DEMOCRATIC AND POPULAR REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

Mohamed Boudiaf University of Msila
Faculty of Mathematics and Computer Sciences
Department of Mathematics

Master memoire

Field : Mathematics and Computer Sciences

Branch : Mathematics

Option : Algebra and Discrete Mathematics

Theme

Fuzzy Relations

Presented by :

Razika Hedjouli

Jury composed of :

LADJELAT Lahcene	M.A.A,	University of M'sila	**President.**
SAADAOUI Kheir	M.C.A,	University of M'sila	**Supervisor.**
FERAHTIA Nassim	M.C.B,	University of M'sila	**Examiner.**

Razika

University year 2022/2023

Fuzzy Relations

Hedjouli Razika

June 22, 2023

Acknowledgements

I cannot begin and finish my work without thanking the greatest And the most powerful "Allah" for blessing me to complete this memoire. I would like to express my sincere gratitude to my supervisor the professor SAADAOUI KHEIR for the continuous support, for his patience, for his guidance helped me the whole time of research and writing of this memoire. All the gratitude to the president of the jury professor LADJELAT LAHCENE and the examiner Professor FERAHTIA NASSIM for devoting their time and thier effort to read and examine my work. I am very grateful to my mother and father. Their prayers, passionate encouragements, and generosities have followed me everywhere to give me alot of power. My sincere thanks to my dear sisters Kaima, Mira, Fatna, Dalila, Noura and brothers. Ahmed, Yahya, Ibrahim, Khalil, Abdelhadi you were the main supporters of me along my study, I am deeply grateful for you. I also sincerely thank to Dear friends Khaoula Nafti, Naila Merzougui, especially Mohamed Saad for his help and support for me, Thanks to all.

Finally, I conclude my thanks and appreciation to all the administrative staff of the Department of Mathematics, who watched for our service.

All the respect and appreciation.

❖ *Razika Hedjouli*

Contents

Introduction

L. A. Zadeh [23] introduced fuzzy set theory in 1965, and is the theoretical basis for fuzzy logic, it is an extension of classical logic.The concept of relation is one of the basic concepts in both pure and applied sciences.Fuzzy relations also generalize the concept of relation in the same way that fuzzy sets generalize and it is the basic idea of sets introduced by Yager [21] who also discussed its application.

In 1983, K.T. Attanassov [4] introduced, the notion of intuitionistic fuzzy set (for short,IFS) as a generalization of the concept of fuzzy set.In fuzzy set theory, the non-membership degree of an element x of the universe is defined as $\nu_A(x) = 1 - \mu_A(x)$, which is fixed, but in intuitionistic fuzzy set theory, the non-membership degree not equal to $1 - \mu_A(x)$ the only condition in this case is $\nu_A(x) + \mu_A(x) \leq 1$.Absolutly, fuzzy sets are intuitionistic fuzzy sets by setting $\nu_A(x) = 1 - \mu_A(x)$.Intuitionistic fuzzy relation concept is an extension to this direction and this notion generalise the notion of fuzzy relations which introduced by Burillo and Bustince [8, 9].

This memoire is divided into three chapters.

In the first chapter, we will mention the notion of fuzzy sets, the basic operations and characteristic on fuzzy sets, the definition of Cartesian product on fuzzy set, and we give the definition of triangular norm and triangular conorm.

In the second chapter, we give the definition of fuzzy relations with some properties, also the operations on fuzzy relations and we will add the composition of fuzzy relations with its properties, and we give definition of fuzzy order relations and their types, followed by defining fuzzy equivalent relations, fuzzy classes equivalent relations and quotient set.

In the last chapter we give the intuitionistic fuzzy sets, its properties, operations on IFSs, the α-level of IFSs, the operators of IFSs, we study the relations in IFSs with a comparison between fuzzy relations and intuitionistic fuzzy relations.

Chapter 1

Fuzzy Sets

1.1 Fuzzy Sets

The notion of fuzzy sets was first introduced by Zadeh [22], and in this section we present an definition of fuzzy sets and we will give examples. See [18].

Definition 1.1. [22] Let X be a non empty set.
A fuzzy set $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ is expression by a membership function

$$\mu_A : X \to [0, 1]$$

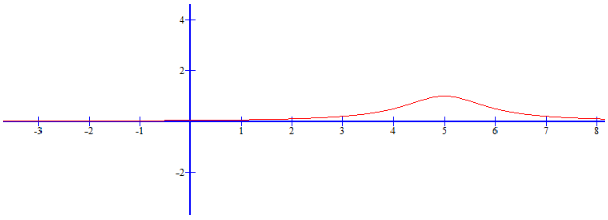
where $\mu_A(x)$ is interpreted as the degree of membership of the element x in the fuzzy subset A for each $x \in X$.

Examples

Finite case:
Let $X = \{1, 2, 3, 4, 5\}$ be universal set.
$A = \{(1, 0.2), (2, 0.6), (3, 0.15), (4, 0.1), (5, 0.2)\}$ a fuzzy subset in X.
Infinite case:

(1) Let $X = \mathbb{R}$, and A fuzzy subset in X, defined by :

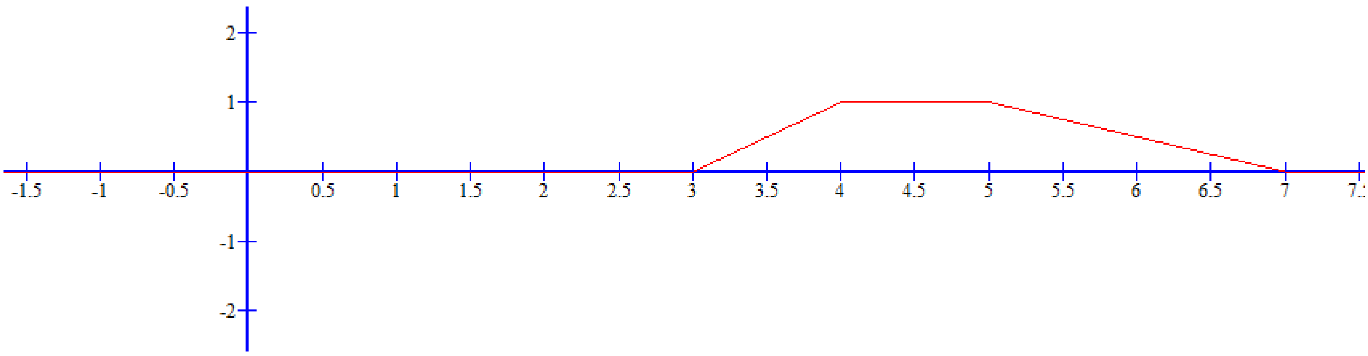
$$\mu_A(x) = \frac{1}{1 + (x - 5)^2}$$



graph of μ_A

(2) Let $X = \mathbb{R}$. We define the fuzzy set A on X by :

$$\mu_A(x) = \begin{cases} 0 & x < 3 \\ (x-3) & 3 \leq x < 4 \\ 1 & 4 \leq x < 5 \\ \left(\frac{7-x}{2}\right) & 5 \leq x < 7 \\ 0 & x \geq 7 \end{cases}$$



graph of μ_A

1.2 Operations on fuzzy sets

Next, we will show definition of operations on fuzzy sets: intersection, union,Containment, equality, complement, Multiplication of two fuzzy subsets and sum, and as well as some examples of them.

Definition 1.2. ***(Intersection).***[22] Let X be a non empty set and let A and B two fuzzy subsets, the intersection defined by for all $x \in X$

$$\mu_{A \cap B}(x) = min \{\mu_A(x), \mu_B(x)\}.$$

Example 1.1. *Let* $X = \{x_1, x_2, x_3\}$*, let* $A = \{(x_1, 0.2), (x_2, 0.8), (x_3, 0.9)\}$
and $B = \{(x_1, 0.5), (x_2, 0.25), (x_3, 0.6)\}$.
Then $A \cap B$=$\{(x_1, 0.2), (x_2, 0.25), (x_3, 0.6)\}$.

Definition 1.3. ***(Union).***[22] Let X be a non empty set and let A and B two fuzzy subsets, the union defined by for all $x \in X$

$$\mu_{A \cup B}(x) = max \{\mu_A(x), \mu_B(x)\}.$$

Example 1.2. *Let* $X = \{a, b, c\}$*, let* $A = \{(a, 0.1), (b, 0.22), (c, 0.58)\}$
and $B = \{(a, 0.12), (b, 0.33), (c, 0.14)\}$.
Then $A \cup B$=$\{(a, 0.12), (b, 0.33), (c, 0.58)\}$.

Definition 1.4. ***(Containment).***[22] Let X be a non empty set and let A and B two fuzzy subsets, we say that $A \subseteq B$, if and only if $\mu_A(x) \leq \mu_B(x)$ for all x in X.

Example 1.3. *Let* $X = \{a, b, c\}$*, let* $A = \{(a, 0.02), (b, 0.28), (c, 0.06)\}$
and $B = \{(a, 0.17), (b, 0.3), (c, 0.15)\}$.
Then $A \subseteq B$ *for all* a, b *and* c *in* X.

Definition 1.5. ***(Equality).***[22] Let X be a non empty set and let A and B two fuzzy subsets, we say that $A = B$, if and only if $\mu_A(x) = \mu_B(x)$ for all $x \in X$.

Example 1.4. *Let* $X = \{1, 2, 3\}$ *and let* $A = \{(1, 0.18), (2, 0.04), (3, 0.77)\}$
and $B = \{(1, 0.18), (2, 0.04), (3, 0.77)\}$.
Then $A = B$ *for all* $1, 2$ *and* 3 *in* X.

Definition 1.6. ***(Complement).***[22] The complement of a fuzzy set A is denoted by $C(A)$ and is defined by for all $x \in X$

$$\mu_{C(A)}(x) = 1 - \mu_A(x).$$

with the complement of a complement fuzzy set A is fuzzy set A i.e.

$$\mu_{C(C(A))}(x) = 1 - \mu_{C(A)}(x) = 1 - (1 - \mu_A(x)) = \mu_A(x).$$

Example 1.5. *Let* $X = \{1, 2, 3\}$ *and let* $A = \{(1, 0.2), (2, 0.9), (3, 1)\}$.
Then $C(A) = \{(1, 0.8), (2, 0.1), (3, 0)\}$ *and* $C(C(A)) = \{(1, 0.2), (2, 0.9), (3, 1)\} = A$.

Remarque 1.1.
$A \cap C(A) \neq \phi$ *(Law of no contradiction);*
$A \cup C(A) \neq X$ *(Law of excluded middle);*

Definition 1.7. ***(Multiplication)***.[22] Let X be a non empty set and let A and B two fuzzy subsets, the product defined by for all $x \in X$

$$\mu_{A\times B}(x) = \mu_A(x)\mu_B(x).$$

Example 1.6. *Let* $X = \{1, 2, 3\}$ *and let* $A = \{(1, 0.5), (2, 0.3), (3, 0.1)\}$
and $B = \{(1, 0.08), (2, 0.2), (3, 0.3)\}$.
Then $A \times B = \{(1, 0.04), (2, 0.06), (3, 0.03)\}$.

