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Title

Binary Operations On Bounded lattices

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Contents

إهداء

السلام و عليكم والصلاة و السلام على أشرف المرسلين أما بعد:

الحمد لله على التمام الحمد لله على البدء والختام، الحمد لله امتنانا وشكرا ما انتهى درب ولا ختم سعي إلا بفضله عز وجل.

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ذبيح كنزة : إلى روح أخي التي لم تفارقني أبدا " **نصير** " رحمه الله وأسكنه فسيح جناته و جميع موتانا و موتى المسلمين .

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Introduction

A binary operation on a non-empoty set S is a mapping of the elements of cartesian product $S \times S \ \ to \ \ S$. The history of binary operations is deeply intertwined with the development of mathematics and computing. Binary operations have ancient roots, with evidence of their use in ancient civilizations such as the Chinese, Babylonians, and Egyptians. However, these early operations often lacked the formalization seen in modern mathematics, the formalization of binary operations began with the work of George Boole in the mid-19th century. Boole developed a symbolic logic system, now known as Boolean algebra, which uses binary variables and binary operations such as AND, OR and XOR. After that, Binary operations gained significant importance with the advent of computing. In the mid-20th century, electronic digital computers became prevalent, relying on binary representation of data and binary operations for computation such as addition, subtraction, multiplication, and division are fundamental operations in computer arithmetic. Also in the mid-20th century, Claude Shannon's groundbreaking work on information theory further highlighted the importance of binary operations in communication and data transmission. Shannon showed how binary digits (bits) could represent information, leading to advances in coding theory and data compression. Today, binary operations continue to be crucial in various fields, including cryptography, data structures and algorithm, digital signal processing, and artificial intelligence. Binary operations form the backbone of modern computing systems, enabling the processing, storage, and transmission of vast amounts of information efficiently. Throughout history, binary operations have evolved from ancient mathematical concepts to the foundation of modern computing and information theory, and they have become the key notion in the definitions of groups, monoids, semigroups, rings, and in more algebraic structures studied in abstract algebra. Binary operations have become essential tools in bounded lattices and its applications, several notions and properties in [2, 5, 6, 7, 8, 12, 17, 18, 19, 20]. Furthermore, it is not surprising that binary operations with specific properties appear in several types. For instance, aggregation operations on bounded lattices see [7, 8, 12]. Also other types such as triangular norms and conorms (shortly t-norms and t-conorms respectively) are special aggregation operators see [2]. Uninorms and nullnorms are special aggregations and they generalize t-norms and t-conorms since their neutral element in the bounded lattice [6, 17, 20]. Also, Uni-nullnorms and null-uninorms are introduced by San et al. [16], which are special classes of 2-uninorms introduced by [1] (wich are comprised of two uninorms). The importance of this types operations is made apparent by their wide use not only in pure mathematics (in theories of fuzzy sets and of functional equations, measure and integration theory), but also in several applied fields such as operations research, computer, economic and social sciences as well as in other experimental areas of physics and natural sciences.

This memory organized on three chapters as follows:

In first chapter, we recall the necessary basic concept and properties of binary relation on a set, and partially ordered sets. Also we give some examples of binary relations in real life.

In the second chapter, we present the necessary basic concepts and general information on lattices, algebraic properties of some lattice classes.

In the third chapter, we give some specific binary operations on bounded lattices, and show their properties and types (aggregation, t-norms, t-conorms, uninorms, nullnorms, and uni-nullnorms, null-uninorms and we mentioned some applications of binary operation in various fields.

Chapter 1

Basic concepts on ordered sets

In this chapter, we recall the necessary basic concepts and properties of binary relations on a set and partially ordered sets. More information on this chapter can be fond in [3, 9, 10, 14, 15]
.

1.1 Notions of binary relations on a set

In this section, we recall the notion of binary relations on a set and analyze several associated properties. Also we mentioned some examples of binary relation in real life.

1.1.1 Binary relations on a set and their properties

Definition 1.1. *(**Sets**) [15, 9] A set S is any well-defined collection of objects called elements or members of the set. We write $x \in S$ to denote that x is an element of S. The notation $x \notin S$ denotes that x is not an element of S.*

Remark 1.1.

- *One way describing a set that has a finite number of elements is by listing the elements of the set between braces. For example:*

  (1) The set V of all vowels in the English alphabet can be written as $V = \{a, e, i, o, u\}$.

  (2) The set of all positive integers that are less than 4 can be written as $\{1, 2, 3\}$.

- *The order in with the elements of a set are listed is not important for example:* (2) *can be written* $\{1,3,2\}$ *or* $\{3,2,1\}$ *or* $\{3,1,2\}$*or* $\{2,1,3\}$ *and* $\{2,3,1\}$. *But the preferred writing is* $\{1,2,3\}$.

Definition 1.2. *(**Cartesian product**)[9] If A and B are two nonempty sets, we define the product between two sets which called the Cartesian product $A \times B$ as the set of all ordered pair (a,b) with $a \in A$ and $b \in B$, i.e,*

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}.$$

In addition, if $A = B$ then $A \times B$ denoted A^2. For example, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

Example 1.1. *[9] Let $A = \{1,2,3\}$ and $B = \{x,y\}$, then*

$$A \times B = \{(1,x),(1,y),(2,x),(2,y),(3,x),(3,y)\}$$

and

$$B \times A = \{(x,1),(x,2),(x,3),(y,1),(y,2),(y,3)\}.$$

In this example, we have $A \times B \neq B \times A$ then we conclude that the Cartesian product of sets does not necessarily commutative.

Theorem 1.1. *[9] For any two finite nonempty sets A and B, $\mid A \times B \mid = \mid A \mid \times \mid B \mid$, such that $\mid A \mid$ is the number of the elements of A, called cardinal of A.*

Definition 1.3. *(**Binary relations**) Let A and B be a nonempty sets, a binary relation R from A to B is a subset of the Cartesian product $A \times B$. If $R \subseteq A \times B$ and $(a,b) \in R$, we say that a is related to b by R and we can write $a\ R\ b$. If a is not related to b by R, we write $a \not R\ b$. Frequently, if A and B are equal, in this case we often say that $R \subseteq A \times A$ is a binary relation on A, instead of a binary relation from A to A.*

Example 1.2. *Let $A = \{1,2,3\}$ and $B = \{x,y\}$ be two finite sets, so $R = \{(1,x),(2,y),(3,x)\}$ is a binary relation from A to B. The fact that $(1,x) \in R$, we can write $1\ R\ x$ and we say that 1 is related to x by the binary relation R.*

Example 1.3. *We define on $A = \{1,2,3,4\}$ the following binary relation R (les then) on A as:*

$$aRb \text{ if and only if } a < b.$$

Then

$$R = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}.$$

Properties of relations:

Let R be a relation on S, we say that:

1. R is reflexive if $\forall x \in S : xRx$;
2. R is irreflexive if $\forall x \in S : x \not R x$;
3. R is symmetric if $\forall x, y \in S : xRy \Longrightarrow yRx$;
4. R is asymmetric if $\forall x, y \in S : xRy \Longrightarrow x \not R y$;
5. R is antisymmetric if $\forall x, y \in S : xRy \wedge yRx \Longrightarrow x = y$;
6. R is transitive if $\forall x, y, z \in S : xRy \wedge yRz \Longrightarrow xRz$.

Example 1.4. *Let $S = \{1, 2, 3, 4\}$, consider the following binary relations on S:*

- $R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$ *;*
- $R_2 = \{(1, 1), (1, 2), (2, 1)\}$*;*
- $R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$*;*
- $R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$*;*
- $R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$*;*
- $R_6 = \{(3, 4)\}$.

We can see that:

1. *R_1 is not reflexive, not irreflexive, not symmtric, not asymmetric and not antisymmetric;*
2. *R_2 is symmetric;*
3. *R_3 is reflexive and symmetric;*
4. *R_4 is irreflexive, antisymmetric and transitive;*
5. *R_5 is reflexive, antisymmetric and transitive;*
6. *R_6 is irreflexive, asymtric and transitive.*

1.1.2 Representation of binary relations

Representing relations using matrices:

Definition 1.4. *[15] A binary relation between two finite sets can be represented using a zero-one matrix. Suppose that R is a binary relation from $A = \{a_1, a_2, ..., a_m\}$ to $B = \{b_1, b_2, ..., b_n\}$. The relation R can be represented by the matrix $M_R = [m_{ij}]$, where*

$$m_{ij} = \begin{cases} 1 & if(a_i, b_i) \in R \\ 0 & if(a_i, b_i) \notin R \end{cases}.$$

In other words, the zero-one matrix representing R has a 1 as its (i, j) entry when a_i is related to b_j, and a 0 in this position if a_i is not related to b_i.

Example 1.5. *The following matrices are the associated matrices of the relations R_1, R_2, R_3, R_4, R_5 and R_6 given in Example 1.4:*

$$M_{R_1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, M_{R_2} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, M_{R_3} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$
$$M_{R_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, M_{R_5} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_{R_6} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Representing relations using digraphs:

Definition 1.5. *[9] If S is a finite set and R is a binary relation on S, we can also represent R pictorially as follows. Draw a small circle for each element of S and label the circle with the corresponding element of S. These circles are called vertexes. Draw an arrow, called an edge, form vertex s_i to vertex s_j if and only if $s_i R s_j$. The resulting pictorial representation of R is called a directed graph or digraph of R. Thus, if R is a binary relation on S, the edges in the digraph of R correspond exactly to the pair in R, and the vertexes correspond exactly to the elements of the set S. Sometimes, when we want to emphasize the geometric nature of* *some property of R, we may refer to the pairs of R themselves as edges and the elements S as vertexes.*

Example 1.6. *Let $S = \{1, 2, 3, 4\}$ be a finite set and R be a binary relation on S such that*

$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (4, 1)\}.$$

The digraph of R is shown in following figure

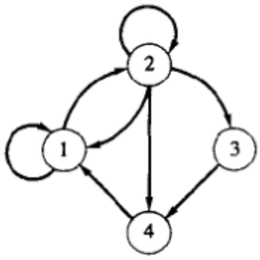


Figure 1.1: The directed graph of a binary relation.

Representing relations using cartesian plane graph:

Definition 1.6. *to represent a binary relation graphically, we can plot the ordered pairs of the relation as points on a coordinate plane, for example: Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4, 5, 6\}$, the relation R between sets A and B such that*

$$R = \{(a, 3), (a, 5), (b, 1), (c, 6), (d, 1), (d, 2), (d, 5), (d, 6)\},$$

we can visualize the relation R by the cartesian plane as follows:

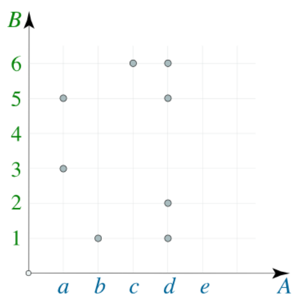


Figure 1.2

1.1.3 Real life examples of binary relations

Binary relation of siblings from the same parents:

For us, this relation is defined by the fact that the elements (individuals) x, y and z have the same parents A, B (see Figure 1.5). We will now examine the properties:

Reflexivity: (xRx) in practice for us this property means "to be our own sibling". Of course, this is not the case, so the relation is not reflexive. At the same time, we can say that this assumption never applies, and therefore the relation is **antireflexive**.

