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Existence and multiplicity of solutions for an impulsive second-order boundary value problem with Dirichlet boundary conditions via variational methods.

Presented by:
Miss: Abir KOUINI

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In front of the jury composed of :

Abdelhak MOKHTARI	Ass. Pr (MCA),	University of M'sla	**President.**
Dahmane BOUAFIA	Ass. Pr (MCA),	University of M'sla	**Supervisor.**
Noureddine DECHOUCHA	Ass. L (MAA),	University of M'sla	**Examiner.**

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Abir Kouini ✍

DEDICATION

With all my heart I dedicate this work :

-To my parents : thanks to their encouragement and their great sacrifices they was able to create a climate conducive to the pursuit of my studies no dedication could express my respect, my gratitude and my deep consideration towards them.

-To my dear brothers and sisters"" Saber, Haroun, Wail, Loubna, Saliha, Manar, Achouak, Amira ".

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Abir Kouini

Contents

Notation

We introduce the necessary notations and definitions that are used later on.

$L^p(\Omega)$	$= \Big\{u : \Omega \to \mathbb{R}$ measurable $: \lvert u\rvert^p \in L^1(\Omega)\Big\}$ with $1 \leq p < \infty$.
$\lVert u\rVert_{L^p}$	$= \big(\int_\Omega \lvert u(t)\rvert^p dt\big)^{\frac{1}{p}}$.
$L^\infty(\Omega)$	$= \{u : \Omega \to \mathbb{R} \text{ measurable and } \exists C > 0 \ : \lvert u(t)\rvert \leq C \ a.e \ on \ \Omega\}$.
$\lVert u\rVert_{L^\infty}$	$= \inf\{C > 0 : \lvert u(t)\rvert \leq C \text{ a.e on } \Omega\}$.
$C[0,T]$	: Space of continuous functions on $[0,T]$.
H	: Hilbert space.
$\langle .,. \rangle$	: Duality bracket.
$(.,.)$	: Inner product.
$H^m(0,1)$	$= \{u \in L^2(0,1) : \lvert u^{(i)}\rvert \in L^2(0,1) \quad \forall i \leq m\}$.
$\lVert u\rVert_{H_0^m}$	$= \Big(\int_0^1 \lvert u^{(m)}(t)\rvert^2 dt\Big)^{\frac{1}{2}}$.
$L^2(0,T)$	$:= \Big\{u$ measurable $: \int\limits_0^T u^2(t)dt < +\infty\Big\}$,
$\lVert u\rVert_\infty$	$= \max\limits_{t \in [0,+T)} \lvert u(t)\rvert$.
$\mathcal{L}(X,Y)$	: Set of continuous linear applications.
$X \hookrightarrow Y$	: We write $X \hookrightarrow Y$ to mean that X is included in Y and the canonical injection of X into Y is continuous.
$X \hookrightarrow\hookrightarrow Y$	: To imply that X is included in Y and that the canonical injection of X into Y is compact.
$u_n \rightharpoonup u_0$	: u_n Converge weakly to u_0.
$i.e.$	That is to say.
$a.e.$	Almost everywhere.
dF	Derivative in the sense of Frechet which is also noted by F'.
$d_G F$	Derivative in the sense of Gateaux which sometimes is also noted by F'.
$\overline{B_M(0)}$	The closed ball with center 0 and radius M.

$(P.S)$	Palais-Smale condition.
$(P.S)_c$	Palais-Smale condition at the level c
$l.s.c$	Lower semi continuous.
$u.s.c$	Semi semi continuous.
$w.u.s.s$	Weakly upper semi continuous.
$w.l.s.c$	Weakly lower semi continuous .
$\Delta u'(t_j)$	$u'(t_j^+) - u'(t_j^-) = \lim\limits_{t\to t_i^+} u'(t) - \lim\limits_{t\to t_i^-} u'(t).$
$\gamma(A)$	Is The genus of A.
$\gamma(A)$	$:= \Big\{k \in \mathbb{N}; \exists h : A \longrightarrow \mathbb{R}^k\{0\}\ such\ that\ his\ continuous\ and\ odd\Big\},$

Introduction

IMpulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. Recent development in this field has been motivated by many applied problems, such as control theory [1, 2], population dynamics [3] and medicine [4].Significant progress has been made in the theory of systems of impulsive differential equations in recent twenty years.We generally consider impulses in the position u and u' for the second-order differential equation $u'' = f(t, u, u')$. For some general and recent works on the theory of impulsive differential equations, we refer the interested reader to [6, 7, 8]. In recent years, critical point theory and variational methods are proved to be very effective in studying the boundary value problem for second-order impulsive differential equations.The study of impulsive differential equation via variational methods was initiated by Nieto and O'Regan [9] Tian and Ge [13]The study of second order impulsive differential equation with derivative dependence ordinary differential equations via variational methods was initiated by Nieto [12]. For some general and recent works on the theory of critical point theory and variational methods we refer the readers to [14, 15]. In this memoire, we will consider the following problem:

$$\begin{cases} -u''(t) + \lambda u(t) = f(t, u(t)), \quad t \in [0, T], \\ \Delta u'(t_j) = I_j(u(t_j)), j = 1, 2, ..., p, \\ u(0) = u(T) = 0, \end{cases} \tag{1}$$

where $T > 0, f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous,$0 = t_0 < t_1 < t_2 < ..... < t_p < t_{p+1} = T$, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \lim\limits_{t \to t_j^+} u'(t) - \lim\limits_{t \to t_j^-} u'(t)$,$I_j : \mathbb{R} \to \mathbb{R}$ are continuous.

In [9], Nieto and O'Regan studied the existence of solution of nonlinear problem (1). They got the following result using the variational method.

Zhang and Yuan [10] extended the results in [9]. They obtained the existence of solutions for problem (1) with a perturbation term. Also, they obtained infinitely many solutions for problem (1) under the assumption that the nonlinearity f is a superlinear case.

Zhou and Li [11] also extended problem (1). They studied and obtained the existence and multiplicity of solutions for the following problem:

$$\begin{cases} -u''(t) + g(t)u(t) = f(t,u), \quad a.e.\, t \in [0,T], \\ \Delta u'(t_j) = I_j(u(t_j)), \quad j = 1,2,....,p, \\ u(0) = u(T) = 0. \end{cases} \tag{2}$$

In [12], Neito studied the following problem:

$$\begin{cases} -u''(t) + g(t)u'(t) + \lambda u(t) = \sigma(t), a.e. \quad t \in [0,T], \\ \Delta u'(t_j) = d_j, \quad j = 1,2,....,p, \\ u(0) = u(T) = 0. \end{cases} \tag{3}$$

He obtained problem (3) has a unique weak solution for $\lambda > \pi^2/T^2$.

Our aim is to revisit problem (1)and study the case that the impulsive functions I_j are superlinear when the nonlinearity f is a superlinear case or a sublinear case. We shall use the variational methods to obtain the existence and multiplicity of solutions for problem (1). Our main results extend the existing results in [9, 10, 11, 12]. To prove the result, we base it on a critical point theory as well as minimization theory, mountain pass theorem , point theorem saddle and symmetric mountain Pass. This memoire was divided into three chapters, as follows:

The first chapter is preliminary which contains some basic tools which are used by the following. We have divided the chapter into sections containing operators on banach spaces like continuity and the properties of semi-continuity of functionality,and critical point theory. We also gave a reminder about the L_p space,finally the conditions (PS) and Mountain pass theorem principle.

In chapter two, we presented results for the existence of solutions for problems in impulsive second-order limits. We use variational structures and some important minimization theorems (we obtained that the functional is coercive and w.l.s.c) to prove that at least one solution exists, we present an application mountain pass theorem to study the existance of two solutions,and check the status of Palais-Smale (PS) as well as the geometric conditions for appropriate functionality.

In the last chapter,and check the status of Palais-Smale (PS) as well as the geometric conditions for appropriate functionality,and we applied the saddle point theorem and symmetric mountain pass theorem ,to Prove that there are an infinite number of solutions to suitable functions.

To obtain the results, we used the following conditions and assumptions:

(H1) There exist $\mu > 2, \delta_j > 0, j = 1, 2, ..., p,$ such that

(i) $I_j(x)x \leq \mu \int_0^x I_j(s)ds < 0$, for $x \in \mathbb{R}\backslash\{0\}$;

(ii) $\int_0^x I_j(s)ds \geq -\delta_j |x|^\mu$, for $x \in \mathbb{R}\backslash\{0\}$;

Finally, we present illustrative examples of second-order boundary value problems impulsive which were studied at the end of each of the second and third chapters.

CHAPTER 1

SOME PRELIMINARIES

1.1 Impulsive differential equation

An impulsive differential equation is a differential equation that contains impulses or jumps in the drive function. More precisely, an impulsive differential equation can be written in the form:

$$-u''(t) = f(t, u(t)), \quad t \in [0, T],$$

with initial conditions $u(0) = u_0$, and impulse conditions of the form:

$$u(t^+) - u(t^-) = I(t, u(t)), \quad t \in [0, T],$$

Or $u(t^+)$ and $u(t^-)$ represent the right and left limits of the function u at time t, and $I(t, u(t))$ is a given function. This condition means that the function u undergoes a jump at time t, and the jump is determined by the function I.

1.2 Operators on Banach spaces.

Let X and Y be two normed Banach spaces.

Definition 1.1. (Linear bounded operator)[16] Let A be a linear operator such that $D(A) = X$ et $R(A) \subset Y$. We say that A is bounded if it is bounded on the unit ball $\overline{B}(0, 1)$, i.e. If the set

$$\Big\{ \|Ax\| : \quad x \in X, \ \|x\| \leq 1 \Big\}$$

is bounded.

According to this definition, if A is bounded, there exists a constant $c > 0$ such that for all x where $\|x\| \leq 1$, we have the inequality

$$\|Ax\| \leq c.$$

Definition 1.2. (Dual space)[16] The set of continuous linear functionals, defined on a normed vector space, constitutes a vector space. It is called dual of the space X and we denote X^*. Equipped X^* with the norm

$$\|f\|_{X^*} = \sup_{u \in X, \|u\| \leq 1} |\langle f, u \rangle|, \quad \forall f \in X^*$$

X provided with this norm is a Banach space and we have the inequality

$$|\langle f, u \rangle| \leq \|f\|_{X^*} \|u\|_X, \quad \forall f \in X^*, \quad \forall u \in X.$$

Definition 1.3. (Weak convergence) It is said that a sequence $(u_n) \in X$ converges weakly to u, if

$$\forall f \in X^*, \langle f, u_n \rangle \longrightarrow \langle f, u \rangle$$

and we write $u_n \rightharpoonup u$.

Proposition 1.1. [17] *Let (u_n) be a sequence of X. We have*

1. *If $u_n \longrightarrow u$, so $u_n \rightharpoonup u$.*
2. *If $u_n \rightharpoonup u$, so (u_n) is bounded and $\|u\| \leq \liminf_{n \longrightarrow \infty} \|u_n\|$.*

Proposition 1.2. [17] *When X is of finite dimension, a sequence (u_n) converges weakly if and only if it converges strongly.*

Corollary 1.1. [17] *If X is a reflexive Banach space then any bounded sequence $\{u_n\} \subset X$ with $\|u_n\| \leq M$, contains a subsequence which converges weakly to an element $u \in X$ satisfying $\|u\| \leq M$.*

Continuity of operators We will consider operators T from X into Y and we will give a definition concerning the properties of the continuity of T.
The easy notion is the following

Definition 1.4. The operator $T : X \longrightarrow Y$ is said to be continuous at u, if for any sequence $\{u_n\} \subset X$ which converges to u, $(T(u_n))_n$ Converges to $T(u)$.
T is called continuous on $\Omega \subset X$ if T is continuous at any point $u \in \Omega$.

Definition 1.5. An operator $T : X \longrightarrow Y$ is said to be compact if it is continuous and has the property, for any (u_n) bounded in X, the sequence $(T(u_n))$ admits a convergent subsequence.

Definition 1.6. Let $(X, \|.\|_1)$ and $(Y, \|.\|_2)$ two Banach spaces. An operator $T : X \to Y$ is said to be completely continuous if it is continuous, and the image of any bounded set of X is relatively compact in Y.

1.2.1 Lower semicontinuous functions

Definition 1.7. 1. The functional $f : X \longrightarrow \mathbb{R}$ is said to be lower semi-continuous $(l.s.c.)$ at the point u_0, if whenever sequence $\{u_n\} \subset X$ such that $u_n \longrightarrow u_0$, it follows

$$f(u_0) \leq \liminf_{n \longrightarrow \infty} f(u_n).$$

2. The functional $f : X \longrightarrow \mathbb{R}$ is said to be weakly lower semi-continuous $(w.l.s.c.)$ at the point u_0, if whenever sequence $\{u_n\} \subset X$ such that $u_n \rightharpoonup u_0$, it follows

$$f(u_0) \leq \liminf_{n \longrightarrow \infty} f(u_n).$$

3. The functional f is said to be upper semi-continuous $(u.s.c.)$ at the point u_0, if whenever sequence $\{u_n\} \subset X$ such that $u_n \longrightarrow u_0$, it follows

$$f(u_0) \geq \limsup_{n \longrightarrow \infty} f(u_n).$$

4. The functional f is said to be weakly upper semi-continuous $(w.u.s.c.)$ at the point u_0, if whenever sequence $\{u_n\} \subset X$ such that $u_n \rightharpoonup u_0$, it follows

$$f(u_0) \geq \limsup_{n \longrightarrow \infty} f(u_n).$$

Example 1.1. Let the functional φ be defined on a Hilbert space H as follows

$$\varphi : u \rightarrow \|u\|^2,$$

then φ is weakly lower semi-continuous $(w.l.s.c.)$.

Indeed, let (u_n), such that $u_n \rightharpoonup u$, show that

$$\|u\|^2 \leq \liminf_{n \to +\infty} \|u_n\|^2.$$

We obtained

$$\begin{aligned} \|u_n - u\|^2 &= (u_n - u, u_n - u) \\ &= \|u_n\|^2 - 2(u_n, u) + \|u\|^2 \geq 0, \end{aligned}$$

We know that $H^* = H$, and $u_n \rightharpoonup u$, so

$$\forall u \in H : (u_n, u) \to (u, u) = \|u\|^2.$$

It follows that

$$\|u\|^2 \leq \liminf_{n \to +\infty} \|u_n\|^2.$$

1.2.2 Differentiability in Banach spaces

we introduce some definitions and fundamental properties **Fréchet differentiability** Let X and Y be two Banach spaces, U an open of X.

