the difference between the approximate solution and the solution exact tends towards 0 when the step of discretization tends towards 0.

Chapter 2

FDM to approximate the fractional derivatives

In this chapter, we present the numerical approaches used for the approximation of fractional derivatives based on finite difference methods, we mainly build algorithms for Riemann-Liouville, Caputo, Caputo-Hadamard and Caputo-Katugampola with a fractional order $\alpha \in]0,1[$ and $\alpha \in]1,2[$. Some ullistrative examples are also given to show the applicability of our results.

2.1 Approximation of Riemann–Liouville and Caputo fractional derivatives

This section concerns the numerical methods for Riemann–Liouville and Caputo fractional derivatives with fractional order $\alpha > 0$. Then, for the finite difference approximation [31], we equally sub-divide the interval $[0,T]$ with $t_i = ih,\ i = 0,1,\dots,n,$ where $h = \dfrac{T}{n}$ is the step size. From the relationship between the Caputo fractional derivative and Riemann–Liouville fractional derivative defined in (1.17) we have

— For $0 < \alpha < 1$, we get

$$\mathcal{D}_{0^+}^{\alpha} u(t) = {}^{C}\mathcal{D}_{0^+}^{\alpha} u(t) + \frac{u(0)}{\Gamma(1-\alpha)} t^{-\alpha}, \tag{2.1}$$

letting $t = t_n$, from (1.15–1.42), the approximate fractional derivative of Caputo is given by

$$\begin{aligned} {}^{C}\mathcal{D}_{0^+}^{\alpha} u(t_n) =& \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} (t_n - s)^{-\alpha} u'(s) ds \\ \approx& \frac{1}{\Gamma(1-\alpha)} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (t_n - s)^{-\alpha} \left(\frac{u_{i+1} - u_i}{t_{i+1} - t_i} \right) ds \\ =& \sum_{i=0}^{n-1} b_{n-i-1} (u_{i+1} - u_i), \end{aligned}$$

where

$$b_i = \frac{h^{-\alpha}}{\Gamma(2-\alpha)}\left[(i+1)^{(1-\alpha)} - i^{(1-\alpha)}\right], \tag{2.2}$$

and

$$^{C}\mathcal{D}^{\alpha}_{0^+}u(t_n) = \sum_{i=0}^{n-1} b_{n-i-1}(u_{i+1}-u_i) + O(h^{2-\alpha}).$$

Therefore, the approximate fractional derivative of Riemann–Liouville is given by

$$\mathcal{D}^{\alpha}_{0^+}u(t_n) \approx \sum_{i=0}^{n-1} b_{n-i-1}(u_{i+1}-u_i) + \frac{u(0)}{\Gamma(1-\alpha)}t_n^{-\alpha}. \tag{2.3}$$

The above scheme (2.3) has the following error estimate [31]

$$\left|\sum_{i=0}^{n-1} b_{n-i-1}(u_{i+1}-u_i) + \frac{u(0)}{\Gamma(1-\alpha)}t_n^{-\alpha} - \mathcal{D}^{\alpha}_{0^+}u(t_n)\right| \leq Ch^{2-\alpha}, \tag{2.4}$$

where C is a positive constant only dependent on α and u.

— For $1 < \alpha < 2$, we get

$$\mathcal{D}^{\alpha}_{0^+}u(t) = {}^{C}\mathcal{D}^{\alpha}_{0^+}u(t) + \frac{u(0)}{\Gamma(1-\alpha)}t^{-\alpha} + \frac{u'(0)}{\Gamma(2-\alpha)}t^{1-\alpha}, \tag{2.5}$$

similarly, letting $t = t_n$ and from (1.15–1.49), the approximate fractional derivative of Caputo is given by

$$\begin{aligned} ^{C}\mathcal{D}^{\alpha}_{0^+}u(t_n) =& \frac{1}{\Gamma(2-\alpha)}\int_0^{t_n}(t_n-s)^{1-\alpha}u''(s)ds \\ =& \frac{1}{\Gamma(2-\alpha)}\int_0^{t_n} s^{1-\alpha}u''(t_n-s)ds. \end{aligned}$$

On each subinterval $[t_i, t_{i+1}]$, one gets

$$\int_{t_i}^{t_{i+1}} s^{1-\alpha}u''(t_n-s)ds \approx \frac{u(t_n-t_{i+1}) - 2u(t_n-t_i) + u(t_n-t_{i-1})}{h^2}\int_{t_i}^{t_{i+1}} s^{1-\alpha}ds. \tag{2.6}$$

Hence, one has

$$\begin{aligned} ^{C}\mathcal{D}^{\alpha}_{0^+}u(t_n) &\approx \frac{1}{\Gamma(2-\alpha)}\frac{u(t_n-t_{i+1}) - 2u(t_n-t_i) + u(t_n-t_{i-1})}{h^2}\int_{t_i}^{t_{i+1}} s^{1-\alpha}ds \\ &= \sum_{i=-1}^{n} W_i u_{n-i}, \end{aligned} \tag{2.7}$$

where

$$W_i = \frac{h^{-\alpha}}{\Gamma(3-\alpha)} \begin{cases} 1,\ i=-1, \\ 2^{2-\alpha}-3,\ i=0, \\ \begin{pmatrix} (i+2)^{(2-\alpha)} - 3(i+1)^{(2-\alpha)} \\ +\,3i^{(2-\alpha)} - (i-1)^{(2-\alpha)} \end{pmatrix},\ 1 \le i \le n-2, \\ -\,2n^{(2-\alpha)} + 3(n-1)^{(2-\alpha)} - (n-2)^{(2-\alpha)},\ i=n-1, \\ n^{(2-\alpha)} - (n-1)^{(2-\alpha)},\ i=n, \end{cases} \tag{2.8}$$

and

$${}^{C}\mathcal{D}_{0+}^{\alpha} u(t_n) = \sum_{i=-1}^{n} W_i u_{n-i} + O(h^{3-\alpha}),$$

which leads to the following scheme for the Riemann-Liouville fractional derivative

$$\mathcal{D}_{0+}^{\alpha} u(t_n) \approx \sum_{i=-1}^{n} W_i u_{n-i} + \frac{u(0)}{\Gamma(1-\alpha)} t_n^{-\alpha} + \frac{u'(0)}{\Gamma(2-\alpha)} t_n^{1-\alpha}, \tag{2.9}$$

The above scheme (2.9) has the following error estimate [31]

$$\left| \sum_{i=-1}^{n} W_i u_{n-i} + \frac{u(0)}{\Gamma(1-\alpha)} t_n^{-\alpha} + \frac{u'(0)}{\Gamma(2-\alpha)} t_n^{1-\alpha} - \mathcal{D}_{0+}^{\alpha} u(t_n) \right| \le C' h^{3-\alpha}, \tag{2.10}$$

where C' is a positive constant only dependent on α and u.

Example 2.1. Consider the function $u(t) = (t+1)$, for $0 < \alpha \le 1$ and $t \in [0,1]$, from (1.12) and (1.15) we have

$$\begin{aligned} {}^{C}\mathcal{D}_{0+}^{\alpha} u(t) &= \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha}, \\ \mathcal{D}_{0+}^{\alpha} u(t) &= \frac{1}{\Gamma(1-\alpha)} t^{-\alpha} + \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha}, \end{aligned}$$

Figures 2.1 and 2.2 represent the comparison between the analytical Caputo and Riemann–Liouville fractional derivatives and its approximation for different values of h.

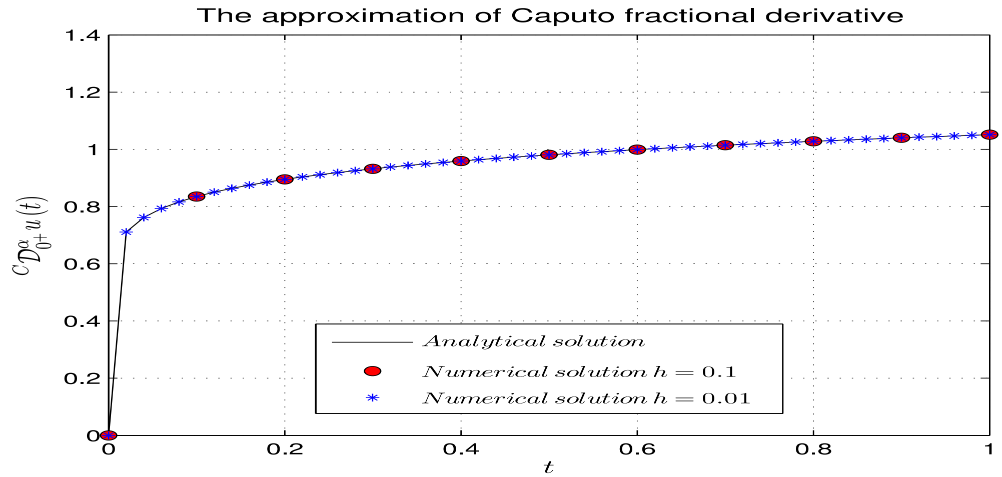

FIGURE 2.1 – Graphical comparison between the analytical Caputo fractional derivative and its approximation with $\alpha = 0.9$, $h = 0.1$ and $h = 0.01$.

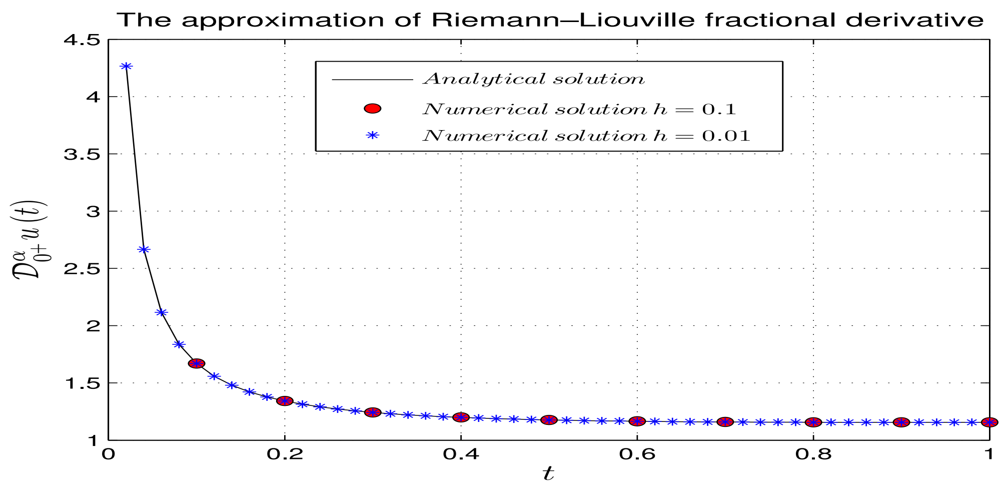

FIGURE 2.2 – Graphical comparison between the analytical Riemann–Liouville fractional derivative and its approximation with $\alpha = 0.9$, $h = 0.1$ and $h = 0.01$.

2.2 Approximation of Caputo–Hadamard fractional derivatives

In this section, we equally sub-divide the intervals $[a, T]$ with $t_i = a + ih$, $i = 0, 1, \ldots, N$, where $h = \dfrac{T-a}{N}$ is the step size.

Let $u : [a, T] \to \mathbb{R}$ be a given function, u_n the numerical approximation to $u(t_n)$ and $f_n = f(t_n)$, our result [10] is presented as follows

Theorem 2.1. *Let $u : [a, T] \to \mathbb{R}$ such that $u \in C^2([a, T] : \mathbb{R})$, α between zero and one, then for*

$N \in \mathbb{N}$, *we have*

$$^{CH}\mathcal{D}^{\alpha}_{a^+} u\left(t_n\right) = {}^{CH}\mathcal{D}^{\alpha}_{a^+} u_n + O\left(h^{1-\alpha}\right),$$

Where $^{CH}\mathcal{D}^{\alpha}_{a^+} u_n$ *is defined as follows :*

$$^{CH}\mathcal{D}^{\alpha}_{a^+} u_n = \frac{1}{h\Gamma(2-\alpha)} \sum_{i=1}^{n} b_i \left(u_i - u_{i-1}\right), \tag{2.11}$$

and

$$b_i = t_i \left(\left(\log \frac{t_n}{t_{i-1}} \right)^{1-\alpha} - \left(\log \frac{t_n}{t_i} \right)^{1-\alpha} \right). \tag{2.12}$$

Proof. For any $N \in \mathbb{N}$ and for each $n \in \{0, 1, \ldots, N\}$, we have

$$\begin{aligned}
^{CH}\mathcal{D}^{\alpha}_{a^+} u\left(t_n\right) =& \frac{1}{\Gamma(1-\alpha)} \int_a^{t_n} \left(\log \frac{t_n}{s} \right)^{-\alpha} \left(s \frac{d}{ds} \right) u(s) \frac{ds}{s} \\
\approx& \frac{1}{\Gamma(1-\alpha)} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_n}{s} \right)^{-\alpha} t_i \left(\frac{u_i - u_{i-1}}{t_i - t_{i-1}} \right) \frac{ds}{s} \\
=& \frac{1}{\Gamma(1-\alpha)} \sum_{i=1}^{n} t_i \left(\frac{u_i - u_{i-1}}{h} \right) \int_{t_{i-1}}^{t_i} \left(\log \frac{t_n}{s} \right)^{-\alpha} \frac{ds}{s} \\
=& \frac{1}{h\Gamma(1-\alpha)} \sum_{i=1}^{n} t_i \left(u_i - u_{i-1}\right) \left[-\frac{\left(\log \frac{t_n}{s} \right)^{1-\alpha}}{(1-\alpha)} \right]_{t_{i-1}}^{t_i} \\
=& \frac{1}{h\Gamma(2-\alpha)} \sum_{i=1}^{n} t_i \left[\left(\log \frac{t_n}{t_{i-1}} \right)^{1-\alpha} - \left(\log \frac{t_n}{t_i} \right)^{1-\alpha} \right] \left(u_i - u_{i-1}\right) \\
=& \frac{1}{h\Gamma(2-\alpha)} \sum_{i=1}^{n} b_i \left(u_i - u_{i-1}\right) \\
=& {}^{CH}\mathcal{D}^{\alpha}_{a^+} u_n .
\end{aligned}$$

Set $E_n = \left| {}^{CH}\mathcal{D}^{\alpha}_{a^+} u\left(t_n\right) - {}^{CH}\mathcal{D}^{\alpha}_{a^+} u_n \right|$ and $M_i = max \left| u^{(i)}\left(t\right) \right|$, $i = 1, 2,$ hence we can obtain

$$E_n \leq \frac{1}{\Gamma(1-\alpha)} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_n}{s} \right)^{-\alpha} \left| s \frac{du}{ds} - t_i \left(\frac{u_i - u_{i-1}}{t_i - t_{i-1}} \right) \right| \frac{ds}{s}.$$

It follows from Taylor's theorem, one has for each $i \in \{0, 1, \ldots, N\}$, with $s \in [t_{i-1}, t_i]$ and

$\eta_1 \in [t_{i-1}, t_i],\ \eta_2 \in [t_{i-1}, s]$

$$\begin{aligned}
\left| s\frac{du}{ds} - t_i\left(\frac{u_i - u_{i-1}}{t_i - t_{i-1}}\right)\right| &= \left| s\frac{du}{ds} - t_i\left(\frac{du\left(t_{i-1}\right)}{ds} - \frac{d^{(2)}u\left(\eta_1\right)}{ds^2}\frac{h}{2!}\right)\right| \\
&\leq \left|\left(s\frac{du}{ds} - t_i\frac{du\left(t_{i-1}\right)}{ds}\right)\right| + M_2\frac{t_i h}{2} \\
&= \left| s\left(\frac{du\left(t_{i-1}\right)}{ds} - t_i\frac{du\left(t_{i-1}\right)}{ds} + \frac{d^{(2)}u\left(\eta_1\right)}{ds^2}\left(s - t_{i-1}\right)\right)\right| + M_2\frac{t_i h}{2} \\
&\leq M_1\left(t_i - t_{i-1}\right) + M_2 t_i\frac{3}{2}h \\
&\leq \left(M_1 + \frac{3T}{2}M_2\right)h.
\end{aligned}$$

Furthermore, for any $0 < \alpha \leq 1$ and $n \in \{0, 1, \ldots, N\}$ with $i \leq n$ and $s \in [t_{i-1}, t_i]$

$$0 \leq \left(\log\frac{t_n}{s}\right)^{-\alpha} \leq \left(\log\frac{t_i}{s}\right)^{-\alpha},$$

therefore, we conclude

$$\begin{aligned}
E_n \leq& \frac{1}{\Gamma(1-\alpha)}\left(M_1 + \frac{3T}{2}M_2\right)h\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_n}{s}\right)^{-\alpha}\frac{ds}{s} \\
\leq& \frac{1}{\Gamma(1-\alpha)}\left(M_1 + \frac{3T}{2}M_2\right)h\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{-\alpha}\frac{ds}{s} \\
\leq& \frac{1}{\Gamma(2-\alpha)}\left(M_1 + \frac{3T}{2}M_2\right)h\sum_{i=1}^{n}\left(\log\frac{t_i}{t_{i-1}}\right)^{1-\alpha} \\
\leq& \frac{1}{\Gamma(2-\alpha)}\left(M_1 + \frac{3T}{2}M_2\right)h\sum_{i=1}^{n}\left(\frac{t_i}{t_{i-1}}\right)^{1-\alpha} \\
\leq& \frac{1}{\Gamma(2-\alpha)}\left(M_1 + \frac{3T}{2}M_2\right)h\sum_{i=1}^{N}h^{1-\alpha}T^{1-\alpha} \\
\leq& \frac{1}{\Gamma(2-\alpha)}\left(M_1 + \frac{3T}{2}M_2\right)T^{1-\alpha}h^{1-\alpha}\sum_{i=1}^{N}h \\
\leq& \frac{1}{\Gamma(2-\alpha)}\left(M_1 + \frac{3T}{2}M_2\right)(T-a)\,T^{1-\alpha}h^{1-\alpha},
\end{aligned}$$

which means

$${}^{CH}\mathcal{D}_{a^+}^{\alpha}u\left(t_n\right) = {}^{CH}\mathcal{D}_{a^+}^{\alpha}u_n + O\left(h^{1-\alpha}\right).$$

□

Example 2.2. Consider the function $u(t) = \log\left(\frac{t}{3}\right)$, for $t \in [1, 2]$, $0 < \alpha \leq 1$ and from (1.24) we have

$$^{CH}\mathcal{D}_{1+}^{\alpha}u\left(t\right) = \frac{1}{\Gamma(2-\alpha)}\left(\log(t)\right)^{1-\alpha}.$$

Figure (2.3) represent the comparison between the analytical Caputo–Hadamard fractional derivative and its approximation for different values of h.

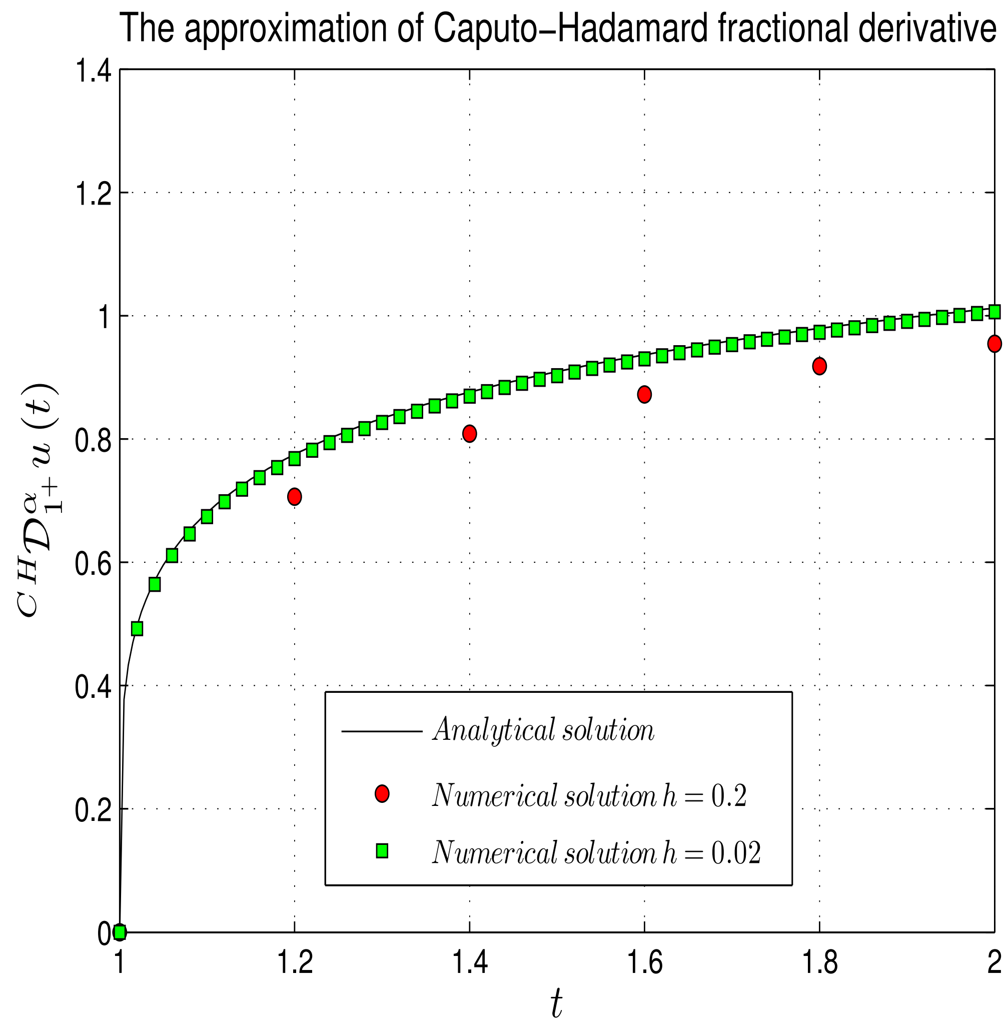

FIGURE 2.3 – Graphical comparison between the analytical Caputo–Hadamard fractional derivative and its approximation with $\alpha = 0.8$, $h = 0.2$ and $h = 0.02$.

Theorem 2.2. *Let $u : [a, T] \to \mathbb{R}$, such that $u \in C^2\left([a, T] : \mathbb{R}\right)$, $1 < \alpha \leq 2$, the approximation of the Caputo–Hadamard fractional derivative $^{CH}\mathcal{D}_{a+}^{\alpha}u$ at a point t_{n+1} is given by the following scheme :*

$$^{CH}\mathcal{D}_{a+}^{\alpha}u_{n+1} = \frac{1}{h^2\Gamma\left(3-\alpha\right)}\sum_{j=0}^{n} b_j\left(t_{j+1}u_{j+1} - \left(t_{j+1}+t_j\right)u_j + t_j u_{j-1}\right), \quad (2.13)$$

where

$$b_j = t_j \left[\left(\log \frac{t_{n+1}}{t_j} \right)^{2-\alpha} - \left(\log \frac{t_{n+1}}{t_{j+1}} \right)^{2-\alpha} \right], \; j = 0, ..., n, \tag{2.14}$$

and

$${}^{CH}\mathcal{D}^{\alpha}_{a^+} u\left(t_{n+1}\right) = {}^{CH}\mathcal{D}^{\alpha}_{a^+} u_{n+1} + C_{2,\alpha} h^{2-\alpha}. \tag{2.15}$$

Proof. The fractional derivative term ${}^{CH}\mathcal{D}^{\alpha}_{a^+} u\left(t\right)$ can be approximated by the following scheme

$$\begin{aligned}
{}^{CH}\mathcal{D}^{\alpha}_{a^+} u\left(t_{n+1}\right) &= \frac{1}{\Gamma(2-\alpha)} \int_{t_0}^{t_{n+1}} \left(\log \frac{t_{n+1}}{s} \right)^{1-\alpha} \left(s \frac{d}{ds} \right)^2 u(x,s) \frac{ds}{s} \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \left(\log \frac{t_{n+1}}{s} \right)^{1-\alpha} \left(s \frac{du(x,s)}{ds} + s^2 \frac{d^2u(x,s)}{ds^2} \right) \frac{ds}{s} \\
&\approx \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{n} \left(t_j \frac{u_{j+1}-u_j}{h} + t_j^2 \frac{u_{j+1}-2u_j+u_{j-1}}{h^2} \right) \left[- \frac{\left(\log \frac{t_{n+1}}{s} \right)^{2-\alpha}}{(2-\alpha)} \right]_{t_j}^{t_{j+1}} \\
&= \frac{1}{h^2 \Gamma(3-\alpha)} \sum_{j=0}^{n} t_j \left[\left(\log \frac{t_{n+1}}{t_j} \right)^{2-\alpha} - \left(\log \frac{t_{n+1}}{t_{j+1}} \right)^{2-\alpha} \right] \\
&\quad \times \left(t_{j+1} u_{j+1} - (t_j + t_{j+1}) u_j + t_j u_{j-1} \right).
\end{aligned}$$

Set $E_n = \left| {}^{CH}\mathcal{D}^{\alpha}_{a^+} u\left(t_{n+1}\right) - {}^{CH}\mathcal{D}^{\alpha}_{a^+} u_{n+1} \right|$ and $M_i = \max\limits_{t \in [t_0, T]} \left| \frac{d^i u(x,t)}{dt^i} \right|, \; i = 1, ...4.$
Then we have

$$\begin{aligned}
E_n \leqslant & \frac{1}{\Gamma\left(2-\alpha\right)} \sum_{j=0}^{n} \int\limits_{t_j}^{t_{j+1}} \left(\log \frac{t_{n+1}}{s} \right)^{1-\alpha} \\
& \times \left| s \frac{du(t_j)}{ds} + s^2 \frac{d^2u(t_j)}{ds^2} - \left(t_j \frac{u_{j+1}-u_j}{h} + t_j^2 \frac{u_{j+1}-2u_j+u_{j-1}}{h^2} \right) \right| \frac{ds}{s}.
\end{aligned}$$

It follows from Taylor's theorem, one has for each $j \in \{0, 1, ..., n\}$ and $s \in [t_j, t_{j+1}]$

$$\begin{aligned}
&\left| s\frac{du(t_j)}{ds} + s^2\frac{d^2u(t_j)}{ds^2} - \left(t_j\frac{u_{j+1} - u_j}{h} + t_j^2\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} \right) \right| \\
&\leqslant \left| \begin{array}{l} s\left(\frac{du(t_j)}{ds} + \frac{d^2u(t_j)}{ds^2}(s - t_j) + \frac{d^3u(t_j)}{ds^3}\frac{(s - t_j)^2}{2!} + \frac{d^4u(\eta_2)}{ds^4}\frac{(s - t_j)^3}{3!} \right) \\ - t_j\left(\frac{du(t_j)}{dt} + \frac{d^{(2)}u(t_j)}{dt^2}\frac{h}{2!} + \frac{d^{(3)}u(t_j)}{dt^3}\frac{h^2}{3!} + \frac{d^{(4)}u(x_i, \eta_1)}{dt^4}\frac{h^3}{4!} \right) \end{array} \right| \\
&+ \left| \begin{array}{l} s^2\left(\frac{d^2u(t_j)}{ds^2} + \frac{d^3u(t_j)}{ds^3}(s - t_j) + \frac{d^4u(\eta_2)}{ds^4}\frac{(s - t_j)^2}{2!} \right) \\ - t_j^2\left(\frac{d^2u(x_i, t_j)}{\partial t^2} + \frac{d^{(4)}u(\eta_1)}{dt^4}\frac{h^2}{12} \right) \end{array} \right| \\
&\leqslant (s - t_j)\, M_1 + \left(s\,(s - t_j) - t_j\frac{h}{2!} \right) M_2 + \left(s\frac{(s - t_j)^2}{2!} - t_j\frac{h^2}{3!} \right) M_3 + s\frac{(s - t_j)^3}{6}M_4 \\
&+ t_j\frac{h^3}{4!}M_4 + \left(s^2 - t_j^2 \right) M_2 + s^2(s - t_j)M_3 + s^2M_4\frac{(s - t_j)^2}{2} + t_j^2M_4\frac{h^2}{12} \\
&\leqslant TM_1 + \frac{3}{2}T^2M_2 + \frac{4}{3}T^3M_3 + \frac{19}{24}T^4M_4,
\end{aligned}$$

where $\eta_2 \in [t_j, s]$, and $\eta_1 \in [t_j, t_{j+1}]$. Therefore, we conclude

$$\begin{aligned}
E_n &\leqslant \frac{1}{\Gamma(2 - \alpha)} \left(TM_1 + \frac{3}{2}T^2M_2 + \frac{4}{3}T^3M_3 + \frac{19}{24}T^4M_4 \right) \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \left(\log\frac{t_{n+1}}{s} \right)^{1-\alpha} \frac{ds}{s} \\
&\leqslant \frac{1}{\Gamma(2 - \alpha)} \left(TM_1 + \frac{3}{2}T^2M_2 + \frac{4}{3}T^3M_3 + \frac{19}{24}T^4M_4 \right) \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \left(\log\frac{t_{j+1}}{s} \right)^{1-\alpha} \frac{ds}{s} \\
&\leqslant \frac{1}{\Gamma(3 - \alpha)} \left(TM_1 + \frac{3}{2}T^2M_2 + \frac{4}{3}T^3M_3 + \frac{19}{24}T^4M_4 \right) \sum_{j=0}^{n} \left(\log\frac{t_{j+1}}{t_j} \right)^{2-\alpha} \\
&\leqslant \frac{1}{\Gamma(3 - \alpha)} \left(TM_1 + \frac{3}{2}T^2M_2 + \frac{4}{3}T^3M_3 + \frac{19}{24}T^4M_4 \right) \sum_{j=0}^{N} T^{2-\alpha}h^{2-\alpha} \\
&\leqslant \frac{TM_1 + \frac{3}{2}T^2M_2 + \frac{4}{3}T^3M_3 + \frac{19}{24}T^4M_4}{\Gamma(3 - \alpha)} T^{2-\alpha}\,(T - a)\, h^{2-\alpha}.
\end{aligned}$$

Hence

$$^{CH}\mathcal{D}^{\alpha}_{a^+}u\,(t_{n+1}) = {}^{CH}\mathcal{D}^{\alpha}_{a^+}u_{n+1} + C_{2,\alpha}h^{2-\alpha}.$$

□

Example 2.3. Consider the function $u(t) = \left(\log\left(\frac{\sqrt{2}}{t} \right) \right)^2$, for $t \in [1, 2]$, $1 < \alpha \leq 2$ and from (1.24) we have

$$^{CH}\mathcal{D}_{1^+}^{\alpha} u\,(t) = \frac{2}{\Gamma(3-\alpha)}\,(\log(t))^{2-\alpha}\,.$$

Figure (2.4)) represent the comparison between the analytical Caputo–Hadamard fractional derivative and its approximation for different values of h.

