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*Analytical solutions for Navier-Stokes Equations with
Caputo Fractional Derivative*

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Contents

Acknowledgement	III
Dedication	IV
List of Symbols	V
List of Figures	VI
General Introduction	VII
1 Basic concepts and elements of fractional calculus	1
1.1 Basic fractional calculus	1
1.1.1 Special functions	2
1.2 Fractional integrations operators	3
1.2.1 Riemann-Liouville fractional integration	3
1.2.2 Hadamard fractional integration	3
1.2.3 Katugampula fractional integration	4
1.2.4 φ -fractional integration	4
1.3 Fractional derivatives operators	5
1.3.1 Riemann-Liouville fractional derivative	5
1.3.2 Hadamard fractional derivative	5
1.3.3 katugampula fractional derivative	5
1.3.4 φ -fractional derivative	6
1.4 Fractional power series (FPS)	7
1.5 Operators on scalar field and vector field	9
1.5.1 Del operator	10
1.5.2 Gradient of a scalar field	10
1.5.3 Divergence of a vector field	10
1.5.4 Directional derivative of a scalar field	10
1.5.5 Directional derivative of a vector field	10
1.5.6 Laplacian operator	11

Contents

1.5.7	Laplacien operator of a scalar field	11
1.5.8	Laplacian operator of a vector field:	11
1.6	Navier-Stokes Equation	11
1.6.1	Classical Model of Navier-Stokes Equation	11
1.6.2	Existence and Smoothness of Solution of Navier-Stokes Equation	13
1.6.3	Time-Fractional Navier-Stokes Equation	14
2	Solving Navier-Stokes Equation by the Residual Power Series Method	15
2.1	RPS Algorithm for Solving Fractional Navier-Stokes equation	15
2.2	Construction of the RPS Approximate Solutions of the NavierStokes Equation	19
3	Applications	25
3.1	Time-Fractional Navier-Stokes Problem	25
3.1.1	Numerical Analysis for the Time-Fractional Navier-Stokes Equation	32
	Conclusion	37
	Future Studies	38
	Bibliographie	39

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Dedication

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List of Symbols

\mathbb{N}	Natural Numbres $\{0, 1, 2, 3, \dots\}$.
\mathbb{N}^*	Nonzero natural numbers $\{1, 2, 3, \dots\}$.
\mathbb{R}	Real numbers $(-\infty, +\infty)$.
\mathbb{R}_+	Positive real numbers $(0, +\infty)$.
\mathbb{R}^*	Nonzero real numbers $(-\infty, 0) \cup (0, +\infty)$.
\mathbb{C}	complex numbers , $z \in \mathbb{C}$, then $z=x+iy$, where $x,y \in \mathbb{R}$,and $i^2=-1$.
$Re(\alpha)$	Real part of complex α .
$[a,b]$	$(-\infty < a < b < +\infty)$ be a finite interval on the real-axis \mathbb{R} .
$C([a,b])$	The BANACH space of all continouis functions from $[a,b]$ into \mathbb{R} .
FDE	Fractional Differential Equation .
FPDE	Fractional Partial Differential Equation.
$\Gamma(\cdot)$	EULER Gamma function.
$B(\cdot, \cdot)$	Beta Function.
$E_\alpha(\cdot)$	Standard Mittag-Leffler function.
$E_{\alpha,\beta}(\cdot)$	Mittag-Laffler function in two arguments α and β .
${}^{RL}I^\alpha u$	RIEMANN-LIOUVILLE'S fractional integral of order α .
${}^{RL}D^\alpha u$	RIEMANN-LIOUVILLE's fractional derivative of order α .
D_t^α	Differential operator $\left(\frac{\partial^\alpha}{\partial t^\alpha}\right)$.
$D_t^\alpha u$	derivative of u in the CAPUTO sense.
${}^C D^\alpha f$	CAPUTO's fractional derivative of order α .
${}^H I^\alpha u$	HADAMARD's fractional integral of order α .
${}^H D^\alpha u$	HADAMARD's fractional derivative of order α .
${}^\rho I^\alpha u$	KATUGAMPOLA's fractional integral of order α .
${}^\rho D^\alpha u$	KATUGAMPOLA's fractional derivative of order α .
$I^{\alpha;\varphi} u$	φ -fractional integral of order α .
$D^{\alpha;\varphi} u$	φ -fractional derivative of order α .

List of Figures

Figure	Name	Page
Figure 1	The velocity \mathbf{u}_x with $x = \pi/4$ and $t = 1.5$ and $0 \leq y \leq 2\pi$ at defferent values of α .	33
Figure 2	The velocity \mathbf{v}_x with $x = \pi/4$ and $t = 1.5$ and $0 \leq y \leq 2\pi$ at defferent values of α .	33
Figure 3	The pressure \mathbf{P}_x with $y = \pi/4$ and $t = 1.5$ and $0 \leq x \leq 2\pi$ and $R = 40$ at defferent values of α .	33
Figure 4	The velocity \mathbf{u}_t with $x = y = \pi/6$ and $1 \leq t \leq 2, 5$ at defferent values of α .	34
Figure 5	The velocity \mathbf{v}_t with $x = y = \pi/6$ and $1 \leq t \leq 2.5$ at defferent values of α .	34
Figure 6	The pressure \mathbf{P}_t with $x = y = \pi/6$ and $1 \leq t \leq 2.5$ and $R = 40$ at defferent values of α .	34
Figure 7	The 4th RPS approximate solution $u(x, y, t)$ of the Navier-Stokes equation in deffrent value of α	35
Figure 8	The 4th RPS approximate solution $v(x, y, t)$ of the Navier-Stokes equation in deffrent value of α	35
Figure 9	The 4th RPS approximate solution $P(x, y, t)$ of the Navier-Stokes equation in deffrent value of α	36

General Introduction

In recent decades, fractional calculus has garnered significant attention from researchers due to its robust theoretical foundation and expanding applications. The origins of this theory trace back to the 17th century when, in 1695, Leibniz first described a derivative of non-integer order ($\alpha = \frac{1}{2}$) in a correspondence with L'Hôpital. Since then, fractional calculus has attracted considerable interest not only from mathematicians but also from physicists, biologists, engineers, and economists, demonstrating its broad interdisciplinary relevance.

Fractional differential equations have gained importance and popularity, mainly due to their demonstrated applications in science and engineering. These equations provide powerful tools for characterizing diverse physical processes, including: model problems in fluid flow, rheology, diffusion, relaxation, oscillation, anomalous diffusion, reaction–diffusion, turbulence, diffusive transport akin to diffusion, electric networks, polymer physics, chemical physics, electrochemistry of corrosion, relaxation processes in complex systems, propagation of seismic waves, dynamical processes in self-similar and porous structures and many other physical processes.

The Classical fractional calculus relies on multiple definitions for fractional integration and derivatives operators, including the: Riemann-Liouville, Hadamard, and Katugampola formulations. As a result, it was necessary to introduce a fractional derivative of a function f with respect to another function is called φ -fractional derivative which generalizes the three derivatives into a single form. This new fractional integral and derivative appeared in several recent articles and books, for example see [[15],[14]]

The primary advantage of employing fractional differential equations in these and various other applications lies in their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is non-local. This implies that the future state of a system is influenced not only by its present condition but also by its entire past history. This is more realistic and it is one reason why fractional calculus has become more and more popular.

Significant research efforts have been devoted to solving fractional ordinary differential equations (ODEs), fractional integral equations (IEs), and fractional partial differential equations (PDEs) arising in physical applications.

Since most fractional differential equations lack exact analytical solutions, approximation methods and numerical techniques are consequently employed extensively.

Some analytical approaches are presented such as :Laplace transform method, Fourier transform method , the variational iteration method and Green function method. In addition, numerical schemes for fractional differential equations have attracted considerable attention from researchers. Significant research efforts have been devoted in recent years to developing both stable numerical methods and analytical approaches for solving fractional differential equations arising in physical applications. The adomian decomposition method, homotopy perturbation method, homotopy analysis method; differential transform method, Residual power series method, are relatively new approaches to provide an analytical approximate solution to linear and nonlinear FDEs [17] .

in this thesis We introduce the residual power series (RPS) method [16] as an effective technique for solving analytical solutions to various types of fractional linear and nonlinear partial differential equations that arise in

mathematics, physics, and engineering applications.

The RPS method was developed as an efficient approach for calculating the coefficients of power series solutions to first-order and second-order fuzzy differential equations [4]. It has been effectively applied to obtain numerical solutions for the generalized Lane-Emden equation, a well-known nonlinear singular differential equation [8], the solution of composite and non-composite fractional differential equations [6], predicting and representing the multiplicity of solutions to boundary value problems of fractional order [7], constructing and predicting the solitary pattern solutions for nonlinear time-fractional dispersive partial differential equations [5], the approximate solution of the nonlinear fractional KdV-Burgers equation [2], the approximate solutions of fractional population diffusion model [3], and the numerical solutions of linear non-homogeneous PDE of fractional order [9].

The effectiveness of the RPS method lies in its ability to generate power series solutions for strongly linear and nonlinear equations without resorting to linearization techniques, perturbation methods, or discretization approaches. Different from the classical power series method, the RPS method does not need to compare the coefficients of the corresponding terms and a recursion relation is not required. This method computes the coefficients of the power series by a chain of algebraic equations of one or more variables. In fact, the RPS method is an alternative procedure for obtaining analytic solutions for partial differential equations of fractional order. By using residual error concept, we get a series solutions in practice truncated series solutions. Moreover, the obtained solutions and all their time-fractional derivatives are applicable for each arbitrary point and each multi-dimensional variable in the given domain. On the other aspect as well, the RPS method does not require any converting while switching from the low-order to the higher-order as a result the method can be applied directly to the given system by choosing an appropriate initial guesses approximations.

In this work, we solve the generalized Navier-Stokes equations with Caputo fractional derivative to determine solutions representing both fluid velocity and pressure fields. The Navier-Stokes equations are commonly used in describing motion of fluids in models relevant to weather, ocean currents, water flow in pipes, and so forth.

Many researchers have employed various analytical and numerical approaches to solve both classical and fractional Navier-Stokes equations, including the fractional step method, exact fractional projection method, Adomian decomposition method, Laplace and finite Hankle transforms, modified Laplace decomposition. More recently, the authors successfully applied the residual power series (RPS) method to obtain solutions for one-dimensional unsteady fractional Navier-Stokes equations.

It is important to note that all previous attempts to solve the fractional NavierStokes equation were limited to one dimensional time-fractional equation with constant pressure.

The primary objective of this thesis is to employ the Residual Power Series (RPS) method to obtain an analytical approximate solution for the two-dimensional time-fractional Navier-Stokes equations with Caputo fractional derivatives with variable pressure. It has been organized as follows:

in the first chapter which is concerned with some basic concepts of fractional calculus, and some definitions and results that are important for the study, also we give definitions and some properties of fractional integrals and fractional derivatives of various types, especially the φ -fractional integral and derivative and its properties.

Thus the fractional power series is a powerful mathematical tool in deriving various existence results.

We will employ in the second chapter, the RPS algorithm to solve the Navier-Stokes equations also we will construct the approximate solutions for these equations using the RPS method.

In the third chapter, we demonstrate the application of the RPS method (presented in Chapter Two) on a two-dimensional benchmark problem. Finally, To evaluate the RPS method's accuracy and efficiency we will examine the numerical solutions of this equation.

Chapter 1

Basic concepts and elements of fractional calculus

In this chapter, we introduce some specific functions and essential tools required for our thesis concerning fractional integration and derivatives and the φ -Caputo fractional derivatives. We also discuss the key properties of these concepts. Additionally, we recall the well-known fractional power series, which will be instrumental in deriving various existence results, then we present some operators on a vector field and function of several variables. Finally we will discuss the concept of Navier-Stokes equation including its components forms, and its generalization into fractional form. .

1.1 Basic fractional calculus

In this section, we shall quickly discuss some basic functions, we will start by giving some special functions and functional spaces as follows:

Let $[a, b]$ be a finite interval on the real-axis \mathbb{R} , and $C([a, b])$, $AC^n([a, b])$, and $C^n([a, b])$ be the spaces of continuous functions, n -times absolutely continuous functions, and n -times continuously differentiable functions on $[a, b]$, respectively.

The space of the continuous functions y on $[a, b]$ with the norm is defined by [15]

$$\|y\| = \max_{a < t < b} |y(t)|$$

On the other hand, the space of n -times absolutely continuous given by [15]:

$$AC^n([a, b]) = \{y : [a, b] \rightarrow \mathbb{R} \mid y^{n-1} \in AC([a, b])\}.$$

1.1.1 Special functions

Here, we give some information on the EULER gamma, the BETA and MITTAG-LEFFLER functions which play the most important role in the theory of fractional calculus.

Definition 1.1 (Euler gamma function[15]). *The Euler Gamma function $\Gamma(z)$ is defined by:*

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt, \quad (\operatorname{Re}(z) > 0, z \in \mathbb{C}),$$

where $t^{z-1} = e^{(z-1)\ln t}$.

Example 1.1. 1. $\Gamma(1) = \int_0^{+\infty} e^{-t} dt = 1$.

2. $\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{\frac{1}{2}-1} e^{-t} dt = \sqrt{\pi}$.

Lemma 1.1. [15]. *For every $z \in \mathbb{C}$, $\operatorname{Re}(z) > 0$, and $n \in \mathbb{N}^*$, we have:*

1. $\Gamma(z+1) = z\Gamma(z)$.

2. $\Gamma(n+1) = (n-1)!$.