Definition 1.8. Let $u \in U$, and $\varphi : U \to Y$. We say that φ is differentiable at the point u if there exists a continuous linear mapping $A \in \mathcal{L}(X, Y)$ such that

$$R(h) = \varphi(u+h) - \varphi(u) - A.h, \quad \text{for } h \in X, \ \text{ is small,}$$

then

$$\frac{R(h)}{\|h\|} \to 0 \quad \text{when } \|h\| \to 0 \quad \text{i.e.}$$

$$\forall \varepsilon > 0, \ \exists \delta > 0 \quad \text{such that if } \|h\|_X \leq \delta, \text{ then } \|R(h)\|_Y \leq \varepsilon \|h\|_X.$$

If such an application $A \in \mathcal{L}(X, Y)$ exists, it is necessarily unique. We note by

$$A = d\varphi(u) \text{ or } A' = \varphi'.$$

It is called differential (in the sense of Fréchet) from φ in u, or a linear application tangent to φ in u. In the absence of differentiable additional precision will mean in the following differentiable in the sense of Fréchet.

Examples 1.1. 1. If $\varphi(u) = c$. Then φ Fréchet differentiable and $d\varphi(u) = 0$, $\forall u$.

2. If $A \in \mathcal{L}(X, Y)$ $\ A(u+h) - A(u) = A.h$ and so $dA(u) = A, \ \forall u \in X$.

3. If $X = H$ is a space of Hilbert and $\varphi(u) = \|u\|^2 = \langle u, u \rangle \ then \ d\varphi(u)h = 2\langle u, h \rangle$.
   Therefore we have all the classical properties.

1.2.3 Gâteaux differentiability

Let us begin by defining the notion of directional derivative

Definition 1.9. Let $\varphi : U \to Y$, $u \in U$, $v \in X$ and $v \neq 0$.
We call the directional derivative in u of φ in the direction v, denoted $\partial_v \varphi(u)$, the limit where it exists

$$\partial_v \varphi(u) = \lim_{t \to 0} \frac{\varphi(u+tv) - \varphi(u)}{t}.$$

The notion of directional derivative is thus an extension of the notion of partial derivative. If φ is Fréchet differentiable, then for all $v \in X$ the directional derivative in the direction v is given by

$$\partial_v \varphi(u) = d\varphi(u)v.$$

Indeed $\quad \varphi(u+tv) = \varphi(u) + d\varphi(u)(tv) + R(tv).$

Then

$$\frac{\varphi(u+tv)-\varphi(u)}{t} = d\varphi(u)(v) + \frac{R(tv)}{t}, \quad \frac{R(tv)}{t} \to 0, \text{ when } t \to 0$$

Definition 1.10. We say that $\varphi : U \to Y$ is Gâteaux differentiable in u (G-differentiable in u), if there exists a continuous linear mapping A from X to Y, ($A \in \mathcal{L}(X,Y)$) such that for any $v \in X$, the directional derivative of φ in u in the direction v exists and is equal to $A(v)$, i.e.

$$\partial_v \varphi(u) = \lim_{t\to 0} \frac{\varphi(u+tv)-\varphi(u)}{t} = A(u), \quad t \to 0, \forall u \in X.$$

It is then verified that, if such an application A exists, it is unique. We notice

$$A = d_G\varphi(u).$$

Proposition 1.3. [18] *If φ is differentiable Fréchet in u, it is Gâteaux differentiable in u and*

$$d_G\varphi(u) = d\varphi(u).$$

The reciprocal is false in general, yet in finite dimension.

Example 1.2. Let $\quad \varphi : \mathbb{R}^2 \to \mathbb{R}$

$$\begin{cases} \varphi(x,y) = \Big[\dfrac{x^2 y}{x^4+y^2}\Big]^2, & if \;\; y \neq 0 \\ \varphi(x,0) = 0 \end{cases}$$

G-differentiability does not imply the continuity of φ ! (while of course differentiability in the sense of Fréchet implies).

On the other hand, if we know that φ is C^1 in the sense of Gâteaux, then φ is Fréchet differentiable on U and the two notions coincide and we have

Theorem 1.1. ([19], [18]) *If $\varphi : U \to Y$ is Gâteaux differentiable in an open neighborhood $U \subset X$, of u and $d_G\varphi : U \to X^*$ is continuous at u, then φ is Fréchet differentiable at u and the two derivatives at u coincide, i.e.,*

$$\varphi'(v) = d_G\varphi(v), \quad \forall v \in U.$$

Definition 1.11. Let $u_0 \in X$. We say that u_0 is a local minimum for φ if there exists $\delta > 0$ such that

$$\varphi(u_0) \leq \varphi(v), \quad \forall v \in B(u_0, \delta), \quad v \neq u_0.$$

Definition 1.12. Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$. We say that φ is coercive if

$$\varphi(u) \to +\infty, \text{ when } \|u\| \to +\infty; \quad \text{i.e.}$$

$$\forall A > 0, \ \exists B > 0, \text{ such that } \|u\| \geq B \text{ leads } \varphi(u) \geq A.$$

1.2.4 Lebesgue and Sobolev spaces

Let Ω an open of $\mathbb{R}$, and $(\Omega, \mathcal{M}, dt)$ denote a measure space, where $\mathcal{M}$ is a $\sigma-$ algebra in Ω.

We assume that $f : \Omega \to \mathbb{R}$ is a measurable function and integrable function. We shall often write $\int f$ instead of $\int_\Omega f$, and we shall also use the notation

$$\|f\|_{L^1(\Omega)} = \int_\Omega |f(t)|dt, \text{ or } \|f\|_{L^1} = \int |f|.$$

Theorem 1.2. ***(Lebesgue's Dominated Convergence Theorem)*[17]** *Let (f_n) be a sequence of functions in L^1 that satisfy*

(a) $(f_n) \to f$ *a.e. on* Ω.

(b) *There is a function $g \in L^1(\Omega)$ such that for all n, $|f_n(t)| \leq g(t)$, a.e. on Ω.*

Then

$$f \in L^1(\Omega) \quad \textit{and} \quad \|f_n - f\|_{L^1} \to 0, \ n \to \infty.$$

Definition 1.13. Let $p \in \mathbb{R}$ with $1 \leq p < \infty$, we put

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R}, \quad \text{measurable and } |f|^p \in L^1(\Omega)\}.$$

We notice $\|f\|_{L^p} = \big(\int_\Omega |f(t)|^p dt\big)^{\frac{1}{p}}$, and $\|.\|_{L^p}$ is a norm.

Definition 1.14. We put

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{R}, \quad \text{measurable and } \exists C : |f(t)| \leq C, \text{ a.e on } \Omega\}.$$

We notice

$$\|f\|_{L^\infty} = \inf\{C : |f(t)| \leq C, \quad \text{a.e on } \Omega\}$$

Theorem 1.3. **[17]** *The space* L^P *is reflexive for* $1 < p < \infty$, *and its dual is* L^q *such that* $\frac{1}{p} + \frac{1}{q} = 1$

Theorem 1.4. ***(Cauchy-Schwarz inequality)*****[17]** *Let* f *and* $g \in L^2(\Omega)$. *Then we have*

$$\int_\Omega |fg| \leq \Big(\int_\Omega |f|^2\Big)^{\frac{1}{2}}\Big(\int_\Omega |g|^2\Big)^{\frac{1}{2}}.$$

Theorem 1.5. ***(Hölder's inequality)*****[17]** *Let* $1 \leq p \leq +\infty$; *we denote by* q *the conjugate exponent,* $\frac{1}{p} + \frac{1}{q} = 1$. *Assume that* $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$ *with* $1 \leq p \leq +\infty$. *Then*

(i) $fg \in L^1(\Omega)$ *and*

(ii) $\int_\Omega |fg| \leq \Big(\int_\Omega |f|^p\Big)^{\frac{1}{p}}\Big(\int_\Omega |g|^q\Big)^{\frac{1}{q}} = \|f\|_p\|g\|_q.$

Theorem 1.6. ***(Rellich's theorem)*****[17]** *If* Ω *is a regular bounded open of class* C^1, *then of any bounded sequence of* $H^1(\Omega)$ *we can extract a convergent subsequence in* $L^2(\Omega)$ *we say that the canonical injection of* $H^1(\Omega)$ *in* $L^2(\Omega)$ *is compact.*

Lemma 1.1. ***(Fatou's lemma)*****[17]** *Let* (f_n) *be a sequence of functions in* L^1 *that satisfy*

1. *for all* n, $f_n \geq 0$ *a.e. on* Ω
2. $\sup_n \int f_n < +\infty.$

For almost all $t \in \Omega$ *we set* $f(t) = \liminf_{n\to\infty} f_n(t) \leq +\infty$. *Then* $f \in L^1$ *and*

$$\int f \leq \liminf_{n\to\infty} \int f_n.$$

Lemma 1.2. **[17]** *Let* $g \in L^1_{loc}(\Omega)$, *for* y_0 *fixed in* Ω , *set*

$$v(t) = \int_{y_0}^t g(t)dt, \quad t \in \Omega.$$

Then $v \in C(\Omega)$ *and*

$$\int_\Omega v\varphi' = -\int_\Omega g\varphi, \quad \forall\varphi \in C_c^1(\Omega).$$

Lemma 1.3. **[17]** *Let* $f \in L^1_{loc}(\Omega)$ *be such that*

$$\int_\Omega f\varphi' = 0, \quad \forall\varphi \in C_c^1(\Omega).$$

Then there exists a constant C *such that* $f = C, a.e.$ *on* Ω.

1.3 Critical point theory

Definition 1.15. Let φ be a differentiable functional from X to $\mathbb{R}$. A point $u \in X$ is said to be critical for φ if and only if

$$d\varphi(u) = 0.$$

Definition 1.16. Let the functional φ, of class C^1 be defined on X. A critical value of φ, is a number $c \in \mathbb{R}$, such that there exists $u \in X$, with

$$\varphi(u) = c, \quad d\varphi(u) = 0.$$

If we are able to determine a critical value, we have proved the existence of a critical point, in some situations it is useful to introduce the following definitions and theorems.

Definition 1.17. (Minimizing sequence) A minimizing sequence of a functional $\varphi : X \longrightarrow]-\infty, +\infty]$ is a sequence (u_n) such that

$$\lim_{n \longrightarrow +\infty} \varphi(u_n) = \inf_{u \in X} \varphi(u)$$

Theorem 1.7. ([16], [14]) *Let φ be a functional defined on a reflexive Banach space X. Assume*

(i) *φ is coercive, i.e., $\varphi(u) \to +\infty$ for $\|u\| \to +\infty$,*

(ii) *φ is weakly lower semi-continuous.*

Then φ is bounded from below on X, and achieved its infimum at some point u_0. If, moreover, φ is Gâteaux-differentiable at u_0, then $\varphi'(u_0) = 0$.

Theorem 1.8. *(**Minimization theorem**)[15, Theorem 38] For the functional $\varphi : M \subseteq X \to [-\infty, +\infty]$ with $M \neq \emptyset$, $\min_{u \in M} \varphi(u) = \alpha$ has a solution in case the following conditions hold :*

(i) X is a real reflexive Banach space,

(ii) M is bounded and weak sequentially closed,

(iii) φ is weak sequentially lower semi-continuous on M.

Theorem 1.9 (the Mazur theorem). *[14] If $\{u_n\}$ is a sequence in a normalized space X such that $u_n \rightharpoonup u$, there exists a sequence of convex combinations:*

$$v_n = \sum_{i=1}^{n} \alpha_{n_i} u_i, \sum_{i=i}^{n} \alpha_{n_i} = 1, \quad \alpha_{n_i} \geq 0 \quad (n \in \mathbb{N}^*),$$

such that $v_n \longrightarrow u$ in X.

Palais-Smale's sequence and condition

Definition 1.18. [20] Let X be a Banach space and $\varphi : X \to \mathbb{R}$ a C^1-functional. We say that φ satisfies the Palais-Smale condition, denoted $(P.S)$, if any sequence (u_n) in X such that

$$(\varphi(u_n)) \text{ is bounded, and } \varphi'(u_n) \to 0, \text{as } n \to +\infty \tag{1.1}$$

admits a convergent subsequence.
Any sequence satisfying (1.1) is called a Palais-Smale sequence.

Definition 1.19. [20] Let X be a Banach space and $\varphi : X \to \mathbb{R}$ a C^1-functional, and $c \in \mathbb{R}$. The functional φ is said to satisfy the (local) Palais-Smale condition at the level c, denoted by $(P.S)_c$, if any sequence (u_n) in X such that

$$\varphi(u_n) \to c, \text{ and } \varphi'(u_n) \to 0, \text{as } n \to +\infty \tag{1.2}$$

admits a convergent subsequence.

Examples 1.2. 1. The identity functional on $X = \mathbb{R}$ satisfies $(P.S)$, while the critical set $K = \Phi$ where K the set of all critical points of φ in X,

2. The functional $\varphi \equiv 0$ on $X = \mathbb{R}$ satisfies neither $(P.S)_0$ nor $(P.S)$, and the set K is the whole space.

3. The functional $\varphi(u) = \sin(u)$ on $X = \mathbb{R}$ satisfies $(P.S)_c$ for all $c \in \mathbb{R} \setminus \{-1, 1\}$ and K is an infinite unbounded set.

Theorem 1.10. ([18], [21]) *Let X be a reflexive Banach space, and φ a functional defined on X such that*

1. *φ is $(w.l.s.c.)$,*

2. *The minimizing sequence of φ is bounded on X,*

Then φ achieves its infimum on X.

Theorem 1.11. [20] *Let X be a Banach space and $\varphi : X \to \mathbb{R}$ a C^1- functional that is bounded from below, and let $c = \inf_X \varphi$. Assume that φ satisfies $(P.S)_c$. Then, there exists $u \in X$ such that*

$$\varphi(u) = \inf_{v \in X} \varphi(v), \quad \textit{and} \quad \varphi'(u) = 0. \tag{1.3}$$

Theorem 1.12. ([14], [20]) *Let X be a Banach space and $\varphi : X \to \mathbb{R}$ a C^1-functional that is bounded from below and satisfies $(P.S)$. Then, there exists a $u_0 \in X$ such that*

$$\varphi(u_0) = \inf_{v \in X} \varphi(v), \text{ and } \varphi'(u_0) = 0.$$

Theorem 1.13 ([14, Theorem 4.10]). *(**Mountain Pass Theorem**) Let $\varphi \in C^1(X, \mathbb{R})$ satisfy the Palais-Smale condition. Assume that there exists $u_0, u_1 \in X$, and a bounded neighborhood Ω of u_0 satisfying $u_1 \notin \overline{\Omega}$ and*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf_{v \in \partial\Omega} \varphi(v).$$

Then there exists a critical value of φ; that is, there exists $u \in X$ such that $\varphi'(u) = 0$ and $\varphi(u) > \max\{\varphi(u_0), \varphi(u_1)\}$.