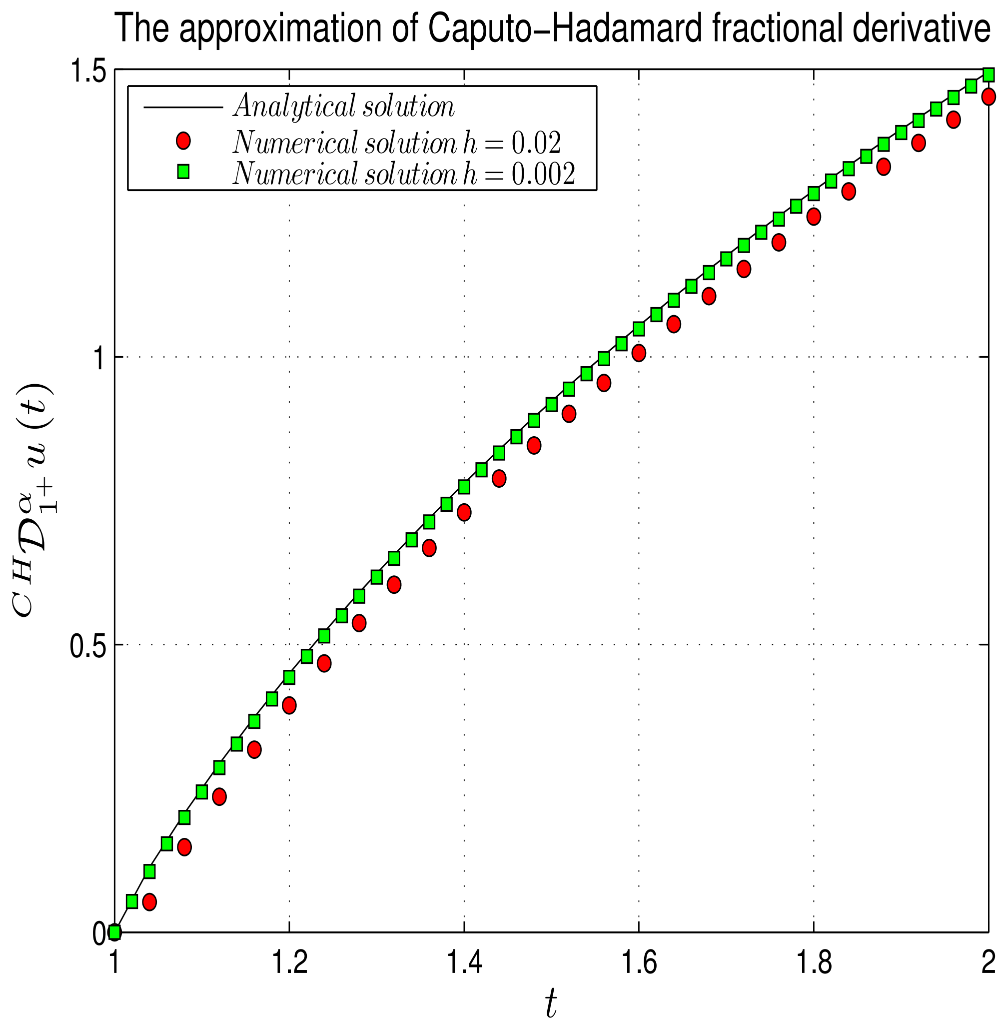

FIGURE 2.4 – Graphical comparison between the analytical Caputo–Hadamard fractional derivative and its approximation with $\alpha = 1.1$, $h = 0.02$ and $h = 0.002$.

2.3 Approximation of Caputo–Katugampola fractional derivatives

In this section, we pay our attention on a numerical approach to generalized Caputo–Katugampola fractional differential equation. In the sequel, for any interval $[a, T]$ with

$a > 0$ and $\rho > 0$, from [49] we denote the step as follows :

$$h = \frac{T^\rho - a^\rho}{N},$$

where $N > 0$ is a given positive integer. In addition, we inherit the following time grid by

$$a = \mathrm{t}_0 < \mathrm{t}_1 < \mathrm{t}_2 < \cdots < \mathrm{t}_{N-1} < \mathrm{t}_N = T,$$

here t_n is defined as

$$\mathrm{t}_n = (a^\rho + nh)^{\frac{1}{\rho}} = \left(a^\rho + \frac{n(T^\rho - a^\rho)}{N}\right)^{\frac{1}{\rho}}, n \in \{0, 1, \ \ldots, \ N\}. \tag{2.16}$$

Remark 2.1. In fact, if $\rho = 1$, then (2.16) reduces to the classical equidistant partition for $[a, T]$

$$\mathrm{t}_n := a + nh = a + \frac{n(T-a)}{N}, \ n \ \in \{0, 1, \ \ldots, \ N\}.$$

However, when $\rho \neq 1$, we readily see that the partition of $[a, \ b]$ is non-equidistance, thus $t_{\mathrm{k}} - t_{\mathrm{k}-1} \neq h$.

Theorem 2.3. *[49]Let $u : [a, T] \to \mathbb{R}$ such that $u \in C^2([a, T] : \mathbb{R})$, $0 < \alpha \leq 1$ and $\rho > 1$, the approximation of the Caputo-Katugampola fractional derivative ${}^C\mathcal{D}_{a^+}^{\alpha,\rho}u$ at a point t_{n+1} is given by the following scheme*

$${}^C\mathcal{D}_{a^+}^{\alpha,\rho}u_{n+1} = \frac{h^{1-\alpha}\rho^{\alpha-1}}{\Gamma(2-\alpha)} \sum_{j=0}^{n} b_j^{\alpha,\rho}(u_{j+1} - u_j), \tag{2.17}$$

where

$$b_j^{\alpha,\rho} = \frac{t_j^{(1-\rho)}}{t_{j+1} - t_j}\left((n-j+1)^{1-\alpha} - (n-j)^{1-\alpha}\right), j = 0, ..., n, \tag{2.18}$$

and

$${}^C\mathcal{D}_{a^+}^{\alpha,\rho}u(t_{n+1}) = {}^C\mathcal{D}_{a^+}^{\alpha,\rho}u_n + c_{\alpha,\rho}h^{1-\alpha}.$$

Example 2.4. Consider the function $u(t) = t^\rho$, for $t \in [1, 2]$, $0 < \alpha \leq 1$ and from 1.36 we have

$${}^C\mathcal{D}_{1^+}^{\alpha,\rho}u(t) = \frac{\rho^\alpha}{\Gamma(2-\alpha)}(t^\rho - 1)^{(1-\alpha)},$$

Figure (2.5) represent the comparison between the analytical Caputo–Katugampola fractional derivative and its approximation for different values of h.

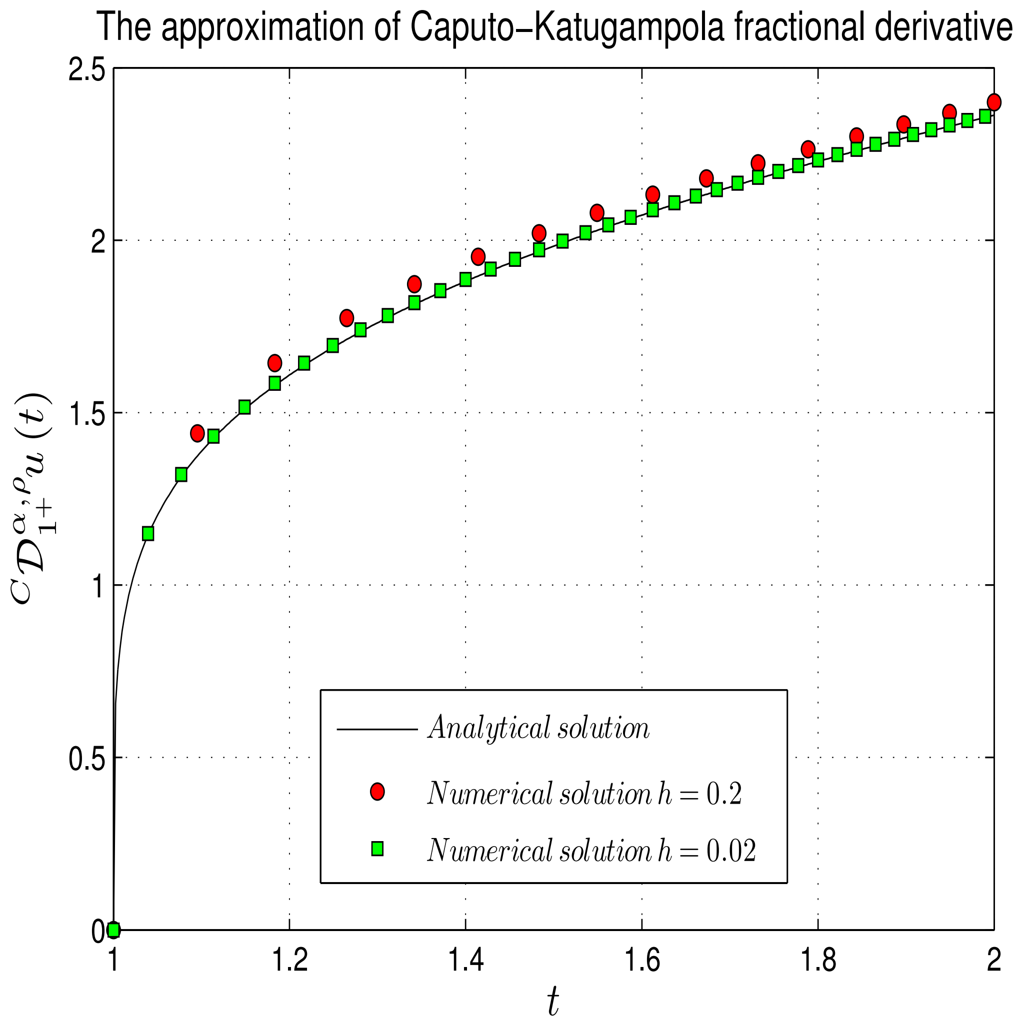

FIGURE 2.5 – Graphical comparison between the analytical Caputo–Katugampola fractional derivative and its approximation with $\alpha = 0.8$, $\rho = 2$, $h = 0.2$ and $h = 0.02$.

Theorem 2.4. *Let* $u : [a, T] \to \mathbb{R}$ *such that* $u \in C^4([a, T] : \mathbb{R})$, $\rho > 1$ *and* $\alpha \in (1, 2\,]$. *Then for each positive integer* $N \in \mathbb{N}$, *we have for each* $n \in \{0, 1, \ldots, N-1\}$, *the approximation of the Caputo-Katugampola fractional derivative is given by the following scheme*

$$ {}^{C}\mathcal{D}_{a^+}^{\alpha,\rho} u\left(t_{n+1}\right) = {}^{C}\, \mathcal{D}_{a^+}^{\alpha,\rho} u_{n+1} + O(h^{2-\alpha}), \tag{2.19}$$

where ${}^{C}\mathcal{D}_{a^+}^{\alpha,\rho} u_{n+1}$ *is defined as follows :*

$$ {}^{C}\mathcal{D}_{a^+}^{\alpha,\rho} u_{n+1} = \frac{h^{2-\alpha}\rho^{\alpha-2}}{\Gamma\left(3-\alpha\right)} \sum_{j=0}^{n} b_j^{\alpha,\rho}\left(u_{j+1} - 2u_j + u_{j-1}\right), \tag{2.20}$$

and

$$b_j^{\alpha,\rho} = \frac{t_j^{2(1-\rho)}}{(t_{j+1}-t_j)^2}\left((n-j+1)^{2-\alpha} - (n-j)^{2-\alpha}\right), j = 0, ..., n, \tag{2.21}$$

where $\{\mathrm{t_j}\}_{\mathrm{j=0}}^{n}$ *is given in (2.16).*

Proof. For any $N \in \mathbb{N},\ \rho > 1$ and $\alpha \in (1, 2\ )$ fixed, we can get the sequence $\{t_n\}_{n=0}^{N-1}$ (see 2.16). Then, we can calculate for each $0 \leq n \leq N-1$

$$\begin{aligned}
{}^{C}\mathcal{D}_{a^+}^{\alpha,\rho} u\left(t_{n+1}\right) &= \frac{\rho^{\alpha-1}}{\Gamma\left(2-\alpha\right)} \int_{t_0}^{t_{n+1}} \frac{s^{\rho-1}}{\left(t_{n+1}^{\rho} - s^{\rho}\right)^{\alpha-1}} s^{2(1-\rho)} \frac{d^2 u(x,s)}{ds^2} ds \\
&\approx \frac{\rho^{\alpha-1}}{\Gamma\left(2-\alpha\right)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \frac{s^{\rho-1}}{\left(t_{n+1}^{\rho} - s^{\rho}\right)^{\alpha-1}} t_j^{2(1-\rho)} \frac{u_{j+1} - 2u_j + u_{j-1}}{(t_{j+1}-t_j)^2} ds \\
&= \frac{\rho^{\alpha-1}}{\Gamma\left(2-\alpha\right)} \sum_{j=0}^{n} t_j^{2(1-\rho)} \frac{u_{j+1} - 2u_j + u_{j-1}}{(t_{j+1}-t_j)^2} \int_{t_j}^{t_{j+1}} \frac{s^{\rho-1}}{\left(t_{n+1}^{\rho} - s^{\rho}\right)^{\alpha-1}} ds \\
&= \frac{\rho^{\alpha-1}}{\Gamma\left(2-\alpha\right)} \sum_{j=0}^{n} t_j^{2(1-\rho)} \frac{u_{j+1} - 2u_j + u_{j-1}}{(t_{j+1}-t_j)^2} \left[\frac{-\left(t_{n+1}^{\rho} - s^{\rho}\right)^{(2-\alpha)}}{\rho(2-\alpha)}\right]_{t_j}^{t_{j+1}} \\
&= \frac{h^{2-\alpha}\rho^{\alpha-2}}{\Gamma\left(3-\alpha\right)} \sum_{j=0}^{n} b_j^{\alpha,\rho}\left(u_{j+1} - 2u_j + u_{j-1}\right) \\
&= {}^{C}\mathcal{D}_{a^+}^{\alpha,\rho} u_{n+1}\ ,
\end{aligned}$$

where

$$b_j^{\alpha,\rho} = \frac{t_j^{2(1-\rho)}}{(t_{j+1}-t_j)^2}\left((n-j+1)^{2-\alpha} - (n-j)^{2-\alpha}\right), \text{ for } j = 0, ..., n. \tag{2.22}$$

Then we have $E_{n+1} = \left| {}^{C}\mathcal{D}_{a^+}^{\alpha,\rho} u\left(t_{n+1}\right) - {}^{C}\mathcal{D}_{a^+}^{\alpha,\rho} u_{n+1} \right|$ and $M_i = \max_{t\in[t_0,T]} \left|\frac{d^i u}{dt^i}\right|,\ i = 1, ...4,$ hence, we can obtain

$$E_{n+1}^{\alpha,\rho} \leqslant \frac{\rho^{\alpha-1}}{\Gamma\left(2-\alpha\right)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \frac{s^{\rho-1}}{\left(t_{n+1}^{\rho} - s^{\rho}\right)^{\alpha-1}} \left| s^{2(1-\rho)} \frac{d^2 u(x_i,s)}{ds^2} - t_j^{2(1-\rho)} \frac{u_{j+1} - 2u_j + u_{j-1}}{(t_{j+1}-t_j)^2} \right| ds.$$

It follows from Taylor's theorem, one has for each $j \in \{0, 1, ..., N\}$ and $s \in [t_j, t_{j+1}]$ and because $\rho > 1$, we have

$$
\begin{aligned}
&\left|s^{2(1-\rho)}\frac{d^2u}{ds^2}-t_j^{2(1-\rho)}\frac{u_{j+1}-2u_j+u_{j-1}}{(t_{j+1}-t_j)^2}\right| \\
&=\left|s^{2(1-\rho)}\frac{d^2u}{ds^2}-t_j^{2(1-\rho)}\left(\frac{d^2u}{dt^2}(t_j)+\frac{d^{(4)}u}{\partial t^4}(\eta_1)\frac{(t_{j+1}-t_j)^2}{12}\right)\right| \\
&\leqslant\left|s^{2(1-\rho)}\left(\frac{d^2u}{ds^2}(t_j)+\frac{d^3u}{ds^3}(t_j)(s-t_j)+\frac{d^{(4)}u}{ds^4}(\eta_2)\frac{(s-t_j)^2}{2}-t_j^{2(1-\rho)}\frac{d^2u}{dt^2}(t_j)\right)\right| \\
&+t_j^{2(1-\rho)}\frac{(t_{j+1}-t_j)^2}{12}M_4 \\
&\leqslant t_j^{2(1-\rho)}M_2+t_j^{2(1-\rho)}(t_{j+1}-t_j)M_3+t_j^{2(1-\rho)}(t_{j+1}-t_j)^2\frac{M_4}{2}+t_j^{2(1-\rho)}\frac{(t_{j+1}-t_j)^2}{12}M_4,
\end{aligned}
$$

finally, we conclude

$$
\left|s^{2(1-\rho)}\frac{d^2u}{ds^2}-t_j^{2(1-\rho)}\frac{u_{j+1}-2u_j+u_{j-1}}{(t_{j+1}-t_j)^2}\right|\leqslant a^{2(1-\rho)}\left(M_2+M_3T+\frac{7}{12}T^2M_4\right), \tag{2.23}
$$

where $\eta_2\in[t_j,s]$ and $\eta_1\in[t_j,t_{j+1}]$.
Furthermore, for any $\alpha\in(1,2)$ and $j,n\in\{0,1,...,N-1\}$, $\rho>1$, with $j<n$ and $s\in[t_j,t_{j+1}]$ we have

$$
0\leqslant\left(t_{n+1}^{\rho}-s^{\rho}\right)^{1-\alpha}\leqslant\left(t_{j+1}^{\rho}-s^{\rho}\right)^{1-\alpha},
$$

therefore, one yields

$$
\begin{aligned}
0&\leqslant\int_{t_j}^{t_{j+1}}\frac{s^{\rho-1}}{\left(t_{n+1}^{\rho}-s^{\rho}\right)^{\alpha-1}}ds \\
&\leqslant\int_{t_j}^{t_{j+1}}\frac{s^{\rho-1}}{\left(t_{j+1}^{\rho}-s^{\rho}\right)^{\alpha-1}}ds \\
&=\frac{\left(t_{j+1}^{\rho}-t_j^{\rho}\right)^{2-\alpha}}{\rho\,(2-\alpha)}.
\end{aligned} \tag{2.24}
$$

According to (2.23) and (2.24), for each $n\in\{0,1,...,N-1\}$, we imply

$$
\begin{aligned}
E_n^{\alpha,\rho}&\leqslant\frac{\rho^{\alpha-2}}{\Gamma(3-\alpha)}a^{2(1-\rho)}\left(M_2+M_3T+\frac{7}{12}T^2M_4\right)\sum_{j=0}^{N}\left(t_{j+1}^{\rho}-t_j^{\rho}\right)^{2-\alpha} \\
&\leqslant\frac{\rho^{\alpha-2}}{\Gamma(3-\alpha)}a^{2(1-\rho)}\left(M_2+M_3T+\frac{7}{12}T^2M_4\right)(T^{\rho}-a^{\rho})h^{2-\alpha}.
\end{aligned}
$$

This means that

$$
{}^{C}\mathcal{D}_{a^+}^{\alpha,\rho}u\left(t_{n+1}\right)={}^{C}\,\mathcal{D}_{a^+}^{\alpha,\rho}u_{n+1}+C_2^{\alpha,\rho}h^{2-\alpha}. \tag{2.25}
$$

□

Example 2.5. Consider the function $u(t)=\dfrac{t^{2\rho}-4t}{2\rho}$, for $t\in[1,2]$, $1<\alpha\leq 2$ and from 1.36 we have

$$^{C}\mathcal{D}_{1^+}^{\alpha,\rho}u\,(t) = \frac{(2\rho-1)\rho^{(\alpha-2)}}{\Gamma(3-\alpha)}(t^{\rho}-1)^{2-\alpha}.$$

Figure (2.6) represent the comparison between the analytical Caputo–Katugampola fractional derivative and its approximation for different values of h.

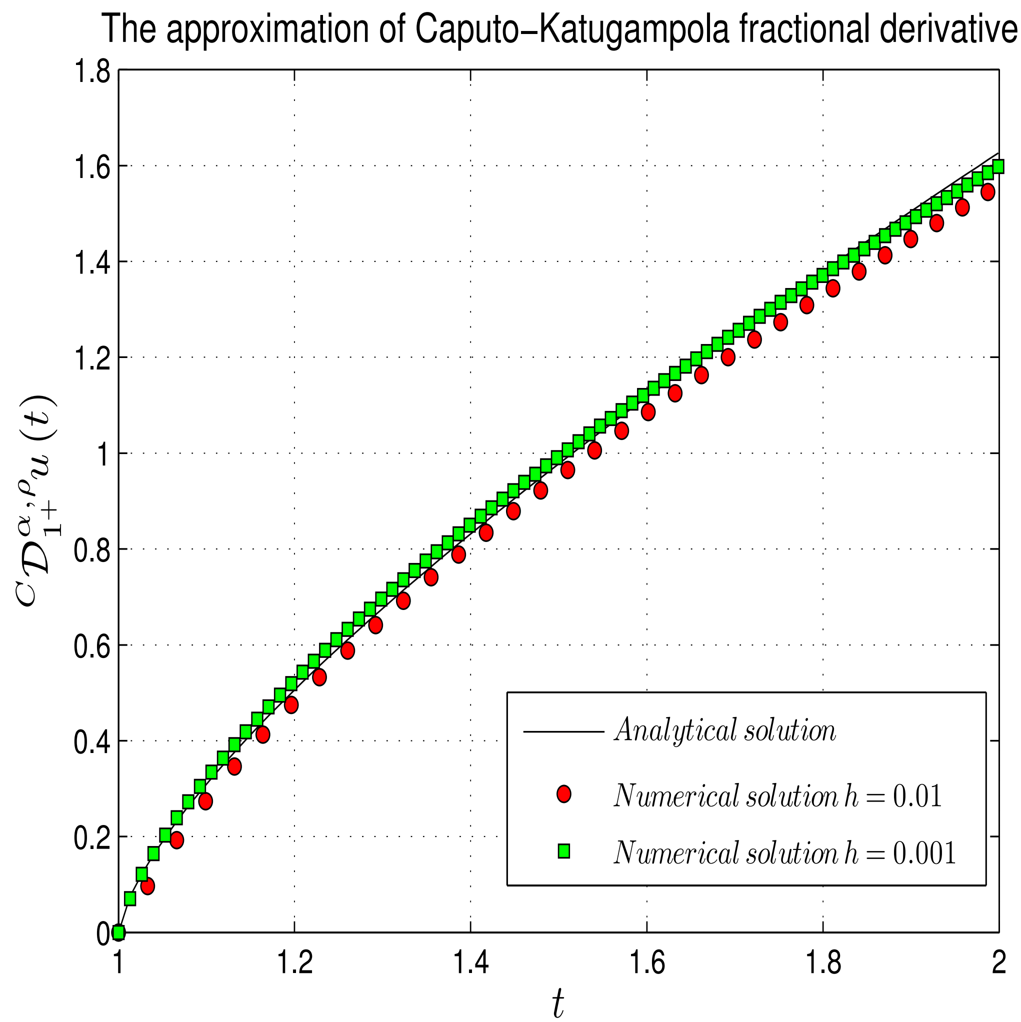

FIGURE 2.6 – Graphical comparison between the analytical Caputo–Katugampola fractional derivative and its approximation with $\alpha = 1.3$, $\rho = 1.2$, $h = 0.01$ and $h = 0.001$.

2.4 Approximation of Riesz–Caputo–Katugampola fractional derivative

This result is found and obtained from [11].

Theorem 2.5. *$u:[a,T]\rightarrow\mathbb{R}$ such that $u\in C^{4}\left([a,T]:\mathbb{R}\right)$, $\rho>1$ and $\alpha\in(1,2\,]$, we have the approximation of the Riesz–Caputo–Katugampola fractional derivative ${}_{a}^{RC}\mathcal{D}_{b}^{\alpha,\rho}u\left(t\right)$ is given by the following scheme*

$$
{}_{a}^{RC}\mathcal{D}_{T}^{\alpha,\rho}u_{m}=\frac{h^{2-\alpha}\rho^{\alpha-2}}{2\Gamma\left(3-\alpha\right)}\left(\sum_{i=0}^{m-1}a_{i,m}^{\alpha,\rho}\left(u_{i+1}-2u_{i}+u_{i-1}\right)+\sum_{i=m}^{N-1}z_{i,m}^{\alpha,\rho}\left(u_{i+1}-2u_{i}+u_{i-1}\right)\right),\tag{2.26}
$$

for $m\in\{1,2,\ldots,N-1\}$, where

$$
\begin{cases}
a_{i,m}^{\alpha,\rho}=\dfrac{t_{i}^{2(1-\rho)}}{(t_{i+1}-t_{i})^{2}}\left[(m-i)^{2-\alpha}-(m-i-1)^{2-\alpha}\right],\ i=0,\ldots,m-1,\\
z_{i,m}^{\alpha,\rho}=\dfrac{t_{i}^{2(1-\rho)}}{(t_{i+1}-t_{i})^{2}}\left[(i+1-m)^{2-\alpha}-(i-m)^{2-\alpha}\right],\ i=m,\ldots,N-1,
\end{cases}\tag{2.27}
$$

and

$$
{}_{a}^{RC}\mathcal{D}_{T}^{\alpha,\rho}u\left(t_{m}\right)={}_{a}^{RC}\mathcal{D}_{T}^{\alpha,\rho}u_{m}+c_{\alpha,\rho}k^{2-\alpha}.
$$

Proof. The fractional derivative term ${}_{a}^{RC}\mathcal{D}_{T}^{\alpha,\rho}u\left(t_{m}\right)$ can be approximated by the following

$$
{}_{a}^{RC}\mathcal{D}_{T}^{\alpha,\rho}u_{m}=\frac{1}{2}\left({}_{a}^{C}\mathcal{D}_{t_{m}}^{\alpha,\rho}\,u_{m}+{}_{t_{m}}^{C}\mathcal{D}_{t_{N}}^{\alpha,\rho}\,u_{m}\right),
$$

where

$$
\begin{aligned}
{}_{a}^{C}\mathcal{D}_{t_{m}}^{\alpha,\rho}\,u(t_{m})&=\frac{\rho^{\alpha-1}}{\Gamma\left(2-\alpha\right)}\int_{a}^{t_{m}}\frac{s^{\rho-1}}{\left(t_{m}^{\rho}-s^{\rho}\right)^{\alpha-1}}s^{2(1-\rho)}\frac{d^{2}u(s)}{ds^{2}}ds\\
&\approx\frac{\rho^{\alpha-1}}{\Gamma\left(2-\alpha\right)}\sum_{i=0}^{m-1}\int_{t_{i}}^{t_{i+1}}\frac{s^{\rho-1}}{\left(t_{m}^{\rho}-s^{\rho}\right)^{\alpha-1}}t_{i}^{2(1-\rho)}\frac{u_{i+1}-2u_{i}+u_{i-1}}{(t_{i+1}-t_{i})^{2}}ds\\
&=\frac{h^{2-\alpha}\rho^{\alpha-2}}{\Gamma\left(3-\alpha\right)}\sum_{i=0}^{m-1}t_{i}^{2(1-\rho)}\frac{u_{i+1}-2u_{i}+u_{i-1}}{(t_{i+1}-t_{i})^{2}}\left[(m-i)^{2-\alpha}-(m-i-1)^{2-\alpha}\right],
\end{aligned}
$$

with $m\in\{1,2,\ldots,N-1\}$.

