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Theme

$W_0^{1,1}$ *Solutions of Elliptic Equations with Degenerate Coercivity*

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Dedication

To my beloved family,

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my gratitude goes beyond words. this is yours as much as mine.

Thank you. Truly.

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Abstract

The Sobolev space $W_0^{1,2}(\Omega)$ is the natural functional framework to find weak solutions of nonlinear elliptic problems of the following type

$$(P) \quad \begin{cases} -\operatorname{div}\left(\frac{a(x)\nabla u}{(1+|u|)^{\theta}}\right) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where the function f belongs to the dual space of $L^m(\Omega)$ with $m > 1$, Ω is a bounded open subset of $\mathbb{R}^N$, with $N > 2$, θ is a real number such that

$$0 \leq \theta \leq 1,$$

and $a : \Omega \to \mathbb{R}$ is a

$$\alpha \leq a(x) \leq \beta,$$

for almost every $x \in \Omega$, where α and β are positive constants.

Keywords: Elliptic equations; $W^{1,1}$ solutions; degenerate equations.

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ملخص

فضاء سوبوليف $W_0^{1,2}(\Omega)$ هو الإطار الوظيفي الطبيعي لإيجاد الحلول الضعيفة للمسائل الإهليلجية غير الخطية من النوع التالي:

$$(P)\begin{cases} -\operatorname{div}\left(a(x)\dfrac{\nabla u}{(1+|u|)^{\theta}}\right) = f(x) \\ u = 0 \end{cases}$$

حيث تنتمي الدالة f إلى الفضاء $L^m(\Omega)$ حيث $m>1$ ، و Ω مجموعة مفتوحة ومحدودة من $\mathbb{R}^N$ ، مع $N>2$ ، و θ عدد حقيقي يحقق:

$$0 \le \theta \le 1.$$

كما أن الدالة $a:\Omega \to \mathbb{R}$ قابلة للقياس و تحقق:

$$\alpha \le a(x) \le \beta,$$

لكل $x \in \Omega$ تقريباً، حيث α و β ثوابت موجبة.

كلمات مفتاحية: معادلات إهليلجية؛ حلول في $W_0^{1,1}$ ؛ معادلات متدهورة.

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Résumé

L'espace de Sobolev $W_0^{1,2}(\Omega)$ constitue le cadre fonctionnel naturel pour la recherche de solutions faibles de problémes elliptiques non linéaires du type

$$(P) \quad \begin{cases} -\operatorname{div}\left(\dfrac{a(x)\nabla u}{(1+|u|)^{\theta}}\right) = f(x) & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega. \end{cases}$$

où la fonction f appartient à l'espace $L^m(\Omega)$ avec $m > 1$. Le domaine Ω est un ouvert borné de $\mathbb{R}^N$ avec $N > 2$. Le paramètre θ est un nombre réel tel que

$$0 \leq \theta \leq 1,$$

et la fonction $a : \Omega \to \mathbb{R}$ est mesurable et vérifie la condition

$$\alpha \leq a(x) \leq \beta,$$

pour presque tout $x \in \Omega$, où α et β sont des constantes positives.

Mots-clés : équations elliptiques ; solutions $W^{1,1}$; équations dégénérées.

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List of Symbols

We introduce the necessary notations and definition which are used in the sequel.

$\mathcal{H}$	Hilbert space.
$\Omega \subset \mathbb{R}^N$	open set in $\mathbb{R}^N$
X'	topological dual of X.
A^c	complement of the set A.
$(.,.)$	scalar product.
$\mathbb{R}^N$	Euclidean space of dimension N, where N is a nonzero natural number.
x	vector in $\mathbb{R}^N$, $x = (x_1, x_2, ..., x_N)$, $x_i \in \mathbb{R}$, $1 \leq i \leq N$
dx	Lebesgue measure in N-dimensional space.
$\partial\Omega = \Gamma$	boundary of Ω.
$\|E\|$	or mes (E) measure of the set E.
$B(x_0, r)$	open ball of radius r centered at x_0.
$\|.\|$	Hilbert norm.
χ_E	characteristic function of set E.
$\langle .,. \rangle$	duality bracket between X and its dual space.
$\int_\Omega f(x)\, dx$	integral of f in Ω with respect to the Lebesgue measure.
supp u	Support of the function u.
$\frac{\partial u}{\partial n}$	outward normal derivative.
$\nabla u = Du$	$= (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N})$ gradient of the function u.
div u	divergence of the vector u, $div\, u = \frac{\partial u}{dx_1} + \frac{\partial u}{dx_2} + ... + \frac{\partial u}{dx_N}$
Δu	$= \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$ Laplacian of u.
$f_n \to f$	denotes that the sequence $\{f_n\}$ converge to f.
$f_n \rightharpoonup f$	denotes that the sequence $\{f_n\}$ converge weakly to f.
$L^p(\Omega)$	$= \{u : \Omega \to \mathbb{R}$ measurable and $\left(\int_\Omega \|u(x)\|^p\right)^{1/p} dx < +\infty$ such that $1 \leq p < \infty\}$.
$\|u\|_p$	$= \left[\int_\Omega \|u(x)\|^p dx\right]^{1/P} = \|u\|_{L^p}$.
$L^\infty(\Omega)$	$= \{u : \Omega \to \mathbb{R}$ measurable, $\exists M > 0 \mid \|u(x)\| \leq M$ a.e.$\}$.
$\|u\|_{L^\infty}$	$= \inf\{C : \|u(x)\| \leq C$ a.e on $\Omega\}$.
q	Hölder conjugate of p : $q = \frac{p}{p-1}$ if $p > 1$ and $q = \infty$ if $p = 1$
$L^q(\Omega) \subset L^p(\Omega)$,	$\forall\ 1 \leq p \leq q < \infty$
$\|u\|_{C_c(\Omega)}$	$= \sup_{x \in \Omega} \|u(x)\| = \max_{x \in \Omega} \|u(x)\|$
$C_c(\Omega)$	space of continuous functions with compact.