Theorem 1.14. *[18](**Saddle point theorem**)*

Let E be an infinite dimensional real Banach space . Let $\varphi \in C^1(E, \mathbb{R})$ be an even functional which satisfies the Palais-Smale condition, and $\varphi(0) = 0$. Suppose that $E = V \oplus X$, where V is of infinite dimensional, and φ satisfies that

(i) There exist $\alpha > 0$ and $\rho > 0$ such that $\varphi(u) \geq \alpha$ for all $u \in X$ with $\|u\| = \rho$.

(ii) For any finite dimensional subspace $W \subset E$ there is $R = R(W)$ such that $\varphi(u) \leq 0$ on $W \setminus B_R(W)$.

Then φ has an unbounded sequence of critical values.

Symmetric Mountain Pass

Definition 1.20 ([18]). Let E be a Banach space and A a subset of E. A is said to be symmetric if $u \in A$ implies $-u \in A$. For a closed symmetric set A which does not contain the origin, we define a genus $\gamma(A)$ of A by the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$. If there does not exist such a k, we define $\gamma(A) = \infty$. Moreover, we set $\gamma(\emptyset) = 0$. Let Γ_k denote the family of closed symmetric subsets A of E such that $0 \notin A$ and $\gamma(A) \geq k$. Here
$\gamma(A) = \Big\{k \in \mathbb{N}; \exists h : A \longrightarrow \mathbb{R}^k\{0\} \text{ such that his continuous and odd}\Big\}$ is the genus of A.

Lemma 1.4 ([18]). *(Symmetric Mountain Pass Theorem) Let E be an infinite dimensional real Banach space and $\varphi \in C^1(E, \mathbb{R})$ be even, satisfying the Palais-Smale condition and $\varphi(0) = 0$. If $E = V \oplus X$*

where V is finite dimensional, and φ satisfies

(A1) *there exist constants $\rho, \alpha > 0$ such that $\varphi\mid_{\partial B_\rho \bigcap X} \geq \alpha$;*

(A2) *for each finite dimensional subspace $V_1 \subset E$, the set $\{x \in V_1 : \varphi(x) \geq 0\}$ is bounded, then φ has an unbounded sequence of critical values.*

CHAPTER 2

STUDY THE EXISTENCE OF AT MOST TWO SOLUTIONS TO AN IMPULSIVE SECOND-ORDER BOUNDARY VALUE PROBLEM

In this chapter, we discuss the existence of at most two solutions for an impulsive second-order equation. To obtain our results, we use the variational method. We used as an example the article [27] which was published by author **Dan Zhang** in **2013**.

2.1 Introduction and a basic hypothesis

We study the following impulsive problem:

$$\begin{cases} -u''(t) + \lambda u(t) = f(t, u(t)), \quad t \in [0, T], \\ \Delta u'(t_j) = I_j(u(t_j)), \;\; j = 1, 2, ..., p, \\ u(0) = u(T) = 0, \end{cases} \tag{2.1}$$

where $T > 0, f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous,$0 = t_0 < t_1 < t_2 < ..... < t_p < t_{p+1} = T$, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \lim\limits_{t \to t_j^+} u'(t) - \lim\limits_{t \to t_j^-} u'(t)$, $I_j : \mathbb{R} \to \mathbb{R}$ are continuous.

We make the following assumption:

(H1) There exist $\mu > 2, \delta_j > 0, j = 1, 2, ..., p$, such that

(i) $I_j(x)x \leq \mu \int_0^x I_j(s)ds < 0$, for $x \in \mathbb{R} \backslash \{0\}$;

(ii) $\int_0^x I_j(s)ds \geq -\delta_j |x|^\mu$, for $x \in \mathbb{R} \backslash \{0\}$;

We always assume that $\lambda > \lambda_1$.

2.2 Variational structure

In the following, we first introduce some notations and some necessary definitions.

Suitable spaces

We now present the Hilbert space $H_0^1(0,T)$ which is suitable for the study of our problem. Let the Hilbert space H^1 be such that

$$H^1([0,T]) = \left\{ u \in L^2([0,T]) : u' \in L^2([0,T]) \right\},$$

let the space $H^1([0,T])$ be defined as

$$H := H_0^1([0,T]) = \{u : [0,T] \to \mathbb{R} | u \text{ is absolutely continuous}, u' \in L^2([0,T]) \ \ and \ \ u(0) = 0\}.$$

and the Hilbert space $H^2([0,T])$ which defines as

$$H^2([0,T]) = \left\{ u \in L^2([0,T]) : u', u'' \in L^2([0,T]) \right\},$$

In the Sobolev space $H_0^1(0,T)$,consider the inner product

$$\langle u, v \rangle = \int_0^T u'(t)v'(t)\ dt$$

which induces the norm

$$\|u\| = \Big(\int_0^T (u'(t))^2\ dt \Big)^{\frac{1}{2}}.$$

The inner product

$$\langle u, v \rangle_{H_0^1([0,T])} = \int_0^T u'(t)v'(t)\ dt + \int_0^T u(t)v(t)\ dt,$$

induces the equivalent norm

$$\|u\|_{H_0^1([0,T])} = \Big(\int_0^T (u'(t))^2\ dt + \int_0^T (u(t))^2\ dt \Big)^{\frac{1}{2}}.$$

Be the space of defined functions and continue on $[0,T]$, :

$$C[0,T] = \{u : [0,T] \to \mathbb{R}, \text{ such that u continues } \},$$

provided with its usual standard norm

$$\|u\|_\infty = \sup_{t \in [0,T]} |u(t)|.$$

We now consider the Lebesgue spaces $L^2(0,T)$which is defined by:

$$L^2(0,T) = \Big\{u \text{ measurable} : \int\limits_0^T u(t)^2 dt < +\infty\Big\},$$

with the associated standard:

$$\|u\|_{L^2} = \left(\int\limits_0^T |u(t)|^2 dt\right)^{\frac{1}{2}}.$$

The eigenvalues of linear boundary problem

We first recall some basic results consisting of the eigenvalue of the linear problem which defined as follows:

$$\begin{cases} -u''(t) = -\lambda u(t), & \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{2.2}$$

Proposition 2.1. *The eigenvalue λ for the problem (2.2) is defined as:*

$$\lambda_k = \Big(\frac{k\pi}{T}\Big)^2, \quad k = 0, 1, ...,$$

Proof. Indeed: We solve the characteristic equation:

$$-r^2 = -\lambda \Longrightarrow r^2 = \lambda,$$

We distinguish two cases:

If $\lambda > 0$. Then we get $r = \mp\sqrt{\lambda}$, we obtained the solution of this problem as:

$$u(t) = C_1 e^{(t\sqrt{\lambda})} + C_2 e^{(t\sqrt{\lambda})}, \quad C_1, C_2 \in \mathbb{R},$$

We apply the conditions to the limits we find:

$$\begin{cases} u(0) = 0 \\ u(T) = 0 \end{cases} \Longrightarrow \begin{cases} C_1 + C_2 = 0 \\ C_1 e^{(T\sqrt{\lambda})} + C_2 e^{(T\sqrt{\lambda})} \end{cases} \Longrightarrow \begin{cases} C_1 = -C_2 \\ C_1 e^{(T\sqrt{\lambda})} - C_1 e^{(T\sqrt{\lambda})} = 0 \end{cases}$$

$$\begin{aligned} &\Longrightarrow C_1 \frac{1}{2}(e^{T\sqrt{\lambda}} - e^{-T\sqrt{\lambda}}) = 0 \\ &\Longrightarrow C_1 \sinh(T\sqrt{\lambda}) = 0 \\ &\Longrightarrow (T\sqrt{\lambda}) = 0 \\ &\Longrightarrow \lambda = 0 \end{aligned}$$

impossible.

If $\lambda < 0$.

Then, we solve the characteristic equation as:

$$-r^2 = \lambda \Longrightarrow r = \mp i\sqrt{\lambda},$$

we obtained the solution of this problem as:

$$u(t) = C_1 \cos(t\sqrt{\lambda}) + C_2 \sin(t\sqrt{\lambda}), \quad C_1, C_2 \in \mathbb{R},$$

we apply the conditions to the limits we find:

$$\begin{cases} u(0) = 0 \\ u(T) = 0 \end{cases} \Longrightarrow \begin{cases} C_1 = 0 \\ u(T) = 0 \end{cases}$$

we obtained the solution of this problem as:

$$u(t) = C_2 \sin(t\sqrt{\lambda}).$$

We apply the conditions to the limits, we have,

$$\begin{aligned} u(T) = 0 &\Longrightarrow C_2 \sin(T\sqrt{\lambda}) = 0 \\ &\Longrightarrow T\sqrt{\lambda} = k\pi \\ &\Longrightarrow \lambda_k = \left(\frac{k\pi}{T}\right)^2, \quad k = 1, 2, ..., \end{aligned}$$

then the first eigenvalue is $\lambda_1 = \dfrac{\pi^2}{T^2}$. □

Remark 2.1. Many research focuses on finding the first eigenvalue, Because of its importance, and for more information, for example, see the references, [23], [24].

2.3 Variational construction of our problem

Remark 2.2. We show that, for all $u, v \in H_0^1(0,T)$ and for all $t \in [0,T]$:

$$-\int_0^T u''(t)v(t)\,dt = \int_0^T u'(t)v'(t)\,dt + \sum_{j=1}^{p} I_j\Big(u(t_j)\Big)v(t_j),$$

for everything $u, v \in H_0^1(0,T)$. So, we have:

$$\begin{aligned} -\int_0^T u''(t)v(t)\,dt &= -\sum_{j=0}^{p} \int_{t_j}^{t_{j+1}} u''(t)v(t)\,dt \\ &= -\int_0^{t_1^-} u''(t)v(t)\,dt - \int_{t_1^+}^{t_2^-} u''(t)v(t)\,dt - ... - \int_{t_{p-1}^+}^{t_p^-} u''(t)v(t)\,dt - \int_{t_p^+}^{T} u''(t)v(t)\,dt. \end{aligned}$$

Using integration by parts, we have that:

$$\begin{aligned} -\int_0^{t_1^-} u''(t)v(t)\,dt &= -\Big[u'(t)v(t)\Big]_0^{t_1^-} + \int_0^{t_1^-} u'(t)v'(t)\,dt \\ &= -u'(t_1^-)v(t_1^-) + u'(0)v(0) + \int_0^{t_1^-} u'(t)v'(t)\,dt, \end{aligned}$$

using the boundary conditions in (2.1), we find

$$-\int_0^{t_1^-} u''(t)v(t)\,dt = -u'(t_1^-)v(t_1^-) + \int_0^{t_1^-} u'(t)v'(t)\,dt.$$

Also:

$$\begin{aligned} -\int_{t_1^+}^{t_2^-} u''(t)v(t)\,dt &= -\Big[u'(t)v(t)\Big]_{t_1^+}^{t_2^-} + \int_{t_1^+}^{t_2^-} u'(t)v'(t)\,dt \\ &= -u'(t_2^-)v(t_2^-) + u'(t_1^+)v(t_1^+) + \int_{t_1^+}^{t_2^-} u'(t)v'(t)\,dt. \end{aligned}$$

And so on,

$$\begin{aligned} -\int_{t_{p-1}^+}^{t_p^-} u''(t)v(t)\,dt &= -\Big[u'(t)v(t)\Big]_{t_{p-1}^+}^{t_p^-} + \int_{t_{p-1}^+}^{t_p^-} u'(t)v'(t)\,dt \\ &= -u'(t_p^-)v(t_p^-) + u'(t_{p-1}^+)v(t_{p-1}^+) + \int_{t_{p-1}^+}^{t_p^-} u'(t)v'(t)\,dt, \end{aligned}$$

And like,

$$\Delta u'(t_1)v(t_1) = \big(u'(t_1^+) - u'(t_1^-)\big)\,v(t_1) ... \Delta u'(t_p)v(t_p) = \big(u'(t_p^+) - u'(t_p^-)\big)\,v(t_p).$$

So by definition of impulsive and the boundary conditions in (2.1), we have

$$\begin{aligned} \Delta u'(t_1)v(t_1) + \Delta u'(t_2)v(t_2) + ... + \Delta u'(t_p)v(t_p) &= \sum_{j=1}^{p} \Delta u'(t_j)v(t_j) \\ &= \sum_{j=1}^{p} I_j\Big(u(t_j)\Big)v(t_j), \end{aligned}$$

Moreover,combining $u(0) - u(T) = 0$, one has

$$\begin{aligned} \int_0^T u''(t)v(t)\,dt &= \sum_{j=0}^{p} u'(t)v(t)\big|_{t_j^+}^{t_{j+1}^-} - \int_0^T u'(t)v'(t)\,dt \\ &= -\sum_{j=1}^{p} \Delta u'(t_j)v(t_j) - u'(0)v(0) + u'(T)v(T) - \int_0^T u'(t)v'(t)\,dt \\ &= -\sum_{j=1}^{p} \Delta u'(t_j)v(t_j) - \int_0^T u'(t)v'(t)\,dt \\ &= -\sum_{j=1}^{p} I_j\Big(u(t_j)\Big)v(t_j) - \int_0^T u'(t)v'(t)\,dt \end{aligned}$$

therefore, it follows that

$$-\int_0^T u''(t)v(t)\ dt = \int_0^T u'(t)v'(t)\ dt - \sum_{j=1}^{p} I_j\Big(u(t_j)\Big)v(t_j).$$

2.3.1 The Euler-Lagrange functional

We now define the Euler-Lagrange function associated with the problem (2.1).

Then φ defined on $H_0^1([0,T])$, by

$$\varphi(u) = \frac{1}{2}\int_0^T (u'(t))^2 dt + \frac{\lambda}{2}\int_0^T (u(t))^2 dt - \int_0^T F(t,u)\ dt + \sum_{j=1}^{p}\int_0^{u(t_j)} I_j(s)ds, \tag{2.3}$$

such that $F(t,u) = \int\limits_0^u f(t,s)ds.$

Proposition 2.2. *The functional φ is continuously differentiable and the Frechet derivative of φ is written in the form*

$$\langle d\varphi(u), v\rangle = \int_0^T u'(t)v'(t)\ dt + \lambda\int_0^T u(t)v(t)\ dt - \int_0^T f(t,u(t))v(t)dt + \sum_{j=1}^{p} I_j\Big(u(t_j)\Big)v(t_j), \tag{2.4}$$

for everything $v \in H_0^1([0,T])$.