Set $E_m = \left| {}_a^C\mathcal{D}_{t_m}^{\alpha,\rho} u(t_m) - {}_a^C\mathcal{D}_{t_m}^{\alpha,\rho} u_m \right|$ and $M_i = \max\limits_{x\in[x_0,L]} \left| \dfrac{d^i u}{dx^i} \right|$, with $i = 1, ...4$, hence, we obtain

$$E_m \leq \frac{\rho^{\alpha-1}}{\Gamma(2-\alpha)} \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \frac{s^{\rho-1}}{(t_m^\rho - s^\rho)^{\alpha-1}} \left| s^{2(1-\rho)} \frac{d^2u}{ds^2} - t_i^{2(1-\rho)} \frac{u_{i+1} - 2u_i + u_{i-1}}{(t_{i+1} - t_i)^2} \right| ds.$$

One has for each $i \in \{0, 1, ..., m-1\}$ and because $\rho > 1$, according to (2.23) and (2.24), we imply

$$E_m \leq \frac{a^{2(1-\rho)}\rho^{\alpha-2}}{\Gamma(3-\alpha)} \left(M_2 + t_m M_3 + \frac{7}{12} t_m^2 M_4 \right) (t_m^\rho - a^\rho)\, h^{2-\alpha}.$$

This means that

$${}_a^C\mathcal{D}_{t_m}^{\alpha,\rho} u_m = {}_a^C\mathcal{D}_{t_m}^{\alpha,\rho} u_m + c_{m,\alpha,\rho} h^{2-\alpha}.$$

Similarly, we find

$${}_{t_m}^{C}\mathcal{D}_{t_N}^{\alpha,\rho} u_m = \frac{h^{2-\alpha}\rho^{\alpha-2}}{\Gamma(3-\alpha)} \sum_{i=m}^{N-1} \frac{t_i^{2(1-\rho)}}{(t_{i+1} - t_i)^2} (u_{i+1} - 2u_i + u_{i-1}) \left[(i+1-m)^{2-\alpha} - (i-m)^{2-\alpha} \right],$$

and

$${}_{t_m}^{C}\mathcal{D}_{t_N}^{\alpha,\rho} u(t_m) = {}_{t_m}^{C}\mathcal{D}_{t_N}^{\alpha,\rho} u_m + c_{N,\alpha,\rho} h^{2-\alpha}.$$

□

CHAPTER 3

FDM FOR SOLVING FDE AND FPDE INVOLVING CAPUTO-HADAMARD DERIVATIVE

This chapter is divided into two sections. In the first section, we investigate the finite difference methods for linear fractional differential equation. In the second section, we construct the time fractional diffusion-wave equation involving the Caputo-Hadamard of order $1 < \alpha \leq 2$.

3.1 Numerical solutions for linear FDE with dependence on the Caputo–Hadamard derivative using FDM

The content of this section has been accepted in "Palestine Journal of Mathematics."
The main objective of this section is to find accurate solutions for linear fractional differential equations involving Caputo-Hadamard fractional derivative of order $\alpha > 0$. Therefore, to achieve this objective, a new method called Finite Fractional Difference Method (FFDM) is employed to find the numerical solution. As such, the convergence and stability of the numerical scheme is discussed and illustrated by solving two linear fractional differential equation problems of order $0 < \alpha \leqslant 1$ to show the validity of our method.
This section concerns the numerical solution for fractional differential equation of Caputo-Hadamard type given by :

$$\begin{cases} {}^{CH}\mathcal{D}^{\alpha}_{a^+} u\,(t) + c(t)u(t) = f(t),\; 0 < a \leq t \leq b < \infty, \\ u(a) = u_0, \end{cases} \tag{3.1}$$

Where ${}^{CH}\mathcal{D}^{\alpha}$ denotes the Caputo-Hadamard fractional derivative operator of order α between zero and one ([4]- [24]).

3.1.1 The finite difference scheme

Now, by using the fractional approximation (2.11–2.12), we obtain the following numerical approximation of the problem (3.1)

$$\frac{1}{h\Gamma(2-\alpha)}\sum_{i=1}^{n} b_i\left(u_i-u_{i-1}\right)+c_nu_n=f_n, \tag{3.2}$$

the resulting equation can be written as

$$\left(\frac{b_n+h\Gamma(2-\alpha)c_n}{h\Gamma(2-\alpha)}\right)u_n=f_n+\frac{1}{h\Gamma(2-\alpha)}b_nu_{n-1}-\frac{1}{h\Gamma(2-\alpha)}\sum_{i=1}^{n-1} b_i\left(u_i-u_{i-1}\right),$$

witch gives

$$u_n=\left(\frac{b_n}{\omega_n}\right)u_{n-1}-\left(\frac{1}{\omega_n}\right)\sum_{i=1}^{n-1} b_i\left(u_i-u_{i-1}\right)+\left(\frac{\lambda}{\omega_n}\right)f_n, \tag{3.3}$$

the above equation can be rewritten as the following form

$$u_n=\frac{b_1}{\omega_n}u_0+\frac{1}{\omega_n}\sum_{i=1}^{n-1}G_iu_i+\frac{\lambda}{\omega_n}f_n, \tag{3.4}$$

with

$$\begin{cases} u_0=u(a),\\ \omega_n=\left(b_n+\lambda c_n\right),\\ G_i=b_{i+1}-b_i,\\ \lambda=h\Gamma(2-\alpha). \end{cases}$$

3.1.2 Stability and convergence of FDM

In this part, we discuss the stability and the convergence of the finite difference scheme (3.4) for the fractional differential equation (3.1). For that, we need the following lemma

Lemma 3.1. *For $n=1,2,\ldots,N$, the coefficient b_n in (2.12) satisfy*

1. $b_n>0,$ *for* $n=1,2,\ldots,N.$
2. $b_n>b_{n-1},$ *for* $i=2,\ldots,N.$

Firstly, we consider the stability of the difference approximation (3.4). We suppose that u_n and $u(t_n)$ is the approximate and the exact solution of (3.4) for $n=1,2,\ldots,N$. Set $\varepsilon^n=u_n-u(t_n)$ then, from (3.4) we have

$$\varepsilon^n=\frac{b_1}{\omega_n}\varepsilon^0+\frac{1}{\omega_n}\sum_{i=1}^{n-2}G_i\varepsilon^i+\left(\frac{b_n-b_{n-1}}{\omega_n}\right)\varepsilon^{n-1},$$

which can be written as

$$E^n = \frac{b_1}{\omega_n}E^0 + \frac{1}{\omega_n}\sum_{i=1}^{n-2} G_i E^i + \left(\frac{b_n - b_{n-1}}{\omega_n}\right) E^{n-1},$$

where $\mathbf{E^n} = \varepsilon^n$. Hence, the following result can be proved.

Lemma 3.2. *The stability condition is equivalent to*

$$|E^n| \leq \left|E^0\right|, \ \textit{for } n = 1, 2, 3, \ldots, N.$$

Proof. We will use mathematical induction to get the above result. For $n = 1$ and becaused $\frac{b_1}{\omega_1} \leq 1$, we have

$$\begin{aligned}\left|E^1\right| = \left|\varepsilon^1\right| &\leq \frac{b_1}{\omega_1}\left|E^0\right| \\ &\leq \left|E^0\right|.\end{aligned}$$

Suppose that $|E^i| \leq |E^0|$ for $i = 1, 2, 3, \ldots, n-1$, using lemma (3.1) we get

$$\begin{aligned}|E^n| = |\varepsilon^n| &\leq \frac{b_1}{\omega_n}\left|E^0\right| + \frac{1}{\omega_n}\sum_{i=1}^{n-2}|G_i|\left|E^i\right| + \left|\frac{b_n - b_{n-1}}{\omega_n}\right|\left|E^{n-1}\right| \\ &\leq \frac{b_1}{\omega_n}\left|E^0\right| + \frac{1}{\omega_n}\sum_{i=1}^{n-2}|b_{i+1} - b_i|\left|E^i\right| + \left|\frac{b_n - b_{n-1}}{\omega_n}\right|\left|E^0\right| \\ &\leq \frac{b_1}{\omega_n}\left|E^0\right| + \frac{1}{\omega_n}\left(b_{n-1} - b_1\right)\left|E^0\right| + \frac{b_n - b_{n-1}}{\omega_n}\left|E^0\right| \\ &\leq \frac{b_n}{\omega_n}\left|E^0\right| \\ &\leq \left|E^0\right|.\end{aligned}$$

Hence, the proof is completed. □

Secondly, we consider the convergence of the difference approximation (3.4). Define $e^n = u(t_n) - u^n$ using $e^0 = 0$, subtiting $u^n = u(t_n) - e^n$ into (3.4) leads to :

$$\begin{aligned}(u(t_n) - e^n) = &\left(\frac{b_n - b_{n-1}}{\omega_n}\right)\left(u(t_{n-1}) - e^{n-1}\right) + \frac{1}{\omega_n}\sum_{i=1}^{n-2} G_i\left(u(t_i) - e^i\right) \\ &+ \frac{b_1}{\omega_n}\left(u(t_0) - e^0\right) + \frac{\lambda}{\omega_n}f_n,\end{aligned}$$

then, we get

$$\begin{aligned}
e^n =& u(t_n) - \frac{b_1}{\omega_n}u(t_0) - \frac{1}{\omega_n}\sum_{i=1}^{n-2} G_i u(t_i) - \left(\frac{b_n - b_{n-1}}{\omega_n}\right) u(t_{n-1}) - \frac{\lambda}{\omega_n} f\left(t_n\right) \\
&+ \left(\frac{b_n - b_{n-1}}{\omega_n}\right) e^{n-1} + \frac{1}{\omega_n}\sum_{i=1}^{n-2} G_i e^i + \frac{b_1}{\omega_n} e^0 \\
=& \left(\frac{b_n - b_{n-1}}{\omega_n}\right) e^{n-1} + \frac{1}{\omega_n}\sum_{i=1}^{n-2} G_i e^i + \frac{b_1}{\omega_n} e^0 + R^n,
\end{aligned}$$

where

$$\begin{aligned}
R^n &= \left(\sum_{i=1}^{n} b_i \left(u(t_i) - u\left(t_{i-1}\right)\right) + \lambda c(t_n) u\left(t_n\right) - \lambda f\left(t_n\right)\right) \\
&= h\Gamma(2-\alpha)\left({}^{CH}\mathcal{D}_{a^+}^{\alpha} u\left(t_n\right) + c(t_n) u\left(t_n\right) - f\left(t_n\right) + c_\alpha h^{1-\alpha}\right) \\
&= c_\alpha \Gamma(2-\alpha) h^{2-\alpha}.
\end{aligned}$$

Hence, there exist c'_α such that

$$|R^n| \leq c'_\alpha h^{2-\alpha}.$$

Consequently, using mathematical induction, we prove

$$|e^n| \leq C_\alpha h^{2-\alpha}.$$

For $n = 1$, we get

$$\begin{aligned}
\left|e^1\right| &\leq \left|R^1\right| \\
&\leq c'_\alpha h^{2-\alpha}.
\end{aligned}$$

Suppose that $|e^i| \leq c'_\alpha h^{2-\alpha}$, for $i = 1, 2, \ldots, n-1$, using Lemma (3.1), we have

$$\begin{aligned}
|e^n| &\leq \left|\frac{b_1}{\omega_n} e^0 + \frac{1}{\omega_n}\sum_{i=1}^{n-2} G_i e^i + \left(\frac{b_n - b_{n-1}}{\omega_n}\right) e^{n-1} + R^n\right| \\
&\leq \frac{1}{\omega_n}\sum_{i=1}^{n-2}\left(b_{i+1} - b_i\right) e^i + \left(\frac{b_n - b_{n-1}}{\omega_n}\right)\left|e^{n-1}\right| + |R^n| \\
&\leq \left(\frac{b_{n-1} - b_1}{\omega_n}\right) c'_\alpha h^{2-\alpha} + \left(\frac{b_n - b_{n-1}}{\omega_n}\right) c'_\alpha h^{2-\alpha} + c'_\alpha h^{2-\alpha} \\
&\leq 2c'_\alpha h^{2-\alpha} \\
&\leq C_\alpha h^{2-\alpha}.
\end{aligned}$$

Hence, the following theorem is obtained and guarantees the stability and convergence of the discretized scheme.

Theorem 3.1. *The obtained approximation sequence u_n, for the discretized scheme (3.4) is stable and convergent, if $C_\alpha h^{2-\alpha}$ tends to zero.*

3.1.3 Numerical examples

In this part, we present two examples to illustrate the usefulness of our main results.

Example 3.1. Let $t \in [1,2]$, $\alpha = 0.8$ and

$$f(t) = \frac{1}{\Gamma(2-\alpha)} (\log t)^{1-\alpha} + t \log\left(\frac{t}{3}\right).$$

Consider the following generalized Caputo–Hadamard fractional differential equation :

$$\begin{cases} {}^{CH}\mathcal{D}_{a^+}^{\alpha} u(t) + tu(t) = f(t), \ 1 \leq t \leq 2, \\ u(1) = \log\left(\frac{1}{3}\right). \end{cases} \tag{3.5}$$

The exact solution of (3.5) is given by :

$$u(t) = \log\left(\frac{t}{3}\right).$$

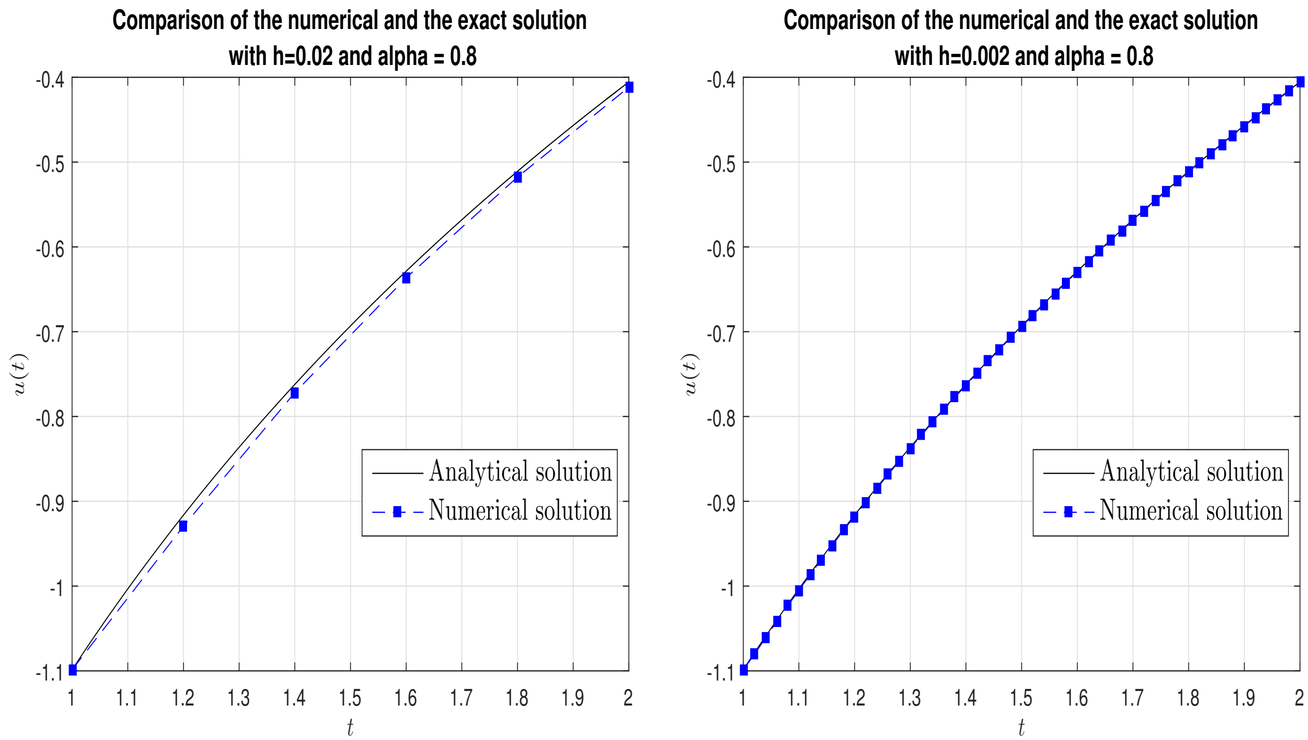

FIGURE 3.1 – Graphical comparison of the numerical and the exact solution.

t	Exact solution	Approx solution	Error for $h = 0.02$
1.0	-1.09861	-1.09861	0.00000e+00
1.1	-1.00330	-1.01907	1.57665e-02
1.2	-0.91629	-0.92935	1.30629e-02
1.3	-0.83625	-0.84738	1.11362e-02
1.4	-0.76214	-0.77186	9.71583e-03
1.5	-0.69315	-0.70180	8.65184e-03
1.6	-0.62861	-0.63646	7.84803e-03
1.7	-0.56798	-0.57522	7.23832e-03
1.8	-0.51083	-0.51760	6.77543e-03
1.9	-0.45676	-0.46318	6.42460e-03
2.0	-0.40547	-0.41162	6.15984e-03

t	Exact solution	Approx solution	Error for $h = 0.002$
1.0	-1.09861	-1.09861	0.00000e+00
1.1	-1.00330	-1.00486	1.56088e-03
1.2	-0.91629	-0.91759	1.29508e-03
1.3	-0.83625	-0.83735	1.10569e-03
1.4	-0.76214	-0.76311	9.66025e-04
1.5	-0.69315	-0.69401	8.61377e-04
1.6	-0.62861	-0.62939	7.82305e-04
1.7	-0.56798	-0.56871	7.22324e-04
1.8	-0.51083	-0.51150	6.76787e-04
1.9	-0.45676	-0.45740	6.42279e-04
2.0	-0.40547	-0.40608	6.16243e-04

TABLE 3.1 – Comparison of the numerical and the exact solutions with $h = 0.02$, $h = 0.002$ and $\alpha = 0.8$.

Example 3.2. Let $t \in [1,3]$, $\alpha = 0.5$ and

$$f(t) = (t+1)\log\left(\frac{\sqrt{2}}{t}\right) - \frac{1}{\Gamma(2-\alpha)}(\log t)^{1-\alpha}.$$

Consider the following generalized Caputo–Hadamard fractional differential equation :

$$\begin{cases} {}^{CH}\mathcal{D}^{\alpha}_{a+}u(t) + (t+1)u(t) = f(t),\ 1 \leq t \leq 3, \\ u(1) = \log\left(\sqrt{2}\right). \end{cases}, \tag{3.6}$$

The exact solution of this problem is given by :

$$u(t) = \log\left(\frac{\sqrt{2}}{t}\right).$$

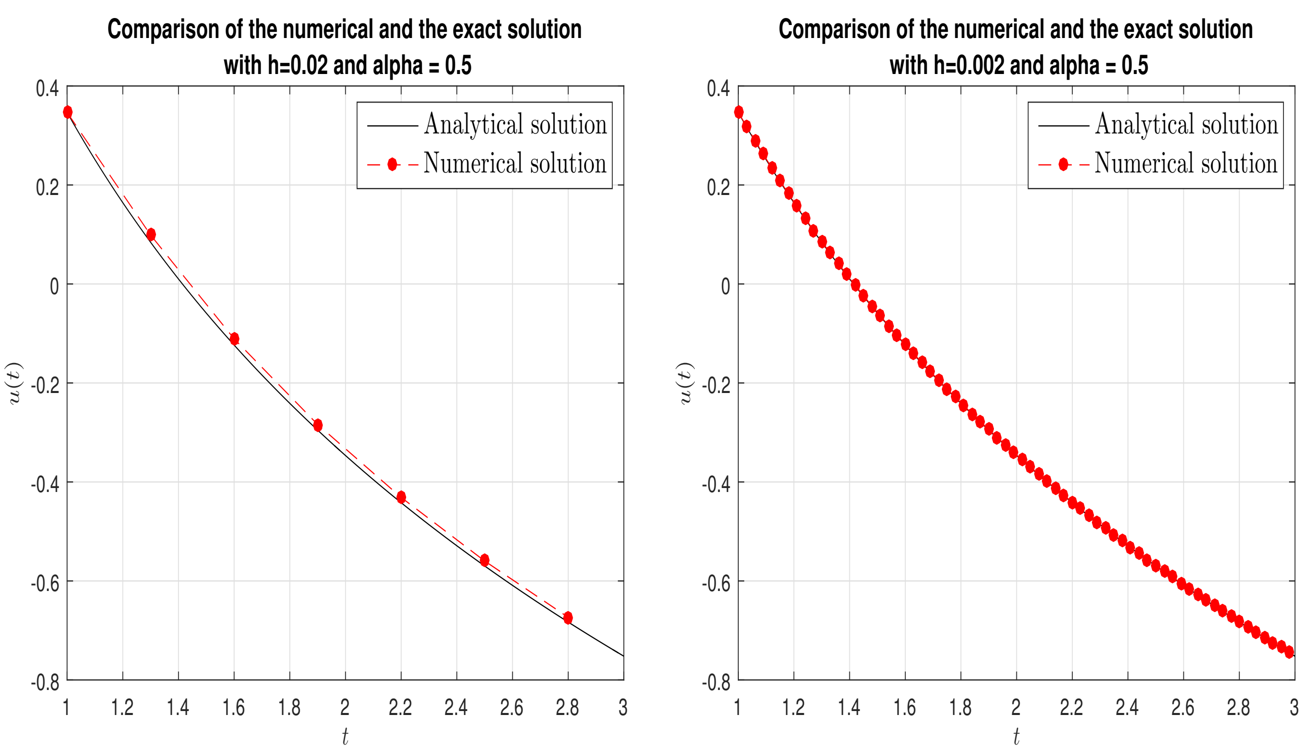

FIGURE 3.2 – Graphical comparison of the numerical and the exact solution.

t	Exact solution	Approx solution	Error for $h = 0.02$
1.0	0.34657	0.34657	0.00000e+00
1.2	0.16425	0.17993	1.56789e-02
1.4	0.01010	0.02393	1.38249e-02
1.6	-0.12343	-0.11080	1.26291e-02
1.8	-0.24121	-0.22943	1.17796e-02
2.0	-0.34657	-0.33544	1.11379e-02
2.2	-0.44188	-0.43125	1.06299e-02
2.4	-0.52890	-0.51868	1.02127e-02
2.6	-0.60894	-0.59908	9.85944e-03
2.8	-0.68305	-0.67349	9.55292e-03
3.0	-0.75204	-0.74276	9.28159e-03

t	Exact solution	Approx solution	Error for $h = 0.002$
1.0	0.34657	0.34657	0.00000e+00
1.2	0.16425	0.16581	1.55983e-03
1.4	0.01010	0.01148	1.37844e-03
1.6	-0.12343	-0.12217	1.26056e-03
1.8	-0.24121	-0.24004	1.17651e-03
2.0	-0.34657	-0.34546	1.11285e-03
2.2	-0.44188	-0.44082	1.06237e-03
2.4	-0.52890	-0.52787	1.02085e-03
2.6	-0.60894	-0.60795	9.85646e-04
2.8	-0.68305	-0.68209	9.55076e-04
3.0	-0.75204	-0.75111	9.27997e-04

TABLE 3.2 – Comparison of the numerical and the exact solutions with $h = 0.02$, $h = 0.002$ and $\alpha = 0.5$.

3.1.4 Dicussion and results

In this section we have developed a fractional finite difference method for a generalized fractional differential equation of Caputo-Hadamard type. Also, we have proved that the approximate solution u_n is stable and convergent. The efficiency of (FDM) has been discussed and illustrated by solving two typical examples (Example 3.1 and Example 3.2). It is found that the approximate solutions produced by this method are in complete agreement with the corresponding exact solutions (Figure 3.1, Figure 3.2). The results obtained show a good global approximation and an improved convergence with an error $C_\alpha h^{2-\alpha}$ reaching to zero. (Table 3.1, Table 3.2).

3.2 FDM for solving TFDWE involving the Caputo-Hadamard time-fractional derivative

This section proposes a finite difference method to obtain the numerical solution of time fractional diffusion-wave equation (TFDWE) involving the Caputo–Hadamard time-fractional derivative operator given by :

$$^{CH}\partial_t^{\alpha} u\left(x,t\right) = \frac{\partial^2 u\left(x,t\right)}{\partial x^2} + f(x,t), \;\; (x,t) \in \left]x_0, L\right[\times \left]t_0, T\right[, \tag{3.7}$$

with Dirichlet-Neumann initial conditions

$$u\left(x,t_0\right) = \varphi_1(x), \; \frac{\partial u}{\partial t}\left(x,t_0\right) = \varphi_2(x), \;\; x \in \left[x_0, L\right], \tag{3.8}$$

and Dirichlet-Neumann boundary conditions

$$u\left(x_0,t\right)=\psi_1(t),\ \frac{\partial u}{\partial x}\left(L,t\right)=\psi_2(t),\ \ t\in\left[t_0,T\right]. \tag{3.9}$$

Where ${}^{CH}\partial_t^\alpha$ denotes the Caputo-Hadamard time-fractional derivative operator of order $1<\alpha\leq 2$ and $t_0>0,\ \varphi_1,\ \varphi_2,\ \psi_1,\ \psi_2$ are continuous functions and $f(x,t)$ is the source term. The Convergence and stability of the given finite difference scheme are obtained and proved. Moreover, numerical examples are given to illustrate the efficiency of the proposed schemes.

3.2.1 Numerical scheme

Let u_i^{n+1} be the numerical approximation to $u(x_i,t_{n+1})$ and $f_i^{n+1}=f(x_i,t_{n+1})$.