3. $\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{4^n n!}$.

property 1.1. [15] *For every $p > 0$, we have:*

$$\Gamma(p) = \lim_{n \rightarrow +\infty} \frac{n! n^p}{p(p+1)(p+2) \cdots (p+n)}.$$

Definition 1.2 (Beta function[15]). *The Beta function is a type of Euler integral, defined by:*

$$B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt, \quad (p, q \in \mathbb{C}, \operatorname{Re}(p) > 0, \operatorname{Re}(q) > 0).$$

For every $p, q \in \mathbb{C}$, with $\operatorname{Re}(p) > 0$ and $\operatorname{Re}(q) > 0$, we have the following relation:

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Definition 1.3 (Mittag-Leffler function[15]). *The Mittag-Leffler function is defined by*

$$E_\alpha(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0.$$

The generalized Mittag-Leffler function is defined by

$$E_{\alpha, \beta}(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0.$$

Theorem 1. [15] *For $\alpha = n \in \mathbb{N}$, $\lambda \in \mathbb{R}$, we have:*

$$\left(\frac{d}{dx}\right)^n E_n(\lambda x^n) = \lambda E_n(\lambda x^n),$$

and

$$\left(\frac{d}{dx}\right)^n x^{\beta-1} E_{n, \beta}(\lambda x^n) = \lambda x^{\beta-n-1} E_n(\lambda x^n).$$

1.2 Fractional integrations operators

1.2.1 Riemann-Liouville fractional integration

Definition 1.4. [15] Let $[a, b]$ be a finite interval on the real-axis \mathbb{R} . The Riemann-Liouville fractional integrals (left-sided and right-sided) of order $\alpha \in \mathbb{R}_+^*$ are defined by:

$${}^{RL}I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(t-x)^{1-\alpha}} dt, \quad x > a, \quad (1.1)$$

and

$${}^{RL}I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt, \quad x < b.$$

respectively.

property 1.2. [15]. For $\alpha > 0$ and $\beta > 0$, we have:

$$1. \left({}^{RL}I_{a+}^{\alpha} (x-a)^{\beta-1} \right) (x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (x-a)^{\alpha+\beta-1}.$$

$$2. \left({}^{RL}I_{b-}^{\alpha} (b-x)^{\beta-1} \right) (x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (b-x)^{\alpha+\beta-1}.$$

1.2.2 Hadamard fractional integration

Definition 1.5. [15]. Let $[a, b]$ be a finite or infinite interval on the half-axis \mathbb{R} . We consider the Hadamard fractional integrals (left-sided and right-sided) of order $\alpha \in \mathbb{C} (\text{Re}(\alpha) > 0)$, are defined by:

$${}^H I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t} \right)^{\alpha-1} \frac{f(t) dt}{t}, \quad a < x < b, \quad (1.2)$$

and

$${}^H I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x} \right)^{\alpha-1} \frac{f(t) dt}{t}, \quad a < x < b.$$

respectively.

property 1.3. [15] If $\text{Re}(\alpha) > 0$, $\text{Re}(\beta) > 0$, and $0 < a < b < +\infty$, then:

$$1. \left({}^H I_{a+}^{\alpha} \left(\ln \frac{t}{a} \right)^{\beta-1} \right) (x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \left(\ln \frac{x}{a} \right)^{\alpha+\beta-1}.$$

$$2. \left({}^H I_{b-}^{\alpha} \left(\ln \frac{b}{t} \right)^{\beta-1} \right) (x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \left(\ln \frac{b}{x} \right)^{\alpha+\beta-1}.$$

1.2.3 Katugampola fractional integration

Definition 1.6. [15] Let $[a, b]$ be a finite interval on the real-axis \mathbb{R} . We consider the Katugampola fractional integrals (left-sided and right-sided) of order $\alpha \in \mathbb{C}$ ($\text{Re}(\alpha) > 0$), are defined by:

$${}^{\rho}I_{a+}^{\alpha} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1} f(t)}{(x^{\rho} - t^{\rho})^{(\alpha-1)}} dt, \quad \rho > 0, a < x < b \quad (1.3)$$

and

$${}^{\rho}I_{b-}^{\alpha} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1} f(t)}{(t^{\rho} - x^{\rho})^{(\alpha-1)}} dt, \quad \rho > 0, a < x < b \quad (1.4)$$

respectively.

1.2.4 φ -fractional integration

Definition 1.7. [15] Let $[a, b]$ be a finite or infinite interval of the real-axis \mathbb{R} , and let $\alpha > 0$. Also, let $\varphi(x)$ be an increasing and positive monotone function on $(a, b]$, having a continuous derivative $\varphi'(x)$ on (a, b) .

The left and right-sided fractional integrals of a function f with respect to another function φ on $[a, b]$, are defined by:

$$I_{a+}^{\alpha;\varphi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \varphi'(t) (\varphi(x) - \varphi(t))^{\alpha-1} f(t) dt, \quad (1.5)$$

and

$$I_{b-}^{\alpha;\varphi} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \varphi'(t) (\varphi(t) - \varphi(x))^{\alpha-1} f(t) dt. \quad (1.6)$$

respectively.

Lemma 1.2. [15] Let $\alpha > 0$ and $\beta > 0$. Then, we have the following semi-group property given by:

$$I_{a+}^{\alpha;\varphi} I_{a+}^{\beta;\varphi} f(x) = I_{a+}^{\alpha+\beta;\varphi} f(x).$$

and

$$I_{b-}^{\alpha;\varphi} I_{b-}^{\beta;\varphi} f(x) = I_{b-}^{\alpha+\beta;\varphi} f(x).$$

Lemma 1.3. [15] Let $\alpha > 0$ and $\delta > 0$.

1. If $f(x) = (\varphi(x) - \varphi(a))^{\delta-1}$, then

$$I_{a+}^{\alpha;\varphi} f(x) = \frac{\Gamma(\delta)}{\Gamma(\delta + \alpha)} (\varphi(x) - \varphi(a))^{\alpha+\delta-1}.$$

2. If $g(x) = (\varphi(b) - \varphi(x))^{\delta-1}$, then

$$I_{b-}^{\alpha;\varphi} g(x) = \frac{\Gamma(\delta)}{\Gamma(\delta + \alpha)} (\varphi(b) - \varphi(x))^{\alpha+\delta-1}.$$

1.3 Fractional derivatives operators

1.3.1 Riemann-Liouville fractional derivative

Definition 1.8. [15] Let $f \in AC^n(a, b)$, where $n - 1 < \alpha < n$, $n \in \mathbb{N}$. The Riemann-Liouville fractional derivatives (left-sided and right-sided) of a function f of order α , are given by

$$D_{a+}^{\alpha} f(x) = \left(\frac{d}{dx} \right)^n {}^{RL}I_{a+}^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx} \right)^n \int_a^x (x-t)^{n-\alpha-1} f(t) dt, \quad (1.7)$$

and

$$D_{b-}^{\alpha} f(x) = (-1)^n \left(\frac{d}{dt} \right)^n {}^{RL}I_{b-}^{n-\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_x^b (t-x)^{n-\alpha-1} f(t) dt. \quad (1.8)$$

respectively.

1.3.2 Hadamard fractional derivative

Definition 1.9. [15] Let $f(x) \in AC^n(a, b)$, where $n - 1 < \alpha < n$, $n \in \mathbb{N}$. The Hadamard fractional derivatives (left-sided and right-sided) of order $\alpha \in \mathbb{C} (\operatorname{Re}(\alpha) > 0)$ on (a, b) , are given by

$${}^H D_{a+}^{\alpha} f(x) = \left(x \frac{d}{dx} \right)^n {}^H I_{a+}^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(x \frac{d}{dx} \right)^n \int_a^x \left(\ln \frac{x}{t} \right)^{(n-\alpha+1)} \frac{f(t) dt}{t}, \quad a < x < b, \quad (1.9)$$

and

$${}^H D_{b-}^{\alpha} f(x) = \left(-x \frac{d}{dx} \right)^n {}^H I_{b-}^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(-x \frac{d}{dx} \right)^n \int_x^b \left(\ln \frac{t}{x} \right)^{(n-\alpha+1)} \frac{f(t) dt}{t}, \quad a < x < b, \quad (1.10)$$

respectively.

property 1.4. [15] If $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\beta) > 0$, and $0 < a < b < +\infty$, then

$$\begin{aligned} 1. \left({}^H D_{a+}^{\alpha} \left(\ln \frac{t}{a} \right)^{\beta-1} \right) (x) &= \frac{\Gamma(\beta)}{\Gamma(\alpha-\beta)} \left(\ln \frac{x}{a} \right)^{\alpha-\beta-1}. \\ 2. \left({}^H D_{b-}^{\alpha} \left(\ln \frac{b}{t} \right)^{\beta-1} \right) (x) &= \frac{\Gamma(\beta)}{\Gamma(\alpha-\beta)} \left(\ln \frac{b}{x} \right)^{\alpha-\beta-1}. \end{aligned}$$

1.3.3 katugampula fractional derivative

Definition 1.10. [15] Let $\alpha, \rho \in \mathbb{R}^+$, and $n = [\alpha] + 1$. The KATUGAMPOLA fractional derivatives corresponding to the KATUGAMPOLA fractional integral (1.3) (resp. (1.4)) are defined by

$${}^{\rho} D_{a+}^{\alpha} f(x) = \left(x^{1-\rho} \frac{d}{dx} \right)^n {}^{\rho} I_{a+}^{n-\alpha} f(x) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(x^{1-\rho} \frac{d}{dx} \right)^n \int_a^x \frac{t^{\rho-1} f(t)}{(x^{\rho}-t^{\rho})^{\alpha-n+1}} dt, \quad (1.11)$$

and

$${}^{\rho} D_{b-}^{\alpha} f(x) = \left(-x^{1-\rho} \frac{d}{dx} \right)^n {}^{\rho} I_{b-}^{n-\alpha} f(x) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(-x^{1-\rho} \frac{d}{dx} \right)^n \int_x^b \frac{t^{\rho-1} f(t)}{(t^{\rho}-x^{\rho})^{\alpha-n+1}} dt. \quad (1.12)$$

respectively.

1.3.4 φ -fractional derivative

Definition 1.11. [15] Let φ be a function defined on $[a, b]$ that satisfies $\varphi'(x) \neq 0$ on $(a, b]$ and $\alpha > 0$. The Riemann-Liouville fractional derivatives (left-sided and right-sided) of a function f with respect to φ of order α corresponding to the Riemann-Liouville fractional integral, are defined by:

$$D_{a+}^{\alpha;\varphi} f(x) = \left(\frac{1}{\varphi'(x)} \frac{d}{dx} \right)^n {}^{RL} I_{a+}^{n-\alpha;\varphi} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\varphi'(x)} \frac{d}{dx} \right)^n \int_a^x \varphi'(t) (\varphi(x) - \varphi(t))^{n-\alpha-1} f(t) dt, \quad (1.13)$$

and

$$D_{b-}^{\alpha;\varphi} f(x) = \left(-\frac{1}{\varphi'(x)} \frac{d}{dx} \right)^n {}^{RL} I_{b-}^{n-\alpha;\varphi} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{1}{\varphi'(x)} \frac{d}{dx} \right)^n \int_x^b \varphi'(t) (\varphi(t) - \varphi(x))^{n-\alpha-1} f(t) dt. \quad (1.14)$$

respectively, where $n = [\alpha] + 1$.

Lemma 1.4. [15] Let $\alpha > 0$ and $\delta > 0$

1. If $f(x) = (\varphi(x) - \varphi(a))^{\delta-1}$, then

$$D_{a+}^{\alpha;\varphi} f(x) = \frac{\Gamma(\delta)}{\Gamma(\delta-\alpha)} (\varphi(x) - \varphi(a))^{\alpha+\delta-1}.$$

2. If $g(x) = (\varphi(b) - \varphi(x))^{\delta-1}$, then

$$D_{b-}^{\alpha;\varphi} g(x) = \frac{\Gamma(\delta)}{\Gamma(\delta-\alpha)} (\varphi(b) - \varphi(x))^{\alpha+\delta-1}.$$

Definition 1.12. [15] Let $\alpha > 0$, $n = [\alpha] + 1$, where $n \in \mathbb{N}$, and $f, \varphi \in C^n([a, b], \mathbb{R})$ two function such that φ is increasing and $\varphi'(x) \neq 0$ for all $x \in I$. The left and right-sided φ -Caputo fractional derivatives of f of order α on $[a, b]$ are given by:

$$({}^C D_{a+}^{\alpha;\varphi} f)(x) = {}^{RL} I_{a+}^{n-\alpha;\varphi} \left(\frac{1}{\varphi'(x)} \frac{d}{dx} \right)^n f(x), \quad (1.15)$$

and

$$({}^C D_{b-}^{\alpha;\varphi} f)(x) = {}^{RL} I_{b-}^{n-\alpha;\varphi} \left(-\frac{1}{\varphi'(x)} \frac{d}{dx} \right)^n f(x). \quad (1.16)$$

respectively.

Theorem 2. [15] Let $f \in C^n([a, b])$ and $\alpha > 0$, then

$$({}^C D_{a+}^{\alpha;\varphi} f)(x) = D_{a+}^{\alpha;\varphi} \left[f(x) - \sum_{k=0}^{n-1} \frac{1}{k!} (\varphi(x) - \varphi(a))^k f_{\varphi}^{(k)}(a) \right]. \quad (1.17)$$

and

$$({}^C D_{b-}^{\alpha;\varphi} f)(x) = D_{b-}^{\alpha;\varphi} \left[f(x) - \sum_{k=0}^{n-1} \frac{1}{k!} (\varphi(b) - \varphi(x))^k f_{\varphi}^{(k)}(b) \right]. \quad (1.18)$$

1.4 Fractional power series (FPS)

In this section, we will introduce some important definitions and theorems that generalize classical power series (CPS) into the fractional case in the sense of a conformable definition. These generalizations have become essential in various areas of analysis, particularly in the study of fractional calculus and its applications.

Definition 1.13. [12](Fractional Power Series)

A power series representation of the form:

$$\sum_{n=0}^{\infty} c_n (\varphi(t) - \varphi(t_0))^{\alpha n} = c_0 + c_1 (\varphi(t) - \varphi(t_0))^{\alpha} + c_2 (\varphi(t) - \varphi(t_0))^{2\alpha} + \dots \quad (1.19)$$

where, $0 \leq m - 1 < \alpha < m$, $m \in \mathbb{N}$ and $\varphi(t) \geq \varphi(t_0)$ is called a Fractional Power Series (FPS) about $\varphi(t_0)$, where $\varphi(t)$ is a variable and the constants c_n are the coefficients of the series.

Remark 1.1. 1. $0 \leq m - 1 < \alpha < m$ means that α is any positive non-integer real numbers.

2. When $\varphi(t_0) = 0$, the expansion

$$\sum_{n=0}^{\infty} c_n \varphi(t)^{n\alpha}$$

is called a fractional Maclaurin Series.

3. When writing out the term corresponding to $n = 0$ in equation (1.19), we have adopted the convention that $(\varphi(t) - \varphi(t_0))^0 = 1$.

4. Even when $\varphi(t) = \varphi(t_0)$, each of the terms of equation (1.19) vanishes for $n \geq 1$.

5. The FPS (1.19) always converges when $\varphi(t) = \varphi(t_0)$.

6. For the sake of simplicity of our notation, we may treat only the case where $\varphi(t_0) = 0$ in some theorems and definitions, which is not a loss of generality since the translation $\varphi(t)' = \varphi(t) - \varphi(t_0)$ reduces the FPS about $\varphi(t_0)$ to the FPS about 0.

Theorem 3. [12] We have the following two cases for the FPS

$$\sum_{n=0}^{\infty} c_n \varphi(t)^{n\alpha}$$

, $\varphi(t) \geq 0$:

1. If the FPS $\sum_{n=0}^{\infty} c_n \varphi(t)^{n\alpha}$ converges when $\varphi(t) = b > 0$, then it converges whenever $0 \leq \varphi(t) < b$.

2. If the FPS $\sum_{n=0}^{\infty} c_n \varphi(t)^{n\alpha}$ diverges when $\varphi(t) = d > 0$, then it diverges whenever $\varphi(t) > d$.

Proof. :

1) Suppose that $\sum c_n b^{n\alpha}$ converges. Then, we have $\lim_{n \rightarrow \infty} c_n b^{n\alpha} = 0$.