Definition 1.8. ***(sum)***.[22] Let X be a non empty set and let A and B two fuzzy subsets, the sum defined by for all $x \in X$

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x).$$

Example 1.7. *Let* $X = \{1, 2, 3\}$ *and let* $A = \{(1, 0.01), (2, 0.4), (3, 0.7)\}$
and $B = \{(1, 0), (2, 0.3), (3, 0.2)\}$.
Then $A + B = \{(1, 0.01), (2, 0.58), (3, 0.76)\}$.

1.2.1 Cartesian product on fuzzy set

The cartesian product of the fuzzy subsets is the minimum of these degrees of belonging.

Definition 1.9. *The cartesian product applied to n fuzzy sets can be defined as follows: Let* $\mu_{A_1}, \mu_{A_2}, ...\mu_{A_n}$*, be membership functions of* $A_1, A_2, ..., A_n$*. Then, the membership degree of* $(x_1, x_2, ...x_n) \in X_1 \times X_2 \times ... \times X_n$ *on the fuzzy set* $A_1 \times A_2 \times ... \times A_n$ *is,*

$$\mu_{A_1\times A_2\times...\times A_n}(x_1, x_2, ..., x_n) = min \{\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_n}(x_n)\}.$$

Example 1.8. *Lets* $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$ *and lets* A_1, A_2 *two fuzzy subsets respectively defined on* X *and* Y *given by:*

$A_1 = \{(x_1, 0.9), (x_2, 0.5), (x_3, 0.02)\}$;
$A_2 = \{(y_1, 0.03), (y_2, 0.8)\}$;
So, we get:

$A_1 \times A_2 = \{((x_1, y_1), 0.03), ((x_1, y_2), 0.8), ((x_2, y_1), 0.03), ((x_2, y_2), 0.5), ((x_3, y_1), 0.02), ((x_3, y_2), 0.02)\}$.

1.3 Characteristics of fuzzy sets

Now, we give definition and example to α-cuts, the strong α-cuts, support, kernel, height and cardinality. Basic properties of α-cuts and proposition of Decomposition theorem with proof.

Definition 1.10. ***(α-cuts).***[22] Let A be a fuzzy set in X and let $\alpha \in]0,1]$, The α-cut of A denoted A_α. We mean all elements of X that belongs to A to a degree of at least α. That is A_α is a classical set defined by

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}.$$

particular cases:

(1) if $\alpha = 0$, then $A_0 = X$.

(2) if $\alpha = 1$, then $A_1 = ker(A)$.

Example 1.9. *Let $X = \{a, b, c, d\}$, and $A = \{(a, 0.3), (b, 0.7), (c, 0), (d, 1)\}$;*
$A_{0.3} = \{a, b, d\}$.

Proposition 1.1. ***[Basic properties of α-cuts].*** *Let A, B are two fuzzy subsets on a universe X and $\alpha, \beta \in [0,1]$*

(1) if $\alpha \leq \beta$ then $A_\beta \subseteq A_\alpha$.

(2) $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$.

(3) $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$.

(4) $A_0 = X$.

(5) $A_1 = ker(A)$.

Proof.

(1) Let $x \in A_\beta$ i.e $\mu_A(x) \geq \beta$

$\mu_A(x) \geq \beta \Rightarrow \mu_A(x) \geq \alpha \quad \text{because} \quad \alpha \leq \beta$

Then $x \in A_\alpha$;

Finally $A_\beta \subseteq A_\alpha$.

(2)
$$\begin{aligned}(A \cap B)_\alpha &= \{x \in X : \mu_{A\cap B}(x) \geq \alpha\} \\ &= \{x \in X : min(\mu_A(x), \mu_B(x)) \geq \alpha\} \\ &= \{x \in X : \mu_A(x) \geq \alpha \wedge \mu_B(x) \geq \alpha\} \\ &= \{x \in X : \mu_A(x) \geq \alpha\} \cap \{x \in X : \mu_B(x)) \geq \alpha\} \\ &= A_\alpha \cap B_\alpha\end{aligned}$$

(3)
$$\begin{aligned}(A \cup B)_\alpha &= \{x \in X : \mu_{A\cup B}(x) \geq \alpha\} \\ &= \{x \in X : max(\mu_A(x), \mu_B(x)) \geq \alpha\} \\ &= \{x \in X : \mu_A(x) \geq \alpha \vee \mu_B(x) \geq \alpha\} \\ &= \{x \in X : \mu_A(x) \geq \alpha\} \cup \{x \in X : \mu_B(x)) \geq \alpha\} \\ &= A_\alpha \cup B_\alpha\end{aligned}$$

(4) $A_0 = \{x \in X : \mu_A(x) \geq 0\} = X$.

(5) $A_1 = \{x \in X : \mu_A(x) \geq 1\} = \{x \in X : \mu_A(x) = 1\} = ker(A)$.

□

Definition 1.11. ***(The strong α-cuts).***[22] For any α of $[0, 1]$ we defined the strong α-cuts of the fuzzy subset A as the subset

$$A_\alpha = \{x \in X \mid \mu_A(x) > \alpha\}.$$

Example 1.10. *Let* $X = \{1, 2, 3, 4\}$, *and let* $A = \{(1, 0.2), (2, 0), (3, 0.25), (4, 0.32)\}$. *Then* $A_{0.2} = \{3, 4\}$.

Definition 1.12. ***(Support).***[22] The support of a fuzzy set A, denoted by $Supp(A)$, we mean all elements of X that belongs to a nonzero degree. That is $S(A)$ is a calssical set defined by

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0\}.$$

Example 1.11. *Let* $X = \{x_1, x_2, x_3, x_4\}$*, and* $A = \{(x_1, 0.22), (x_2, 0), (x_3, 0.87), (x_4, 1)\}$;
$Supp(A) = \{x_1, x_3, x_4\}$.

Definition 1.13. ***(Kernel).***[22] The ker of a fuzzy set A, denoted by $ker(A)$, we mean all elements of X that belongs to a equal one. That is $ker(A)$ is a calssical set defined by

$$ker(A) = \{x \in X \mid \mu_A(x) = 1\} .$$

Example 1.12. *Let* $X = \{a, b, c, d\}$*, and* $A = \{(a, 0.03), (b, 0), (c, 1), (d, 1)\}$;
$ker(A) = \{c, d\}$.

Definition 1.14. ***(Height).***[22] The height of a fuzzy set A is the largest membership grade of any element in A.

$$H(A) = Max\mu_A(x).$$

Example 1.13. *Let* $X = \{a, b, c, d\}$*, and* $A = \{(a, 0.97), (b, 0.49), (c, 0), (d, 0.13)\}$;
$H(A) = 0.97$.

Definition 1.15. ***(Cardinality).***[22] Cardinality of a finite fuzzy set A, denoted $\mid A \mid$ is defined as

$$\mid A \mid = \sum_{x \in X} \mu_A(x).$$

Cardinality of a infinite fuzzy set A, denoted $\mid A \mid$ is defined as

$$\mid A \mid = \int_{x \in X} \mu_A(x).$$

Example 1.14. .

Finite case:

Let $X = \{a, b, c, d\}$*, and* $A = \{(a, 0.07), (b, 0.05), (c, 0.03), (d, 0.1)\}$

$|A| = 0.25.$

Infinite case:

Let $X = [0, 10]$*, and* A *fuzzy subset in* X*, defined by :*

$$\mu_A(x) = \frac{1}{1 + x}$$

$|A| = \int_0^{10} \frac{1}{1 + x} = [\ln(1 + x)]_0^{10} = \ln(11) - \ln(1) = \ln 11.$

1.4 T-norm and T-conorm

In the fuzzy sets theory, there are two types of operators: T-norm and T-conorm, they are often called Triangular norm and Triangular conorm respectively.

Definition 1.16. ***(Triangular norm).***[19] Triangular norm is a binary operation T on the unit interval $[0,1]$, i.e. it is a function $T : [0,1]^2 \longrightarrow [0,1]$: the following four axioms are satisfied for all x, y, z and $w \in [0,1]$:

(T1) Commutativity i.e. $T(x,y) = T(y,x)$;

(T2) Associativity i.e. $T(x, T(y,z)) = T(T(x,y), z)$;

(T3) Monotonicity i.e. $T(x,y) \leq T(z,w)$ whenever $x \leq z$ and $y \leq w$;

(T4) Boundary condition i.e. $T(x,1) = T(1,x) = x$.

Example 1.15. *Let* $T(x,y) = x.y$

$T : [0,1]^2 \longrightarrow [0,1]$

(T1) Commutativity

Let $x, y \in [0,1]$

$$\begin{aligned} T(x,y) &= x.y \\ &= y.x \\ &= T(y,x) \end{aligned}$$

.

(T2) Associativity

Let x, y *and* $z \in [0,1]$

$$\begin{aligned} T(x, T(y,z)) &= T(x, y.z) \\ &= x.y.z \\ &= T(x.y, z) \\ &= T(T(x,y), z). \end{aligned}$$

(T3) Monotonicity

Let x, y, z *and* $w \in [0,1]$

For $x \leq z$ *and* $y \leq w$ *then* $x.y \leq z.w$,

Hence $T(x,y) \leq T(z,w)$.

(T4) Boundary condition

$$\begin{aligned} &\textit{Let } x \in [0,1] \\ &T(x,1) = x.1 \\ &\qquad\quad = 1.x \\ &\qquad\quad = T(1,x) \\ &\qquad\quad = x \end{aligned}$$

.

Definition 1.17. ***(Triangular conorm).***[19] A triangular conorm is a binary operation S on the unit interval $[0,1]$, i.e. it is a function $S : [0,1]^2 \longrightarrow [0,1]$: the following four axioms are satisfied for all x, y, z and $w \in [0,1]$:

(S1) Commutativity : $S(x,y) = S(y,x)$;

(S2) Associativity : $S(x, S(y,z)) = S(S(x,y), z)$;

(S3) Monotonicity : $S(x,y) \leq S(z,w)$ whenever $x \leq z$ and $y \leq w$;

(S4) Boundary condition : $S(x,0) = S(0,x) = x$.

Example 1.16. . *Let* $S(x,y) = x + y - x.y$

$S : [0,1]^2 \longrightarrow [0,1]$

(S1) Commutativity

$$\begin{aligned} &\textit{Let } x, y \in [0,1] \\ &S(x,y) = x + y - x.y \\ &\qquad\quad = y + x - y.x \\ &\qquad\quad = S(y,x) \end{aligned}$$

.

