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Weak type estimate of some commutators on variable Herz- type Hardy spaces

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Notation

- For $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$, we write $|\alpha| = \alpha_1 + ... + \alpha_n$.
- The Euclidean scalar product of $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ is given by $xy = x_1y_1 + ... + x_ny_n$.
- The expression $f \lesssim g$ means that $f \leqslant cg$ for some independent constant c (and non-negative functions f and g).
- $f \approx g$ means $f \lesssim g \lesssim f$.
- As usual for any $x \in \mathbb{R}$, $[x]$ stands for the largest integer smaller than or equal to x.
- supp$f$ is the support of the function f , i.e., the closure of its non-zero set.
- If $Q \subset \mathbb{R}^n$ is a measurable set, then $|Q|$ stands for the (Lebesgue) measure of Q.
- χ_E denotes its characteristic function.
- $\mathcal{S}(\mathbb{R}^n)$ is used in place of set of all Schwartz functions on $\mathbb{R}^n$.
- $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$.

Introduction

It is well known that Herz spaces play an important role in Harmonic Analysis. After they have been introduced in [15], the theory of these spaces had a remarkable development in part due to its usefulness in applications. For instance, they appear in the characterization of multipliers on Hardy spaces [3], in the summability of Fourier transforms [12] and in regularity theory for elliptic equations in divergence form [27].

In recent years, there has been growing interest in generalizing classical spaces such as Lebesgue, Herz spaces and Sobolev spaces to the case with either variable integrability or variable smoothness. The motivation for the increasing interest in such spaces comes not only from theoretical purposes, but also from applications to fluid dynamics [29], image restoration and PDE with non-standard growth conditions.

Herz spaces $K^{\alpha}_{p(\cdot),q}(\mathbb{R}^n)$ and $\dot{K}^{\alpha}_{p(\cdot),q}(\mathbb{R}^n)$ with variable exponent p but fixed $\alpha \in \mathbb{R}$ and q were recently studied by Izuki [17, 18]. These spaces with variable exponents $\alpha(\cdot)$ and $p(\cdot)$ were studied in [2], where they gave the boundedness results for a wide class of classical operators on these function spaces. The spaces $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, were first introduced by Izuki and Noi in [20].

Based on the papers [9], [32] and [33] we will study the weak type estimates of commutators on Herz type spaces with variable exponent.

Our work is divided in to three chapters.

In the first one, we collects fundamental notation and concepts. We also give some key results needed in the proofs of main statements.

In the second chapter we define the BMO space and present the weak type estimate of commutators on variable Herz–type Hardy spaces.
In the last chapter we study the weak type estimate of commutators on variable Herz–type Hardy spaces with Lipschitz function.

Chapter 1

Variable Herz spaces

In this chapter, we present some fundamental proprieties of variable Herz spaces. We also give some key technical results needed in the proofs of the main results of this theses.

1.1 Modular space

In this section we recall some properties of semi-modular functional space.

1.1.1 Definition and basic properties

Definition 1.1 *Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$.. A function $\varrho : X \longrightarrow [0, +\infty]$ is called a semi-modular on X if it satisfies the following condition:*

1. $\varrho(0) = 0$.
2. $\varrho(\lambda x) = \varrho(x)$ *for all* $x \in X$, *and for all scalar* λ *with* $|\lambda| = 1$.
3. ϱ *is quasi-convex.*
4. $\varrho(\lambda x) = 0$ *for all* $\lambda > 0$ *implies* $x = 0$.
5. ϱ *is left-continuous on* $[0, +\infty)$ *for every* $x \in X$.

   A semi-modular ϱ is called a modular if

6. $\varrho(x) = 0$ *implies* $x = 0$.

A semi-modular ϱ *is called continuous if*

7. *the mapping* $\lambda \longrightarrow \varrho(\lambda x)$ *is continuous on* $[0, +\infty)$ *for every* $x \in X$.

Example 1.2 *Let* Ω *be a Lebesgue measurable subset of* $\mathbb{R}^n$. *If* $1 \leq p < \infty$, *then*

$$\varrho_p(f) = \int_\Omega |f(x)|^p dx$$

defines a continuous modular on the space of all measurable functions on Ω. *If* $1 \leq p < \infty$, *then*

$$\varrho_p((x_j)) = \sum_{j=0}^{\infty} |x_j|^p$$

defines a continuous modular on $\mathbb{R}^n$.

Definition 1.3 *If* ϱ *be a semimodular or modular on* X, *then*

$$X_\varrho = \{x \in X : \lim_{\lambda \longrightarrow 0} \varrho(\lambda x) = 0\}$$

is called a semimodular space or modular space, respectively.

Proposition 1.4 *If* ϱ *is a semimodular or modular on* X, *then*

$$X_\varrho = \{x \in X, \exists \lambda > 0 : \varrho(\lambda x) < \infty\}$$

is called a semimodular space or modular space, respectively.

Theorem 1.5 *Let* ϱ *be a semimodular on* X. *Then* X_ϱ *is a quasi-normed space. The quasi-norm, called the Luxemburg quasi-norm, is defined by*

$$\|x\|_\varrho := \inf \left\{ \lambda > 0 : \varrho(\frac{x}{\lambda}) \leqslant 1 \right\}.$$

Lemma 1.6 *(Norm-modular unit ball property). Let* ϱ *be semi-modular on* X. *Then*

$$\|x\|_\varrho \leq 1 \Leftrightarrow \varrho(x) \leq 1.$$

If ϱ *is continuous, then also*

$\|x\|_\varrho < 1 \Leftrightarrow \varrho(x) < 1$, *and :* $\|x\|_\varrho = 1 \Leftrightarrow \varrho(x) = 1$.

Corollary 1.7 *Let ϱ be a semi-modular on X and $x \in X_\varrho$.*

(a) *If $\|x\|_\varrho \leq 1$, then $\varrho(x) \leq \|x\|_\varrho$.*

(b) *If $1 < \|x\|_\varrho$, then $\|x\|_\varrho \leq \varrho(x)$.*

(c) $\|x\|_\varrho \leq \varrho(x) + 1$.

Remark 1.8 *The proof of the above results can be found in [8].*

1.2 Variable Lebesgue spaces

In this section we recall and present some properties of variable Lebesgue spaces. Given an open set $\Omega \subset \mathbb{R}^n$. We put

$$\mathcal{P}_0(\Omega) := \{p \ \text{: measurable} : p(\cdot) : \Omega \longrightarrow [c, \infty[\ \text{: for some } c > 0\}.$$

The elements of $\mathcal{P}_0(\Omega)$ are called exponent functions. In order to distinguish between variable and constant exponents, we will always denote exponent functions by $p(\cdot)$. We denote by

$$\mathcal{P}(\Omega) := \{p \ \text{: measurable} :: p(\cdot) : \Omega \subset \mathbb{R}^n \longrightarrow [1, \infty[\}.$$

Given $p \in \mathcal{P}_0(\Omega)$ and a set $E \subseteq \Omega$, let

$$p^-(E) = \operatorname*{ess\,inf}_{x \in E} p(x), \qquad p^+(E) = \operatorname*{ess\,sup}_{x \in E} p(x).$$

If the domain $E = \Omega = \mathbb{R}^n$, then we will simply write

$$p^- = p^-(\mathbb{R}^n), \qquad p^+ = p^+(\mathbb{R}^n).$$

Definition 1.9 *Given Ω and $p \in \mathcal{P}_0(\Omega)$. The variable Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by*

$$L^{p(\cdot)}(\Omega) := \left\{ f \ \textit{measurable} : \exists \lambda > 0 : \lim_{\lambda \longrightarrow 0} \varrho_{L^{p(\cdot)}(\Omega)}(\lambda f) = 0 \right\},$$

equipped with the following quasi-norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\{\lambda > 0 : \varrho_{L^{p(\cdot)}(\Omega)}(f/\lambda) \leqslant 1\}.$$

Definition 1.10 *Given* $\Omega \subset \mathbb{R}^n$ *open and* $p \in \mathcal{P}_0(\Omega)$, *define* $L^{p(\cdot)}_{loc}(\Omega)$ *by*

$$L^{p(\cdot)}_{loc}(\Omega) := \{f \text{ measurable} : f \in L^{p(\cdot)}(K), : \text{for every compact set} : K \subset \Omega\}.$$

Definition 1.11 *Let* $p \in \mathcal{P}_0(\mathbb{R}^n)$. *The weak Lebesgue space with variable exponent* $L^{p(\cdot),\infty}(\mathbb{R}^n)$ *(or* $WL^{p(\cdot)}(\mathbb{R}^n)$*)consists of all Lebesgue measurable function* f *satisfying*

$$\big\|f\big\|_{L^{p(\cdot),\infty}} := \sup_{\lambda>0} \lambda \big\|\chi_{\{x\in\mathbb{R}^n:|f(x)>\lambda\}}\big\|_{p(\cdot)} < \infty.$$

Definition 1.12 *We say that a function* $g : \mathbb{R}^n \to \mathbb{R}$ *is* locally log-Hölder continuous, *if there exists a constant* $c_{\log} > 0$ *such that*

$$|g(x) - g(y)| \leq \frac{c_{\log}}{\ln(e + 1/|x-y|)}$$

for all $x, y \in \mathbb{R}^n$. *If*

$$|g(x) - g(0)| \leq \frac{c_{\log}}{\ln(e + 1/|x|)}$$

for all $x \in \mathbb{R}^n$, *then we say that* g *is* log-Hölder continuous at the origin *(or has a* log decay at the origin *). If, for some* $g_\infty \in \mathbb{R}$ *and* $c_{\log} > 0$, *there holds*

$$|g(x) - g_\infty| \leq \frac{c_{\log}}{\ln(e + |x|)}$$

for all $x \in \mathbb{R}^n$, *then we say that* g *is* log-Hölder continuous at infinity *(or has a* log *decay at infinity).*

1.2.1 The mixed Lebesgue-sequence space

Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$ -functions by the modular

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) := \sum_v \inf\Big\{\lambda_v > 0 : \varrho_{p(\cdot)}\Big(\frac{f_v}{\lambda_v^{1/q(\cdot)}}\Big) \leq 1\Big\}.$$

The (quasi)-norm is defined from this as usual:

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} := \inf\left\{\mu > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\frac{1}{\mu}(f_v)_v\right) \leqslant 1\right\}.$$

1. If $q^+ < \infty$, then

$$\inf\left\{\lambda > 0 : \varrho_{p(\cdot)}\left(\frac{f}{\lambda^{1/q(\cdot)}}\right) \leqslant 1\right\} = \||f|^{q(\cdot)}\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

2. If p and q are constants, then $\ell^{q(\cdot)}(L^{p(\cdot)}) = \ell^p(L^q)$.

Theorem 1.13 *Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a quasi-norm on the mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$.*
Let $p, q \in \mathcal{P}(\mathbb{R}^n)$. If $p(x) \geqslant 1$ is constant almost everywhere (a.e.) on $\mathbb{R}^n$ and $q \geqslant 1$, or if $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} \leqslant 1$ a.e. on $\mathbb{R}^n$, or if $1 \leqslant q(\cdot) \leqslant p(\cdot) < \infty$ a.e. on $\mathbb{R}^n$, then $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a norm on the mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$.

For the proof see [1] and [21].

1.3 The spaces $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$

In this section, we give the definition of Herz spaces with variable exponent. Also, we present a useful properties for these function spaces. For convenience, we set

$$B_k := B(0, 2^k), \quad R_k := B_k \setminus B_{k-1} \quad \text{and} \quad \chi_k = \chi_{R_k}, \quad k \in \mathbb{Z}.$$

Definition 1.14 *Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The* inhomogeneous Herz space $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *consists of all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ such that*

$$\|f\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \|f\,\chi_{B_0}\|_{p(\cdot)} + \left\|\left(2^{k\alpha(\cdot)}f\,\chi_k\right)_{k\geq 1}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty. \tag{1.1}$$

Similarly, the homogeneous Herz space $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *is defined as the set of all $f \in L^{p(\cdot)}(\mathbb{R}^n \setminus \{0\})$ such that*

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \left\|\left(2^{k\alpha(\cdot)}f\,\chi_k\right)_{k\in\mathbb{Z}}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty. \tag{1.2}$$

If p, q and α are constant, then $K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = K_{p,q}^{\alpha}(\mathbb{R}^n)$ and $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = \dot{K}_{p,q}^{\alpha}(\mathbb{R}^n)$ are the classical Herz spaces.