According to the definition of limit of sequences, there is a positive integer N such that $|c_n b^{n\alpha}| < \varepsilon$ whenever $n > N$. Take $\varepsilon = 1$ then $|c_n b^{n\alpha}| < 1$. Thus, for $n > N$, we have $|c_n \varphi(t)^{n\alpha}| = \left| \frac{c_n b^{n\alpha} \varphi(t)^{n\alpha}}{b^{n\alpha}} \right| = |c_n b^{n\alpha}| \cdot$

$\left|\frac{\varphi(t)}{b}\right|^{n\alpha} < \left|\frac{\varphi(t)}{b}\right|^{n\alpha}$. Since $0 \leq \varphi(t) < b$, then $\left|\frac{\varphi(t)}{b}\right|^{n\alpha} < 1$, so $\sum \left|\frac{\varphi(t)}{b}\right|^{n\alpha}$ is a convergent geometric series and since $\sum |c_n \varphi(t)^{n\alpha}| < \sum \left|\frac{\varphi(t)}{b}\right|^{n\alpha}$. Therefore, by the comparison test, the series is convergent. Thus the series $\sum |c_n \varphi(t)^{n\alpha}|$ is absolutely convergent and therefore convergent.

2) Suppose that $\sum c_n d^{n\alpha}$ diverges. Now, if $\varphi(t)$ is any number such that $\varphi(t) > d > 0$, then $\sum c_n \varphi(t)^{n\alpha} > \sum c_n d^{n\alpha}$, this imply $\sum c_n \varphi(t)^{n\alpha}$ diverges by the comparison test. \square

Theorem 4. [12] For the FPS $\sum_{n=0}^{\infty} c_n \varphi(t)^{n\alpha}$, $\varphi(t) \geq 0$, there are only three possibilities:

1. The series converges only when $\varphi(t) = 0$.
2. The series converges for each $\varphi(t) \geq 0$.
3. There is a positive real number R such that the series converges whenever $0 \leq \varphi(t) < R$ and diverges whenever $\varphi(t) \geq R$.

Proof. :

Suppose that neither case 1 nor case 2 are true. Then, there are nonzero numbers b and d such that $\sum c_n \varphi(t)^{n\alpha}$ converges for $\varphi(t) = b$ and diverges for $\varphi(t) = d$. Therefore, the set $S = \{\varphi(t) \mid \sum c_n \varphi(t)^{n\alpha} \text{ converges}\}$ is not empty. By the preceding theorem, the series diverges if $\varphi(t) > d$, so if $\varphi(t) \in S$ it must $0 < \varphi(t) \leq d$ and here. This says that d is an upper bound for S . Thus, by the completeness axiom, S has a least upper bound R such that, If $\varphi(t) > R$, then $\varphi(t) \notin S$, so $\sum c_n \varphi(t)^{n\alpha}$ diverges. If, $0 \leq \varphi(t) < R$, then $\varphi(t)$ is not an upper bound for S and so there exists $b \in S$ such that $b > \varphi(t)$. Since $b \in S$ and $\sum c_n \varphi(t)^{n\alpha}$ converges, so by the preceding theorem $\sum c_n \varphi(t)^{n\alpha}$ converges, so the proof of the theorem is complete. \square

Remark 1.2. The number R in Theorem (4) is called the radius of convergence of the FPS, where in case 1 in the previous theorem $R = 0$, and in case 2 $R = \infty$.

Theorem 5. [12] The CPS $\sum_{n=0}^{\infty} c_n \varphi(t)^n$, $-\infty < \varphi(t) < \infty$ has radius of convergence R if and only if the FPS $\sum_{n=0}^{\infty} c_n \varphi(t)^{n\alpha}$ has radius of convergence $R^{\frac{1}{\alpha}}$.

Proof. : If we make the change of variable $\varphi(t) = x^\alpha$, $x \geq 0$, then the CPS $\sum_{n=0}^{\infty} c_n \varphi(t)^n$ becomes $\sum_{n=0}^{\infty} c_n X^{n\alpha}$. This series converges for $0 < X^\alpha < R$, that is for $0 < x < R^{\frac{1}{\alpha}}$, and so the FPS $\sum_{n=0}^{\infty} c_n X^{n\alpha}$ has radius of convergence $R^{\frac{1}{\alpha}}$. Conversely, if we make the change of variable $\varphi(t) = x^{\frac{1}{\alpha}}$, $x \geq 0$, then the FPS $\sum_{n=0}^{\infty} c_n \varphi(t)^{n\alpha}$ becomes $\sum_{n=0}^{\infty} c_n x^n$. In fact, this series converges for $0 < X^{\frac{1}{\alpha}} < R^{\frac{1}{\alpha}}$ that is for $0 < x < R$. Since the two series $\sum_{n=0}^{\infty} c_n x^n$, $x \geq 0$ and $\sum_{n=0}^{\infty} c_n x^n$, $-\infty < x < \infty$ have the same radius of convergence $R = \lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right|$, the radius of convergence for the CPS $\sum_{n=0}^{\infty} c_n x^n$, $-\infty < x < \infty$ is R , so the proof of the theorem is complete. \square

Theorem 6. [1] Suppose that f has a FPS representation at $\varphi(t_0) = 0$ of the form

$$f(t) = \sum_{n=0}^{\infty} c_n \varphi(t)^{n\alpha}, 0 < \varphi(t) < R^{\frac{1}{\alpha}}, \quad R > 0. \quad (1.20)$$

and suppose that f is an infinitely conformable α -differentiable function, for some $0 \leq m - 1 < \alpha < m$ in a neighborhood of a point $\varphi(t_0) = 0$. Then the coefficients c_n in (1.20) will take the form $c_n = \frac{f^{(n\alpha)}(0)}{\alpha^n n!}$. where $f^{(n\alpha)}(\varphi(t))$ means the application of the conformable fractional derivative n times.

Proof. :

$f(t) = c_0 + c_1\varphi(t)^\alpha + c_2\varphi(t)^{2\alpha} + c_3\varphi(t)^{3\alpha} + \dots$, $0 < \varphi(t) < R^{\frac{1}{\alpha}}$, $R > 0$, Then, $f(0) = c_0$, and since $f^{(\alpha)}(\varphi(t)) = \alpha c_1 + 2\alpha c_2\varphi(t)^\alpha + 3\alpha c_3\varphi(t)^{2\alpha} + 4\alpha c_4\varphi(t)^{3\alpha} + \dots$, then $f^{(\alpha)}(0) = \alpha c_1$, implies $c_1 = \frac{f^{(\alpha)}(0)}{\alpha}$ then by applying second α -th derivative on f and evaluating $f^{(2\alpha)}(0) = 2\alpha^2 c_2$, implies $c_2 = \frac{f^{(2\alpha)}(0)}{2\alpha^2}$, when applying α -th derivative on f - times and evaluating at $\varphi(t_0) = 0$ we see that $f^{(n\alpha)}(0) = c_n \alpha(2\alpha)(3\alpha) \dots (n\alpha) = c_n \alpha^n n!$ and hence $c_n = \frac{f^{(n\alpha)}(0)}{\alpha^n n!}$. \square

Definition 1.14. [12] A power series of the form $\sum_{n=0}^{\infty} f_n(x)\varphi(t)^\alpha$, is called a multiple fractional power series about $\varphi(t_0) = 0$, where $\varphi(t)$ is a variable and $f_n(x)$ are functions of x called the coefficients of the series.

Theorem 7. Suppose that $u(x, \varphi(t))$ has a multiple FPS representation at $\varphi(t_0) = 0$ of the form $u(x, \varphi(t)) = \sum_{n=0}^{\infty} f_n(x)\varphi(t)^{n\alpha}$, $0 \leq m - 1 < \alpha < m$, $x \in I$, $0 \leq \varphi(t) \leq R^{\frac{1}{\alpha}}$. If $u_{\varphi(t)}^{(n\alpha)}(x, \varphi(t))$, $n = 0, 1, 2, \dots$ are continuous on $I \times (0, R^{\frac{1}{\alpha}})$, then $f_n(x) = \frac{u_{\varphi(t)}^{(n\alpha)}(x, 0)}{\alpha^n n!}$.

Proof. :

Assume that $u(x, \varphi(t))$ is a function of two variables that can be represented by a multiple FPS. Then: $u(x, \varphi(t)) = f_0(x) + f_1(x)\varphi(t)^\alpha + f_2(x)\varphi(t)^{2\alpha} + f_3(x)\varphi(t)^{3\alpha} + \dots$, put $\varphi(t) = 0$, then $f_0(x) = u(x, 0)$, and since $u_{\varphi(t)}^{(\alpha)}(x, \varphi(t)) = \alpha f_1(x) + 2\alpha f_2(x)\varphi(t)^\alpha + \dots$, then $u_{\varphi(t)}^{(\alpha)}(x, 0) = \alpha f_1(x)$, implies $f_1(x) = \frac{u_{\varphi(t)}^{(\alpha)}(x, 0)}{\alpha}$, then by applying second α -derivative of $\varphi(t)$ on $u(x, \varphi(t))$ we evaluate $u_{\varphi(t)}^{(2\alpha)}(x, 0) = 2\alpha^2 f_2(x)$, implies $f_2(x) = \frac{u_{\varphi(t)}^{(2\alpha)}(x, 0)}{2\alpha^2}$, when applying α -th derivative of $\varphi(t)$ on $u(x, \varphi(t))$ n -times and evaluating at $\varphi(t_0) = 0$ we see that $u_{\varphi(t)}^{(n\alpha)}(x, 0) = f_n(x) \cdot \alpha(2\alpha)(3\alpha) \dots (n\alpha) = f_n(x) \alpha^n n!$, and hence $f_n(x) = \frac{u_{\varphi(t)}^{(n\alpha)}(x, 0)}{\alpha^n n!}$. \square

Remark 1.3. : From the previous theorem, it is clear that if u is an n -dimensional function has a multiple FPS representation at $\varphi(t_0) = 0$, then it can be derived in the same manner, which will be illustrated in the next corollary.

Corollary 1.1. [4] Suppose that $u(x, y, \varphi(t))$ has a multiple FPS representation at $\varphi(t_0) = 0$ of the form $u(x, y, \varphi(t)) = \sum_{n=0}^{\infty} f_n(x, y)\varphi(t)^{n\alpha}$, $0 \leq m - 1 < \alpha < m$, $(x, y) \in I_1 \times I_2$, $0 \leq \varphi(t) \leq R^{\frac{1}{\alpha}}$. If $u_{\varphi(t)}^{(n\alpha)}(x, y, \varphi(t))$ are continuous on $I_1 \times I_2 \times (0, R^{\frac{1}{\alpha}})$. Then $f_n(x, y) = \frac{u_{\varphi(t)}^{(n\alpha)}(x, y, 0)}{\alpha^n n!}$.

1.5 Operators on scalar field and vector field

Definition 1.15. A scalar field f is a function that takes a point in space and assigns a real number to it (f : From points in the space \rightarrow To real numbers) $f : (x, y, z) \rightarrow \mathbb{R}$.

Definition 1.16. A vector field $\mathbf{U} = \langle u, v, w \rangle$ in space is a function that takes any point in space and assigns a vector to it : (\mathbf{U} : From points in the space \rightarrow To vectors in the space) or $\mathbf{U} : (x, y, z) \rightarrow \langle u(x, y, z), v(x, y, z), w(x, y, z) \rangle$.

We notice that the component of vector field is a scalar field.

1.5.1 Del operator

Definition 1.17. :The del operator is:

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle. \quad (1.21)$$

So the del operator is a vector.

1.5.2 Gradient of a scalar field

Definition 1.18. : The gradient of a given scalar field $f = f(x, y, z)$ is a vector field denoted by $(\text{grad } f)$ or (∇f) and it is defined by apply del operator on f as follows :

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle. \quad (1.22)$$

1.5.3 Divergence of a vector field

Definition 1.19. : The divergence of a given vector field $\mathbf{U} = \langle u, v, w \rangle$ is denoted by $(\text{div } \mathbf{U})$ or $(V \cdot \mathbf{U})$ and it is defined as follows :

$$\nabla \cdot \mathbf{U} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}. \quad (1.23)$$

where (\cdot) is the dot (scalar) product. We notice that the divergence of a vector field is a scalar field.

1.5.4 Directional derivative of a scalar field

From Vector Calculus we know that the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ give the rates of change of the scalar field $f(x, y, z)$ in the directions of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively. It seems natural to ask what is the rate of change in any other direction. This is what the directional derivatives represent.

Definition 1.20. The directional derivative of scalar field $f(x, y, z)$ in the direction of a unit vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is denoted by $(D_{\mathbf{a}}f)$ or $(\mathbf{a} \cdot \nabla f)$ and it is defined as follows

$$\mathbf{a} \cdot \nabla f = a_1 \frac{\partial f}{\partial x} + a_2 \frac{\partial f}{\partial y} + a_3 \frac{\partial f}{\partial z}. \quad (1.24)$$

where (\cdot) is the dot (scalar) product.

1.5.5 Directional derivative of a vector field

Definition 1.21. The directional derivative of vector field $\mathbf{U} = \langle u, v, w \rangle$ in the direction of a unit vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is denoted by $(D_{\mathbf{a}}\mathbf{U})$ or $(\mathbf{a} \cdot \nabla)\mathbf{U}$ and it is defined as follows:

$$(\mathbf{a} \cdot \nabla) \mathbf{U} = \left\langle a_1 \frac{\partial u}{\partial x} + a_2 \frac{\partial u}{\partial y} + a_3 \frac{\partial u}{\partial z}, a_1 \frac{\partial v}{\partial x} + a_2 \frac{\partial v}{\partial y} + a_3 \frac{\partial v}{\partial z}, a_1 \frac{\partial w}{\partial x} + a_2 \frac{\partial w}{\partial y} + a_3 \frac{\partial w}{\partial z} \right\rangle \quad (1.25)$$

If we want to find the directional derivative of vector field $\mathbf{U} = \langle u, v, w \rangle$ in the direction of \mathbf{U} itself we obtain a similar equation which is :

$$(\mathbf{U} \cdot \nabla) \mathbf{U} = \left\langle u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}, u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z}, u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right\rangle \quad (1.26)$$

1.5.6 Laplacian operator

Definition 1.22. The Laplacian operator is denoted by (∇^2) or (Δ) and it is defined as follows:

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (1.27)$$

. where (\cdot) is The dot (scalar) product .

1.5.7 Laplacien operator of a scalar field

Definition 1.23. The Laplacian operator of a scalar field $f(x, y, z)$ is denoted by $(\nabla^2 f)$ or (Δf) and it is defined as follows:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \quad (1.28)$$

1.5.8 Laplacian operator of a vector field:

Definition 1.24. : The Laplacian operator of a vector field $\mathbf{U} = \langle u, v, w \rangle$ is denoted by $(\nabla^2 \mathbf{U})$ or $(\Delta \mathbf{U})$ and it is defined as follows :

$$\nabla^2 \mathbf{U} = \left\langle \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}, \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right\rangle \quad (1.29)$$

1.6 Navier-Stokes Equation

The Navier-Stokes Equation is a famous governing equation of motion of viscous fluid flow has been derived in 1822 [18]. The equation can be regarded as Newton's second law of motion for fluid substances, and is a combination of momentum equation, continuity equation and energy equation .