(S2) Associativity

$$\begin{aligned} &\textit{Let } x, y \textit{ and } z \in [0,1] \\ &S(x, S(y,z)) = S(x, y + z - y.z) \\ &\qquad\qquad\quad = x + y + z - y.z - x(y + z - y.z) \\ &\qquad\qquad\quad = x + y + z - y.z - x.y - x.z + x.y.z \\ &\qquad\qquad\quad = x + y - x.y + z - x.z - y.z + x.y.z \end{aligned}$$

$$= S(S(x,y),z)$$

.

(S3) Monotonicity

Let x, y, z *and* $w \in [0,1]$

For $x \leq z$ *and* $y \leq w$ *then* $x + y - x.y \leq z + w - z.w$,

Hence $S(x,y) \leq S(z,w)$.

(S4) Boundary condition

Let $x \in [0,1]$

$$\begin{aligned} S(x,0) &= x + 0 - x.0 \\ &= 0 + x - 0.x \\ &= S(0,x) \\ &= x \end{aligned}$$

.

Proposition 1.2. *A T-norm and T-conorm are dual if and only if:*

(1) $1 - T(x,y) = S(1-x, 1-y)$*;*

(2) $1 - S(x,y) = T(1-x, 1-y)$*;*

Proof.

$$(1)\ \ 1 - T(x,y) = 1 - 1 \times \begin{cases} x & if \ \ x \leq y \\ y & if \ \ x > y \end{cases}$$

$$= 1 + \begin{cases} -x & if \ \ x \leq y \\ -y & if \ \ x > y \end{cases}$$

$$= 1 + \begin{cases} -x & if \ \ -x \geq -y \\ -y & if \ \ -x < -y \end{cases}$$

$$= \begin{cases} 1-x & if \ \ 1-x \geq 1-y \\ 1-y & if \ \ 1-x < 1-y \end{cases}$$

Then $1 - T(x,y) = max(1-x, 1-y)$;

Hense $1 - T(x,y) = S(1-x, 1-y)$.

$$(2)\ \ 1-S(x,y) = 1-1\times\begin{cases} x \ \ if \ \ x \geq y \\ y \ \ if \ \ x < y \end{cases}$$

$$= 1+\begin{cases} -x \ \ if \ \ x \geq y \\ -y \ \ if \ \ x < y \end{cases}$$

$$= 1+\begin{cases} -x \ \ if \ \ -x \leq -y \\ -y \ \ if \ \ -x > -y \end{cases}$$

$$= \begin{cases} 1-x \ \ if \ \ 1-x \leq 1-y \\ 1-y \ \ if \ \ 1-x > 1-y \end{cases}$$

Then $1-S(x,y) = min(1-x, 1-y)$;

Hense $1-S(x,y) = T(1-x, 1-y)$. □

Chapter 2

Fuzzy relations and compositions

2.1 Fuzzy relations

In the section, we introduce the definition of fuzzy relations, examples and their basic properties.

Definition 2.1. [23] Fuzzy relation from X to Y is a fuzzy subset of $X \times Y$ characterized by a membership function $\mu_R : X \times Y \longrightarrow [0,1]$, which associates with each pair (x,y) its grade of membership $\mu_R(x,y)$ in the intervale $[0,1]$.

$$R = \{< (x,y), \mu_R(x,y) > |(x,y) \in X \times Y\}.$$

Particular cases:

If $X = Y$, then $R = \{< (x,y), \mu_R(x,y) > |(x,y) \in X \times X\}$.

Examples

Finite case:

Let $X = \{a,b,c\}$ and $Y = \{x,y,z\}$, the fuzzy relation R defined on $X \times Y$ by

$$R = \{< (x,y), \mu_R(x,y) > |(x,y) \in X \times Y\}.$$

R	x	y	z
a	0.09	0.28	0.47
b	0.3	1	0.17
c	0.2	0.1	0.15

Then R is a fuzzy relation on $X \times Y$.

Infinite case:

Let $X = Y = \mathbb{R}$, we defined the fuzzy relation $R \subseteq \mathbb{R} \times \mathbb{R}$ by:

$$\mu_R(x, y) = \begin{cases} 0, & x \leq y \\ \dfrac{|x-y|}{11.|y|}, & y < x \leq 12.y \\ 1, & x > 12.y \end{cases}$$

2.1.1 Properties of fuzzy relations

(1) Reflexive

$\forall x \in X : R(x, x) = 1.$

(2) No reflexive

$\exists x \in X : R(x, x) = 0.$

(3) Antireflexive

$\forall x \in X : R(x, x) = 0.$

(4) Symetrical

$\forall (x, y) \in X \times Y : R(x, y) = R(y, x).$

(5) Asymetrical

$R(x, y) \wedge R(y, x) = 0, \forall x, y \in X$ and $x \neq y$.

(6) Antisymerical

$R(x, y) \wedge R(y, x) \neq 0 \Rightarrow x = y.$

(7) Transitive

$R(x, y) \wedge R(y, z) \leq R(x, z), \forall x, y$ and $z \in X$.

(8) $R \cap R^c \neq \phi$;

(9) $R \cup R^c \neq X$;

2.2 Operations on fuzzy relations

Below, we will define the operations on fuzzy relations with some examples.

Definition 2.2. ***(Intersection).***[14] Let X and Y be two non empty sets and let R and S be two fuzzy relations, the intersection defined by for all $(x, y) \in X \times Y$

$$\mu_{R \cap S} = min\{\mu_R(x, y), \mu_S(x, y)\}.$$

Example 2.1. *Let R and S be tow fuzzy relations on $X \times X$ such that $X = \{a, b, c, d\}$, represented by the following tables*

R	a	b	c	d
a	0.5	0.6	0.01	0.25
b	0.14	0.09	0.05	0.81
c	1	0.06	0.03	0.95
d	0.07	0.35	0.14	0.04

S	a	b	c	d
a	0.07	0.4	0.22	0.54
b	0.33	1	0.07	0.73
c	0.13	1	0.15	0.88
d	0.12	0.18	0.25	0.55

The intersection relations defined by

$R \cap S$	a	b	c	d
a	0.07	0.4	0.01	0.25
b	0.14	0.09	0.05	0.73
c	0.13	0.06	0.03	0.88
d	0.07	0.18	0.14	0.04

Definition 2.3. ***(union).***[14] Let X and Y be two non empty sets and let R and S be two fuzzy relations, the union defined by for all $(x, y) \in X \times Y$

$$\mu_{R \cup S}(x, y) = max\{\mu_R(x, y), \mu_S(x, y)\}.$$

Example 2.2. *Let R and S be tow fuzzy relations on $X \times X$ such that $X = \{a, b, c, d\}$, represented by the following tables*

R	a	b	c	d
a	0.01	0.02	0.03	0.04
b	0.2	0.6	0.3	0.5
c	0.8	0.9	0.17	0.78
d	0.08	1	0	0.97

S	a	b	c	d
a	0.08	1	0.97	1
b	0.23	0.44	0.69	0.42
c	0.6	1	0.3	0.04
d	1	0	0.5	0.01

The union relations defined by

$R \cup S$	a	b	c	d
a	0.08	1	0.97	1
b	0.23	0.6	0.69	0.5
c	0.8	1	0.3	0.78
d	1	1	0.5	0.97

Definition 2.4. ***(Containment).***[15] Let X and Y be two non empty sets and let R be a fuzzy relation, the Containment defined by for all $(x, y) \in X \times Y$

$$\mu_R(x, y) \leq \mu_S(x, y).$$

Example 2.3. *Let R and S be tow fuzzy relations on $X \times X$ such that $X = \{a, b, c, d\}$, represented by the following table*

R	a	b	c	d
a	0.01	0.03	0.06	0.25
b	0	0.09	0.02	0.22
c	0.5	0	0.03	0.05
d	0.07	0.04	0.49	0.38

S	a	b	c	d
a	0.02	0.04	0.08	0.5
b	0	1	0.05	0.3
c	0.6	0	0.2	0.18
d	0.27	0.11	0.52	0.48

Then $\mu_R(x, y) \leqslant \mu_S(x, y)$ $\forall a, b, c$ and $d \in X$;
Hense $R \subseteq S$.

Definition 2.5. ***(Complement).***[14] Let X and Y be two non empty sets and let R be a fuzzy relation, the copmlement defined by for all $(x, y) \in X \times Y$

$$\mu_{R^c}(x, y) = 1 - \mu_R(x, y).$$

Example 2.4. *Let R be a fuzzy relation on $X \times X$ such that $X = \{a, b, c, d\}$, represented by the following table*

R	a	b	c	d
a	*0.12*	*0.5*	*0.95*	*0.34*
b	*0.09*	*0.55*	*0.16*	*0.85*
c	*1*	*0.35*	*0.13*	*0.77*
d	*0.44*	*0.49*	*0.66*	*0.68*

The coplement relation defined by

R^c	a	b	c	d
a	*0.88*	*0.5*	*0.05*	*0.66*
b	*0.91*	*0.45*	*0.84*	*0.15*
c	*0*	*0.65*	*0.87*	*0.23*
d	*0.56*	*0.51*	*0.34*	*0.32*

Definition 2.6. ***(Product).***[25] Let X and Y be two non empty sets and let R and S be two fuzzy relations, the product defined by for all $(x, y) \in X \times Y$

$$\mu_{R \times S}(x, y) = \mu_R(x, y)\mu_S(x, y).$$

Example 2.5. *Let R and S be tow fuzzy relations on $X \times X$ such that $X = \{a, b, c, d\}$, represented by the following tables*

R	a	b	c	d
a	*0.1*	*0*	*0.3*	*0.8*
b	*0.5*	*0.6*	*0.5*	*0.3*
c	*0*	*0.4*	*0.2*	*0.18*
d	*0.02*	*0.1*	*0.6*	*0.5*

S	a	b	c	d
a	*0.1*	*0.23*	*0.2*	*0.2*
b	*0.5*	*0*	*0.7*	*0.9*
c	*0.5*	*0.6*	*0*	*1*
d	*0*	*0.8*	*1*	*0.7*

The product relations defined by

$R \times S$	a	b	c	d
a	*0.01*	*0*	*0.06*	*0.16*
b	*0.25*	*0*	*0.35*	*0.27*
c	*0*	*0.24*	*0*	*0.18*
d	*0*	*0.08*	*0.6*	*0.35*

Definition 2.7. ***(Inverse).***[14] Let $R \subseteq X \times Y$ be a fuzzy relation, the inverse relation R^{-1} is defined by for all $(x, y) \subseteq X \times Y$:

$$\mu_R^{-1}(x, y) = \mu_R(y, x).$$

Example 2.6. *Let R be a fuzzy relation on $X \times X$ such that $X = \{a, b, c, d\}$, represented by the following table*

R	a	b	c	d
a	0.12	0.5	0.95	0.34
b	0.09	0.55	0.16	0.85
c	1	0.35	0.13	0.77
d	0.44	0.49	0.66	0.68