Let us denote

$$\|\{g_k\}\|_{\ell^q_>(L^{p(\cdot)})} := \left(\sum_{k=0}^{\infty} \|g_k\|_{p(\cdot)}^q\right)^{1/q}$$

and

$$\|\{g_k\}\|_{\ell^q_<(L^{p(\cdot)})} := \left(\sum_{k=-\infty}^{-1} \|g_k\|_{p(\cdot)}^q\right)^{1/q}$$

for sequences $\{g_k\}_{k\in\mathbb{Z}}$ of measurable functions (with the usual modification if $q = \infty$).

Proposition 1.15 *Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. If α and q are* log*-Hölder continuous at infinity, then*

$$K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = K_{p(\cdot),q_\infty}^{\alpha_\infty}(\mathbb{R}^n).$$

Additionally, if α and q have a log *decay at the origin, then*

$$\|f\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \approx \|\{2^{k\alpha(0)} f\,\chi_k\}\|_{\ell^{q(0)}_<(L^{p(\cdot)})} + \|\{2^{k\alpha_\infty} f\,\chi_k\}\|_{\ell^{q_\infty}_>(L^{p(\cdot)})}. \tag{1.3}$$

Proof. *Step 1.* We will prove that

$$K_{p(\cdot),q_\infty}^{\alpha_\infty}(\mathbb{R}^n) \hookrightarrow K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n),$$

which is equivalent to

$$\|f\|_{K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \lesssim \|f\|_{K_{p(\cdot),q_\infty}^{\alpha_\infty}}$$

for any $f \in K_{p(\cdot),q_\infty}^{\alpha_\infty}(\mathbb{R}^n)$. By the scaling argument, we see that it suffices to consider the case $\|f\|_{K_{p(\cdot),q_\infty}^{\alpha_\infty}} = 1$ and show that the modular of f on the left-hand side is bounded. In particular, we will show that

$$\sum_{k=1}^{\infty} \left\| \left| c\, 2^{k\alpha(\cdot)} f\,\chi_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \le 1 \tag{1.4}$$

for some constant $c > 0$. Since α has logarithmic decay at infinity, then for $k \geq 1$ and $x \in R_k$ we have

$$k|\alpha(x) - \alpha_\infty| \lesssim \frac{k}{\ln(e + |x|)} \lesssim 1.$$

Therefore, $2^{k\alpha(x)} \approx 2^{k\alpha_\infty}$ with constants independent of k and x, and hence

$$\sum_{k=1}^{\infty} \left\| \left| c\ 2^{k\alpha(\cdot)} f\, \chi_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \approx \sum_{k=1}^{\infty} \left\| \left| c\ 2^{k\alpha_\infty} f\, \chi_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} .$$

Our estimate (1.4), clearly follows from the inequality

$$\left\| \left| c\ 2^{k\alpha_\infty} f\, \chi_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq \left\| 2^{k\alpha_\infty} f\, \chi_k \right\|_{p(\cdot)}^{q_\infty} + 2^{-k} = \delta. \tag{1.5}$$

This claim can be reformulated as

$$\left\| \delta^{-1} \left| c\ 2^{k\alpha_\infty} f\, \chi_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1,$$

which is equivalent to

$$\left\| c\ \delta^{-\frac{1}{q(\cdot)}} 2^{k\alpha_\infty} f\, \chi_k \right\|_{p(\cdot)} \leq 1.$$

For any $x \in R_k$, we have

$$\delta^{-\frac{1}{q(x)}} = (2^k\delta)^{\frac{1}{q_\infty} - \frac{1}{q(x)}} 2^{k(\frac{1}{q(x)} - \frac{1}{q_\infty})} \delta^{-\frac{1}{q_\infty}} .$$

Since q has logarithmic decay at infinity, then for $k \geq 1$ and $x \in R_k$ we have

$$\frac{k|q(x) - q_\infty|}{q_\infty q(x)} \leq \frac{k|q(x) - q_\infty|}{q_\infty q^-} \lesssim \frac{k}{\ln(e + |x|)} \lesssim 1.$$

Therefore, $2^{k(\frac{1}{q_\infty} - \frac{1}{q(x)})} \approx 1$ with constants independent of k and x. Also, since $1 < 2^k\delta < 2^{k+1}$,

$$(2^k\delta)^{\frac{1}{q_\infty} - \frac{1}{q(x)}} \leq (2^{k+1})^{\left|\frac{1}{q_\infty} - \frac{1}{q(x)}\right|} \lesssim 1.$$

Hence, with an appropriate choice of $c > 0$

$$\left\| c\ \delta^{-\frac{1}{q(\cdot)}} 2^{k\alpha_\infty} f\, \chi_k \right\|_{p(\cdot)} \leq \left\| \delta^{-\frac{1}{q_\infty}} 2^{k\alpha_\infty} f\, \chi_k \right\|_{p(\cdot)} \leq 1,$$

since of

$$\left\| 2^{k\alpha_\infty} f\, \chi_k \right\|_{p(\cdot)} \leq \delta^{\frac{1}{q_\infty}} .$$

Step 2. We will prove that

$$K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} (\mathbb{R}^n) \hookrightarrow K_{p(\cdot),q_\infty}^{\alpha_\infty} (\mathbb{R}^n) ,$$

which is equivalent to

$$\|f\|_{K^{\alpha_\infty}_{p(\cdot),q_\infty}} \lesssim \|f\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$$

for any $f \in K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. By the scaling argument, we see that it suffices to consider the case $\|f\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = 1$ and show that

$$\sum_{k=1}^{\infty} \left\|c\, 2^{k\alpha_\infty} f\, \chi_k\right\|_{p(\cdot)}^{q_\infty} \lesssim 1 \tag{1.6}$$

for some constant $c > 0$. As before, we have for $k \geq 1$

$$\left\|2^{k\alpha_\infty} f\, \chi_k\right\|_{p(\cdot)}^{q_\infty} \lesssim \left\|2^{k\alpha(\cdot)} f\, \chi_k\right\|_{p(\cdot)}^{q_\infty}.$$

Now, our estimate (1.6), clearly follows from the inequality

$$\left\|c\, 2^{k\alpha(\cdot)} f\, \chi_k\right\|_{p(\cdot)}^{q_\infty} \leq \left\|\left|2^{k\alpha(\cdot)} f\, \chi_k\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} + 2^{-k} = \delta. \tag{1.7}$$

This claim can be reformulated as

$$\left\|c\, \delta^{-\frac{1}{q_\infty}} 2^{k\alpha(\cdot)} f\, \chi_k\right\|_{p(\cdot)} \leq 1.$$

From above, $\delta^{-\frac{1}{q_\infty}} \lesssim \delta^{-\frac{1}{q(x)}}$, then with an appropriate choice of $c > 0$

$$\left\|c\, \delta^{-\frac{1}{q_\infty}} 2^{k\alpha(\cdot)} f\, \chi_k\right\|_{p(\cdot)} \leq \left\|\delta^{-\frac{1}{q(\cdot)}} 2^{k\alpha(\cdot)} f\, \chi_k\right\|_{p(\cdot)}.$$

The left-hand side is less than or equal to 1 if and only if

$$\left\|\left|\delta^{-\frac{1}{q(\cdot)}} 2^{k\alpha(\cdot)} f\, \chi_k\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1.$$

We see that the right-hand side can be rewritten as

$$\delta^{-1}\left\|\left|2^{k\alpha(\cdot)} f\, \chi_k\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1$$

which follows immediately from the definition of δ.

Step 3. Let us prove that

$$\|\{2^{k\alpha(0)} f\, \chi_k\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})} + \|\{2^{k\alpha_\infty} f\, \chi_k\}\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})} \lesssim \|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

We suppose that $\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq 1$. If, in addition, α has a log decay at the origin, then we also have $2^{k\alpha(x)} \approx 2^{k\alpha(0)}$ for $k < 0$ and $x \in R_k$. Thus

$$\|\{2^{k\alpha(0)} f \, \chi_k\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})} \approx \|\{2^{k\alpha(\cdot)} f \, \chi_k\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})}.$$

As in Step 2 we can prove that

$$\left\|c \; 2^{k\alpha(\cdot)} f \, \chi_k\right\|^{q(0)}_{p(\cdot)} \leq \left\| \left|2^{k\alpha(\cdot)} f \, \chi_k\right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} + 2^k$$

for any $k < 0$ and for some constant $c > 0$. Then $\|\{2^{k\alpha(0)} f \, \chi_k\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})} \lesssim 1$. Using the estimate (1.7) we obtain

$$\|\{2^{k\alpha_\infty} f \, \chi_k\}\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})} \lesssim 1.$$

Therefore,

$$\|\{2^{k\alpha(0)} f \, \chi_k\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})} + \|\{2^{k\alpha_\infty} f \, \chi_k\}\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})} \lesssim 1.$$

The desired estimate can be obtained by the scaling argument.

Finally

$$\|\{2^{k\alpha(0)} f \, \chi_k\}\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})} \leq 1,$$

and

$$\|\{2^{k\alpha_\infty} f \, \chi_k\}\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})} \leq 1.$$

As in Step 1 we have for any $k < 0$ and for some constant $c > 0$

$$\left\| \left|c \; 2^{k\alpha(\cdot)} f \, \chi_k\right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq \left\|2^{k\alpha(0)} f \, \chi_k\right\|^{q(0)}_{p(\cdot)} + 2^k$$

by using (1.5) we obtain

$$\sum_{k=-\infty}^{\infty} \left\| \left|2^{k\alpha(\cdot)} f \, \chi_k\right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \lesssim 1.$$

Therefore,

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim 1$$

and the result follows by the scaling argument. ■

By $\mathcal{P}_0^{\ln}(\mathbb{R}^n)$ and $\mathcal{P}_\infty^{\ln}(\mathbb{R}^n)$ we denote the class of all exponents $p \in \mathcal{P}(\mathbb{R}^n)$ which have a log decay at the origin and at infinity, respectively. The notation $\mathcal{P}^{\ln}(\mathbb{R}^n)$ is used for all those exponents $p \in \mathcal{P}(\mathbb{R}^n)$ which are locally log-Hölder continuous and have a log decay at infinity, with $p_\infty := \lim_{|x|\to\infty} p(x)$. Obviously we have

$$\mathcal{P}^{\ln}(\mathbb{R}^n) \subset \mathcal{P}_0^{\ln}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\ln}(\mathbb{R}^n).$$

Note that $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ if and only if $p' \in \mathcal{P}^{\ln}(\mathbb{R}^n)$, and since $(p')_\infty = (p_\infty)'$ we write only p'_∞ for any of these quantities. The next lemma is a Hardy-type inequality which is easy to prove.

Lemma 1.16 *Let $0 < a < 1$ and $0 < q \le \infty$. Let $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ be a sequence of positive real numbers, such that*

$$\left\|\{\varepsilon_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} = I < \infty.$$

Then the sequences $\left\{\delta_k : \delta_k = \sum_{j\le k} a^{k-j}\varepsilon_j\right\}_{k\in\mathbb{Z}}$ and $\left\{\eta_k : \eta_k = \sum_{j\ge k} a^{j-k}\varepsilon_j\right\}_{k\in\mathbb{Z}}$ belong to ℓ^q, and

$$\left\| \{\delta_k\}_{k\in\mathbb{Z}} \right\|_{\ell^q} + \left\| \{\eta_k\}_{k\in\mathbb{Z}} \right\|_{\ell^q} \le c\, I,$$

with $c > 0$ only depending on a and q.

Chapter 2

Weak type estimates of commutators I

In this chapter, we study the weak type estimate of some commutators with BMO function on Herz-type Hardy spaces with variable exponent.

2.1 Variable Herz-type Hardy space

Let $k \in \mathbb{Z}$ and $\lambda > 0$. We set $A_k(\lambda, f) := \{x \in R_k : |f(x)| > \lambda\}$ and $\tilde{A}_0(\lambda, f) := \{x \in B(0,1) : |f(x)| > \lambda\}$.