$\mathscr{D}(\Omega)$	space of infinitely differentiable functions on Ω with compact support in Ω.
$W^{1,p}(\Omega)$	$= \left\{ u \in L^p(\Omega) \mid \nabla u \in (L^p(\Omega))^N \right\}$.
$W_0^{1,p}(\Omega)$	$= \left\{ u \in W^{1,p}(\Omega), \quad \text{with} \quad u = 0 \quad \text{on} \quad \partial\Omega \right\}$.
a.e.	almost everywhere.

Introduction

This master's thesis is devoted to the study of the existence and regularity of weak solutions for a class of nonlinear elliptic equations with degenerate coercivity. More precisely, we consider the following boundary value problem:

$$(P) \quad \begin{cases} -\operatorname{div}\left(a(x)\dfrac{\nabla u}{(1+|u|)^{\theta}}\right) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with $N > 2$, θ is a real number satisfying $0 \le \theta \le 1$, and the source term f belongs to $L^m(\Omega)$. The coefficient $a(x)$ is measurable and satisfies

$$\alpha \le a(x) \le \beta \quad \text{almost everywhere in } \Omega,$$

for some positive constants α and β, and the function f belongs to various Lebesgue spaces $L^m(\Omega)$ for some $m \ge 1$. The regularity of solutions in terms of the summability of the datum f can be summarized as follows (see Boccardo & Croce [2]):

- If $f \in L^m(\Omega)$ with $m > \frac{N}{2}$, then $u \in H_0^1(\Omega) \cap L^{\infty}(\Omega)$.
- If $f \in L^m(\Omega)$ with $\frac{2N}{N+2-\theta(N-2)} \le m < \frac{N}{2}$, then $u \in H_0^1(\Omega) \cap L^r(\Omega)$, where $r = \frac{Nm(1-\theta)}{N-2m}$.
- If $f \in L^m(\Omega)$ with $\frac{N(2-\theta)}{N+2-\theta(N-2)} \le m < \frac{2N}{N+2-\theta(N-2)}$, then $u \in W_0^{1,q}(\Omega)$, where $q = \frac{Nm(1-\theta)}{N-m(1+\theta)}$.
- If $f \in L^m(\Omega)$ with $\frac{N}{N+1-\theta(N-1)} < m < \frac{N(2-\theta)}{N+2-\theta(N-2)}$, then $u \in W_0^{1,q}(\Omega)$, where $q = \frac{Nm(1-\theta)}{N-m(1+\theta)}$.

This work is organized into three chapters.

The first chapter deals with Sobolev spaces, which constitute one of the most relevant functional settings for the treatment of boundary value problems. These spaces also allow us to study the existence results presented in Chapters 2 and 3. We refer to Brezis [5] and Lions [7] for the theory of these spaces. In this chapter, we recall:

- **Lebesgue spaces** $L^p(\Omega)$ and their main inequalities.

- **Convergence theorems** (dominated convergence, Fatou's lemma).
- **Sobolev spaces** $W_0^{1,p}(\Omega)$ and Sobolev embeddings.
- **Properties of operators**: boundedness, hemicontinuity, monotonicity, coercivity, and pseudo-monotonicity.
- **Truncation functions** (Stampacchia), which are essential for handling degenerate problems.

In the second chapter of this thesis, we study an existence and uniqueness theorem for the solution of elliptic problems with irregular data, associated to the differential operator $Au = -\operatorname{div}(a(x)\nabla u)$. We consider two cases:

- If $f \in L^\infty(\Omega)$, we apply the pseudo-monotone operator method to prove the existence of a unique weak solution in $H_0^1(\Omega)$.
- If $f \in L^1(\Omega)$, we use an approximation scheme together with truncation techniques (Stampacchia) to obtain a weak solution in $W_0^{1,q}(\Omega)$ for all

$$q \in \left[1, q_c = \frac{N}{N-1}\right[.$$

.

Moreover, we give more details about the regularity of the solutions. The principal and most interesting subject of this chapter concerns the passage from regular to irregular data. We recall that similar problems have been studied by many authors, including Arcoya and Boccardo [2].