The inverse relation defined by

R^{-1}	a	b	c	d
a	0.12	0.09	1	0.44
b	0.5	0.55	0.35	0.49
c	0.95	0.16	0.13	0.66
d	0.34	0.85	0.77	0.68

Proposition 2.1. *Let R, S and Q be three fuzzy relations of $X \times Y$ then:*

(1) $R \subseteq S \Rightarrow R^{-1} \subseteq S^{-1}$;

(2) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$;

(3) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$;

(4) $(R^{-1})^{-1} = R$;

(5) $R \cup (S \cap Q) = (R \cup S) \cap (R \cup Q)$;

(6) $R \cap (S \cup Q) = (R \cap S) \cup (R \cap Q)$;

2.3 α-cuts of fuzzy relation

Definition 2.8. ***(α-cuts of fuzzy relation).***[14]
Let $R \subseteq X^2$ be a fuzzy relation, and R_α is a α -cut relation. Then

$$R_\alpha = \{(x, y) \in X^2 | \mu_R(x, y) \geq \alpha\}.$$

Example 2.7. *Let $X = \{x_1, x_2, x_3\}$ and let R be a fuzzy relation represented by the following table*

R	x_1	x_2	x_3
x_1	*0.1*	*0.8*	*0.4*
x_2	*0.6*	*0.2*	*0.5*
x_3	*0.3*	*1*	*0*

$$R_{0.5} = \{(x, y) \in X^2 | \mu_R(x, y) \geq 0.5\} = \{(x_1, x_2), (x_2, x_1), (x_2, x_3), (x_3, x_2)\}.$$

2.4 Composition of fuzzy relations

In this section, we will learn about the types of copmosition in fuzzy relations. See [13].

Definition 2.9. ***(Max-min composition).***[14] Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be two fuzzy relations the max-min composition of R and S denoted $R \circ S$ is then fuzzy set such that

$$\mu_{R \circ S}(x, z) = max_y[min\{\mu_R(x, y), \mu_S(y, z)\}].$$

Example 2.8. *Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be two fuzzy relations as follows:*

R	y_1	y_2	y_3
x_1	*0.12*	*0.05*	*0.9*
x_2	*0.13*	*0.02*	*0.17*
x_3	*0.14*	*0.6*	*0.3*

and

S	z_2	z_2	z_3
y_1	*0.01*	*0.5*	*0.07*
y_2	*0.2*	*0*	*0.03*
y_3	*0.02*	*0.6*	*1*

The composition max-min is $R \circ S$

$R \circ S$	z_1	z_2	z_3
x_1	*0.05*	*0.6*	*0.9*
x_2	*0.02*	*0.17*	*0.17*
x_3	*0.2*	*0.3*	*0.3*

Definition 2.10. ***(Max-product composition).***[14] Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be two fuzzy relations the max product composition of R and S denoted $R \circ S$ is then fuzzy set such that

$$\mu_{R \circ S}(x, z) = max\{\mu_R(x, y) \times \mu_S(y, z)\}.$$

Example 2.9. *Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be two fuzzy relations as follows:*

R	y_1	y_2	y_3
x_1	*0.1*	*0.4*	*0.7*
x_2	*0.2*	*0.5*	*0.8*
x_3	*0.3*	*0.6*	*0.9*

and

S	z_1	z_2	z_3
y_1	*0*	*0.7*	*1*
y_2	*1*	*0.5*	*0.6*
y_3	*0*	*0.6*	*1*

The composition max product is $R \circ S$

$R \circ S$	z_1	z_2	z_3
x_1	*0.4*	*0.42*	*0.7*
x_2	*0.5*	*0.48*	*0.8*
x_3	*0.6*	*0.54*	*0.9*

Definition 2.11. ***(max average composition).***[11][25] Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be two fuzzy relations the max average composition of R and S denoted $R \circ S$ is then fuzzy set such that

$$\mu_{R \circ S}(x, z) = \frac{1}{2}[max\{\mu_R(x, y) + \mu_S(y, z)\}].$$

Example 2.10. *Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be two fuzzy relations as follows:*

R	y_1	y_2	y_3
x_1	*0.7*	*0.6*	*0.5*
x_2	*0.8*	*0.3*	*0.4*
x_3	*0.4*	*0.7*	*0.6*

and

S	z_1	z_2	z_3
y_1	*0.1*	*0.2*	*0*
y_2	*1*	*0.5*	*0.3*
y_3	*0.8*	*0*	*1*

The composition max-average is $R \circ S$

$R \circ S$	z_1	z_2	z_3
x_1	*0.8*	*0.55*	*0.75*
x_2	*0.65*	*0.5*	*0.7*
x_3	*0.85*	*0.6*	*0.8*

2.5 Properties of composition

(1) The max-min composition is associative:

$(R_1 \circ R_2) \circ R_3 = R_1 \circ (R_2 \circ R_3)$

(2) If $R_1 = R_2 = R_3$ we can write:

$R^2 = R \circ R$

$R^3 = R \circ R \circ R$

Also, we can defined $R^n, \forall n \in \mathbb{N}$.

Definition 2.12. ***(Closest formal relation).***[25] Let R be a fuzzy relation on X, a closest formal relation of R denoted $R^{'}$ defined by

$$\mu_{R'}(x,y) = \begin{cases} 0 \;\; if \;\; \mu_R(x,y) < 0.5 \\ 1 \;\; if \;\; \mu_R(x,y) \geq 0.5 \end{cases}$$

Definition 2.13. ***(Resemblance relation).***[25] Let R be a fuzzy relation on X, we say that R is a resemblance relation if and only if

(1) R is reflexive

$\mu_R(x,x) = 1, \forall x \in X;$

(2) R is symetrical

$\mu_R(x,y) = \mu_R(y,x), \forall (x,y) \in X^2;$

Example 2.11. *Let $X = \{a,b,c,d,e\}$, and let R be a resemblance relation represented by the following table*

R	a	b	c	d	e
a	*1*	*0.1*	*0.2*	*0*	*0.4*
b	*0.1*	*1*	*0.5*	*0.6*	*0.7*
c	*0.2*	*0.5*	*1*	*0.8*	*0.9*
d	*0*	*0.6*	*0.8*	*1*	*0.1*
e	*0.4*	*0.7*	*0.9*	*0.1*	*1*

Definition 2.14. ***(Dissemblance relation).***[25] Let R be a fuzzy relation on X, we say that R is a dissemblance relation if and only if

(1) R is antireflexive

$\mu_R(x, x) = 0, \forall x \in X;$

(2) R is symetrical

$\mu_R(x, y) = \mu_R(y, x), \forall (x, y) \in X^2;$

Example 2.12. *Let* $X = \{a, b, c, d, e\}$*, and let* R *be a dissemblance relation represented by the following table*

R	a	b	c	d	e
a	*0*	*0.1*	*0.2*	*1*	*0.4*
b	*0.1*	*0*	*0.5*	*0.6*	*0.7*
c	*0.2*	*0.5*	*0*	*0.8*	*0.9*
d	*1*	*0.6*	*0.8*	*0*	*0.1*
e	*0.4*	*0.7*	*0.9*	*0.1*	*0*

Definition 2.15. ***(Similarity relation).***[14] A similarity relation R on X is a fuzzy relation which is

(1) Reflexive

$\mu_R(x, x) = 1, \forall x \in X;$

(2) Symmetric

$\mu_R(x, y) = \mu_R(y, x), \forall (x, y) \in X^2;$

(3) Transitive(max-min)

$\forall x, y$ and $z \in X : max[min[\mu_R(x, y), \mu_R(y, z)]] \leq \mu_R(x, z);$

Example 2.13. *Let $X = \{a, b, c, d, e\}$, and let R be a similarity relation represented by the following table*

R	a	b	c	d	e
a	1	0.8	0.7	1	0.9
b	0.8	1	0.7	0.8	0.8
c	0.7	0.7	1	0.7	0.7
d	1	0.8	0.7	1	0.9
e	0.9	0.8	0.7	0.9	1

Definition 2.16. ***(Dissimilitude relation).***[14] A dissimilitude relation R on X is a fuzzy relation which is

(1) Antireflexive

$\mu_R(x, x) = 0, \forall x \in X.$

(2) Symmetric

$\mu_R(x, y) = \mu_R(y, x), \forall (x, y) \in X^2.$

(3) Transitive(max-min)

$\forall x, y$ and $z \in X : max[min[\mu_R(x, y), \mu_R(y, z)]] \leq \mu_R(x, z).$

Example 2.14. *Let $X = \{a, b, c, d, e\}$, and let R be a dissimilitude relation represented by the following table*

R	a	b	c	d	e
a	0	0.2	0.3	0	0.1
b	0.2	0	0.3	0.2	0.2
c	0.3	0.3	0	0.3	0.3
d	0	0.2	0.3	0	0.1
e	0.1	0.2	0.3	0.1	0

Definition 2.17. ***(Fuzzy Pre-order relation).***[14] Given fuzzy relation R in set X, if the followings are well kept for all x, y and $z \in X$ this relation is called pre-order(pseudo-ordre) relation.

(1) Reflexive relation

$\forall x \in X \Rightarrow \mu_R(x, x) = 1;$

(2) Transitive relation

$\forall x, y$ and $z \in X$: $max[min[\mu_R(x, y), \mu_R(y, z)]] \leq \mu_R(x, z)$.

Example 2.15. *Let $X = \{a, b, c\}$, and let R be a Fuzzy Pre-order relation represented by the following table*

R	x_1	x_2	x_3
x_1	*1*	*0.6*	*0.3*
x_2	*0.8*	*1*	*0.2*
x_3	*0.4*	*0.3*	*1*

2.6 Fuzzy order relations

Next, we recal the definition of the fuzzy order relation and we see some examples.

Definition 2.18. [14] If fuzzy relation R satisfies the followings $\forall x, y$ and $z \in X$, it is called fuzzy order relation.

(1) Reflexive relation

$\forall x \in X \Rightarrow \mu_R(x, x) = 1$;

(2) Antisymmetric relation

$\forall (x, y) \in X \times X$: $\mu_R(x, y) \neq \mu_R(y, x)$ or $\mu_R(x, y) = \mu_R(y, x) = 0$;

(3) Transitive relation

$\forall x, y$ and $z \in X$: $max[min[\mu_R(x, y), \mu_R(y, z)]] \leq \mu_R(x, z)$.

Example 2.16. *Let $X = \{a, b, c\}$, then the fuzzy relation R defined on X by*

$$R = \{< (x, y), \mu_R(x, y) > |(x, y) \in X^2\}$$

$\mu_R(x, y)$	a	b	c
a	*1*	*0.2*	*0.4*
b	*0*	*1*	*0.1*
c	*0*	*0*	*1*

Then R is a fuzzy order relation on X.