1. The general two-order center difference scheme of the integer-order derivative $\frac{\partial^2 u(x_i,t_{n+1})}{\partial x^2}$ in (3.7) is given by

$$\frac{\partial^2 u(x_i,t_{n+1})}{\partial x^2}=\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{k^2}+C_1k^2. \tag{3.10}$$

2. The initial boundary conditions (3.8) and (3.9) are discretized as

$$\begin{aligned}
&u(x_i,t_0)=u_i^0=\varphi_1^i,\\
&u_t(x_i,t_0)=\varphi_2^i,\\
&u(x_0,t_{n+1})=u_0^{n+1}=\psi_1^{n+1},\\
&u_x(L,t_{n+1})=\psi_2^{n+1}.
\end{aligned} \tag{3.11}$$

3. From (2.13) and (2.14), the time fractional derivative term ${}^{CH}\partial_t^\alpha u\left(x,t\right)$ can be approximated by

$${}^{CH}\partial_t^\alpha u_i^{n+1}=\frac{1}{h^2\Gamma\left(3-\alpha\right)}\sum_{j=0}^{n}b_j\left(t_{j+1}u_i^{j+1}-\left(t_{j+1}+t_j\right)u_i^j+t_ju_i^{j-1}\right). \tag{3.12}$$

Now, By using the time fractional approximation (3.10), (3.11) and (3.12), we obtain the following numerical approximation of equation (3.7)

$$\begin{aligned}
\frac{1}{h^2\Gamma\left(3-\alpha\right)}\sum_{j=0}^{n}b_j\left(t_{j+1}u_i^{j+1}-\left(t_{j+1}+t_j\right)u_i^j+t_ju_i^{j-1}\right)=&\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{k^2}\\
&+f_i^{n+1},
\end{aligned} \tag{3.13}$$

the above equation can be rewritten as the following form

$$\begin{aligned}-\lambda u_{i+1}^{n+1}+\left(2\lambda+t_{n+1}b_n\right)u_i^{n+1}-\lambda u_{i-1}^{n+1}=&b_n\left(t_n+t_{n+1}\right)u_i^n-b_nt_nu_i^{n-1}\\&-\sum_{j=0}^{n-1}b_j\left(t_{j+1}u_i^{j+1}-\left(t_{j+1}+t_j\right)u_i^j+t_ju_i^{j-1}\right)\\&+h^2\Gamma\left(3-\alpha\right)f_i^{n+1}.\end{aligned}$$

For each $n\in\{0,1,...,N-1\}$ and $i\in\{0,1,...,M\}$, where $\lambda=\dfrac{h^2\Gamma\left(3-\alpha\right)}{k^2}$.
Case (1) : **for** $n=0$

$$-\lambda u_{i+1}^1+\left(2\lambda+t_1b_0\right)u_i^1-\lambda u_{i-1}^1=t_1b_0u_i^0+t_0b_0h\varphi_2^i+h^2\Gamma\left(3-\alpha\right)f_i^1, \tag{3.14}$$

Case (2) : **for** $n>0$

$$\begin{aligned}-\lambda u_{i+1}^{n+1}+\left(2\lambda+t_{n+1}b_n\right)u_i^{n+1}-\lambda u_{i-1}^{n+1}=&\sum_{j=1}^{n-1}c_ju_i^j+\left(\left(t_n+t_{n+1}\right)b_n-t_nb_{n-1}\right)u_i^n\\&+t_1\left(b_0-b_1\right)u_i^0+t_0b_0h\varphi_2^i\\&+h^2\Gamma\left(3-\alpha\right)f_i^{n+1}.\end{aligned} \tag{3.15}$$

Where $c_j=(-t_{j+1}b_{j+1}+(t_j+t_{j+1})b_j-t_jb_{j-1})$, $j=1,...,n-1$. Thus, the finite difference scheme in the matrix form is given by :

$$\begin{cases}\mathbf{U^0}=\varphi_\mathbf{1},\\ A_0\mathbf{U^1}=t_1b_0\mathbf{U^0}+b_0t_0h\varphi_\mathbf{2}+\mathbf{F^1},\\ A_n\mathbf{U^{n+1}}=t_1\left(b_0-b_1\right)\mathbf{U^0}+\displaystyle\sum_{j=1}^{n-1}c_j\mathbf{U^j}+\left(\left(t_n+t_{n+1}\right)b_n-t_nb_{n-1}\right)\mathbf{U^n},\\ +t_0b_0h\varphi_\mathbf{2}+\mathbf{F^{n+1}}.\end{cases} \tag{3.16}$$

where, for $n=0,1,...,N-1$

$$A_n=\begin{pmatrix}(2\lambda+t_{n+1}b_n)&-\lambda&0&\cdots&0\\-\lambda&(2\lambda+t_{n+1}b_n)&-\lambda&\ddots&\vdots\\0&\ddots&\ddots&\ddots&0\\\vdots&\cdots&\ddots&\ddots&-\lambda\\0&\cdots&0&-\lambda&(\lambda+t_{n+1}b_n)\end{pmatrix},$$

and

$$\mathbf{U^n}=\begin{pmatrix}u_1^n\\u_2^n\\\vdots\\u_{M-1}^n\end{pmatrix},\boldsymbol{\varphi_1}=\begin{pmatrix}\varphi_1^1\\\varphi_1^2\\\vdots\\\varphi_1^{M-1}\end{pmatrix},\boldsymbol{\varphi_2}=\begin{pmatrix}\varphi_2^1\\\varphi_2^2\\\vdots\\\varphi_2^{M-1}\end{pmatrix},$$

$$\mathbf{F^{n+1}} = \begin{pmatrix}h^2\Gamma\left(3-\alpha\right)f_1^{n+1}+\lambda\psi_1^{n+1}\\h^2\Gamma\left(3-\alpha\right)f_2^{n+1}\\\vdots\\h^2\Gamma\left(3-\alpha\right)f_{M-2}^{n+1}\\h^2\Gamma\left(3-\alpha\right)f_{M-1}^{n+1}+\lambda k\psi_2^{n+1}\end{pmatrix}.$$

Obviously, the matrix A_n are symmetric, strictly diagonally dominant, the difference equation (3.16) has only one solution. In the next section, we will prove its stability and convergence with the help of analysis to the coefficient matrix.

3.2.2 Stability and convergence of the scheme

In this part, we discuss the stability and Convergence of the solution of time fractional finite difference scheme (3.14) and (3.15) for the time fractional diffusion equation (3.7). For that, we need the following lemma

Lemma 3.3. *For $j=0,1,2\ldots,n$, the coefficients b_j in (2.14) satisfy*

1. $b_j>0,\ j=0,1,2\ldots,n$,
2. $b_{j+1}>b_j$,
3. $\lim\limits_{n\to\infty}\frac{(t_{n+1}b_n)^{-1}}{\left(\frac{t_0}{h}+n\right)^2}=0.$

Firstly, we consider the stability of the difference approximation (3.14) and (3.15). We suppose that $\widetilde{u}_i^n,\ (i=0,1,2,\ldots,M;\ n=0,1,2,\ldots,N)$ is the approximate solution of (3.14) and (3.15). Set $\varepsilon_i^n=\widetilde{u}_i^n-u_i^n$ then, From (3.14), (3.15) we have

$$-\lambda\varepsilon_{i+1}^1+\left(2\lambda+t_1b_0\right)\varepsilon_i^1-\lambda\varepsilon_{i-1}^1=t_1b_0\varepsilon_i^0,\qquad \text{for } n=0, \tag{3.17}$$

and

$$\begin{aligned}-\lambda\varepsilon_{i+1}^{n+1}+\left(2\lambda+t_{n+1}b_n\right)\varepsilon_i^{n+1}-\lambda\varepsilon_{i-1}^{n+1}&=t_1\left(b_0-b_1\right)\varepsilon_i^0+\sum_{j=1}^{n-1}c_j\varepsilon_i^j\\&\quad+\left(\left(t_n+t_{n+1}\right)b_n-t_nb_{n-1}\right)\varepsilon_i^n,\qquad \text{for } n>0.\end{aligned} \tag{3.18}$$

Which can be written as

$$A_0\mathbf{E^1} = t_1 b_0 \mathbf{E^0}, \quad \text{for } n = 0, \tag{3.19}$$

and

$$\begin{aligned} A_n\mathbf{E^{n+1}} = t_1\left(b_0 - b_1\right)\mathbf{E^0} + \sum_{j=1}^{n-1} c_j \mathbf{E^j} \\ + \left(\left(t_n + t_{n+1}\right)b_n - t_n b_{n-1}\right)\mathbf{E^n}, \ \text{for } n > 0, \end{aligned} \tag{3.20}$$

where $\mathbf{E^n} = \left(\varepsilon_1^n, \varepsilon_2^n, \ldots, \varepsilon_{M-1}^n\right)^t$. Hence, the following result can be proved using mathematical induction. proved.

Lemma 3.4.

$$\left\|\mathbf{E^{n+1}}\right\|_\infty \leqslant C\left\|\mathbf{E^0}\right\|_\infty, \ n = 0, 1, 2, \ldots, N. \tag{3.21}$$

Proof. $\|\mathbf{E^{n+1}}\|_\infty = \left|\varepsilon_l^{n+1}\right| = \max_{1\leq i\leq M-1}\left|\varepsilon_i^{n+1}\right|$.

1. **For** $n = 0$, we have

$$\begin{aligned} t_1 b_0\left|\varepsilon_l^1\right| &= -\lambda\left|\varepsilon_l^1\right| + (2\lambda + t_1 b_0)\left|\varepsilon_l^1\right| - \lambda\left|\varepsilon_l^1\right| \\ &\leqslant -\lambda\left|\varepsilon_{l+1}^1\right| + (2\lambda + t_1 b_0)\left|\varepsilon_l^1\right| - \lambda\left|\varepsilon_{l-1}^1\right| \\ &\leqslant \left|-\lambda\varepsilon_{l+1}^1 + (2\lambda + t_1 b_0)\varepsilon_l^1 - \lambda\varepsilon_{l-1}^1\right| \\ &\leqslant t_1 b_0\left|\varepsilon_l^0\right| \\ \text{hence, } \left|\varepsilon_l^1\right| &\leqslant \left|\varepsilon_l^0\right| \\ &\leqslant \left\|\mathbf{E}^0\right\|_\infty, \end{aligned}$$

its follows

$$\left\|\mathbf{E^1}\right\|_\infty \leqslant \left\|\mathbf{E^0}\right\|_\infty. \tag{3.22}$$

2. **For** $n > 0$, we assume that $\left\|\mathbf{E^j}\right\|_\infty \leqslant c\left\|\mathbf{E^0}\right\|_\infty, \ j = 1, 2, \ldots, n$

$$\begin{aligned} t_{n+1}b_n\left|\varepsilon_l^{n+1}\right| &= -\lambda\left|\varepsilon_l^{n+1}\right| + (2\lambda + t_{n+1}b_n)\left|\varepsilon_l^{n+1}\right| + \lambda\left|\varepsilon_l^{n+1}\right| \\ &\leqslant -\lambda\left|\varepsilon_{l+1}^{n+1}\right| + (2\lambda + t_{n+1}b_n)\left|\varepsilon_l^{n+1}\right| - \lambda\left|\varepsilon_{l-1}^{n+1}\right| \\ &\leqslant \left|-\lambda\varepsilon_{l+1}^{n+1} + (2\lambda + t_{n+1}b_n)\varepsilon_l^{n+1} - \lambda\varepsilon_{l-1}^{n+1}\right| \\ &\leqslant \left|\sum_{j=1}^{n-1} c_j\varepsilon_l^j + \left(\left(t_n + t_{n+1}\right)b_n - t_n b_{n-1}\right)\varepsilon_l^n + t_1\left(b_0 - b_1\right)\varepsilon_l^0\right| \\ &\leqslant \left|\sum_{j=1}^{n-1} c_j\right| c\left\|\mathbf{E}^0\right\|_\infty + \left|\left(\left(t_n + t_{n+1}\right)b_n - t_n b_{n-1}\right)\right| c\left\|\mathbf{E}^0\right\|_\infty \\ &\quad + \left|t_1\left(b_0 - b_1\right)\right| c\left\|\mathbf{E}^0\right\|_\infty. \end{aligned}$$

Following lemma 3.3 (b), b_j are increasing, we have

$$\begin{aligned}
&\left|\sum_{j=1}^{n-1} c_j\right| c\left\|\mathbf{E}^0\right\|_\infty + \left|\left(\left(t_n+t_{n+1}\right) b_n - t_n b_{n-1}\right)\right| c\left\|\mathbf{E}^0\right\|_\infty + \left|t_1\left(b_0-b_1\right)\right| c\left\|\mathbf{E}^0\right\|_\infty \\
\leqslant &\left[-\sum_{j=1}^{n-1}\left(t_j b_j - t_{j+1} b_{j+1}\right) + \sum_{j=1}^{n-1}\left(t_{j+1} b_j - t_j b_{j-1}\right)\right] c\left\|\mathbf{E}^0\right\|_\infty \\
&+\left[t_{n+1} b_n + t_n\left(b_n-b_{n-1}\right) - t_1\left(b_0-b_1\right)\right] c\left\|\mathbf{E}^0\right\|_\infty \\
&\leqslant \left(3 t_{n+1} b_n - 2 t_1 b_0\right) c\left\|\mathbf{E}^0\right\|_\infty, \\
&\leqslant 3 t_{n+1} b_n c\left\|\mathbf{E}^0\right\|_\infty,
\end{aligned}$$

Finally, we find

$$\left\|\mathbf{E}^{\mathbf{n+1}}\right\|_\infty \leqslant C\left\|\mathbf{E}^{\mathbf{0}}\right\|_\infty .$$

□

Consequently, the following theorem is obtained.

Theorem 3.2. *The solution of the discretized scheme (3.14) and (3.15) for the time fractional diffusion equation (3.7) is unconditionally stable.*

Secondly, we consider the convergence of the difference approximation (3.14) and (3.15). Let $u\left(x_i, t_n\right)$ be the exact solution of the time fractional diffusion equation (3.7) at mesh points $\left(x_i, t_n\right)$, where $i=0,1,2,\ldots,M$ and $n=0,1,2,\ldots,N$. Define $e_i^n = u\left(x_i, t_n\right) - u_i^n$ with $\mathbf{e^n} = \left(e_1^n, e_2^n, \ldots, e_{M-1}^n\right)^t$. Using $\mathbf{e^0} = (0,0,\ldots,0)^t$. substituting $u_i^n = u\left(x_i, t_n\right) - e_i^n$ into (3.14) and (3.15) leads to :

1. **For** $n=0$, the approximate scheme (3.14) gives

$$\begin{aligned}
&-\lambda\left(u\left(x_{i+1}, t_1\right) - e_{i+1}^1\right) + \left(2\lambda + t_1 b_0\right)\left(u\left(x_i, t_1\right) - e_i^1\right) - \lambda\left(u\left(x_{i-1}, t_1\right) - e_{i-1}^1\right) \\
&= t_1 b_0\left(u\left(x_i, t_0\right) - e_0^1\right) + b_0 t_0 h \varphi_2^i + h^2 \Gamma(3-\alpha) f_i^1 .
\end{aligned}$$

Its follow

$$\begin{aligned}
-\lambda e_{i+1}^1 + \left(2\lambda + t_1 b_0\right) e_i^1 - \lambda e_{i-1}^1 = &-\lambda u\left(x_{i+1}, t_1\right) + \left(2\lambda + t_1 b_0\right) u\left(x_i, t_1\right) \\
&-\lambda u\left(x_{i-1}, t_1\right) - t_1 b_0 u\left(x_i, t_0\right) - b_0 t_0 h \varphi_2^i \\
&-h^2 \Gamma(3-\alpha) f_i^1 \\
= &R_i^1 .
\end{aligned}$$

2. **For** $n > 0$, the approximate scheme (3.15) gives

$$\begin{aligned}-\lambda e_{i+1}^{n+1} + (2\lambda + t_{n+1}b_n)\, e_i^{n+1} - \lambda e_{i-1}^{n+1} =& -\lambda u\left(x_{i+1}, t_{n+1}\right) + \left(2\lambda + t_{n+1}b_n\right) u\left(x_i, t_{n+1}\right) \\ & - \lambda u\left(x_{i-1}, t_{n+1}\right) - \sum_{j=1}^{n-1} c_j u\left(x_i, t_j\right) + \sum_{j=1}^{n-1} c_j e_i^j \\ & - \left(\left(t_n + t_{n+1}\right) b_n - t_n b_{n-1}\right)\left(u\left(x_i, t_n\right) - e^n\right) \\ & - t_1\left(b_0 - b_1\right)\left(u\left(x_i, t_0\right) - e_i^0\right) \\ & - b_0 t_0 h \varphi_2^i - h^2 \Gamma\left(3-\alpha\right) f_i^{n+1} \\ =& \sum_{j=1}^{n-1} c_j e_i^j + \left(\left(t_n + t_{n+1}\right) b_n - t_n b_{n-1}\right) e^n \\ & + t_1\left(b_0 - b_1\right) e_i^0 + R_i^{n+1},\end{aligned}$$

where

$$\begin{aligned}R_i^{n+1} =& \sum_{j=0}^{n} b_j\left(t_j u\left(x_i, t_{j-1}\right) - \left(t_j + t_{j+1}\right) u\left(x_i, t_j\right) + t_{j+1} u\left(x_i, t_{j+1}\right)\right) \\ & - \lambda\left(u\left(x_{i+1}, t_{n+1}\right) + 2u\left(x_i, t_{n+1}\right) - u\left(x_{i-1}, t_{n+1}\right)\right) - h^2 \Gamma\left(3-\alpha\right) f_i^{n+1}.\end{aligned}$$

From (3.7), we have

$$\begin{aligned}R_i^{n+1} &= h^2 \Gamma\left(3-\alpha\right)\left({}^{CH}\partial_t^{\alpha}(x_i, t_{n+1}) - \frac{\partial^2 u(x_i, t_{n+1})}{\partial x^2} - f_i^{n+1} - C_{2,\alpha} h^{2-\alpha} + C_1 k^2\right) \\ &= h^2 \Gamma\left(3-\alpha\right)\left(-C_{2,\alpha} h^{2-\alpha} + C_1 k^2\right).\end{aligned}$$

Hence, there exist $C_{3,\alpha} > 0$, such that

$$\left|R_i^{n+1}\right| \leqslant C_{3,\alpha} h^2\left(h^{2-\alpha} + k^2\right),$$

Consequently, we obtain the following lemma

Lemma 3.5.

$$\left\|\mathbf{e^{n+1}}\right\|_\infty \leqslant C_{4,\alpha}(t_{n+1}b_n)^{-1} h^2\left(h^{2-\alpha} + k^2\right), \;\; n = 0, 1, 2, \ldots, N-1,$$

where $C_{4,\alpha} = C_{3,\alpha}$ *if* $n = 0$ *and* $C_{4,\alpha} = \left(4t_{n+1}b_n - 2t_1 b_0\right)\left(t_1 b_0\right)^{-1} C_{3,\alpha}$, *if* $n > 0$.

Proof. Let $\|\mathbf{e^{n+1}}\|_\infty = \left|e_l^{n+1}\right| = \max\limits_{1 \le i \le M-1}\left|e_i^{n+1}\right|$.

1. **For** $n = 0$, we get

$$\begin{aligned} t_1 b_0 \left|e_l^1\right| &= -\lambda \left|e_l^1\right| + (2\lambda + t_1 b_0) \left|e_l^1\right| - \lambda \left|e_l^1\right| \\ &\leqslant -\lambda \left|e_{l+1}^1\right| + (2\lambda + t_1 b_0) \left|e_l^1\right| - \lambda \left|e_{l-1}^1\right| \\ \text{imply, } \left|e_l^1\right| &\leqslant (t_1 b_0)^{-1} \left|R_i^1\right| \\ &\leqslant C_{3,\alpha} (t_1 b_0)^{-1} h^2 \left(h^{2-\alpha} + k^2\right). \end{aligned}$$

2. **For** $n > 0$, suppose that $|e_l^n| \leqslant C_{3,\alpha} (t_n b_{n-1})^{-1} h^2 \left(h^{2-\alpha} + k^2\right)$, then

$$\left|e_l^{n+1}\right| \leqslant C_{3,\alpha} \left(4t_{n+1} b_n - 2t_1 b_0\right) (t_1 b_0)^{-1} (t_{n+1} b_n)^{-1} h^2 \left(h^{2-\alpha} + k^2\right),$$

finally, we find

$$\left\|\mathbf{e^{n+1}}\right\|_\infty \leqslant C_{4,\alpha} (t_{n+1} b_n)^{-1} h^2 \left(h^{2-\alpha} + k^2\right).$$

□

Theorem 3.3. *Let u_i^n be the approximate value of $u(x_i, t_n)$ computed by use of the difference scheme (3.14) and (3.15). Then there is a positive constant C_α such that*

$$\left|u_i^n - u\left(x_i, t_n\right)\right| \leqslant C_\alpha \left(h^{2-\alpha} + k^2\right).$$

Proof. From (c) in lemma 3.3, there is a constant $\zeta > 0$, such that $\dfrac{(t_{n+1} b_n)^{-1}}{\left(\dfrac{t_0}{h} + n\right)^2} \leqslant \zeta,$ then we obtained

$$\begin{aligned} \left|e_l^{n+1}\right| &\leqslant C_{4,\alpha} \zeta \left(\frac{t_0}{h} + n\right)^2 h^2 \left(h^{2-\alpha} + k^2\right) \\ &\leqslant C_{4,\alpha} \zeta T^2 \left(h^{2-\alpha} + k^2\right) \\ &\leqslant C_\alpha \left(h^{2-\alpha} + k^2\right). \end{aligned}$$

□

3.2.3 Numerical examples

In this part, we present some examples to illustrate the usefulness of our main results.

Example 3.3. Let $(x, t) \in [0, 1] \times [1, 2]$, $\alpha = 1.9$ and

$$f(x, t) = \frac{2}{\Gamma\left(3 - \alpha\right)} \sin(2\pi x) \left(\log t\right)^{(2-\alpha)} + 4\pi^2 \sin(2\pi x) \left(\log\left(\frac{t}{3}\right)\right)^2,$$

consider the following time fractional diffusion-wave equation

$$\begin{cases} {}^{CH}\partial_t^{\alpha} u(x,t) = \dfrac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \\ u(x,1) = \sin(2\pi x)\left(\log\left(\dfrac{1}{3}\right)\right)^2, \partial_t u(x,1) = 2\sin(2\pi x)\log\left(\dfrac{1}{3}\right), \\ u(0,t) = 0, \partial_x u(1,t) = 2\pi\left(\log\left(\dfrac{t}{3}\right)\right)^2. \end{cases} \tag{3.23}$$

The exact solution of the problem (3.23) is given by

$$u(x,t) = \sin(2\pi x)\left(\log\left(\frac{t}{3}\right)\right)^2.$$

We apply the discretization method described in section 3.2.1 by taking $h = 0.01$, $h = 0.001$ respectively and $n = 99$. The obtained results are shown in Figure 3.3 and table 3.3.

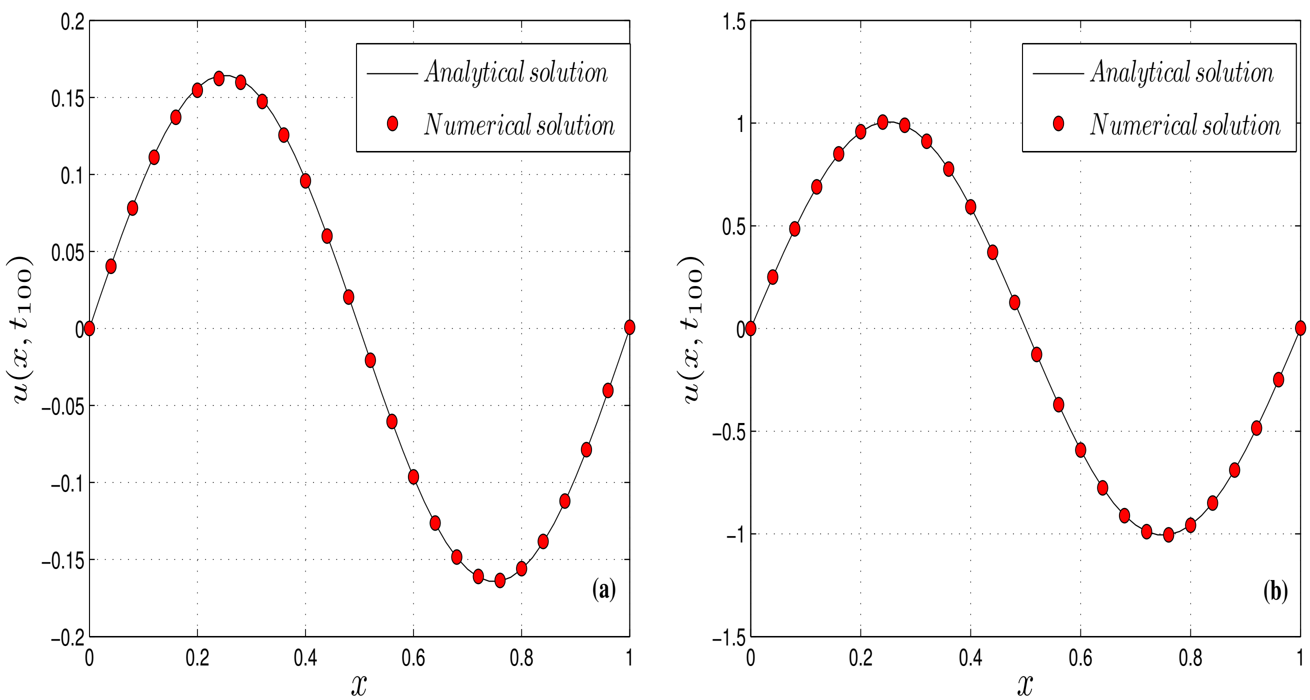

FIGURE 3.3 – Graphical comparison of the numerical and the exact solutions with (a) $h = 0.01$, (b) $h = 0.001$, $M = 50$ and $n = 99$.

x	Exa_sol	App_sol	Error for $h = 0.01$
0.00	0.00000	0.00000	0.00000e+000
0.10	0.09663	0.09542	1.21104e-003
0.20	0.15636	0.15457	1.78313e-003
0.30	0.15636	0.15484	1.51214e-003
0.40	0.09663	0.09582	8.10907e-004
0.50	0.00000	-0.00015	1.53516e-004
0.60	-0.09663	-0.09637	2.61104e-004
0.70	-0.15636	-0.15592	4.34278e-004
0.80	-0.15636	-0.15588	4.77860e-004
0.90	-0.09663	-0.09609	5.40587e-004
1.00	-0.00000	0.00074	7.40805e-004

x	Exa_sol	App_sol	Error for $h = 0.001$
0.00	0.00000	0.00000	0.00000e+000
0.10	0.59167	0.59198	3.10896e-004
0.20	0.95735	0.95785	5.03040e-004
0.30	0.95735	0.95785	5.03040e-004
0.40	0.59167	0.59198	3.10896e-004
0.50	0.00000	0.00000	8.91201e-015
0.60	-0.59167	-0.59198	3.10896e-004
0.70	-0.95735	-0.95785	5.03040e-004
0.80	-0.95735	-0.95785	5.02554e-004
0.90	-0.59167	-0.59156	1.18501e-004
1.00	-0.00000	0.00196	1.96282e-003

TABLE 3.3 – Comparison of the numerical and the exact solutions with $h = 0.01$, $h = 0.001$, $M = 50$ and $n = 99$.