Symmetry: $(xRy \Rightarrow yRx)$ if individual x is a sibling of y, then individual y is a sibling of x. This assumption holds true, which also leads us to the fact that the session is **symmetric**.

Transitivity: $(xRy \wedge yRz \Rightarrow xRz)$ So if x is a sibling of y and at the same time y is a sibling of z, then it must hold that x is a sibling of z. therefore, we can say that this relation is also **transitive**. So, in this case we got a relation that is **antireflexive**, **symmetric** and **transitive**.

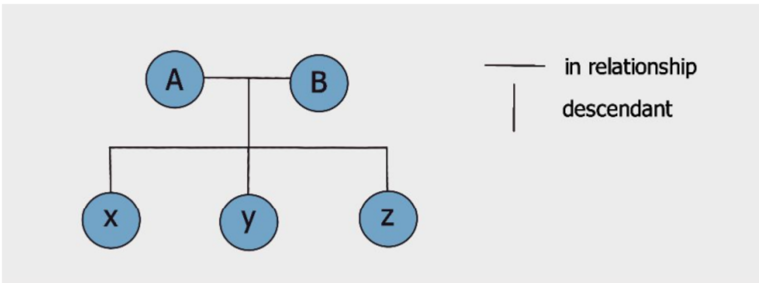


Figure 1.3

Public transport:

Under the term public transport, we therefore imagine a system of public passenger transport lines. In the Pilsen region, there would be 3 tram lines, 9 trolleybus lines, 27 bus lines and 9-night lines. This system then uses exactly 312 stops spread throughout the city and in its suburban areas. A relation defined by the term "able to transfer from line x to line y". Let us now determine the properties of this session.

Reflexivity: (xRx) so we are interested if we go some line x, then we can change to the same line x at some stop. Of course, as soon as we get off at a stop on line x, we can definitely get on the same line again at the same place (either if we got out of the car just to let the other passengers out, or if we want to wait for the next flight). For example, if we take line 12 and get off at the Mikulasska stop. We can take line 12 again at the same stop. The session is therefore **reflexive**.

Symmetry: ($xRy \Rightarrow yRx$) in our case, the principle is that if we can change from line x to line y, then we can also change from line y to line x at the same place. So as soon as line x stops at a stop where another line y stops, we can switch between them at will. There are 183 such stops where more than one line stops. So, for example, if we took tram line 4 and got off at the pod Zahorskem stop, we could change to tram line 1 and we could also change from line 1 to line 4 at the Pod Zahorskem stop. The relation is therefore **symmetric**.

Transitivity: ($xRy \wedge yRz \Rightarrow xRz$) we are interested in whether we can change from line x to line y and from line y to line z, if we can also switch from line x to line z. We refute this assumption if, for example, we choose bus line 32 for x, trolleybus line 16 for y and tram line 2 for z. We can change from line 32 to line 16 at two stops (U Luny and U Teplarny). From line 16 we can then change to tram 2 at the Hlavni nardrazi stop, but from line 32 we can never change directly to tram 2. The session is therefore not **transitive**. The relation is therefore **reflexive**, **symmetric** but not **transitive**.

Note this relation could be transitive if, for example, the predicate was "able to transfer from line x to line y at the same stop". In that case, all three elements would have to stop at the same stop and the session would be transitive. In this case, the relation would also be an equivalence relation. There are 123 such stops (including night lines) in Pilsen, where more than three lines stop.

Binary relation of game rock scissors paper:

The first game that demonstrates a binary relation is the old classic game "Rock, scissors, paper". It is a game for two or more players. The history of this game can be traced back to the 19th century in Japan. This game works on the principle of elimination. Players will show one of the symbols in sync with the countdown (stone - clenched fist, scissors - close hand with two raised fingers (index finger, middle finger), paper - open palm (see Figure 2). The rules are as follows: the stone reloads the scissors (blunts them), the scissors reload the paper (cuts it), and the paper reloads the stone (wraps it). One round beat, so the whole round is canceled, and all players remain playing.

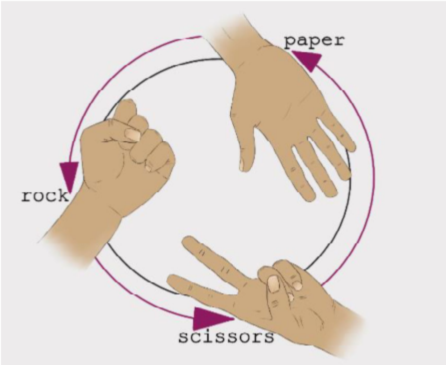


Figure 1.4

Now let's see what properties this binary relation has (x overrides y).

Reflexivity: (xRx) **Antisymetry:** $(xRy \wedge yRx \Rightarrow x = y)$ that is, the relation is symmetric only when the elements x and y are equal. But right part of this implication cannot happen. From the logic functions we know, that 0 (cannot happen) implicates 1 (true), which in the and means that this relation is **antisymetric**.

Transitivity: $(xRy \wedge yRz \Rightarrow xRz)$ if x overloads y and at the same time y overloads z, then it follows that x overloads z. As we mentioned, it is one-way circle, and therefore this property is also invalid. For example, paper overwhelms stone and stone overwhelms scissors, but paper does not overwhelm scissors. As a result, the "stone, scissors, paper" relation is a relation that is **antireflexive**, **antisymmetric** and not **transitive**.

1.2 Partially ordred sets

Definition 1.7. *[3] Let P be a set. An **order** (or **partial order**) on P is a binary relation $\leqslant$ on P such that, $\forall x, y, z \in P$ is:*

1. *reflexive* $(\forall x \in S : x \leqslant x)$*;*

2. *antisymmetric* $(\forall x, y \in S : x \leqslant y \wedge y \leqslant x \Longrightarrow x = y)$*;*

3. *transitive* $(\forall x, y, z \in S : x \leqslant y \wedge y \leqslant z \Longrightarrow x \leqslant z)$*.*

A set P equipped with an order relation $\leqslant$ is said to be an ***ordered set*** *(or* ***partially ordered set****), denoted by $(P, \leqslant)$. Some authors use the shorthand* ***poset****.*

Example 1.7. *In Example 1.4, we can see that R_5 is reflexive, antisymmetric and transitive, so it is an order on the set S and (S, R_5) is a poset.*

Example 1.8. *Let $\mathbb{Z}^+$ be the set of positive integers, the usual relation $\leqslant$ (less than or equal to) is an order on $\mathbb{Z}^+$. So $(\mathbb{Z}^+, \leqslant)$ is a poset.*

Example 1.9. *The divisibility relation $|$ is an order on the set of positive integers, because it is reflexive, antisymmetric and transitive. Then we have that $(\mathbb{Z}^+, |)$ is a poset.*

Example 1.10. *We show that the inclusion relation $\subseteq$ is an order on the power set of a set S, because $A \subseteq A$ whenever A is a subset of S, thus $\subseteq$ is reflexive. It is antisymmetric because if $A \subseteq B$ and $B \subseteq A$, then $A = B$. Finally, $\subseteq$ is transitive, because $A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C$. Hence, $(\mathcal{P}(S), \subseteq)$ is a poset.*

Definition 1.8 (**Hasse diagram**). *[10] A finite poset $(P, \leqslant)$ can be graphically represented by a Hasse diagram. The elements of P are represented as points in the plane and if $a \leqslant b$ with $a \neq b$ (in which case we write $a < b$), we draw b higher up than a, and connect a and b with line segment.*

Example 1.11. *The Hasse diagram of the poset $(\mathcal{P}(\{a, b, c\}), \subseteq)$ is shown in the bellow figure 1.5, where $\mathcal{P}(S)$ denoted the power set of S, i.e., the set of all subset of S.*

1.2.1 Particular elements of ordered sets

Definition 1.9. *Let $(P; \leqslant)$ be a poset and A be a subset of P. An element $x_0 \in P$ is called a lower bound of A if $x_0 \leqslant x$, for any $x \in P$. x_0 is called the greatest lower bound (or the infimum) of A if x_0 is a lower bound of A and $m \leqslant x_0$, for any lower bound m of A. Upper bound and least upper bound (or the supremum) are defined dually.*

Example 1.12. *Let $(P(\{a, b, c\}), \subseteq)$ be the poset given in Example 1.11. So the greatest lower bound of this poset is the subset $\{a, b, c\}$ and $\varnothing$ is its least upper bound.*

Definition 1.10. *A poset $(P, \leqslant)$ is called bounded, if it has a least and a greatest element respectively denoted by 0 and 1, i.e., $0 \leqslant x \leqslant 1$, for any $x \in P$. Usually, the notation $(P, \leqslant, 0, 1)$ is used to describe a bounded poset.*

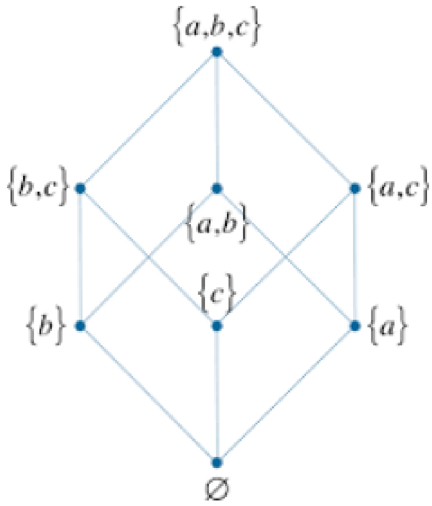


Figure 1.5: The Hasse diagram of $(\mathcal{P}(a,b,c),\subseteq)$.

Example 1.13. *Let $(P(\{a,b,c\}),\subseteq)$ be the poset given in Example 1.11, this poset has $\varnothing$ as the least element and $\{a,b,c\}$ as the greatest element. Indeed, $\varnothing$ contained in all the elements of $P(\{a,b,c\})$, and any element of $P(\{a,b,c\})$ contained in $\{a,b,c\}$. Thus the structure $(P(\{a,b,c\}),\subseteq,\varnothing,\{a,b,c\})$ is a bounded poset.*

1.2.2 Morphisms of ordered sets

Definition 1.11. *Let $(P_1,\leqslant_1)$, $(P_2,\leqslant_2)$ be posets and $f: P_1 \longrightarrow P_2$ be a map between P_1 and P_2. The map f is called an order-preserving (order-morphism) form P_1 to P_2 if for any $x,y \in P_1$:*

$$x \leqslant_1 y \Rightarrow f(x) \leqslant_2 f(y).$$

Example 1.14. *Let $(P_1 = \{1,2,3,6\},\leqslant_1)$ and $(P_2 = \{\varnothing,\{a\},\{b\},\{a,b\}\},\leqslant_2)$ be two posets such that $\leqslant_1$ is the divisibility order " $\mid$ " and $\leqslant_2$ is the inclusion order " $\subseteq$ ". Their Hasse diagrams are showen in the Figure 3.1. Let $f: P_1 \longrightarrow P_2$ be a mapping defined by the following table:*

x	1	2	3	6
$f(x)$	$\varnothing$	$\{a\}$	$\{b\}$	$\{a,b\}$

Thus, it is not difficult to see that f is an order-isomorphism (order-morphism and bejective).