Definition 2.1 *Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The* inhomogeneous weak Herz space $WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *consists of all measurable functions f such that*

$$\|f\|_{WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \sup_{\lambda>0} \lambda \left\| \left(2^{k\alpha(\cdot)} \chi_{A_k(\lambda,f)}\right)_{k\geq 0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty,$$

where A_0 is replaced by $\tilde{A}_0$. Similarly, the homogeneous weak Herz space $W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *is defined as the set of all measurable functions f such that such that*

$$\|f\|_{W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \sup_{\lambda>0} \lambda \left\| \left(2^{k\alpha(\cdot)} \chi_{A_k(\lambda,f)}\right)_{k\in\mathbb{Z}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

Proposition 2.2 *Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. If α and q are* log*-Hölder continuous at infinity, then*

$$WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = WK^{\alpha_\infty}_{p(\cdot),q_\infty}(\mathbb{R}^n).$$

Additionally, if α and q have a log *decay at the origin, then*

$$\|f\|_{W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \sup_{\lambda>0} \big(\lambda \big\|\{2^{k\alpha(0)}\chi_{A_k(\lambda,f)}\}\big\|_{\ell^{q(0)}_<(L^{p(\cdot)})} + \lambda \big\|\{2^{k\alpha_\infty}\chi_{A_k(\lambda,f)}\}\big\|_{\ell^{q_\infty}_>(L^{p(\cdot)})}\big). \quad (2.1)$$

For the proof see [4].

Let $G_N f$ be the grand maximal function of f defined by

$$G_N f(x) := \sup_{\varphi \in \mathcal{A}_N} |\varphi^*_N(f)(x)|,$$

where $\mathcal{A}_N := \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|\le N, |\beta|\le N} |x^\alpha \partial^\beta \varphi(x)| \le 1\}$ and

$$\varphi^*_N(f)(x) := \sup_{t>0} |\varphi_t * f(x)|,$$

with $\varphi_t := t^{-n}\varphi(\frac{\cdot}{t})$.

Definition 2.3 *Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$ and $N > n+1$. The* inhomogeneous Herz-type Hardy space *$HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $G_N f \in K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and we define*

$$\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \|G_N f\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Similarly, the homogeneous Herz-type Hardy space *$H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $G_N f \in \dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and we define*

$$\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \|G_N f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Definition 2.4 *Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p \in \mathcal{P}(\mathbb{R}^n)$, $q \in \mathcal{P}_0(\mathbb{R}^n)$ and $s \in \mathbb{N}_0$. A function a is said to be a central $(\alpha(\cdot), p(\cdot))$-atom, if*

(i) $\mathrm{supp} a \subset \overline{B(0,r)} = \{x \in \mathbb{R}^n : |x| \le r\}, r > 0.$

(ii) $\|a\|_{p(\cdot)} \le |\overline{B(0,r)}|^{-\alpha(0)/n}, \qquad 0 < r < 1.$

(iii) $\|a\|_{p(\cdot)} \le |\overline{B(0,r)}|^{-\alpha_\infty/n}, \qquad r \ge 1.$

(iv) $\int_{\mathbb{R}^n} x^\beta a(x) dx = 0, \quad |\beta| \le s.$

A function a on $\mathbb{R}^n$ is said to be a central $(\alpha(\cdot), p(\cdot))$-atom of restricted type, if it satisfies the conditions (iii), (vi) above and supp$a \subset B(0,r), r \geq 1$.

Now we come to the atomic decomposition theorems.

Theorem 2.5 *Let α and q are* log*-Hölder continuous at infinity and $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$. For any $f \in HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, we have*

$$f = \sum_{k=0}^{\infty} \lambda_k a_k, \tag{2.2}$$

where the series converges in the sense of distributions, $\lambda_k \geq 0$,each a_k is a central $(\alpha(\cdot), p(\cdot))$-atom of restricted type with supp$a \subset B_k$ *and*

$$\Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty} \Big)^{1/q_\infty} \leq c \|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Conversely,if $\alpha_\infty \geq n(1 - \frac{1}{p_\infty})$ and $s \geq [\alpha_\infty + n(\frac{1}{p_\infty} - 1)]$, and if (2.2) holds, then $f \in HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, and

$$\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \inf \Big\{ \Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty} \Big)^{1/q_\infty} \Big\},$$

where the infimum is taken over all the decompositions of f as above.

Theorem 2.6 *Let α and q are be* log*-Hölder continuous, both at the origin and at infinity and $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$. For any $f \in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, we have*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k, \tag{2.3}$$

where the series converges in the sense of distributions, $\lambda_k \geq 0$, each a_k is a central $(\alpha(\cdot), p(\cdot))$-atom with supp$a \subset B_k$ *and*

$$\Big(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \Big)^{1/q(0)} + \Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty} \Big)^{1/q_\infty} \leq c \|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Conversely,if $\alpha(\cdot) \geq n(1-\frac{1}{p^-})$ *and* $s \geq [\alpha^+ + n(\frac{1}{p^-}-1)]$*, and if* (2.3) *holds, then* $f \in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$*, and*

$$\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \inf\Big\{\Big(\sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)}\Big)^{1/q(0)} + \Big(\sum_{k=0}^{\infty}|\lambda_k|^{q_\infty}\Big)^{1/q_\infty}\Big\},$$

where the infimum is taken over all the decompositions of f *as above.*

Definition 2.7 *Let* $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ *and* $\alpha : \mathbb{R}^n \to \mathbb{R}$ *with* $\alpha \in L^\infty(\mathbb{R}^n)$ *and* $N > n+1$*. The* inhomogeneous weak Herz-type Hardy space $WHK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *consists of all* $f \in \mathcal{S}'(\mathbb{R}^n)$ *such that* $G_N f \in WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *and we define*

$$\|f\|_{WHK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \|G_N f\|_{WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Similarly, the homogeneous weak Herz-type Hardy space $WH\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *is defined as the set of all* $f \in \mathcal{S}'(\mathbb{R}^n)$ *such that* $G_N f \in W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *and we define*

$$\|f\|_{WH\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \|G_N f\|_{W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

2.2 Key results

Recall that the space $BMO(\mathbb{R}^n)$ consists of all locally integrable functions f such that

$$\|f\|_{BMO} := \sup_{\mathrm{Q}} \frac{1}{|\mathrm{Q}|}\int_{\mathrm{Q}} |f(x) - f_{\mathrm{Q}}|\, dx < \infty,$$

where

$$f_{\mathrm{Q}} = \frac{1}{|\mathrm{Q}|}\int_{\mathrm{Q}} f(y)\, dy$$

and the supremum is taken over all cubes $\mathrm{Q} \subset \mathbb{R}^n$ with sides parallel to the coordinate axes.

Lemma 2.8 *Let* $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$*, let* k *be a positive integer, and let* B *be ball in* $\mathbb{R}^n$*. Then, for all* $b \in BMO(\mathbb{R}^n)$ *and all* $i, j \in \mathbb{Z}$ *with* $j > i$*, the following inequality is true*

$$\frac{1}{c}\|b\|^k_{BMO} \leq \sup_B \frac{1}{\|\chi_B\|_{p(\cdot)}}\|(b - b_B)^k \chi_B\|_{p(\cdot)} \leq \|b\|^k_{BMO},$$

$$\left\| \left(b - b_{B_i}\right)^k \chi_{B_j} \right\|_{p(\cdot)} \leq c\,(j-i)^k \, \|b\|_{BMO}^k \|\chi_{B_j}\|_{p(\cdot)}.$$

For the proof see [19].

Lemma 2.9 *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$. There exist constants c_1, c_2 such that for any $f \in BMO\left(\mathbb{R}^n\right), \gamma > 0$ and any $B \subset \mathbb{R}^n$, we have*

$$\left\|\chi_{\left\{x\in B:\, |f(x)-f_B|>\gamma\right\}}\right\|_{p(\cdot)} \leq c_1 e^{-c_2\gamma/\|f\|_{BMO}} \left\|\chi_B\right\|_{p(\cdot)},$$

where $f_B = \frac{1}{|B|}\int_B f\left(x\right) dx$.

For the proof see [16]. It is said that $b \in BMO\left(\mathbb{R}^n\right)$ satisfies the condition $\mathcal{L}$, if for any $j, k \in \mathbb{Z}$ with $k \leq j-3$ and any $x \in R_k$ there exists a constant $c > 0$ only dependent on n; such that

$$\left|b\left(x\right) - b_{B_k}\right| \leq c\left|b\left(x\right) - b_{B_j}\right|.$$

Let $b \in BMO(\mathbb{R}^n)$. The commutator of the maximal operator and the fractional maximal operator are defined, respectively by

$$M_b f\left(x\right) = \sup_{r>0} \frac{1}{\left|B\left(x,r\right)\right|} \int_{B(x,r)} \left|b\left(x\right) - b\left(y\right)\right| \left|f\left(y\right)\right| dy$$

and

$$M_b^v f\left(x\right) = \sup_{r>0} \frac{1}{\left|B\left(x,r\right)\right|^{\frac{n-v}{n}}} \int_{B(x,r)} \left|b\left(x\right) - b\left(y\right)\right| \left|f\left(y\right)\right| dy,$$

where $0 < v < n$.

Let $b \in BMO\left(\mathbb{R}^n\right)$ and T be a linear operator. The commutator $[b, T]$ generated by b and T is defined by

$$[b, T]\left(f\right)\left(x\right) := b\left(x\right) Tf\left(x\right) - T\left(bf\right)\left(x\right).$$

2.3 Main results and their proofs

The main results of this chapter are the following.

Theorem 2.10 *Let α and q are* log*-Hölder continuous at infinity and $p\in\mathcal{P}^{\log}(\mathbb{R}^n)$ with $0<p^-\le p^+<\infty$ and $0<q^-\le q^+\le 1$ such that*

$$\alpha_\infty = n(1-\frac{1}{p_\infty}).$$

If $b\in BMO\left(\mathbb{R}^n\right)$ then for any $f\in HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}\left(\mathbb{R}^n\right)$ and any $\lambda>0$, we have

$$\big\|\{2^{k\alpha_\infty}\chi_{A_k(\lambda,M_bf)}\}\big\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})}\le \frac{c\big\|f\big\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}}{\lambda}\Big(1+\log^+\frac{c\big\|f\big\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}}{\lambda}\Big),$$

with $c>0$ only dependent on f and λ.

Proof. Let $b\in BMO\left(\mathbb{R}^n\right)$ and $f\in HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$. By using Theorem 2.5, we can assume that

$$f=\sum_{k=0}^{\infty}\lambda_k a_k,$$

where the series converges in the sense of distributions, each a_k is a central $(\alpha(\cdot),p(\cdot))$-atom with supp$a_k\subset R_k$ and

$$\Big(\sum_{k=0}^{\infty}|\lambda_k|^{q_\infty}\Big)^{1/q_\infty}\le c\big\|f\big\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Observe that

$$\begin{aligned}
\big\|\{2^{j\alpha_\infty}\chi_{A_j(\lambda,M_bf)}\}\big\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})} &\le c\Big(\sum_{j=0}^{3}2^{j\alpha_\infty q_\infty}\big\|\chi_{\left\{x\in R_j:\,\left|M_bf(x)\right|>\lambda\right\}}\big\|^{q_\infty}_{p(\cdot)}\Big)^{\frac{1}{q_\infty}}\\
&\quad+c\Big(\sum_{j=4}^{\infty}2^{j\alpha_\infty q_\infty}\big\|\chi_{\left\{x\in R_j:\,\left|\sum_{k=0}^{j-3}\lambda_kM_ba_k(x)\right|>\frac{\lambda}{2}\right\}}\big\|^{q_\infty}_{p(\cdot)}\Big)^{\frac{1}{q_\infty}}\\
&\quad+c\Big(\sum_{j=4}^{\infty}2^{j\alpha_\infty q_\infty}\big\|\chi_{\left\{x\in R_j:\,\left|\sum_{k=j-2}^{\infty}\lambda_kM_ba_k(x)\right|>\frac{\lambda}{2}\right\}}\big\|^{q_\infty}_{p(\cdot)}\Big)^{\frac{1}{q_\infty}}\\
&=:F_1+F_2+F_3.
\end{aligned}$$

Estimate of F_1 and F_3. Using the boundedness of M_b on $L^{p(\cdot)}(\mathbb{R}^n)$ yield that

$$\begin{aligned}F_1 &\leq \frac{c}{\lambda}\Big(\sum_{j=0}^{3} 2^{j\alpha_\infty q_\infty}\big\|f\big\|_{p(\cdot)}^{q_\infty}\Big)^{\frac{1}{q_\infty}}\\ &\leq \frac{c}{\lambda}\Big(\sum_{k=0}^{\infty}|\lambda_k|\,\big\|a_k\big\|_{p(\cdot)}\Big)\\ &\leq \frac{c}{\lambda}\sup_{k\in\mathbb{N}_0}|\lambda_k|\,\Big(\sum_{k=0}^{\infty}2^{-k\alpha_\infty}\Big)\\ &\leq \frac{c}{\lambda}\Big(\sum_{k=0}^{\infty}|\lambda_k|^{q_\infty}\Big)^{1/q_\infty} \leq \frac{c}{\lambda}\big\|f\big\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.\end{aligned}$$

We easily obtain that

$$\begin{aligned}F_3 &\leq \frac{c}{\lambda}\Big(\sum_{j=0}^{\infty} 2^{j\alpha_\infty q_\infty}\Big(\sum_{k=j-2}^{\infty}|\lambda_k|\big\|a_k\big\|_{p(\cdot)}\Big)^{q_\infty}\Big)^{\frac{1}{q_\infty}}\\ &\leq \frac{c}{\lambda}\Big(\sum_{j=0}^{\infty}\Big(\sum_{k=j-2}^{\infty}2^{(j-k)\alpha_\infty}|\lambda_k|\Big)^{q_\infty}\Big)^{\frac{1}{q_\infty}}\\ &\leq \frac{c}{\lambda}\Big(\sum_{j=0}^{\infty}|\lambda_j|^{q_\infty}\Big)^{1/q_\infty} \leq \frac{c}{\lambda}\big\|f\big\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},\end{aligned}$$

where we used Lemma 1.16.