This equation describes many physical things such as ocean currents, liquid flow in pipes , blood flow and air flow around the wings of an aircraft [13].

1.6.1 Classical Model of Navier-Stokes Equation

The Navier-Stokes equation is defined as follows [19]:

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} + \frac{1}{\rho} \nabla \mathbf{P} - \nu \nabla^2 \mathbf{U} = \mathbf{F}, \Omega \times (0, T] \quad (1.30)$$

This system is subject to the following conditions:

$$\nabla \cdot \mathbf{U} = 0, \text{ in } \Omega \times (0, T] \text{ with } \mathbf{x} \in \Omega \quad \text{Mass conservation condition} \quad (1.31)$$

$$\mathbf{U} = \mathbf{U}(\mathbf{x}), \text{ on } \Gamma \times (0, T] \quad \text{Boundary conditions} \quad (1.32)$$

$$\mathbf{U}(\mathbf{x}, \mathbf{0}) = \mathbf{U}_0(\mathbf{x}) \quad \text{in } \Omega \quad \text{Initial conditions} \quad (1.33)$$

where:

$\mathbf{U} = \langle u, v, w \rangle$ is a vector velocity field for fluid element which moves through space, and the components u, v and w are scalar fields depend on both Cartesian space ($\mathbf{x} = (x, y, z) \in \Omega \subseteq \mathbb{R}^3$) and time $\varphi(t) \in (0, T]$.

\mathbf{P} is the pressure scalar field which also depends on both Cartesian space \mathbf{x} and time $\varphi(t)$.

∇ is the Dell operator.

∇^2 is the Laplacian operator.

\mathbf{F} is the force scalar field of components of the external forces ($\mathbf{F} = \langle F_x, F_y, F_z \rangle$).

Γ is the boundary of space Ω .

ν is the kinematics viscosity constant of fluid.

ρ is the density of fluid.

The Physical analysis of (1.30) as follows :

$\frac{\partial \mathbf{U}}{\partial t}$: (Acceleration term) which expresses Change in velocity of fluid.

$(\mathbf{U} \cdot \nabla) \mathbf{U}$: (Advection term) which expresses the force exerted on a particle of fluid by the other particles of fluid surrounding it, or in other words how the fluid pushes itself around.

$\nu \nabla^2 \mathbf{U}$: (Diffusion term) which describes how fluid motion is damped.

$\frac{1}{\rho} \nabla \mathbf{P}$: (Pressure follows term) which expresses the fluid movement from high-pressure areas to low-pressure areas.

\mathbf{F} : (External forces term) represents external forces that act on the fluid for example the gravity, wind, etc....

$\nabla \cdot \mathbf{U} = 0$ or $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ (Mass conservation condition) As is clear the velocity vector has zero divergence which means the net mass change of any sub-region is zero and the fluid is Incompressible.

By using (1.22), (1.23) ,(1.26) and (1.29) we can write the Navier-Stokes equation in Cartesian coordinates in three dimensions as:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + \frac{1}{\rho} \frac{\partial P}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} - \nu \frac{\partial^2 u}{\partial y^2} - \nu \frac{\partial^2 u}{\partial z^2} = f_x \quad (1.33.a)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + \frac{1}{\rho} \frac{\partial P}{\partial y} - \nu \frac{\partial^2 v}{\partial x^2} - \nu \frac{\partial^2 v}{\partial y^2} - \nu \frac{\partial^2 v}{\partial z^2} = f_y \quad (1.33.b)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} + \frac{1}{\rho} \frac{\partial P}{\partial z} - \nu \frac{\partial^2 w}{\partial x^2} - \nu \frac{\partial^2 w}{\partial y^2} - \nu \frac{\partial^2 w}{\partial z^2} = f_z \quad (1.33.c)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{(Mass conservation condition)} \quad (1.33.d)$$

and in the Cartesian coordinates in two dimensions as:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial P}{\partial x} - v \frac{\partial^2 u}{\partial x^2} - v \frac{\partial^2 u}{\partial y^2} = f_x \quad (1.34.a)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial P}{\partial y} - v \frac{\partial^2 v}{\partial x^2} - v \frac{\partial^2 v}{\partial y^2} = f_y \quad (1.34.b)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (\text{Mass conservation condition}) \quad (1.34.c)$$

Remark 1.4.

1. Navier-Stokes equation can be written also in the cylindrical and spherical coordinate systems [21].
2. Even though there are other forms of the Navier-Stokes equation the reader may face in other references, we will use the form mentioned above, and that is not a loss of generality since various forms resulted only from some scale invariance or change of variables of the equation [19].
3. During our work in this thesis we will consider the external forces to be zero.

1.6.2 Existence and Smoothness of Solution of Navier-Stokes Equation

At the beginning we need to introduce the following definitions:

Definition 1.25. : [20] (Differentiable functions of class C^k):

Let $U \subset \mathbb{R}$ be open, and let $f : U \rightarrow \mathbb{R}$, we say that f is of class $C^k, k = 0, 1, 2, \dots$ if the derivatives $f', f'', \dots, f^{(k)}$ exist and all are continuous except for $f^{(k)}$.

Definition 1.26. : The function f is said to be smooth if it is of class C^∞ .

Definition 1.27. : The function f is said to be analytic if it is smooth and it equals its Taylor's series expansion around any point in its domain.

The Navier-Stokes equation describes the motion of a fluid in \mathbb{R}^n ($n = 2$ or 3), and this equation is to be solved for an unknown velocity vector $\mathbf{U}(\mathbf{x}, \varphi(t)) = u_i(\mathbf{x}, \varphi(t))_{1 \leq i \leq n} \in \mathbb{R}^n$, and pressure $\mathbf{P}(\mathbf{x}, \varphi(t)) \in \mathbb{R}^1$ defined for position $\mathbf{x} \in \mathbb{R}^n$ and time $\varphi(t) \geq 0$.

In two dimensions ($n = 2$) the existence and smoothness of a solution to the Navier-Stokes equation have been known for a long time [10], but that is not the case in three dimensions equation since the solution often includes turbulent flows which means the flow is disordered in space and time, and that property remains one of the greatest unsolved problems in physics, and because of that, mathematicians have not yet proved that smooth solutions always exist, or that if they do exist.

This is called the "Navier-Stokes Existence and Smoothness Problem", which has been considered as one of seven problems called "Millennium Prize Problems" in mathematics that have been developed in May 2000, by the Clay Mathematics Institute, which offered a one million US Dollar prize to the first person providing a solution for a specific statement of the problem. [10]

1.6.3 Time-Fractional Navier-Stokes Equation

Recently, several researchers have extended the classical Navier-Stokes equations to a fractional-order form by replacing the time derivative with a fractional derivative of order α , $0 < \alpha \leq 1$ as in [11], and that form is called time-fractional Navier-Stokes equation.

However, in our work we analyze the generalized form of the fractional Navier-Stokes equations with Caputo fractional derivatives, formulated as follows:

$$\frac{\partial^\alpha \mathbf{U}}{\partial t^\alpha} + (\mathbf{U} \cdot \nabla) \mathbf{U} - \nu \nabla^2 \mathbf{U} + \frac{1}{\rho} \nabla \mathbf{P} = 0 \quad (1.35)$$

Subject to the same conditions mentioned in equations (1.31), (1.32) and (1.33), where $0 < \alpha \leq 1$

Chapter 2

Solving Navier-Stokes Equation by the Residual Power Series Method

In this chapter, we employ the RPS algorithm to solve the Navier-Stokes equations in the first section. Subsequently, in the following section, we construct approximate solutions for these equations using the RPS method.

2.1 RPS Algorithm for Solving Fractional Navier-Stokes equation

consider the following unsteady two-dimensional incompressible fractional Navier-Stokes equation in the Cartesian coordinat system:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + Ru \frac{\partial u}{\partial x} + Rv \frac{\partial u}{\partial y} + \frac{\partial P}{\partial x} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \quad (2.1.a)$$

$$\frac{\partial^\alpha v}{\partial t^\alpha} + Ru \frac{\partial v}{\partial x} + Rv \frac{\partial v}{\partial y} + \frac{\partial P}{\partial y} - \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = 0 \quad (2.1.b)$$

Subject to the following conditions:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \text{incompressibility condition} \quad (2.2.a)$$

$$\mathbf{U}(x_b, y_b, \varphi(t)) = \mathbf{U}_b(x_b, y_b) \in \Gamma \quad \text{Boundary conditions} \quad (2.2.b)$$

$$\mathbf{U}(x, y, 0) = \mathbf{f}_i(x, y), \quad \text{Initial conditions} \quad (2.2.c)$$

where $\mathbf{U} = (u, v) = (u(x, y, T), v(x, y, T))$, $(x, y) \in \Omega \subseteq \mathbb{R}^2$, Γ is the boundary of Ω $i=1,2$ and $0 < \alpha \leq 1$.

It is important to note that although the solution obtained by the RPS method satisfies the incompressibility condition, we will not adopt this method in our approach.

The RPS method assumes the solution of the system (2.1.a)(2.1.b) as a fractional power series about the initial point $\varphi(t) = 0$, as follows:

$$u(x, y, t) = \sum_{n=0}^{\infty} f_n(x, y) \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(1 + n\alpha)} \quad (2.3.a)$$

$$v(x, y, t) = \sum_{n=0}^{\infty} g_n(x, y) \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(1 + n\alpha)} \quad (2.3.b)$$

$$P(x, y, t) = \sum_{n=0}^{\infty} h_n(x, y) \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(1 + n\alpha)} \quad (2.3.c)$$

where $0 < \alpha \leq 1$, $(x, y) \in \Omega$, $0 \leq \varphi(t) < R^{\frac{1}{\alpha}}$

It is clear that $u(x, y, t)$ and $v(x, y, t)$ satisfy the initial conditions (2.2.c) which can be rewritten as:

$$u(x, y, 0) = f(x, y) \quad (2.4.a)$$

$$v(x, y, 0) = g(x, y) \quad (2.4.b)$$

Hence, we can obtain the initial guess approximation of $u(x, y, t)$ and $v(x, y, t)$ as:

$$u_0(x, y, 0) = f_0(x, y) = f(x, y) \quad (2.5.a)$$

$$v_0(x, y, 0) = g_0(x, y) = g(x, y) \quad (2.5.b)$$

So equations (2.3.a) and (2.3.b) could be reformulated as:

$$u(x, y, t) = f(x, y) + \sum_{n=1}^{\infty} f_n(x, y) \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(1 + n\alpha)} \quad (2.6.a)$$

$$v(x, y, t) = g(x, y) + \sum_{n=1}^{\infty} g_n(x, y) \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(1 + n\alpha)} \quad (2.6.b)$$

$$P(x, y, t) = \sum_{n=1}^{\infty} h_{n-1}(x, y) \frac{(\varphi(t) - \varphi(t_0))^{(n-1)\alpha}}{\Gamma(1 + (n-1)\alpha)} \quad (2.6.c)$$

The initial conditions are

$$u(x, y, 0) = -\cos x \sin y \quad (2.7.a)$$

$$v(x, y, 0) = \sin x \cos y \quad (2.7.b)$$

The boundary conditions are

$$u(x, 0, t) = 0 \quad (2.8.a)$$

$$v(0, y, t) = 0 \quad (2.8.b)$$

The RPS method will take the following technique:

Assume that the solution of the problem takes the following form

$$u(x, y, t) = \sum_{n=0}^{\infty} f_n(x, y) \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \quad (2.9.a)$$

$$v(x, y, t) = \sum_{n=0}^{\infty} g_n(x, y) \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \quad (2.9.b)$$

$$P(x, y, t) = \sum_{n=0}^{\infty} h_n(x, y) \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \quad (2.9.c)$$

To find the initial guess for $n = 0$ using initial conditions (2.7.a)(2.7.b), and note that the pressure has no initial condition which needs to make a shift of its index, we get

$$u(x, y, t) = -\cos x \sin y + \sum_{n=1}^{\infty} f_n(x, y) \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(1 + n\alpha)} \quad (2.10.a)$$

$$v(x, y, t) = \sin x \cos y + \sum_{n=1}^{\infty} g_n(x, y) \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(1 + n\alpha)} \quad (2.10.b)$$

$$P(x, y, t) = \sum_{n=1}^{\infty} h_{n-1}(x, y) \frac{(\varphi(t) - \varphi(t_0))^{(n-1)\alpha}}{\Gamma(1 + (n-1)\alpha)} \quad (2.10.c)$$

Now we will construct the truncated series of the proposed solutions as follows

$$u_k(x, y, t) = -\cos x \sin y + \sum_{n=1}^k f_n(x, y) \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(1 + n\alpha)} \quad (2.11.a)$$

$$v_k(x, y, t) = \sin x \cos y + \sum_{n=1}^k g_n(x, y) \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(1 + n\alpha)} \quad (2.11.b)$$

$$P_k(x, y, t) = \sum_{n=1}^k h_{n-1}(x, y) \frac{(\varphi(t) - \varphi(t_0))^{(n-1)\alpha}}{\Gamma(1 + (n-1)\alpha)} \quad (2.11.c)$$

Now we define the residual functions Res_u and Res_v for equations (2.1.a)(2.1.b), as follows:

$$Res_u(x, y, t) = \frac{\partial^\alpha u}{\partial t^\alpha} + Ru \frac{\partial u}{\partial x} + Rv \frac{\partial u}{\partial y} + \frac{\partial P}{\partial x} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} \quad (2.12.a)$$

$$Res_v(x, y, t) = \frac{\partial^\alpha v}{\partial t^\alpha} + Ru \frac{\partial v}{\partial x} + Rv \frac{\partial v}{\partial y} + \frac{\partial P}{\partial y} - \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} \quad (2.12.b)$$

Therefore, the k-th truncated residual functions are:

$$Res_{u,k}(x, y, t) = \frac{\partial^\alpha u_k}{\partial t^\alpha} + Ru_k \frac{\partial u_k}{\partial x} + Rv_k \frac{\partial u_k}{\partial y} + \frac{\partial P_k}{\partial x} - \frac{\partial^2 u_k}{\partial x^2} - \frac{\partial^2 u_k}{\partial y^2} \quad (2.13.a)$$

$$Res_{v,k}(x, y, t) = \frac{\partial^\alpha v_k}{\partial t^\alpha} + Ru_k \frac{\partial v_k}{\partial x} + Rv_k \frac{\partial v_k}{\partial y} + \frac{\partial P_k}{\partial y} - \frac{\partial^2 v_k}{\partial x^2} - \frac{\partial^2 v_k}{\partial y^2} \quad (2.13.b)$$