In Chapter 3, we study the question of existence of weak solutions for the degenerate problem (P) data f belonging to $L^m(\Omega)$ for some $m \geq 1$. We will only be concerned with the case where $m = \frac{N}{N+1-\theta(N-1)}$ and $\frac{1}{N-1} < \theta < 1$.

Chapter 1

Preliminaries and basic tools

In this chapter, we introduce a set of fundamental concepts and analytical tools from the theory of Sobolev spaces that will be used throughout this work. We also present the main definitions and notations required for the subsequent analysis.

1.1 Reminders and some definition

Throughout this chapter Ω will denote a bounded domain in $\mathbb{R}^N$,$N \geq 1$, i.e.a connected and bounded open subset of $\mathbb{R}^N$. Its boundary will be denoted by Γ or $\partial\Omega$ and its closure by $\overline{\Omega}$.

Let $u = u(x_1, ..., x_N)$ be a function defined in $\Omega \subset \mathbb{R}^N$. Assuming that exists, the gradient of u at point x is defined as the vector:

$$\nabla u(x) = \left(\frac{\partial u}{\partial x_1}(x), ..., \frac{\partial u}{\partial x_N}(x)\right)$$

The Euclidean norm of ∇u is denoted by $|\nabla u|$:

$$|\nabla u| = \left(\left|\frac{\partial u}{\partial x_1}\right|^2 + ... + \left|\frac{\partial u}{\partial x_N}\right|^2\right)^{1/2}$$

Definition 1.1 (**Dual space**). *[4] The topological dual of a vector space E over the field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or$\mathbb{C}$ is defined as the set of continuous linear forms on E,i.e. continuous linear maps form E to $\mathbb{K}$. We denote by $E' = \mathbb{L}(E, \mathbb{K})$.*

E' equipped with the norm $\|.\|$ defined by :

$$\|u\|_{E'} = \sup_{x\in E, \|x\|=1} |u(x)| = \sup_{x\in E, \|x\|\leq 1} |u(x)| = \sup_{x\in E, \|x\|\neq 0} \frac{|u(x)|}{\|x\|}$$

Definition 1.2 (**reflexive space**). *[4] The space E is said to be reflexive if the canonical injection $i : E \to E''$ is surjective , i.e. $i(E) = E''$.*

1.2 Functional spaces

1.2.1 Lebesgue Spaces

Let Ω be an open subset of $\mathbb{R}^N$ and let p be a real number greater than or equal to 1. The Lebesgue Space $L^p(\Omega)$ is defined as:

$$L^p(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \quad \text{is measurable and} \quad \int_\Omega \mid u(x) \mid^p < \infty \Big\}.$$

We equip it with the following norm

$$\| u \|_{L_p(\Omega)} = \| u \|_p = \Big(\int_\Omega |u(x)|^p \Big)^{\frac{1}{p}}, \quad p \geq 1.$$

By $L^\infty(\Omega)$ we denote the set

$$L^\infty(\Omega) = \{ u : \Omega \to \mathbb{R} \, \text{measurable}, \exists M > 0 \mid |u(x)| \leq M \quad \text{a.e.} \}$$

It is equipped with the following norm:

$$\| u \|_{\infty,\Omega} = \inf \quad \{ M > 0 : \mid u \mid \leq M \quad \text{a.e.in} \quad \Omega \}$$

The space $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{2,\Omega} = \int_\Omega uv dx.$$

Theorem 1.1. *The space $L^p(\Omega)$ is a Banach space for any $1 \leq p \leq \infty$ and separable for $1 \leq p < \infty$. Let $1 \leq p \leq \infty$. be denote by p' the conjugate exponent of p:*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

1.2.2 Main Inequalities

Theorem 1.2 (**Holder's inquality**). *[5] Let $1 \leq p \leq \infty$ and p' be the conjugate exponent of p, Assume that $f \in L^p(\Omega)$,and $g \in L^{p'}(\Omega)$. Then $f \cdot g \in L^1(\Omega)$ and:*

$$\| f \cdot g \|_{L^1(\Omega)} \leq \| f \|_{L^p(\Omega)} \| g \|_{L^{p'}(\Omega)}.$$

The case $p = p' = 2$ is known as the *Schwarz inequality*:

$$\int_\Omega u(x)\, v(x)\, dx \leq \left(\int_\Omega |u(x)|^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega |v(x)|^2 \, dx \right)^{\frac{1}{2}}.$$

Theorem 1.3 (**Young's inequality**). *[5] Let $1 < p < \infty$. Then we have:*

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \tag{1.1}$$

Theorem 1.4. *(**Divergence and Green's formolas**).*

Let Ω be a domain in $\mathbb{R}^N$, and $n(x)$ its exterior normal. Let u and v be two regular functions, and w be a vector field defined on Ω. Then,

$$\int_\Omega \operatorname{div} w \, dx = \int_{\partial\Omega} w \cdot n \, d\sigma \quad \textit{(divergence formula).}$$

$$\int_\Omega (\Delta u)\, v \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, d\sigma \quad \textit{(Green formula).}$$

Theorem 1.5. *(**Poincare inequality**). Suppose that $1 \leq p < \infty$ and Ω is a bounded open set. Then there exists a constant $C > 0$ (depending on Ω and p) such that*

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}. \tag{1.2}$$

Moreover, by the density of $\mathscr{D}(\Omega)$ in $W_0^{1,p}(\Omega)$, the inequality (1.2) stay true for any function $u \in W_0^{1,p}(\Omega)$.