Proof. Step 1: Let us show that φ is Gâteaux differentiable. For everything $v \in H_0^1([0,T])$ and $0 < \mu < T$, we have

$$\begin{aligned}
\varphi(u+\mu v) - \varphi(u) &= \frac{1}{2}\int_0^T \Big((u+\mu v)'\Big)^2\ dt + \frac{\lambda}{2}\int_0^T (u+\mu v)^2\ dt - \int_0^T F(u+\mu v)\ dt + \sum_{j=1}^{p}\int_0^{u+\mu v} I_j(s)\ ds \\
&\quad -\frac{1}{2}\int_0^T (u)^2\ dt - \frac{\lambda}{2}\int_0^T (u)^2\ dt + \int_0^T F(u)\ dt - \sum_{j=1}^{p}\int_0^{u} I_j(s)\ ds \\
&= \frac{1}{2}\int_0^T (u')^2\ dt + \frac{1}{2}\mu^2\int_0^T (v')^2\ dt + \mu\int_0^T u'v'\ dt + \frac{\lambda}{2}\int_0^T u^2\ dt + \frac{\lambda}{2}\mu^2\int_0^T v^2\ dt \\
&\quad +\lambda\mu\int_0^T uv\ dt - \int_0^T F(u+\mu v)dt + \sum_{j=1}^{p}\int_0^{u+\mu v} I_j(s)ds \\
&\quad -\frac{1}{2}\int_0^T (u')^2\ dt - \frac{\lambda}{2}\int_0^T u^2\ dt + \int_0^T F(u)dt - \sum_{j=1}^{p}\int_0^{u} I_j(s)ds \\
&= \frac{1}{2}\mu^2\int_0^T (v')^2\ dt + \mu\int_0^T u'v'\ dt + \frac{\lambda}{2}\mu^2\int_0^T v^2\ dt + \lambda\mu\int_0^T uv\ dt \\
&\quad -\int_0^T \Big[F(u+\mu v) - F(u)\Big]dt + \sum_{j=1}^{p}\Big[\int_0^{u+\mu v} I_j(s)\ ds - \int_0^{u} I_j(s)\ ds\Big],
\end{aligned}$$

using the increment finite theorem, we have

$$F(u+\mu v)-F(u)=\mu\int_0^T f(u+\mu\theta v)v\ dt,$$

and

$$\int_0^{u+\mu v} I_j(s)\ ds-\int_0^{u} I_j(s)\ ds=\mu v(t_j)I_j\Big(u(t_j)+\theta v(t_j)\Big),$$

or $0<\theta<T$ and then

$$\begin{aligned}\frac{\varphi(u+\mu v)-\varphi(u)}{\mu} &= \frac{1}{2}\mu\int_0^T (v')^2\ dt+\int_0^T u'v'\ dt+\frac{\lambda}{2}\mu\int_0^T v^2\ dt+\lambda\int_0^T uv\ dt\\ &\qquad -\int_0^T f(u+\mu\theta v)vdt+\sum_{j=1}^{p} v(t_j)I_j\Big(u(t_j)+\theta v(t_j)\Big),\end{aligned}$$

either $\mu\longrightarrow 0$, so

$$\langle d\varphi(u),v\rangle=\int_0^T u'v'\ dt+\lambda\int_0^T uv\ dt-\int_0^T f(t,u)v\ dt-\sum_{j=1}^{p} I_j(s)v\ ds.$$

Step 2: $d\varphi(u)$ is continuous. Indeed, let u_n be a sequence in H_0^1 such that $u_n\longrightarrow u$, so

$$\langle\varphi'(u_n)-\varphi'(u),v\rangle=\int_0^T\Big[(u_n'-u')v'+\lambda(u_n-u)v\Big]\ dt-\int_0^T\Big(f_\varepsilon(u_n)-f(u)\Big)v\ dt+\sum_{j=1}^{p} I_j(u_n-u)v\ dt,$$

from theorem 1.2, we obtain

$$\begin{aligned}\lim_{n\to+\infty}\int_0^T f(u_n)v\ dt &= \int_0^T f(u)v\ dt.\\ \lim_{n\to+\infty}\int_0^T u_n'v'\ dt &= \int_0^T u'v'\ dt.\\ \lim_{n\to+\infty}\int_0^T \lambda u_n v\ dt &= \int_0^T \lambda uv\ dt.\end{aligned}$$

And according to the continuity of f, we pass to the limit in $\langle\varphi'(u_n)-\varphi'(u),v\rangle$ when $n\to+\infty$ we obtain that $\varphi'(u_n)\to\varphi(u)$ in $H^{-1}(0,T)$. So φ' is continuous. Finally, φ is Gâteaux differentiable and continuous. Then φ is continuously differentiable. □

Definition 2.1. We say that $u\in H_0^1(0,T)$ is a weak solution of the problem (2.1), if for all $v\in H_0^1(0,T)$, we have

$$\begin{aligned}\langle d\varphi(u),v\rangle &= \int_0^T u'(t)v'(t)\ dt+\lambda\int_0^T u(t)v(t)\ dt+\sum_{j=1}^{p} I_j\Big(u(t_j)\Big)v(t_j)\\ &= \int_0^T f(t,u(t))v(t)dt.\end{aligned}$$

Lemma 2.1. *If a function $u \in H_0^1$ is a critical point of a differentiable Fréchet function φ, then u is a weak solution of problem (2.1).*

Remark 2.3. Since the nonlinear term f is continuous, then a classical solution of the problem (2.1) is a weak solution.

2.4 Some necessary lemmas

Lemma 2.2 ([13, Lemma 2.6])**.** *The space H is injected continuously into $C[0, T]$, there exists $c > 0$ such that if $u \in H$ then*

$$\|u\|_\infty \leq c\|u\|,$$

Proof. Either $u \in H$, we have

$$\begin{aligned} |u(t)| &= \left| \int_0^t u'(s)ds \right| \\ &\leq \int_0^t |u'(s)|\, ds \leq \int_0^T |u'(t)|\, dt, \end{aligned}$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |u(t)| &\leq \left(\int_0^T dt \right)^{1/2} \left(\int_0^T |u'(t)|^2\, dt \right)^{1/2} \\ \max_{t\in[0,T]} |u(t)| &\leq \sqrt{T}\|u\|. \end{aligned}$$

From where

$$\|u\|_\infty \leq \sqrt{T}\|u\|.$$

Thus, we can choose $c = \sqrt{T}$ such that the lemma is verified. □

Lemma 2.3 ([28])**.** *The embedding $H_0^1(0, T) \hookrightarrow\hookrightarrow C[0, T)$ is compact.*

Remark 2.4. We know that the injection canonical of $H_0^1(0, T)$ in $C(0, T)$, is compact according to Rellich's theorem (1.6)(see chapter 1), and we have

$$\begin{cases} u_n \rightharpoonup u, & in\ H_0^1(0, T), \\ u_n \to u, & in\ C[0, T], \\ u_n(t) \to u(t), & a.e.\ in\ (0, T). \end{cases}$$

Lemma 2.4. *$C[0, T)$ is continuously embedded in $L^2(0, T)$.*

Lemma 2.5 ([24, Lemma 1.4]). *λ_1 is positive and is achieved for some positive function $\varphi_1 \in H_0^1(0,T)\backslash\{0\}$. And*

$$\lambda_1 = \inf_{u \in H_1^0 \backslash\{0\}} \frac{\|u\|^2}{\|u\|_{L^2}^2}$$

Proof. We proceed as in [23]. For $u \in H_1^0(0,T)$, let $I_1(u) = \|u\|^2, I_2(u) = \|u\|_{L^2}^2$, and define the quotient functional $Q : H_1^0 \setminus \{0\} \to \mathbb{R}$ by

$$Q(u) = \frac{I_1(u)}{I_2(u)}. \text{ Then } \lambda_1 = \inf_{u \in H_1^0 \backslash\{0\}} Q(u).$$

Let $u \in H_1^0(0,T)$. We have that $\lambda_1 = \inf_{u \in H_1^0 \backslash\{0\}} \frac{\|u\|^2}{\|u\|_{L^2}^2} \geq \frac{1}{T(T+1)} > 0$. Indeed, for $t > 0$, note

$$\begin{aligned} |u(t)|^2 &= \left| \int_0^t u'(s)ds \right|^2 \leq \left| \int_0^T u'(s)ds \right|^2 \leq \left(\int_0^T u'^2(s)ds \right) \left(\int_0^T ds \right) \\ &\leq T \left(\int_0^T u'^2(s)ds \right) + \left(\int_0^T u^2(s)ds \right) \\ &\leq (T+1) \left(\int_0^T u'^2(s)ds + \int_0^T u^2(s)ds \right) = (T+1)\|u\|^2, \end{aligned}$$

which yields

$$\|u\|_{L^2}^2 \leq T(T+1)\|u\|^2,$$

and

$$\lambda_1 = \inf_{u \in H_0^1 \backslash\{0\}} \frac{\|u\|^2}{\|u\|_{L^2}^2} \geq \frac{1}{T(T+1)} > 0\cdot$$

Let (u_n) be a minimizing sequence. Since$(|u_n|)$ is a minimizing sequence for Q, we may suppose that $u_n(t) \geq 0$, for $t \in [0,T)$. Moreover the functional Q satisfies $Q(\alpha u) = Q(u)$, for every $\alpha \in \mathbb{R}$. By setting $\tilde{u_n} = \frac{u_n}{\|u\|_{L^2}}$, for every n, we can assume that $\|u\|_{L^2} = 1$. Note $\lim_{n \to +\infty} Q(u_n) = \inf_{u \in H_1^0 \backslash\{0\}} Q(u) = \lambda_1$, so the sequence $(Q(u_n))$ is bounded. From this and since $Q(u_n) = \|u_n\|^2$, we deduce that (u_n) is bounded in H_0^1.From 2.3 and the reflexivity and separability of H_0^1, there exists a subsequence (u_{n_k}) of (u_n) such that, as $k \to +\infty$,

$$\begin{cases} u_{n_k} \rightharpoonup \bar{u}, & in\ H_0^1(0,T), \\ u_{n_k} \to \bar{u}, & in\ C[0,T], \end{cases}$$

so $u_{n_k}(t) \rightharpoonup \bar{u}(t)$, for all $t \in [0,T)$. From (2.4) (u_{n_k}) converges in norm to $\bar{u}$ in L^2. Thus $\|\bar{u}\|_{L^2} = 1$ and $\bar{u}(t) \geq 0$, for $t \in [0,T)$. Finally, the weak lower semi-continuity of the norm guarantees that

$$Q(\bar{u}) = I_1(\bar{u}) \leq \liminf_k I_1(u_{n_k}) = \liminf_k Q_1(u_{n_k}) = \lambda_1,$$

so $\bar{u} \in H_1^0 \backslash \{0\}$ and $Q(\bar{u}) = \lambda_1$ □

Lemma 2.6. *$H_0^1(0,T)$ is compactly embedded in $L^2(0,T)$, and*

$$\Big(\int_0^T (u(t))^2 dt\Big)^{1/2} \leq \frac{1}{\sqrt{\lambda_1}}\Big(\int_0^T (u'(t))^2 dt\Big)^{1/2}. \tag{2.5}$$

Such that $\lambda_1 = \dfrac{\pi^2}{T^2}$ is the first eigenvalue of the previous linear problem.

Proof. Using lemma(2.5) we have

$$\lambda_1 = \inf_{u \in H_1^0 \backslash \{0\}} \frac{\|u\|^2}{\|u\|_{L^2}^2}$$

then

$$\lambda_1 \leq \frac{\|u\|^2}{\|u\|_{L^2}^2} \Leftrightarrow \|u\|_{L^2}^2 \leq \frac{1}{\lambda_1}\|u\|^2 \Leftrightarrow \|u\|_{L^2} \leq \frac{1}{\sqrt{\lambda_1}}\|u\|.$$

Therefore, we have,

$$\Big(\int_0^T (u(t))^2 dt\Big)^{1/2} \leq \frac{1}{\sqrt{\lambda_1}}\Big(\int_0^T (u'(t))^2 dt\Big)^{1/2}.$$

□

Lemma 2.7 ([10, Proposition 1.1])**.** *If $\lambda > -\lambda_1$, there exist constants $\theta_2 > \theta_1 > 0$ such that*

$$\theta_1 \|u\|^2 \leq L(u,u) \leq \theta_2 \|u\|^2, \ \forall u \in H_0^1(0,T). \tag{2.6}$$

With

$$L(u,u) = \frac{1}{2}\int_0^T (u'(t))^2 dt + \frac{\lambda}{2}\int_0^T (u(t))^2 dt$$

In fact, it is sufficient to take $\theta_1 = \dfrac{1}{2}$, for $\lambda \geq 0$ and $\theta_1 = \dfrac{1}{2}\Big(1 - \dfrac{\lambda}{\lambda_1}\Big)$ for $0 > \lambda > -\lambda_1, \theta_2 = \dfrac{1}{2}\Big(1 + \dfrac{|\lambda|}{\lambda_1}\Big)$.