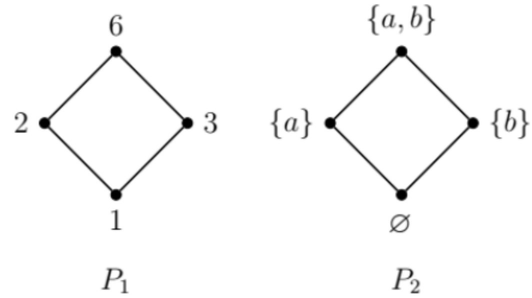


Figure 1.6

Chapter 2

Lattices

In this chapter, we recall necessary concepts and generalites on lattices and algebraic properties of some lattice classes. Further information on lattices can be found in [3, 9, 10, 14]

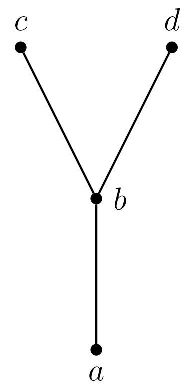
2.1 Generalities on Lattices

In this section, we give the notion of a lattice, ideal and filter in lattices, sub-lattices and lattice-morphisms.

2.1.1 Algebraic Structures of Lattices

Definition 2.1 (meet-semilattice). *[14] A poset $(P, \leq)$ is said to be a meet-semilattice, if for any pair of elements x and y of P, the greatest lower bound (infimum) of x and y exists in P and denoted $x \wedge y$.*

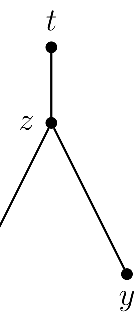
Example 2.1. *Let $(P = \{a, b, c, d\}, \leq)$ be a finite poset represented by its following Hasse diagram.*



This poset is a meet-semilattice, because the infimum $x \wedge y$ is exist in P, for any $x, y \in P$.

Definition 2.2 (join-semilattice). *[14] A poset $(P, \leq_p)$ is said to be a join-semilattice if for any pair of elements x and y of P, the least upper bound (supermum) of x and y exists and denoted $x \vee y$.*

Example 2.2. *Let $(P = \{a, b, c, d\}, \leqslant)$ be a finite poset represented by its following Hasse diagram.*



This poset is a join-semilattice, because the supermum $x \vee y$ is exist in P, for any $x, y \in P$.

Example 2.3. *Let $(P_1 = \{\{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \subseteq)$ and $(P_2 = \{\{1\}, \{2\}, \{1, 2, 3\}, \{1, 2, 4\}, \subseteq$ $)$ be two posts orders by the inclusion order and given by the Hasse diagrams in Figure 2.1. One can easily verify that $(P_1, \subseteq)$ is meet-semilattice but $(P_2, \subseteq)$ is not. Indeed, the two elements $\{1\}$ and $\{2\}$ have not an infimum in P_2 with respect to the inclusion order.*

Definition 2.3 (lattice structure). *[14] A poset $(P, \leq)$ is said to be a lattice if it is a join and meet-semilattice, i.e., for any pair of elements x and y of P, the greatest lower bound (infimum) $x \wedge y$ and the least upper bound (supermum) $x \vee y$ of x and y exist.*

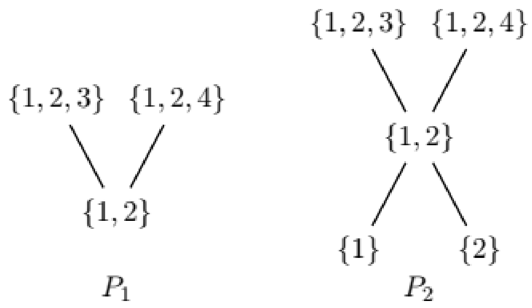


Figure 2.1: The Hasse diagram of P_1 and P_2.

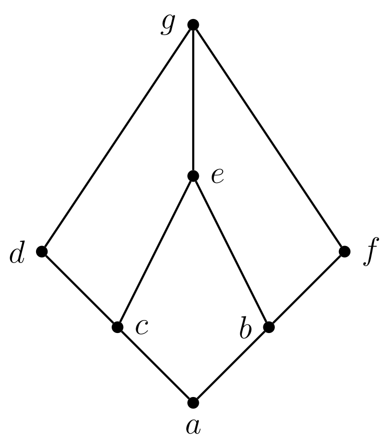
Corollary 2.1 (lattice structure). *A lattice structure is a poset $(P, \leq)$ in with every subset $\{x, y\}$ consisting of two elements x and y has a least upper bound denoted $x \vee y$ and a greatest lower bound denoted $x \wedge y$. Usually, the structure $(P, \leq, \wedge, \vee)$ is used to describe a lattice.*

Example 2.4. *1) The poset $(\mathbb{N}, \leqslant)$ ordered by the usual order is a lattice, where $x \wedge y = min(x, y)$ and $x \vee y = max(x, y)$, for any $x, y \in \mathbb{N}$.*

2) The poset $(\mathbb{N}^*, |)$ ordered by the divisibility order is a lattice, where $x \wedge y = gcd(x, y)$ and $x \vee y = lcm(x, y)$, for any $x, y \in \mathbb{N}^*$.

3) Let $(P(E); \subseteq)$ be the poset of all parts of the set E. This poset forms a lattice. Indeed, for all $A, B \in P(E)$, we have $A \wedge B = A \cap B$ and $A \vee B = A \cup B$.
The empty set is the smallest element of $P(E)$ and the set E is the greatest element of $P(E)$.

4) Let $P = (\{a, b, c, d, e, f, g\}, \leqslant)$ be the poset given by the bellow Hasse diagram. This poset has the structure of a lattice.



Theorem 2.1. *[10] Let* $(L, \leqslant, \wedge, \vee)$ *be a lattice. The operations* $\wedge$ *and* $\vee$ *have the following algebraic properties:*

- ***Idempotence:***

  $x \vee x = x$ *and* $x \wedge x = x$.

- ***The commutativity****:*

  $x \vee y = y \vee x$ *and* $x \wedge y = y \wedge x$.

- ***Associativity:***

  $x \vee (y \vee z) = (x \vee y) \vee z$ *and* $x \wedge (y \wedge z) = (x \wedge y) \wedge z$.

- ***Absorption lows****:*

  $x \vee (x \wedge y) = x$ *and* $x \wedge (x \vee y) = x$.

Proof.

- **Idempotence:** $x \wedge x = inf\{x, x\} = x$ and $x \vee x = sup\{x, x\} = x$.

  Then $x \wedge x = x = x \vee x$.

- **The commutativity**:

  $x \wedge y = inf\{x, y\} = inf\{y, x\} = y \wedge x$ and

  $x \vee y = sup\{x, y\} = sup\{y, x\} = y \vee x$.

- **Associativity:**

  We have to prove: $x \vee (y \vee z) = (x \vee y) \vee z$.

  Suppose that $T = x \vee (y \vee z)$ so $x \leq T$ and $y \vee z \leq T$, then $x \leq T, y \leq T$ and $z \leq T$, i.e., $x \vee y \leq T$ and $z \leq T$. so T is an upper bound of $\{(x \vee y), z\}$.

  Let M an upper bound of $\{(x \vee y), z\}$, so $(x \vee y) \leq M$ and $z \leq M$. Then $x \leq M, y \leq M$ and $z \leq M$. Thus $x \leq M$ and $(y \vee z) \leq M$. Hence, M is an upper bound of $\{x, (y \vee z)\}$. The fact that T is the smallest upper bound of $\{x, (y \vee z)\}$ implies that $T \leq M$. Then T is the smallest upper bound of $\{(x \vee y), z\}$. So $T = (x \vee y) \vee z$, therefore $x \vee (y \vee z) = (x \vee y) \vee z$.

  In similar way, we prove that $x \wedge (y \wedge z) = (x \wedge y) \wedge z$.

- **Absorption lows**:

  we prove that: $x \wedge (x \vee y) = x$.

  Obvious that $x \wedge (x \vee y) \leq x$ because $x \wedge (x \vee y)$ is a lower bound of $\{x, (x \vee y)\}$. Also,

$x \leq x$ and $x \leq x \vee y$, so x is a lower bound of $\{x, (x \vee y)\}$. Since $x \wedge (x \vee y)$ is the greatest lower bound of $\{x, (x \vee y)\}$. Then $x \leq x \wedge (x \vee y)$. Thus $x \wedge (x \vee y) = x$. In similar way, we prove that $x \vee (x \wedge y) = x$.

□

Definition 2.4. *[10] An algabraic lattice $(L, \wedge, \vee)$ is a set L equipped with two binary operations $\wedge$ (**meet**) and $\vee$ (**join**) which satisfy the following laws for all $x, y, z \in L$:*

Commutative law:

1) $x \wedge y = y \wedge x$ *and* $x \vee y = y \vee x$

   Associative law:

2) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ *and* $x \vee (y \vee z) = (x \vee y) \vee z$.

   Absorption law:

3) $x \wedge (y \vee z) = x$ *and* $(x \vee y) \wedge z = x$.

   Idempotent law:

4) $x \wedge x = x$ *and* $x \vee x = x$.

In this case, the unique ordered relation with respect to those binary operations "$\wedge$" and "$\vee$" on L is defined as

$$x \leqslant y \text{ if an only if } x \wedge y = x \text{ if an only if } x \vee y = y.$$

2.1.2 Ideals and filters of lattices

Definition 2.5 (Filters). *[3] Let $(L, \leqslant, \wedge, \vee)$ be a lattice. We call a filter of L any non-empty subset F of L verifying:*

1. *If $x \in F$ and $x \leqslant y$, then $y \in F$;*

2. *If $x \in F$ and $y \in F$, then $x \wedge y \in F$.*

Definition 2.6 (Ideals). *[3] Let $(L, \leqslant, \wedge, \vee)$ be a lattice. We call an ideal of L any non-empty subset I of L verifying:*

1. *If $x \in I$ and $y \leqslant x$, then $y \in I$;*

2. *If $x \in I$ and $y \in I$, then $x \vee y \in I$.*

Remark 2.1. *An ideal or filter is called **proper** if it does not coincide with L. It is a very easy to show that an ideal I of a lattice with 1 is proper if and only if $1 \notin I$. Dually, a filter F of a lattice with 0 is proper if and only if $0 \notin F$.*

Definition 2.7. *For each $a \in L$:*

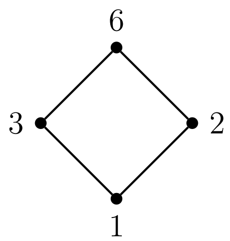
1. *the set $\downarrow a = \{x \in L \mid x \leqslant a\}$ is an ideal, it is known as the **principal ideal** generated by a;*

2. *the set $\uparrow a = \{x \in L \mid a \leqslant x\}$ is a filter, it is known as the **principal filter** generated by a.*

Remark 2.2. *[3] In a finite lattice, all its ideals and filters are principals.*

Example 2.5. *Let $(D(6), |, gcd, lcm)$ be the lattice of the positive divisors of 6 ordered by the divisibility order. Then it holds that*

- *The principal filter generated by 3 is $F_3 = \{3, 6\}$.*
- *The principal ideal generated by 3 is $I_3 = \{1, 3\}$.*



Example 2.6. *Let $(D(30), |, gcd, lcm)$ be the lattice of the positive divisors of 30 ordered by the divisibility order, and represented by the Hasse diagram in Figure.*

1. *Let $F_1 = \{2, 6, 10, 30\}$ and $F_2 = \{2, 10, 30\}$ be to subsets of $D(30)$. Then it holds that*
   - *F_1 is a filter of $D(30)$.*
   - *F_2 is not a filter of $D(30)$. Indeed, $2 \in F_2$ and $2 \mid 6$, but $6 \notin F_2$.*

2. *Let $I_1 = \{1, 3, 5, 15\}$ and $I_2 = \{5, 10, 15, 30\}$ be two subsets of $D(30)$. Then it holds that*
   - *$I_1 = \{1, 3, 5, 15\}$ is an ideal of (30).*

- $I_2 = \{5, 10, 15, 30\}$ *is not ideal of* (30). *Indeed, we have* $10 \in I_2$ *and* $2 \mid 10$, *but* $2 \notin I_2$.