1.2.3 Convergence Theorem

In this section, we present some definitions and results on the convergence sequences of measurable functions.

Lemma 1.1. *.[1] Let (u_n) be a sequence in $L^p(\Omega)$ with $1 < p < +\infty$. Suppose that*

- *(u_n) is bounded in $L^p(\Omega)$;*
- *$u_n \to u$ a.e. on Ω.*

Then, $u_n \to u$ in $L^q(\Omega)$, for all $1 \leq q < p$ and weakly in $L^p(\Omega)$, i.e.,

$$\lim_{n\to+\infty} \int_\Omega u_n v \, dx = \int_\Omega uv \, dx, \quad \forall\, v \in L^{p'}(\Omega).$$

Theorem 1.6. *(**Fatou's Lemma**) .[5] Let (u_n) be a sequence of non-negative measurable functions on Ω. Then*

$$\int_\Omega \liminf_{n\to+\infty} u_n(x) \, dx \leq \liminf_{n\to+\infty} \int_\Omega u_n(x) \, dx.$$

Theorem 1.7. *(**Lebesgue's Dominated Convergence Theorem**).[5] Let $\Omega \subset \mathbb{R}^N$ be open set and (u_n) be a sequence of functions in $L^1(\Omega)$ such that:*

- *$u_n(x) \to u(x)$ a.e in Ω as $n \to \infty$,*
- *there exists $g \in L^1(\Omega)$ such that for all n,*

$$|u_n(x)| \leq g(x) \quad \text{a.e. in } \Omega.$$

Then $u \in L^1(\Omega)$ and $u_n \to u$ in $L^1(\Omega)$, that is,

$$\|u_n - u\|_{L^1(\Omega)} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Theorem 1.8. *(**Lebesgue's Dominated Convergence Inverse Theorem**).[5] Let $\Omega \subset \mathbb{R}^N$ be open and (u_n) be a sequence of functions in $L^p(\Omega)$ and $p \in [1, \infty)$ such that:*

$$u_n \to u \quad \text{in } L^p(\Omega) \text{ as } n \to \infty.$$

Then there exist a subsequence (u_{n_k}) and a function $h \in L^p(\Omega)$ such that:

- *$u_{n_k}(x) \to u(x)$ a.e. in Ω as $k \to \infty$,*
- *for all k, $|u_{n_k}(x)| \leq h(x)$ a.e. in Ω.*

Theorem 1.9 (**Giuseppe Vitali convergence theorem**). *Let $p \in [1, \infty[$ and (f_n) be a sequence of functions in $L^p(\Omega)$ such that*

- *$f_n \to f$ a.e. in Ω*
- *(f_n) equal-integration in Ω, i.e. $\forall \varepsilon > 0, \exists \delta > 0, \forall E \subset \Omega$ with $|E| \leq \delta$, such that*

$$\int_E |f_n(x)|^p dx \leq \varepsilon, \quad \forall n \in \mathbb{N}$$

Or

$$\limsup_{|E| \to 0} \int_E |f_n(x)|^p dx = 0, \quad \forall n \in \mathbb{N}$$

Then

$$f \in L^p(\Omega) \quad \text{and} \quad f_n \to f \quad \text{in} \quad L^p(\Omega) \quad \text{strongly.}$$

1.2.4 Some notation of convergence

Definition 1.3. .*(**Strong convergence**) Convergence of a sequence $(x_n)_{n\in\mathbb{N}}$ in a normed vector space $(E,\|\cdot\|_E)$ to an element $x\in E$, defined in the following way:*

$$\forall\varepsilon>0,\ \exists n_0\in\mathbb{N},\ \forall n\in\mathbb{N}:\ n\geq n_0\Rightarrow\|x_n-x\|_E\leq\varepsilon.$$

or

$$\lim_{n\to+\infty}\|x_n-x\|_E=0.$$

We denote by $x_n\to x$ as $n\to+\infty$.

Definition 1.4. . *(**Weak convergence**) Let E be a Banach space, E' its dual and $\langle\cdot,\cdot\rangle$ the duality bracket EE'. We say that the sequence (x_n) in E weakly converges to $x\in E$ if and only if:*

$$\langle g,x_n\rangle\xrightarrow[n\to+\infty]{}\langle g,x\rangle,\quad\forall g\in E'.$$

We denote this as

$$x_n\rightharpoonup x\quad\text{in } E.$$

Proposition 1.1.