2.6.1 Types of fuzzy order relations

There are two types of fuzzy order relations, the first type is the fuzzy order partial relation (fuzzy order relation), and the second type is the fuzzy order total relation, and we will present a definition for the latter with an example of both.

Definition 2.19. *An fuzzy order R on a universe X is called **total** if for any $x, y \in X$ it holds that*

$$\mu_R(x,y) > 0 \quad or \quad \mu_R(y,x) > 0$$

Example 2.17. *Let $X = \{a, b, c\}$, then the fuzzy relations R and S defined on X by*

$$R = \{< (x,y), \mu_R(x,y) > |(x,y) \in X\}$$

$$S = \{< (x,y), \mu_S(x,y) > |(x,y) \in X\}$$

$\mu_R(x,y)$	a	b	c
a	1	0.2	0.4
b	0	1	0
c	0	0	1

$\mu_S(x,y)$	a	b	c
a	1	0.2	0.4
b	0	1	0.2
c	0	0	1

Then R is a fuzzy order partial relation on X, because $\mu_R(b,c) = 0$ and $\mu_R(c,b) = 0$. And S is a fuzzy order total relation on X.

2.7 Fuzzy equivalent relations

In the followings we shows that the definition of fuzzy equivalent relation and example.

Definition 2.20. [14] If a fuzzy relation $R \subseteq X \times X$ satisfies the following conditions, we call it a "fuzzy equivalent relation"

(1) Reflexive relation
$\forall x \in X \Rightarrow \mu_R(x,x) = 1;$

(2) Symmetric relation
$\forall (x,y) \in X \times X : \mu_R(x,y) = \mu_R(y,x);$

(3) Transitive relation

$\forall x, y$ and $z \in X$: $max[min[\mu_R(x,y), \mu_R(y,z)]] \leq \mu_R(x,z)$.

Example 2.18.

Let $X = \{x_1, x_2, x_3\}$, *then the fuzzy relation* R *defined on* X *by*

$$R = \{< (x,y), \mu_R(x,y) > |(x,y) \in X\}$$

.

$\mu_R(x,y)$	x_1	x_2	x_3
x_1	*1*	*0.2*	*0.4*
x_2	*0.2*	*1*	*0*
x_3	*0.4*	*0*	*1*

R is fuzzy equivalent relation on X.

2.7.1 Fuzzy classes equivalent relations

Definition 2.21. [12][20] Let R be a fuzzy relation on a nonempty set X and $x \in X$. Then $B(x) = \{y \in X : \mu_{X\times X}(y,x) \geq 0.5\}$ is called the set of all element which has strong bond with x.

Definition 2.22. [12][20] Let R be a fuzzy equivalence relation on a nonempty set X and $x \in X$. Then

$$\tilde{[x]} = \{(y, \mu_{\tilde{[x]}}(y)) : y \in X\}$$

Where

$$\mu_{\tilde{[x]}}(y) = \begin{cases} 1 & if \;\; y \in B(x); \\ min\{\mu_R(z,y) : z \in B(x)\} & if \;\; y \notin B(x). \end{cases}$$

is called the fuzzy equivalence class determined by x.

Example 2.19. *Let* $X = \{a, b, c\}$ *and* R *be the fuzzy equivalent relation given by*

R	a	b	c
a	*1*	*0.8*	*0.4*
b	*0.8*	*1*	*0.1*
c	*0.4*	*0.1*	*1*

$B(a) = \{a, b\}, B(b) = \{a, b\}, B(c) = \{c\}$.

The fuzzy equivalent class determined by a is

$[\tilde{a}] = \{(a, 1), (b, 1), (c, 0.1)\}$.

The fuzzy equivalent class determined by b is

$[\tilde{b}] = \{(a, 1), (b, 1), (c, 0.1)\}$.

The fuzzy equivalent class determined by c is

$[\tilde{c}] = \{(a, 0.4), (b, 0.4), (c, 1)\}$.

2.7.2 Quotient set

Definition 2.23. [12][20] The set $[\tilde{X}] = \{[\tilde{x}] : x \in X\}$ is called the set of all fuzzy equivalence classes.

Example 2.20. *Let* $X = \{a, b, c\}$ *and* R *be the fuzzy equivalent relation given by*

R	a	b	c
a	*1*	*0.8*	*0.4*
b	*0.8*	*1*	*0.1*
c	*0.4*	*0.1*	*1*

and let the fuzzy equivalent class determined by a, b *and* c *defined by*

$[\tilde{a}] = \{(a, 1), (b, 1), (c, 0.1)\}$;

$[\tilde{b}] = \{(a, 1), (b, 1), (c, 0.1)\}$;

$[\tilde{c}] = \{(a, 0.4), (b, 0.1), (c, 1)\}$;

Then $[\tilde{X}] = \{[\tilde{a}], [\tilde{b}], [\tilde{c}]\} = \{(a, 1), (a, 0.4), (b, 1), (b, 0.1), (c, 0.1), (c, 1)\}$.

Chapter 3

Generalities on intuitionistic fuzzy sets

3.1 Intuitionistic fuzzy sets

In the following, we recall the definition of an intuitionistic fuzzy sets and example.For more information see [1], [2] and [3].

Definition 3.1. [5] Let X be a nonempty set.
An intuitionistic fuzzy set (IFS, for short) A on X is an object of the form
$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$.
Where function $\mu_A : X \to [0,1]$.and $\nu_A : X \to [0,1]$.Define the degree of membership and the degree of non-membership of the element $x \in X$ to the set A, respectively, and for every $x \in X$

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1$$

Obviously, every ordinary fuzzy set has the from $\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$.

Example 3.1. *Let* $X = \{x_1, x_2, x_3\}$ *be universal set.*
$A = \{(1, 0.1, 0.3), (2, 0.5, 0.4), (3, 1, 0)\}$ *a intuitionistic fuzzy subset in* X*.*

3.2 Operations on Intuitionistic fuzzy sets

In the section, we need following defintion of intersection, union, equality,containment,complement and sum of an intuitionistic fuzzy set with some examples.

Definition 3.2. ***(Intersection).***[5] Let X be a non empty set and let A and B be two intuitionistic fuzzy subsets, the intersection defined by

$$A \cap B = \{\langle x, min\{\mu_A(x), \mu_B(x)\}, max\{\nu_A(x), \nu_B(x)\}\rangle \mid x \in X\}.$$

Example 3.2. *Let* $X = \{a, b, c\}$*.*
Let $A = \{(a, 0.3, 0.2), (b, 0.9, 0.1), (c, 0.6, 0.4)\}$ *and* $B = \{(a, 0.1, 0.8), (b, 0.24, 0.12), (c, 0.59, 0.31)\}$*.*
Then $A \cap B$=$\{(a, 0.1, 0.8), (b, 0.24, 0.12), (c, 0.59, 0.4)\}$.

Definition 3.3. ***(Union).***[5] Let X be a non empty set and let A and B be two intuitionistic fuzzy subsets, the union defined by

$$A \cup B = \{\langle x, max\{\mu_A(x), \mu_B(x)\}, min\{\nu_A(x), \nu_B(x)\}\rangle \mid x \in X\}.$$

Example 3.3. *Let* $X = \{x, y, z\}$*, and let* $A = \{(x, 0.13, 0.56), (y, 0.14, 0.22), (z, 0.15, 0.25)\}$ *and* $B = \{(x, 0.22, 0.38), (y, 0.77, 0.15), (z, 0.66, 0.34)\}$*.*
Then $A \cup B$=$\{(x, 0.22, 0.38), (y, 0.77, 0.15), (z, 0.66, 0.25)\}$.

Definition 3.4. ***(Equality).***[5] Let X be a non empty set and let A and B be two intuitionistic fuzzy subsets, we say that $A = B$, if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ for all $x \in X$.

Example 3.4. *Let* $X = \{1, 2, 3\}$*, and let* $A = \{(1, 0.19, 0.17), (2, 0.05, 0.18), (3, 0.33, 0.02)\}$ *and* $B = \{(1, 0.19, 0.17), (2, 0.05, 0.18), (3, 0.33, 0.02)\}$*.*
Then $A = B$ *for all* $1, 2$ *and* 3 *in* X*.*

Definition 3.5. ***(Containment).***[5] Let X be a non empty set and let A and B be two intuitionistic fuzzy subsets, we say that $A \subseteq B$, if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for any $x \in X$.

Example 3.5. *Let* $X = \{a, b, c\}$*, and let* $A = \{(a, 0.03, 0.17), (b, 0.27, 0.25), (c, 0.05, 0.4)\}$ *and* $B = \{(a, 0.16, 0.15), (b, 0.33, 0.22), (c, 0.18, 0.38)\}$*.*
Then $A \subseteq B$ *for all* a, b *and* c *in* X*.*

Definition 3.6. ***(Complement).***[2] The complement of an intuitionistic fuzzy set A is denoted by $C(A)$ and is defined by

$$C(A) = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}.$$

Example 3.6. *Let* $X = \{1, 2, 3\}$.
Let $A = \{(1, 0.6, 0.2), (2, 0, 1), (3, 0.2, 0.8)\}$.
Then $C(A)= \{(1, 0.2, 0.6), (2, 1, 0), (3, 0.8, 0.2)\}$.

Definition 3.7. ***(Product).***[5] Let X be a non empty set and let A and B be two intuitionistic fuzzy subsets, the product defined by

$$A \times B = \{\langle x, \mu_A(x).\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x).\nu_B(x)\rangle \mid x \in X\}.$$

Example 3.7. *Let* $X = \{1, 2, 3\}$.
Let $A = \{(1, 0.2, 0.1), (2, 0.4, 0.6), (3, 0.9, 0.1)\}$ *and* $B = \{(1, 0.5, 0.3), (2, 0.25, 0.1), (3, 0.6, 0.2)\}$.
Then $A \times B= \{(1, 0.1, 0.37), (2, 0.1, 0.64), (x_3, 0.54, 0.28)\}$.

Definition 3.8. ***(Sum).***[5] Let X be a non empty set and let A and B be two intuitionistic fuzzy subsets, the sum defined by

$$A + B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x), \nu_A(x).\nu_B(x)\rangle \mid x \in X\}.$$

Example 3.8. *Let* $X = \{1, 2, 3\}$.
Let $A = \{(1, 0.5, 0.1), (2, 0.25, 0.6), (3, 0.6, 0.1)\}$ *and* $B = \{(1, 0.2, 0.3), (2, 0.4, 0.1), (3, 0.9, 0)\}$.
Then $A + B = \{(1, 0.6, 0.03), (2, 0.55, 0.06), (3, 0.96, 0)\}$.

3.2.1 Necessity and possibility operators

This section contains the necessity and possibility operators. For more see [1].