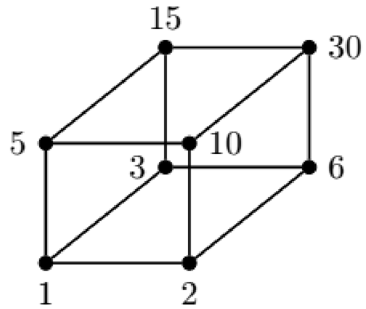


Figure 2.2: The Hasse diagram of $D(30)$.

Definition 2.8 (**maximal filters**). *[14] Let* $(L, \leqslant, \wedge, \vee)$ *be a lattice. A proper filter* F *of* L *(i.e.,* $F \subsetneq L$*) is said to be maximal (or ultra filter) if for any proper filter* F' *of* L*, it has* $F \subseteq F'$ *implies* $F' = F$.

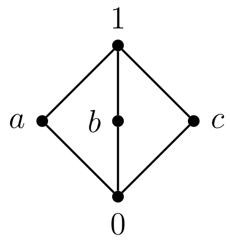
Definition 2.9 (**maximal ideals**). *[14] Let* $(L, \leqslant, \wedge, \vee)$ *be a lattice. A proper ideal* I *of* L *(i.e.,* $I \subsetneq L$*) is said to be maximal if for any proper ideal* I' *of* L*, it has* $I \subseteq I'$ *implies* $I' = I$.

Example 2.7. *Let* $D(30) = \{1, 2, 3, 5, 6, 10, 15, 30\}$ *be the lattice given in Example 2.6. Then* $D(30)$ *has three maximal filters and three maximal ideals are:*

- $F_2 = \{x \in D(30) : 2 \mid x\} = \{2, 6, 10, 30\}$;
- $F_3 = \{x \in D(30) : 3 \mid x\} = \{3, 6, 15, 30\}$;
- $F_5 = \{x \in D(30) : 5 \mid x\} = \{5, 10, 15, 30\}$;
- $I_6 = \{x \in D(30) : x \mid 6\} = \{1, 2, 3, 6\}$;
- $I_{10} = \{x \in D(30) : x \mid 10\} = \{1, 2, 5, 10\}$;
- $I_{15} = \{x \in D(30) : x \mid 15\} = \{1, 3, 5, 15\}$.

Definition 2.10. *[10] Let* $(L, \leqslant, \wedge, \vee, 0)$ *be a lattice with a smallest element* 0*. An element* $a \in L$ *is called atom if it covers* 0*, i.e., if* $0 < b \leqslant a$ *implies* $b = a$*, for any* $b \in L$.

Example 2.8. *Let $L = \{0, a, b, c, 1\}$ be the lattice given by the bellow Hasse diagram. This lattice has three atoms are a, b and c.*



Theorem 2.2. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $\alpha \in L$ be an atom. Then the principal filter F_α generated by α is maximal.*

Proof. Let α be an atom of L and F_α be the principal filter generated by α. We would like to prove that the filter F_α is maximal. Let F be a filter of L such that $F_\alpha \subsetneq F$. Then there exists $\beta \in F$ and $\beta \notin F_\alpha$. So we have two possible cases are $\beta < \alpha$ implies $\beta = 0$, so $F = L$ or β is also an atom, then $\alpha \wedge \beta = 0 \in F$. Thus $F = L$. Therefore, F_α is maximal. □

Definition 2.11 (Prime filters). *[14] A proper filter F of a lattice $(L, \leqslant, \wedge, \vee)$ is said to be prime if:*

$$a \vee b \in F \text{ implies } a \in F \text{ or } b \in F, \text{ for any } a, b \in L.$$

Definition 2.12 (Prime ideals). *[14] A proper ideal I of a lattice $(L, \leqslant, \wedge, \vee)$ is said to be prime if:*

$$a \wedge b \in I \text{ implies } a \in I \text{ or } b \in I, \text{ for any } a, b \in L.$$

Example 2.9. *Let $F_3 = \{3, 6, 15, 30\}$ and $F_{15} = \{15, 30\}$ be two filters of $D(30)$. Let $I_{15} = \{1, 3, 5, 15\}$ and $I_2 = \{1, 2\}$ be two ideals of $D(30)$. Then it holds that:*

- *F_3 is a prime filter of $D(30)$;*
- *F_{15} is not a prime filter of $D(30)$. Indeed, $3 \vee 5 = lcm(3, 5) = 15 \in F_{15}$, but $3 \notin F_{15}$ and $5 \notin F_{15}$;*
- *I_{15} is a prime ideal of $D(30)$;*
- *I_2 is not a prime ideal of $D(30)$. Because $6 \wedge 10 = gcd(6, 10) = 2 \in I_2$, but $6 \notin I_2$ and $10 \notin I_2$;*

Proposition 2.1 (**Characterization of ultrafilter**). *[13]*
Let $(L, \leqslant, \wedge, \vee)$ be a lattice with a smallest element 0 and F be a proper filter of L. Then the following two assertions are equivalent:

1. *F is an ultra filter.*
2. *For all $x \notin F$, there exists $y \in F$ such that $x \wedge y = 0$.*

Proof.
- For the direct implication assume that F is an ultra filter of L. Suppose that there exists $x \notin F$ such that for all $y \in F, x \wedge y \neq 0$.
  Let $G = F \cup \{x\}$ and $a_1, ..., a_n$ be elements of F. We put $a = x \wedge a_1 \wedge ... \wedge a_n$. Then $a = x \wedge y$ with $y = a_1 \wedge ... \wedge a_n \in F$. Thus $a \notin F$. By assuming a proper filter F_G we have $F \subsetneq G \subseteq F_G$ which contradicts the maximization of F.
- Conversely, we suppose that F is not an ultra filter and for all $x \notin F$, there exists $y \in F$ such that $x \wedge y = 0$. Then there exists F' a proper filter of L such that $F \subsetneq F'$. Hence there exists $x \in F'$ and $x \notin F$. Therefore there exists $y \in F$ such that $x \wedge y = 0$ with x and y belong to $F^{'}$. Thus $0 \in F^{'}$ which contradicts the fact that F is proper. Consequently, F is maximal.

□

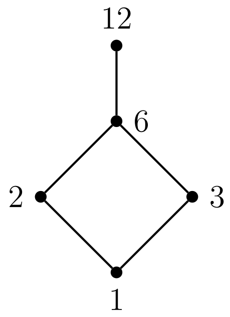
Proposition 2.2 (**Characterization of maximal ideals**). *[13]*
Let $(L, \leqslant, \wedge, \vee)$ be a lattice with a greatest element 1 and I be a proper ideal of L. Then the following two assertions are equivalent:

1. *I is a maximal ideal.*
2. *For all $x \notin I$, there exists $y \in I$ such that $x \vee y = 1$.*

2.1.3 Sub-lattices and lattice morphisms

Definition 2.13 (**Sublattices**). *[14] A sublattice of a lattice L is a subset S such that $x \wedge y$ and $x \vee y$ are in S for all $x, y \in S$.*

Example 2.10. *Let $L = \{1, 2, 3, 6, 12\}$ be the lattice given by the Hasse diagram in the above* *figure ordered by the divisibility order.*



Let $S_1 = \{1, 2, 3, 6\}$ *and* $S_2 = \{1, 2, 3, 12\}$ *be two subsets of* L. *Then* S_1 *is a sublattice of* L, *but* S_2 *is not. Because* $2, 3 \in S_2$ *and* $2 \vee 3 = 6 \notin S_2$.

Definition 2.14 (lattice morphisms). *[14] Let* $(L_1, \leqslant_1, \wedge_1, \vee_1)$ *and* $(L_2, \leqslant_2, \wedge_2, \vee_2)$ *be two lattices. A mapping* $f : L_1 \longrightarrow L_2$ *verifies the two conditions*

$$f(a \wedge_1 b) = f(a) \wedge_2 f(b) \text{ and } f(a \vee_1 b) = f(a) \vee_2 f(b), \text{ for any } a, b \in L_1$$

is called a lattice morphism.

Example 2.11. *[4] Let* $D_6 = \{1, 2, 3, 6\}$ *and* $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$ *be two lattices defined by ordered by the divisibility order, and* $f : D_6 \longmapsto D_{30}$ *be a mapping defined in the following table.*

x	1	2	3	6
f(x)	1	2	5	10

The mapping f *is a lattice-morphism. Indeed, let* $x, y \in L$, *we have:*

(1) if $(x, y) = (1, 1)$,*then*

$$f(1 \wedge 1) = f(1) \wedge f(1) = 1 \wedge 1 = 1 \text{ and } f(1 \vee 1) = f(1) \vee f(1) = 1 \vee 1 = 1$$

(2) if $(x, y) = (1, 2)$, *then*

$$f(1 \wedge 2) = f(1) = 1 = f(1) \wedge f(2) = 1 \wedge 2 \text{ and } f(1 \vee 2) = f(2) = 2 = f(1) \vee f(2) = 1 \vee 2;$$

(3) if $(x, y) = (1, 3)$, *then*

$$f(1 \wedge 3) = f(1) = 1 = f(1) \wedge f(3) = 1 \wedge 5 \textit{ and } f(1 \vee 3) = f(3) = 5 = f(1) \vee f(3) = 1 \vee 5;$$

(4) *if* $(x, y) = (1, 6)$, *then*

$$f(1 \wedge 6) = f(1) = 1 = f(1) \wedge f(6) = 1 \wedge 10 \textit{ and}$$
$$f(1 \vee 6) = f(6) = 10 = f(1) \vee f(6) = 1 \vee 10;$$

(5) *if* $(x, y) = (2, 2)$, *then*

$$f(2 \wedge 2) = f(2) = 2 = f(2) \wedge f(2) = 2 \wedge 2 \textit{ and } f(2 \vee 2) = f(2) = 2 = f(2) \vee f(2) = 2 \vee 2;$$

(6) *if* $(x, y) = (2, 3)$, *then*

$$f(2 \wedge 3) = f(1) = 1 = f(2) \wedge f(3) = 2 \wedge 5 \textit{ and}$$
$$f(2 \vee 2) = f(6) = 10 = f(2) \vee f(3) = 2 \vee 5;$$

(7) *if* $(x, y) = (2, 6)$, *then*

$$f(2 \wedge 6) = f(2) = 2 = f(2) \wedge f(6) = 2 \wedge 10 = 2 \textit{ and}$$
$$f(2 \vee 6) = f(6) = 10 = f(2) \vee f(6) = 2 \vee 10;$$

(8) *if* $(x, y) = (3, 3)$, *then*

$$f(3 \wedge 3) = f(3) = 5 = f(3) \wedge f(3) = 5 \wedge 5 = 5 \textit{ and}$$
$$f(3 \vee 3) = f(3) = 5 = f(3) \vee f(3) = 5 \vee 5;$$

(9) *if* $(x, y) = (3, 6)$, *then*

$$f(3 \wedge 6) = f(3) = 5 = f(3) \wedge f(6) = 5 \wedge 10 = 5 \textit{ and}$$
$$f(3 \vee 6) = f(6) = 10 = f(3) \vee f(6) = 5 \vee 10.$$

2.2 Algebraic properties of some classes of lattices

In this section, we recall some algebraic properties of some lattice classes as distributive, modular and bounded lattices.