Example 3.4. Let $(x,t) \in [1,2] \times [1,2]\,, \alpha = 1.5$ and

$$f(x,t) = \frac{2}{\Gamma(3-\alpha)} \log(x+1)\,(\log t)^{2-\alpha} + \frac{1}{(x+1)^2}\left(\log\left(\frac{\sqrt{2}}{t}\right)\right)^2,$$

consider the following time fractional diffusion-wave equation

$$\begin{cases} {}^{CH}\partial_t^{\alpha} u\,(x,t) = \dfrac{\partial^2 u\,(x,t)}{\partial x^2} + f(x,t), \\ u\,(x,1) = \log(x+1)\left(\log\left(\sqrt{2}\right)\right)^2, \partial_t u\,(x,1) = (-2)\log(x+1)\log\left(\sqrt{2}\right), \\ u\,(1,t) = \log(2)\left(\log\left(\dfrac{\sqrt{2}}{t}\right)\right)^2, \partial_x u\,(2,t) = \left(\dfrac{1}{3}\right)\left(\log\left(\dfrac{\sqrt{2}}{t}\right)\right)^2. \end{cases} \tag{3.24}$$

The exact solution of (3.24) is given by

$$u(x,t) = \log(x+1)\left(\log\left(\frac{\sqrt{2}}{t}\right)\right)^2.$$

Here, we use the discretization method in section(3.2.1), with $n = 30$ and $h = 0.005$, $h = 0.0005$ respectively. The numerical results of problem of Example 3.24 are shown in Figure 3.4 and Table 3.4

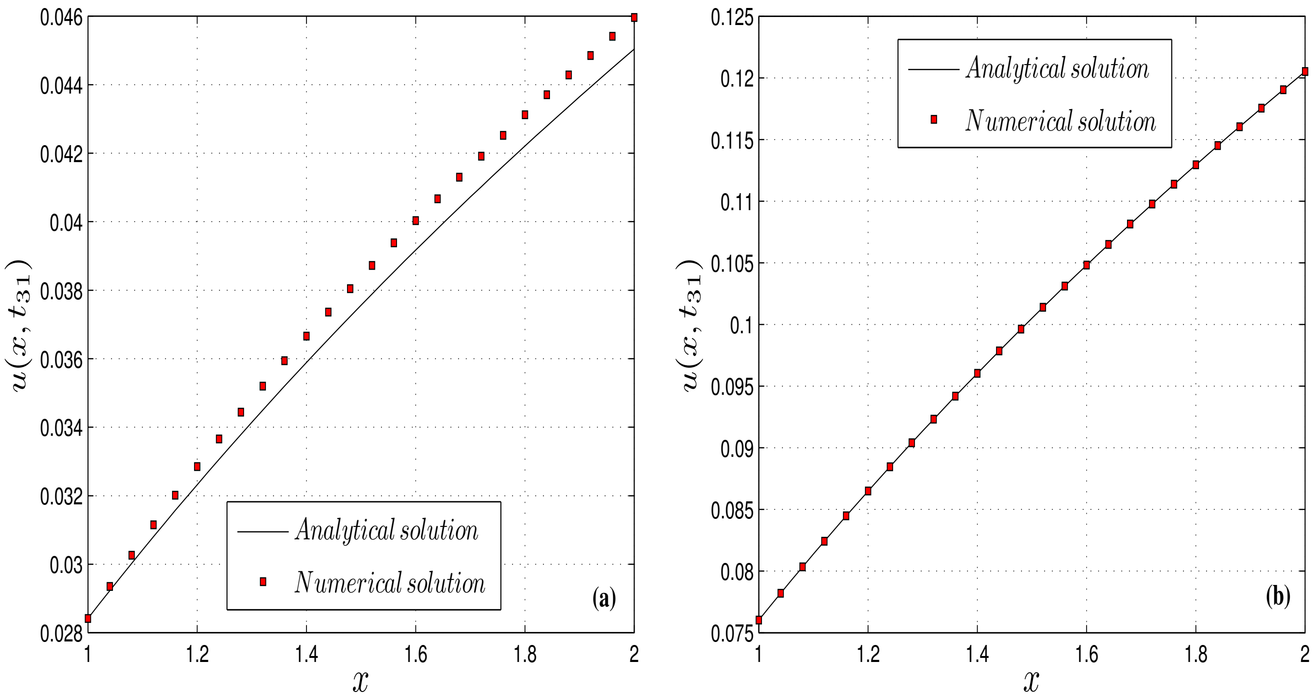

FIGURE 3.4 – Graphical comparison of the numerical and the exact solutions with $(a)\, h = 0.005$, $(b)\, h = 0.0005, M = 50$ and $n = 30$.

x	Exa_sol	App_sol	Error for $h = 0.005$
1.00	0.02842	0.02842	0.00000e+000
1.10	0.03042	0.03071	2.97115e-004
1.20	0.03232	0.03285	5.26043e-004
1.30	0.03415	0.03483	6.79842e-004
1.40	0.03589	0.03666	7.69962e-004
1.50	0.03756	0.03839	8.22838e-004
1.60	0.03917	0.04003	8.60234e-004
1.70	0.04072	0.04161	8.89899e-004
1.80	0.04221	0.04312	9.11598e-004
1.90	0.04365	0.04457	9.23489e-004
2.00	0.04504	0.04596	9.23893e-004

x	Exa_sol	App_sol	Error for $h = 0.0005$
1.00	0.07603	0.07603	0.00000e+000
1.10	0.08138	0.08139	7.57486e-006
1.20	0.08648	0.08649	8.09990e-006
1.30	0.09136	0.09137	8.55705e-006
1.40	0.09603	0.09604	8.99470e-006
1.50	0.10051	0.10052	9.41444e-006
1.60	0.10481	0.10482	9.81769e-006
1.70	0.10895	0.10896	1.02057e-005
1.80	0.11294	0.11295	1.05796e-005
1.90	0.11679	0.11680	1.08234e-005
2.00	0.12051	0.12051	3.89216e-006

TABLE 3.4 – Comparison of the numerical and the exact solutions with $h = 0.005$, $h = 0.0005$, $M = 50$ and $n = 30$.

Example 3.5. Let $(x, t) \in [0, 1] \times [1, 3]\,,\ \alpha = 1.1$ and

$$f(x,t) = \frac{2}{\Gamma(3-\alpha)} x^3 \left(\log t\right)^{2-\alpha} - 6x \left(\log\left(\frac{t}{2}\right)\right)^2,$$

consider the following time fractional diffusion-wave equation

$$\begin{cases} {}^{CH}\partial_t^{\alpha} u(x,t) = \dfrac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \\ u(x,1) = x^3 \left(\log\left(\dfrac{1}{2}\right)\right)^2, \partial_t u(x,1) = 2x^3 \log\left(\dfrac{1}{2}\right), \\ u(0,t) = 0, \partial_x u(1,t) = 3\left(\log\left(\dfrac{t}{2}\right)\right)^2. \end{cases} \tag{3.25}$$

The exact solution of the given problem (4.16) is given by

$$u(x,t) = x^3 \left(\log\left(\frac{t}{2}\right)\right)^2.$$

In this example, taking $n = 70$ and applying the same method described previously, the results are shown in Figure 4.3 and Table 3.5 for various values of h.

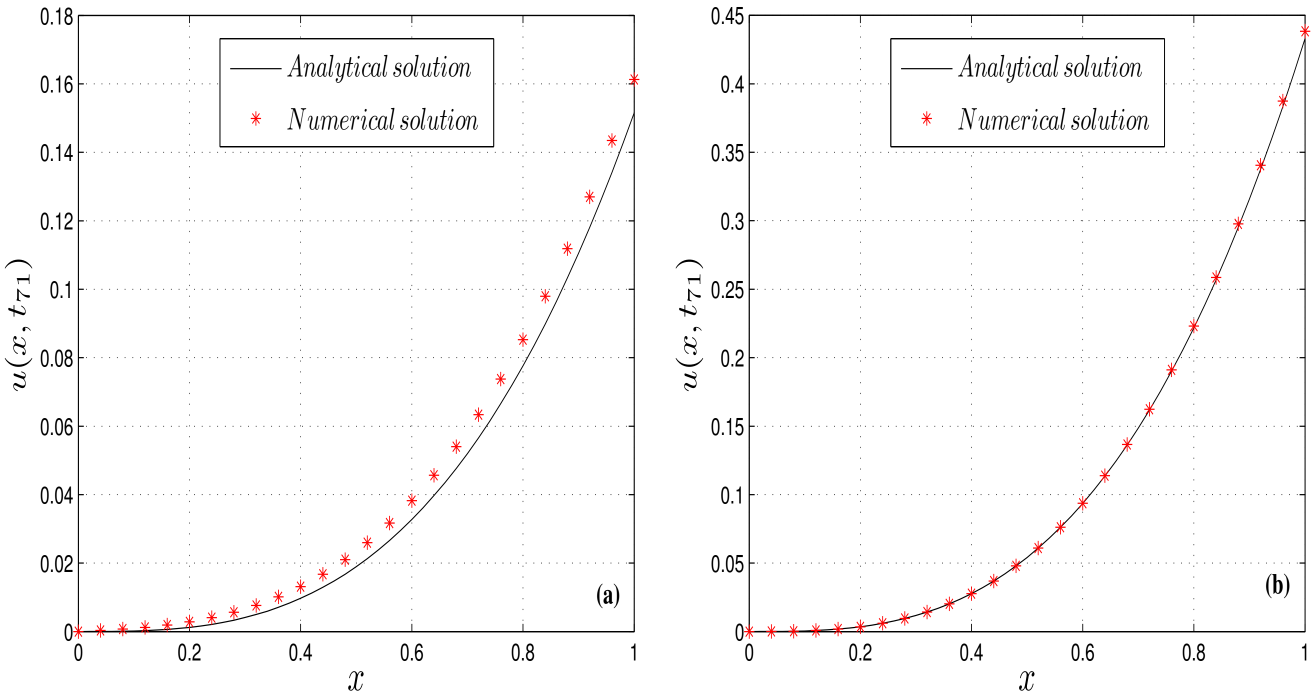

FIGURE 3.5 – Graphical comparison of the numerical and the exact solutions with $(a)\, h = 0.005$, $(b)\, h = 0.0005$, $M = 50$ and $n = 70$.

x	Exa_sol	App_sol	Error for $h = 0.005$
0.00	0.00000	0.00000	0.00000e+000
0.10	0.00015	0.00095	8.00015e-004
0.20	0.00121	0.00284	1.62470e-003
0.30	0.00409	0.00659	2.49530e-003
0.40	0.00970	0.01313	3.42643e-003
0.50	0.01895	0.02337	4.42328e-003
0.60	0.03274	0.03822	5.47936e-003
0.70	0.05200	0.05857	6.57502e-003
0.80	0.07761	0.08529	7.67678e-003
0.90	0.11051	0.11925	8.73752e-003
1.00	0.15159	0.16129	9.69752e-003

x	Exa_sol	App_sol	Error for $h = 0.0005$
0.00	0.00000	0.00000	0.00000e+000
0.10	0.00043	0.00043	2.50137e-007
0.20	0.00347	0.00347	8.12610e-007
0.30	0.01170	0.01170	2.70622e-006
0.40	0.02773	0.02774	1.06369e-005
0.50	0.05416	0.05421	4.36057e-005
0.60	0.09359	0.09376	1.60976e-004
0.70	0.14863	0.14913	5.00119e-004
0.80	0.22185	0.22314	1.28656e-003
0.90	0.31588	0.31864	2.76059e-003
1.00	0.43331	0.43834	5.03401e-003

TABLE 3.5 – Comparison of the numerical and the exact solutions with $h = 0.005$, $h = 0.0005$, $M = 50$ and $n = 70$.

3.2.4 Dicussion and results

In this section we have discussed the numerical solutions of the time-fractional diffusion wave equation (TFDWE) with Dirichlet-Neumann initial and boundary conditions. The differential operator was defined in Caputo-Hadamard sense. Also, the convergence and stability of the scheme are proved. The major goal of this work is to find accurate approximate solutions for (TFDWE) of order $1 < \alpha \leq 2$. Hence, we carry out this goal by using the finite difference method (FDM). The efficiency of (FDM) was discussed and illustrated by solving some examples of (TFDWE). We found that our method is powerful and efficient in finding numerical solutions for those equations. Moreover, the error in the examples we have treated is approximately of order $\mathcal{O}(h^{2-\alpha} + k^2)$ which supports our theoretical analysis.

CHAPTER 4

FDM FOR SOLVING FPDE INVOLVING CAPUTO-KATUGAMPOLA DERIVATIVE

This chapter is divided into two sections. In the first section, we investigate the finite difference methods for the time-fractional equation in one spatial dimension. In the second section, we construct the finite difference methods for the space-fractional equations in one spatial dimension involving Caputo–Katugampola fractional derivative and Riesz–Caputo–Katugampola fractional derivative respectively.

4.1 Approximate solution of TFDWE using the Caputo–Katugampola time-fractional derivative

In this section we proposes a FDM to obtain the numerical solution of time fractional diffusion-wave equation (TFDWE) involving the Caputo–Katugampola time-fractional derivative operator given by

$$ {}^{C}\partial_t^{\alpha,\rho} u\,(x,t) = \frac{\partial^2 u\,(x,t)}{\partial x^2} + f(x,t),\ 0 \le x \le L,\ t_0 \le t \le T, \tag{4.1} $$

with Dirichlet-Neumann initial and boundary conditions

$$ u\,(x,t_0) = \varphi_1(x),\ \frac{\partial u}{\partial t}\,(x,t_0) = \varphi_2(x),\ \ 0 \le x \le L, \tag{4.2} $$

$$ u\,(0,t) = \psi_1(t),\ \frac{\partial u}{\partial x}\,(L,t) = \psi_2(t),\ \ t_0 \le t \le T, \tag{4.3} $$

where $T > 0$, $L > 0$, $\rho > 1$, $1 < \alpha \leqslant 2$ and ${}^{C}\partial_t^{\alpha,\rho} u\,(x,t)$ is the fractional order Caputo-Katugampola sense derivative. φ_1, φ_2, ψ_1 and ψ_2 are continuous functions and $f(x,t)$ is the source term, whereas u is unknown and needs to be determined. An explicit difference approximation for the TFDWE is presented. Stability and convergence of the method are discussed using mathematical induction. Finally, a numerical examples are given. The numerical results are in excellent agreement with our theoretical analysis.

4.1.1 The finite difference scheme

From (2.16), (2.20) and (2.21), the time fractional approximation of ${}^{C}\partial_t^{\alpha,\rho}u\,(x,t)$ given by

$$ {}^{C}\partial_t^{\alpha,\rho}u\,(x,t_{n+1}) = \frac{h^{2-\alpha}\rho^{\alpha-2}}{\Gamma\,(3-\alpha)}\sum_{j=0}^{n} b_j^{\alpha,\rho}\left(u_i^{j+1}-2u_i^{j}+u_i^{j-1}\right) \tag{4.4} $$

By using the time fractional approximation (4.4) and (3.10), we obtain the following numerical approximation to equation 4.1

$$ \frac{h^{2-\alpha}\rho^{\alpha-2}}{\Gamma\,(3-\alpha)}\sum_{j=0}^{n} b_j^{\alpha,\rho}\left(u_i^{j+1}-2u_i^{j}+u_i^{j-1}\right) = \frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{k^2}+f_i^{n+1}, \tag{4.5} $$

The resulting equation can be written as the following form :

$$ \begin{aligned} -\lambda u_{i+1}^{n+1}+\left(2\lambda+b_n^{\alpha,\rho}\right)u_i^{n+1}-\lambda u_{i-1}^{n+1} = &-b_n^{\alpha,\rho}u_i^{n-1}+2b_n^{\alpha,\rho}u_i^{n} \\ &-\sum_{j=0}^{n-1} b_j^{\alpha,\rho}\left(u_i^{j+1}-2u_i^{j}+u_i^{j-1}\right) \\ &+h^2\Gamma\,(3-\alpha)\,f_i^{n+1}, \end{aligned} $$

For each $n\in\{0,1,...,N-1\}$ and $i\in\{0,1,...,M\}$, where $\lambda=\dfrac{h^{\alpha-2}\Gamma\,(3-\alpha)}{\rho^{\alpha-2}k^2}$, then

1. For $n=0$ and $i=1,2,...,M-1$

$$ -\lambda u_{i+1}^{1}+\left(2\lambda+b_0^{\alpha,\rho}\right)u_i^{1}-\lambda u_{i-1}^{1} = b_0^{\alpha,\rho}(u_i^{0}+h\varphi_2^{i})+\frac{h^{\alpha-2}\Gamma\,(3-\alpha)}{\rho^{\alpha-2}}f_i^{1}, \tag{4.6} $$

2. For $n>0$, $i=1,2,...,M-1$

$$ \begin{aligned} -\lambda u_{i+1}^{n+1}+\left(2\lambda+b_n^{\alpha,\rho}\right)u_i^{n+1}-\lambda u_{i-1}^{n+1} = &\sum_{j=1}^{n-1} c_j u_i^{j}+\left(2b_n^{\alpha,\rho}-b_{n-1}^{\alpha,\rho}\right)u_i^{n}+\left(b_0-b_1\right)u_i^{0} \\ &+b_0^{\alpha,\rho}h\varphi_2^{i}+\frac{h^{\alpha-2}\Gamma\,(3-\alpha)}{\rho^{\alpha-2}}f_i^{n+1}, \end{aligned} \tag{4.7} $$

where $c_j=\left(-b_{j+1}^{\alpha,\rho}+2b_j^{\alpha,\rho}-b_{j-1}^{\alpha,\rho}\right)$ for $j=1,2,\cdots,n-1$.

The above equation (4.6) and (4.7) can be written as

$$\begin{cases} \mathbf{U}^0 = \boldsymbol{\varphi}_1 \\ A_0 \mathbf{U}^1 = b_0^{\alpha,\rho}(\mathbf{U}^0 + h\boldsymbol{\varphi}_2) + \mathbf{F}^1 \\ A_n \mathbf{U}^{n+1} = \left(b_0^{\alpha,\rho} - b_1^{\alpha,\rho}\right)\mathbf{U}^0 + \sum_{j=1}^{n-1} c_j \mathbf{U}_i^j + \left(2b_n^{\alpha,\rho} - b_{n-1}^{\alpha,\rho}\right)\mathbf{U}^n + b_0^{\alpha,\rho} h \boldsymbol{\varphi}_2 + \mathbf{F}^{n+1} \end{cases}, \tag{4.8}$$

where, for $n = 0, 1, ..., N-1$,

$$A_n = \begin{pmatrix} 2\lambda + b_n^{\alpha,\rho} & -\lambda & 0 & \cdots & 0 \\ -\lambda & 2\lambda + b_n^{\alpha,\rho} & -\lambda & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & \ddots & \ddots & -\lambda \\ 0 & \cdots & 0 & -\lambda & \lambda + b_n^{\alpha,\rho} \end{pmatrix},$$

and

$$\mathbf{U}^n = \begin{pmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{M-1}^n \end{pmatrix}, \ \boldsymbol{\phi}_1 = \begin{pmatrix} \varphi_1^1 \\ \varphi_1^2 \\ \vdots \\ \varphi_1^{M-1} \end{pmatrix}, \ \boldsymbol{\phi}_2 = \begin{pmatrix} \varphi_2^1 \\ \varphi_2^2 \\ \vdots \\ \varphi_2^{M-1} \end{pmatrix}$$

$$\mathbf{F}^{n+1} = \begin{pmatrix} \lambda \psi_1^{n+1} + \frac{h^{\alpha-2}\Gamma(3-\alpha)}{\rho^{\alpha-2}} f_1^{n+1} \\ \frac{h^{\alpha-2}\Gamma(3-\alpha)}{\rho^{\alpha-2}} f_2^{n+1} \\ \vdots \\ \frac{h^{\alpha-2}\Gamma(3-\alpha)}{\rho^{\alpha-2}} f_{M-2}^{n+1} \\ \lambda k \psi_2^{n+1} + \frac{h^{\alpha-2}\Gamma(3-\alpha)}{\rho^{\alpha-2}} f_{M-1}^{n+1} \end{pmatrix}.$$

Remark 4.1. The tridiagonal matrix A is symmetric, with strictly dominant diagonal, so the system (4.8) admits a unique solution.

Lemma 4.1. *The coefficient $b_j^{\alpha,\rho}$, $j = 0, 1, 2, \ldots, n$, in (2.21) satisfy*

(a) $b_j^{\alpha,\rho} > 0, \ j = 0, 1, 2, \ldots, n.$

(b) $b_j^{\alpha,\rho} < b_{j+1}^{\alpha,\rho}, \ j = 0, 1, 2, \ldots, n.$

Proof. Using the properties of functions

$f(x) = (-x)^{2-\alpha}, \ (x \leqslant 0)$ and $h(x) = \dfrac{t_j^{2(1-\rho)}}{(t_{j+1}-t_j)^2}\left[(1-x)^{2-\alpha} - (-x)^{2-\alpha}\right]$, where $x = j-n$. □

4.1.2 Stability and convergence of the approximate scheme

Now, we analyze the stability via mathematical induction method, we suppose that $\widetilde{u}_i^n$, for $i = 0,1,2,\ldots,M$ and $n = 0,1,2,\ldots,N$ is the approximate solution of (4.6) and (4.7), the error

$$\varepsilon_i^n = \widetilde{u}_i^n - u_i^n,$$

from (4.6), (4.7) we have

$$-\lambda\varepsilon_{i+1}^1 + \left(2\lambda + b_0^{\alpha,\rho}\right)\varepsilon_i^1 - \lambda\varepsilon_{i-1}^1 = b_0^{\alpha,\rho}\varepsilon_i^0, \quad \text{for } n=0, \tag{4.9}$$

$$\begin{aligned}-\lambda\varepsilon_{i+1}^{n+1} + \left(2\lambda + b_n^{\alpha,\rho}\right)\varepsilon_i^{n+1} - \lambda\varepsilon_{i-1}^{n+1} =& \sum_{j=1}^{n-1} c_j\varepsilon_i^j + \left(2b_n^{\alpha,\rho} - b_{n-1}^{\alpha,\rho}\right)\varepsilon_i^n \\ &+ \left(b_0^{\alpha,\rho} - b_1^{\alpha,\rho}\right)\varepsilon_i^0, \quad \text{for } n>0,\end{aligned} \tag{4.10}$$

which can be written as

$$\begin{cases} A_0\mathbf{E}^1 = b_0^{\alpha,\rho}\mathbf{E}^0, \quad \text{for } n=0, \\ A_n\mathbf{E}^{n+1} = \displaystyle\sum_{j=1}^{n-1} c_j\mathbf{E}^j + \left(2b_n - b_{n-1}\right)\mathbf{E}^n + \left(b_0 - b_1\right)\mathbf{E}^0, \quad \text{for } n>0, \end{cases} \tag{4.11}$$

where $\mathbf{E}^n = \left(\varepsilon_1^n, \varepsilon_2^n, \ldots, \varepsilon_{M-1}^n\right)^T$. Hence, the following result can be proved.

Lemma 4.2. *We have*

$$\left\|\mathbf{E}^{n+1}\right\|_\infty \leqslant C\left\|\mathbf{E}^0\right\|_\infty, \ n = 0,1,2,\ldots,N. \tag{4.12}$$

Proof. Let $\|\mathbf{E}^{n+1}\|_\infty = \left|\varepsilon_l^{n+1}\right| = \max\limits_{1\leqslant i\leqslant M-1}\left|\varepsilon_i^{n+1}\right|$, then

1. For $n=0$, we have

$$\begin{aligned} b_0^{\alpha,\rho}\left|\varepsilon_l^1\right| &= -\lambda\left|\varepsilon_l^1\right| + \left(2\lambda + b_0^{\alpha,\rho}\right)\left|\varepsilon_l^1\right| - \lambda\left|\varepsilon_l^1\right| \\ &\leqslant -\lambda\left|\varepsilon_{l+1}^1\right| + \left(2\lambda + b_0^{\alpha,\rho}\right)\left|\varepsilon_l^1\right| - \lambda\left|\varepsilon_{l-1}^1\right| \\ &\leqslant \left|-\lambda\varepsilon_{l+1}^1 + \left(2\lambda + b_0^{\alpha,\rho}\right)\varepsilon_l^1 - \lambda\varepsilon_{l-1}^1\right| \\ &\leqslant b_0^{\alpha,\rho}\left|\varepsilon_l^0\right| \\ \text{hence, } \left|\varepsilon_l^1\right| &\leqslant \left|\varepsilon_l^0\right|, \end{aligned}$$

Its follows

$$\left\|\mathbf{E}^1\right\|_\infty \leqslant \left\|\mathbf{E}^0\right\|_\infty.$$

2. For $n > 0$, we assum that $\|\mathbf{E}^j\|_\infty \leqslant c\,\|\mathbf{E}^0\|_\infty\,, \quad j = 1, 2, \ldots, n$

$$\begin{aligned}
b_n^{\alpha,\rho}\left|\varepsilon_l^{n+1}\right| &= -\lambda\left|\varepsilon_l^{n+1}\right| + \left(2\lambda + b_n^{\alpha,\rho}\right)\left|\varepsilon_l^{n+1}\right| - \lambda\left|\varepsilon_l^{n+1}\right| \\
&\leqslant -\lambda\left|\varepsilon_{l+1}^{n+1}\right| + \left(2\lambda + b_n^{\alpha,\rho}\right)\left|\varepsilon_l^{n+1}\right| - \lambda\left|\varepsilon_{l-1}^{n+1}\right| \\
&\leqslant \left|-\lambda\varepsilon_{l+1}^{n+1} + \left(2\lambda + b_n^{\alpha,\rho}\right)\varepsilon_l^{n+1} - \lambda\varepsilon_{l-1}^{n+1}\right| \\
&\leqslant \left|\sum_{j=1}^{n-1} c_j\varepsilon_l^n + \left(2b_n^{\alpha,\rho} - b_{n-1}^{\alpha,\rho}\right)\varepsilon_l^n + \left(b_0^{\alpha,\rho} - b_1^{\alpha,\rho}\right)\varepsilon_l^0\right| \\
&\leqslant \left|\sum_{j=1}^{n-1} c_j\right| c\left\|\mathbf{E}^0\right\|_\infty + \left|\left(2b_n^{\alpha,\rho} - b_{n-1}^{\alpha,\rho}\right)\right| c\left\|\mathbf{E}^0\right\|_\infty \\
&\quad + \left|\left(b_0^{\alpha,\rho} - b_1^{\alpha,\rho}\right)\right| c\left\|\mathbf{E}^0\right\|_\infty,
\end{aligned}$$

because $b_j^{\alpha,\rho}$ is increasing (using lemma 4.1 (2)) we have

$$\begin{aligned}
&\left|\sum_{j=1}^{n-1} c_j\right| c\left\|\mathbf{E}^0\right\|_\infty + \left|\left(2b_n^{\alpha,\rho} - b_{n-1}^{\alpha,\rho}\right)\right| c\left\|\mathbf{E}^0\right\|_\infty + \left|\left(b_0^{\alpha,\rho} - b_1^{\alpha,\rho}\right)\right| c\left\|\mathbf{E}^0\right\|_\infty \\
&\leqslant \left[-\sum_{j=1}^{n-1}\left(b_j^{\alpha,\rho} - b_{j+1}^{\alpha,\rho}\right) + \sum_{j=1}^{n-1}\left(b_j^{\alpha,\rho} - b_{j-1}^{\alpha,\rho}\right) + \left(2b_n^{\alpha,\rho} - 2b_{n-1}^{\alpha,\rho}\right) + b_{n-1}\right] c\left\|\mathbf{E}^0\right\|_\infty \\
&\quad + \left(b_1^{\alpha,\rho} - b_0^{\alpha,\rho}\right) c\left\|\mathbf{E}^0\right\|_\infty \\
&\leqslant \left[\left(b_n^{\alpha,\rho} - b_1^{\alpha,\rho}\right) + \left(b_{n-1}^{\alpha,\rho} - b_0^{\alpha,\rho}\right) + \left(2b_n^{\alpha,\rho} - 2b_{n-1}^{\alpha,\rho}\right) + b_{n-1}^{\alpha,\rho}\right] c\left\|\mathbf{E}^0\right\|_\infty \\
&\quad + \left(b_1^{\alpha,\rho} - b_0^{\alpha,\rho}\right) c\left\|\mathbf{E}^0\right\|_\infty \\
&\leqslant \left(3b_n^{\alpha,\rho} - 2b_0^{\alpha,\rho}\right) c\left\|\mathbf{E}^0\right\|_\infty \\
&\leqslant 3b_n^{\alpha,\rho} c\left\|\mathbf{E}^0\right\|_\infty,
\end{aligned}$$

finally, we find that

$$\|\mathbf{E}^{n+1}\|_\infty \leqslant C\,\|\mathbf{E}^0\|_\infty\,.$$

□

Hence, the following theorem is obtained.