Definition 3.9. ***(Necessity).***[5] Let A be an intuitionistic fuzzy set on X, the necessity of A denoted by $\Box A$ is defined by

$$\Box A(x) = \{(x, \mu_A(x), \mu_A^c(x)) \mid \mu_A^c(x) = 1 - \mu_A(x)\}.$$

Example 3.9. *Let* $X = \{a, b, c, d\}$, *and let* A *be intuitionistic fuzzy set given by:*
$A = \{(a, 0.12, 0.25), (b, 0.55, 0.33), (c, 0.17, 0), (d, 0.9, 0.05)\}$
Then $\Box A(x) = \{(a, 0.12, 0.88), (b, 0.55, 0.45), (c, 0.17, 0.83), (d, 0.9, 0.1)\}$.

Definition 3.10. ***(Possibility).***[5] Let A be an intuitionistic fuzzy set on X, the possibility of A denoted by $\diamond A$ is defined by

$$\diamond A(x) = \{(x, \nu_A^c(x), \nu_A(x)) \mid \nu_A^c(x) = 1 - \nu_A(x)\}.$$

Example 3.10. *Let* $X = \{x_1, x_2, x_3, x_4\}$*, and let* A *be intuitionistic fuzzy set given by:*
$A = \{(x_1, 0.5, 0.4), (x_2, 0.6, 0.2), (x_3, 0, 1), (x_4, 0.1, 0.8)\}$
Then $\diamond A(x) = \{(x_1, 0.6, 0.4), (x_2, 0.8, 0.2), (x_3, 0, 1), (x_4, 0.2, 0.8)\}$.

3.3 Characteristics of intuitionistic fuzzy sets

In the section we defined the support, the kernel and the (α, β)-cut of an intuitionistic fuzzy set with examples.

Definition 3.11. ***(Support).***[21] Let A be an intuitionistic fuzzy set on X, the support of A denoted by $Supp(A)$ is defined by

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0 \ \ or \ \ (\mu_A(x) = 0 \ \ and \ \ \nu_A(x) < 1)\}.$$

Example 3.11. *Let* $X = \{x, y, z, t\}$*, and* $A = \{(x, 0, 0.3), (y, 1, 0), (z, 0, 0.42), (t, 0, 1)\}$;
$Supp(A) = \{x, y, z\}$.

Definition 3.12. ***(Kernel).***[21] Let A be an intuitionistic fuzzy set on X, the kernel of A denoted by $Ker(A)$ is defined by

$$Ker(A) = \{x \in X \mid \mu_A(x) = 1 \ \ and \ \ \nu_A(x) = 0\}.$$

Example 3.12. *Let* $X = \{x_1, x_2, x_3\}$*, and* $A = \{(x_1, 1, 0), (x_2, 1, 0), (x_3, 0.22, 0.58)\}$*;*
$ker(A) = \{x_1, x_2\}$.

Definition 3.13. ***($(\alpha, \beta) - cut$).***[21] Let A be an intuitionistic fuzzy set on a set X. The (α, β)-cut of A is a crisp subset

$$A_{(\alpha,\beta)} = \{x \in X \mid \mu_A(x) \geqslant \alpha \ \ and \ \ \nu_A(x) \leq \beta\},$$

where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$.

Example 3.13. *Let* $X = \{x, y, z, t\}$*, and* $A = \{(x, 0.02, 0.5), (y, 0, 0.8), (z, 0.15, 0.25), (t, 0.12, 0.28)\}$*;*
$A_{(0.1,0.3)} = \{z, t\}$.

3.3.1 Cartesian product on intuitionistic fuzzy set

The cartesian product of the intuitionistic fuzzy subsets is the minimum of these degrees of belonging and the maximum of these degrees of non-belonging.

Definition 3.14. *The cartesian product applied to n intuitonistic fuzzy sets can be defined as follows:*
Let $\mu_{A_1}, \mu_{A_2}, ...\mu_{A_n}$, *be membership functions of* $A_1, A_2, ..., A_n$.*Then, the membership degree of* $(x_1, x_2, ...x_n) \in X_1 \times X_2 \times ... \times X_n$ *on the intuitionistic fuzzy set* $A_1 \times A_2 \times ... \times A_n$ *is,*

$$\mu_{A_1 \times A_2 \times ... \times A_n}(x_1, x_2, ..., x_n) = min\left\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_n}(x_n)\right\}.$$

and the non-membership degree is,

$$\nu_{A_1 \times A_2 \times ... \times A_n}(x_1, x_2, ..., x_n) = max\left\{\nu_{A_1}(x_1), \nu_{A_2}(x_2), ..., \nu_{A_n}(x_n)\right\}.$$

Example 3.14. *Lets* $X = \{a, b, c\}$, $Y = \{\alpha, \beta\}$ *and lets* A_1, A_2 *two intuitionistic fuzzy subsets respectively defined on* X *and* Y *given by:*
$A_1 = \{(a, 0.1, 0.2), (b, 0.4, 0.1), (c, 0.03, 0.54)\}$;
$A_2 = \{(\alpha, 0.02, 0.16), (\beta, 0.7, 0.3)\}$;
So, we get:
$A_1 \times A_2 = \{((a, \alpha), 0.02, 0.2), ((a, \beta), 0.1, 0.3), ((b, \alpha), 0.02, 0.16), ((b, \beta), 0.4, 0.3),$
$((c, \alpha), 0.02, 0.54), ((c, \beta), 0.03, 0.54)\}$

3.4 Intuitionistic fuzzy relations

In the following we give a definition and an example of intuitionistic fuzzy relations. More details in [24].

Definition 3.15. [8] Let X and Y be two non-empty sets.
An intuitionistic fuzzy relation from X to Y (IFR, for short) is an intuitionistic fuzzy subset of $X \times Y$, i.e. is an expression R given by

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle \mid (x, y) \in X \times Y\}.$$

Where

$$\mu_A : X \times Y \to [0,1]$$

and

$$\nu_A : X \times Y \to [0,1]$$

satisfy the condition $0 \leq \mu_R(x,y) + \nu_R(x,y) \leq 1$, for every $(x,y) \in X \times Y$.
The value $\mu_R(x,y)$ is called the degree of membership of (x,y) in R and $\nu_R(x,y)$ is called the degree of non-membership of (x,y) in R.

Example 3.15. *Let $X = \{a,b,c\}$ and $Y = \{x,y,z\}$ be two nonempty sets, and let R be an intuitionistic fuzzy relation defined by*

μ_R	x	y	z
a	*0.9*	*0.7*	*0.2*
b	*0.89*	*0.93*	*0.97*
c	*1*	*0.5*	*0.55*

ν_R	x	y	z
a	*0.1*	*0.3*	*0.8*
b	*0.11*	*0.07*	*0.03*
c	*0*	*0.5*	*0.45*

3.5 Operations on intuitionistic fuzzy relations

In the section we defined the intersection, union, containment and the complement of intuitionistic fuzzy relation plus some examples. More details in [24].

Definition 3.16. ***(Intersection).***[8] Let X and Y be two non empty sets and let R and S be two intuitionistic fuzzy relations, the intersection defined by for all $(x,y) \in X \times Y$

$$R \cap S = \{\langle (x,y), min\{\mu_R(x,y), \mu_S(x,y)\}, max\{\nu_R(x,y), \nu_S(x,y)\}\rangle\}.$$

Example 3.16. *Let R and S be tow intuitionistic fuzzy relations on $X \times X$ such that $X = \{a,b,c\}$, represented by the following tables*

μ_R	a	b	c
a	*0.1*	*0.2*	*0.3*
b	*0.27*	*0.65*	*0.38*
c	*0.81*	*0.92*	*0*

ν_R	a	b	c
a	*0.9*	*0.8*	*0.7*
b	*0.73*	*0.35*	*0.42*
c	*0.19*	*0*	*1*

μ_S	a	b	c
a	0.51	0.62	0.73
b	0.4	0.45	0.58
c	0.14	0.25	0.32

ν_S	a	b	c
a	0.49	0.38	0.27
b	0.6	0.55	0.22
c	0.41	0.73	0

The intersection relations defined by

$\mu_{R\cap S}$	a	b	c
a	0.1	0.2	0.3
b	0.27	0.45	0.38
c	0.14	0.25	0

$\nu_{R\cap S}$	a	b	c
a	0.9	0.8	0.7
b	0.73	0.55	0.42
c	0.41	0.73	1

Definition 3.17. ***(Union).***[8] Let X and Y be two non empty sets and let R and S be two intuitionistic fuzzy relations, the union defined by for all $(x, y) \in X \times Y$

$$R \cup S = \{\langle (x, y), max\{\mu_R(x, y), \mu_S(x, y)\}, min\{\nu_R(x, y), \nu_S(x, y)\}\rangle\}.$$

Example 3.17. *Let R and S be tow intuitionistic fuzzy relations on $X \times X$ such that $X = \{a, b, c\}$, represented by the following tables*

μ_R	a	b	c
a	0.12	0.22	0.4
b	0.03	0.15	0.14
c	0.9	0.7	0.8

ν_R	a	b	c
a	0.01	0.02	0.03
b	0.06	0.05	0.04
c	0.08	0.07	0.09

μ_S	a	b	c
a	0.11	0.12	0.13
b	0.16	0.05	0.86
c	0.17	0.19	0.18

ν_S	a	b	c
a	0.2	0.3	0.4
b	0.5	0.6	0.07
c	0.8	0.01	0.11

The union relations defined by

$\mu_{R\cup S}$	a	b	c
a	0.12	0.22	0.4
b	0.16	0.15	0.86
c	0.9	0.7	0.8

$\nu_{R\cup S}$	a	b	c
a	0.01	0.02	0.03
b	0.06	0.05	0.04
c	0.08	0.01	0.09

Definition 3.18. ***(Containment).***[8] Let X and Y be two non empty sets and let R and S be two intuitionistic fuzzy relations, we say that $R \subseteq S$, if and only if $\mu_R(x, y) \leq \mu_S(x, y)$ and $\nu_R(x, y) \geq \nu_S(x, y)$ for any $(x, y) \in X \times Y$.

Example 3.18. *Let R and S be tow intuitionistic fuzzy relations on $X \times X$ such that $X = \{a, b, c\}$, represented by the following tables*

μ_R	a	b	c
a	0.1	0.12	0.02
b	0.16	0.22	0.09
c	0	0.02	0.12

ν_R	a	b	c
a	0.21	0.23	0.34
b	0.15	0.6	0.72
c	0.98	0.52	0.55

μ_S	a	b	c
a	0.27	0.4	0.03
b	0.7	0.4	0.1
c	0.01	0.03	0.14

ν_S	a	b	c
a	0.1	0.03	0.27
b	0.11	0.4	0.6
c	0.18	0.37	0.26

Then $\mu_R(x, y) \leqslant \mu_S(x, y)$ and $\nu_R(x, y) \geq \nu_S(x, y)$ $\forall a, b$ and $c \in X$;
Hense $R \subseteq S$.