Proof. Firstly, when $\lambda \geq 0$, by the Poincaré inequality we have

$$\begin{aligned} L(u,u) &= \frac{1}{2}\int_0^T (u'(t))^2 dt + \frac{\lambda}{2}\int_0^T (u(t))^2 dt \\ &\leq \frac{1}{2}\|u\|^2 + \frac{\lambda}{2\lambda_1}\int_0^T (u'(t))^2 dt \\ &\leq \frac{1}{2}\|u\|^2 + \frac{\lambda}{2\lambda_1}\|u\|^2 \leq \Big(\frac{1}{2} + \frac{\lambda}{2\lambda_1}\Big)\|u\|^2. \end{aligned}$$

and

$$L(u,u)=\frac{1}{2}\int_0^T (u'(t))^2dt+\frac{\lambda}{2}\int_0^T (u(t))^2dt$$
$$\geq \frac{1}{2}\int_0^T (u'(t))^2dt \geq \frac{1}{2}\|u\|^2.$$

thus, $\theta_1=\frac{1}{2}, \theta_2=\frac{1}{2}\Big(1+\frac{\lambda}{\lambda_1}\Big)$. Secondly, when $0>\lambda>-\lambda_1$, by using Poincare inequality, on has

$$L(u,u)=\frac{1}{2}\int_0^T (u'(t))^2dt+\frac{\lambda}{2}\int_0^T (u(t))^2dt$$
$$\geq \frac{1}{2}\|u\|^2-\frac{\lambda}{2\lambda_1}\int_0^T (u'(t))^2dt \geq \Big(\frac{1}{2}-\frac{\lambda}{2\lambda_1}\Big)\|u\|^2.$$

thus, $\theta_1=\frac{1}{2}\Big(1-\frac{\lambda}{\lambda_1}\Big)$. □

Lemma 2.8. *The functional φ which be defined by* (2.3) *is continuous, and sequentially weakly lower semi-continuous.*

Proof. Using the continuity of f and $I_j(i=1,2,..,p)$. It is easy to prove that the function φ is continuous and differentiable (see proof of proposition (2.2)).To show thatφ is sequentially weakly lower semi-continuous, let $\{u_k\}$ be a weakly convergent sequence to u in $H_0^1(0,T)$,then according to **(w.l.s.c)** of the norm (see example (1.1) in chapter 1), we have $\|u\|\leq \liminf_{k\to\infty}\|u_k\|$. And as $\{u_k\}$ converges uniformly to u in $C[0,T]$ (according to the compact injection of $H_0^1(0,T)$ in $C[0,T]$). Then

$$\lim_{k\to\infty}\left[\frac{\lambda}{2}\int_0^T (u_k(t))^2dt-\int_0^T F(t,u_k(t))dt+\sum_{j=1}^p\int_0^{u_k(t_j)} I_j(s)ds\right]$$
$$=\frac{\lambda}{2}\int_0^T (u(t))^2dt-\int_0^T F(t,u(t))dt+\sum_{j=1}^p\int_0^{u_k(t_j)} I_j(s)ds,$$

which implies that

$$\liminf_{k\to\infty}\varphi(u_k)=\liminf_{k\to\infty}\left[\frac{1}{2}\|u_k\|^2+[\frac{\lambda}{2}\int_0^T (u_k(t))^2dt-\int_0^T F(t,u_k(t))dt+\sum_{j=1}^p\int_0^{u_k(t_j)} I_j(s)ds\right]$$
$$\geq \|u\|^2+\frac{\lambda}{2}\int_0^T (u(t))^2dt-\int_0^T F(t,(t))dt+\sum_{j=1}^p\int_0^{u_k(t_j)} I_j(s)ds=\varphi(u).$$

Thus, by definition (1.7) , φ is weakly inferiorly semi-continuous. □

Lemma 2.9. *If **(H1)**$-$**(H2)** hold, then φ satisfies the Palais-Smale condition.*

Proof. Let $\{u_k\}$ be a sequence in $H_0^1(0,T)$ such that $\{\varphi(u_k)\}$ is bounded and $(u_k) \to 0$ as $k \to \infty$. Then there exists a constant C_1 such that

$$|\varphi(u_k)| \leq C_1, \|\varphi'(u_k)\| \leq C_1. \tag{2.7}$$

We first prove that $\{u_k\}$ is bounded. By the assumption **(H1)** and (2.3), we have

$$\begin{aligned} &\int_0^T (u_k'(t))^2 dt + \lambda \int_0^T (u_k(t))^2 dt \\ &= 2\varphi(u_k) + 2\int_0^T F(t, u_k(t))dt - 2\sum_{j=1}^p \int_0^{u_k(t_j)} I_j(s)ds \\ &\leq 2C_1 + 2\int_0^T F(t, u_k(t))dt - \frac{2}{\mu}\sum_{j=1}^p I_j(u_k(t_j))u_k(t_j). \end{aligned}$$

Combing (3.3), the above inequality implies that

$$\begin{aligned} &\left(1 - \frac{2}{\mu}\right)\left(\int_0^T (u_k'(t))^2 dt + \lambda \int_0^T (u_k(t))^2 dt\right) \\ &\leq 2C_1 + 2\int_0^T F(t, u_k(t))dt - \frac{2}{\mu}\varphi'(u_k(t))u_k \\ &\quad -\frac{2}{\mu}\int_0^T f(t, u_k(t))u_k(t)dt. \end{aligned} \tag{2.8}$$

According to the assumption **(H2)** and the inequalities (2.6) and (2.7), one has

$$2\theta_1\left(1 - \frac{2}{\mu}\right)\|u_k\|^2 \leq 2C_1 + \frac{2}{\mu}C_1\|u_k\| + \left(2 + \frac{2}{\mu}\right)T\left[ac\|u_k\| + \frac{b}{\gamma+1}c^{\gamma+1}\|u_k\|^{\gamma+1}\right].$$

Since $\theta_1 > 0$ and $\mu > 2$ it follows that $\{u_k\}$ is bounded in $H_0^1(0,T)$. Hence there exists a subsequence of $\{u_k\}$ (for simplicity denoted again by $\{u_k\}$) such that $\{u_k\}$ weakly concerges to some u in $H_0^1(0,T)$. Then the sequence $\{u_k\}$ converges uniformly to u in $C[0,T]$. Hence

$$\begin{aligned} &(\varphi'(u_k) - \varphi'(u))(u_k - u) \to 0, \\ &\int_0^T [f(t, u_k(t)) - f(t, u(t))]((u_k(t) - u(t))dt \to 0, \\ &\sum_{j=1}^p [I_j(u_k(t_j)) - I_j(u(t_j))]((u_k(t_j) - u(t_j)) \to 0, \end{aligned}$$

as $k \to +\infty$. Moreover, an easy computation shows that

$$\begin{aligned}(\varphi'(u_k) - \varphi'(u))(u_k - u) &= \int_0^T (u_k'(t) - u'(t))^2 dt + \lambda \int_0^T (u_k(t) - u(t))^2 dt \\ &- \int_0^T [f(t, u_k(t)) - f(t, u(t))]((u_k(t) - u(t))dt \\ &+ \sum_{j=1}^{p} [I_j(u_k(t_j)) - I_j(u(t_j))]((u_k(t_j) - u(t_j)) \\ &\geq 2\theta_1 \|u_k - u\|^2 - \int_0^T [f(t, u_k(t)) - f(t, u(t))]((u_k(t) - u(t))dt \\ &+ \sum_{j=1}^{p} [I_j(u_k(t_j)) - I_j(u(t_j))]((u_k(t_j) - u(t_j)).\end{aligned}$$

so $\|u_k - u\| \to 0$ as $k \to +\infty$. That is, $\{u_k\}$ converges strongly to u in $H_0^1(0, T)$. □

2.5 Study the existence of a unique solution

We now prove the existence of at least one solution in the following two theorems. **First result of existence**

Theorem 2.1. *Suppose that f is bounded and that the impulsive functions I_j are bounded. Then there is a critical point of φ, and (2.1) has at least one solution.*

Proof. Take $M > 0$ and $M_j > 0, j = 1, 2, ..., p$ such that

$$|f(t, u)| \leq M \quad for\ every \quad (t, u) \in [0, T] \times \mathbb{R},$$

and

$$|I_j(u)| \leq M_j$$

for every $u \in \mathbb{R}, j = 1, 2, ...p$. Using that $\lambda > -\lambda_1$ there exists $\theta_1 > 0$ such that for any

$u \in H_0^1(0,T)$ By lemma (2.2) and (2.7), one has

$$\begin{aligned}
\varphi(u) &= L(u,u) + \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(s)ds - \int_0^T F(t,u(t))dt \\
&\geq \theta_1 \|u\|^2 + \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(s)ds - \int_0^T F(t,u(t))dt \\
&\geq \theta_1 \|u\|^2 - \sum_{j=1}^{p} M_j |u(t_j)| - \int_0^T F(t,u(t))dt \\
&\geq \theta_1 \|u\|^2 - \sum_{j=1}^{p} M_j |u(t_j)| - M \int_0^T |u(t)|dt \\
&\geq \theta_1 \|u\|^2 - \sum_{j=1}^{p} M_j \|u\|_\infty - MT\|u\|_\infty \\
&\geq \theta_1 \|u\|^2 - \sum_{j=1}^{p} M_j \|u\|_\infty - MT\|u\|_\infty \\
&\geq \theta_1 \|u\|^2 - \sum_{j=1}^{p} cM_j \|u\| - cMT\|u\|,
\end{aligned}$$

for some $c = \sqrt{T} > 0$. This implies that $\lim_{\|u\|\to\infty} \varphi(u) = +\infty$, φ is coercive,and φ is $(w.l.s.c)$(see lemma 2.8). Hence, theorem(1.7) φ has a minimum, which is a critical point of φ, then (2.1) has at least one solution. □

Second existence result

Theorem 2.2. *Assume that following conditions hold.*

(d1) *The impulsive functions I_j have sublinear growth, i.e., there exist constants $a_j > 0$, $b_j > 0$ and $\gamma_j \in [0,1)$, $j = 1,2,...,p$, such that*

$$|I_j(t)| \leq a_j + b_j\,|t|^{\gamma_j}\,, for\ every \quad t \in \mathbb{R}, j = 1,2,....,p.$$

(d2) *f is sublinear growth, i.e., there exist constants $a > 0$,$b > 0$ and $\gamma \in [0,1)$ such that*

$$|f(t,u)| \leq a + b\,|u|^{\gamma}\,, for\ every \quad (t,u) \in [0,T] \times \mathbb{R}.$$

Then the impulsive problem (2.1) *has at least one solutions for $\lambda > -\pi^2/T^2$.*

Proof. Like $f(t,0) \not\equiv 0$ there is $a > 0, \delta_1 \leq 1$ and, $\forall (t,u) \in [0,T] \times \mathbb{R}$

$$|f(t,u)| \leq a, \quad |u| \leq \delta_1 \leq 1, \quad \forall t \in [0,T], \tag{2.9}$$

and $b > 0, \delta_2 \geq 1$, and $\gamma \in [0, 1)$, then according to sublinearity we have:

$$|f(t, u)| \leq b|u|^\gamma, \quad |u| \geq \delta_2 \geq 1, \quad \forall t \in [0, T], \tag{2.10}$$

then we find according to (2.9) and (2.10)

$$|f(t, u)| \leq a + b|u|^\gamma, \quad \forall (t, u) \in [0, T] \times \mathbb{R}. \tag{2.11}$$

We have,

$$\begin{aligned}|F(t, u)| &= \int_0^u |f(t, s)|\, ds \leq \int_0^u (a + b|u|^\gamma)\; ds \\ &\leq a \int_0^u\; ds + \frac{b}{\gamma + 1}|u|^{\gamma+1} \leq a|u| + \frac{b}{\gamma + 1}|u|^{\gamma+1}\end{aligned}$$

So, we obtain,

$$\begin{aligned}\int_0^T |F(t, u)|\, dt &\leq \int_0^T \left(a|u| + \frac{b}{\gamma + 1}|u|^{\gamma+1}\right)\; dt \\ &\leq T\left(a|u| + \frac{b}{\gamma + 1}|u|^{\gamma+1}\right),\end{aligned}$$

In the same way with $I_j(u)$, we take $a_j, b_j > 0, \delta_{1j} \leq 1, \delta_{2j} \geq 1$ and $\gamma_j, \in [0, 1)$, $j = 1, 2, \ldots, p$, we find:

$$|I_j(u)| \leq a_j + b_j|u|^{\gamma_j}, \quad \forall (t, u) \in [0, T] \times \mathbb{R}. \tag{2.12}$$

We have,

$$\begin{aligned}\int_0^{u(t_j)} |I_j(u)|\, dt &\leq \int_0^{u(t_j)} (a_j + b_j|u|^{\gamma_j})\; dt \\ &\leq a_j \int_0^{u(t_j)}\; dt + \frac{b_j}{\gamma_j + 1}|u|^{\gamma_j+1} \\ &\leq a_j|u| + \frac{b_j}{\gamma_j + 1}|u|^{\gamma_j+1}.\end{aligned}$$

By lemma 2.2 and 2.7 and the same methods as in the proof above,

$$\begin{aligned}\varphi(u) &= L(u, u) + \sum_{j=1}^p \int_0^{u(t_j)} I_j(s)ds - \int_0^T F(t, u(t))dt \\ &\geq \theta_1\|u\|^2 - \sum_{j=1}^p (a_j|u| + \frac{b_j}{\gamma_j + 1}|u|^{\gamma_j+1}) - \int_0^T (a|u| + \frac{b}{\gamma + 1}|u|^{\gamma+1}) \\ &\geq \theta_1\|u\|^2 - \sum_{j=1}^p (a_j\|u\|_\infty + \frac{b_j}{\gamma_j + 1}\|u\|_\infty^{\gamma_j+1}) - T(a\|u\|_\infty + \frac{b}{\gamma + 1}\|u\|_\infty^{\gamma+1}) \\ &\geq \theta_1\|u\|^2 - \sum_{j=1}^p (ca_j\|u\| + \frac{b_j}{\gamma_j + 1}c^{\gamma_j+1}\|u\|^{\gamma_j+1}) - T(ca\|u\| + \frac{b}{\gamma + 1}c^{\gamma+1}\|u\|^{\gamma+1}),\end{aligned}$$

and like $-\|u\| \geq -\|u\|^{\gamma+1}$ so:

$$\geq \theta_1\|u\|^2 - \|u\|^{\gamma+1}\Big[\sum_{j=1}^{p}(ca_j + \frac{b_j}{\gamma_j+1}c^{\gamma_j+1}) + T(ca + \frac{b}{\gamma+1}c^{\gamma+1})\Big],$$

then finally There exists $\eta > 0$ such that

$$\varphi(u) \geq \theta_1\|u\|^2 - \eta\|u\|^{\gamma+1},$$

with :

$$\eta = \sum_{j=1}^{p}(ca_j + \frac{b_j}{\gamma_j+1}c^{\gamma_j+1}) + T(ca + \frac{b}{\gamma+1}c^{\gamma+1}).$$

Which implies that $\liminf_{\|u\|\to\infty} \varphi(u) = +\infty$, thus φ is coercive. φ admits a minimum, which is a critical point of φ,and problem(2.1) has at least one solution. □

Example 2.1. Let $T = \pi, \lambda > -1, t_1 \in (0, \pi)$, and $\sigma \in C[0, \pi]$. The nonlinear Dirichlet impulsive problem

$$\begin{cases} -u''(t) + \lambda u(t) = \sqrt{|u(t)|} + \sigma(t), \quad t \in [0, \pi], \\ -\Delta u'(t_1) = 1 + \sqrt[3]{u(t_1)}, \\ u(0) = u(T) = 0, \end{cases} \tag{2.13}$$

It is easy to know that $f(t, u) = \sqrt{|u(t)|} + \sigma(t)$. So problem (2.1) has at least one solution.