Substituting Eq. (2.11.a),(2.11.b),(2.11.c) in Eq. 2.13.a,2.13.b gives

$$\begin{aligned}
Res_{u,k}(x, y, t) = & D_{\varphi(t)-\varphi(t_0)}^\alpha \left(\sum_{n=1}^k f_n \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \right) - R \cos x \sin y \sum_{n=0}^k \frac{\partial f_n}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \\
& + R \sin x \sin y \sum_{n=1}^k f_n \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \\
& + R \left(\sum_{n=1}^k f_n \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \right) \left(\sum_{n=0}^k \frac{\partial f_n}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \right) \\
& + R \sin x \cos y \sum_{n=0}^k \frac{\partial f_n}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} - R \cos x \cos y \sum_{n=1}^k g_n \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \\
& + R \left(\sum_{n=1}^k g_n \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \right) \left(\sum_{n=0}^k \frac{\partial f_n}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \right) \\
& + \sum_{n=1}^k \frac{\partial h_{n-1}}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^{(n-1)\alpha}}{\Gamma(1 + (n-1)\alpha)} - 2 \cos x \sin y - R \sin x \cos x \\
& - \sum_{n=0}^k \frac{\partial^2 f_n}{\partial x^2} \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} - \sum_{n=0}^k \frac{\partial^2 f_n}{\partial y^2} \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \tag{2.14.a}
\end{aligned}$$

$$\begin{aligned}
Res_{v,k}(x, y, t) = & D_{\varphi(t)-\varphi(t_0)}^\alpha \left(\sum_{n=1}^k g_n \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \right) - R \cos x \sin y \sum_{n=0}^k \frac{\partial g_n}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \\
& + R \cos x \cos y \sum_{n=1}^k f_n \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \\
& + R \left(\sum_{n=1}^k f_n \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \right) \left(\sum_{n=0}^k \frac{\partial g_n}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \right) \\
& + R \sin x \cos y \sum_{n=0}^k \frac{\partial g_n}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} - R \sin x \sin y \sum_{n=1}^k g_n \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \\
& + R \left(\sum_{n=1}^k g_n \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \right) \left(\sum_{n=0}^k \frac{\partial g_n}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \right) \\
& + \sum_{n=1}^k \frac{\partial h_{n-1}}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^{(n-1)\alpha}}{\Gamma(1 + (n-1)\alpha)} + 2 \sin x \cos y \\
& - R \sin y \cos y - \sum_{n=0}^k \frac{\partial^2 g_n}{\partial x^2} \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \\
& - \sum_{n=0}^k \frac{\partial^2 g_n}{\partial y^2} \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \tag{2.14.b}
\end{aligned}$$

As in [4] [6] [3] $Res(x, y, t) = 0$ and $\lim_{k \rightarrow \infty} Res_k(x, y, t) = Res(x, y, t)$ for each $\varphi(t) \in [\varphi(t_0), \varphi(t_0) + R^{\frac{1}{\alpha}}]$ and $(x, y) \in \mathbb{R}$, where R is a non negative real number represents the radius of convergence.

Therefore $\frac{\partial^{r\alpha}}{\partial t^{r\alpha}} Res(x, y, t) = 0$, since the fractional derivative of a constant function in the conformable sense is zero, and in the mean time the fractional derivative $\frac{\partial^{r\alpha}}{\partial t^{r\alpha}}$ of $Res(x, y, t)$ and $Res_k(x, y, t)$ are matching at $\varphi(t) = \varphi(t_0)$ for each $r = 0, 1, 2, \dots$. Now, if we substitute $\varphi(t_0) = 0, r = k - 1$ we obtain :

$$\frac{\partial^{(k-1)\alpha}}{\partial t^{(k-1)\alpha}} \text{Res}_{u,k}(x, y, 0) = 0 \quad (2.15.a)$$

$$\frac{\partial^{(k-1)\alpha}}{\partial t^{(k-1)\alpha}} \text{Res}_{v,k}(x, y, 0) = 0 \quad (2.15.b)$$

To get the form of the required coefficients $f_n(x, y), g_n(x, y)$ or $h_{n-1}(x, y)$, where $n = 1, 2, 3, \dots, k$ in equations (2.11.a)(2.11.b)(2.11.c), we apply the following RPS technique:

First we substitute the k -th truncated series of $u(x, y, t), v(x, y, t)$ and $P(x, y, t)$ into equations (2.9.a)(2.9.b)(2.9.c). Second we find the fractional derivative formula for $\frac{\partial^{(k-1)\alpha}}{\partial t^{(k-1)\alpha}}$ of both $\text{Res}_{u,k}(x, y, \varphi(t))$ and $\text{Res}_{v,k}(x, y, t)$, $k = 1, 2, 3, \dots$, and finally we solve the obtained algebraic system (2.10.a)(2.10.b)(2.10.c).

2.2 Construction of the RPS Approximate Solutions of the Navier-Stokes Equation

To get the form of the required coefficients $f_n(x, y), g_n(x, y)$ and $h_{n-1}(x, y)$, where $n = 1, 2, 3, \dots, k$ in equations (2.11.a)(2.11.b)(2.11.c), we apply the following steps:

Step one:

To determine $f_1(x, y), g_1(x, y)$ and $h_0(x, y)$, we consider $k = 1$ in (2.13.a)(2.13.b), to have:

$$\text{Res}_{u,1}(x, y, t) = \frac{\partial^\alpha u_1}{\partial t^\alpha} + Ru_1 \frac{\partial u_1}{\partial x} + Rv_1 \frac{\partial u_1}{\partial y} + \frac{\partial P_1}{\partial x} - \frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_1}{\partial y^2} \quad (2.16.a)$$

$$\text{Res}_{v,1}(x, y, t) = \frac{\partial^\alpha v_1}{\partial t^\alpha} + Ru_1 \frac{\partial v_1}{\partial x} + Rv_1 \frac{\partial v_1}{\partial y} + \frac{\partial P_1}{\partial y} - \frac{\partial^2 v_1}{\partial x^2} - \frac{\partial^2 v_1}{\partial y^2} \quad (2.16.b)$$

and by substituting $k = 1$ in (2.11.a)(2.11.b)(2.11.c), we get

$$u_1(x, y, t) = -\cos x \sin y + f_1(x, y) \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \quad (2.17.a)$$

$$v_1(x, y, t) = \sin x \cos y + g_1(x, y) \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \quad (2.17.b)$$

$$P_1(x, y, t) = h_0(x, y) \quad (2.17.c)$$

Then we have to substitute Eq (2.17.a)(2.17.b)(2.17.c) in Eq (2.14.a)(2.14.b) to find the first residual functions as follows:

$$\begin{aligned}
\text{Res}_{u,1}(x, y, t) = & f_1 - R \cos x \sin y \frac{\partial f_1}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} + R \sin x \\
& \times \sin y f_1 \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} + R f_1 \frac{\partial f_1}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma^2(\alpha + 1)} \\
& + R \sin x \cos y \frac{\partial f_1}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} - R \cos x \\
& \times \cos y g_1 \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} + R g_1 \frac{\partial f_1}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma^2(\alpha + 1)} \\
& + \frac{\partial h_0}{\partial x} - 2 \cos x \sin y - R \sin x \cos x - \frac{\partial^2 f_1}{\partial x^2} \\
& \times \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} - \frac{\partial^2 f_1}{\partial y^2} \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)}
\end{aligned} \tag{2.18.a}$$

$$\begin{aligned}
\text{Res}_{v,1}(x, y, t) = & g_1 - R \cos x \sin y \frac{\partial g_1}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} + R \cos x \\
& \times \cos y f_1 \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} + R f_1 \frac{\partial g_1}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma^2(\alpha + 1)} \\
& + R \sin x \cos y \frac{\partial g_1}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} - R \sin x \\
& \times \sin y g_1 \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} + R g_1 \frac{\partial g_1}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma^2(\alpha + 1)} \\
& + \frac{\partial h_0}{\partial y} + 2 \sin x \cos y - R \sin y \cos y - \frac{\partial^2 g_1}{\partial x^2} \\
& \times \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} - \frac{\partial^2 g_1}{\partial y^2} \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)}
\end{aligned} \tag{2.18.b}$$

Then we will calculate the truncated residual functions at $t = 0$, to get

$$\text{Res}_{u,1}(x, y, 0) = f_1 - 2 \cos x \sin y - R \sin x \cos x + \frac{\partial h_0}{\partial x} \tag{2.19.a}$$

$$\text{Res}_{v,1}(x, y, 0) = g_1 + 2 \sin x \cos y - R \sin y \cos y + \frac{\partial h_0}{\partial y} \tag{2.19.b}$$

And by Eq. (2.15.a)(2.15.b) we know that:

$$\text{Res}_{u,1}(x, y, 0) = 0 \quad \text{and} \quad \text{Res}_{v,1}(x, y, 0) = 0 \tag{2.20}$$

Solving these equations for f_1 and g_1 we get

$$f_1(x, y) = 2 \cos x \sin y + R \sin x \cos x - \frac{\partial h_0}{\partial x} \tag{2.21.a}$$

$$g_1(x, y) = -2 \sin x \cos y + R \sin y \cos y - \frac{\partial h_0}{\partial y} \tag{2.21.b}$$

These are preliminary results since we still did not find $h_0(x, y)$, and in order to do so we must use the boundary conditions.

$$u_1(x, 0, t) = 0 \quad \text{and} \quad v_1(0, y, t) = 0 \quad (2.22)$$

After we substitute Eq. (2.21.a)(2.21.b) in Eq. (2.17.a)(2.17.b)(2.17.c), and applying the boundary conditions (2.22), we have:

$$\frac{\partial h_0}{\partial x} = R \sin x \cos x \quad (2.23.a)$$

$$\frac{\partial h_0}{\partial y} = R \sin y \cos y \quad (2.23.b)$$

Integrating Eq. (2.23.a) with respect to x , we get

$$h_0(x, y) = -\frac{R}{4} \cos 2x + \theta(y) \quad (2.24)$$

where $\theta(y)$ is a function of y only, and to find it, derive the last equation with respect to y :

$$\frac{\partial h_0}{\partial y} = \theta'(y) \quad (2.25)$$

Substitute this equation in 2.23.b, then integrate the resultant equation with respect to y , we get

$$\theta(y) = -\frac{R}{4} \cos 2y \quad (2.26)$$

Now substitute this equation in (2.24), we have:

$$h_0(x, y) = -\frac{R}{4} (\cos 2x + \cos 2y) \quad (2.27)$$

Therefore the functions $f_1(x, y)$ and $g_1(x, y)$ will have their final forms as:

$$f_1(x, y) = 2 \cos x \sin y \quad (2.28.a)$$

$$g_1(x, y) = -2 \sin x \cos y \quad (2.28.b)$$

And finally the first RPS approximate solution would be:

$$u_1(x, y, t) = -\cos x \sin y + 2 \cos x \sin y \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \quad (2.29.a)$$

$$v_1(x, y, t) = \sin x \cos y - 2 \sin x \cos y \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \quad (2.29.b)$$

$$P_1(x, y, t) = -\frac{R}{4} (\cos 2x + \cos 2y) \quad (2.29.c)$$

Step Two:

To obtain $f_2(x, y)$, $g_2(x, y)$ and $h_1(x, y)$ we substitute the second truncated series:

At first we have to find the truncated series (2.11.a) (2.11.b) (2.11.c) for $k = 2$, to get

$$u_2(x, y, t) = -\cos x \sin y + f_1(x, y) \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} + f_2(x, y) \frac{\alpha^{2\alpha}}{\Gamma(2\alpha + 1)} \quad (2.30.a)$$

$$v_2(x, y, t) = \sin x \cos y + g_1(x, y) \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} + g_2(x, y) \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} \quad (2.30.b)$$

$$P_2(x, y, t) = h_0(x, y) + h_1(x, y) \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \quad (2.30.c)$$

Then substitute these equations in Eq.(2.14.a) (2.14.b), to get the following residual truncated functions.

$$\begin{aligned} \text{Res}_{u,2}(x, y, t) = & f_1 + f_2 \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} - R \cos x \sin y \frac{\partial f_1}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \\ & - R \cos x \sin y \frac{\partial f_2}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} + R \sin x \sin y f_1 \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \\ & + R f_1 \frac{\partial f_1}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(\alpha + 1)} + R f_1 \frac{\partial f_2}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^{3\alpha}}{\Gamma(\alpha + 1)\Gamma(2\alpha + 1)} \\ & + R \sin x \sin y f_2 \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} + R f_2 \frac{\partial f_1}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^{3\alpha}}{\Gamma(\alpha + 1)\Gamma(2\alpha + 1)} \\ & + R f_2 \frac{\partial f_2}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^{4\alpha}}{\Gamma^2(2\alpha + 1)} + R \sin x \cos y \frac{\partial f_1}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \\ & + R \sin x \cos y \frac{\partial f_2}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} - R \cos x \cos y g_1 \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \\ & + R g_1 \frac{\partial f_1}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma^2(\alpha + 1)} + R g_1 \frac{\partial f_2}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^{3\alpha}}{\Gamma(\alpha + 1)\Gamma(2\alpha + 1)} \\ & - R \cos x \cos y g_2 \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} + R g_2 \frac{\partial f_1}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^{3\alpha}}{\Gamma(\alpha + 1)\Gamma(2\alpha + 1)} \\ & + R g_2 \frac{\partial f_2}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^{4\alpha}}{\Gamma^2(2\alpha + 1)} - R \sin x \cos x - 2 \cos x \sin y + \frac{\partial h_0}{\partial x} \\ & + \frac{\partial h_1}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} - \frac{\partial^2 f_1}{\partial x^2} \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} - \frac{\partial^2 f_2}{\partial x^2} \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} \\ & - \frac{\partial^2 f_1}{\partial y^2} \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} - \frac{\partial^2 f_2}{\partial y^2} \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} \end{aligned} \quad (2.31.a)$$

$$\begin{aligned}
Res_{v,2}(x, y, t) = & g_1 + g_2 \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} - R \cos x \sin y \frac{\partial g_1}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \\
& - R \cos x \sin y \frac{\partial g_2}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} + R \cos x \cos y f_1 \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \\
& + R f_1 \frac{\partial g_1}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma^2(\alpha + 1)} + R f_1 \frac{\partial g_2}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^{3\alpha}}{\Gamma(\alpha + 1)\Gamma(2\alpha + 1)} \\
& + R \cos x \cos y f_2 \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} + R f_2 \frac{\partial g_1}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^{3\alpha}}{\Gamma(\alpha + 1)\Gamma(2\alpha + 1)} \\
& + R f_2 \frac{\partial g_2}{\partial x} \frac{(\varphi(t) - \varphi(t_0))^{4\alpha}}{\Gamma^2(2\alpha + 1)} + R \sin x \cos y \frac{\partial g_1}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \\
& + R \sin x \cos y \frac{\partial g_2}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} - R \sin x \sin y g_1 \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \\
& + R g_1 \frac{\partial g_1}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma^2(\alpha + 1)} + R g_1 \frac{\partial g_2}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^{3\alpha}}{\Gamma(\alpha + 1)\Gamma(2\alpha + 1)} \\
& - R \sin x \sin y g_2 \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} + R g_2 \frac{\partial g_1}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^{3\alpha}}{\Gamma(\alpha + 1)\Gamma(2\alpha + 1)} \\
& + R g_2 \frac{\partial g_2}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^{4\alpha}}{\Gamma^2(2\alpha + 1)} - R \sin y \cos y + 2 \sin x \cos y + \frac{\partial h_0}{\partial y} \\
& + \frac{\partial h_1}{\partial y} \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} - \frac{\partial^2 g_1}{\partial x^2} \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} - \frac{\partial^2 g_2}{\partial x^2} \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} \\
& - \frac{\partial^2 g_1}{\partial y^2} \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} - \frac{\partial^2 g_2}{\partial y^2} \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} \tag{2.31.b}
\end{aligned}$$