2.2.1 Distributive lattices

Definition 2.15. *[14] A lattice $(L, \wedge, \vee)$ is distributive if one of the two equivalent conditions is satisfied, for any $x, y, z \in L$:*

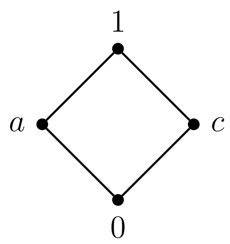
$$\text{(D1) } x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$
$$\text{(D2) } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Example 2.12. *1) The lattice $(\mathbb{N}, \leqslant)$ ordered by the usual order is distributive.*

2) The lattice $(\mathbb{N}^*, |)$ ordered by the divisibility order is distributive.

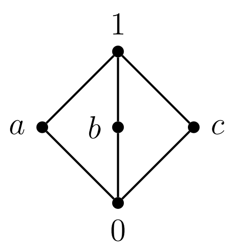
3) Let $(P(E); \subseteq)$ be the lattice of all parts of the set E. This lattice is distributive.

4) Let $(L = \{0, a, c, 1\}, \leqslant, \wedge, \vee)$ be a lattice. Then L is distributive.

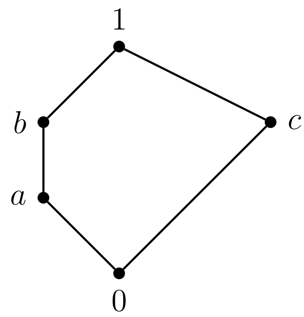


Example 2.13. *Here we give examples of non-distributive lattices.*

1. *Let $(M_3 = \{0, a, b, c, 1\}, \leqslant, \wedge, \vee)$ be the diamond lattice, then L is not distributive. Indeed, $a \wedge (b \vee c) = a \wedge 1 = a$, but $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$. So $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$.*



2. *Let $(N_5 = \{0, a, b, c, 1\}, \leqslant, \wedge, \vee)$ be the pentagon lattice is also not distributive, because $b \wedge (a \vee c) = b \wedge 1 = b$ and $(b \wedge a) \vee (b \wedge c) = a \vee 0 = a$. Then $b \wedge (a \vee c) \neq (b \wedge a) \vee (b \wedge c)$.*



Remark 2.3. *Here, we show that the two conditions* $(D1)$ *and* $(D2)$ *are equivalent. Indeed, suppose that* $(D2)$ *is verified and we prove that* $(D1)$ *is also verified. We can write:*

$$\begin{aligned}(x \vee y) \wedge (x \vee z) &= ((x \vee y) \wedge x) \vee ((x \vee y) \wedge z) \\ &= x \vee ((x \vee y) \wedge z) \quad (\textit{absorption laws}) \\ &= x \vee (x \wedge z) \vee (y \wedge z) \\ &= x \vee (y \wedge z) \quad (\textit{absorption laws}).\end{aligned}$$

The reciprocal, suppose that (2) *holds, we can write:*

$$\begin{aligned}(x \wedge y) \vee (x \wedge z) &= ((x \wedge y) \vee x) \wedge ((x \wedge y) \vee z) \\ &= x \wedge (z \vee (x \wedge y)) \quad (\textit{absorption law}) \\ &= x \wedge (z \vee x) \wedge (z \vee y) \\ &= x \wedge (y \vee z) \quad (\textit{absorption law}).\end{aligned}$$

Theorem 2.3. *[3] Let* $(L, \wedge, \vee)$ *be a lattice and* a, b, c *be three elements of* L*. Then*

(i) $a \wedge (b \vee c) \geqslant (a \wedge b) \vee (a \wedge c)$;

(ii) $a \vee (b \wedge c) \leqslant (a \vee b) \wedge (a \vee c)$;

(iii) $a \geqslant c$ *implies* $a \wedge (b \vee c) \geqslant (a \wedge b) \vee c$;

(iv) $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leqslant (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$.

Theorem 2.4. *[3] Let* $(L, \wedge, \vee)$ *be a lattice. Then* L *is distributive if and only if it has not a copy of one of the two sub-lattices* M_3 *or* N_5*.*

Theorem 2.5. *[3] Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice with a greatest element 1. Then every maximal ideal in L is prime. Dually, in a distributive lattice with smallest element 0 every ultrafilter is prime.*

Proof. Let I be a maximal ideal in L and $a, b \in L$. Assume $a \wedge b \in I$ and $a \notin I$. Then we require $b \in I$. Define $I' = \{a \vee x \mid x \in L\}$. Thus I' is an ideal containing a, i.e., $a = a \vee 1$. because $I$is maximal, we have $I_a = L$. In particular $1 \in I$ So $1 = a \vee d$ for some $d \in I$. Then $I \ni (a \wedge b) \vee d = (a \vee d) \wedge (b \vee d) = b \vee d$. Sine $b \leqslant b \vee d. we have b \in I$. □

2.2.2 Modular lattices

Definition 2.16. *[14] A lattice $(L, \wedge, \vee)$ is said to be modular if for all $x, y, z \in L$:*

$$x \leqslant z \text{ implies } x \vee (y \wedge z) = (x \vee y) \wedge z.$$

(Note for any lattice, if $x \leqslant z$, then $x \vee (y \wedge z) \leqslant (x \vee y) \wedge z)$).

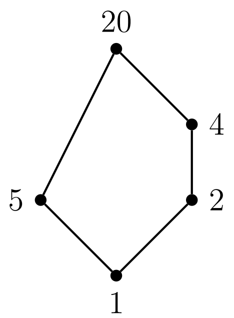
Theorem 2.6. *[14] Every distributive lattice is modular.*

Proof. Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and $a, b, c \in L$ such that $a \leqslant c$, we have $a \vee c = c$. Then

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = (a \vee b) \wedge c.$$

Hence L is modular. □

Example 2.14. *(1) Let $N_5 = \{1, 2, 4, 5, 20\}$ be the pentagon lattice ordered by the divisibility order. The fact that $2 \leq 4$ but $2 \vee (5 \wedge 4) = 2 \vee 1 = 2$ and $(2 \vee 5) \wedge 4 = 20 \wedge 4 = 4$. Then $2 \vee (5 \wedge 4) \neq (2 \vee 5) \wedge 4$. Thus, N_5 is not modular.*

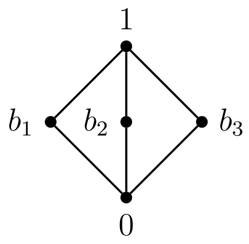


(2) Let $(M_3 = \{0, b_1, b_2, b_3, 1\}, \leqslant, \wedge, \vee)$ *be the diamond lattice. Then L is not distributive but it is modular. Indeed, let* $a = 0, c = b_1$

Then $a \vee (b \wedge c) = 0 \vee (b \wedge b_1) = b \wedge b_1$

and $(a \vee b) \wedge c = (0 \vee b) \wedge b_1 = b \wedge b_1$

Let $a = b_1, c = 1$, *then* $a \vee (b \wedge c) = b_1 \vee (b \wedge b1) = b_1 \vee b$ *and* $(a \vee b) \wedge c = (b_1 \vee b) \wedge c = b_1 \vee b$.



Theorem 2.7. *[3] Let* $(L, \wedge, \vee)$ *be a lattice. Then L is modular if and only if it has not a copy of the pentagon sub-lattice* N_5.

Definition 2.17. *[10] A lattice* $(L, \leqslant, \wedge, \vee)$ *is called bounded if it has a smallest element* 0 *and a greatest element* 1, *i.e.,* $0 \leqslant x \leqslant 1$, *for all* $x \in L$.

Theorem 2.8. *Any finite lattice is bounded.*

Proof. Let $(L, \leqslant, \wedge, \vee)$ be a finite bounded lattice, then $L = \{x_1, x_2, \cdots, x_n\}$. Thus, $x_1 \wedge x_2 \wedge \cdots \wedge x_n$ is the smallest element of L and $x_1 \vee x_2 \vee \cdots \vee x_n$ is the greatest element of L. □

Chapter 3

Binary operations on bounded lattices

In this chapter, we present some properties of a given binary operation on a lattice. Importantly, aggregation operations, triangular norms and conorms, uninorms and nullnorms then uni-nullnorms and null-uninorms on bounded lattices. We finish this chapter by providing some applications of binary operations in various fields. Additional basic concepts can be fond in [18, 19, 2, 17, 20, 6, 5, 8, 7, 12].

3.1 Definitions and properties

In this section, we present some basic definitions and properties of binary operations on bounded lattices. More information, we refer to [18, 19].