- *if $x_n\to x$ strongly then $x_n\rightharpoonup x$ in E;*
- *if $x_n\rightharpoonup x$ in E then $\|x_n\|_E$ is bounded;*
- *if*

$$\begin{cases}x_n\rightharpoonup x & \text{weakly in } E,\\ g_n\to g & \text{strongly in } E',\end{cases}$$

then

$$\langle g_n,x_n\rangle\to\langle g,x\rangle\quad\text{as } n\to+\infty.$$

Theorem 1.10. . *Let E be a reflexive Banach space. Then, for any bounded sequence $(x_n)_n\subset E$, there exists a subsequence $(x_{n_k})_k$ that converges weakly in E.*

Weak* convergence

Definition 1.5. . *We say that the sequence (f_n) in E' weakly* converges to $f\in E'$ if and only if:*

$$\langle f_n,g\rangle\xrightarrow[n\to+\infty]{}\langle f,g\rangle,\quad\forall g\in E.$$

We denote this as

$$f_n \overset{*}{\rightharpoonup} f \quad \text{in } E'.$$

Proposition 1.2.

$$f_n \to f \text{ strongly} \quad \Longrightarrow \quad f_n \overset{*}{\rightharpoonup} f \text{ in } E'.$$

For more details, you can refer to the book [5] .

1.3 Sobolev Spaces

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

Definition 1.6. . *The Sobolev space $W^{1,p}(\Omega)$ is defined by*

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \, ; \, \nabla u \in \left(L^p(\Omega)\right)^N \right\}.$$

Remarks 1.1. *The space $W^{1,p}(\Omega)$ is equipped with the norm*

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_p + \|\nabla u\|_p. \tag{1.3}$$

For $p = \infty$

$$\|u\|_{W^{1,\infty}} = \max\left(\|u\|_{L^\infty}, \|\nabla u\|_{L^\infty}\right).$$

For $p = 2$ we denote by $H^1 = W^{1,2}$ which is a Hilbert[1] space equipped with the scalar product

$$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)}.$$

The associated norm

$$\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$$

is a norm equivalent to that of $W^{1,2}(\Omega)$.

Proposition 1.3.

- *The space $W^{1,p}$ is a Banach space for $1 \leq p \leq \infty$.*
- *The space $W^{1,p}$ is a separable space for $1 \leq p < \infty$.*
- *The space $W^{1,p}$ is a reflexive space for $1 < p < \infty$.*
- *The space $W_0^{1,p}$ is a separable space. Also reflexive for $1 < p < \infty$.*

- *The space H_0^1 is a separable Hilbert space.*

Definition 1.7. .*[4] For $1 \leq p < \infty$, we define*

$$W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega) \; ; \; u = 0 \text{ on } \partial\Omega \right\}.$$

Proposition 1.4. *The space $\mathscr{D}(\Omega)$ denotes the set of functions in $C^\infty(\Omega)$ with compact support in Ω. Since, for $1 \leq p < \infty$, the space $\mathscr{D}(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, we can identify the dual of $W_0^{1,p}(\Omega)$ with a subspace of the space of distributions $\mathscr{D}'(\Omega)$. We denote this as*

$$W^{-1,p'}(\Omega) = \left(W_0^{1,p}(\Omega) \right)'.$$

1.3.1 Sobolev embeddings

Definition 1.8. . *Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ Banach spaces.*

- *E is injected continuously into F, means that the canonical injection $j : E \longrightarrow F$ is continuous i.e, $\exists c > 0$, $\forall x \in E$:*

$$\|x\|_F \leq c\|x\|_E,$$

and we denote by $E \hookrightarrow F$.

- *E is injected in compact into F means that the canonical injection $j : E \longrightarrow F$ is compact i.e for all sequence bounded u_n in E we can extract subsequence u_{n_k} convergent in F and we denote by $E \hookrightarrow_c F$.*

If $1 \leq p < \infty$, the Sobolev exponent of p defined by :

$$p^* = \frac{Np}{N-p} \quad or \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}.$$

Theorem 1.11. ***(continuous injection)**.[6] Let $1 \leq p < \infty$, we suppose that Ω is on open set class C^1 a bounded frontier, and we take $\Omega = \mathbb{R}_+^N$.*

- $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [1, p^*] \quad$ *if $p < N$.*

- $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [p, \infty] \quad$ *if $p = N$.*

[1] David Hilbert (1862–1943) is a German mathematician. He is often considered one of the greatest mathematicians of the 20th century.

- $W^{1,p}(\Omega) \hookrightarrow L^{\infty}(\Omega) \quad$ *if* $p > N$.

Theorem 1.12. *(**compact injection**[Rellich–Kondrachon]).[6] Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain of class* C^1.

- $W^{1,p}(\Omega) \hookrightarrow_c L^q(\Omega) \quad \forall q \in [1, p^*] \quad$ *if* $p < N$.
- $W^{1,p}(\Omega) \hookrightarrow_c L^q(\Omega) \quad \forall q \in [p, \infty] \quad$ *if* $p = N$.
- $W^{1,p}(\Omega) \subset C(\overline{\Omega}) \quad$ *if* $p > N$.