Now we must apply the operator $D_{\varphi(t)-\varphi(t_0)}^\alpha$ on Eq. (2.31.a) (2.31.b) then substitute $t = 0$ to have.

$$\begin{aligned}
D_{\varphi(t)-\varphi(t_0)}^\alpha Res_{u,2}(x, y, 0) = & f_2 - R \cos x \sin y \frac{\partial f_1}{\partial x} + R \sin x \sin y f_1 \\
& + R \sin x \cos y \frac{\partial f_1}{\partial y} - R \cos x \cos y g_1 \\
& - \frac{\partial^2 f_1}{\partial x^2} - \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial h_1}{\partial x} \tag{2.32.a}
\end{aligned}$$

$$\begin{aligned}
D_{\varphi(t)-\varphi(t_0)}^\alpha Res_{v,2}(x, y, 0) = & g_2 - R \cos x \sin y \frac{\partial g_1}{\partial x} + R \cos x \cos y f_1 \\
& + R \sin x \cos y \frac{\partial g_1}{\partial y} - R \sin x \sin y g_1 \\
& - \frac{\partial^2 g_1}{\partial x^2} - \frac{\partial^2 g_1}{\partial y^2} + \frac{\partial h_1}{\partial y} \tag{2.32.b}
\end{aligned}$$

We know from Eq. (2.15.a)(2.15.b) that:

$$D_{\varphi(t)-\varphi(t_0)}^\alpha Res_{u,2}(x, y, 0) = 0 \quad \text{and} \quad D_{\varphi(t)-\varphi(t_0)}^\alpha Res_{v,2}(x, y, 0) = 0 \tag{2.33}$$

Using this fact and substituting f_1 and g_1 and their partial derivatives in Eq. (2.32.a)(2.32.b), we get the preliminary forms of $f_2(x, y, t)$ and $g_2(x, y, t)$ as follows:

$$f_2(x, y) = -4R \sin x \cos x - 4 \cos x \sin y - \frac{\partial h_1}{\partial x} \tag{2.34.a}$$

$$g_2(x, y) = -4R \sin y \cos y + 4 \sin x \cos y - \frac{\partial h_1}{\partial y} \tag{2.34.b}$$

Now to find $h_1(x, y)$ we need to use the following boundary conditions:

$$u_2(x, 0, t) = 0 \quad \text{and} \quad v_2(0, y, t) = 0 \quad (2.35)$$

Substituting Eq. (2.34.a)(2.34.b) in Eqs.(2.30.a) and (2.30.b) then applying the boundary conditions on the resultant equations gives:

$$\frac{\partial h_1}{\partial x} = -4R \sin x \cos x \quad (2.35.a)$$

$$\frac{\partial h_1}{\partial y} = -4R \sin y \cos y \quad (2.35.b)$$

Integrating Eq. (2.35.a) with respect to x gives

$$h_1(x, y) = R \cos 2x + \theta(y) \quad (2.36)$$

Again $\theta(y)$ is a function of y only, and to find it, we derive the last equation with respect to y :

$$\frac{\partial h_1}{\partial y} = \theta'(y) \quad (2.37)$$

Substituting this equation in (2.35.b) and integrating the result with respect to y , we get:

$$\theta(y) = R \cos 2y \quad (2.38)$$

Substituting $\theta(y)$ in Eq. (2.36), we get

$$h_1(x, y) = R(\cos 2x + \cos 2y) \quad (2.39)$$

And the final forms of $f_2(x, y)$ and $g_2(x, y)$ become:

$$f_2(x, y) = -4 \cos x \sin y \quad (2.40.a)$$

$$g_2(x, y) = 4 \sin x \cos y \quad (2.40.b)$$

Finally, the second RPS approximate solution would be:

$$\begin{aligned} u_2(x, y, t) = & -\cos x \sin y + 2 \cos x \sin y \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \\ & - 4 \cos x \sin y \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} \end{aligned} \quad (2.41.a)$$

$$\begin{aligned} v_2(x, y, t) = & \sin x \cos y - 2 \sin x \cos y \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \\ & + 4 \sin x \cos y \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} \end{aligned} \quad (2.41.b)$$

$$\begin{aligned} P_2(x, y, t) = & -\frac{R}{4}(\cos 2x + \cos 2y) \\ & + R(\cos 2x + \cos 2y) \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \end{aligned} \quad (2.41.c)$$

Until the arbitrary order coefficients of the multiple FPS solution of the system (2.1) and (2.2) are found, this process can be repeated.

Moreover, higher accuracy can be achieved by evaluating more components of the solution, and as we will see later, if there is a pattern in the series coefficients, then calculating few terms in the series is sufficient to reach the solution.

Chapter 3

Applications

In this chapter, we apply the RPS method (introduced in Chapter 2) to a two-dimensional test problem from the literature. Specifically, we focus on the following case

Our test problem, widely studied in the literature [16,17], consists of Equation (1.30) with $\mathbf{F} = \mathbf{0}$, and Ω is the square $0 \leq x \leq 2\pi$ and $0 \leq y \leq 2\pi$.

The initial conditions are:

$$u(x, y, t) = -\cos x \sin y \quad (3.1.a)$$

$$v(x, y, t) = \sin x \cos y \quad (3.1.b)$$

The boundary conditions are:

$$u(x, y, t) = -\cos x \sin y e^{-2t} \quad (3.2.a)$$

$$v(x, y, t) = \sin x \cos y e^{-2t} \quad (3.2.b)$$

and the exact solution of the problem is:

$$u(x, y, t) = -\cos x \sin y e^{-2vt} \quad (3.3.a)$$

$$v(x, y, t) = \sin x \cos y e^{-2vt} \quad (3.3.b)$$

$$P(x, y, t) = -\frac{R}{4}(\cos 2x + \cos 2y) e^{-4vt} \quad (3.3.c)$$

3.1 Time-Fractional Navier-Stokes Problem

In this section, we employ the RPS method (as presented in the preceding chapter) to solve a time-fractional Navier-Stokes problem.

If we replace the time derivative in equation (1.30) by a fractional time derivative to the arbitrary order α , where $0 < \alpha \leq 1$, and rewrite the equation in the cartesian coordinates subject to the conditions (3.1.a)(3.1.b) and (3.2.a)(3.2.b), we will get the following fractional problem:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + Ru \frac{\partial u}{\partial x} + Rv \frac{\partial u}{\partial y} + \frac{\partial P}{\partial x} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0, \quad (3.4.a)$$

$$\frac{\partial^\alpha v}{\partial t^\alpha} + Ru \frac{\partial v}{\partial x} + Rv \frac{\partial v}{\partial y} + \frac{\partial P}{\partial y} - \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = 0. \quad (3.4.b)$$

where, $0 < \alpha \leq 1$, $0 < x \leq 2\pi$ and $0 < y \leq 2\pi$,

P is the pressure [Units:Pa].

u and v are the velocity components.

R is The Reynold's number $R = \frac{\rho v L}{\mu} = \frac{v L}{\nu}$

The initial conditions are:

$$u(x, y, 0) = -\cos x \sin y, \quad (3.5.a)$$

$$v(x, y, 0) = \sin x \cos y. \quad (3.5.b)$$

The boundary conditions are:

$$u(x, 0, t) = 0, \quad (3.6.a)$$

$$v(0, y, t) = 0. \quad (3.6.b)$$

The RPS method will take the following technique:

As in (2.3.a)(2.3.b)(2.3.c) we assume that the solution of the problem takes the following form

$$u(x, y, t) = \sum_{n=0}^{\infty} f_n(x, y) \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \quad (3.7.a)$$

$$v(x, y, t) = \sum_{n=0}^{\infty} g_n(x, y) \frac{(\varphi(t) - \varphi(t_0))^{n\alpha}}{\Gamma(n\alpha + 1)} \quad (3.7.b)$$

$$P(x, y, t) = \sum_{n=1}^{\infty} h_n(x, y) \frac{(\varphi(t) - \varphi(t_0))^{(n\alpha)}}{\Gamma(n\alpha + 1)} \quad (3.7.c)$$

where $0 < \alpha \leq 1$, $(x, y) \in \Omega$, $0 \leq (\varphi(t) - \varphi(t_0)) < R^{\frac{1}{\alpha}}$

It is clear that $u(x, y, t)$ and $v(x, y, t)$ satisfy the initial conditions (3.5.a)(3.5.b). Hence, we can obtain the initial guess approximation of $u(x, y, t)$ and $v(x, y, t)$ as:

$$u(x, y, 0) = f(x, y) = -\cos x \sin y \quad (3.8.a)$$

$$v(x, y, 0) = g(x, y) = \sin x \cos y \quad (3.8.b)$$

Now from (3.7.a)(3.7.b)(3.7.c) the first approximate RPS solutions are:

$$u_1(x, y, t) = -\cos x \sin y + f_1(x, y) \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \quad (3.9.a)$$

$$v_1(x, y, t) = \sin x \cos y + g_1(x, y) \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \quad (3.9.b)$$

$$P_1(x, y, t) = h_0(x, y) \quad (3.9.c)$$

where:

$$f_1 = -Rf \frac{\partial f}{\partial x} - Rg \frac{\partial f}{\partial y} - \frac{\partial h_0}{\partial x} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \quad (3.10.a)$$

$$g_1 = -Rf \frac{\partial g}{\partial x} - Rg \frac{\partial g}{\partial y} - \frac{\partial h_0}{\partial y} + \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} \quad (3.10.b)$$

Thus:

$$\begin{aligned} f_1 &= R(\cos x \sin y)(\sin x \sin y) - R(\sin x \cos y)(-\cos x \cos y) \\ &\quad - \frac{\partial h_0}{\partial x} + (\cos x \sin y) + (\cos x \sin y) \\ &= R \sin x \cos y + 2(\cos x \sin y) - \frac{\partial h_0}{\partial x} \end{aligned} \quad (3.11.a)$$

$$\begin{aligned} g_1 &= R(\cos x \sin y)(\cos x \cos y) - R(\sin x \cos y)(-\sin x \sin y) \\ &\quad - \frac{\partial h_0}{\partial y} + (-\sin x \cos y) + (-\sin x \cos y) \\ &= R \sin y \cos y - 2(\sin x \cos y) - \frac{\partial h_0}{\partial y} \end{aligned} \quad (3.11.b)$$

Then the equation (3.9.a)(3.9.b) becomes:

$$u_1(x, y, t) = -\cos x \sin y + \left[R \sin x \cos x + 2(\cos x \sin y) - \frac{\partial h_0}{\partial x} \right] \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \quad (3.12.a)$$

$$v_1(x, y, t) = \sin x \cos y + \left[R \sin y \cos y - 2(\sin x \cos y) - \frac{\partial h_0}{\partial y} \right] \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \quad (3.12.b)$$

These are preliminary results since we still did not find $h_{x_0}(x, y)$, and in order to do so we must use the boundary conditions:

$$u_1(x, 0, t) = 0 \quad \text{and} \quad v_1(0, y, t) = 0 \quad (3.13)$$

After we substitute in equations (3.11.a)(3.11.b) in Eq (3.9.a)(3.9.b)(3.9.c), and apply the boundary conditions (3.13), we have:

$$\frac{\partial h_0}{\partial x} = R \sin x \cos x \quad (3.14.a)$$

$$\frac{\partial h_0}{\partial y} = R \sin y \cos y \quad (3.14.b)$$

Integrating equation (3.14.a) with respect to x , we get

$$h_0(x, y) = -\frac{R}{4} \cos 2x + \theta(y) \quad (3.15)$$

where $\theta(y)$ is a function of y only, and to find it, derive the last equation with respect to y

$$\frac{\partial h_0}{\partial y} = \theta'(y) \quad (3.16)$$

Substitute this equation in (3.14.b), then integrate the resultant equation with respect to y , we get

$$\theta(y) = -\frac{R}{4} \cos 2y \quad (3.17)$$

Now substitute this equation in (3.15), we have:

$$h_0(x, y) = -\frac{R}{4} (\cos 2x + \cos 2y) \quad (3.18)$$

Therefore the functions $f_1(x, y, t)$ and $g_1(x, y, t)$ in (3.11.a)(3.11.b) will have their final forms as:

$$f_1(x, y) = 2 \cos x \sin y \quad (3.19.a)$$

$$g_1(x, y) = -2 \sin x \cos y \quad (3.19.b)$$

and finally the first RPS approximate solution would be:

$$u_1(x, y, t) = -\cos x \sin y + 2 \cos x \sin y \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \quad (3.20.a)$$

$$v_1(x, y, t) = \sin x \cos y - 2 \sin x \cos y \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \quad (3.20.b)$$

$$P_1(x, y, t) = -\frac{R}{4} (\cos 2x + \cos 2y) \quad (3.20.c)$$

Now to find the second RPS approximate solution from (3.7.a)(3.7.b)(3.7.c) we have:

$$u_2(x, y, t) = -\cos x \sin y + f_1(x, y) \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} + f_2(x, y) \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} \quad (3.21.a)$$

$$v_2(x, y, t) = \sin x \cos y + g_1(x, y) \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} + g_2(x, y) \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} \quad (3.21.b)$$

$$P_2(x, y, t) = h_0 + h_1(x, y) \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \quad (3.21.c)$$

where:

$$f_2 = -Rf \frac{\partial f_1}{\partial x} - Rf_1 \frac{\partial f}{\partial x} - Rg \frac{\partial f_1}{\partial y} - Rg_1 \frac{\partial f}{\partial y} - \frac{\partial h_1}{\partial x} + \frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} \quad (3.22.a)$$

$$g_2 = -Rf \frac{\partial g_1}{\partial x} - Rf_1 \frac{\partial g}{\partial x} - Rg \frac{\partial g_1}{\partial y} - Rg_1 \frac{\partial g}{\partial y} - \frac{\partial h_1}{\partial y} + \frac{\partial^2 g_1}{\partial x^2} + \frac{\partial^2 g_1}{\partial y^2} \quad (3.22.b)$$