2.6 Existence of two solutions

In this section, we deduce the conditions under which the problem(2.1) admits two solutions.

Theorem 2.3. *Suppose that **(H1)** and the following conditions hold:*

***(H2)** f is sublinear growth, i.e., there exist constants $a > 0, b > 0$ and $\gamma \in [0, 1)$ such that*

$$|f(t, u)| \leq a + b|u|^{\mu}$$

***(H3)** $F(t, u) \leq 0$ for all $(t, u) \in [0, T] \times \mathbb{R}$.*

Then the impulsive problem (2.1) *has at least two weak solutions.*

Proof. In our case it is clear that $\varphi(0) = 0$. Lemma 2.9 has shown that φ satisfies the Palais-Smale condition.

Step 1 We will show that there exists $M > 0$ such that the functional φ has a local minimum $u_0 \in B_M = \left\{u \in H^1(0, T) : \|u\| < M\right\}$.

Let $M > 0$ which will be determined later. First we claim that $\overline{B}_M$is a bounded and weak sequentially closed.

In fact, let $\{u_n\} \subseteq \overline{B}_M$ and $u_n \rightharpoonup u$ as $n \to +\infty$, by the Mazur Theorem [14], there exists a sequence of convex combinations

$$v_n = \sum_{j=1}^{n} \alpha_{n_j} u_j, \sum_{j=1}^{n} \alpha_{n_j} = 1, \alpha_{n_j} u_j \geq 0, j \in N,$$

such that $v_n \to u$ in $H_0^1(0,T)$. Since $\overline{B}_M$ is a closed convex set, we have $v_n \subseteq \overline{B}_M$ and $u \in \overline{B}_M$. Noting that φ is weak sequentially lower semi-continuous on $\overline{B}_M$ and $H_0^1(0,T)$ is a reflexive Banach space. Then by Theorem 1.8 we can know that φ has a local minimum $u_0 \in \overline{B}_M$. Without loss of generality, we assume that $\varphi(u_0) = \min_{u \in \overline{B}_M} \varphi(u)$. Now we will show that

$$\varphi(u_0) < \inf_{u \in \partial B_M} \varphi(u). \tag{2.14}$$

In fact, choose $M = \epsilon > 0$ satisfying

$$\theta_1 \epsilon^2 - \sum_{j=1}^{p} \delta_j c^{\mu} \epsilon^{\mu} > 0$$

where c is defined in lemma (2.2) and θ_1 is defined in lemma (2.7). For any $u = \epsilon\omega$ with $\omega \in H_0^1(0,T)$ and $\|\omega\| = 1$,$\|u\| = \|\epsilon\omega\| = \epsilon\|\omega\| = \epsilon$, then $u \in \partial B_M$, by **(H3)** we have $\int_0^T F(t, \epsilon\omega(t))dt \leq 0.$

By lemma 2.7

$$\begin{aligned}
\varphi(u) = \varphi(\epsilon\omega) =& \frac{1}{2} \int_0^T [(\epsilon\omega'(t))^2 + \lambda(\epsilon\omega(t))^2]dt - \int_0^T F(t, \epsilon\omega(t))dt \\
&+ \sum_{j=1}^{p} \int_0^{\epsilon\omega(t_j)} I_j(s)ds \\
&= L(\epsilon\omega, \epsilon\omega) + \sum_{j=1}^{p} \int_0^{\epsilon\omega(t_j)} I_j(s)ds - \int_0^T F(t, \epsilon\omega(t))dt \\
&\geq \theta_1 \|\epsilon\omega\|^2 + \sum_{j=1}^{p} \int_0^{\epsilon\omega(t_j)} I_j(s)ds \geq \theta_1 \epsilon^2 \|\omega\|^2 + \sum_{j=1}^{p} \int_0^{\epsilon\omega(t_j)} I_j(s)ds \\
&\geq \theta_1 \epsilon^2 - \sum_{j=1}^{p} \delta_j |\epsilon\omega|^{\mu} \geq \theta_1 \epsilon^2 - \sum_{j=1}^{p} \delta_j c^{\mu} \epsilon^{\mu} > 0.
\end{aligned}$$

So $\varphi(u) > 0 = \varphi(0) \geq \varphi(u_0)$ for $u \in \partial B_M$. Hence (2.14) holds and $u_0 \in B_M$.

Step 2: We will show that there exists u_1 with $u_1 > M$ such that

$$\varphi(u_1) < \inf_{u \in \partial B_M} \varphi(u)$$

First, from (i) of **(H1)**, we have

$$\frac{I_j(x)}{\int_0^x I_j(s)ds} \geq \frac{\mu}{x}, \quad for \quad x > 0, \tag{2.15}$$

$$\frac{I_j(x)}{\int_0^x I_j(s)ds} \leq \frac{\mu}{x}, \quad for \quad x < 0. \tag{2.16}$$

Integrating (2.15) and (2.16) from 1 to x and x to -1 respectively, we have

$$\ln \frac{\int_0^x I_j(s)ds}{\int_0^1 I_j(s)ds} \geq \mu \ln x, \quad for \quad x > 1, \tag{2.17}$$

$$\ln \frac{\int_0^{-1} I_j(s)ds}{\int_0^t I_j(s)ds} \leq \mu \ln \frac{1}{-x}, \quad for \quad x < -1, \tag{2.18}$$

So

$$\int_0^x I_j(s)ds \leq x^\mu \int_0^1 I_j(s)ds, \quad for \quad x > 1, \tag{2.19}$$

$$\int_0^x I_j(s)ds \leq (-x)^\mu \int_0^{-1} I_j(s)ds, \quad for \quad x < -1, \tag{2.20}$$

Noting that $\int_0^1 I_j(s)ds < 0$ and $\int_0^{-1} I_j(s)ds < 0$. Let

$$k_j = \min\left\{ |\int_0^1 I_j(s)ds|, |\int_0^{-1} I_j(s)ds| \right\} > 0$$

then we have

$$\int_0^t I_j(s)ds \leq -k_j|x|^\mu, |x| \geq 1. \tag{2.21}$$

Since $\int_0^t I_j(s)ds$ is continuous on $[-1, 1]$, then there exists a constant $K > 0$ such that

$$\int_0^x I_j(s)ds \leq K, |x| \leq 1. \tag{2.22}$$

It follows from (2.21) and (2.22) that

$$\int_0^t I_j(s)ds \leq -k_j|x|^\mu + K, \forall u \in \mathbb{R}. \tag{2.23}$$

Now, let $\tilde{e}(t) = \varphi_1(t) \in H_0^1(0, T)$ and $u_1 = r\tilde{e}, r > 0$, where φ_1 corresponding to λ_1 is the first eigenfunction of (3.2) and $\varphi_1 = 1$, where c is defined in lemma 2.2 and θ_2 is defined in lemma

2.7 and 2.23. Then

$$
\begin{aligned}
\varphi(u_1) = \varphi(r\tilde{e}) &= \frac{1}{2}\int_0^T ((r\tilde{e}'(t))^2 + \lambda(r\tilde{e}(t))^2)dt - \int_0^T F(t, r\tilde{e}(t))dt \\
&\quad + \sum_{j=1}^{p}\int_0^{r\tilde{e}(t_j)} I_j(s)ds \\
&= r^2 L(\tilde{e},\tilde{e}) - \int_0^T F(t, r\tilde{e}(t))dt + \sum_{j=1}^{p}\int_0^{r\tilde{e}(t_j)} I_j(s)ds \\
&\leq \theta_2 r^2 - \int_0^T (a|r\tilde{e}| + \frac{b}{\gamma+1}|r\tilde{e}|^{\gamma+1}) + \sum_{j=1}^{p}\int_0^{r\tilde{e}(t_j)} I_j(s)ds \\
&\leq \theta_2 r^2 + T(a\|r\tilde{e}\|_\infty + \frac{b}{\gamma+1}\|r\tilde{e}\|_\infty^{\gamma+1}) - \sum_{j=1}^{p} k_j|r\tilde{e}(t_j)|^{\mu} + Kp \\
&\leq \theta_2\|r\tilde{e}\|^2 + T(ca\|r\tilde{e}\| + \frac{b}{\gamma+1}c^{\gamma+1}\|r\tilde{e}\|^{\gamma+1}) - \sum_{j=1}^{p} k_j|r\tilde{e}(t_j)|^{\mu} + Kp \\
&\leq \theta_2 r^2\|\tilde{e}\|^2 + T(acr\|\tilde{e}\| + \frac{b}{\gamma+1}c^{\gamma+1}r^{\gamma+1}\|\tilde{e}\|^{\gamma+1}) - r^{\mu}\sum_{j=1}^{p} k_j|\tilde{e}(t_j)|^{\mu} + Kp \\
&\leq \theta_2 r^2 + T(acr + \frac{b}{\gamma+1}c^{\gamma+1}r^{\gamma+1}) - r^{\mu}\sum_{j=1}^{p} k_j|\tilde{e}(t_j)|^{\mu} + Kp.
\end{aligned}
$$

So there exists sufficiently large $r > M = \epsilon > 0$ such that $\varphi(r\tilde{e}) < 0$. Therefore, by Step 1 and Step 2, we have

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf_{u\in\partial B_M}\varphi(u)$$

Then theorem 1.13 gives the critical point u^*. Therefore, u_0, u^* are two critical points of φ, which are two solutions of (2.1). □

Example 2.2. Let $T > 0, a_1 > 0, b_1 > 0, k > 0, t_1 \in (0, T)$. Consider the following problem:

$$
\begin{cases}
-u''(t) + \lambda u(t) = -\cos(u(t))e^{\sin(u(t))} - u^{\frac{1}{5}}(t), & t \in [0, T], \\
\Delta u'(t_1) = -ku^3(t_1), \\
u(0) = u(T) = 0.
\end{cases}
\tag{2.24}
$$

It is easy to know that $f(t,u) = -\cos(u(t))e^{\sin(u(t))} - u^{\frac{1}{5}}(t) = -ku^3$. Let $\mu = 4, \delta_1 = \dfrac{1}{4}k$,the condition **(H1)** holds. And $|f(t,u)| \leq e + |u|^{\frac{1}{5}}$ $F(t,u) = -e^{\sin(u(t))} - \dfrac{5}{6}u^{\frac{6}{5}}(t) \leq 0$, the conditions **(H2)** and **(H3)** hold. So problem (2.24) has at least two weak solutions.

Remark 2.5. Since $I_1(u) = -ku^3$ does not satisfy the sublinear growth condition **(d1)**, so example (2.24) cannot determine the existence of solution in the paper [9, 10, 11, 12].

Corollary 2.1. Suppose that **(H1)** and **(H3)** hold. If f is bounded, then the impulsive problem (2.1) has at least two weak solutions.

CHAPTER 3

EXISTENCE AND MULTIPLICITY OF SOLUTIONS TO AN IMPULSIVE SECOND-ORDER BOUNDARY VALUE PROBLEM

In this chapter, we discuss the existence and multiplicity of solutions for an impulsive second-order equation (3.1) that was published in the article [27].

We study the following impulsive problem,

$$\begin{cases} -u''(t) + \lambda u(t) = f(t, u(t)), \quad t \in [0, T], \\ \Delta u'(t_j) = I_j(u(t_j)), \;\; j = 1, 2, ..., p, \\ u(0) = u(T) = 0, \end{cases} \tag{3.1}$$

where $T > 0, f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous, $0 = t_0 < t_1 < t_2 < ..... < t_p < t_{p+1} = T$, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \lim_{t \to t_j^+} u'(t) - \lim_{t \to t_j^-} u'(t)$, $I_j : \mathbb{R} \to \mathbb{R}$ are continuous.

During this chapter, we assume that the following assumptions are satisfied:

(H1) There exist $\mu > 2, \delta_j > 0, j = 1, 2, ..., p$, such that

(i) $I_j(x)x \leq \mu \int_0^x I_j(s)ds < 0$, for $x \in \mathbb{R} \backslash \{0\}$;

(ii) $\int_0^x I_j(s)ds \geq -\delta_j |x|^\mu$, for $x \in \mathbb{R} \backslash \{0\}$;

We always assume that $\lambda > \lambda_1$.

3.1 The results of existence and multiplicity

We first recall the Euler-Lagrange function associated with the problem (3.1)

$$\varphi(u) = L(u,u) - \int_0^T F(t,u)\ dt + \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(s)ds, \tag{3.2}$$

such that $F(t,u) = \int\limits_0^u f(t,s)ds$. And L defined by,

$$L(u,u) = \frac{1}{2}\int_0^T (u'(t))^2 dt + \frac{\lambda}{2}\int_0^T (u(t))^2 dt, \ \ \forall u \in H_0^1(0,T).$$

Proposition 3.1. *the functional φ is continuously differentiable. The Fréchet derivative of φ in the form:*

$$\langle \varphi'(u), v\rangle = \int_0^T u'(t)v'(t)\ dt + \lambda \int_0^T u(t)v(t)\ dt - \int_0^T f(t,u(t))v(t)dt + \sum_{j=1}^{p} I_j\Big(u(t_j)\Big)v(t_j), \tag{3.3}$$

3.2 Some lemmas and theorems used

We now present the Hilbert space $H_0^1(0,T)$ which is suitable for the study of our problem.

Theorem 3.1 ([11, Theorem 3.3])**.** *Suppose that **(d1)** and the following conditions hold.*

(d5) *$f(t,u)$ is odd in u.*

(d6) *There exist constants $a > 0$,$b > 0$ and $\gamma \in (1,+\infty)$ such that*

$$f(t,u) \leq a + b\,|u|^{\gamma} \quad for\ every \quad (t,u) \in [0,T] \times \mathbb{R}$$

(d7) *There exist constants $\beta > 2, r > 0$ such that*

$$0 < \beta F(t,u) \leq uf(t,u) \quad for\ every \quad t \in [0,T], u \in \mathbb{R}\ with\ \ |u| \geq r,$$

where $F(t,u) = \int_0^u f(t,s)ds$; moreover, assume that $f(t,u) = o(u)$ as $u \to 0$ uniformly in t.