1.4 The p-Laplacian operator

The p-Laplacian is the second-order nonlinear differential operator defined as

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right).$$

Here, $p > 1$, and of course, when $p = 2$, the p-Laplacian reduces to the usual Laplacian.

$$\Delta_p u = \Delta u$$

1.4.1 Monotone Operators

Definition 1.9. *The operator* $A : V \to V'$ *is said to be:*

- ***monotone*** *if* $\quad \langle Au - Av, u - v\rangle \geq 0, \quad \forall\, u, v \in V.$
- ***strictly monotone*** *if* $\quad \langle Au - Av, u - v\rangle > 0, \quad \forall\, u, v \in V,\ u \neq v.$

Exemple 1.1. *The operator* $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ *defined by* $\quad Au = -\Delta u$ *is monotone, where* $H_0^1(\Omega)$ *is equipped with the norm of the gradient. Indeed, for* $u, v \in H_0^1(\Omega)$, *and we use theorem1.4, we have*

$$\begin{aligned}\langle Au - Av, u - v\rangle &= \int_\Omega \nabla(u - v) \cdot \nabla(u - v)\, dx \\ &= \|u - v\|^2_{H_0^1(\Omega)} \\ &\geq 0.\end{aligned}$$

1.4.2 Bounded Operator

Definition 1.10. *Let V and W be two Banach spaces and let $A : V \to W$ be an operator. We say that A is bounded if it maps every bounded set in V to a bounded set in W; i.e*

$$\forall \rho > 0, \quad \exists C_\rho > 0 : \quad A(B_V(0,\rho)) \subset B_W(0, C_\rho).$$

where $B_V(0,\rho)$ designates the open ball in V with center 0 and radius $\rho > 0$ and $B_W(0, C_\rho)$ designates the open ball in W with center 0 radius $C_\rho > 0$.

Exemple 1.2. *The operator $Au = -\triangle_p$ is bounded from $W_0^{1,p}(\Omega)$ in $W^{-1,p'}(\Omega)$*

From the expression of the norm in a dual space, let $\rho > 0$, for $u \in B_V(0,\rho)$, we can write:

$$\| Au \|_{V'} = \sup_{\substack{\varphi \in V \\ \|\varphi\| \leq 1}} |\langle Au, \varphi \rangle| = \sup_{\substack{\varphi \in V \\ \|\varphi\| \leq 1}} \left| \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla \varphi \right|.$$

So,

$$\begin{aligned} \left| \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla \varphi \right| &\leq \int_\Omega |\nabla u|^{p-1} . |\nabla \varphi| \\ &\leq \left(\int_\Omega |\nabla u||^p \right)^{\frac{1}{p'}} \left(\int_\Omega |\nabla \varphi|^p \right)^{\frac{1}{p}} \\ &= \| u \|_V^{p-1} \| \varphi \|_V \\ &\leq \rho^{p-1} \end{aligned}$$

Hence $\| Au \|_{V'} \leq \rho^{p-1}$. this shows that $A(B_V(0,\rho)) \subset B_{V'}(0, \rho^{p-1})$

1.4.3 Hemicontnuous Operators

Definition 1.11. *Let X is a reflexive and separable Banach space, and A is an application from $X \to X'$. We say that:*

A is hemicontinuous if for all $u, v \in X$, the application $t \mapsto \langle A(u + tv), v \rangle$ is continuous from $\mathbb{R}$ to $\mathbb{R}$

Exemple 1.3. *The operator $A : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ defined by $Au = -\Delta u$ is hemicontinuous. Indeed, let $u, v \in H_0^1(\Omega)$ and $t \in \mathbb{R}$:*

$$\begin{aligned} \langle A(u + tv), v \rangle &= \int_\Omega \nabla(u + tv) . \nabla v \\ &= \int_\Omega \nabla u . \nabla v + t \int_\Omega \nabla v \nabla v \\ &= a + bt. \end{aligned}$$

This shows that $t \longrightarrow \langle A(u + tv), v \rangle$ is continuous.

1.4.4 Coercive Operators

Definition 1.12. *An operator $A : V \to V'$ is said to be* coercive *if:*

$$\lim_{\|v\|_V \to +\infty} \frac{\langle Av, v\rangle}{\|v\|_V} = +\infty.$$

Exemple 1.4. *The operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ defined by*

$$Au = -\operatorname{div}(\nabla u) + a(x)u = -\Delta u + a(x)u, \quad \text{where } a(x) \geq 0,\ a(x) \in L^\infty(\Omega),$$

is coercive. Indeed, we have:

$$\frac{\langle Av, v\rangle}{\|v\|_{H_0^1}} = \frac{\displaystyle\int_\Omega |\nabla v|^2\, dx + \int_\Omega a(x) v^2\, dx}{\|v\|_{H_0^1}}.$$

Since $a(x) \geq 0$, we obtain:

$$\frac{\langle Av, v\rangle}{\|v\|_{H_0^1}} \geq \alpha \|v\|_{H_0^1} \longrightarrow +\infty \quad \text{as } \|v\|_{H_0^1} \to +\infty.$$

1.5 Pseudo-monotone Operators

Definition 1.13. *Let V be a separable and reflexive Banach space. An operator $A : V \to V'$ is pseudo-monotone if:*

1. *A is bounded, that is, the image of a bounded subset of V is a bounded subset of V'.*
2. *whenever $u_j \rightharpoonup u$ in V weak and if*

$$\limsup_{j \to +\infty} \langle A(u_j), u_j - u\rangle \leq 0$$

then

$$\liminf_{j \to +\infty} \langle A(u_j), u_j - v\rangle \geq \langle A(u), u - v\rangle, \quad \forall v \in V. \tag{1.4}$$

where $\langle \cdot, \cdot \rangle$ refers to the duality product between V' and V.