Definition 3.19. ***(Complement).***[8] The complement of an intuitionistic fuzzy relation R is denoted by R^c and is defined by

$$R^c = \{\langle (x, y), \nu_R(x, y), \mu_R(x, y) \rangle \mid (x, y) \in X \times Y\}.$$

Example 3.19. *Let R be an intuitionistic fuzzy relation on $X \times X$ such that $X = \{a, b, c\}$, represented by the following tables*

μ_R	a	b	c
a	0.15	0.16	0.2
b	0.4	0.3	0.5
c	0.05	0.03	0.17

ν_R	a	b	c
a	0.35	0.42	0.54
b	0.32	0.66	0.43
c	0.9	0.82	0.69

The coplement relation defined by

μ_{R^c}	a	b	c
a	0.35	0.42	0.54
b	0.32	0.66	0.43
c	0.9	0.82	0.69

ν_{R^c}	a	b	c
a	0.15	0.16	0.2
b	0.4	0.3	0.5
c	0.05	0.03	0.17

Proposition 3.1. [8] Let R, P and Q be three intuitionistic fuzzy relations of $X \times Y$ then:

(1) $R \subseteq P \Rightarrow R^{-1} \subseteq P^{-1}$;

(2) $(R \cup P)^{-1} = R^{-1} \cup P^{-1}$;

(3) $(R \cap P)^{-1} = R^{-1} \cap P^{-1}$;

(4) $(R^{-1})^{-1} = R$;

(5) $R \cup (P \cap Q) = (R \cup P) \cap (R \cup Q)$;

(6) $R \cap (P \cup Q) = (R \cap P) \cup (R \cap Q)$;

3.5.1 $(\alpha, \beta) - cut$ of intuitionistic fuzzy relation

In the following, we extend the $(\alpha, \beta) - cut$ of intuitionistic fuzzy relation.

Definition 3.20. [21] Let R be an intuitionistic fuzzy relation on a set X. The (α, β)-cut of R is a crisp subset

$$R_{(\alpha,\beta)} = \{(x, y) \in X^2 \mid \mu_R(x, y) \geqslant \alpha \ \ and \ \ \nu_R(x, y) \leq \beta\}.$$

Example 3.20. *Let $X = \{x_1, x_2, x_3\}$, and let R be a intuitionistic fuzzy relation represented by the following table*

μ_R	x_1	x_2	x_3
x_1	*0.1*	*0.8*	*0.4*
x_2	*0.6*	*0.2*	*0.5*
x_3	*0.3*	*1*	*0*

ν_R	x_1	x_2	x_3
x_1	*0.9*	*0.2*	*0.6*
x_2	*0.4*	*0.8*	*0.5*
x_3	*0.7*	*0*	*1*

$R_{(0.1,0.8)} = \{(x, y) \in X^2 \mid \mu_R(x, y) \geqslant 0.1 \ \ and \ \ \nu_R(x, y) \leq 0.8\} = \{(x_1, x_2), (x_1, x_3), (x_2, x_1), (x_2, x_2), (x_2, x_3), (x_3, x_1), (x_3, x_2)\}$.

3.6 Composition of intuitionistic fuzzy relations

In the section, we will study the composition of intuitionistic fuzzy relation.

Definition 3.21. [16] Let R and S be two intuitionistic fuzzy relations, one defines the composition " $\circ$ " by $\forall x, y \in X$

$$\mu_{R\circ S}(x,y) := max_z[min(\mu_R(x,z), \mu_S(z,y))],$$

$$\nu_{R\circ S}(x,y) := min_z[max(\nu_R(x,z), \nu_S(z,y))].$$

Example 3.21. *Let R and S be tow intuitionistic fuzzy relations on $X \times X$ such that $X = \{a, b, c\}$, represented by the following tables*

μ_R	a	b	c
a	0.12	0.22	0.4
b	0.03	0.15	0.14
c	0.9	0.7	0.8

ν_R	a	b	c
a	0.01	0.02	0.03
b	0.06	0.05	0.04
c	0.08	0.07	0.09

μ_S	a	b	c
a	0.2	0.3	0.4
b	0.5	0.6	0.07
c	0.8	0.01	0.11

ν_S	a	b	c
a	0.11	0.12	0.13
b	0.16	0.05	0.86
c	0.17	0.19	0.18

The composition $R \circ S$ is defined by

$\mu_{R\circ S}$	a	b	c
a	0.4	0.22	0.12
b	0.15	0.15	0.11
c	0.8	0.6	0.4

$\nu_{R\circ S}$	a	b	c
a	0.11	0.05	0.13
b	0.11	0.05	0.13
c	0.11	0.07	0.13

3.7 Intuitionistic fuzzy order relations

We will study now the intuitionistic fuzzy order relation.

Definition 3.22. [9][10] Let R be an intuitionistic fuzzy relation on the set X we say that R is an intuitionistic fuzzy order relation if anf only if

(1) **Reflexive**

$\forall x \in X : \mu_R(x,x) = 1$ and $\nu_R(x,x) = 0$.

(2) **Antisymetrical**

$\forall (x,y) \in X^2$ and $x \neq y$: $\mu_R(x,y) \neq \mu_R(y,x)$ and $\nu_R(x,y) \neq \nu_R(y,x)$.

(3) **transitive**

$\forall (x,y,z) \in X^3$: $max[min[\mu_R(x,y), \mu_R(y,z)]] \leq \mu_R(x,z)$ and

$min[max[\nu_R(x,y), \nu_R(y,z)]] \geq \nu_R(x,z)$.

Example 3.22. *Let $X = \{a,b,c\}$. Then the intuitionistic fuzzy relation R defined on X by*

$$R = \{\langle (x,y), \mu_R(x,y), \nu_R(x,y) \mid (x,y) \in X^2\}$$

where μ_R and ν_R given by the following tables :

μ_R	a	b	c
a	1	0	0
b	0	1	0
c	0	0	1

ν_R	a	b	c
a	0	1	0.4
b	0.3	0	0.2
c	1	1	0

is intuitionistic fuzzy ordering on X.

Definition 3.23. [24] An intuitionistic fuzzy order R on a universe X is called complete (or total) if for any $(x,y) \in X^2$ it holds that

$$[\mu_R(x,y) > 0 \;\; or \;\; (\mu_R(x,y) = 0 \;\; and \;\; \nu_R(x,y) < 1)]$$

or

$$[\mu_R(y,x) > 0 \;\; or \;\; (\mu_R(y,x) = 0 \;\; and \;\; \nu_R(y,x) < 1)].$$

3.8 Intuitionistic fuzzy equivalent relations

Next, we will study now the intuitionistic fuzzy equivalent relation.

Definition 3.24. [9][10] Let R be an intuitionistic fuzzy relation on the set X we say that R is an intuitionistic fuzzy equivalent relation if anf only if

(1) **Reflexive**

$\forall x \in X : \mu_R(x,x) = 1$ and $\nu_R(x,x) = 0$.

(2) **symetrical**

$\forall(x, y) \in X^2$: $\mu_R(x, y) = \mu_R(y, x)$ and $\nu_R(x, y) = \nu_R(y, x)$.

(3) **transitive**

$\forall(x, y, z) \in X^3$: $max[min[\mu_R(x, y), \mu_R(y, z)]] \leq \mu_R(x, z)$ and $min[max[\nu_R(x, y), \nu_R(y, z)]] \geq \nu_R(x, z)$.

Example 3.23. *Let $X = \{a, b, c\}$. Then the intuitionistic fuzzy relation R defined on X by*

$$R = \{\langle(x, y), \mu_R(x, y), \nu_R(x, y) \mid (x, y) \in X^2\}$$

where μ_R and ν_R given by the following tables :

μ_R	a	b	c
a	1	0.8	0.7
b	0.8	1	0.7
c	0.7	0.7	1

ν_R	a	b	c
a	0	0.1	0.2
b	0.1	0	0.2
c	0.2	0.2	0

is intuitionistic fuzzy equivalent on X.

3.8.1 Intuitionistic fuzzy classes equivalent relations

Definition 3.25. *Let R be an intuitionistic fuzzy relation on a nonempty set X and $x \in X$. Then $B(x) = \{y \in X : \mu_{X \times X}(y, x) \geq 0.5 \;\; and \;\; \nu_{X \times X}(y, x) < 0.5\}$ is called the set of all element which has strong bond with x.*

Definition 3.26. *Let R be an intuitionistic fuzzy equivalence relation on a nonempty set X and $x \in X$. Then*

$$\tilde{[x]} = \{(y, \mu_{\tilde{[x]}}(y), \nu_{\tilde{[x]}}(y)) : y \in X\}$$

Where

$$\mu_{\tilde{[x]}}(y) = \begin{cases} 1 & if \;\; y \in B(x); \\ min\{\mu_R(z, y) : z \in B(x)\} & if \;\; y \notin B(x). \end{cases}$$

and

$$\nu_{\tilde{[x]}}(y) = \begin{cases} 0 & if \;\; y \in B(x); \\ max\{\nu_R(z, y) : z \in B(x)\} & if \;\; y \notin B(x). \end{cases}$$

is called the intuitionistic fuzzy equivalence class determined by x.

Example 3.24. *Let $X = \{a, b, c\}$ and R be the intuitionistic fuzzy equivalent relation given by*

μ_R	a	b	c
a	1	0.8	0.4
b	0.8	1	0.1
c	0.4	0.1	1

ν_R	a	b	c
a	0	0.2	0.6
b	0.2	0	0.9
c	0.6	0.9	0

$B(a) = \{a, b\}, B(b) = \{a, b\}, B(c) = \{c\}$

The intuitionistic fuzzy equivalent class determined by a is

$\tilde{[a]} = \{(a, 1, 0), (b, 1, 0), (c, 0.1, 0.9)\}$.

The intuitionistic fuzzy equivalent class determined by b is

$\tilde{[b]} = \{(a, 1, 0), (b, 1, 0), (c, 0.1, 0.9)\}$.

The intuitionistic fuzzy equivalent class determined by c is

$\tilde{[c]} = \{(a, 0.4, 0.6), (b, 0.1, 0.9), (c, 1, 0)\}$.

3.8.2 Quotient set

Definition 3.27. *The set $\tilde{[X]} = \{\tilde{[x]} : x \in X\}$ is called the set of all intuitionistic fuzzy equivalence classes.*

Example 3.25. *Let $X = \{a, b, c\}$ and R be the intuitionistic fuzzy equivalent relation given by*

μ_R	a	b	c
a	1	0.8	0.4
b	0.8	1	0.1
c	0.4	0.1	1

ν_R	a	b	c
a	0	0.2	0.6
b	0.2	0	0.9
c	0.6	0.9	0

and let the intuitionistic fuzzy equivalent class determined by a, b and c defined by

$\tilde{[a]} = \{(a, 1, 0), (b, 1, 0), (c, 0.1, 0.9)\}$;

$\tilde{[b]} = \{(a, 1, 0), (b, 1, 0), (c, 0.1, 0.9)\}$;

$\tilde{[c]} = \{(a, 0.4, 0.6), (b, 0.1, 0.9), (c, 1, 0)\}$;

Then $\tilde{[X]} = \{\tilde{[a]}, \tilde{[b]}, \tilde{[c]}\} = \{(a, 1, 0), (a, 0.4, 0.6), (b, 1, 0), (b, 0.1, 0.9), (c, 0.1, 0.9), (c, 1, 0)\}$.