Estimate of F_2. From the vanishing moment of a_k and Hölder's inequality we obtain

$$\begin{aligned}&\Big|\sum_{k=0}^{j-3}\lambda_k M_b a_k(x)\Big|\\ \leq\;& c\sum_{k=0}^{j-3}|\lambda_k|\,|M_b a_k(x)|\\ \leq\;& c\sum_{k=0}^{j-3}|\lambda_k|\,2^{-jn}\int_{R_k}|b(x)-b(y)|\,|a_k(y)|\,dy\\ \leq\;& c\sum_{k=0}^{j-3}|\lambda_k|\,2^{-jn}\Big(|b(x)-b_{B_k}|\int_{R_k}|a_k(y)|\,dy+\int_{R_k}|a_k(y)|\,|b_{B_k}-b(y)|\,dy\Big)\\ \leq\;& c\sum_{k=0}^{j-3}|\lambda_k|\,2^{-jn}\big\|a_k\big\|_{p(\cdot)}\Big(|b(x)-b_{B_k}|\,\big\|\chi_{B_k}\big\|_{p'(\cdot)}+\big\|(b-b_{B_k})\chi_{B_k}\big\|_{p'(\cdot)}\Big) \qquad (2.4)\end{aligned}$$

for any $x \in R_j, y \in R_k$ with $0 \le k \le j-3$. In view of the well-known that

$$\left|b_{B_j} - b_{B_k}\right| \le (j-k)\left\|b\right\|_{BMO} \qquad \text{for any } j > k, \tag{2.5}$$

and Lemma 2.8, the last expression is bounded by

$$\le c\sum_{k=0}^{j-3}|\lambda_k|\,2^{-jn}\big\|a_k\big\|_{p(\cdot)}\big\|\chi_{B_k}\big\|_{p'(\cdot)}\Big(\left|b(x)-b_{B_j}\right| + (j-k)\left\|b\right\|_{BMO}\Big),$$

since a_k 's are $(\alpha(\cdot), p(\cdot))$-atom, this term is bounded by

$$c\sum_{k=0}^{j-3}|\lambda_k|\,2^{-jn}2^{-k\alpha_\infty}\big\|\chi_{B_k}\big\|_{p'(\cdot)}\Big(\left|b(x)-b_{B_j}\right| + j\big\|b\big\|_{BMO}\Big).$$

So F_2 is bounded by

$$\begin{aligned}
&c\Big(\sum_{j=4}^{\infty}2^{j\alpha_\infty q_\infty}\big\|\chi_{\left\{x\in R_j: c\left|b(x)-b_{B_j}\right|\sum_{k=0}^{j-3}|\lambda_k|2^{-jn}2^{-k\alpha_\infty}\left\|\chi_{B_k}\right\|_{p'(\cdot)}>\frac{\lambda}{4}\right\}}\big\|_{p(\cdot)}^{q_\infty}\Big)^{\frac{1}{q_\infty}}\\
&+\Big(\sum_{j=4}^{\infty}2^{j\alpha_\infty q_\infty}\big\|\chi_{\left\{x\in R_j: cj\left\|b\right\|_{BMO}\sum_{k=0}^{j-3}|\lambda_k|2^{-jn}2^{-k\alpha_\infty}\left\|\chi_{B_k}\right\|_{p'(\cdot)}>\frac{\lambda}{4}\right\}}\big\|_{p(\cdot)}^{q_\infty}\Big)^{\frac{1}{q_\infty}}\\
:&= F_2^1 + F_2^2
\end{aligned}$$

For F_2^1. Observe that $\sum_{k=0}^{j-3}|\lambda_k| \lesssim \|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$, from Lemma 2.9 we deduce

$$\begin{aligned}
&\big\|\chi_{\left\{x\in R_j: c\left|b(x)-b_{B_j}\right|\sum_{k=0}^{j-3}|\lambda_k|2^{-jn}2^{-k\alpha_\infty}\left\|\chi_{B_k}\right\|_{p'(\cdot)}>\frac{\lambda}{4}\right\}}\big\|_{p(\cdot)}\\
\le\ & c\big\|\chi_{\left\{x\in R_j: c\left|b(x)-b_{B_j}\right|2^{-j\left(\alpha_\infty+\frac{n}{p_\infty}\right)}\left\|f\right\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}>\frac{\lambda}{4}\right\}}\big\|_{p(\cdot)}\\
\le\ & c\big\|\chi_{\left\{x\in R_j: \left|b(x)-b_{B_j}\right|>\frac{\lambda 2^{j\left(\alpha_\infty+\frac{n}{p_\infty}\right)}}{c4\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}}\right\}}\big\|_{p(\cdot)}\\
\le\ & c\exp\Big(-\frac{c\lambda 2^{j\left(\alpha_\infty+\frac{n}{p_\infty}\right)}}{\|b\|_{BMO}\left\|f\right\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}}\Big)\big\|\chi_j\big\|_{p(\cdot)}\\
\le\ & c2^{\frac{jn}{p_\infty}}\exp\Big(-\frac{c\lambda 2^{j\left(\alpha_\infty+\frac{n}{p_\infty}\right)}}{\|b\|_{BMO}\left\|f\right\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}}\Big),
\end{aligned}\tag{2.6}$$

where we have used the condition $\alpha_\infty = n(1-\frac{1}{p_\infty})$. On the other hand we have

$$
\begin{aligned}
&\sum_{j=-\infty}^{\infty} 2^{j\left(\alpha_\infty+\frac{n}{p_\infty}\right)q_\infty} \exp\Big(-\frac{c\lambda 2^{j\left(\alpha_\infty+\frac{n}{p_\infty}\right)}}{\|b\|_{BMO}\,\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}}\Big)\\
\leq\ & c\int_0^\infty s^{q_\infty-1}\exp\Big(-\frac{c\lambda s}{\|b\|_{BMO}\,\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}}\Big)ds\\
\leq\ & \left(\frac{c}{\lambda}\|b\|_{BMO}\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}\right)^{q_\infty}\Big(\int_0^\infty t^{q_\infty-1}e^{-t}dt\Big)\\
\leq\ & \left(\frac{c}{\lambda}\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}\right)^{q_\infty}. \qquad (2.7)
\end{aligned}
$$

In view of (2.6) and (2.7), we find that

$$F_2^1 \leq \frac{c}{\lambda}\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

For F_2^2. We use a well-known inequality $\log_2^x \leq \frac{x}{2}$ whenever $x > 2$ to get the following claim:

Claim If there exists a $z > 1$ such that $\frac{2^x}{x} \leq z$ holds for $x \geq 2$, then $2^x \leq cz\log_2^z$. we use this claim to estimate for F_2^2. For $j \geq 4$, if

$$\Big|\Big\{x\in R_j : cj\|b\|_{BMO}\sum_{k=0}^{j-2}|\lambda_k|\,2^{-jn}2^{-k\alpha_\infty}\|\chi_{B_k}\|_{p'(\cdot)} > \frac{\lambda}{4}\Big\}\Big| \neq 0.$$

then

$$\frac{\lambda}{4} < cj2^{-j\left(\alpha_\infty+\frac{n}{p_\infty}\right)}\|b\|_{BMO}\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},$$

where we have used the condition $\alpha_\infty = n(1-\frac{1}{p_\infty})$ and $\sum_{k=0}^{j-2}|\lambda_k| \lesssim \|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$. Since $j\left(\alpha_\infty+\frac{n}{p_\infty}\right) = jn \geq 4$, we deduce

$$1 < \frac{2^{j\left(\alpha_\infty+\frac{n}{p_\infty}\right)}}{j\left(\alpha_\infty+\frac{n}{p_\infty}\right)} \leq \frac{c}{\lambda}\|b\|_{BMO}\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Therefore,

$$
\begin{aligned}
2^{j\alpha_\infty}\|\chi_j\|_{p(\cdot)} &\leq c2^{j\left(\alpha_\infty+\frac{n}{p_\infty}\right)}\\
&\leq \frac{c\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}}{\lambda}\Big(1+\log^+\frac{c\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}}{\lambda}\Big)
\end{aligned}
$$

Let j_λ be the maximal positive integer which satisfies this estimate, then we find that

$$\begin{aligned}
F_2^2 &\leq c\Big(\sum_{j=0}^{j_\lambda} 2^{j\alpha_\infty q_\infty}\big\|\chi_j\big\|_{p(\cdot)}^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\
&\leq c\Big(\sum_{j=0}^{j_\lambda} 2^{j\left(\alpha_\infty+\frac{n}{p_\infty}\right)q_\infty}\Big)^{\frac{1}{q_\infty}} \\
&\leq 2^{j_\lambda\left(\alpha_\infty+\frac{n}{p_\infty}\right)} \\
&\leq \frac{c\big\|f\big\|_{HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}}{\lambda}\Big(1+\log^{+}\frac{c\big\|f\big\|_{HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}}{\lambda}\Big).
\end{aligned}$$

This completes the proof. ■

Theorem 2.11 *Let $0 < v < n$, $p_1 \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ and $q \in \mathcal{P}_0(\mathbb{R}^n)$ with $1 < p_1^- \leq p_1^+ < \frac{n}{v}$, $0 < q^- \leq q^+ \leq 1$, $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{v}{n}$ and let α and q are be* log*-Hölder continuous, at infinity such that $\alpha \in L^\infty(\mathbb{R}^n)$, and*

$$\alpha_\infty = n\Big(1-\frac{1}{(p_1)_\infty}\Big).$$

If $b \in BMO(\mathbb{R}^n)$ then for any $f \in HK_{p_1(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and any $\lambda > 0$, we have

$$\big\|\{2^{k\alpha_\infty}\chi_{A_k(\lambda,M_b^v f)}\}\big\|_{\ell_{>}^{q_\infty}(L^{p_2(\cdot)})} \leq \frac{c\big\|f\big\|_{HK_{p_1(\cdot),q(\cdot)}^{\alpha(\cdot)}}}{\lambda}\Big(1+\log^{+}\frac{c\big\|f\big\|_{HK_{p_1(\cdot),q(\cdot)}^{\alpha(\cdot)}}}{\lambda}\Big),$$

with $c > 0$ only dependent on f and λ.

We omit the proof of Theorem 2.11 since they essentially similar to the proof of Theorem 2.10.

Theorem 2.12 *Let α and q are be* log*-Hölder continuous at infinity and $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ \leq 1$ such that*

$$\alpha_\infty = n(1-\frac{1}{p_\infty}).$$

If $b \in BMO(\mathbb{R}^n)$ satisfies the condition $\mathcal{L}$, then M_b is bounded from $HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ into $WK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$.

Theorem 2.13 *Let* α *and* q *are be* log-*Hölder continuous, both at the origin and at infinity and* $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ *with* $1 < p^- \le p^+ < \infty$ *and* $0 < q^- \le q^+ \le 1$ *such that*

$$\alpha(\cdot) \ge n(1 - \frac{1}{p^-}), \quad \alpha(0) = n(1 - \frac{1}{p(0)}), \quad \text{and } \alpha_\infty = n(1 - \frac{1}{p_\infty}).$$

If $b \in BMO(\mathbb{R}^n)$ *satisfies the condition* $\mathcal{L}$*, then* M_b *is bounded from* $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ *into* $W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$.