Definition 3.1. *[18] A binary operation on a non-empty set X is any function from the Cartesian product $X \times X$ into X, i.e., $F : X^2 \to X$. A binary operation F on X is called:*

(1) commutative, if $F(x, y) = F(y, x)$, for any $x, y \in X$;

(2) associative, if $F(x, F(y, z)) = F(F(x, y), z)$, for any $x, y, z \in X$;

An element $e \in X$ is called:

(i) a left-neutral (resp. right-neutral) element of F, if $F(e, x) = x$ (resp. $F(x, e) = x$), for any $x \in X$;

(ii) a neutral element of F, if it is left- and right-neutral element, i.e., $F(e, x) = F(x, e) = x$, for any $x \in X$;

(iii) *a left- (resp. right-) absorbing element of* F*, if* $F(e,x) = e$ *(resp.* $F(x,e) = e$*), for any* $x \in X$*;*

(iv) *an absorbing element of* F*, if it is left- and right-absorbing element, i.e.,* $F(e,x) = F(x,e) = e$*, for any* $x \in X$*.*

Example 3.1. *[18] On the set of real numbers* $\mathbb{R}$*, the binary operations of addition* $(+)$ *and multiplication* $(\times)$ *are commutative and associative. Where,* 0 *is the neural element of* $(+)$ *and* 1 *is the neural element of* $(\times)$*. Also,* 0 *is the absorbing element of* $(\times)$*. But* $(+)$ *has not an absorbing element.*

Definition 3.2. *[18] Let* $(L, \leqslant, \wedge, \vee)$ *be a lattice and* F *be a binary operation on* L*.* F *is called:*

(1) *idempotent, if* $F(x,x) = x$*, for any* $x \in L$*;*

(2) *conjunctive (resp. disjunctive), if* $F(x,y) \leqslant x \wedge y$ *(resp.* $x \vee y \leqslant F(x,y)$*), for any* $x, y \in L$*;*

(3) *left-increasing (resp. right-increasing), if* $x \leqslant y$*, implies* $F(x,z) \leqslant F(y,z)$ *(resp.* $F(z,x) \leqslant F(z,y)$*), for any* $x, y, z \in L$*;*

(4) *increasing, if* $x_1 \leqslant x_2$ *and* $y_1 \leqslant y_2$ *imply* $F(x_1,y_1) \leqslant F(x_2,y_2)$*, for any* $x_1, y_1, x_2, y_2 \in L$*.*

We note that F *is increasing if and only if it is left- and right-increasing. Also, if* F *is commutative, then* F *is left-increasing if and only if* F *is right-increasing.*

Example 3.2. *[18] Let* $(L, \leqslant, \wedge, \vee)$ *be a lattice. The meet* $(\wedge)$ *and the join* $(\vee)$ *operations of* L *are idempotent and increasing. Moreover,* $\wedge$ *is conjunctive and* $\vee$ *is disjunctive.*

Proposition 3.1. *[19] (**Uniqueness**) Let* $(L, \leqslant, \wedge, \vee)$ *be a lattice and* F *be a conjunctive or a disjunctive binary operation on* L*. If* F *has a left- (resp. right-) neutral element* $e \in L$*, then* e *is unique.*

Proof. Let $e_1, e_2 \in L$ be two left- (resp. right-) neutral elements of F . Here, we discuss the following two possible cases:

(i) if F is conjunctive, then it holds that $e_2 = F(e_1, e_2) \leqslant e_1$ *and* $e_1 = F(e_2, e_1) \leqslant e_2$ (resp.$e_2 = F(e_2, e_1) \leqslant e_1$ *and* $e_1 = F(e_1, e_2) \leqslant e_2$). *Hence*, $e_1 = e_2$.

(ii) if F is disjunctive, the proof is analogous to that of (i)

□

Proposition 3.2. *[19] Let $(L, \leqslant, \wedge, \vee, 0, 1)$ be a bounded lattice and F be a binary operation on L.*

(i) If F is conjunctive and has a left- or a right-neutral element $e \in L$, then $e = 1$;

(ii) If F is disjunctive and has a left- or a right-neutral element $e \in L$, then $e = 0$.

Proof. (i) Suppose that F has a left- (resp. a right-) neutral element $e \in L$. Since F is conjunctive, we get $1 = F(e, 1) \leqslant e(resp.1 = F(1, e) \leqslant e)$. Hence, $e = 1$.

(ii) The proof is a similar to that of (i).

□

In view of Proposition 3.2, we obtain the following corollary.

Corollary 3.1. *[19] Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be a binary operation on L. If $(L, \leqslant, \wedge, \vee)$ has not a greatest (resp. least) element and F is conjunctive (resp. disjunctive), then F has not a left- and a right-neutral element.*

The following result characterizes the conjunction and the disjunction properties of a binary operation on a bounded lattice in terms of its neutral element.

Proposition 3.3. *[19] Let $(L, \leqslant, \wedge, \vee, 0, 1)$ be a bounded lattice and F be a binary operation on L. If F is increasing and having a neutral element $e \in L$, then the following statements hold:*

(i) F is conjunctive if and only if $e = 1$;

(ii) F is disjunctive if and only if $e = 0$.

Proof. (i) Proposition 3.2 guarantees that if F is conjunctive, then $e = 1$. Conversely, suppose that $e = 1$. Let $x, y \in L$, then $x \leqslant e$ *and* $y \leqslant e$. Since F is increasing, it follows that $F(x, y) \leqslant F(x, e) = x$ *and* $F(x, y) \leqslant F(e, y) = y$. Hence, $F(x, y) \leqslant x \wedge y$. Thus, F is conjunctive.

(ii) The proof is a similar to that of (i)

□

The following propositions list some conditions for the existence and the uniqueness of a left- (resp. right-) neutral element of a binary operation on a lattice.

Proposition 3.4. ***(Existence and uniqueness)****[19] Let $(L, \leqslant, \wedge, \vee, 1)$ be a lattice with a greatest element $1 \in L$ and F be a binary operation on L. If F is idempotent, conjunctive and left-increasing (resp. right-increasing), then 1 is the unique left-neutral (resp. the unique right-neutral) element of F.*

Proof. Since F is idempotent, conjunctive and left-increasing (resp. right increasing), we have $x = F(x,x) \leqslant F(1,x) \leqslant x$ (resp. $x = F(x,x) \leqslant F(x,1) \leqslant x$), for any $x \in L$. This implies that $F(1,x) = x$ (resp.$F(x,1) = x$), for any $x \in L$. Hence, 1 is a left-neutral (resp. right-neutral) element of F . Moreover, since F is conjunctive, it holds from Proposition 3.1 that 1 is the unique left-neutral (resp. the unique right-neutral) element of F . □

Dually, we get the following proposition.

Proposition 3.5. ***(Existence and uniqueness)****[19] Let $(L, \leqslant, \wedge, \vee, 0)$ be a lattice with a least element $0 \in L$ and F be a binary operation on L. If F is idempotent, disjunctive and left-increasing (resp. right-increasing), then 0 is the unique left-neutral (resp. the unique right-neutral) element of F.*

3.2 Types of binary operations

Here we give some important types of binary operations on bounded lattices, for example: aggregation operations, triangular norms and conorms, uninorms and nullnorms then uni-nullnorms and null-uninorms .

3.2.1 Aggregation operations

Definition 3.3. *[12] Let $(L, \leqslant, \wedge, \vee, 0, 1)$ be a bounded lattice. An aggregation operator on L is a binary operation A on L which is increasing and satisfies the boundary conditions*

$$A(0,0) = 0 \text{ and } A(1,1) = 1.$$

Example 3.3. *[7] Let $L = \{0, a, 1\}$ be a bounded lattice and A be a binary operation on L defined by the following table:*

A	0	a	1
0	0	0	a
a	0	0	1
1	a	1	1

It is not difficult to see that A is an aggregation on L which is an example of an aggregation of a 3 element chain is shown in the following figure.

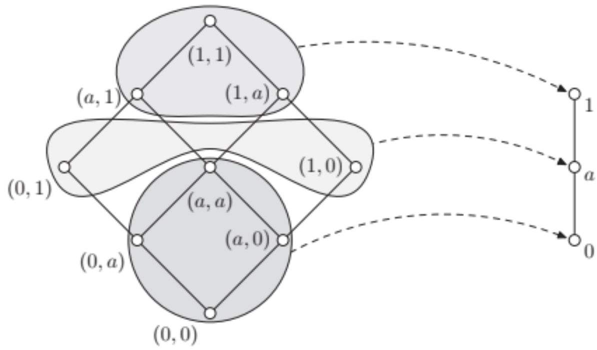


Figure 3.1

Definition 3.4. *[8] Denote by $A_2(L)$ the set of all binary aggregations. Consider $A_2(L)$ with this order*

$$A_1 \leqslant A_2 \Leftrightarrow A_1(x_1, y_1) \leqslant A_2(x_2, y_2)$$

For all $A_1, A_2 \in A_2(L)$ and $(x_1, y_1), (x_2, y_2) \in L^2$.

Proposition 3.6. *[8] Let $(L, \leqslant, \wedge, \vee, 0, 1)$ be a bounded lattice and let the following binary operations defined by:*

$$A_\perp = \begin{cases} 1 & if (x, y) = (1, 1) \\ 0 & otherwise \end{cases} ;$$

$$A_\top = \begin{cases} 0 & if (x, y) = (0, 0) \\ 1 & otherwise \end{cases} .$$

$A_\perp$ is the smallest aggregation and $A_\top$ is the greatest aggregation in $A_2(L)$. So

$$A_\perp \leqslant A \leqslant A_\top, \ for\ any \ \ A \in A_2(L)$$

3.2.2 Triangular norms and co-norms

All information of this section from [2].

Triangular norms:

Definition 3.5. *Let* $(L, \leqslant, \wedge, \vee, 0, 1)$ *be a bounded lattice. A triangular norm (t-norm for short) on* L *is a binary operation* T*, i.e., it is a function* $T : L^2 \to L$ *such that for all* $x, y, z \in L$*, the following four axioms are satisfied:*

$T1:\ T(x, y) = T(y, x)$ *(commutativity);*

$T2:\ T(x, T(y, z)) = T(T(x, y), z)$ *(associativity);*

$T3:\ T(x, y) \leq T(x, z)$ *whenever* $y \leq z$ *(monotonicity);*

$T4:\ T(x, 1) = x$ *(boundary condition).*

In this section we focus our attention on the triangular norms of the unit interval $[0, 1]$, which is a particular bounded lattice.

Example 3.4. *The following are the four basic t-norms* $T_M, T_P, T_L,$ *and* T_D *given by, respectively:*

$T_M(x, y) = min((x, y)$	(*Minimum*)
$T_P(x, y) = x \cdot y$	(*Product*)
$T_L(x, y) = max(x + y - 1, 0)$	(*Lukasiewicz t-norm*)
$T_D(x, y) = \begin{cases} 0 \text{ if } (x, y) \in [0, 1[^2 \\ min(x, y) \text{ otherwise} \end{cases}$	(*Drastic product*)

Proposition 3.7. *Any t-norm* T *satisfies* $T(0, x) = T(x, 0) = 0$*, for all* $x \in [0, 1]$*.*

Proof. We know that $T(x, 0) \in [0, 1]$, so $T(x, 0) \geq 0$, and we use the axiom $(T3)$ (monotonicity), we obtain that $T(x, 0) \leq T(1, 0) = 0$. □

Proposition 3.8. *Let* A *be a set with* $]0, 1[\ \subseteq A \subseteq [0, 1]$*, and assume that* $F : A^2 \to A$ *is a binary operation on* A *such that for all* $x, y, z \in A$ *the properties* $(T1) - (T3)$ *and* $F(x, y) \leq$

min (x, y) *are satisfied. Then the function* $T : [0,1]^2 \to [0,1]$ *defined by*

$$T(x,y) = \begin{cases} F(x,y) \text{ if } (x,y) \in (A - \{1\})^2, \\ min(x,y) \text{ otherwise.} \end{cases}$$

is a t-norm.