3.8.3 Comparison between fuzzy relations and intuitionistic fuzzy relations

Definition

- **Fuzzy relations:** A fuzzy relation is a generalization of a crisp binary relation where the membership degree of an element in the relation is represented by a value between 0 and 1. It allows for degrees of membership and provides a quantitative measure of the relation between elements.

- **Intuitionistic fuzzy relations:** An intuitionistic fuzzy relation is an extension of a fuzzy relation that introduces a second value, called degree of non membership, that is, it also measures the degree of non membership and provides the non-existent quantity of the relation between the elements.

Example 3.26. *Let $X = \{x_1, x_2\}$ be a finite set and R_1 , R_2 be two relations defined as:*
$R_1 = \{<((x_1, x_1), 1)>, <((x_1, x_2), 0.2)>, <((x_2, x_1), 0.3)>, <((x_2, x_2), 1)\}$
and
$R_2 = \{<((x_1, x_1), 1, 0)>, <((x_1, x_2), 0.3, 0.5)>, <((x_2, x_1), 0.4, 0.2)>, <((x_2, x_2), 1, 0)\}$
Then R_1 is a fuzzy relation, but R_2 is an intuitionistic fuzzy relation.

Representation :

- **Fuzzy relations:** Fuzzy relations are typically represented using matrices or graphs, where each entry represents the degree of membership or similarity between two elements. Fuzzy relations can also be represented using fuzzy logic operators.

- **Intuitionistic fuzzy relations:** are often represented using two matrices or tables, where each entry contains two values: the degree of membership and degree of non membership, the sum of the two values is not necessarily 1.

Example 3.27. *Let $X = \{x_1, x_2\}$ be a finite set. We represented the fuzzy relation R_1 by table*

R_1	x_1	x_2
x_1	*1*	*0.2*
x_2	*0.3*	*1*

And the intuitionistic fuzzy relation R_2 by the followings tables

μ_{R_2}	x_1	x_2
x_1	*1*	*0.3*
x_2	*0.4*	*1*

ν_{R_2}	x_1	x_2
x_1	*0*	*0.5*
x_2	*0.2*	*0*

Applications :

- **Fuzzy relations:** Fuzzy relations find applications in fields such as decision-making, pattern recognition, control systems, image processing, and expert systems. They are particularly useful in situations where precise boundaries are difficult to define.
- **Intuitionistic fuzzy relations:** Intuitionistic fuzzy relations have applications in areas like information retrieval, knowledge representation, multi-criteria decision-making, and fuzzy clustering. They are suitable for modeling situations where uncertainty and incomplete information are prevalent.

Operations :

- **Fuzzy relations:** Fuzzy relations support various operations such as composition, union, intersection, and complementation. These operations are defined based on fuzzy logic operators like min, max, and complement.
- **Intuitionistic fuzzy relations:** Intuitionistic fuzzy relations also support operations like composition, union, and intersection. However, the operations are defined differently, taking into degree of membership and degree of non membership.

Example 3.28.

- At the intersection of two fuzzy relations in Definition 2.2 we just calculated degree of membership.

$$\mu_{R\cap S} = min\{\mu_R(x,y), \mu_S(x,y)\}.$$

- At the intersection of two intuitionistic fuzzy relations in Definition 3.16 we just calculated the degree of membership and the degree of non membership.

$$R \cap S = \{\langle (x, y), min\{\mu_R(x, y), \mu_S(x, y)\}, max\{\nu_R(x, y), \nu_S(x, y)\}\rangle\}.$$

In summary, while both fuzzy relations and intuitionistic fuzzy relations deal with uncertainty, fuzzy relations focus on degrees of membership or similarity, whereas intuitionistic fuzzy relations capture degrees of membership and degrees of non membership. The choice between the two depends on the specific requirements and characteristics of the problem at hand.

Remarque 3.1. *Every fuzzy relations is intuitionistic fuzzy relations because we can give the degree of non-membership a value of zero. Not every intuitionistic fuzzy relation is a fuzzy relation because we cannot neglect the degree of non-membership, because it is necessary in defining the intuitionistic fuzzy relation.*
This means
Fuzzy relation $\Rightarrow$ *intuitionistic fuzzy relation*
Intuitionistic fuzzy relation $\nRightarrow$ *fuzzy relation*

Conclusion

In this memoire, we have given a definition of fuzzy relations with some operations on them. Then, we have give definition of intuitionistic fuzzy sets and intuitionistic fuzzy relations by studied some properties, and we concluded the study with a comparison between fuzzy relations and intuitionistic fuzzy relations.

In summary, intuitionistic fuzzy relation is a more general concept than fuzzy relation, and we think so that this result will open the door in front of many useful studies and researches about this scientific point in the future.

Bibliography

[1] K. T. Atanassov, Intuitionistic fuzzy sets. Fuzzy Sets and Systems 20, 1 (1986), 87–96.

[2] K. T. Atanassov, Intuitionistic fuzzy sets. In Intuitionistic fuzzy sets. Springer, 1999, pp. 1–137.

[3] K. T. Atanassov, Intuitionistic Fuzzy Sets, Springer-Verlag, Heidelberg/New York, (1999).

[4] K. T. Atanassov, Intuitionistic fuzzy sets, vii itkr's session, sofia deposed in central sci. Technical Library of Bulg. Acad. of Sci 1697 (1983), 84.

[5] K. T. Atanassov, On Intuitionistic Fuzzy Sets Theory, Springer, Berlin, (2012).

[6] K. Atanassov, P. Vassilev, A new intuitionistic fuzzy definiteness norm. Notes on Intuitionistic Fuzzy Sets 26, 3 (2020), 52–60.

[7] R. Biswas, On fuzzy sets and intuitionistic fuzzy sets. Notes on Intuitionistic Fuzzy Sets 3, 1 (1997).

[8] P. Burillo, H. Bustince, Intuitionistic fuzzy relations (Part I), Mathware and Soft Com- puting, 2 (1995), 5-38.

[9] P. Burillo, H. Buistince, Intuitionistic fuzzy relations. (Part II), Mathware and Soft Computing 2 (1995), 117-148.

[10] H. Bustince, P. Burillo, Structures on intuitionistic fuzzy relations, Fuzzy Sets and Systems 78 (1996), 293-303.

[11] T. Chaira, Fuzzy Set And Its Extension, Wiley, India, (2019).

[12] M. Ciric, J. Ignjatovic, S. Bogdanovic, Fuzzy equivalence relations and their equivalence classes, Fuzzy Sets and Systems 158 (2007) 1295-1313.

[13] B. De Baets, E. Kerre, Fuzzy relational compositions, Fuzzy Sets and Systems 60 (1993) 109–120.

[14] K. H. Lee, First Course on Fuzzy Theory and Applications, Advances in Intelligent and Soft Computing, Springer-Verlag Berlin Heidelberg, 27 (2004).

[15] H. R. Lin, B. Y. Cao, and Y.Z. Liao. Fuzzy Sets Theory Preliminary. Springer, 2018.

[16] M. Panigrahi, S. Nanda, Intuitionistic Fuzzy Relations Over Intuitionistic Fuzzy Sets, Kharagpur-721302, West Bengal, INDIA.

[17] E. Sanchez "Eigen Fuzzy sets and Fuzzy Relations. Journal of Mathematical Analysis and Applications" 81 (1981)399-421.

[18] W. Pedrycz, F. Gomide, An introduction to fuzzy sets, Analysis and Design, A Bradford Book, Cambridge, London, England, (1998).

[19] B. Schweizer, Associative functions and statistical tringle inequalities. Puble. Math. Debrecen, 8(1961), 169-186.

[20] M. Shakhatreh, M. Hayajneh, Fuzzy equivalence classes and fuzzy partitions, international journal of Applied Mathematics, V(21) no4, 2008, 607-613.

[21] R. R. Yager, Level sets and the representation theorem for intuitionistic fuzzy sets, Soft Computing, 14 (2010) 1-7.

[22] L. A. Zadeh, Fuzzy sets. Information and Control, 8 (1965), 338–353.

[23] L. A. Zadeh. Similarity relations and fuzzy orderings. Information sciences, 3(2):177–200, 1971.

[24] L. Zedam, A. Amroune, B, Davvaz, Szpilrajn theorem for intuitionistic fuzzy orderings, Annals of Fuzzy Mathematics and Informatics, 9 (2015) 703-718.

[25] L. Zedam, Cour de la logique flou, Universite de msila, 2019.

ملخص

في هذه المذكرة ، قدمنا تعريف للعلاقات الضبابية ومن ثم تطرقنا إلى مفهوم المجموعات الحدسية كتعميم مع بعض العمليات عليها كما أدرجنا الخصائص ، النتائج والأنواع كعلاقة الترتيب و التكافؤ . ومن ثم تطرقنا إلى تعريف المجموعات و العلاقات الحدسية من خلال دراسة بعض الخواص، المفاهيم والعمليات كتعميم للعلاقات الضبابية . واختتمنا الدراسة بالمقارنة بين العلاقتين الضبابية و الحدسية .

كلمات مفتاحية

مجموعات ضبابية، علاقات ضبابية، مجموعات حدسية، علاقات حدسية.

Abstract

In this memoire , we have given a definition of fuzzy relations then we touched on concept of intuitionistic fuzzy sets as a generalization with some operations on them we also included the characteristics, results and types as the relation of order and equivalence. Then, we gived definition of intuitionistic fuzzy sets and intuitionistic fuzzy relations by studied some properties, concepts and operations as a generalization of fuzzy relations, and we concluded the study with a comparison between the fuzzy relation and intuitionistic fuzzy relation.

Key words

Fuzzy sets, Fuzzy relations, Intuitionistic fuzzy sets, Intuitionistic fuzzy relations.

Résumé

Dans cette mémoire, nous avons donné une définition des relations floues puis nous avons abordé le concept de ensembles flous intuitionnistes comme généralisation avec quelques opérations sur celles-ci nous avons également inclus les propriétés, résultats et types comme la relation d'ordre et d'équivalence. Et puis nous avons abordé la définition des ensembles et des relations intuitionniste flou en étudiant certaines propriétés, notions et opérations comme une généralisation des relations flous. Et nous avons conclu l'étude en comparant les deux relations floues et intuitionnistes.

Mot-clés

Ensembles flou, Relations flou, Ensembles intuitionniste flou, Relations intuitionniste flou.