Theorem 4.1. *The solution of the discretised scheme (4.6) and (4.7) for the time fractional diffusion equation (4.1) is unconditionally stable.*

Then, the convergence analysis of the approximate scheme (4.6) and (4.7).is discussed

Theorem 4.2. *Let $u\,(x_i, t_n)$ be the exact solution of the time fractional diffusion equation (4.1) at mesh points (x_i, t_n) where $i = 0, 1, 2, \ldots, M,\ n = 0, 1, 2, \ldots, N$ and u_i^n the approximate value of $u(x_i, t_n)$ computed using the difference scheme (4.6) and (4.7). Then there is a positive constant $C^{\alpha,\rho}$, such that*

$$|u_i^n - u\,(x_i, t_n)| \leqslant C^{\alpha,\rho}\left(h^{2-\alpha} + k^2\right).$$

Proof. Define $e_i^n = u(x_i, t_n) - u_i^n$, where $\mathbf{e}^n = \left(e_1^n, e_2^n, \ldots, e_{M-1}^n\right)^T$.
Using $\mathbf{e}^0 = (0, 0, \ldots, 0)$. Substitution $u_i^n = u(x_i, t_n) - e_i^n$ in to (4.6) and (4.7) leads to :

1. For $n = 0$, the approximate scheme (4.6) gives
$$\begin{aligned}
-\lambda e_{i+1}^1 + \left(2\lambda + b_0^{\alpha,\rho}\right) e_i^1 - \lambda e_{i-1}^1 =& -\lambda u(x_{i+1}, t_1) + \left(2\lambda + b_0^{\alpha,\rho}\right) u(x_i, t_1) \\
& - \lambda u(x_{i-1}, t_1) - b_0^{\alpha,\rho}(u(x_i, t_0) - h\varphi_2^i) \\
& - \frac{h^{\alpha-2}\Gamma(3-\alpha)}{\rho^{\alpha-2}} f_i^1 \\
=& R_i^1.
\end{aligned}$$

2. For $n > 0$, the approximate scheme (4.7) gives
$$\begin{aligned}
-\lambda e_{i+1}^{n+1} + \left(2\lambda + b_n^{\alpha,\rho}\right) e_i^{n+1} - \lambda e_{i-1}^{n+1} =& \sum_{j=1}^{n-1} c_j e_i^j + \left(2b_n^{\alpha,\rho} - b_{n-1}^{\alpha,\rho}\right) e^n \\
& + \left(b_0^{\alpha,\rho} - b_1^{\alpha,\rho}\right) e_i^0 + R_i^{n+1},
\end{aligned}$$
where
$$\begin{aligned}
R_i^{n+1} =& -\lambda u(x_{i+1}, t_{n+1}) + \left(2\lambda + b_n^{\alpha,\rho}\right) u(x_i, t_{n+1}) - \lambda u(x_{i-1}, t_{n+1}) \\
& - \sum_{j=1}^{n-1} c_j u(x_i, t_j) - \left(2b_n^{\alpha,\rho} - b_{n-1}^{\alpha,\rho}\right) u(x_i, t_n) - \left(b_0^{\alpha,\rho} - b_1^{\alpha,\rho}\right) u(x_i, t_0) \\
& - b_0^{\alpha,\rho} h\varphi_2^i - \frac{h^{\alpha-2}\Gamma(3-\alpha)}{\rho^{\alpha-2}} f_i^{n+1} \\
=& \sum_{j=0}^{n} b_j^{\alpha,\rho} \left(u(x_i, t_{j-1}) - 2u(x_i, t_j) + u(x_i, t_{j+1})\right) \\
& - \lambda \left(u(x_{i+1}, t_{n+1}) - 2u(x_i, t_{n+1}) + u(x_{i-1}, t_{n+1})\right) - \frac{h^{\alpha-2}\Gamma(3-\alpha)}{\rho^{\alpha-2}} f_i^{n+1}.
\end{aligned}$$
From (4.1), we have
$$\begin{aligned}
R_i^{n+1} =& \frac{h^{\alpha-2}\Gamma(3-\alpha)}{\rho^{\alpha-2}} \left({}^{C}\partial_t^{\alpha,\rho}(x_i, t_{n+1}) - \frac{\partial^{(2)} u(x_i, t_{n+1})}{\partial x^2} - f_i^{n+1} - C_2^{\alpha,\rho} h^{2-\alpha} + C_1 k^2\right) \\
=& \frac{h^{\alpha-2}\Gamma(3-\alpha)}{\rho^{\alpha-2}} \left(-C_2^{\alpha,\rho} h^{2-\alpha} + C_1 k^2\right).
\end{aligned} \tag{4.13}$$
Hence, there exist $C_3^{\alpha,\rho} > 0$, such that
$$\left|R_i^{n+1}\right| \leqslant C_3^{\alpha,\rho} h^{\alpha-2} \left(h^{2-\alpha} + k^2\right).$$

Consequently, using mathematical induction, we will prove for $n = 0, 1, 2, \ldots, N$

$$\left\|\mathbf{e}^{n+1}\right\|_\infty \leqslant C_4^{\alpha,\rho} \left(b_n^{\alpha,\rho}\right)^{-1} h^{\alpha-2} \left(h^{2-\alpha} + k^2\right).$$

we have

Let $\left\|\mathbf{e}^{n+1}\right\|_\infty = \left|e_l^{n+1}\right| = \max_{1\leqslant i\leqslant M-1} \left|e_i^{n+1}\right|,$

1. For $n = 0$, we get

$$\begin{aligned} b_0^{\alpha,\rho}\left|e_l^1\right| &= -\lambda\left|e_l^1\right| + \left(2\lambda + b_0^{\alpha,\rho}\right)\left|e_l^1\right| - \lambda\left|e_l^1\right| \\ &\leqslant -\lambda\left|e_{l+1}^1\right| + \left(2\lambda + b_0^{\alpha,\rho}\right)\left|e_l^1\right| - \lambda\left|e_{l-1}^1\right| \\ \text{imply, } \left|e_l^1\right| &\leqslant \left(b_0^{\alpha,\rho}\right)^{-1}\left|R_i^1\right| \\ &\leqslant C_3^{\alpha,\rho}\left(b_0^{\alpha,\rho}\right)^{-1} h^{\alpha-2}\left(h^{2-\alpha} + k^2\right). \end{aligned}$$

2. For $n > 0$, suppose that $\left|e_l^j\right| \leqslant C_3^{\alpha,\rho}\left(b_{j-1}^{\alpha,\rho}\right)^{-1} h^{\alpha-2}\left(h^{2-\alpha} + k^2\right),\ (j = 1, \ldots, n)$, because $\left(b_{j-1}^{\alpha,\rho}\right)^{-1} \leqslant \left(b_0^{\alpha,\rho}\right)^{-1}$ for $j = 1, 2, \ldots, n$, using lemma 4.1 we get

$$\left|e_l^{n+1}\right| \leqslant C_3^{\alpha,\rho}\left(\left(3b_n^{\alpha,\rho} + 1\right)\left(b_0^{\alpha,\rho}\right)^{-1}\right)\left(b_n^{\alpha,\rho}\right)^{-1} h^{\alpha-2}\left(h^{2-\alpha} + k^2\right),$$

then

$$\left\|\mathbf{e}^{n+1}\right\|_\infty \leqslant C_4^{\alpha,\rho}\left(b_n^{\alpha,\rho}\right)^{-1} h^{\alpha-2}\left(h^{2-\alpha} + k^2\right),$$

where $C_4^{\alpha,\rho} = C_3^{\alpha,\rho}$ if $n = 0$ and $C_4^{\alpha,\rho} = C_3^{\alpha,\rho}\left(\left(3b_n^{\alpha,\rho} + 1\right)\left(b_0^{\alpha,\rho}\right)^{-1}\right)$ if $n = 1, 2, \ldots, N$. We can prove that

$$\lim_{n\to\infty} \frac{\left(b_n^{\alpha,\rho}\right)^{-1}}{\left(\frac{t_0^\rho}{h} + n\right)^{\alpha-2}} = 0,$$

therefor, there exist a constant $\zeta > 0$ such that

$$\left\|\mathbf{e}^{n+1}\right\|_\infty \leqslant C_4^{\alpha,\rho}\zeta\left(\frac{t_0^\rho}{h} + n\right)^{\alpha-2} h^{\alpha-2}\left(h^{2-\alpha} + k^2\right).$$

Because

$$\begin{aligned} \left(\frac{t_0^\rho}{h} + n\right)^{\alpha-2} h^{\alpha-2} &= t_n^{\rho(\alpha-2)} \\ &\leqslant T^{\rho(\alpha-2)}, \end{aligned}$$

is finite, we have

$$\left|u_i^n - u\left(x_i, t_n\right)\right| \leqslant C^{\alpha,\rho}\left(h^{2-\alpha} + k^2\right).$$

□

4.1.3 Numerical examples

In this part, we present some examples to illustrate the usefulness of our main results.

Example 4.1. Let $(x,t) \in [0,1] \times \left[1, \sqrt{3}\right]$, $\alpha = 1,1$, $\rho = 2$ and

$$f(x,t) = \frac{(2\rho - 1)\,\rho^{\alpha-2}}{\Gamma(3-\alpha)} x^2 \left(t^\rho - 1\right)^{2-\alpha} - 2\left(\frac{t^{2\rho}+2}{2\rho}\right),$$

consider the following time fractional diffusion-wave equation

$$\begin{cases} {}^{C}\partial_t^{\alpha,\rho} u(x,t) = \dfrac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \\ u(x,1) = x^2 \left(\dfrac{3}{2\rho}\right), \ \partial_t u(x,1) = x^2, \\ u(0,t) = 0, \ \partial_x u(1,t) = 2\left(\dfrac{t^{2\rho}+2}{2\rho}\right). \end{cases} \tag{4.14}$$

The exact solution of (4.14) is given by

$$u(x,t) = x^2 \left(\frac{t^{2\rho}+2}{2\rho}\right).$$

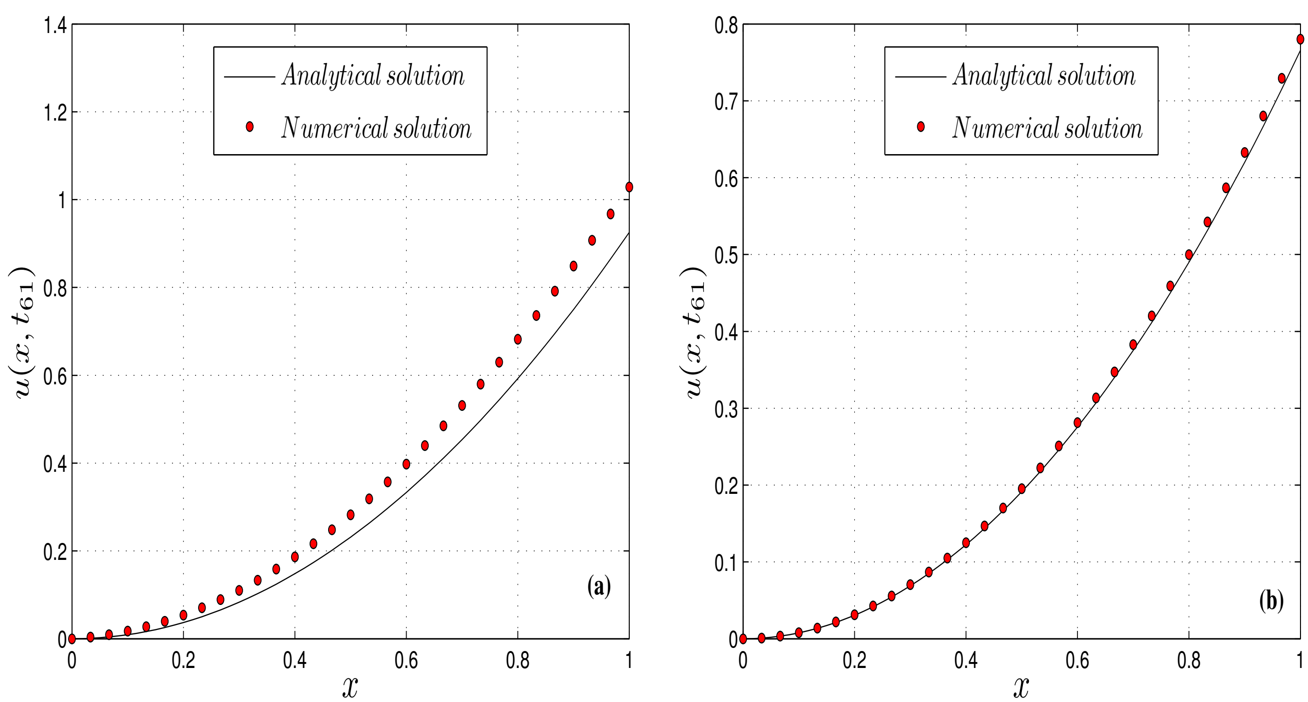

FIGURE 4.1 – Graphical comparison of the numerical and the exact solutions with (a) $h = 0.005$, (b) $h = 0.0005$, $\alpha = 1.1$, $\rho = 2$, $M = 60$ and $n = 60$.

TABLE 4.1 – Comparison of the numerical and the exact solutions with $(a)\, h = 0.005$, $(b)\, h = 0.0005$, $\alpha = 1.1,\ \rho = 2$, $M = 60$ and $n = 60$.

x	Exact sol	Approx solu	Error for h=0.005
0.0	0.00000	0.00000	0.00000e+00
0.1	0.00926	0.01736	8.10400e-03
0.2	0.03703	0.05396	1.69311e-02
0.3	0.08332	0.11027	2.69484e-02
0.4	0.14812	0.18645	3.83310e-02
0.5	0.23144	0.28238	5.09430e-02
0.6	0.33327	0.39759	6.43145e-02
0.7	0.45362	0.53124	7.76158e-02
0.8	0.59248	0.68212	8.96335e-02
0.9	0.74986	0.84861	9.87512e-02
1.0	0.92576	1.02870	1.02939e-01

Table 1(a)

x	Exact sol	Approx sol	Error for h=0.0005
0.0	0.00000	0.00000	0.00000e+00
0.1	0.00765	0.00793	2.71287e-04
0.2	0.03062	0.03137	7.54324e-04
0.3	0.06889	0.07042	1.52565e-03
0.4	0.12248	0.12508	2.60191e-03
0.5	0.19137	0.19536	3.98545e-03
0.6	0.27557	0.28125	5.67729e-03
0.7	0.37509	0.38277	7.68407e-03
0.8	0.48991	0.49993	1.00216e-02
0.9	0.62004	0.63262	1.25816e-02
1.0	0.76548	0.78004	1.45541e-02

Table 1(b)

Example 4.2. Let $(x,t) \in [0,1] \times [1,2]\,,\ \ \alpha = 1,5$, $\rho = 3$ and

$$f(x,t) = \frac{(2\rho - 1)\,\rho^{\alpha-2}}{\Gamma\left(3-\alpha\right)} \sin\left(2\pi x\right)\left(t^{\rho}-1\right)^{2-\alpha} + 4\pi^2 \sin 2\pi x \left(\frac{t^{2\rho}+3}{2\rho}\right),$$

consider the following time fractional diffusion-wave equation

$$\begin{cases} {}^{c}\partial_t^{\alpha,\rho} u\left(x,t\right) = \dfrac{\partial^2 u\left(x,t\right)}{\partial x^2} + f(x,t), \\ u\left(x,1\right) = \sin\left(2\pi x\right)\left(\dfrac{2}{\rho}\right),\ \ \partial_t u\left(x,1\right) = \sin\left(2\pi x\right), \\ u\left(0,t\right) = 0,\ \ \partial_x u\left(1,t\right) = 2\pi\left(\dfrac{t^{2\rho}+3}{2\rho}\right). \end{cases} \tag{4.15}$$

The exact solution of (4.15) is given by

$$u(x,t) = \sin\left(2\pi x\right)\left(\frac{t^{2\rho}+3}{2\rho}\right).$$

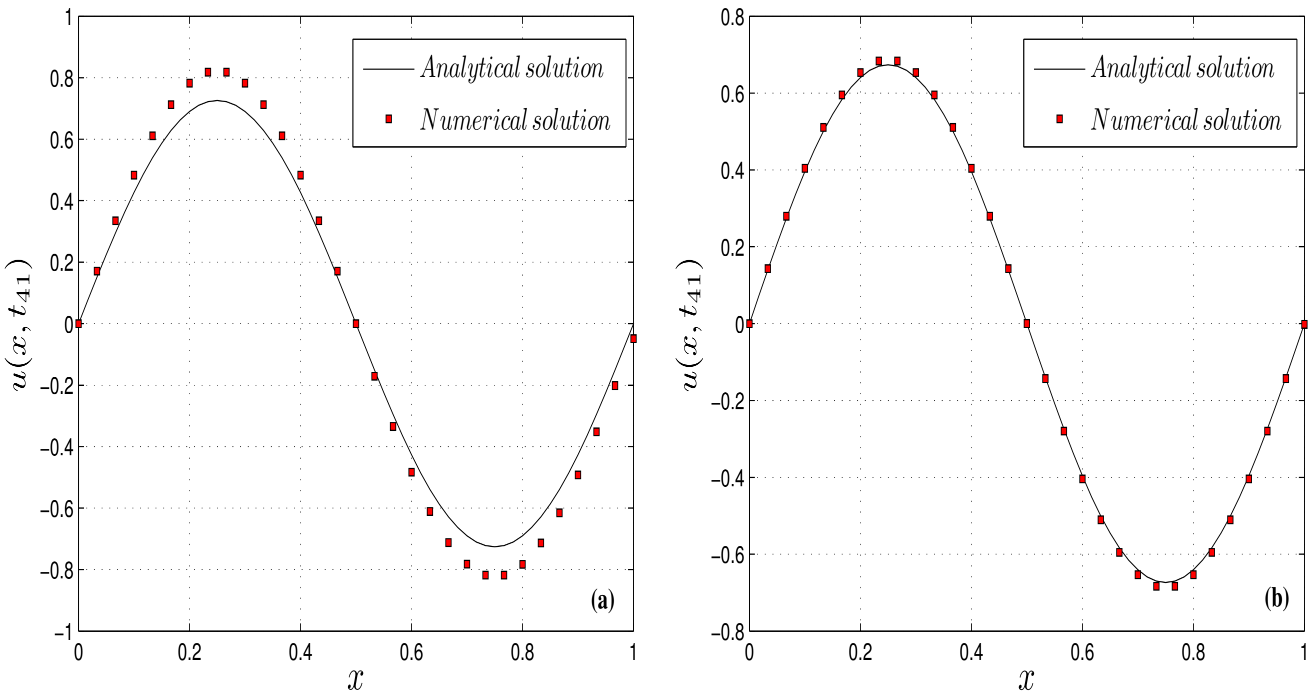

FIGURE 4.2 – Graphical comparison of the numerical and the exact solutions with $(a)\, h = 0.004$, $(b)\, h = 0.0005$, $\alpha = 1.5$, $\rho = 3$, $M = 60$ and $n = 40$.

.

x	Exact sol	Approx solu	Error for h=0.004
0.0	0.00000	0.00000	0.00000e+00
0.1	0.42662	0.48337	5.67474e-02
0.2	0.69029	0.78211	9.18193e-02
0.3	0.69029	0.78211	9.18193e-02
0.4	0.42662	0.48337	5.67474e-02
0.5	0.00000	-0.00000	4.00916e-11
0.6	-0.42662	-0.48337	5.67475e-02
0.7	-0.69029	-0.78212	9.18272e-02
0.8	-0.69029	-0.78266	9.23694e-02
0.9	-0.42662	-0.49246	6.58402e-02
1.0	-0.00000	-0.04919	4.91903e-02

Table 2(a)

x	Exact sol	Approx sol	Error for h=0.0005
0.0	0.00000	0.00000	0.00000e+00
0.1	0.39591	0.40393	8.01260e-03
0.2	0.64060	0.65357	1.29646e-02
0.3	0.64060	0.65357	1.29646e-02
0.4	0.39591	0.40393	8.01260e-03
0.5	0.00000	0.00000	4.09782e-16
0.6	-0.39591	-0.40393	8.01260e-03
0.7	-0.64060	-0.65357	1.29646e-02
0.8	-0.64060	-0.65357	1.29646e-02
0.9	-0.39591	-0.40393	8.01261e-03
1.0	-0.00000	-0.00187	1.86546e-03

Table 2(b)

TABLE 4.2 – Comparison of the numerical and the exact solutions with $(a)\, h = 0.004$, $(b)\, h = 0.0005$, $\alpha = 1.5$, $\rho = 3$, $M = 60$ and $n = 40$.

Example 4.3. Let $(x, t) \in [0, 3] \times \left[1, \sqrt{2}\right],\ \alpha = 1.7,\ \rho = 2$ and

$$f(x,t) = \frac{(2\rho - 1)\,\rho^{\alpha-2}}{\Gamma\,(3-\alpha)} \exp(x)\,(t^{\rho} - 1)^{2-\alpha} - \exp(x)\left(\frac{t^{2\rho} + t}{2\rho}\right),$$

consider the following time fractional diffusion-wave equation

$$\begin{cases} {}^{C}\partial_t^{\alpha,\rho} u\left(x,t\right) = \dfrac{\partial^2 u\left(x,t\right)}{\partial x^2} + f(x,t), \\ u\left(x,1\right) = \exp(x)\left(\dfrac{1}{\rho}\right),\ \partial_t u\left(x,1\right) = \exp(x)\left(1+\dfrac{1}{2\rho}\right), \\ u\left(0,t\right) = \left(\dfrac{t^{2\rho}+t}{2\rho}\right),\ \partial_x u\left(3,t\right) = \exp(3)\left(\dfrac{t^{2\rho}+t}{2\rho}\right). \end{cases} \tag{4.16}$$

The exact solution of (4.16) is given by

$$u(x,t) = \exp(x)\left(\frac{t^{2\rho}+t}{2\rho}\right).$$

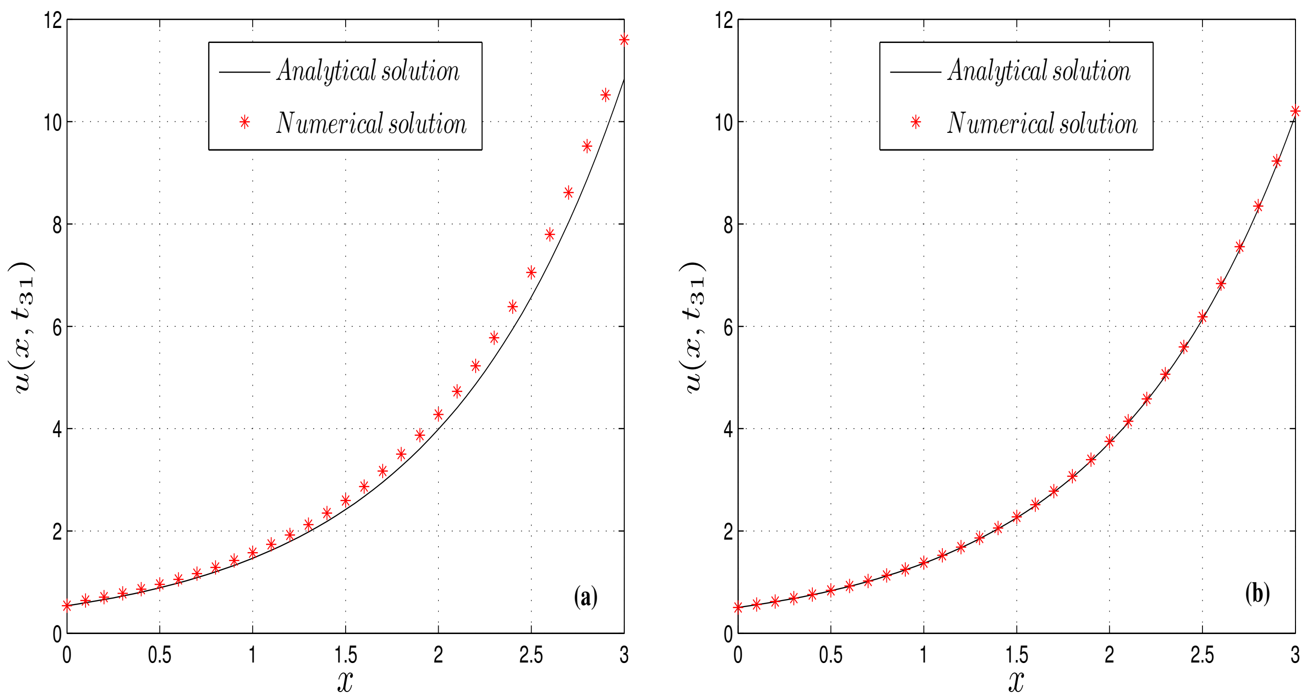

FIGURE 4.3 – Graphical comparison of the numerical and the exact solutions with (a) $h = 0.002$, (b) $h = 0.0002$, $\alpha = 1.7$ $\rho = 2$, $M = 60$ and $n = 30$.

x	Exact sol	Approx sol	Error for h=0.002
0.0	0.53959	0.53959	0.00000e+00
0.3	0.72838	0.78159	5.32104e-02
0.6	0.98321	1.05503	7.18265e-02
0.9	1.32719	1.42414	9.69557e-02
1.2	1.79152	1.92239	1.30876e-01
1.5	2.41829	2.59496	1.76664e-01
1.8	3.26436	3.50283	2.38472e-01
2.1	4.40642	4.72832	3.21904e-01
2.4	5.94804	6.38257	4.34525e-01
2.7	8.02902	8.61557	5.86547e-01
3.0	10.8380	11.6026	7.64639e-01

Table 3(a)

x	Exact sol	Approx sol	Error for h=0.0002
0.0	0.50388	0.50388	0.00000e+00
0.3	0.68017	0.68541	5.23950e-03
0.6	0.91814	0.92521	7.07259e-03
0.9	1.23935	1.24890	9.54700e-03
1.2	1.67295	1.68584	1.28871e-02
1.5	2.25825	2.27564	1.73957e-02
1.8	3.04832	3.07180	2.34818e-02
2.1	4.11480	4.14649	3.16971e-02
2.4	5.55440	5.59718	4.27867e-02
2.7	7.49765	7.55541	5.77560e-02
3.0	10.1207	10.2075	8.67568e-02

Table 3(b)

TABLE 4.3 – Comparison of the numerical and the exact solutions with $(a)\, h = 0.002$, $(b)\, h = 0.0002$, $\alpha = 1.7,\ \rho = 2, M = 60$ and $n = 30$.

4.1.4 Dicussion and results

The numerical scheme (finite difference scheme) for the time-fractional diffusion wave equation has been presented. Unconditional stability which can be provided through Caputo–Katugampola synthesis with fractional derivative operator $1 < \alpha \leq 2$ is used. Different examples have been investigated to assess the validity of the approach, showing good overall approximation and improved convergence with $C^{\alpha,\rho}\left(h^{2-\alpha} + k^2\right)$ reaching to zero.

4.2 The numerical solution of the STFDE involving the Caputo-Katugampola fractional derivative

(The content of this section has been published in : Journal of NUMERICAL ALGEBRA, CONTROL AND OPTIMIZATION)

In this section, we have appreciated the finite difference method to give the numerical solution of the space-time fractional diffusion problem defined by

$$\begin{cases} {}^{C}\partial_t^{\alpha,\rho} u\left(x,t\right) = \dfrac{\partial^{\beta,\rho} u\left(x,t\right)}{\partial\left|x\right|^{\beta}} + f(x,t),\ \ (x,t) \in \left]x_0, L\right[\times \left]t_0, T\right[\,, \\ u\left(x,t_0\right) = u_0(x),\ x \in \left[x_0, L\right], \\ \dfrac{\partial u}{\partial x}\left(x_0,t\right) = \psi(t), u\left(x_0,t\right) = \phi(t),\ u\left(L,t\right) = \varphi(t),\ \ t \in \left[t_0, T\right]. \end{cases} \tag{4.17}$$

where ${}^{C}\partial_t^{\alpha,\rho}$ and $\dfrac{\partial^{\beta,\rho}}{\partial\left|x\right|^{\beta}}$ denotes the Caputo–Katugampola fractional derivative and Riesz–

Caputo–Katugampola fractional derivative of order $0 < \alpha \leq 1$ and $1 \leq \beta \leq 2$ respectively, with $\rho > 1,\ t_0, x_0 > 0$ and $f(x,t)$ is the source term and $u_0(x),\ \psi,\ \phi,\ \varphi$ are continuous functions. Stability and convergence of the proposed scheme are discussed using mathematical induction. Finally, the proposed method is validated through numerical simulation results of different examples.