Thus:

$$\begin{aligned}
f_2(x, y) &= R(\cos x \sin y)(-2 \sin x \sin y) - R(2 \cos x \sin y)(\sin x \sin y) \\
&\quad - R(\sin x \cos y)(2 \cos x \cos y) - R(-2 \sin x \cos y)(-\cos x \cos y) \\
&\quad + (-2 \cos x \sin y) + (-2 \cos x \sin y) - \frac{\partial h_1}{\partial x} \\
&= -4 \sin x \cos x - 4 \cos x \sin y - \frac{\partial h_1}{\partial x}
\end{aligned} \tag{3.23.a}$$

$$\begin{aligned}
g_2(x, y) &= R(\cos x \sin y)(-2 \cos x \cos y) - R(2 \cos x \sin y)(\cos x \cos y) \\
&\quad - R(\sin x \cos y)(2 \sin x \sin y) - R(-2 \sin x \cos y)(-\sin x \sin y) \\
&\quad + (2 \sin x \cos y) + (2 \sin x \cos y) - \frac{\partial h_1}{\partial y} \\
&= -4R \sin y \cos y + 4 \sin x \cos y - \frac{\partial h_1}{\partial y}
\end{aligned} \tag{3.23.b}$$

Now to find $h_1(x, y)$ we need to use the following boundary conditions:

$$u_2(x, 0, t) = 0 \quad \text{and} \quad v_2(0, y, t) = 0 \tag{3.24}$$

Substituting equations (3.23.a)(3.23.b) in equations (3.21.a) and (3.21.b) then applying the boundary conditions to the resultant equations give:

$$\frac{\partial h_1}{\partial x} = -4R \sin x \cos x \tag{3.25.a}$$

$$\frac{\partial h_1}{\partial y} = -4R \sin y \cos y \tag{3.25.b}$$

Integrating equation (3.25.a) with respect to x gives

$$h_1(x, y) = R \cos 2x + \theta(y) \tag{3.26}$$

Again $\theta(y)$ is a function of y only, and to find it derive the last equation with respect to y , we have:

$$\frac{\partial h_1}{\partial y} = \theta'(y) \tag{3.27}$$

Substituting this equation in (3.25.b) and integrating the result with respect to y , we get:

$$\theta(y) = R \cos 2y \tag{3.28}$$

Substituting $\theta(y)$ in equation (3.26), we get:

$$h_1(x, y) = R(\cos 2x + \cos 2y) \tag{3.29}$$

and the final form of $f_2(x, y)$ and $g_2(x, y)$ becomes:

$$f_2(x, y) = -4 \cos x \sin y \quad (3.30.a)$$

$$g_2(x, y) = 4 \sin x \cos y \quad (3.30.b)$$

Finally the second RPS approximate solution would be:

$$u_2(x, y, t) = -\cos x \sin y + 2 \cos x \sin y \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} - 4 \cos x \sin y \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} \quad (3.31.a)$$

$$v_2(x, y, t) = \sin x \cos y - 2 \sin x \cos y \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} + 4 \sin x \cos y \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} \quad (3.31.b)$$

$$P_2(x, y, t) = -\frac{R}{4}(\cos 2x + \cos 2y) + R(\cos 2x + \cos 2y) \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \quad (3.31.c)$$

Now to find the third RPS approximate solution from (3.7.a)(3.7.b)(3.7.c) we have

$$u_3(x, y, t) = -\cos x \sin y + f_1 \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} + f_2 \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} + f_3 \frac{(\varphi(t) - \varphi(t_0))^{3\alpha}}{\Gamma(3\alpha + 1)} \quad (3.32.a)$$

$$v_3(x, y, t) = \sin x \cos y + g_1 \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} + g_2 \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} + g_3 \frac{(\varphi(t) - \varphi(t_0))^{3\alpha}}{\Gamma(3\alpha + 1)} \quad (3.32.b)$$

$$P_3(x, y, t) = h_0 + h_1 \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} + h_2 \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} \quad (3.32.c)$$

When we apply the same procedure in chapter 2 we will get the preliminary forms of $f_3(x, y, t)$ and $g_3(x, y, t)$, then applying the following boundary conditions:

$$u_3(x, 0, t) = 0 \quad \text{and} \quad v_3(0, y, t) = 0 \quad (3.33)$$

would give us the partial derivatives of the function $h_2(x, y)$, and following the same procedures as in the previous steps, we find the following results:

$$f_3(x, y) = 8 \cos x \sin y \quad (3.34.a)$$

$$g_3(x, y) = -8 \sin x \cos y \quad (3.34.b)$$

$$h_2(x, y) = -4R(\cos 2x + \cos 2y) \quad (3.34.c)$$

Hence, the third approximate RPS solution would be given by:

$$u_3(x, y, t) = -\cos x \sin y + 2 \cos x \sin y \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} - 4 \cos x \sin y \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} + 8 \cos x \sin y \frac{(\varphi(t) - \varphi(t_0))^{3\alpha}}{\Gamma(3\alpha + 1)} \quad (3.35.a)$$

$$\begin{aligned}
v_3(x, y, t) &= \sin x \cos y - 2 \sin x \cos y \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} + 4 \sin x \cos y \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} \\
&\quad - 8 \sin x \cos y \frac{(\varphi(t) - \varphi(t_0))^{3\alpha}}{\Gamma(3\alpha + 1)}
\end{aligned} \tag{3.35.b}$$

$$\begin{aligned}
P_3(x, y, t) &= -\frac{R}{4} (\cos 2x + \cos 2y) + R (\cos 2x + \cos 2y) \frac{(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} \\
&\quad - 4R (\cos 2x + \cos 2y) \frac{(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)}
\end{aligned} \tag{3.35.c}$$

If we repeat the same procedures for $k = 4, 5, 6, \dots$, we will get the following final solutions of our time-fractional problem:

$$\begin{aligned}
u(x, y, t) &= -\cos x \sin y \left(1 - \frac{2(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} + \frac{4(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} - \frac{8(\varphi(t) - \varphi(t_0))^{3\alpha}}{\Gamma(3\alpha + 1)} + \dots \right) \\
&= -\cos x \sin y \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{2(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(k\alpha + 1)} \right)^k
\end{aligned} \tag{3.36.a}$$

$$\begin{aligned}
v(x, y, t) &= \sin x \cos y \left(1 - \frac{2(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} + \frac{4(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} - \frac{8(\varphi(t) - \varphi(t_0))^{3\alpha}}{\Gamma(3\alpha + 1)} + \dots \right) \\
&= \sin x \cos y \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{2(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(k\alpha + 1)} \right)^k
\end{aligned} \tag{3.36.b}$$

$$\begin{aligned}
P(x, y, t) &= -\frac{R}{4} (\cos 2x + \cos 2y) \left(1 - \frac{4(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(\alpha + 1)} + \frac{16(\varphi(t) - \varphi(t_0))^{2\alpha}}{\Gamma(2\alpha + 1)} - \frac{64(\varphi(t) - \varphi(t_0))^{3\alpha}}{\Gamma(3\alpha + 1)} + \dots \right) \\
&= -\frac{R}{4} (\cos 2x + \cos 2y) \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{4(\varphi(t) - \varphi(t_0))^\alpha}{\Gamma(k\alpha + 1)} \right)^k
\end{aligned} \tag{3.36.b}$$

And now we will discuss the solutions of the equation based on Riemann-Liouville, Katogampula and Hadamard derivatives.

If $\varphi(t) = t$ and $\varphi(t_0) = 0$ with $t_0 = 0$, we will get the solutions with Riemann-Liouville derivative:

$$u(x, y, t) = -\cos x \sin y \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{2t^\alpha}{\Gamma(k\alpha + 1)} \right)^k \tag{3.37.a}$$

$$v(x, y, t) = \sin x \cos y \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{2t^\alpha}{\Gamma(k\alpha + 1)} \right)^k \tag{3.37.b}$$

$$P(x, y, t) = -\frac{R}{4} (\cos 2x + \cos 2y) \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{4t^\alpha}{\Gamma(k\alpha + 1)} \right)^k \tag{3.37.c}$$

If $\varphi(t) = \frac{t^\beta}{\beta}$ and $\varphi(t_0) = 0$ with $t_0 = 0$, we will get the solutions with Katogampula derivative:

$$u(x, y, t) = -\cos x \sin y \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{2 \left(\frac{t^\beta}{\beta} \right)^\alpha}{\Gamma(k\alpha + 1)} \right)^k \quad (3.38.a)$$

$$v(x, y, t) = \sin x \cos y \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{2 \left(\frac{t^\beta}{\beta} \right)^\alpha}{\Gamma(k\alpha + 1)} \right)^k \quad (3.38.b)$$

$$P(x, y, t) = -\frac{R}{4} (\cos 2x + \cos 2y) \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{4 \left(\frac{t^\beta}{\beta} \right)^\alpha}{\Gamma(k\alpha + 1)} \right)^k \quad (3.38.c)$$

And if $\varphi(t) = \ln(t)$ and $\varphi(t_0) = 0$ with $t_0 = 1$, we will get the solutions with Hadamard derivative:

$$u(x, y, t) = -\cos x \sin y \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{2(\ln(t))^\alpha}{\Gamma(k\alpha + 1)} \right)^k \quad (3.39.a)$$

$$v(x, y, t) = \sin x \cos y \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{2(\ln(t))^\alpha}{\Gamma(k\alpha + 1)} \right)^k \quad (3.39.b)$$

$$P(x, y, t) = -\frac{R}{4} (\cos 2x + \cos 2y) \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{2(\ln(t))^\alpha}{\Gamma(k\alpha + 1)} \right)^k \quad (3.39.c)$$

3.1.1 Numerical Analysis for the Time-Fractional Navier-Stokes Equation

In this section we will study the solutions of the time fractional Hadamard Navier-Stokes equation with Caputo fractional derivative numerically in order to validate the efficiency and accuracy of the RPS method. We will demonstrate the plots of the three solutions $u(x, y, t)$, $v(x, y, t)$ and $P(x, y, t)$ for the exact solutions at $\alpha = 1$ then for $\alpha = 0.9, 0.7, 0.5, 0.2$ where $0 \leq x \leq \pi$, $0 \leq y \leq \pi$, $1 \leq t \leq 2$

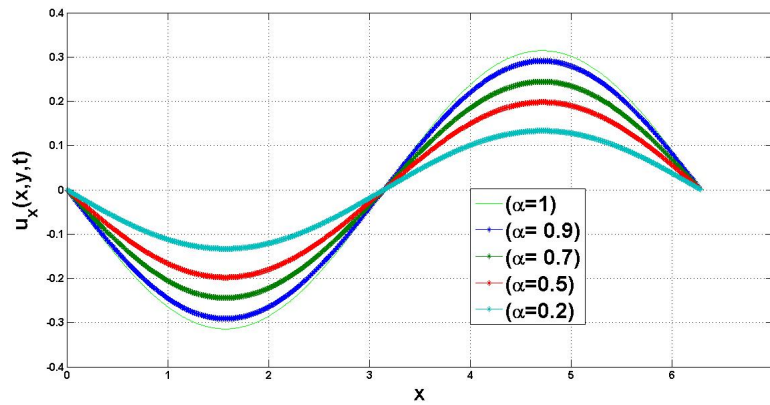


Figure.1. The velocity u_x with $x = \pi/4$ and $t = 1.5$ and $0 \leq y \leq 2\pi$ at different values of α

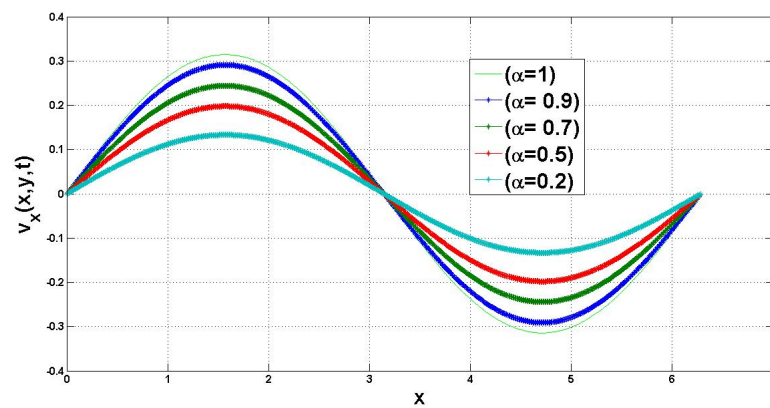


Figure.2. The velocity v_x with $x = \pi/4$ and $t = 1.5$ and $0 \leq y \leq 2\pi$ at different values of α .

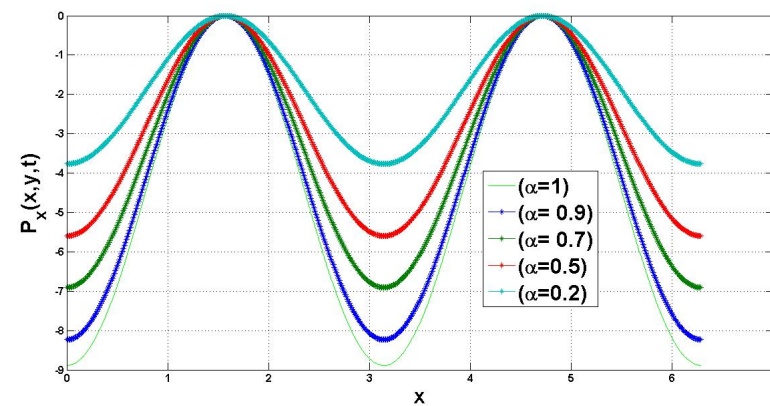


Figure.3. The pressure P_x with $y = \pi/4$ and $t = 1.5$ and $0 \leq x \leq 2\pi$ and $R = 40$ at different values of α .

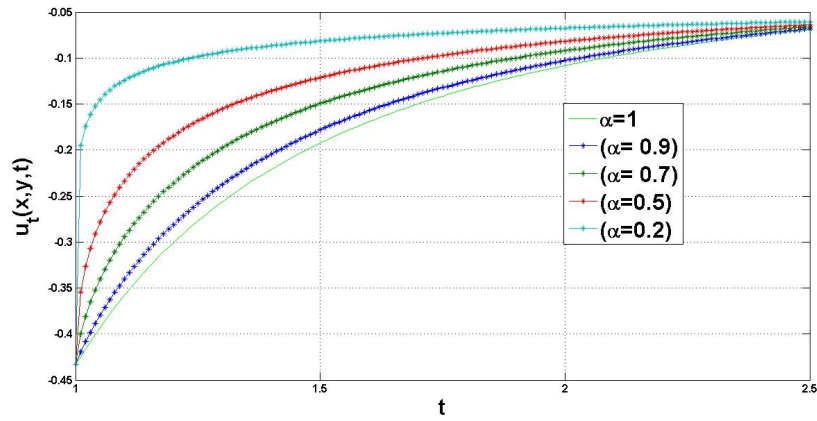


Figure.4. The velocity u_t with $x = y = \pi/6$ and $1 \leq t \leq 2,5$ at different values of α .