Proposition 1.5. *If $A : V \to V'$ is bounded, hemicontinuous and monotonic, then A is pseudo-monotone.*

proof: **a)** Let $\{u_j\}$ be a sequence weakly converging to u in V. Suppose that

$$\limsup_{j\to+\infty}\langle Au_j, u_j - u\rangle \leq 0.$$

If A is monotone, we have

$$\lim_{j\to+\infty}\langle Au_j, u_j - u\rangle \to 0. \tag{1.5}$$

Indeed, the monotonicity of A and the weak convergence of $\{u_j\}$ towards u implies that

$$\langle Au_j, u_j - u\rangle \geq \langle Au, u_j - u\rangle \to 0 \quad \text{as } j \to +\infty.$$

And so

$$0 \geq \limsup_{j\to+\infty}\langle Au_j, u_j - u\rangle \geq \liminf_{j\to+\infty}\langle Au_j, u_j - u\rangle \geq 0.$$

Hence (1.5).

b) For $v \in V$ and $t \in]0,1[$, let $w = (1-t)u + tv$. We have $\langle Au_j - Aw, u_j - w\rangle \geq 0$ so that

$$t\langle Au_j, u - v\rangle \geq -\langle Au_j, u_j - u\rangle + \langle Aw, u_j - u\rangle - t\langle Aw, v - u\rangle.$$

From which, thanks to (1.5),

$$t\liminf_{j\to+\infty}\langle Au_j, u - v\rangle \geq -t\langle Aw, v - u\rangle.$$

Dividing by t and taking into account (1.5), we obtain

$$\liminf_{j\to+\infty}\langle Au_j, u_j - v\rangle \geq \langle Aw, u - v\rangle, \quad w = (1-t)u + tv, \ \forall t \in]0,1[. \tag{1.6}$$

By making t tend towards 0 in (1.6), and using hemicontinuity, we deduce

$$\liminf_{j\to+\infty}\langle Au_j, u_j - v\rangle \geq \langle Au, u - v\rangle, \quad \forall v \in V.$$

Which means that A is pseudo-monotone.

Chapter 2

Elliptic Problem with Irregular Data

In this chapter, we establish the mathematical framework for elliptic problems involving irregular data. We introduce the problem under consideration, specify the fundamental assumptions, and provide the definition of a weak solution. Furthermore, the analytical tools necessary for investigating the existence and regularity results are briefly discussed.

2.1 The problem (P_0)

We are concerned with elliptic boundary value problems of the form

$$(P_0) \qquad \begin{cases} Au = -\operatorname{div}(a(x)\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N > 2)$, with boundary $\partial\Omega$. f belongs to $L^1(\Omega)$ and the function $a(x)$ is measurable and satisfies

$$0 < \alpha \leq a(x) \leq \beta,$$

for some positive constants α and β, and $\theta > 0$. we differentiate between two cases.

2.2 the case: $f \in L^\infty(\Omega)$

2.2.1 General result

Let V be a reflexive and separable Banach space, and $A : V \to V'$ be an operator.

Theorem 2.1 (**Existence theorem**). *Suppose that $A : V \longrightarrow V'$ is an operator :*

- *bounded*
- *hemicontinuous*

- *coercive, meaning that*

$$\lim_{|v|_V \to +\infty} \frac{\langle Av, v\rangle}{\|v\|_V} = +\infty$$

- *monotone*

Let $f \in V'$. Then, there exists a function $u \in V$ and only one such that:

$$Au = f$$

Theorem 2.2. *Let $f \in L^\infty(\Omega)$. Then the problem (P_0) has a unique solution $u \in H_0^1(\Omega)$.*

proof

(1) Proof of Coercivity

We compute the inner product of the operator A with u in the space $H_0^1(\Omega)$:

$$\langle Au, u\rangle = \int_\Omega Au.u\,dx = \int_\Omega a(x)\nabla u \cdot \nabla u\,dx.$$

Since $a(x)$ satisfies the ellipticity condition $a(x) \geq \alpha > 0$, we obtain the following estimate:

$$\langle Au, u\rangle \geq \alpha \int_\Omega |\nabla u|^2\,dx = \alpha \|u\|^2_{H_0^1(\Omega)}.$$

Thus,

$$\frac{\langle Au, u\rangle}{\|u\|_{H_0^1(\Omega)}} \geq \alpha \|u\|_{H_0^1(\Omega)},$$

which implies

$$\lim_{\|u\|_{H_0^1(\Omega)} \to +\infty} \frac{\langle Au, u\rangle}{\|u\|_{H_0^1(\Omega)}} = +\infty.$$

as $\|u\|_{H_0^1(\Omega)} \to +\infty$ since $\alpha > 0$. Hence, the operator A is **coercive**.