Proof. The commutativity $(T1)$ and the boundary condition $(T4)$ are satisfied by definition. Concerning the associativity $(T2)$, observe that for $x, y, z \in A \setminus 0, 1$ we have $T(T(x,y),z) = T(x, T(y,z))$ as a A consequence of the associativity of F, If $0 \in \{x, y, z\}$ then we get $T(x, T(y,z)) = 0 = T(T(x,y), z)$, and if $1 \in \{x, y'z\}$ then $T(T(x,y),z) = T(x, T(y,z))$ follows from $(T4)$. Concerning the monotonicity $(T3)$, suppose $y \leq z$. In the cases $x, y, z \in A \setminus 1$ or $x \in \{0; 1\}$ or $y = 0$, the inequality $T(x,y) \leq T(x,z)$ is inherited from the monotonicity of F and min. The only non-trivial case is when $x, y \in A \setminus \{1\}$ and $z = 1$, in which case $T(x,y) \leq T(x,z)$ follows from $F(x,y) \leq min(x,y)$. □

Definition 3.6. *A function* $f : [0,1]^2 \to [0,1]$ *which satisfies, for all* $x, y, z \in [0,1]$*, the properties* $(T1) - (T3)$ *and* $f(x,y) \leq min(x,y)$ *is called a t-subnorm.*

Example 3.5.

1. $f(x,y) = 0.$
2. $f(x,y) = \dfrac{x \times y}{3}.$
3. $f(x,y) = x \times y.$

Remark 3.1. *Clearly, each t-norm is a t-subnorm, but not the conserve does not necessary hold. For example, the function* $f : [0,1]^2 \to [0,1]$ *given by* $f(x,y) = 0$ *, is a t-subnorm but not a t-norm because* $(T4)$ *is not satisfies* $(\forall x \in [0,1], f(x,1) = 0 \neq x)$.

Corollary 3.2. *If f is a t-subnorm then the function* $T : [0,1]^2 \to [0,1]$ *defined by*

$$T(x,y) = \begin{cases} f(x,y) \text{ if } (x,y) \in [0,1[^2, \\ min(x,y) \text{ otherwise.} \end{cases}$$

T is a triangular norm.

Comparison of t-norms

Definition 3.7. *(i) If, for two t-norms T_1 and T_2, the inequality $T_1(x,y) \leq T_2(x,y)$ holds for all $(x,y) \in [0;1]^2$, then we say that T_1 is weaker than T_2 or, equivalently, that T_2 is stronger than T_1, and we write in this case $T_1 \leq T_2$.*

(ii) We shall write $T_1 < T_2$ whenever $T_1 \leq T_2$ and $T_1 \neq T_2$, i.e., if $T_1 \leq T_2$ and for some $(x_0, y_0) \in [0,1]^2$, we have $T_1(x_0,y_0) < T_2(x_0,y_0)$.

Lemma 3.1. *(i) The minimum T_M is the greatest t-norm ($T \leq T_M$, for every t-norm T).*

(ii) The drastic product T_D is the smallest t-norm ($T_D \leq T$, for any t-norm T).

Proof. (i) For each t-norm T and for each $(x,y) \in [0,1]^2$, we have both $T(x,y) \leq T(x,1) = x$ and $T(x,y) \leq T(1,y) = y$, so $T(x,y) \leq min(x,y) = T_M(x,y)$.

(ii) All t-norms coincide on the boundary of $[0,1]^2$ and for all $(x,y) \in]0,1[^2$ we trivially have $T(x,y) \geq 0 = T_D(x,y)$.

□

Example 3.6.
- $T_0(x,y) = \begin{cases} 0 \ \text{if} \ (x,y) \in [0,1[^2, \\ min(x,y) \ \text{otherwise}. \end{cases}$ *(Drastic product of weber);*
- $T_1(x,y) = max(x+y-1, 0)$ *(Lukasiewicz);*
- $T_{1,5}(x;y) = \dfrac{xy}{2-x-y+xy}$ *(Einstein);*
- $T2(x,y) = xy$ *(Algebraic or probaliste);*
- $T_{2,5}(x;y) = \dfrac{xy}{x+y-xy}$ *(Hamacher);*
- $T_3(x;y) = min(x;y)$ *(Zadeh);*

  We have: $T_0 \leq T_1 \leq T_{1,5} \leq T_2 \leq T_{2,5} \leq T_3$.

Proposition 3.9. *(1) The only t-norm T satisfying $T(x,x) = x$ for all $x \in [0,1]$ is the minimum T_M.*

(2) The only t-norm T satisfying $T(x,x) = 0$ for all $x \in [0,1[$ is the drastic product T_D.

Proof. (1) If for a t-norm T we have $T(x,x) = x$ for each $x \in [0,1]$, then for all $(x,y) \in [0,1]^2$ with $y \leq x$ the monotonicity (T3) implies $y = T(y,y) \leq T(x,y) \leq T_M(x,y) = y$, which, together with (T1), means $T = T_M$.

(2) Assume $T(x,x)=0$ for each $x\in[0,1[$. Then for all $(x,y)\in[0,1[^2$ with $y\leq x$ we have $0\leq T(x,y)\leq T(x,x)=0$, hence, together with (T1) and (T4), yielding $T=T_D$.

□

Triangular conorms:

Definition 3.8. *Let $(L,\leqslant,\wedge,\vee,0,1)$ be a bounded lattice. A triangular conorm (t-conorm for short) on L is a binary operation S, i.e., it is a function $S:L^2\to L$ such that for all $x,y,z\in L$, the following four axioms are satisfied:*

$S1:\ S(x,y)=S(y,x)$ *(commutativity);*

$S2:\ S(x,S(y,z))=S(S(x,y),z)$ *(associativity);*

$S3:\ S(x,y)\leq S(x,z)$ *whenever* $y\leq z$ *(monotonicity);*

$S4:\ S(x,0)=x$ *(boundary condition).*

Here also we present the triangular conorms of the unit interval $[0,1]$, which is a particular bounded lattice.

Example 3.7. *The following are the four basic t-conorms S_M, S_P, S_L, and S_D given by, respectively:*

$S_M(x,y)=max(x,y)$	$(maximum)$
$S_P(x,y)=x+y-x\cdot y$	$(probabilistic\ sum)$
$S_L(x,y)=min(x+y,1)$	$(Lukasiewicz\ t\text{-}conorm,\ bounded\ sum)$
$S_D(x,y)=\begin{cases}1\ if\ (x,y)\in]0,1]^2,\\ max(x,y)\ otherwise.\end{cases}$	$(Drastic\ sum\ )$

Proposition 3.10. *Any t-conorm S satisfies $S(1,x)=S(x,1)=1$, for all $x\in[0,1]$.*

Proof. We know that $S(x,1)\in[0,1]$, so $S(x,1)\leq 1$, and we use the axiom $(S3)$ (monotonicity), we obtient $S(x,1)\geq S(0,1)=1$. □

Proposition 3.11. *A function $S:[0,1]^2\to[0,1]$ is a t-conorm if and only if there exists a t-norm T such that for all $(x,y)\in[0,1]^2$*

$$S(x,y)=1-T(1-x,1-y).(*)$$

Proof. If T is a t-norm then obviously the operation S defined by $(*)$ satisfies $(S1)-(S3)$ and $(S4)$ $(S1)\, S(x,y) = 1-T(1-x,1-y) = 1-(1-y,1-x) = S(y,x)\,, (S2)\, S(x,S(y,z)) = 1-T(1-x,1-S(y,z)) = 1-T(1-x,1-(1-T(1-y,1-z))) = 1-T(1-x,T(1-y,1-z))\,,$ $S(S(x,y),z) == 1-T(1-s(x,y),1-z) = 1-T(1-(1-T(1-x,1-y)),1-z) = 1-T(T(1-x,1-y),1-z) = 1-T(1-x,T(1-y,1-z))\,, (S3)\, S(x,y) = 1-T(1-x,1-y) \leq 1-T(1-x,1-z) = S(x,z)$ whenever $y \leq z$, $(S4)\, S(x,0) = 1-T(1-x,1) = 1-(1-x) = x$, and is, therefore, a t-conorm. On the other hand, if S is a t-conorm, then define the function $T : [0,1]^2 \to [0,1]$ by

$$T(x,y) = 1 - S(1-x,1-y)\,. \,(**)$$

Again, it is trivial to T is a t-norm and that $(*)$ holds. □

Remark 3.2. *1. The t-conorm given by $(*)$ is called the dual t-conorm of T and, analogously, the t-norm given by $(**)$ is said to be the dual t-norm of S.*

2. The proof of Proposition makes it clear that also each t-norm is the dual operation of some t-conorm. Note that $(T_M, S_M)\,, (T_P, S_P)\,, (T_L, S_L)\,,$ and (T_D, S_D) are pairs of t-norms and t-conorms which are mutually dual to each other.

Definition 3.9. *Let T be a t-norm and S be a t-conorm. Then we say that T is distributive over S if for all $x, y, z \in [0,1]$*

$$T(x, S(y,z)) = S(T(x,y), T(x,z))\,.$$

and that S is distributive over T if for all $x, y, z \in [0,1]$

$$S(x, T(y,z)) = T(S(x,y), S(x,z))\,.$$

Remark 3.3. *If T is distributive over S and S is distributive over T, then (T,S) is called a distributive pair (of t-norms and t-conorms).*

Proposition 3.12. *Let T be a t-norm and S a t-conorm. Then we have:*

(i) S is distributive over T if and only if $T = T_M$.

(ii) T is distributive over S if and only if $S = S_M$.

(iii) (T,S) is a distributive pair if and only if $T = T_M$ and $S = S_M$.

Proof. Obviously, each t-conorm is distributive over T_M because of the monotonicity (S3) of the t-conorm. $(\subseteq)$ we have

$$S(x, T_M(y, z)) \leq S(x, y)\,(a)$$

$$S(x, T_M(y, z)) \leq S(x, z)\,(b)$$

$(a)\,and\,(b)$ given that $S(x, T(y, z)) \leq T_M(S(x, y), S(x, z))$. $(\supseteq)$ ..... Conversely, if S is distributive over$T$ then for all$x \in [0, 1]$ we have $x = S(x, T(0, 0)) = T(S(x, 0), S(x, 0)) = T(x, x)$, and we obtain $T = T_M$. An analogaus argument proves (ii), and (iii) is just the combination of (i) and (ii). $\square$

3.2.3 Uni-norm and Null-norm

Uni-norm and Null-norm on bounded lattices are special binary operations generalize t-norms and t-conorms.