(d8) *$I_j(j = 1,2,....,p)$ are odd and nondecreasing.*

Then the problem (2) *(See Introduction) has infinite many nontrivial solutions.*

Lemma 3.1. *If **(H1)** and **(H4)** hold, then φ satisfies the Palais-Smale condition.*

Proof. Let $\{u_k\}$ be a sequence in $H_0^1(0,T)$ such that $\{\varphi(u_k)\}$ is bounded and $\varphi'(u_k) \to 0$ as $k \to \infty$. We first prove that $\{u_k\}$ is bounded. By (3.2) and (3.3), one has

$$\begin{aligned}\beta\varphi(u_k) - \varphi'(u_k)u_k =& \frac{\beta}{2}\left(\int_0^T (u_k'(t))^2 dt + \lambda\int_0^T (u_k(t))^2 dt\right) \\ &- \beta\int_0^T F(t,u_k(t))dt + \beta\sum_{j=1}^{p}\int_0^{u(t_j)} I_j(s)ds \\ &- \left[\int_0^T (u_k'(t))^2 dt + \lambda\int_0^T (u_k(t))^2 dt\right] \\ &+ \int_0^T f(t,u_k(t))u_k(t)dt - \sum_{j=1}^{p} I_j(u_k(t_j))u_k(t_j).\end{aligned}$$

By **(H1)** and **(H4)**, noting that $\beta \in (2,\mu]$, we can deduce that

$$\int_0^T f(t,u_k(t))u_k(t)dt - \beta\int_0^T F(t,u_k(t))dt \geq 0, \tag{3.4}$$

and

$$\beta\sum_{j=1}^{p}\int_0^{u(t_j)} I_j(s)ds - \sum_{j=1}^{p} I_j(u_k(t_j))u_k(t_j) \geq (\beta-\mu)\sum_{j=1}^{p}\int_0^{u(t_j)} I_j(s)ds \geq 0 \tag{3.5}$$

From (3.4) and (3.5), we conclude that

$$\beta\varphi(u_k) - \varphi'(u_k)u_k \geq (\beta-2)\theta_1\|u_k\|^2$$

Since $\theta_1 > 0$ and $\beta > 2$ it follows $\{u_k\}$ is bounded in $H_0^1(0,T)$. The following proof of (PS) condition is similar to that in Lemma (2.9). We omit it here. □

Lemma 3.2. *[10, Lemma 2.2] Assume that* **(H4)** *holds. Then, for every $t \in [0,T]$, the following inequalities hold, we have*

(i) $F(t,u) \leq M|u|^\mu$, *if* $|u| < 1$,

(ii) for every finite-dimensional subspace $W \in H$ and every $u \in W$, there exists a constant $A, B > 0$, such that $\int_0^T F(t,u)dt \geq mB^\mu\|u\|^\mu - AT$.

where $M = \max_{t\in[0,T],|u|=1} F(t,u)$, $m = \min_{t\in[0,T],|u|=1} F(t,u)$.

Proof. For all $t \in [0,T]$, we have

$$0 < \beta F(t,u) \le f(t,u)u, \quad \forall u \in \mathbb{R} \setminus \{0\},\ t \in [0,T].$$

If $|u| \ge 1$ then, we integrate between 1 and $|u|$,

$$\int_1^{|u|} \frac{\beta}{s} ds \le \int_1^{|u|} \frac{f(t,s)}{F(t,s)} ds$$
$$\ln|u|^\beta - \ln(1)^\beta \le \ln F(t,|u|) - \ln F(t,1)$$
$$\ln|u|^\beta \le \ln\Big(\frac{F(t,|u|)}{F(t,1)}\Big)$$
$$F(t,|u|) \ge F(t,1)|u|^\beta.$$

if $|u| \le 1$

$$\int_{|u|}^{1} \frac{\beta}{s} ds \le \int_{|u|}^{1} \frac{f(t,s)}{F(t,s)} ds$$
$$-\ln|u|^\beta \le \ln F(t,1) - \ln F(t,|u|)$$
$$\frac{1}{|u|^\beta} \le \frac{F(t,1)}{F(t,|u|)}$$
$$F(t,|u|) \le F(t,1)|u|^\beta.$$

So, we find the two inequalities

$$F(t,u) \le F(t,1)|u|^\beta \quad if\ 0 < |u| \le 1,$$

$$F(t,u) \ge F(t,1)|u|^\beta \quad if\ |u| \ge 1,$$

using the hypotheses, we obtain:

$$F(t,u) \le M|u|^\beta \quad if\ |u| \le 1. \tag{3.6}$$

$$F(t,u) \ge m|u|^\beta \quad if|u| \ge 1. \tag{3.7}$$

Like the real numbers, $M > 0$ and $m > 0$, and since $F(t,u) - m|u|^\beta$ is a function continues on $[0,T] \times [-1,1]$, therefore, there exists a constant $A > 0$ such that

$$F(t,u) \ge m|u|^\beta - A \quad \forall (t,u) \in [0,T] \times [-1,1], \tag{3.8}$$

By integration of equations (3.8) on $[0, T]$, we find:

$$F(t, u) \geq m|u|^\beta - A \quad \forall (t, u) \in [0, T] \times \mathbb{R}. \tag{3.9}$$

It is well known that for every finite-dimensional subspace $W \subset H_0^1(0, T)$ and every $u \in W$, there exists a constant $B > 0$ such that

$$\|u\|_p = \Big(\int_0^T |u(t)|^p dt \Big)^{\frac{1}{p}} \geq d\|u\|, \quad p \geq 1.$$

Then, for all $u \in W$, and according to (3.9) there exists a constant $B > 0$, such that

$$\int_0^T F(t, u(t))dt \geq \int_0^T (m|u|^\beta - A) \geq m\|u\|_\beta^\beta - AT$$
$$\geq mB^\beta \|u\|^\beta - AT.$$

□

Lemma 3.3. *Set* $m := \inf\{F(t, u) : t \in [0, T], |u| = 1\}$. *Then for every* $\xi \in \mathbb{R} \setminus \{0\}$ *and* $u \in H_0^1(0, T) \setminus \{0\}$ *we have*

$$\int_0^T F(t, \xi u(t))dt \geq m|\xi|^\mu \int_0^T |u(t)|^\mu dt - Tm. \tag{3.10}$$

Proof. Fix $\xi \in \mathbb{R} \setminus \{0\}$ and $u \in H_0^1(0, T) \setminus \{0\}$. Set $A = \{t \in [0, T] : |\xi u(t)| \leq 1\}$ and $B = \{t \in [0, T] : |\xi u(t)| \geq 1\}$ from the second inequality in (**??**) and (**??**), we obtain

$$\begin{aligned}
\int_0^T F(t, \xi u(t))dt &\geq \int_B F(t, \xi u(t))dt \\
&\geq \int_B F(t, \frac{\xi u(t)}{|\xi u(t)|})dt|\xi u(t)|^\mu dt \\
&\geq \int_B |\xi u(t)|^\mu dt = m \int_0^T |\xi u(t)|^\mu dt - m \int_A |\xi u(t)|^\mu dt \\
&\geq m|\xi|^\mu \int_0^T |u(t)|^\mu dt - Tm.
\end{aligned}$$

□

3.3 The existence of infinite solutions

In this section, we derive some sufficient conditions under which the function φ possesses infinitely many critical points, therefore,the impulsive problem (3.1) has infinitely many weak solutions.

Theorem 3.2. Suppose that **(H1)—(H3)** hold. If $f(t,u)$ and $I_j(u)$ are odd about u, then the impulsive problem (3.1) has infinitely many weak solutions.

Proof. Using the continuity of f and $I_j, j = 1, 2, ..., p$, we obtain that $\varphi(u)$ is continuously and differentiable.

In view of (3.2), it is obvious that $\varphi(0) = 0$ and $\varphi(u)$ is even. Indeed

If we put $s = -v, ds = -dv$. So,

$$\begin{aligned} F(t,-u) &= \int_0^{-u} f(t,v)dv = -\int_0^u f(t,-s)ds \\ &= \int_0^u f(t,s)ds = F(t,u), \quad \text{(because } f \text{ is odd)}. \end{aligned}$$

Also, we pose $A(u) = \displaystyle\int_0^u I_i(s)ds$, then we have

$$\begin{aligned} A(-u) &= \int_0^{-u} I_i(s)ds = -\int_0^u I_i(-v)dv \\ &= \int_0^u I_i(v)v = A(u). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \varphi(-u) = \frac{1}{2}\|-u\|^2 + \frac{\lambda}{2}\int_0^T (-u(t))^2 dt + \sum_{j=1}^p \int_0^{-u(t_j)} I_j(s)ds \\ - \int_0^T F(t,-u(t))dt \\ = \frac{1}{2}\|u\|^2 + \frac{\lambda}{2}\int_0^T u(t)^2 dt + \sum_{j=1}^p \int_0^{u(t_j)} I_j(s)ds - \int_0^T F(t,u(t))dt \\ = \varphi(u). \end{aligned}$$

By lemma (2.9), $\varphi(u)$ satisfies the (PS) condition. Combing (3.2) and the conditions **(H1)—(H3)**, one has

$$\begin{aligned}\varphi(u) &= L(u,u) + \sum_{j=1}^{p}\int_0^{u(t_j)} I_j(s)ds - \int_0^T F(t,u(t))dt \\ &\geq \theta_1\|u\|^2 - \sum_{j=1}^{p}\delta_j|u|^{\mu} - \int_0^T F(t,u(t))dt \\ &\geq \theta_1\|u\|^2 - \sum_{j=1}^{p}\delta_j\|u\|_{\infty}^{\mu} - \int_0^T F(t,u(t))dt \\ &\geq \theta_1\|u\|^2 - \sum_{j=1}^{p}\delta_j c^{\mu}\|u\|^{\mu}.\end{aligned}$$

Since $\mu > 2$, the above inequality implies that we can choose $\rho > 0$ small enough such that $\varphi(u) \geq \alpha > 0$ with $\|u\| = \rho$.

For every $r > 0$ and any finite dimensional subspace $W \in H_0^1(0,T)$, where c is defined in lemma (2.2) and θ_2 is defined in lemma (2.7).we have

$$\begin{aligned}\varphi(ru) &= L(ru,ru) + \sum_{j=1}^{p}\int_0^{ru(t_j)} I_j(s)ds - \int_0^T F(t,ru(t))dt \\ &\leq \theta_2\|ru\|^2 + \sum_{j=1}^{p}\int_0^{ru(t_j)} I_j(s)ds - \int_0^T (a|ru| + \frac{b}{\gamma+1}|ru|^{\gamma+1}) \\ &\leq \theta_2 r^2\|u\|^2 + \sum_{j=1}^{p}\int_0^{ru(t_j)} I_j(s)ds + T(ac\|ru\| + \frac{b}{\gamma+1}c^{\gamma+1}\|ru\|^{\gamma+1}) \qquad (3.11) \\ &\leq \theta_2 r^2\|u\|^2 - \sum_{j=1}^{p}k_j\|ru\|^{\mu} + T(acr\|u\| + \frac{b}{\gamma+1}c^{\gamma+1}r^{\gamma+1}\|u\|^{\gamma+1}) \\ &\leq \theta_2 r^2\,\|u\|^2 + T(acr\|u\| + \frac{b}{\gamma+1}c^{\gamma+1}r^{\gamma+1}\|u\|^{\gamma+1}) - r^{\mu}\sum_{j=1}^{p}k_j|u|^{\mu}\end{aligned}$$

for every $u \in W$. Taking $\tilde{e}(t) \in W$ such that $\|\tilde{e}(t)\| = 1$ where c is defined in lemma (2.2) and θ_2

is defined in lemma 2.7, one has

$$
\begin{aligned}
\varphi(r\tilde{e}(t)) &= L(r\tilde{e}, r\tilde{e}) + \sum_{j=1}^{p}\int_{0}^{r\tilde{e}(t_j)} I_j(s)ds - \int_0^T F(t, r\tilde{e}(t))dt \\
&\leq \theta_2\|r\tilde{e}\|^2 + \sum_{j=1}^{p}\int_{0}^{r\tilde{e}(t_j)} I_j(s)ds - \int_0^T (a|r\tilde{e}| + \frac{b}{\gamma+1}|r\tilde{e}|^{\gamma+1}) \\
&\leq \theta_2 r^2\|\tilde{e}\|^2 + \sum_{j=1}^{p}\int_{0}^{r\tilde{e}(t_j)} I_j(s)ds + T(ac\|r\tilde{e}\| + \frac{b}{\gamma+1}c^{\gamma+1}\|r\tilde{e}\|^{\gamma+1}) \\
&\leq \theta_2 r^2\|\tilde{e}\|^2 - \sum_{j=1}^{p} k_j\|r\tilde{e}\|^{\mu} + T(acr\|\tilde{e}\| + \frac{b}{\gamma+1}c^{\gamma+1}r^{\gamma+1}\|\tilde{e}\|^{\gamma+1}) \\
&\leq \theta_2 r^2\left\|\tilde{e}\right\|^2 + T(acr\left\|\tilde{e}\right\| + \frac{b}{\gamma+1}c^{\gamma+1}r^{\gamma+1}\left\|\tilde{e}\right\|^{\gamma+1}) - r^{\mu}\sum_{j=1}^{p} k_j|\tilde{e}|^{\mu} \\
&\leq \theta_2 r^2 + T(acr + bc^{\gamma+1}r^{\gamma+1}) - r^{\mu}\sum_{j=1}^{p} k_j|\tilde{e}(t)|^{\mu}
\end{aligned}
\tag{3.12}
$$

Since $\mu > 2, k_j > 0$ and $0 < \gamma < 1$, (3.12) implies that there exists $r_\epsilon > 0$ such that $\|r\tilde{e}(t)\| = r > \rho$ and $\varphi(r\tilde{e}(t)) < 0$ for every $r \geq r_\epsilon > 0$. Since W is a finite dimensional subspace, we can choose an $R = R(W) > 0$ such that $\varphi(u) \leq 0$ for any $u \in W$ with $\|u\| \geq R$.