Proof. Suppose $f \in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$. By Theorem 2.6, we have

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k,$$

where the series converges in the sense of distributions, each a_k is a central $(\alpha(\cdot), p(\cdot))$-atom with supp$a_k \subset R_k$ and

$$\Big(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\Big)^{1/q(0)} + \Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\Big)^{1/q_\infty} \le c\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Using Proposition 2.2 we have

$$\begin{aligned}
\|f\|_{W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} &\approx \sup_{\lambda>0} \Big(\lambda\big\|\{2^{k\alpha(0)}\chi_{A_k(\lambda, M_b f)}\}\big\|_{\ell^{q(0)}_<(L^{p(\cdot)})} \\
&\quad +\lambda\big\|\{2^{k\alpha_\infty}\chi_{A_k(\lambda, M_b f)}\}\big\|_{\ell^{q_\infty}_>(L^{p(\cdot)})}\Big) \\
&\lesssim \sup_{\lambda>0}\{G_1 + G_2 + G_3 + G_4\},
\end{aligned}$$

where

$$G_1 := \lambda\Big\|\Big\{2^{j\alpha(0)}\chi_{A_j(\frac{\lambda}{2}, M_b(\sum_{k=-\infty}^{j-3}\lambda_k a_k))}\Big\}\Big\|_{\ell^{q(0)}_<(L^{p(\cdot)})}$$

$$G_2 := \lambda\Big\|\Big\{2^{j\alpha_\infty}\chi_{A_j(\frac{\lambda}{2}, M_b(\sum_{k=-\infty}^{j-3}\lambda_k a_k))}\Big\}\Big\|_{\ell^{q_\infty}_>(L^{p(\cdot)})}$$

$$G_3 := \lambda\Big\|\Big\{2^{j\alpha(0)}\chi_{A_j(\frac{\lambda}{2}, M_b(\sum_{k=j-2}^{\infty}\lambda_k a_k))}\Big\}\Big\|_{\ell^{q(0)}_<(L^{p(\cdot)})}$$

and

$$G_4 := \lambda\Big\|\Big\{2^{j\alpha_\infty}\chi_{A_j(\frac{\lambda}{2}, M_b(\sum_{k=j-2}^{\infty}\lambda_k a_k))}\Big\}\Big\|_{\ell^{q_\infty}_>(L^{p(\cdot)})}$$

To complete the proof, it suffices to show that

$$G_i \lesssim \|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}, \quad i=1,2,3,4.$$

The estimation of G_2 is similar to G_1. So we omit the detail for the estimation of G_2. By (2.4) and Lemma 2.8 and since b satisfies the condition $\mathcal{L}$, we have

$$\begin{aligned} & \Big|\sum_{k=-\infty}^{j-3} \lambda_k M_b a_k(x)\Big| \\ \leq \ & c\sum_{k=-\infty}^{j-3} 2^{-jn}\|a_k\|_{p(\cdot)}\|\chi_{B_k}\|_{p'(\cdot)}\Big(\big|b(x)-b_{B_j}\big| + \|b\|_{BMO}\Big), \end{aligned}$$

for any $x\in R_j, y\in R_k$ with $k\leq j-3$. For $j<0$, the last term is bounded by

$$\begin{aligned} \Big|\sum_{k=-\infty}^{j-3} \lambda_k M_b a_k(x)\Big| \ \leq \ & c\big|b(x)-b_{B_j}\big|\sum_{k=-\infty}^{j-3}|\lambda_k|\,2^{-jn}2^{-k\alpha(0)}\|\chi_{B_k}\|_{p'(\cdot)} \\ & +c\|b\|_{BMO}\sum_{k=-\infty}^{j-3}|\lambda_k|\,2^{-jn}2^{-k\alpha(0)}\|\chi_{B_k}\|_{p'(\cdot)}. \end{aligned}$$

So G_1, is bounded by

$$\begin{aligned} & c\lambda\Big(\sum_{j=-\infty}^{-1} 2^{j\alpha(0)q(0)}\Big\|\chi_{\left\{x\in R_j: c\left|b(x)-b_{B_j}\right|\sum_{k=-\infty}^{j-3}|\lambda_k|2^{-j\left(\alpha(0)+\frac{n}{p(0)}\right)}>\frac{\lambda}{4}\right\}}\Big\|_{p(\cdot)}^{q(0)}\Big)^{\frac{1}{q(0)}} \\ & +c\lambda\Big(\sum_{j=-\infty}^{-1} 2^{j\alpha(0)q(0)}\Big\|\chi_{\left\{x\in R_j: c\|b\|_{BMO}\sum_{k=-\infty}^{j-3}|\lambda_k|2^{-jn}2^{-k\alpha(0)}\left\|\chi_{B_k}\right\|_{p'(\cdot)}>\frac{\lambda}{4}\right\}}\Big\|_{p(\cdot)}^{q(0)}\Big)^{\frac{1}{q(0)}}. \quad (2.8) \end{aligned}$$

Observing that $\sum_{k=-\infty}^{j-3}|\lambda_k| \lesssim \left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)}\right)^{1/q(0)} \lesssim \|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$, then the first term in (2.8) is bounded by

$$\begin{aligned} & c\lambda\Big(\sum_{j=-\infty}^{\infty} 2^{j\alpha(0)q(0)}\Big(\exp\Big(-\frac{c\lambda 2^{j\left(\alpha(0)+\frac{n}{p(0)}\right)}}{\|b\|_{BMO}\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}}\Big)\|\chi_j\|_{p(\cdot)}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}} \\ \lesssim \ & \lambda\Big(\int_0^\infty s^{q(0)-1}\exp\Big(-\frac{c\lambda s}{\|b\|_{BMO}\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}}\Big)ds\Big)^{\frac{1}{q(0)}} \\ \lesssim \ & \|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}. \end{aligned}$$

For any fixed $\lambda > 0$, if

$$\Big|\Big\{x \in R_j : c\big\|b\big\|_{BMO}\sum_{k=-\infty}^{j-3}|\lambda_k|\,2^{-jn}2^{-k\alpha(0)}\big\|\chi_{B_k}\big\|_{p'(\cdot)} > \frac{\lambda}{4}\Big\}\Big| \neq 0.$$

Then

$$\begin{aligned}\lambda &\leq 4c\big\|b\big\|_{BMO}2^{-j\left(\alpha(0)+\frac{n}{p(0)}\right)}\sum_{k=-\infty}^{j-3}|\lambda_k| \\ &\leq 4c\big\|b\big\|_{BMO}\big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}2^{-jn}, \quad j \leq -1.\end{aligned}$$

We consider two cases. The first is the case $4c\lambda^{-1}\big\|b\big\|_{BMO}\big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \geq 2^{-2n}$. So the second sum in (2.8) is bounded by

$$\begin{aligned}c\lambda\Big(\sum_{j=-\infty}^{-1}2^{j\alpha(0)q(0)}\big\|\chi_j\big\|^{q(0)}_{p(\cdot)}\Big)^{\frac{1}{q(0)}} &\leq c\lambda\Big(\sum_{j=-\infty}^{-1}2^{j\left(\alpha(0)+\frac{n}{p(0)}\right)q(0)}\Big)^{\frac{1}{q(0)}} \\ &\leq c\lambda \lesssim \big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.\end{aligned}$$

We now consider another case $4c\lambda^{-1}\big\|b\big\|_{BMO}\big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} < 2^{-2n}$. For any fixed $\lambda > 0$ we put

$$j_\lambda = \left[\frac{1}{n}\log_2(4C\lambda^{-1}\big\|b\big\|_{BMO}\big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}})\right].$$

The advantage of this choice consists in the fact that

$$\Big\{x \in R_j : c\,\|b\|_{BMO}\sum_{k=-\infty}^{j-3}|\lambda_k|\,2^{-jn}2^{-k\alpha(0)}\big\|\chi_{B_k}\big\|_{p'(\cdot)} > \frac{\lambda}{4}\Big\} = \emptyset,$$

if $-1 \geq j \geq j_\lambda + 1$. Hence the second sum in (2.8) is bounded by

$$\begin{aligned}&c\lambda\Big(\sum_{j=-\infty}^{j_\lambda}2^{j\alpha(0)q(0)}\big\|\chi_j\big\|^{q(0)}_{p(\cdot)}\Big)^{\frac{1}{q(0)}} \\ \leq\ &c\lambda\Big(\sum_{j=-\infty}^{j_\lambda}2^{j(\alpha(0)+\frac{n}{p(0)})q(0)}\Big)^{\frac{1}{q(0)}} \\ \leq\ &c\lambda 2^{j_\lambda n} \lesssim \big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.\end{aligned}$$

For G_3. The boundedness of M_b on $L^{p(\cdot)}(\mathbb{R}^n)$ yield that

$$\begin{aligned}\lambda 2^{j\alpha(0)}\left\|\chi_{A_j(\frac{\lambda}{2},M_b(\sum_{k=j-2}^{\infty}\lambda_k a_k))}\right\|_{p(\cdot)} &\lesssim 2^{j\alpha(0)}\sum_{k=j-2}^{\infty}|\lambda_k|\left\|a_k\right\|_{p(\cdot)}\\ &\lesssim \sum_{k=j-2}^{-1}...+\sum_{k=0}^{\infty}....\end{aligned} \tag{2.9}$$

Since $\left\|a_k\right\|_{p(\cdot)} \lesssim 2^{-k\alpha(0)}, k<0$. The first sum in (2.9) is bounded by

$$c\sum_{k=j-2}^{-1}|\lambda_k|\,2^{(j-k)\alpha(0)},$$

an application of Lemma 1.16 yields that the $\ell_<^{q(0)}$-norm of this expression is bounded by $\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)}\right)^{1/q(0)}$. Now if $k\geq 0$ then $\left\|a_k\right\|_{p(\cdot)} \lesssim 2^{-k\alpha_\infty}$. The second sum in (2.9) is bounded by

$$c2^{j\alpha(0)}\sum_{k=0}^{\infty}|\lambda_k|\,2^{-k\alpha_\infty} \lesssim 2^{j\alpha(0)}\Big(\sum_{k=0}^{\infty}|\lambda_k|^{q_\infty}\Big)^{1/q_\infty}.$$

The $\ell_<^{q(0)}$-norm of this expression is bounded by $\left(\sum_{k=0}^{\infty}|\lambda_k|^{q_\infty}\right)^{1/q_\infty}$.

For G_4. As above we have

$$\lambda 2^{j\alpha_\infty}\left\|\chi_{A_j(\frac{\lambda}{2},M_b(\sum_{k=j-2}^{\infty}\lambda_k a_k))}\right\|_{p(\cdot)} \lesssim \sum_{k=j-2}^{\infty}|\lambda_k|\,2^{(j-k)\alpha_\infty}$$

we apply Lemma 1.16 and get

$$G_4 \lesssim \Big(\sum_{j=0}^{\infty}|\lambda_j|^{q_\infty}\Big)^{1/q_\infty} \lesssim \left\|f\right\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},$$

which completes the proof. ■

Theorem 2.14 *Let* $0<v<n$, $p_1\in\mathcal{P}^{\ln}(\mathbb{R}^n)$ *and* $q\in\mathcal{P}_0(\mathbb{R}^n)$ *with* $1<p_1^-\leq p_1^+<\frac{n}{v}$, $0<q^-\leq q^+\leq 1$, $\frac{1}{p_1(\cdot)}-\frac{1}{p_2(\cdot)}=\frac{v}{n}$ *and let* α *and* q *are be* log*-Hölder continuous, at infinity such that* $\alpha\in L^\infty(\mathbb{R}^n)$, *and*

$$\alpha_\infty = n\Big(1-\frac{1}{(p_1)_\infty}\Big).$$

If $b\in BMO(\mathbb{R}^n)$ *satisfies the condition* $\mathcal{L}$, *then* M_b^v *is bounded from* $HK^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n)$ *into* $WK^{\alpha(\cdot)}_{p_2(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Theorem 2.15 *Let* $0 < v < n$, $p_1 \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ *and* $q \in \mathcal{P}_0(\mathbb{R}^n)$ *with* $1 < p_1^- \leq p_1^+ < \frac{n}{v}$, $0 < q^- \leq q^+ \leq 1$, $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{v}{n}$ *and let* α *and* q *are be* log-*Hölder continuous, both at the origin and at infinity such that* $\alpha \in L^{\infty}(\mathbb{R}^n)$, *and*

$$\alpha(\cdot) \geq n(1 - \frac{1}{p_1^-}), \quad \alpha(0) = n\Big(1 - \frac{1}{p_1(0)}\Big), \quad \alpha_\infty = n\Big(1 - \frac{1}{(p_1)_\infty}\Big).$$