Definition 3.10. ***(Uni-norm)***[17] *Let $(L, \leqslant, 0, 1)$ be a bounded lattice. A binary operation U on L is called a uni-norm if it is commutative, associative and increasing and has a neutral element $e \in L$.*

Remark 3.4. *If $e = 1$, then U is a t-norm and if $e = 0$ so U is a t-conorm. Thus, every t-norm (resp. t-conorm) on a bounded lattice is a uni-norm. But the conserve does not necessary hold.*

Definition 3.11. *[20] For all $x \in L$, a uninorm U is called:*

- *conjunctive if $U(0, x) = 0$ (i.e., 0 is an absorbing element of U);*
- *disjunctive if $U(1, x) = 1$ (i.e., 1 is an absorbing element of U).*

Example 3.8. *[6] Let $(L, \leqslant, 0, 1)$ be bounded lattice and $e \in L$. We define two binary operations U_1 and U_2 as follows:*

$$U_1 = \begin{cases} max(x, y) & if(x, y) \in [e, 1]^2, \\ min(x, y) & elsewhere. \end{cases}$$

And

$$U_2 = \begin{cases} min(x, y) & if(x, y) \in [0, e]^2 \\ max(x, y) & elsewhere. \end{cases}$$

Then U_1 and U_2 are uni-norms on L and they have the same neutral element e, satisfying also $U_1(0,1) = U_1(1,0) = 0$ and $U_2(0,1) = U_2(1,0) = 1$

Definition 3.12. *[17]**(Null-norm)** Let $(L, \leqslant, 0, 1)$ be a bounded lattice. A binary operation $V : L^2 \to L$ is called a null-norm if it is commutative, associative and increasing and it has an absorbing element $a \in L$ ($V(x,a) = a$ for all $x \in L$), and*

$$V(x,0) = x, \text{ for all } x \leqslant a \text{ and } V(x,1) = x, \text{ for all } x \geqslant a.$$

Remark 3.5. *We note that a null-norm V is a t-norm if $a = 0$ and it is a t-conorm S if $a = 1$*

Example 3.9. *Let $(L, \leqslant, 0, 1)$ be a bounded lattice, S be a t-conorm on $[0,a]$ and T be t-norm on $[a,1]$ such that $a \in L$. Based on Remark 3.5, we define a binary operation V on L, for all $x, y \in L$ by:*

$$V(x,y) = \begin{cases} S(x,y) & if (x,y) \in [0,a]^2 \\ T(x,y) & if (x,y) \in [a,1]^2 \\ a & otherwise. \end{cases} .$$

This binary operation V is a null-norm.

3.2.4 Uni-Nullnorms and Null-Uninorms

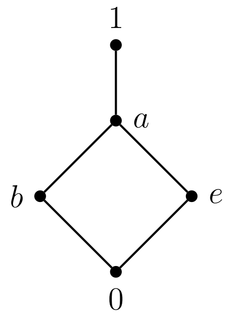
An other special classes of aggregations operations are Uni-Nullnorms and Null-Uninorms which satisfy commutative and associative properties. Further, Uni-Nullnorms and Null-Uninorms operations are important generalization of triangular norms and conorms. In 2017, the notation of Uni-Nullnorms and Null-Uninorms has been introduced by San et al. [16], which are special case of 2-uninorms (see [1]).

Definition 3.13. *[5] **2-neutral element** Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $e_1, a, e_2 \in L$. Let F be a binary operation on L, then $\{e_1, e_2\}_a$ with $0 \leqslant e_1 \leqslant a \leqslant e_2 \leqslant 1$ is called a 2-neutral element of F if $F(e_1, x) = x$ for all $x \leqslant a$ and $F(e_2, x) = x$ for all $x \geqslant a$ where $0 < a < 1$.*

Definition 3.14. *[1] **(2-uninorm)** Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $e_1, a, e_2 \in L$ with $e_1 \leqslant a \leqslant e_2$. A commutative, associative, increasing binary operation on L having a 2-neutral element $\{e_1, e_2\}_a$ is called a 2-uninorm on L.*

Definition 3.15. *[16] **(Uni-nullnorm)** Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $e, a \in L$ with $e \leqslant a$. A uni-nullnorm is a 2-uninorm having 2-neutral elements $\{e, 1\}_a$ where $e_1 = e$ and $e_2 = 1$.*

Example 3.10. *[5] Let $(L = \{0, e, b, a, 1\}, \leqslant, \wedge, \vee)$ be the lattice given by the following Hasse diagram*



The binary operation F given in the bellow table is an uni-nullnorm on L:

F	0	e	b	a	1
0	0	0	b	b	b
e	0	e	b	a	a
b	b	b	b	b	b
a	b	a	b	a	a
1	b	a	b	a	1

Proposition 3.13. *[5] Let $(L, \leqslant, 0, 1)$ be a bounded lattice, $e, a \in L$ with $e \leqslant a$. F be a uni-nullnorm with 2-neutral element $\{e, 1\}_a$.*

(i) $U^* = F\,|_{[0,a]^2} \colon [0, a]^2 \to [0, a]$ is a uninorm on $[0, a]$.

(ii) $V^* = F\,|_{[e,1]^2} \colon [e, 1]^2 \to [e, 1]$ is a nullnorm on $[e, 1]$.

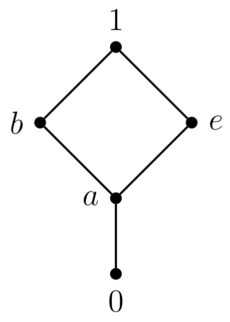
(iii) $T_1^* = F\,|_{[0,e]^2} \colon [0; e]^2 \to [0, e]$ is a t-norm on $[0, e]$.

(iv) $S^* = F\,|_{[e,a]^2} \colon [e; a]^2 \to [e, a]$ is a t-conorm on $[e, a]$.

(v) $T_2^* = F\,|_{[a,1]^2} \colon [a, 1]^2 \to [a, 1]$ is a t-norm on $[a, 1]$.

Definition 3.16. *[16] **(Null-uninorm)** Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $e, a \in L$ with $e \leqslant a$. A null-uninorm is a 2-uninorm having a 2-neutral elements $\{0, e\}_a$ where $e_1 = 0$ and $e_2 = e$.*

Example 3.11. *[5] let* $(L = \{0, a, b, e, 1\}, \leqslant, \wedge, \vee)$ *be the lattice given by the following Hasse diagram*



The binary operation F *given in the bellow table is a null-uninorm on* L*:*

F	0	a	b	e	1
0	0	a	b	a	b
a	a	a	b	a	b
b	b	b	b	b	b
e	a	a	b	e	1
1	b	b	b	1	1

Proposition 3.14. *[5] Let* $(L, \leqslant, 0, 1)$ *be a bounded lattice,* $e, a \in L$ *with* $a \leqslant e$ *and* F *be a null-uninorm having a* 2*-neutral element* $\{0, e\}_a$ *with* a *being an annihilator over* $[0, e]$*. Then,*

(i) $V^* = F \mid^{[0,e]^2} : [0, e]^2 \to [0, e]$ *is a nullnorm on* $[0, e]$*.*

(ii) $U^* = F \mid_{[a,1]^2} : [a, 1]^2 \to [a, 1]$ *is a uninorm on* $[a; 1]$*.*

(iii) $S_1^* = F \mid_{[0,a]^2} : [0, a]^2 \to [0, a]$ *is a t-conorm on* $[0, a]$*.*

(iv) $T^* = F \mid_{[a,e]^2} : [a, e]^2 \to [a, e]$ *is a t-norm on* $[a, e]$*.*

(v) $S_2^* = F \mid_{[e,1]^2} : [e, 1]^2 \to [e, 1]$ *is a t-conorm on* $[e, 1]$*.*

3.3 Applications of binary operations

We finish this chapter by providing some applications of binary operations in various fields.

Binary operations are fundamental mathematical operations that involve two operands and produce a single result. While they are often studied in the context of abstract algebra and mathematics, binary operations also have practical applications in everyday life. Here are some examples of how binary operation are used in various fields :

1. **Digital computers**:

   Binary operations are essential in the field of computer science and digital electronics. Computers process data using binary operations such as addition, subtraction, multiplication, and division. Boolean operations like AND, OR, and NOT are also fundamental to computer programming and logic circuits.

2. **Internet and Telecommunication**:

   Binary operations are used in data transmission and networking protocols. Data is transmitted in binary form (0s and 1s), and operations like bitwise operations are used for error detection and correction, encryption, and data compression.

3. **Finance and Economics**:

   Binary operations are used in financial calculations and economic modeling. For example, in options trading, binary operations are used to calculate profit and loss scenarios. Binary decision trees are also used in financial risk assessment and investment analysis.

4. **Genetics and Bionformatics**:

   In genetics, binary operations are used to represent genetic sequences and perform operations like sequence alignment and comparison. Bioinformatics tools use binary operations to analyze DNA, RNA, and protein sequences

5. **Digital Image Processing**:

   In image processing, binary operations are used for tasks like image segmentation, edge detection, and morphological operations. Operations like dilation, erosion, opening, and closing are commonly used in binary image processing.

6. **Cryptography**:

   Binary operations play a crucial role in encryption algorithms and cryptographic protocols. Operations like XOR (exclusive OR) are used in encryption schemes to secure data and communications.

7. **Automotive Industry**:

   Binary operations are used in automotive systems for tasks like engine control, sensor data processing, and vehicle diagnostics. Binary operations help control various functions in modern vehicles, such as fuel injection timing and anti-lock braking systems.

8. **Robotics and Automation**:

   Binary operations are used in robot control systems and automation processes. Binary logic is used to make decisions and control the movement and operation of robots in manufacturing, logistics, and other industries.

   These are just a few examples of how binary operations are applied in everyday life across various fields and industries. Their versatility and efficiency make them essential tools for solving complex problems and optimizing processes in different domains.

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ملخص

في هذه المذكرة، قمنا أولاً بعرض المفاهيم والخصائص الأساسية الضرورية للعلاقات الثنائية على مجموعة والمجموعات المرتبة. ثم قدمنا خصائص ومعلومات عامة عن الشبكات. كما قمنا بعرض مفهوم وخصائص العمليات الثنائية على الشبكات المحدودة. بتعبير أدق، لقد عرضنا بعض الأنواع المشهورة للعمليات الثنائية، كعمليات التجميع والقواعد الثلاثية وأنواع عمليات ثنائية جديدة. وأخيراًدعمنا هذه المذكرة بذكر بعض تطبيقات العمليات الثنائية في مختلف المجالات.

كلمات مفتاحية:

شبكات المحدودة، العلاقات الثنائية، العمليات ثنائية، المجموعات المرتبة، عمليات التجميع، القواعد الثلاثية.

Abstract

In this memory, we first have recalled the necessary basic concepts and properties of binary relations on a set and partially ordered sets. Then, we have provided general properties and information on lattices. Also, we have presented binary operations and their properties on bounded lattices. More precisely, we have given some spacial classes of binary operations as aggregation operations, triangular norms and conorms. Moreover, we have expanded their concept to uni-norms and null-norms. After that we have presented how they lead to the new mathematical concepts of uni-nullnorms and null-uninorms. Finally, we supported this memory by mentioning some applications of binary operations in various fields.

Key words :

Bounded lattice, binary operations, aggregation, t-norms, t-conorms, uni-norms, null-norms, uni-nullnorms, null-uninorms.

Résumé

Dans ce mémoire, nous avons d'abord rappelé les concepts de base et les propriétés nécessaires des relations binaires sur un ensemble et des ensembles partiellement ordonnés. Ensuite, nous avons fourni des propriétés générales et des informations sur les treillis. Nous avons également présenté la notion des opérations binaires et leurs propriétés sur treillis bornés. Plus précisément, nous avons donné quelques classes spatiales d'opérations binaires, d'agrégation, de normes et conormes triangulaires. De plus, nous avons élargi leur concept aux uni-norms et null-norms. Nous avons ensuite présenté comment ils conduisent aux nouveaux concepts mathématiques des uni-nullnorms and null-uninorms. Enfin, nous avons soutenu ce mémoire en évoquant quelques applications des opérations binaires dans divers domaines.