4.2.1 The finite difference scheme

In this part, for the finite difference approximation we denote u_i^{n+1} the numerical approximation to $u(x_i, t_{n+1})$ and $f_i^{n+1} = f(x_i, t_{n+1})$ and

1. The initial boundary conditions of (4.17), are discretized as

$$\begin{cases} u(x_i, t_0) = u_i^0, \\ u_x(x_0, t_{n+1}) = \psi^{n+1}, \\ u(x_0, t_{n+1}) = \phi^{n+1}, \\ u(L, t_{n+1}) = \varphi^{n+1}. \end{cases}$$

2. From (2.16), (2.26) and (2.27), the space fractional derivative term $\dfrac{\partial^{\beta,\rho} u\left(x_i, t_{n+1}\right)}{\partial \left|x\right|^{\beta}}$ can be approximated by

$$\frac{\partial^{\beta,\rho} u\left(x_i, t_{n+1}\right)}{\partial \left|x\right|^{\beta}} = \frac{k^{2-\beta}\rho^{\beta-2}}{2\Gamma\left(3-\beta\right)} \begin{pmatrix} \displaystyle\sum_{i=0}^{m-1} a_{i,m}^{\beta,\rho}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) \\ \displaystyle + \sum_{i=m}^{M-1} z_{i,m}^{\beta,\rho}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) \end{pmatrix}, \tag{4.18}$$

for $m \in \{1, 2, \ldots, M-1\}$, where

$$\begin{cases} a_{i,m}^{\beta,\rho} = \dfrac{x_i^{2(1-\rho)}}{\left(x_{i+1} - x_i\right)^2}\left[\left(m-i\right)^{2-\beta} - \left(m-i-1\right)^{2-\beta}\right],\ i = 0, \ldots, m-1, \\ z_{i,m}^{\beta,\rho} = \dfrac{x_i^{2(1-\rho)}}{\left(x_{i+1} - x_i\right)^2}\left[\left(i+1-m\right)^{2-\beta} - \left(i-m\right)^{2-\beta}\right],\ i = m, \ldots, M-1, \end{cases} \tag{4.19}$$

Now, By using the space-time fractional approximation (4.18) and (2.17) we obtain the following numerical approximation to equation the (4.17),

$$\begin{aligned}&\frac{-k^{2-\beta}\rho^{\beta-2}}{2\Gamma\left(3-\beta\right)}\left(\sum_{i=0}^{m-1}a_i^{\beta,\rho}\left(u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}\right)+\sum_{i=m}^{M-1}z_{i,m}^{\beta,\rho}\left(u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}\right)\right)\\=&\frac{-h^{1-\alpha}\rho^{\alpha-1}}{\Gamma\left(2-\alpha\right)}\sum_{j=0}^{n}b_j^{\alpha,\rho}\left(u_i^{j+1}-u_i^j\right)+f_i^{n+1},\end{aligned}$$

Then, for each $n=0,1,...,N-1$, and $m=1,...,M-1$, setting $\lambda=\dfrac{\Gamma\left(2-\alpha\right)k^{2-\beta}h^{\alpha-1}}{2\Gamma\left(3-\beta\right)\rho^{\alpha-\beta+1}}$, we obtain the following difference approximation for $l\in\{1,2,\ldots,M-1\}$

$$\sum_{i=1}^{M-1}\omega_{i,m}u_i^{n+1}+b_nu_l^{n+1}=b_0^{\alpha,\rho}u_l^0+\sum_{j=1}^{n}G_ju_l^j+V_l^{n+1}, \tag{4.20}$$

with $\mathbf{V}^{n+1}=\lambda\left(\left(a_1^{\beta,\rho}-a_0^{\beta,\rho}\right)\phi^{n+1}+z_{M-1,m}^{\beta,\rho}\varphi^{n+1}-ka_0^{\beta,\rho}\psi^{n+1}\right)+\dfrac{h^{\alpha-1}\Gamma\left(2-\alpha\right)}{\rho^{\alpha-1}}f^{n+1},\ G_j=\left(b_j^{\alpha,\rho}-b_{j-1}^{\alpha,\rho}\right)$ and

$$w_{i,m}=\begin{cases}\lambda\left(-a_{i+1,m}^{\beta,\rho}+2a_{i,m}^{\beta,\rho}-a_{i-1,m}^{\beta,\rho}\right), \text{ if } 1\le i\le m-2,\\ \lambda\left(-z_{m,m}^{\beta,\rho}+2a_{m-1,m}^{\beta,\rho}-a_{m-2,m}^{\beta,\rho}\right), \text{ if } i=m-1,\\ \lambda\left(-z_{m+1,m}^{\beta,\rho}+2z_{m,m}^{\beta,\rho}-a_{m-1,m}^{\beta,\rho}\right), \text{ if } i=m,\\ \lambda\left(-z_{i+1,m}^{\beta,\rho}+2z_{i,m}^{\beta,\rho}-z_{i-1,m}^{\beta,\rho}\right), \text{ if } m+1\le i\le M-2,\\ \lambda\left(2z_{M-1,m}^{\beta,\rho}-z_{M-2,m}^{\beta,\rho}\right), \text{ if } i=M-1.\end{cases}$$

So, for $n=0$ and $l\in\{1,2,\ldots,M-1\}$ we have

$$\sum_{i=1}^{M-1}\omega_{i,m}u_i^1+b_0^{\alpha,\rho}u_l^1=b_0^{\alpha,\rho}u_l^0+V_l^1, \tag{4.21}$$

then, with $n>0$ and $l\in\{1,2,\ldots,M-1\}$ we obtain

$$\sum_{i=1}^{M-1}\omega_{i,m}u_i^{n+1}+b_n^{\alpha,\rho}u_l^{n+1}=b_0^{\alpha,\rho}u_l^0+\sum_{j=1}^{n}G_ju_l^j+V_l^{n+1}, \tag{4.22}$$

Thus, we have the difference scheme in the matrix form

$$\begin{cases}\mathbf{U}^0=u_i^0, \text{ for } i=1,...,M-1,\\ \mathbf{A}^1\mathbf{U}^1=b_0\mathbf{U}^0+\mathbf{V}^1,\\ \mathbf{A}^n\mathbf{U}^{n+1}=b_0\mathbf{U}^0+G_1\mathbf{U}^1+G_2\mathbf{U}^2+\ldots+G_n\mathbf{U}^n+\mathbf{V}^{n+1},\end{cases},$$

with

$$\begin{cases} \mathbf{U}^0 = \left[u_1^0, u_2^0, \ldots, u_{M-1}^0\right]^T, \\ \mathbf{U}^n = \left[u_1^n, u_2^n, \ldots, u_{M-1}^n\right]^T, \\ \mathbf{V}^{n+1} = \left[V_1^{n+1}, V_2^{n+1}, \ldots, V_{M-1}^{n+1}\right]^T, \end{cases}$$

and $\mathbf{A}^n$ is square matrix of dimension $(M-1)\times(M-1)$ of coefficients :

$$\mathbf{A}^n_{(i,j)} = \begin{cases} \omega_{j,m}, & \text{if } i \neq j, \\ \omega_{i,m} + b_n^{\alpha,\rho}, & \text{if } i = j, \end{cases}$$

Lemma 4.3. *The coefficients $a_i^{\beta,\rho}$, $b_j^{\alpha,\rho}$ and $z_{i,m}^{\beta,\rho}$ in (4.19),(2.18) satisfy :*

1. $a_{i,m}^{\beta,\rho} > 0$, $z_{i,m}^{\beta,\rho} > 0$, *and* $b_j^{\alpha,\rho} > 0$, *for* $i = 0, \ldots, m-1$, $i = m, \ldots, M-1$ *and* $j = 0, \ldots, n$.
2. $a_i^{\beta,\rho} > a_{i-1}^{\beta,\rho}$ *and* $b_j^{\alpha,\rho} > b_{j-1}^{\alpha,\rho}$, *for* $i = 1, \ldots, m-1$ *and* $j = 1, \ldots, n$.
3. $z_{i+1,m}^{\beta,\rho} < z_{i,m}^{\beta,\rho}$, *for* $i = m, \ldots, M-1$.

4.2.2 Stability and convergence analysis of finite difference scheme for FDE

In the following, we discuss the stability and convergence of finite difference schemes (4.21) and (4.22). Firstly, we consider the stability of finite difference schemes (4.21) and (4.22). We suppose that $\widetilde{u}_l^n$ is the approximate solution of (4.21) and (4.22), the error $\varepsilon_l^n = \widetilde{u}_l^n - u_l^n$, for $l \in \{1, 2, \ldots, M-1\}$ and $n \in \{1, 2, \ldots, N-1\}$ satisfies

$$\begin{aligned} &\sum_{i=1}^{M-1} \omega_{i,m}\varepsilon_i^1 + b_0^{\alpha,\rho}\varepsilon_l^1 = b_0^{\alpha,\rho}\varepsilon_l^0, \\ &\sum_{i=1}^{M-1} \omega_{i,m}\varepsilon_i^{n+1} + b_n^{\alpha,\rho}\varepsilon_l^{n+1} = b_0^{\alpha,\rho}\varepsilon_l^0 + \sum_{j=1}^{n} G_j\varepsilon_l^j + V_l^{n+1}. \end{aligned} \tag{4.23}$$

So, $n = 1, 2, \ldots, N-1$, the above formula can be written in the matrix form as :

$$\begin{cases} \mathbf{A}^0\mathbf{E}^1 = b_0^{\alpha,\rho}\mathbf{E}^0, \\ \mathbf{A}^n\mathbf{E}^{n+1} = b_0^{\alpha,\rho}\mathbf{E}^0 + G_1\mathbf{E}^1 + G_2\mathbf{E}^2 + \ldots + G_n\mathbf{E}^n, \\ \mathbf{E}^0 = 0, \end{cases}$$

where $\mathbf{E}^n = \left(\varepsilon_1^n, \varepsilon_2^n, \ldots, \varepsilon_{M-1}^n\right)^T$. Hence, the stability analysis of the difference approximation is studied via mathematical induction method.

Let $\|\mathbf{E}^1\|_\infty = |\varepsilon_l^1| = \max\limits_{1\le i\le M-1} |\varepsilon_i^1|$ and

$$\gamma_{M,m} = \lambda\left(2a_{m-1,m}^{\beta,\rho} - a_{0,m}^{\beta,\rho} - a_{1,m}^{\beta,\rho} + 2\left(z_{m,m}^{\beta,\rho} - z_{M-2,m}^{\beta,\rho}\right)\right).$$

Then, for $n=0$, note that $a_{i,m}^{\beta,\rho}$ is increasing and $z_{i,m}^{\beta,\rho}$ is decreasing (Lemma 4.3), we have

$$\begin{aligned}\left(\gamma_{M,m}+b_0^{\alpha,\rho}\right)\left|\varepsilon_l^1\right| \leq &\left(\gamma_{m,M}+\lambda z_{M-1,m}^{\beta,\rho}+b_0^{\alpha,\rho}\right)\left|\varepsilon_l^1\right| \\ =&\left|\sum_{i=1}^{M-1}\omega_{i,m}\varepsilon_l^1+b_0^{\alpha,\rho}\varepsilon_l^1\right| \\ \leq & b_0^{\alpha,\rho}\left|\varepsilon_l^0\right|,\end{aligned}$$

hence, $\left|\varepsilon_l^1\right| \leq \dfrac{b_0^{\alpha,\rho}}{\left(\gamma_{M,m}+b_0^{\alpha,\rho}\right)}\left|\varepsilon_l^0\right|$. Its follows

$$\left\|\mathbf{E}^1\right\|_\infty \leq \left\|\mathbf{E}^0\right\|_\infty .$$

Let $\|\mathbf{E}^{n+1}\|_\infty=\left|\varepsilon_l^{n+1}\right|=\max\limits_{1\leq i\leq M-1}\left|\varepsilon_i^{n+1}\right|$, we assum that $\|\mathbf{E}^j\|_\infty \leq c\,\|\mathbf{E}^0\|_\infty$, $(j=1,2,\ldots,n)$, using Lemma 4.3, we also have

$$\begin{aligned}\left(\gamma_{m,M}+b_n^{\alpha,\rho}\right)\left|\varepsilon_l^{n+1}\right| \leq &\left|\sum_{i=1}^{M-1}w_{i,m}\varepsilon_l^{n+1}+b_n^{\alpha,\rho}\varepsilon_l^{n+1}\right| \\ \leq &\left|b_0^{\alpha,\rho}\varepsilon_l^0+\sum_{j=1}^{n}G_j\varepsilon_l^j\right| \\ \leq & b_0^{\alpha,\rho}\left|\varepsilon_l^0\right|+\left|\sum_{j=1}^{n}G_j\right|\left|\varepsilon_l^j\right| \\ \leq & b_0^{\alpha,\rho}\left|\varepsilon_l^0\right|+\left|\sum_{j=1}^{n}b_j^{\alpha,\rho}-b_{j-1}^{\alpha,\rho}\right|\left|\varepsilon_l^j\right| \\ \leq & b_0^{\alpha,\rho}\left|\varepsilon_l^0\right|+\left(b_n^{\alpha,\rho}-b_0^{\alpha,\rho}\right)\left|\varepsilon_l^0\right|\end{aligned}$$

finally, we find

$$\left|\varepsilon_l^{n+1}\right| \leq \frac{b_n^{\alpha,\rho}}{\left(\gamma_{M,m}+b_n^{\alpha,\rho}\right)}\left\|E^0\right\|_\infty ,$$

imply,

$$\left\|\mathbf{E}^{n+1}\right\|_\infty \leq \left\|\mathbf{E}^0\right\|_\infty .$$

Hence, the following theorem holds.

Theorem 4.3. *The finite difference schemes (4.21) and (4.22) for the FDE (4.17) are unconditionally stable.*

Secondly, we discuss the convergence of the approximate scheme (4.21) and (4.22). Let $u\left(x_i,t_n\right)$ be the exact solution of the fractional diffusion equation (4.17) at mesh points $\left(x_i,t_n\right)$ where $i=0,1,2,\ldots,M$ and $n=0,1,2,\ldots,N$.

Define $e_i^n = u(x_i, t_n) - u_i^n$ and $\mathbf{e}^n = \left(e_1^n, e_2^n, \ldots, e_{M-1}^n\right)^T$. Using $\mathbf{e}^0 = 0$, substituting $u_i^n = u(x_i, t_n) - e_i^n$ into (4.21) and (4.22) leads to :

1. For $n = 0$, and $l \in \{1, 2, \ldots, M-1\}$, we have

$$\begin{aligned}\sum_{i=1}^{M-1} \omega_{i,m} e_i^1 + b_0^{\alpha,\rho} e_l^1 &= \sum_{i=1}^{M-1} \omega_{i,m} u(x_i, t_1) + b_0^{\alpha,\rho} u(x_l, t_1) - b_0^{\alpha,\rho}\left(u(x_l, t_0) - e_l^0\right) - V_l^1 \\ &= R_l^1.\end{aligned}$$

2. For $n > 0$, and $l \in \{1, 2, \ldots, M-1\}$, the approximate scheme becomes

$$\begin{aligned}\sum_{i=1}^{M-1} \omega_{i,m} e_i^{n+1} + b_n^{\alpha,\rho} e_l^{n+1} &= \sum_{i=1}^{M-1} \omega_{i,m} u(x_i, t_{n+1}) + b_n^{\alpha,\rho} u(x_l, t_{n+1}) \\ &\quad - b_0^{\alpha,\rho}\left(u(x_l, t_0) - e_l^0\right) - \sum_{j=1}^{n} G_j\left(u(x_l, t_j) - e_l^j\right) - V_l^{n+1} \\ &= \sum_{j=1}^{n} G_j e_l^j + R_l^{n+1},\end{aligned}$$

where

$$\begin{aligned}R_l^{n+1} =& \sum_{j=0}^{n} b_j^{\alpha,\rho}\left(u(x_l, t_{j+1}) - u(x_l, t_j)\right) \\ &-\lambda\left(\begin{array}{l}\displaystyle\sum_{i=0}^{m-1} a_{i,m}^{\beta,\rho}\left(u(x_{i+1}, t_{n+1}) - 2u(x_i, t_{n+1}) + u(x_{i-1}, t_{n+1})\right) \\ \displaystyle+\sum_{i=m}^{M-1} z_{i,m}^{\beta,\rho}\left(u(x_{i+1}, t_{n+1}) - 2u(x_i, t_{n+1}) + u(x_{i-1}, t_{n+1})\right)\end{array}\right) \\ &-\frac{h^{\alpha-1}\Gamma(2-\alpha)}{\rho^{\alpha-1}} f_l^{n+1}.\end{aligned}$$

From (4.17), we have

$$\begin{aligned}R_l^{n+1} &= \frac{h^{\alpha-1}\Gamma(2-\alpha)}{\rho^{\alpha-1}} \\ &\quad \times \left({}^{C}\partial_t^{\alpha,\rho}(x_l, t_{n+1}) - \frac{\partial^{\beta,\rho} u(x_l, t_{n+1})}{\partial |x|^{\beta}} - f_l^{n+1} - c_{\alpha,\rho} h^{1-\alpha} + c_{\beta,\rho} k^{2-\beta}\right) \\ &= \frac{h^{\alpha-1}\Gamma(2-\alpha)}{\rho^{\alpha-1}}\left(-c_{\alpha,\rho} h^{1-\alpha} + c_{\beta,\rho} k^{2-\beta}\right).\end{aligned} \tag{4.24}$$

Hence, there exist $c_{\alpha,\beta,\rho} > 0$, such that

$$\left|R_i^{n+1}\right| \leq c_{\alpha,\beta,\rho}\left(1+h^{\alpha-1}k^{2-\beta}\right),\ i=1,2,\ldots,M-1,\ n=0,1,\ldots,N-1.$$

Consequently, using mathematical induction, we prove

$$\left\|\mathbf{e}^{n+1}\right\|_\infty \leq \left(b_n^{\alpha,\rho}\right)^{-1} C_{\alpha,\beta,\rho}\left(1+h^{\alpha-1}k^{2-\beta}\right).$$

Let $\|\mathbf{e}^{n+1}\|_\infty = \left|e_l^{n+1}\right| = \max\limits_{1\leq i\leq M-1}\left|e_i^{n+1}\right|$, then

1. For $n=0$ and $i\in\{1,2,\ldots,M-1\}$ we get

$$\begin{aligned}\left(\gamma_{M,m}+b_0^{\alpha,\rho}\right)\left|e_l^1\right| &\leq \left(\gamma_{m,M}+\lambda z_{M-1,m}^{\beta,\rho}+b_0^{\alpha,\rho}\right)\left|e_l^1\right| \\ &= \left|\sum_{i=1}^{M-1}\omega_{i,m}e_l^1+b_0^{\alpha,\rho}e_l^1\right| \\ &\leq \left|R_i^1\right|,\end{aligned}$$

imply,

$$\begin{aligned}\left|e_l^1\right| &\leq \left(\gamma_M+b_0^{\alpha,\rho}\right)^{-1}\left|R_i^1\right| \\ &\leq \left(b_0^{\alpha,\rho}\right)^{-1} c_{\alpha,\beta,\rho}\left(1+h^{\alpha-1}k^{2-\beta}\right).\end{aligned}$$

2. For $n>0$ and $i\in\{1,2,\ldots,M-1\}$, suppose that $\left|e_l^j\right| \leq \left(b_{j-1}^{\alpha,\rho}\right)^{-1} c_{\alpha,\beta,\rho}\left(1+h^{\alpha-1}k^{2-\beta}\right)$, for $j=1,\ldots,n$, we have $\left(b_{j-1}^{\alpha,\rho}\right)^{-1} \leq \left(b_0^{\alpha,\rho}\right)^{-1}$ (Lemma 4.3), we get

$$\begin{aligned}\left|e_l^{n+1}\right| &\leq \left(\gamma_M+b_n^{\alpha,\rho}\right)^{-1}\left|\sum_{j=1}^{n}\left(b_j^{\alpha,\rho}-b_{j-1}^{\alpha,\rho}\right)\right|\left|e_i^j\right| \\ &+\left(\gamma_M+b_n^{\alpha,\rho}\right)^{-1}\left|R_i^{n+1}\right| \\ &\leq \left(b_n^{\alpha,\rho}\right)^{-1} C'_{\alpha,\beta,\rho}\left(1+h^{\alpha-1}k^{2-\beta}\right).\end{aligned}$$

We can prove that

$$\lim_{n\to\infty}\frac{1}{b_n^{\alpha,\rho}\left(\dfrac{t_0^\rho}{h}+n\right)^{\alpha-1}}=0,$$

therefor, there exist a constant $\zeta>0$ suth that

$$\left\|e_l^{n+1}\right\|_\infty \leq \zeta\left(\frac{t_0^\rho}{h}+n\right)^{\alpha-1} C'_{\alpha,\beta,\rho}\left(1+h^{\alpha-1}k^{2-\beta}\right),$$

then

$$
\begin{aligned}
\left|e_l^{n+1}\right| &\leq C'_{\alpha,\beta,\rho}\zeta\left(\frac{t_0^\rho}{h}+n\right)^{(\alpha-1)} h^{\alpha-1}\left(h^{1-\alpha}+k^{2-\beta}\right)\\
&\leq C'_{\alpha,\beta,\rho}\zeta t_n^{\rho(\alpha-1)}\left(h^{1-\alpha}+k^{2-\beta}\right)\\
&\leq C'_{\alpha,\beta,\rho}\zeta T^{\rho(\alpha-1)}\left(h^{1-\alpha}+k^{2-\beta}\right),
\end{aligned}
$$

is finite, we have

$$
\left\|\mathbf{e}^{n+1}\right\|_\infty \leq C_{\alpha,\beta,\rho}\left(h^{1-\alpha}+k^{2-\beta}\right).
$$

Then, the convergence of the finite difference scheme is given by the following theorem :

Theorem 4.4. *Let u_i^n be the approximate value of $u(x_i,t_n)$, then there is a positive constant $C_{\alpha,\beta,\rho}$, such that*

$$
\left|u_i^n-u\left(x_i,t_n\right)\right| \leq C_{\alpha,\beta,\rho}\left(h^{1-\alpha}+k^{2-\beta}\right),\ i=1,2,\ldots,M-1,\ n=1,2,\ldots,N.
$$

4.2.3 Numerical examples

In this part, we present some examples to illustrate the usefulness of our main results.

Example 4.4. Let $(x,t)\in[1,2]\times[1,2]$ and

$$
\begin{aligned}
f(x,t) &= \left(\frac{x^{2\rho}-3}{2\rho}\right)\frac{\rho^\alpha}{\Gamma(2-\alpha)}\left(t^\rho-1\right)^{1-\alpha}\\
&\quad - t^\rho\frac{(2\rho-1)\,\rho^{\beta-2}}{2\Gamma(3-\beta)}\left(\left(x^\rho-1\right)^{2-\beta}+\left(2^\rho-x^\rho\right)^{2-\beta}\right).
\end{aligned}
$$

Consider the following space-time fractional diffusion equation

$$
\begin{cases}
{}^C\partial_t^{\alpha,\rho}u(x,t) = \dfrac{\partial^{\beta,\rho}u(x,t)}{\partial|x|^\beta}+f(x,t),\\
u(x,1) = \left(\dfrac{x^{2\rho}-3}{2\rho}\right),\\
\partial_x u(1,t) = t^\rho, u(1,t) = t^\rho\left(\dfrac{-1}{\rho}\right),\ u(2,t) = t^\rho\left(\dfrac{2^{2\rho}-3}{2\rho}\right).
\end{cases}
\tag{4.25}
$$

The exact solution for this problem is

$$
u(x,t) = t^\rho\left(\frac{x^{2\rho}-3}{2\rho}\right).
$$

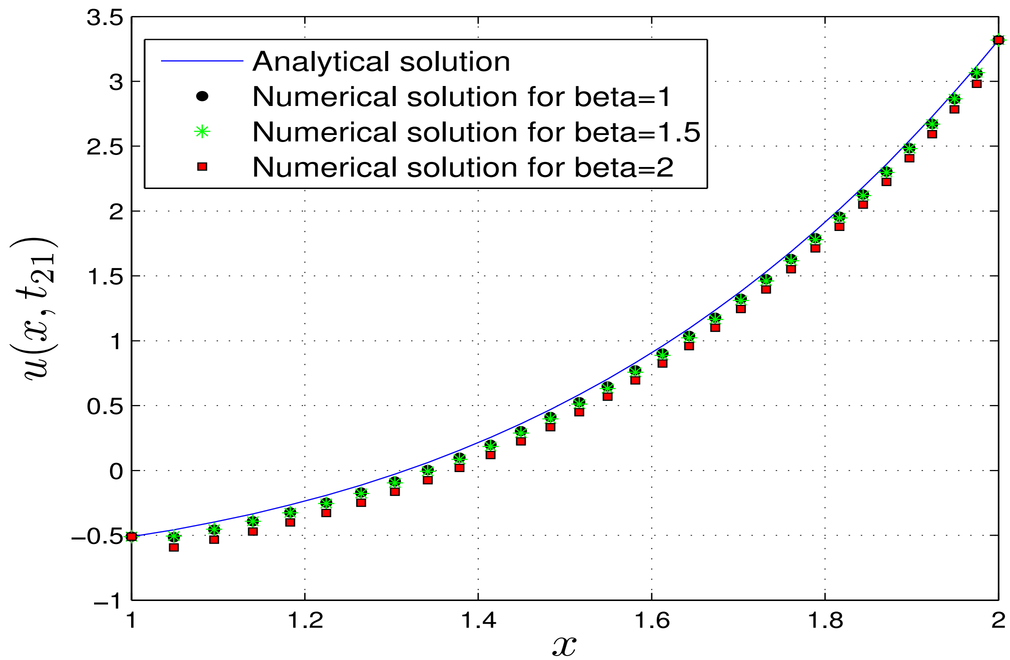

FIGURE 4.4 – Graphical comparison of the numerical and the exact solution with $h = 0.001$, $k = 0.1$, $\rho = 2$, $\alpha = 0.7$, $n = 20$ and $m = 25$.

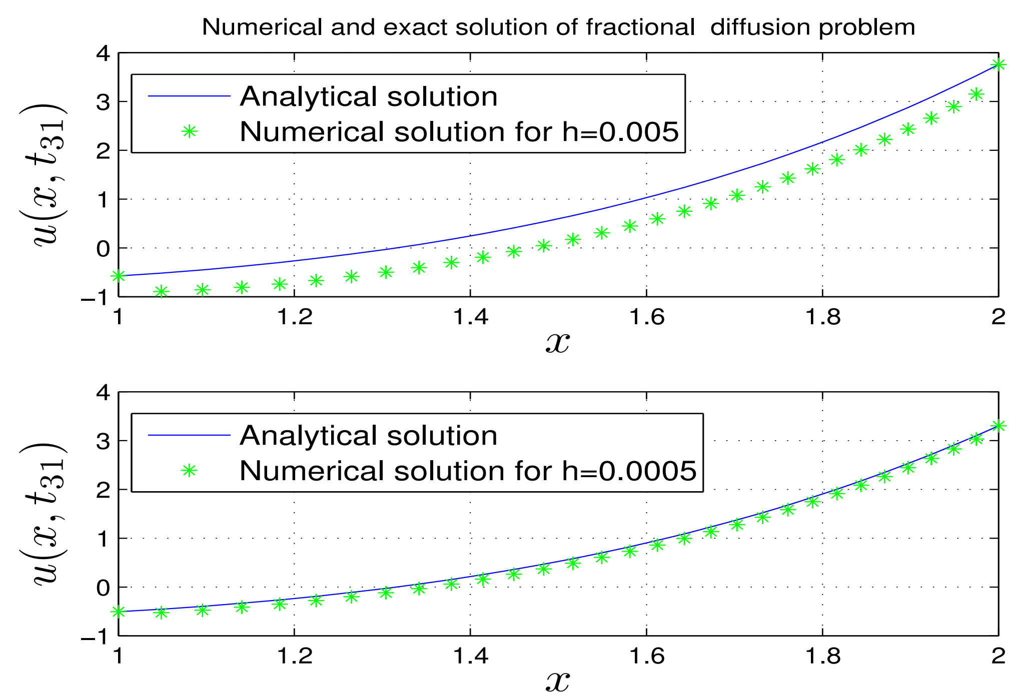

FIGURE 4.5 – Graphical comparison of the numerical and the exact solution with $k = 0.1$, $\rho = 2$, $\alpha = 0.6$, $\beta = 1.8$, $n = 30$ and $m = 25$.