If $b \in BMO(\mathbb{R}^n)$ *satisfies the condition* $\mathcal{L}$, *then* M_b^v *is bounded from* $H\dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n)$ *into* $W\dot{K}^{\alpha(\cdot)}_{p_2(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Theorem 2.16 *Let* α *and* q *are* log-*Hölder continuous at infinity and* $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ *with* $0 < p^- \leq p^+ < \infty$ *and* $0 < q^- \leq q^+ \leq 1$ *such that*

$$\alpha_\infty = n(1 - \frac{1}{p_\infty}).$$

Let $b \in BMO(\mathbb{R}^n)$ *and* T *be a linear operator. Suppose that the commutator* $[b, T]$ *is bounded on* $WL^{p(\cdot)}(\mathbb{R}^n)$ *and* T *satisfies the local size condition*

$$|Tf(x)| \leq c\,|x|^{-n} \int_{\mathbb{R}^n} |f(y)|\,dy$$

for $f \in L^1(\mathbb{R}^n)$, $\mathrm{supp} f \subset R_k$ *and* $|x| \geq 2^{k+1}$ *with* $k \in \mathbb{Z}^n$. *Then for any* $f \in HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *and any* $\lambda > 0$, *we have*

$$\big\|\{2^{k\alpha_\infty}\chi_{A_k(\lambda,[b,T]f)}\}\big\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})} \leq \frac{c\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}}{\lambda}\Big(1 + \log^+ \frac{c\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}}{\lambda}\Big),$$

with $c > 0$ *only dependent on* f *and* λ.

Theorem 2.17 *Let* $0 < l < n$, $p_1 \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ *and* $q \in \mathcal{P}_0(\mathbb{R}^n)$ *with* $1 < p_1^- \leq p_1^+ < \frac{n}{l}$, $0 < q^- \leq q^+ \leq 1$, $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{l}{n}$ *and let* α *and* q *are be* log-*Hölder continuous, at infinity such that* $\alpha \in L^{\infty}(\mathbb{R}^n)$, *and*

$$\alpha_\infty = n\Big(1 - \frac{1}{(p_1)_\infty}\Big).$$

Let $b \in BMO(\mathbb{R}^n)$ *and* T_l *be a linear operator. Suppose that the commutator* $[b, T_l]$ *is bounded from* $L^{p_1(\cdot)}(\mathbb{R}^n)$ *into* $L^{p_2(\cdot)}(\mathbb{R}^n)$ *and* T_l *satisfies the local size condition*

$$|Tf(x)| \leq c\,|x|^{-(n-l)} \int_{\mathbb{R}^n} |f(y)|\,dy$$

for $f \in L^1(\mathbb{R}^n)$, $\operatorname{supp} f \subset R_k$ *and* $|x| \geq 2^{k+1}$ *with* $k \in \mathbb{Z}^n$. *Then for any* $f \in HK^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n)$ *and any* $\lambda > 0$, *we have*

$$\left\|\{2^{k\alpha_\infty}\chi_{A_k(\lambda,[b,T_l]f)}\}\right\|_{\ell^{q_\infty}_{>}(L^{p_2(\cdot)})} \leq \frac{c\|f\|_{HK^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}}}{\lambda}\Big(1+\log^+\frac{c\|f\|_{HK^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}}}{\lambda}\Big),$$

with $c > 0$ *only dependent on* f *and* λ.

Theorem 2.18 *Let* α *and* q *are be* log*-Hölder continuous at infinity and* $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ *with* $1 < p^- \leq p^+ < \infty$ *and* $0 < q^- \leq q^+ \leq 1$ *such that*

$$\alpha_\infty = n(1-\frac{1}{p_\infty}).$$

Let b, T *and* $[b,T]$ *be as in Theorem 2.16. If* b *satisfies the condition* $\mathcal{L}$, *then* $[b,T]$ *is bounded from* $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *into* $WK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Theorem 2.19 *Let* α *and* q *are be* log*-Hölder continuous, both at the origin and at infinity and* $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ *with* $1 < p^- \leq p^+ < \infty$ *and* $0 < q^- \leq q^+ \leq 1$ *such that*

$$\alpha(\cdot) \geq n(1-\frac{1}{p^-}), \quad \alpha(0) = n(1-\frac{1}{p(0)}), \quad \text{and } \alpha_\infty = n(1-\frac{1}{p_\infty}).$$

Let b, T *and* $[b,T]$ *be as in Theorem 2.16. If* b *satisfies the condition* $\mathcal{L}$, *then* $[b,T]$ *is bounded from* $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *into* $W\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Theorem 2.20 *Let* $0 < l < n$, $p_1 \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ *and* $q \in \mathcal{P}_0(\mathbb{R}^n)$ *with* $1 < p_1^- \leq p_1^+ < \frac{n}{l}$, $0 < q^- \leq q^+ \leq 1$, $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{l}{n}$ *and let* α *and* q *are be* log*-Hölder continuous, at infinity such that* $\alpha \in L^\infty(\mathbb{R}^n)$, *and*

$$\alpha_\infty = n\Big(1-\frac{1}{(p_1)_\infty}\Big).$$

Let b, T_l *and* $[b,T_l]$ *be as in Theorem 2.17. If* b *satisfies the condition* $\mathcal{L}$, *then* $[b,T_l]$ *is bounded from* $HK^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n)$ *into* $WK^{\alpha(\cdot)}_{p_2(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Theorem 2.21 *Let* $0 < l < n$, $p_1 \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ *and* $q \in \mathcal{P}_0(\mathbb{R}^n)$ *with* $1 < p_1^- \le p_1^+ < \frac{n}{l}$, $0 < q^- \le q^+ \le 1$, $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{l}{n}$ *and let* α *and* q *are be* log*-Hölder continuous, both at the origin and at infinity such that* $\alpha \in L^\infty(\mathbb{R}^n)$, *and*

$$\alpha(\cdot) \ge n(1 - \frac{1}{p_1^-}), \quad \alpha(0) = n\Big(1 - \frac{1}{p_1(0)}\Big), \quad \alpha_\infty = n\Big(1 - \frac{1}{(p_1)_\infty}\Big).$$

Let b, T_l *and* $[b, T_l]$ *be as in Theorem 2.17. If* b *satisfies the condition* $\mathcal{L}$, *then* $[b, T_l]$ *is bounded from* $H\dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n)$ *into* $W\dot{K}^{\alpha(\cdot)}_{p_2(\cdot),q(\cdot)}(\mathbb{R}^n)$.

We omit the proofs of Theorems 2.20 and 2.21 since are they essentially similar to the proof of Theorem 2.19.

Corollary 2.22 *If we replace the size condition of* T *in Theorems 2.18, 2.19 by*

$$|Tf(x)| \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy, \quad x \notin suppf, \tag{2.10}$$

for integrable and compactly supported functions f. Then the conclusion of Theorems 2.18, 2.19 is also true.

Corollary 2.23 *If we replace the size condition of* T *in Theorems 2.20, 2.21 by*

$$|T_\lambda f(x)| \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\lambda}} dy, \quad x \notin suppf, \tag{2.11}$$

for integrable and compactly supported functions f. Then the conclusion of Theorems 2.20, 2.21 is also true.

Remark 2.24 *The results of this chapter with* α *and* q *are fixed are given in [33]. All results of this chapter are taken from [5].*

Chapter 3

Weak type estimates of commutators II

In this chapter, we study the weak type estimate of some commutators with Lipschitz function on Herz-type Hardy spaces with variable exponent.

3.1 Preliminaries

For $0 < \beta \leq 1$, the Lipschitz space $Lip_\beta(\mathbb{R}^n)$ is defined as

$$Lip_\beta(\mathbb{R}^n) := \left\{ f : \|f\|_{Lip_\beta} = \sup_{y,t \in \mathbb{R}^n; x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\beta} < \infty \right\}.$$

Let $b \in Lip_\beta(\mathbb{R}^n)$, and let T be a Calderón–Zygmund singular integral operator that is,

$$Tf(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy,$$

$p.v.$ or (principal value integrals) where $\Omega \in c^2(S^{n-1})$ is homogeneous of degree zero and has mean value zero on the unit sphere. The commutator $[b,T]$ generated by b and T is defined by

$$[b,T](f)(x) := b(x)Tf(x) - T(bf)(x).$$

Let $0 < \beta < n$, The fractional integral operator I_β is defined by

$$I_\beta(f)(x) := \frac{1}{\gamma(\beta)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta}} dy,$$

where

$$\gamma(\beta) := \frac{\pi^{n/2} 2^{\beta} \Gamma(\beta/2)}{\Gamma((n-\beta)/2)}.$$

Let $b \in Lip_{\beta}(\mathbb{R}^n)$, $0 < l < n$ and let I_l denote the fractional integral operator. The commutator of fractional integral operator

$$[b, I_l] f(x) := \int_{\mathbb{R}^n} \frac{(b(x) - b(y))}{|x-y|^{n-l}} f(y) \, dy.$$

3.2 Main results and their proofs

The main results of this chapter are the following.

Theorem 3.1 *Let $0 < \beta \leq 1$, $s \geq [\beta]$, $p_1 \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ and $q \in \mathcal{P}_0(\mathbb{R}^n)$ with $1 < p_1^- \leq p_1^+ < \frac{n}{\beta}$, $0 < q^- \leq q^+ \leq 1$, $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\beta}{n}$ and let α and q are be log-Hölder continuous, at infinity such that $\alpha \in L^{\infty}(\mathbb{R}^n)$ and*

$$\alpha_{\infty} = \beta + n\Big(1 - \frac{1}{(p_1)_{\infty}}\Big).$$

If $b \in Lip_{\beta}(\mathbb{R}^n)$, then $[b, T]$ is bounded from $HK^{\alpha(\cdot)}_{p_1(\cdot), q(\cdot)}(\mathbb{R}^n)$ into $WK^{\alpha(\cdot)}_{p_2(\cdot), q(\cdot)}(\mathbb{R}^n)$.

Theorem 3.2 *Let $0 < \beta \leq 1$, $s \geq [\beta]$, $p_1 \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ and $q \in \mathcal{P}_0(\mathbb{R}^n)$ with $1 < p_1^- \leq p_1^+ < \frac{n}{\beta}$, $0 < q^- \leq q^+ \leq 1$, $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\beta}{n}$ and let α and q are be log-Hölder continuous, both at the origin and at infinity such that $\alpha \in L^{\infty}(\mathbb{R}^n)$, and*

$$\alpha(\cdot) \geq n(1 - \frac{1}{p_1^-}), \quad \alpha(0) = \beta + n\Big(1 - \frac{1}{p_1(0)}\Big), \quad \alpha_{\infty} = \beta + n\Big(1 - \frac{1}{(p_1)_{\infty}}\Big).$$

If $b \in Lip_{\beta}(\mathbb{R}^n)$, then $[b, T]$ is bounded from $H\dot{K}^{\alpha(\cdot)}_{p_1(\cdot), q(\cdot)}(\mathbb{R}^n)$ into $W\dot{K}^{\alpha(\cdot)}_{p_2(\cdot), q(\cdot)}(\mathbb{R}^n)$.