According to theorem (1.14), the functional $\varphi(u)$ possesses infinitely many critical points, i.e., the impulsive problem (3.1) has infinitely many weak solutions. □

Example 3.1. Let $T > 0, k > 0, t_1 \in (0, T)$. Consider the following problem:

$$
\begin{cases}
-u''(t) + \lambda u(t) = -(1+3t)u^{1/3}, \quad t \in [0, T], \\
\Delta u'(t_1) = -ku^5(t_1), \\
u(0) = u(T) = 0,
\end{cases}
\tag{3.13}
$$

It is easy to know that $f(t, u) = -(1+3t)u^{\frac{1}{3}}, I_1(u) = -ku^5$. Let $\mu = 6, \delta_1 = \frac{1}{6}k$, the condition **(H1)** holds. $|f(t, u)| = |(1+3t)u^{\frac{1}{3}}| \leq (1+3T)|u|^{\frac{1}{3}}, F(t, u) \leq 0$ the condition **(H2)** and **(H3)** hold. Moreover, $f(t, u)$ and $I_j(u)$ are odd about u, so problem (3.13) has infinitely many weak solutions.

Remark 3.1. Since $I_1(u) = -ku^5$ does not satisfy the sublinear growth condition **(d1)**, so example 3.1 cannot determine the existence of solution in the paper [9, 10, 11, 12].

Theorem 3.3. Suppose that **(H1)** and the following condition hold.

(H4) There exists a constant $\beta \in (2, \mu]$ such that

$$0 < \beta F(t, u) \le u f(t, u)$$

for every $t \in [0, T]$ and $u \in R \backslash \{0\}$, where $F(t, u) = \int_0^u f(t, s) ds.$

Moreover, $f(t, u)$ and I_j are odd about u, then the impulsive problem (3.1) has infinitely many weak solutions.

Proof. Step1: Using the continuity of f and $I_j, j = 1, 2, \ldots, p$, we obtain that $\varphi(u)$ is continuously and differentiable.

In view of (3.2), it is obvious that $\varphi(u)$ is even and $\varphi(0) = 0$.

By Lemma 3.1, $\varphi(u)$ satisfies the (PS) condition.

Step 2: φ satisfies the first geometric condition:

For any $u \in H$, we know that $\|u\| \le \frac{1}{c} = \frac{1}{\sqrt{T}}$ implies $\|u\|_\infty \le 1$ by lemma 2.2, so when $\|u\| \le \frac{1}{c} = \frac{1}{\sqrt{T}}$, we have the following inequality,on has

$$\begin{aligned} \int_0^T F(t, u(t)) dt &\le \int_0^T M |u|^\beta \le M \int_0^T \|u\|_\infty^\beta \, dt \\ &\le M T c^\beta \|u\|^\beta, \quad \|u\| \le \frac{1}{c}. \end{aligned} \tag{3.14}$$

Combing (3.16) and condition **(H1)**, by lemma 2.7, one has

$$\begin{aligned} \varphi(u) &= L(u, u) + \sum_{j=1}^p \int_0^{u(t_j)} I_j(s) ds - \int_0^T F(t, u(t)) dt \\ &\ge \theta_1 \|u\|^2 - \sum_{j=1}^p \delta_j |u|^\mu - \int_0^T F(t, u(t)) dt \\ &\ge \theta_1 \|u\|^2 - \sum_{j=1}^p \delta_j \|u\|_\infty^\mu - \int_0^T F(t, u(t)) dt \\ &\ge \theta_1 \|u\|^2 - \sum_{j=1}^p \delta_j c^\mu \|u\|^\mu - M T c^\beta \|u\|^\beta. \end{aligned}$$

which implies that we can choose $\rho > 0$ small enough such that $\varphi(u) \ge \alpha > 0$ with $\|u\| = \rho$. Thus φ satisfies condition (i) of theorem 1.14. In the following, it is a question of verifying condition (i) of theorem 1.14.

Step3: φ satisfies the second geometric condition:

By **(H1)**, we have

$$\sum_{j=1}^{p}\int_0^{u(t_j)} I_j(s)ds \leq 0$$

In fact, we can obtain the following inequality by lemma 2.7,

$$\begin{aligned}\varphi(u) &= L(u,u) + \sum_{j=1}^{p}\int_0^{u(t_j)} I_j(s)ds - \int_0^T F(t,u(t))dt \\ &\leq \theta_2\|u\|^2 - \int_0^T F(t,u(t))dt \\ &\leq \theta_2\|u\|^2 - m(B^\beta\|u\|^\beta - AT) \leq \theta_2\|u\|^2 - mB^\beta\|u\|^\beta + AT.\end{aligned}$$

Noting that $\beta > 2$, the above inequality implies that $\varphi(u) \to -\infty$ as $\|u\| \to \infty$ with $u \in W$. Therefore, there exists $R = R(W)$ such that $\varphi(u) \leq 0$ on $W \setminus B_R$.

According to theorem1.14, the functional $\varphi(u)$ possesses infinitely many critical points, i.e., the impulsive problem (3.1) has infinitely many solutions. □

Example 3.2. Let $T > 0, a(t), b(t) \in C([0,T],(0,+\infty)), k > 0, t_1 \in (0,T)$. Consider the following problem:

$$\begin{cases} -u''(t) + \lambda u(t) = a(t)u^3(t) + b(t)u^5(t), \quad t \in [0,T], \\ \Delta u'(t_1) = -ku^9(t_1), \\ u(0) = u(T) = 0, \end{cases} \tag{3.15}$$

It is easy to know that $f(t,u) = a(t)u^3(t) + b(t)u^5(t), I_1(u) = -ku_9$. Let $\beta = 4, \mu = 10, \delta_1 = \frac{1}{10}k$ the condition **(H1)** and **(H4)** holds. $f(t,u), I_1(u)$ are odd about u, so problem (3.15) has infinitely many weak solutions.

Remark 3.2. Since $I_1(u) = -ku^9$ does not satisfy the sublinear growth condition **(d1)**, so example (3.2) cannot determine the existence of solution in the paper [9, 10, 11, 12].

Theorem 3.4 ([10, Theorem 1.3])**.** *Suppose that **(d1)** and the following conditions hold.*

***(d3)** There exists a constant $\beta > 2$ such that*

$$0 < \beta F(t,u) \leq uf(t,u) \ for\ every \quad t \in [0,T], u \in \mathbb{R}\setminus\{0\}$$

where $F(t,u)=\int_0^u f(t,s)ds.$

(d4) $\theta_1-\frac{T^2M}{\pi^2}-\sum_{j=1}^{p}Ta_j-\sum_{j=1}^{p}Tb_j>0,$

where $M=\sup\{F(t,u):t\in[0,T],|u|=1\}$; $\theta_1=\frac{1}{2}$ *for* $\lambda>0$ *and*
$\theta_1=\frac{1}{2}(1-\frac{T^2}{\pi^2}\lambda)$ *for* $-\frac{\pi^2}{T^2}<\lambda<0.$

Moreover, $f(t,u)$ *and the impulsive functions* I_j *are odd about* u*. Then the impulsive problem* (3.1) *has infinitely many weak solutions for* $\lambda>-\pi^2/T^2$.

Proof. The condition that f and I_j are odd functions implies that φ is even defined by 3.2. Moreover, by the assumptions of Theorem 3.4, we know that $\varphi\in C^1(H_0^1(0,T),\mathbb{R}),\varphi(0)=0$ (see proposition 2.2) and φ satisfies the Palais-Smale condition(See lemma 3.1).

To apply symmetric Mountain Pass Theorem (see Lemma 1.4), it suffices to prove that φ satisfies the conditions **(A1)** and **(A2)**.

Step 1: φ satisfies the first geometric condition **(A1)**

For any $u\in H$, we know that $\|u\|\leq\frac{1}{c}=\frac{1}{\sqrt{T}}$ implies $\|u\|_\infty\leq 1$ by lemma 2.2, we have the following inequality, on has

$$\int_0^T F(t,u(t))dt\leq M\int_0^T\|u\|_\infty^\beta\,dt\leq MTc^\beta\|u\|^\beta,\|u\|\leq\frac{1}{c}. \tag{3.16}$$

Combing (3.16) and condition **(H1)**, by lemma (2.7), one has

$$\begin{aligned}\varphi(u)&=L(u,u)+\sum_{j=1}^{p}\int_0^{u(t_j)}I_j(s)ds-\int_0^T F(t,u(t))dt\\&\geq\theta_1\|u\|^2-\sum_{j=1}^{p}\delta_j|u|^\mu-\int_0^T F(t,u(t))dt\\&\geq\theta_1\|u\|^2-\sum_{j=1}^{p}\delta_j\|u\|_\infty^\mu-\int_0^T F(t,u(t))dt\\&\geq\theta_1\|u\|^2-\sum_{j=1}^{p}\delta_jc^\mu\|u\|^\mu-MTc^\beta\|u\|^\beta.\end{aligned}$$

which implies that we can choose $\rho>0$ small enough such that $\varphi(u)\geq\alpha>0$ with $\|u\|=\rho$. Thus φ satisfies condition **(A1)** of lemma (1.4). In the following, it is a question of verifying condition **(A1)** of lemma (1.4).

Step2: φ satisfies the second geometric condition **(A2)**

It remains to prove that, for each finite dimensional subspace $V_1 \subset H_0^1(0,T)$, the set $\{x \in V_1 : \varphi(x) \geq 0\}$ is bounded. By (2.6) and (3.10), we have that for every $\xi \in \mathbb{R} \setminus \{0\}$ and $u \in V_1 \setminus \{0\}$, the following inequality

$$\begin{aligned}
\varphi(\xi u) =& \frac{1}{2}\int_0^T [(\xi u)^2 + \lambda(\xi u)^2]dt - \int_0^T F(t, \xi u)dt + \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(s)ds \\
&\leq \theta_2 \xi^2 \|u\|^2 - \int_0^T F(t, \xi u)dt + \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(s)ds \\
&\leq \theta_2 \xi^2 \|u\|^2 + |\xi|\sqrt{T}\|u\| \sum_{j=1}^{p} \left(a_j + b_j |\xi|^{\gamma_j} \sqrt{T^{\gamma_j}} \|u\|^{\gamma_j}\right) - m|\xi|^{\mu} \int_0^T |u(t)|^{\mu} dt + Tm
\end{aligned} \tag{3.17}$$

holds. Take $Q \in V_1$ such that $\|Q\| = 1$, since $\mu > 2, m > 0$ and $0 < \gamma_j < 1$, (3.17)implies that there exists $\xi_Q > 0$ such that $\|\xi Q\| > \rho$ and $\varphi(\xi Q) < 0$ for every $\xi \geq \xi_Q > 0$. Since V_1 is a finite dimensional subspace, we can choose an $r = r(V_1) > 0$ such that

$$\varphi(q) < 0, \forall q \in V_1 \setminus B_r.$$

According to lemma (1.4), the functional φ possesses infinitely many critical points, i.e. the impulsive problem (3.1) has infinitely many solutions. □

Remark 3.3. Study of the existence and multiplicity of solutions using the mountain pass lemma for somme boundary problems, whether on a bounded domain or not, is in diffusion among many researchers, where let us suppose that the member non-linear is verified the Ambrosetti-Rabinowitz condition in order to prove the Palais-Smale condition (PS), and for more information you can look at the references see the references, [18], [20] [25] and [26]

Conclusion

In this work, we studied some theories of critical points, and we studied an impulsive second order boundary problem posed on the bounded interval $[0, T]$ by variational methods, minimization theorem, mountain pass theorem, saddle point theorem and symmetric mountain pass theorem.

In the first chapter, we gave some basic tools, which are necessary for this work, such as the theory of critical points, the differentiability of an operator in the space of Banach, and mountain pass theorem, and also we have given some minimization theorems.

In second goal our objective was to show the existence of at most two solutions of impulsive second order equation by minimization theorm and mountain pass theorem.

Finally, we studied the existence and multiplicity of solutions for the problems with impulsive limits by the saddle point theorem and symmetric mountain pass theorem .

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Abstract

In this memoiry, we have studied an impulsive second-order boundary value problem on the bounded domain $[0, T]$, as well as the theory of critical points, the mountain pass theorem and the saddle point theorem. Our goal in this study was to apply the critical point theory, Mountain pass theorem, saddle point theorem and Symmetric Mountain Pass Theorem to verify the existence and multiplicity of solutions to the following impulsive second-order boundary value problem :

$$\begin{cases} -u''(t) + \lambda u(t) = f(t, u(t)), \quad t \in [0, T], \\ \Delta u'(t_j) = I_j(u(t_j)), \quad j = 1, 2, ..., p, \\ u(0) = u(T) = 0, \end{cases}$$

Key words : Variational method. Critical point.Existence of solutions. Second order. Impulsive differential equation. Mountain pass theorem. Palace-Smale Condition. Saddle point theorem.Symmetric Mountain Pass Theorem.

Résumé

Dans ce mémoire, nous avons étudié un problème aux limites de seconde ordre impulsif sur le domaine borné $[0, T]$, ainsi que la théorie des points critiques, lemme du col et théorème de points selle. Notre objectif dans cette étude était d'appliquer la théorie des points critiques, lemme du col , théorème du points selle et théorème du col symétrique pour vérifier l'existence et la multiplicité des solutions au problème aux limites du second ordre impulsif suivant :

$$\begin{cases} -u''(t) + \lambda u(t) = f(t, u(t)), \quad t \in [0, T], \\ \Delta u'(t_j) = I_j(u(t_j)), \quad j = 1, 2, ..., p, \\ u(0) = u(T) = 0, \end{cases}$$

Mots clés : Méthode variationnelle. Point critique. Existence de solution. Seconde ordre. Impulsif. Lemme de Col. Condition de Palais-Smale. Théorème de point selle.théorème du col symétrique.

ملخص

في هذه الذاكرة ، درسنا مسألة قيمة حد من الدرجة الثانية اندفاعية على المجال المحدود $[0, T]$, بالإضافة إلى نظرية النقاط الحرجة، توطئة ممر الجبل، ونظرية نقطة السرج. كان هدفنا في هذه الدراسة هو تطبيق نظرية النقاط الحرجة، و توطئة ممر الجبل، وونظرية نقطة السرج،و نظرية ممر الجبل المتماثل للتحقق من وجود وتعدد الحلول لمسألة قيمة حدية من الدرجة الثانية اندفاعية الآتية:

$$\begin{cases} -u''(t) + \lambda u(t) = f(t, u(t)), \quad t \in [0, T], \\ \Delta u'(t_j) = I_j(u(t_j)), \quad j = 1, 2, ..., p, \\ u(0) = u(T) = 0, \end{cases}$$

كلمات مفتاحية: طريقة التغاير .نقطة حرجة . وجود الحلول. الدرجة الثانية. المعادلة التفاضلية المندفعة. توطئة ممر الجبل. نظرية شرط بالي-سمال. نقطة السرج.نظرية ممر الجبل المتماثل.