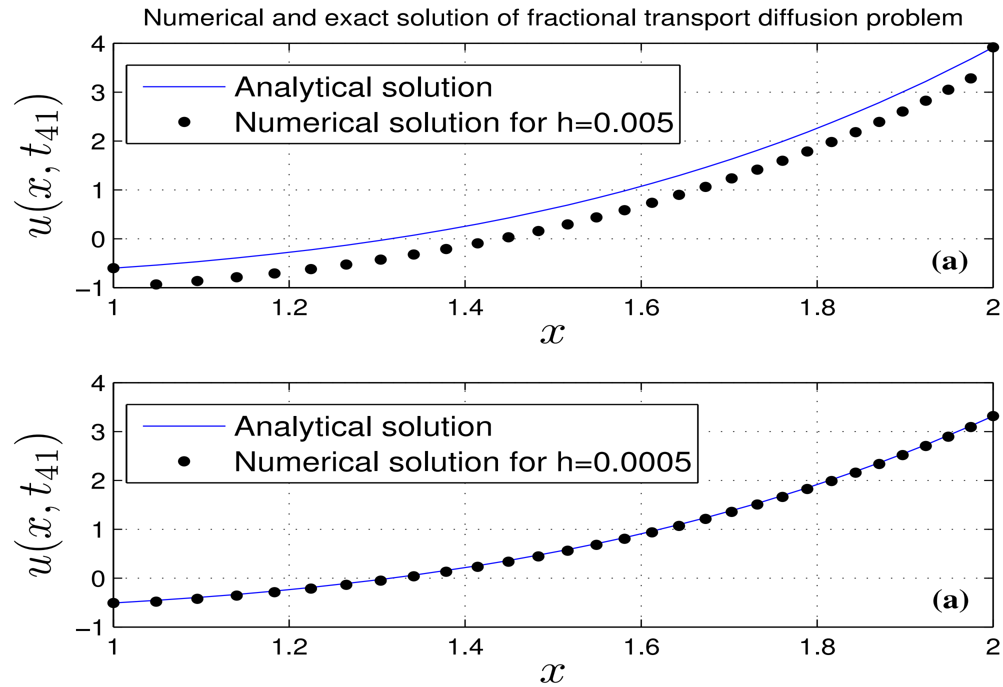

FIGURE 4.6 – Graphical comparison of the numerical and the exact solution with $k = 0.1$, $\rho = 2$, $\alpha = 0.9$, (a) $\beta = 1$ and $m = 15$.

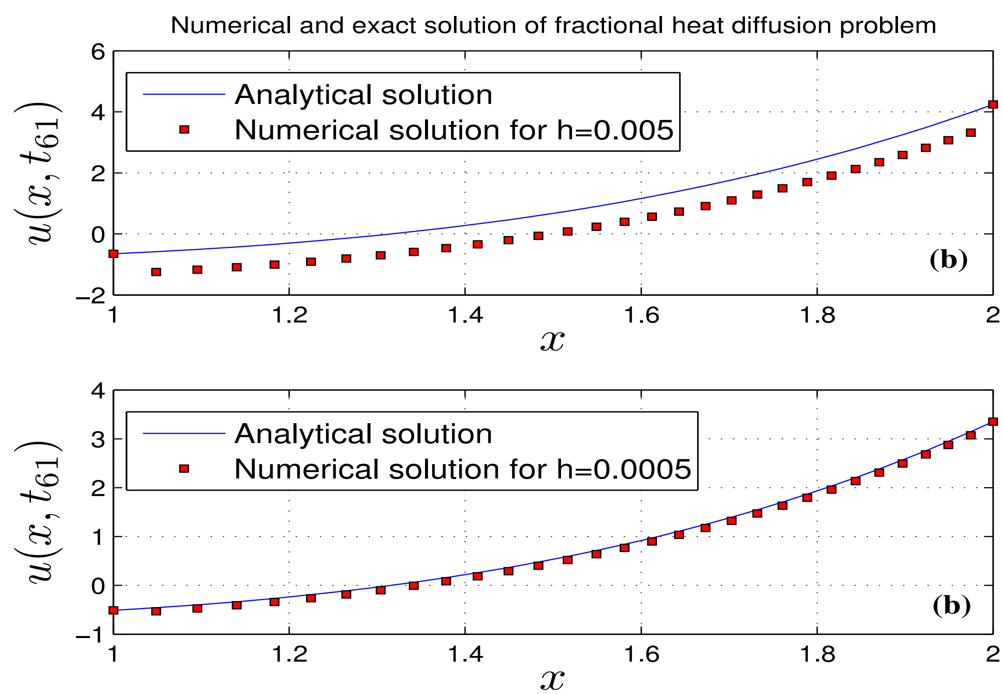

FIGURE 4.7 – Graphical comparison of the numerical and the exact solution with $k = 0.1$, $\rho = 2$, $\alpha = 0.9$, (b) $\beta = 2$ and $m = 15$.

Example 4.5. Let $(x,t) \in [1,2] \times [1,2]$ and

$$f(x,t) = \left(\frac{-x^{2\rho}+x}{2\rho}\right)\frac{\rho^{\alpha}}{\Gamma(2-\alpha)}(t^{\rho}-1)^{1-\alpha} + \frac{(2\rho-1)\,\rho^{\beta-2}}{2\Gamma(3-\beta)}(t^{\rho}+1)\left((x^{\rho}-1)^{2-\beta}+(2^{\rho}-x^{\rho})^{2-\beta}\right).$$

Consider the following space-time fractional diffusion equation

$$\begin{cases} {}^{C}\partial_t^{\alpha,\rho} u(x,t) = \dfrac{\partial^{\beta,\rho} u(x,t)}{\partial |x|^{\beta}} + f(x,t), \\ u(x,1) = 2\left(\dfrac{-x^{2\rho}+x}{2\rho}\right), \\ \partial_x u(1,t) = \left(\dfrac{-2\rho+1}{2\rho}\right)(t^{\rho}+1), u(1,t)=0, u(2,t) = (t^{\rho}+1)\left(\dfrac{-2^{2\rho-1}+1}{\rho}\right). \end{cases} \tag{4.26}$$

The exact solution of the given problem is given by

$$u(x,t) = (t^{\rho}+1)\left(\frac{-x^{2\rho}+x}{2\rho}\right).$$

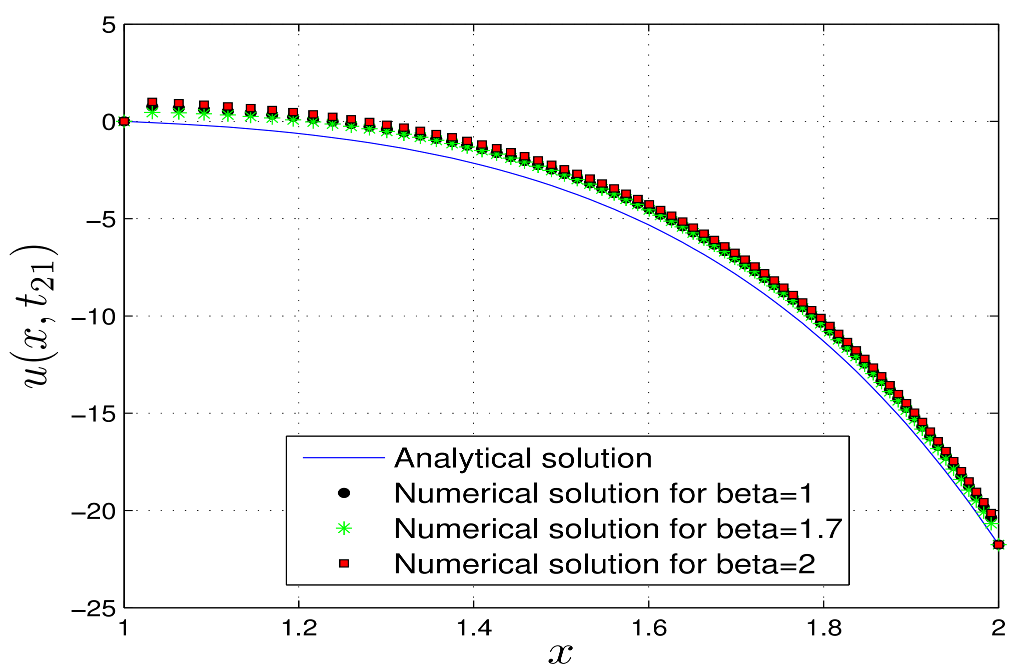

FIGURE 4.8 – Graphical comparison of the numerical and the exact solution with $h = 0.005$, $k = 0.1$, $\rho = 3$, $\alpha = 0.7$, $n = 20$ and $m = 15$.

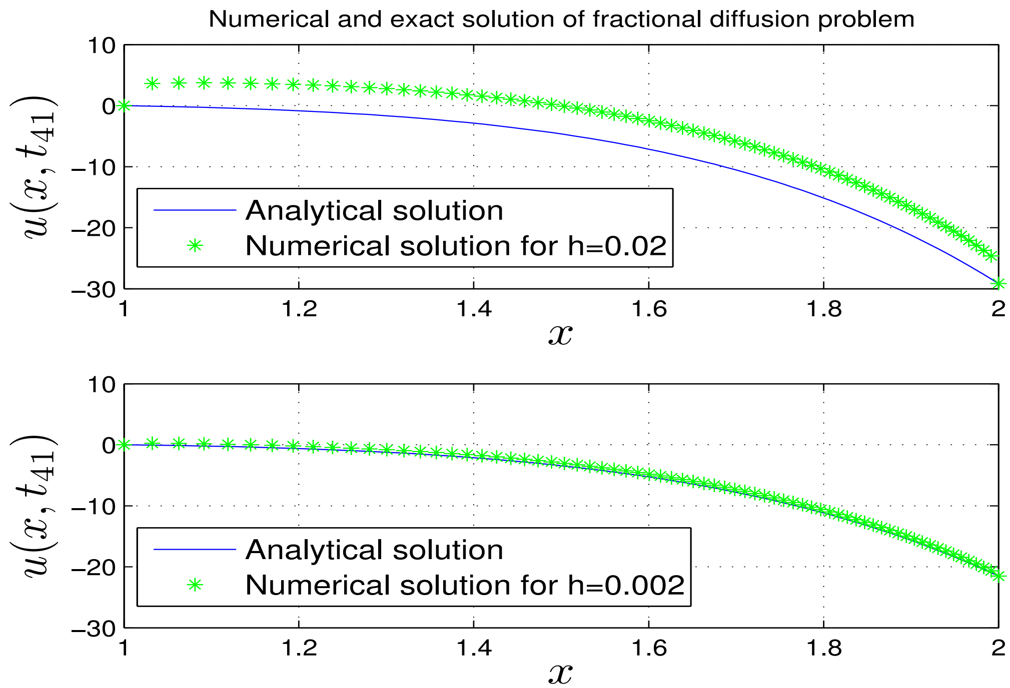

FIGURE 4.9 – Graphical comparison of the numerical and the exact solution with $k = 0.1$, $\rho = 3$, $\alpha = 0.8$, $\beta = 1.8$, $n = 40$ and $m = 15$.

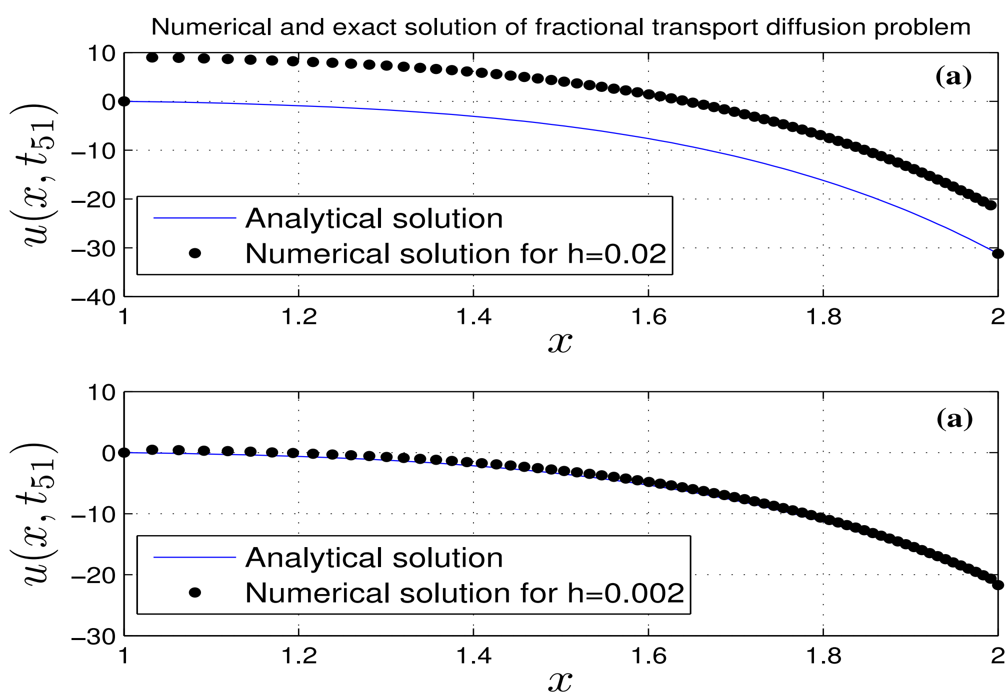

FIGURE 4.10 – Graphical comparison of the numerical and the exact solution with $k = 0.1$, $\rho = 3$, $\alpha = 0.9$, (a) $\beta = 1$ and $m = 15$.

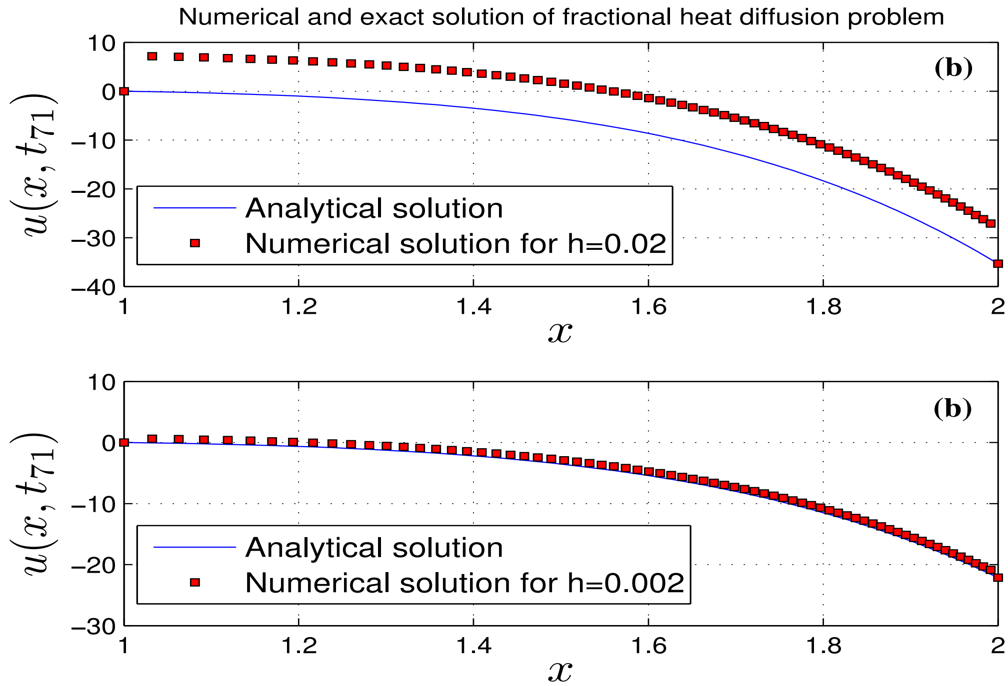

FIGURE 4.11 – Graphical comparison of the numerical and the exact solution with $k = 0.1$, $\rho = 3$, $\alpha = 0.9$, (b) $\beta = 2$ and $m = 15$.

4.2.4 Dicussion and results

In this chapter we have discussed a new numerical method for solving space-time fractional partial differential equations. Moreover, various results were obtained for different values of the parameters β, α and ρ. So, in the case of $\beta = 1$, we obtain the numerical solution of the fractional transport equation, (Figure 4.6, Figure 4.10). However, if $\beta = 2$, we obtain the numerical solution of fractional heat–diffusion equation (Figure 4.7, Figure 4.11). Eventually, different values for h and k have been tested on examples 4.4 and 4.5 to evaluate the validity of the approach, the results obtained show a good global approximation and an improved convergence with an error $C_{\alpha,\beta,\rho}(h^{1-\alpha} + k^{2-\beta})$ reaching to zero.

Conclusion générale

The work contained in this thesis is essentially composed of two main parts. In the first part 3 which is divided into two sub-paties, first we considered the problems of fractional differential equation (FDE) of order α ($0 < \alpha \leq 1$), with the initial condition of Cauchy, second, we considered the time fractional diffusion-wave equation (TFDWE) of order α ($1 < \alpha \leq 2$), with Dirichlet-Neumann initial conditions and Dirichlet-Neumann boundary conditions, this equation is a generalization of the classical diffusion equation by replacing the first order derivative in time or the first and second order derivative in time and space (respectively) by fractional derivatives. The fractional derivative in both cases is described in the Caputo–Hadamard sense. We used the fractional finite difference method (FDM) to compute the numerical solution. Thus, the convergence and stability of the numerical scheme for both problems are discussed and illustrated by solving several examples of linear fractional differential equations for different values of h and α to show the validity of our method. moreover, we found that the convergence error depends on the discretization step in time h, the error in this case is of order $\mathcal{O}(h^{2-\alpha})$ in the first problem ; and it depends on h and k in the second problem, where k is the discretization step in space, the error in this case of the order $\mathcal{O}(h^{2-\alpha} + k^2)$.
in the second part 4 which is also divided into two sub-parts, first we considered the time fractional diffusion-wave equation (TFDWE) of order α ($1 < \alpha \leq 2$), with Dirichlet-Neumann initial and boundary conditions, second, we considered the space-time fractional diffusion problem. The fractional derivatives in both cases are described in Caputo-Katugampola and Riesz–Caputo–Katugampola sense of order $0 < \alpha \leq 1$ and $1 \leq \beta \leq 2$ respectively. In the same way, we used the fractional finite difference method (FDM) to compute the numerical solution. Thus, the convergence and stability of the numerical scheme for both case are discussed. Different examples have been investigated to assess the validity of the approach showing good overall approximation and improved convergence.Thus, we have shown that the error is of order $\mathcal{O}\left(h^{2-\alpha} + k^2\right)$ and $\mathcal{O}(h^{1-\alpha} + k^{2-\beta})$.

Bibliographie

[1] R. Almeida, T. Odzijewicz and Agnieszka B. Malinowska, Fractional differential Equations with dependence on the Caputo-Katugampola derivative, *J. Comput. Nonlinear Dynam.* **11**(6) (2016), (11 pages), DOI : 10.1115/1.4034432.

[2] Y. Arioua, Initial Value Problem for a Coupled System of Katugampola-Type Fractional Differential Equations, *Adv. Dyn. Syst. Appl.* **14**(1) (2019), 29–47.

[3] Y. Arioua, B. basti and N. Benhamidouche, Initial value problem for nonlinear implicit fractional differential equations with Katugampola derivative. *Appl. Math. E-Notes.* **19** (2019), 397-412.

[4] Y. Arioua and N. Benhamidouche, Boundary value problem for Caputo-Hadamard fractional differential equations, *Surv. Math. Appl.* **12** (2017), 103–115.

[5] Y. Arioua and N. Benhamidouche, Traveling profiles solutions to heat equation with a power-law nonlinearity, *Adv. Dyn. Syst. Appl.* **7**(4) (2015), 1–6.

[6] Y. Arioua and M. Li, On Criteria of Existence for Nonlinear Katugampola Fractional Differential Equations with p–Laplacian Operator, *Fract. Differ. Calc.* **11** (1) (2021), 55–68.

[7] B. basti, Y. Arioua and N. Benhamidouche, Existence and uniqueness of solutions for nonlinear Katugampola fractional differential equations, *J. Math. Appl.* **42** (2019), 1–18.

[8] B. basti, Y. Arioua and N. Benhamidouche, Existence results for nonlinear Katugampola fractional differential equations with an integral condition, *Acta Math. Univ. Comenian.* **89** (2) (2020), 243–260.

[9] R. B Albadarneha, I. M Batihab and M. Zurigatb, Numerical solutions for linear fractional differential equations of order $1 < \alpha < 2$ using finite difference method (FFDM), *Int. J. Math.Comput. Sci.* **16**(1) (2016), 103–111.

[10] K. Bouchama, A. Merzougui and Y. Arioua, Numerical solutions for linear fractional differential equation with dependence on the Caputo-Hadamard derivative using finite difference method, *Palestine Journal of Mathematics*. (accepted) 1–9.

[11] K. Bouchama, Y. Arioua and A. Merzougui, The numerical solution of the space-time fractional diffusion equation involving the Caputo-Katugampola fractional derivative, *Numerical Algebra, Control and Optimization*, doi :10.3934/naco.2021026, (2021).

[12] C. Chang-ming, F. Liu and K. Burrage, Finite difference methods and a fourier analysis for fractional reaction–subdiffusion equation, *Appl. Math. Comput.* **198**(2) (2008), 754–769.

[13] J. F Cheng and Y. M Chu, Solution to the linear fractional differential equation using Adomian decomposition method, *Math. Probl. Eng.* **2011** (2011), 1–14, https ://doi.org/10.1155/2011/587068.

[14] P.L. Butzer, A.A. Kilbas and J.J. Trujillo. Compositions of hadamard-type fractional integration operators and the semigroup property, *J. Math. Anal. Appl*. **269**(2) (2002), 387–400.

[15] H.-f. Ding and Y.-x. Zhang, Notes on Implicit finite difference approximation for time fractional diffusion equations, *Comput. Math.Appl.* **61**(9) (2011), 2924–2928.

[16] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer Science Berlin, New York, 2010.

[17] Gambo et al, On Caputo modification of the Hadamard fractional derivatives, *Adv. Difference Equ.* **2014**(1) (2014), 1–12.

[18] N. Faraz, Y. Khan and D.S. Sankar, Decomposition-transform method for fractional differential equations, *Int. J. Nonl. Sci. Num. Sim*, **11** (2010), 305–310.

[19] A. Fortin, *numérique pour ingénieurs.* Presses inter Polytechnique, 2008.

[20] Q. Guan and M. Gunzburger, $\theta-$ schemes for finite element discretization of the space-time fractional diffusion equations, *J. Comput. Appl. Math.* **288** (2015), 264–273.

[21] J. Hadamard, Essai sur l'étude des fonctions données par leur développment de Taylor, *J. Math. Pures Appl.* **8** (1892), 101-186.

[22] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[23] M. Janaki, K. Kanagarajan and D. Vivek, Analytic Study on Fractional Implicit Differential Equations with Impulses via Katugampola Fractional Derivative, *Int. J. Math. And Appl.* **6(2–A)** (2018), 53–62.

[24] F. Jarad, D. Baleanu and A. Abdeljawad. Caputo-type modification of the Hadamard fractional derivatives, *Adv. Differ. Equ.* **142**(1) (2012), 1–8.

[25] U.N. Katugampola, New approach to a generalized fractional integral, *Appl. Math. Comput.* **218**(3) (2011), 860-865.

[26] U.N. Katugampola, *A new approach to generalized fractional derivatives*, *J. Math. Anal. Appl*, **6** (2014), 1-15.

[27] M.M. Khader, On the numerical solutions for the fractional diffusion equation, *Commun Nonlinear Sci Numer Simulat.* **16**(6) (2011), 2535–2542.

[28] A. A. Kilbas. Hadamard-type fractional calculus. Journal of Korean Mathematical Society, *J. Korean Math. Soc.* **38**(6) (2011), 1191–1204.

[29] A.A. Kilbas, H.H. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V, Amsterdam, 2006.

[30] V. Kiryakova, A brief story about the operators of the generalized fractional calculus. *Frac. Calc. Appl. Anal.* **11**(2) (2008), 203–220.

[31] C. LI and F.ZENG, *Numerical methods for fractional calculus*, Chapman and Hall/CRC, 2019.

[32] R.L. Magin, *Fractional Calculus in Bioengineering*, Begell House Publishers, 2006.

[33] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity : An Introduction to Mathematical Models*, Imperial College Press, Singapore, 2010.

[34] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[35] C.A. Monje, Y.Q. Chen, B.M. Vinagre, D. Xue and V. Feliu, *Fractional-Order Systems and Controls, Advances in Industrial Control*, Springer Science and Business Media, 2010.

[36] D.A. Murio, Implicit finite difference approximation for time fractional diffusion equations, *Comput. Math. Appl.* **56**(4) (2008), 1138–1145.

[37] I. Petras, *Fractional-Order Nonlinear Systems*, Springer Science and Business Media, New York, 2011.

[38] V.G. Pimenov and A.S. Hendy, A numerical solution for a class of time fractional diffusion equations with delay, *Int. J. Appl. Math. Comput. Sci.* **27**(3) (2017), 477–488.

[39] I. Podlubny, *Fractional Differential Equations, Mathematics in Science and Engineering*, Academic Press, New York,1999.

[40] J. Ren and Z.Z. Sun, Efificient Numerical Solution of the Multi-Term Time Fractional Diffusion Wave Equation, *East Asian J.Appl. Math.* **5**(1) (2015), 1–28.

[41] J. Sabatier, O.P. Agrawal and J.A. Tenreiro Machado (Eds.), *Advances in Fractional Calculus : Theoretical Developments and Applications in Physics and Engineering*, Springer, 2007.

[42] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integral and Derivatives (Theory and Applications)*, Gordon and Breach, Switzerland, 1993.

[43] S. Shen, F. Liu and V. Anh, Numerical approximations and solution techniques for the space-time Riesz Caputo fractional advection-diffusion equation, *Numer. Algorithms*. **56**(3) (2011), 383–403.

[44] H. Sheng, Y.Q. Chen and T.S. Qiu, *Fractional Processes and Fractional-order Signal Processing,* Springer, London, 2012.

[45] B.R. Sontakke, A.S. Shelke, Approximate Scheme for Time Fractional Diffusion Equation and Its Applications, *Glob. J. Pure Appl. Math.* **13**(8) (2017) 4333–4345.

[46] H. Sun, Z.Z. Sun and G.H. Gao, Some Temporal Second Order Difference Schemes for Fractional Wave Equations, *Numerical Methods for Partial Differential Equations.* **32**(3) (2016), 970–1001.

[47] X. C Shi, L. L. Huang and Y Zeng, Fast Adomian decomposition method for the Cauchy problem of the time–fractional reaction diffusion equation, *Advances in Mechanical Engineering*. **8**(2) (2016), 1–5.

[48] B.J. West, M. Bologna and P. Grigolini, *Physics of Fractal Operators*, 1^{st} edition, Springer, New York, 2003.

[49] S. Zeng, D. Baleanu and Y. Bai, G. Wu, Fractional differential equations of Caputo–Katugampola type and numerical solutions, *Appl. Math. Comput.* **315** (2017), 549–554.

ملخص

الهدف الرئيسي من هذه الأطروحة هو تقديم طريقة الفروق المحدودة لتقريب المشتقات الكسرية، من اجل إيجاد حلول عددية للمعادلات التفاضلية الخطية و ذات المشتقات الجزئية الكسرية التي تتضمن المشتق الجزئي لكابوتو-هادامارد وكابوتو-كاتوجامبولا. وعلى هذا النحو، تمت برهنة تقارب واستقرار المخططات العددية باستخدام الاستنتاج الرياضي، كما تم تقديم أمثلة توضيحية لإظهار نجاعة و فعالية الطريقة.

كلمات مفتاحية: المعادلات التفاضلية الكسرية، طرق الفروق المحدودة، المشتق الكسري كابوتو-هادامارد، المشتق الكسري كبوتو-كاتوجامبولا، التقارب، الاستقرار.

Absract

The main objective of this thesis is to present the finite difference method to approximate fractional derivatives, in order to find numerical solutions to linear fractional differentials equations and partial differentials equations, involving Caputo-Hadamard and Caputo-Katugampola fractional derivative. As such, the convergence and stability of the numerical schemes are proved using mathematical induction. illustrative examples have been presented to show the effectiveness and validity of our method.

Keywords: Fractional differential equations, Finite difference methods, Caputo-Hadamard fractional derivative, Caputo-Katugampola derivative, Convergence, Stability.

Résumé

L'objectif principal de cette thèse est de présenter la méthode des différences finies pour approximer les dérivés fractionnaires, dans le but de trouver des solutions pour les équations différentielles et aux dérivées partielles fractionnaires linéaires, impliquant les dérivées fractionnaires de Caputo-Hadamard et Caputo-Katugampola. Ainsi, la convergence et la stabilité des schémas numériques sont prouvées en utilisant l'induction mathématique. Des exemples illustratifs ont été présentés pour montrer l’efficacité et la validité de notre méthode.

Mots clés: Equations différentielles fractionnaires, Méthodes des différences finies, Dérivée fractionnaire de Caputo-Hadamard, Dérivée de Caputo-Katugampola, Convergence, Stabilité.