Proof. Suppose $f \in H\dot{K}^{\alpha(\cdot)}_{p_1(\cdot), q(\cdot)}$. By Theorem 2.6, we have

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k,$$

where the series converges in the sense of distributions, each a_k is a central $(\alpha(\cdot), p_1(\cdot))$-atom with supp$a_k \subset R_k$ and

$$\Big(\sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)}\Big)^{1/q(0)}+\Big(\sum_{k=0}^{\infty}|\lambda_k|^{q_\infty}\Big)^{1/q_\infty}\leq c\big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}}.$$

Let $\lambda > 0$, using Proposition 2.2 we have

$$\begin{aligned}\|\,[b,T]\,\|_{W\dot{K}^{\alpha(\cdot)}_{p_2(\cdot),q(\cdot)}} &\approx \sup_{\lambda>0}\Big(\lambda\big\|\{2^{k\alpha(0)}\chi_{A_k(\lambda,[b,T])}\}\big\|_{\ell^{q(0)}_<(L^{p_2(\cdot)})}\\ &\quad+\lambda\big\|\{2^{k\alpha_\infty}\chi_{A_k(\lambda,[b,T])}\}\big\|_{\ell^{q_\infty}_>(L^{p_2(\cdot)})}\Big)\\ &\lesssim \sup_{\lambda>0}\{E_1+E_2+E_3+E_4+E_5+E_6\},\end{aligned}$$

where

$$E_1 := \lambda\Big\|\Big\{2^{j\alpha(0)}\chi_{A_j(\frac{\lambda}{3},\sum_{k=-\infty}^{j-2}\lambda_k(b-b(0))Ta_k)}\Big\}\Big\|_{\ell^{q(0)}_<(L^{p_2(\cdot)})}$$

$$E_2 := \lambda\Big\|\Big\{2^{j\alpha_\infty}\chi_{A_j(\frac{\lambda}{3},\sum_{k=-\infty}^{j-2}\lambda_k(b-b(0))Ta_k)}\Big\}\Big\|_{\ell^{q_\infty}_>(L^{p_2(\cdot)})}$$

$$E_3 := \lambda\Big\|\Big\{2^{j\alpha(0)}\chi_{A_j(\frac{\lambda}{3},\sum_{k=-\infty}^{j-2}\lambda_kT(b-b(0))a_k)}\Big\}\Big\|_{\ell^{q(0)}_<(L^{p_2(\cdot)})}$$

$$E_4 := \lambda\Big\|\Big\{2^{j\alpha_\infty}\chi_{A_j(\frac{\lambda}{3},\sum_{k=-\infty}^{j-2}\lambda_kT(b-b(0))a_k)}\Big\}\Big\|_{\ell^{q_\infty}_>(L^{p_2(\cdot)})}$$

$$E_5 := \lambda\Big\|\Big\{2^{j\alpha(0)}\chi_{A_j(\frac{\lambda}{3},[b,T](\sum_{k=j-1}^{\infty}\lambda_ka_k))}\Big\}\Big\|_{\ell^{q(0)}_<(L^{p_2(\cdot)})}$$

and

$$E_6 := \lambda\Big\|\Big\{2^{j\alpha_\infty}\chi_{A_j(\frac{\lambda}{3},[b,T](\sum_{k=j-1}^{\infty}\lambda_ka_k))}\Big\}\Big\|_{\ell^{q_\infty}_>(L^{p_2(\cdot)})}$$

To complete the proof, it suffices to show that

$$\sup_{\lambda>0}E_i \lesssim \big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}},\quad i=1,2,3,4,5,6.$$

Estimate of E_1 and E_2. From the vanishing moment of a_k we obtain

$$\begin{aligned}|b(x)-b(0)\,Ta_k(x)| &\leq c\,|b(x)-b(0)|\int_{R_k}\Big|\frac{\Omega(x-y)}{|x-y|^n}-\sum_{|\gamma|\leq[\beta]}\frac{1}{\gamma!}\partial^\gamma\frac{\Omega(x)}{|x|^n}(-y)^\gamma\Big|\,|a_k(y)|\,dy\\ &\leq c\,\|b\|_{Lip_\beta}|x|^\beta\int_{R_k}\frac{|y|^{[\beta]+1}}{|x|^{n+[\beta]+1}}\,|a_k(y)|\,dy,\end{aligned}$$

for any $y \in R_k, x \in B_j$ with $k \leq j-2$. After applying Hölder's inequality, the last expression is bounded by

$$\leq c \|b\|_{Lip_\beta} 2^{j(\beta-n-[\beta]-1)} 2^{k([\beta]+1)} \|a_k\|_{p_1(\cdot)} \|\mathcal{X}_k\|_{p_1'(\cdot)}. \tag{3.1}$$

Note that

$$I_\beta(B_j)(x) \geq \frac{1}{\gamma(\beta)} \int_{B_j} \frac{dy}{|x-y|^{n-\beta}} \mathcal{X}_k(x) \geq c 2^{j\beta} \mathcal{X}_{B_j}(x), \tag{3.2}$$

From this and the fact that I_β maps $L^{p_1(\cdot)}(\mathbb{R}^n)$ into $L^{p_2(\cdot)}$, we obtain

$$\left\|\mathcal{X}_{B_j}\right\|_{p_2(\cdot)} \leq c 2^{-j\beta} \left\|I_\beta\left(\mathcal{X}_{B_j}\right)\right\|_{p_2(\cdot)} \leq c 2^{-j\beta} \left\|\mathcal{X}_{B_j}\right\|_{p_1(\cdot)}, \quad j \in \mathbb{Z}. \tag{3.3}$$

So for any $j < 0$, we have

$$\begin{aligned}
& 2^{j\alpha(0)} \left\|\chi_{A_j\left(\frac{\lambda}{3}, \sum_{k=-\infty}^{j-2} \lambda_k (b-b(0)) T a_k\right)}\right\|_{p_2(\cdot)} \\
\leq\ & \frac{c}{\lambda} \|b\|_{Lip_\beta} \sum_{k=-\infty}^{j-2} |\lambda_k| 2^{j\alpha(0)} 2^{-k\alpha(0)} 2^{j(\beta-n-[\beta]-1)} 2^{k([\beta]+1)} \|\mathcal{X}_k\|_{p_1'(\cdot)} \left\|\mathcal{X}_{B_j}\right\|_{p_2(\cdot)} \\
\leq\ & \frac{c}{\lambda} \|b\|_{Lip_\beta} \sum_{k=-\infty}^{j-2} |\lambda_k| 2^{j\alpha(0)} 2^{-k\alpha(0)} 2^{j(-n-[\beta]-1)} 2^{k([\beta]+1)} \|\mathcal{X}_k\|_{p_1'(\cdot)} \left\|\mathcal{X}_{B_j}\right\|_{p_1(\cdot)} \\
\leq\ & \frac{c}{\lambda} \|b\|_{Lip_\beta} \sum_{k=-\infty}^{j-2} |\lambda_k| 2^{(k-j)\left([\beta]+1+n-\frac{n}{p_1(0)}-\alpha(0)\right)} \\
\leq\ & \frac{c}{\lambda} \|b\|_{Lip_\beta} \sum_{k=-\infty}^{j-2} |\lambda_k| 2^{(k-j)([\beta]+1-\beta)}
\end{aligned}$$

where we have used the fact a_k is a central $(\alpha(\cdot), p_1(\cdot))$-atom and the condition $\alpha(0) = \beta + n\Big(1 - \frac{1}{p_1(0)}\Big)$. Therefore,

$$\begin{aligned}
E_1 &\leq c\Big(\sum_{j=-\infty}^{-1} \Big(\sum_{k=-\infty}^{j-2} |\lambda_k| 2^{(k-j)([\beta]+1-\beta)}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}} \\
&\leq c\Big(\sum_{j=-\infty}^{-1} |\lambda_j|^{q(0)}\Big)^{\frac{1}{q(0)}} \lesssim \|f\|_{H\dot{K}^{\alpha(\cdot)}_{p_1(\cdot), q(\cdot)}}.
\end{aligned}$$

where we have used Lemma 1.16.

Again, for any $j \geq 0$. As above, we have

$$
\begin{aligned}
& 2^{j\alpha_\infty} \left\| \chi_{A_j\left(\frac{\lambda}{3}, \sum_{k=-\infty}^{j-2} \lambda_k (b-b(0)) T a_k\right)} \right\|_{p_2(\cdot)} \\
\leq\ & \frac{c}{\lambda} 2^{j\alpha_\infty} \|b\|_{Lip_\beta} \sum_{k=-\infty}^{j-2} |\lambda_k| 2^{j(\beta-n-[\beta]-1)} 2^{k([\beta]+1)} \|a_k\|_{p_1(\cdot)} \|\mathcal{X}_k\|_{p_1'(\cdot)} \|\mathcal{X}_{B_j}\|_{p_2(\cdot)} \\
\leq\ & \frac{c}{\lambda} 2^{j\alpha_\infty} \|b\|_{Lip_\beta} \sum_{k=-\infty}^{j-2} |\lambda_k| 2^{(-n-[\beta]-1)} 2^{k([\beta]+1)} \|a_k\|_{p_1(\cdot)} \|\mathcal{X}_k\|_{p_1'(\cdot)} \|\mathcal{X}_{B_j}\|_{p_1(\cdot)} \\
\leq\ & \frac{c}{\lambda} \|b\|_{Lip_\beta} 2^{-j\left([\beta]+1+n-\frac{n}{(p_1)_\infty}-\alpha_\infty\right)} \sum_{k=-\infty}^{-1} |\lambda_k| 2^{k\left([\beta]+1+n-\frac{n}{p_1(0)}-\alpha(0)\right)} \\
& + \frac{c}{\lambda} \|b\|_{Lip_\beta} \sum_{k=0}^{j-2} |\lambda_k| 2^{(k-j)\left([\beta]+1+n-\frac{n}{(p_1)_\infty}-\alpha_\infty\right)} \\
\leq\ & \frac{c}{\lambda} \|b\|_{Lip_\beta} 2^{-j([\beta]+1-\beta)} \sup_{k<0} |\lambda_k| \sum_{k=-\infty}^{-1} 2^{k([\beta]+1-\beta)} + \frac{c}{\lambda} \|b\|_{Lip_\beta} \sum_{k=0}^{j-2} |\lambda_k| 2^{(k-j)([\beta]+1-\beta)} \\
\leq\ & \frac{c}{\lambda} \|b\|_{Lip_\beta} \left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \right)^{1/q(0)} 2^{-j([\beta]+1-\beta)} + \frac{c}{\lambda} \|b\|_{Lip_\beta} \sum_{k=0}^{j-2} 2^{(k-j)([\beta]+1-\beta)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E_2 \leq\ & c \|b\|_{Lip_\beta} \left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \right)^{1/q(0)} \left(\sum_{j=0}^{\infty} 2^{-j([\beta]+1-\beta) q_\infty} \right)^{\frac{1}{q_\infty}} \\
& + c \|b\|_{Lip_\beta} \left(\sum_{j=0}^{\infty} \left(\sum_{k=0}^{j-2} |\lambda_k| 2^{(k-j)([\beta]+1-\beta)} \right)^{\frac{1}{q_\infty}} \right. \\
\leq\ & c \|b\|_{Lip_\beta} \left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \right)^{1/q(0)} + c \|b\|_{Lip_\beta} \left(\sum_{j=0}^{\infty} |\lambda_j|^{q_\infty} \right)^{1/q_\infty} \\
\lesssim\ & \|f\|_{H\dot{K}^{\alpha(\cdot)}_{p_1(\cdot), q(\cdot)}}.
\end{aligned}
$$

Estimate of E_3. For $y \in R_k, x \in R_j$ with $k+2 \leq j < 0$, we have

$$
\begin{aligned}
|(b(x) - b(0)) T a_k(x)| &= \left| \int_{R_k} \frac{\Omega(x-y)}{|x-y|^n} (b(y) - b(0)) a_k(y) \, dy \right| \\
&\leq c \|b\|_{Lip_\beta} \int_{R_k} \frac{|y|^\beta}{|x-y|^n} |a_k(y)| \, dy,
\end{aligned}
$$

by the Hölder's inequality the last expression is bounded by

$$c\,\|b\|_{Lip_\beta}\,|x|^{-n}\,2^{k\beta}2^{-k\alpha(0)}\,\|\mathcal{X}_k\|_{p_1'(\cdot)} \lesssim \|b\|_{Lip_\beta}\,2^{-jn}2^{k\left(\beta-\alpha(0)+n-\frac{n}{p_1(0)}\right)} \lesssim \|b\|_{Lip_\beta}\,2^{-jn}.$$

Thus,

$$\begin{aligned}\Big|\sum_{k=-\infty}^{j-2}\lambda_k T\left(b-b\left(0\right)\right)a_k\left(x\right)\Big| &\leq c\,\|b\|_{Lip_\beta}\,2^{-jn}\sum_{k=-\infty}^{j-2}|\lambda_k| \\ &\leq c\,\|b\|_{Lip_\beta}\,2^{-jn}\Big(\sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)}\Big)^{1/q(0)} \\ &\leq c\,\|b\|_{Lip_\beta}\,2^{-jn}\big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}}. \qquad (3.4)\end{aligned}$$