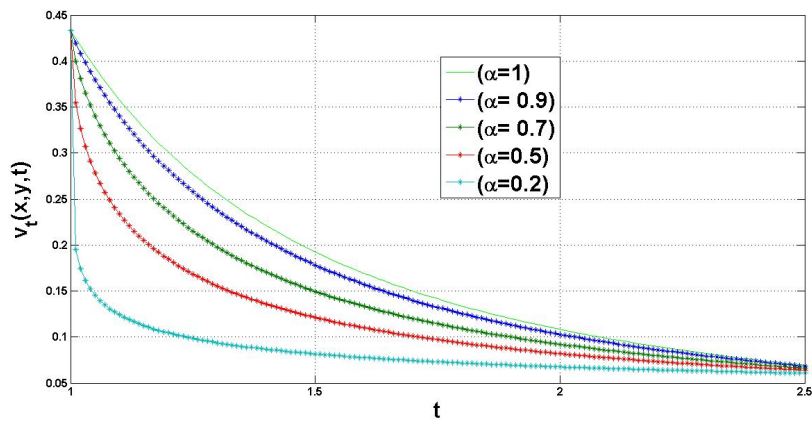


Figure.5. The velocity v_t with $x = y = \pi/6$ and $1 \leq t \leq 2,5$ at different values of α .

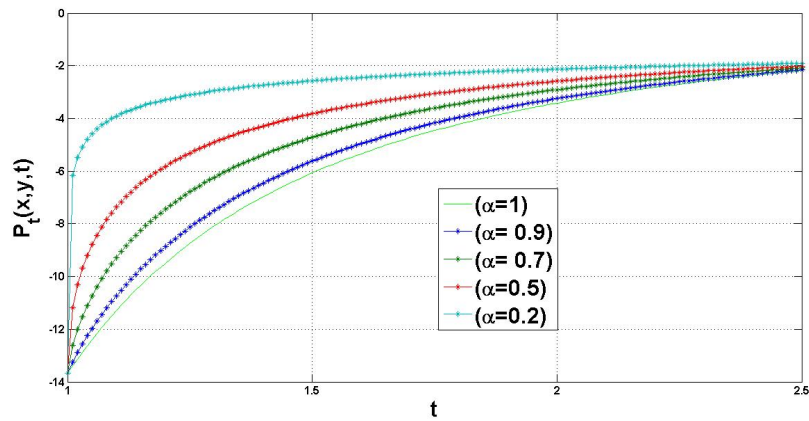


Figure.6. The pressure P_t with $x = y = \pi/6$ and $1 \leq t \leq 2,5$ and $R = 40$ at different values of α .

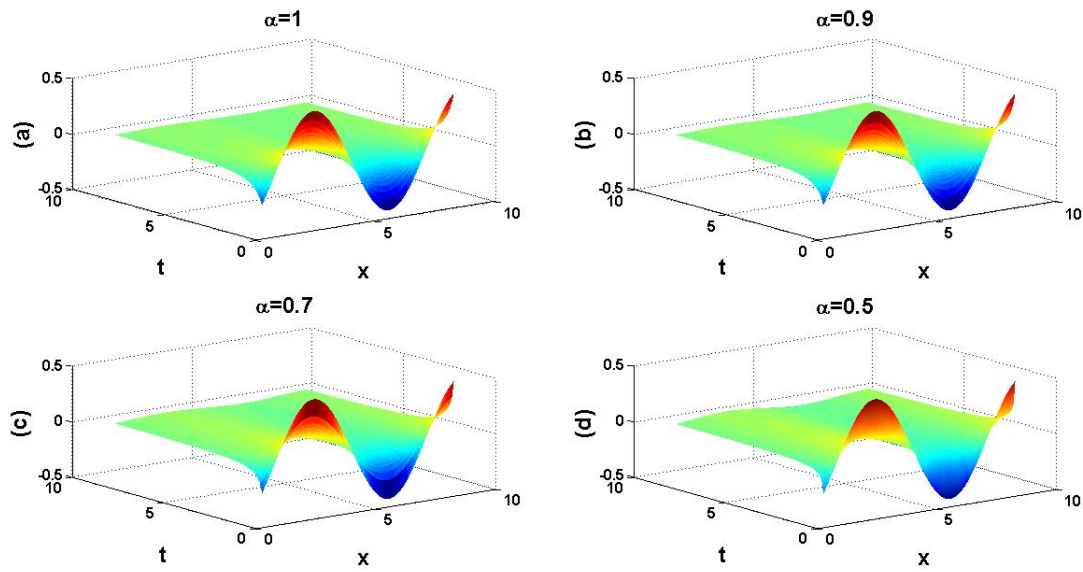


Figure.7.The 4th RPS approximate solution $u(x, y, t)$ of the Navier-Stokes equation in deffrent value of α with $y = \pi/6$: (a) $u(x, y, t)$ at $\alpha = 1$ (b) $u(x, y, t)$ at $\alpha = 0.9$ (c) $u(x, y, t)$ at $\alpha = 0.7$ (d) $u(x, y, t)$ at $\alpha = 0.5$

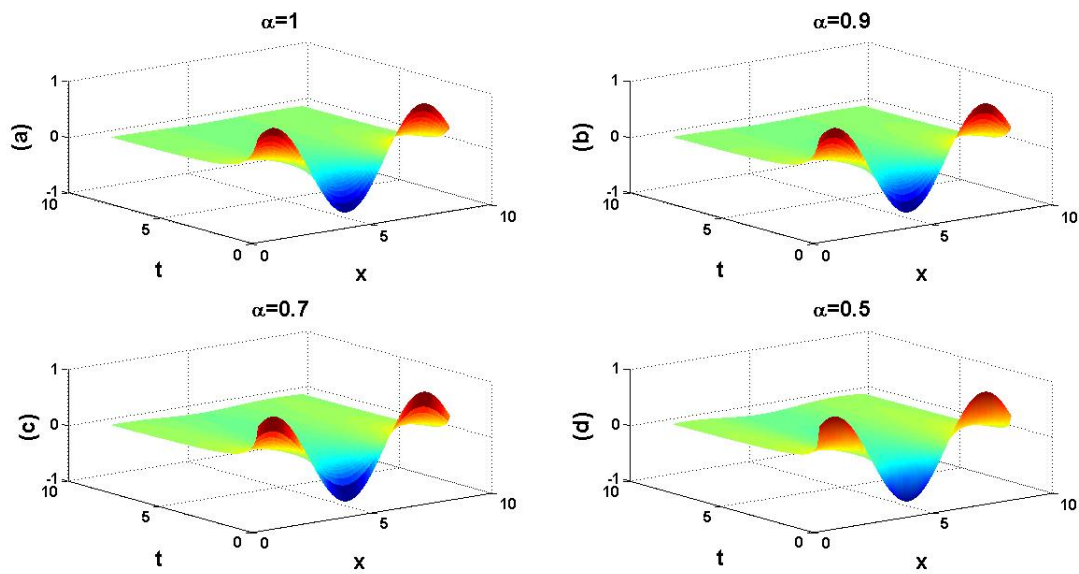


Figure.8.The 4th RPS approximate solution $v(x, y, t)$ of the Navier-Stokes equation in deffrent value of α with $y = \pi/6$: (a) $v(x, y, t)$ at $\alpha = 1$ (b) $v(x, y, t)$ at $\alpha = 0.9$ (c) $v(x, y, t)$ at $\alpha = 0.7$ (d) $v(x, y, t)$ at $\alpha = 0.5$

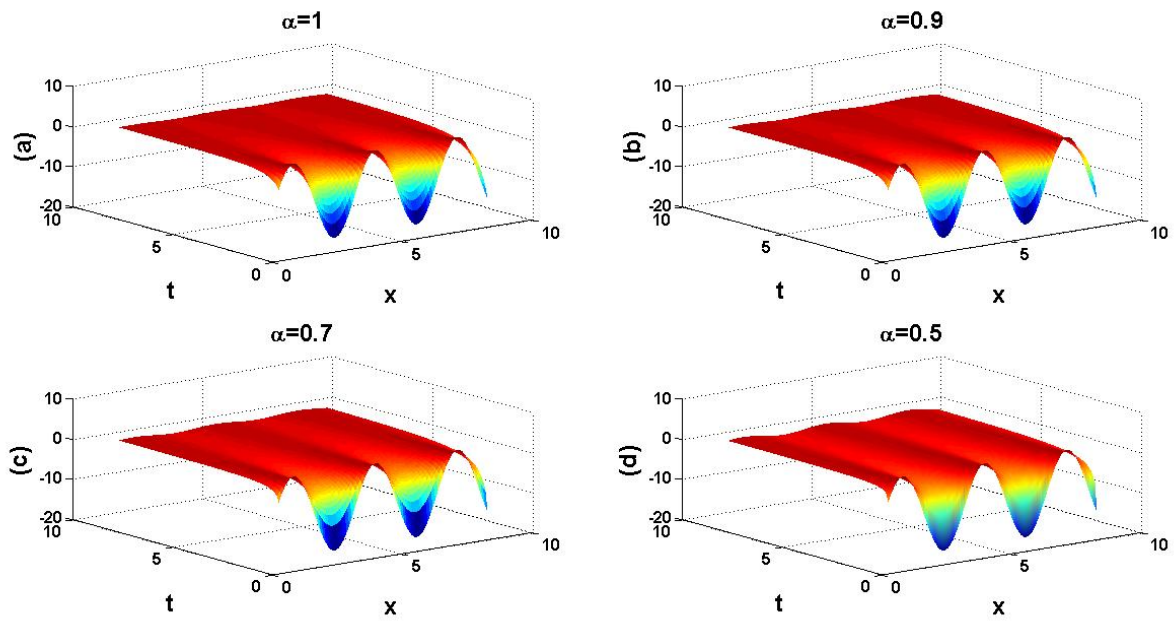


Figure.9. The 4th RPS approximate solution $P(x, y, t)$ of the Navier-Stokes equation in different value of α with $y = \pi/6$ and $R = 40$: (a) $P(x, y, t)$ at $\alpha = 1$ (b) $P(x, y, t)$ at $\alpha = 0.9$ (c) $P(x, y, t)$ at $\alpha = 0.7$ (d) $P(x, y, t)$ at $\alpha = 0.5$.

Conclusion

In This work we have successfully used the Residual power series method to solve the time-fractional Navier-Stokes equation with Caputo fractional derivative in two dimensions, the proposed technique provides solutions in terms of rapidly convergent series with easily computable components. Numerical results demonstrate that these solutions exhibit exceptional agreement with the exact solutions.

The first chapter introduced fundamental concepts of fractional calculus, along with key definitions and results essential for this research. It also covered the definitions and properties of various types of fractional integrals and derivatives, with a particular focus on φ -fractional integral and derivative operator and its characteristics. We also presented the fractional power series, which is an effective mathematical tool in deriving various existence results. Additionally, the chapter explored the Navier-Stokes equations, their components, and their extension to the fractional framework.

The second chapter employed the Residual power series algorithm to solve the two-dimensional time-fractional Navier-Stokes equations with Caputo fractional derivatives, as well as to construct approximate solutions to these equations.

The third chapter demonstrates the application of the Residual power series method—introduced in Chapter 2—to a two-dimensional benchmark problem. Finally, to assess the precision and effectiveness of the Residual power series method, we analyzed the numerical solutions of the given equation. Graphical representations show that the proposed method works greatly in terms of efficiency and simplicity. Hence, we conclude that the Residual power series method is a very powerful and efficient tool for solving linear and nonlinear fractional partial differential equations.

Future Studies

While this thesis has successfully demonstrated the efficacy of the RPS method in solving the two-dimensional unsteady Navier-Stokes equations, several potential avenues for future research remain unexplored. The following section outlines some of these promising directions for extending the scope of this work:

Solving the Steady Navier-Stokes equation:

- 1) In this work, the RPS method has been employed to solve the fractional-time Navier-Stokes equations. The solutions are constructed in the form of fractional power series expansions, with the time variable t serving as the series variable.
- 2) The problem addressed in this work is inherently "unsteady", as it incorporates time-dependent derivatives. In contrast, the "steady" form of the Navier-Stokes equations omits temporal variations, resulting in a time-independent formulation.
- 3) we anticipate that this work is a step towards extending applications of the RPS method to solve fractional problems with boundary conditions at infinity which I believe that this method is applicable for, but still needs to verify accuracy of solutions, which will be discussed in detail in further work.

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ملخص

في هذا العمل، سنستخدم طريقة سلسلة الطاقة المتبقية لتوليد حل معادلة نافيه ستوكس الكسرية غير الخطية مع المشتقة الكسرية كابوتو في بعدين في شكل سلسلة متقاربة بسرعة، حيث اعتمدنا تعريف مشغلي حساب التفاضل والتكامل الكسري المعمم الذي يندرج ضمنه مشغل ريمان ليوفيل وهادامارد وكاتوغامبولاً من نوع كابوتو. قدمنا تطبيقاً توضيحياً لإثبات فعالية الطريقة المقترحة والاستفادة منها، حيث تكشف النتائج العددية والتمثيلات الرسومية أن الطريقة المقترحة تعمل بشكل كبير حسناً من حيث الكفاءة والبساطة ويمكن استخدامها لحل المزيد من المشاكل في مجال المعادلات التفاضلية الكسرية غير الخطية.

كلمات مفتاحية :

معادلات نافيه ستوكس الكسرية، مشتقة كابوتو الكسرية، سلسلة الطاقة الكسرية، دالة ميتاغ-لافلر.

Abstract :

In this work, we will use the residual power series (RPS) method to generate the solution of the nonlinear time fractional Navier-Stokes equation with Caputo fractional derivative in two dimensions in the form of a rapidly convergent series, where we adopted the definition of generalized Fractional calculus operators, including those of Riemann-Liouville, Hadamard, Katugampula of the Caputo type. We gave an illustrative application to demonstrate the effectiveness and leverage of the proposed method, where numerical results and graphical representations reveal that the proposed method performs extremely well in terms of efficiency and simplicity and it can be utilized to solve more problems in the field of non-linear fractional differential equations (FDEs).

Keywords :

Fractional Navier-Stokes equations, Caputo fractional derivative, Fractional power series, Mittag-Leffler function.

Résumé :

Dans ce travail, nous utilisons la méthode des séries de puissances résiduelles (RPS) pour générer la solution de l'équation non linéaire de Navier-Stokes fractionnaire avec la dérivée fractionnaire de Caputo en temps dans un espace à deux dimensions, sous la forme d'une série à convergence rapide. Notre approche s'appuie sur les définitions des opérateurs du calcul fractionnaire généralisé, y compris Riemann-Liouville, Hadamard, Katugampola du type Caputo. Une application illustrative est présentée pour démontrer l'efficacité et la pertinence de la méthode proposée. Les résultats numériques et les représentations graphiques montrent que cette méthode offre d'excellentes performances en termes d'efficacité et de simplicité. Elle peut ainsi être utilisée pour résoudre une gamme plus large de problèmes dans le domaine des équations différentielles fractionnaires non linéaires (EDES).

Mot-clés :

Équations fractionnaire de Navier-Stokes, Dérivé fractionnaire de Caputo, Séries de puissance fractionnée, Fonction Mittag-Leffler.