Assume that

$$\Big|\Big\{x\in R_j:\Big|\sum_{k=-\infty}^{j-2}\lambda_k T\left(b-b\left(0\right)\right)a_k\left(x\right)\Big|>\frac{\lambda}{3}\Big\}\Big|\neq 0.$$

Then

$$\lambda\leq 3c\,\|b\|_{Lip_\beta}\,2^{-jn}\big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}},\quad j<0.$$

We consider two cases. The first is the case $3c\lambda^{-1}\,\|b\|_{Lip_\beta}\,\big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}}\geq 2^{-2n}$. By (3.3), we get

$$\begin{aligned}E_3 &\leq c\lambda\Big(\sum_{j=-\infty}^{-1}2^{j\alpha(0)q(0)}\big\|\chi_{\left\{x\in R_j:\,\left|\sum_{k=-\infty}^{j-2}\lambda_k T(b-b(0))a_k(x)\right|>\frac{\lambda}{3}\right\}}\big\|^{q(0)}_{p_2(\cdot)}\Big)^{\frac{1}{q(0)}} \\ &\leq c\lambda\Big(\sum_{j=-\infty}^{-1}2^{j\alpha(0)q(0)}\big\|\chi_j\big\|^{q(0)}_{p_2(\cdot)}\Big)^{\frac{1}{q(0)}} \\ &\leq c\lambda\Big(\sum_{j=-\infty}^{-1}2^{j(\alpha(0)-\beta)q(0)}\big\|\mathcal{X}_{B_j}\big\|^{q(0)}_{p_1(\cdot)}\Big)^{\frac{1}{q(0)}} \\ &\leq c\lambda\Big(\sum_{j=-\infty}^{-1}2^{j(\alpha(0)-\beta+\frac{n}{p_1(0)})q(0)}\Big)^{\frac{1}{q(0)}} \\ &\leq c\lambda\lesssim\big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}}.\end{aligned}$$

We now consider another case $3c\lambda^{-1}\|b\|_{Lip_\beta}\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}} < 2^{-2n}$. For any fixed $\lambda > 0$ we put

$$j_\lambda = \left[\frac{1}{n}\log_2(3c\lambda^{-1}\|b\|_{Lip_\beta}\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}})\right].$$

The advantage of this choice consists in the fact that

$$\Big\{x \in R_j : \Big|\sum_{k=-\infty}^{j-2}\lambda_k T\left(b - b\left(0\right)\right)a_k\left(x\right)\Big| > \frac{\lambda}{3}\Big\} = \emptyset,$$

if $-1 \geq j \geq j_\lambda + 1$. Hence by (3.3), we obtain

$$\begin{aligned}
E_3 &\leq c\lambda\Big(\sum_{j=-\infty}^{j_\lambda} 2^{j\alpha(0)q(0)}\big\|\chi_j\big\|^{q(0)}_{p_2(\cdot)}\Big)^{\frac{1}{q(0)}} \\
&\leq c\lambda\Big(\sum_{j=-\infty}^{j_\lambda} 2^{j(\alpha(0)-\beta)}\big\|\mathcal{X}_{B_j}\big\|^{q(0)}_{p_1(\cdot)}\Big)^{\frac{1}{q(0)}} \\
&\leq c\lambda\Big(\sum_{j=-\infty}^{j_\lambda} 2^{jq(0)(\alpha(0)-\beta+\frac{n}{p_1(0)})}\Big)^{\frac{1}{q(0)}} \\
&\leq c\lambda 2^{j_\lambda n} \lesssim \big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}}.
\end{aligned}$$

Estimate of E_4. Similar to E_3 from (3.4) and (3.3) we get

$$\begin{aligned}
E_4 &\leq c\lambda\Big(\sum_{j=0}^{j_\lambda} 2^{j\alpha_\infty q_\infty}\big\|\chi_{\{x\in R_j:\,|\sum_{k=-\infty}^{j-2}\lambda_k T(b-b(0))a_k(x)|>\frac{\lambda}{3}\}}\big\|^{q_\infty}_{p_2(\cdot)}\Big)^{\frac{1}{q_\infty}} \\
&\leq c\lambda\Big(\sum_{j=0}^{j_\lambda} 2^{j\alpha_\infty q_\infty}\big\|\chi_j\big\|^{q_\infty}_{p_2(\cdot)}\Big)^{\frac{1}{q_\infty}} \\
&\leq c\lambda\Big(\sum_{j=0}^{j_\lambda} 2^{j(\alpha_\infty-\beta)q_\infty}\big\|\mathcal{X}_{B_j}\big\|^{q_\infty}_{p_1(\cdot)}\Big)^{\frac{1}{q_\infty}} \\
&\leq c\lambda\Big(\sum_{j=0}^{j_\lambda} 2^{j(\alpha_\infty-\beta+\frac{n}{(p_1)_\infty})q_\infty}\Big)^{\frac{1}{q_\infty}} \lesssim \big\|f\big\|_{H\dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}},
\end{aligned}$$

Estimate of E_5. Using the fact that $[b,T]$ maps $L^{p_1(\cdot)}(\mathbb{R}^n)$ into $L^{p_2(\cdot)}(\mathbb{R}^n)$, we have

$$\begin{aligned}
2^{j\alpha(0)}\left\|\chi_{A_j(\frac{\lambda}{3},[b,T](\sum_{k=j-1}^{\infty}\lambda_k a_k))}\right\|_{p_2(\cdot)} &\leq \frac{c}{\lambda}2^{j\alpha(0)}\sum_{k=j-1}^{\infty}|\lambda_k|\left\|[b,T]\,a_k\chi_j\right\|_{p_2(\cdot)}\\
&\leq \frac{c}{\lambda}2^{j\alpha(0)}\sum_{k=j-1}^{\infty}|\lambda_k|\left\|a_k\right\|_{p_1(\cdot)}\\
&\leq \frac{c}{\lambda}\sum_{k=j-1}^{-1}|\lambda_k|\,2^{(j-k)\alpha(0)}+\frac{c}{\lambda}2^{j\alpha(0)}\sum_{k=0}^{\infty}|\lambda_k|\,2^{-k\alpha_\infty}\\
&\leq \frac{c}{\lambda}\sum_{k=j-1}^{-1}|\lambda_k|\,2^{(j-k)\alpha(0)}+\frac{c}{\lambda}2^{j\alpha(0)}\sup_{k\geq 0}|\lambda_k|\\
&\leq \frac{c}{\lambda}\sum_{k=j-1}^{-1}|\lambda_k|\,2^{(j-k)\alpha(0)}+\frac{c}{\lambda}2^{j\alpha(0)}\Big(\sum_{k=0}^{\infty}|\lambda_k|^{q_\infty}\Big)^{1/q_\infty}.
\end{aligned}$$

Then we see that

$$\begin{aligned}
E_5 &\leq c\Big(\sum_{j=-\infty}^{-1}\Big(\sum_{k=j-1}^{-1}|\lambda_k|\,2^{(j-k)\alpha(0)}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}}\\
&\quad +c\Big(\sum_{k=0}^{\infty}|\lambda_k|^{q_\infty}\Big)^{1/q_\infty}\Big(\sum_{j=-\infty}^{-1}2^{j\alpha(0)q(0)}\Big)^{\frac{1}{q(0)}}\\
&\leq c\Big(\sum_{j=-\infty}^{-1}|\lambda_j|^{q(0)}\Big)^{\frac{1}{q(0)}}+c\Big(\sum_{k=0}^{\infty}|\lambda_k|^{q_\infty}\Big)^{1/q_\infty}\lesssim\left\|f\right\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},
\end{aligned}$$

where we have used Lemma 1.16.

Estimate of E_6. As above we have

$$2^{j\alpha_\infty}\left\|\chi_{A_j(\frac{\lambda}{3},[b,T](\sum_{k=j-1}^{\infty}\lambda_k a_k))}\right\|_{p_2(\cdot)}\leq\frac{c}{\lambda}\sum_{k=j-1}^{\infty}|\lambda_k|\,2^{(j-k)\alpha_\infty},$$

again by Lemma 1.16, we obtain

$$\begin{aligned}
E_6 &\leq c\Big(\sum_{j=0}^{\infty}\Big(\sum_{k=j-1}^{\infty}|\lambda_k|\,2^{(j-k)\alpha_\infty}\Big)^{q_\infty}\Big)^{\frac{1}{q_\infty}}\\
&\leq c\Big(\sum_{j=0}^{\infty}|\lambda_j|^{q_\infty}\Big)^{1/q_\infty}\lesssim\left\|f\right\|_{H\dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}}.
\end{aligned}$$

■

Theorem 3.3 *Let* $0 < \beta \leq 1$, $0 < l < n-\beta$, $s \geq [\beta]$, $p_1 \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ *and* $q \in \mathcal{P}_0(\mathbb{R}^n)$ *with* $1 < p_1^- \leq p_1^+ < \frac{n}{\beta+l}$, $0 < q^- \leq q^+ \leq 1$, $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\beta+l}{n}$ *and let* α *and* q *are be log-Hölder continuous, at infinity such that* $\alpha \in L^\infty(\mathbb{R}^n)$ *and*

$$\alpha_\infty = \beta + n\Big(1 - \frac{1}{(p_1)_\infty}\Big).$$

If $b \in Lip_\beta(\mathbb{R}^n)$, *then* $[b, I_l]$ *is bounded from* $HK^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n)$ *into* $WK^{\alpha(\cdot)}_{p_2(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Theorem 3.4 *Let* $0 < \beta \leq 1$, $0 < l < n-\beta$, $s \geq [\beta]$, $p_1, p_2 \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ *and* $q \in \mathcal{P}_0(\mathbb{R}^n)$ *with* $1 < p_1^- \leq p_1^+ < \frac{n}{\beta+l}$, $0 < q^- \leq q^+ \leq 1$, $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\beta+l}{n}$, *and let* α *and* q *are be log-Hölder continuous, both at the origin and at infinity such that* $\alpha \in L^\infty(\mathbb{R}^n)$ *and*

$$\alpha(\cdot) \geq n(1 - \frac{1}{p_1^-}), \quad \alpha(0) = \beta + n\Big(1 - \frac{1}{p_1(0)}\Big), \quad \alpha_\infty = \beta + n\Big(1 - \frac{1}{(p_1)_\infty}\Big).$$

If $b \in Lip_\beta(\mathbb{R}^n)$, *then* $[b, I_l]$ *is bounded from* $H\dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n)$ *into* $W\dot{K}^{\alpha(\cdot)}_{p_2(\cdot),q(\cdot)}(\mathbb{R}^n)$.

We omit the proofs of Theorems 3.1, 3.3 and 3.4 since are they essentially similar to the proof of Theorem 3.2.

Remark 3.5 *The results of this chapter with* α *and* q *are fixed are given in [33]. All results of this chapter are taken from [5].*

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Abstract

In this memory of Master option functional analysis, we will *establish the weak type BMO estimates of the commutators of the maximal operator, the fractional maximal operator and some commutators related to linear operators on the Herz type spaces with variable exponent. Subsequently the weak type Lipschitz estimates of Calderón–Zygmund singular integral commutator and fractional integral commutator from Herz-type Hardy spaces with variable exponent to weak Herz spaces with variable exponent are obtained.*

ملخص

في هده المذكرة المتعلقة بالطور الماستر تخصص تحليل دالي سنقوم بإنشاء تقديرات من النوعBMO الضعيف لمبدلات المشغل الاقصى و المشغل الاقصى الكسري وبعض المحولات المتعلقة بالمشغلين الخطيين في مسافات نوع هيرتز ذات الاس المتغير .بعد ذلك تم الحصول على تقديرات ليبشيتز من النوع الضعيف لمحول كالديرون-زيجموند ومبدل متكامل جزئي من مسافات هاردي من نوع هيرتز مع الاس المتغير الى مساحات هيرتز الضعيفة ذات الاس المتغير.

Résumé

Dans ce mémoire de Master option analyse fonctionnelle .Nous établirons les estimations BMO de type faible des commutateurs de l’opérateur maximal, de l’opérateur maximal fractionnaire et de certains commutateurs liés aux opérateurs linéaire sue les espaces de type Herz à exposant variable. Par la suite , les estimation de Lipschitz de type faible du commutateur intégral singulier de Calderon –Zygmund et du commutateur intégral fractionnaire des espaces de Hardy de type Herz avec un exposant variable aux espaces de Herz faible avec un exposant variable sont obtenues.