(2) Proof of Pseudo-Monotonicity

(i) A is bounded:

Let $\rho > 0$ and let $u \in B_{H_0^1(\Omega)}(0, \rho)$, which implies $\|u\|_{H_0^1(\Omega)} \leq \rho$. To show that A is a bounded operator, we estimate its dual norm in $H^{-1}(\Omega)$. For any $v \in H_0^1(\Omega)$ with $\|v\|_{H_0^1(\Omega)} \leq 1$, we consider the duality product:

$$\langle Au, v\rangle = \int_\Omega a(x)\nabla u \cdot \nabla v\,dx.$$

By applying the absolute value and using the assumption that $0 < \alpha \leq a(x) \leq \beta$ a.e. in Ω, we obtain:

$$|\langle Au, v\rangle| \leq \int_\Omega |a(x)||\nabla u||\nabla v|\, dx \leq \beta \int_\Omega |\nabla u||\nabla v|\, dx.$$

Applying the **Cauchy-Schwarz inequality** in $L^2(\Omega)$, we have:

$$\int_\Omega |\nabla u||\nabla v|\, dx \leq \left(\int_\Omega |\nabla u|^2 dx\right)^{1/2} \left(\int_\Omega |\nabla v|^2 dx\right)^{1/2} = \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}.$$

Substituting this back into the estimate, we get:

$$|\langle Au, v\rangle| \leq \beta \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}.$$

Given that $\|u\|_{H_0^1(\Omega)} \leq \rho$ and $\|v\|_{H_0^1(\Omega)} \leq 1$, it follows that:

$$|\langle Au, v\rangle| \leq \beta\rho.$$

By taking the supremum over all v such that $\|v\|_{H_0^1(\Omega)} \leq 1$, we define the norm in the dual space $H^{-1}(\Omega)$:

$$\|Au\|_{H^{-1}(\Omega)} = \sup_{\|v\|\leq 1} |\langle Au, v\rangle| \leq \beta\rho.$$

Thus, for every $\rho > 0$, the image of the ball under the mapping A satisfies:

$$A(B_{H_0^1(\Omega)}(0, \rho)) \subset B_{H^{-1}(\Omega)}(0, \beta\rho).$$

This concludes that the operator A is bounded.

(ii)

Suppose that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ as $n \to +\infty$, and assume that for any $v \in H_0^1(\Omega)$, we have:

$$\limsup_{n\to+\infty} \langle Au_n, u_n - u\rangle \leq 0.$$

We aim to show that for all $v \in H_0^1(\Omega)$, the following inequality holds:

$$\liminf_{n\to+\infty} \langle Au_n, u_n - v\rangle \geq \langle Au, u - v\rangle.$$

By the compact embedding theorem (see Theorem 1.11), it follows that $u_n \to u$ strongly in $L^2(\Omega)$ for a subsequence (still denoted by (u_n)). Furthermore, we can assume that $u_n \to u$ a.e. in Ω. Since $\nabla u_n \rightharpoonup \nabla u$ weakly in $(L^2(\Omega))^N$, and by the definition of weak convergence combined with the boundedness of $a(x)$, we obtain:

$$a(x)\nabla u_n \rightharpoonup a(x)\nabla u \quad \text{weakly in } (L^2(\Omega))^N. \tag{2.1}$$

Consequently, for any test function $v \in H_0^1(\Omega)$, we have:

$$\lim_{n\to+\infty}\int_\Omega a(x)\nabla u_n\cdot\nabla v\,dx = \int_\Omega a(x)\nabla u\cdot\nabla v\,dx. \tag{2.2}$$

Next, applying **Fatou's Lemma** to the quadratic form (noting that $a(x) \geq \alpha > 0$), we get:

$$\liminf_{n\to+\infty}\int_\Omega a(x)\nabla u_n\cdot\nabla u_n\,dx \geq \int_\Omega a(x)\nabla u\cdot\nabla u\,dx. \tag{2.3}$$

By combining the results from (2.2) and (2.3), we conclude:

$$\begin{aligned}\liminf_{n\to+\infty}\langle Au_n, u_n - v\rangle &= \liminf_{n\to+\infty}\left(\int_\Omega a(x)\nabla u_n\cdot\nabla u_n\,dx - \int_\Omega a(x)\nabla u_n\cdot\nabla v\,dx\right)\\ &\geq \int_\Omega a(x)\nabla u\cdot\nabla u\,dx - \int_\Omega a(x)\nabla u\cdot\nabla v\,dx\\ &= \langle Au, u - v\rangle.\end{aligned}$$

This completes the proof that the operator A is **pseudo-monotone**. According to the Existence Theorem 2.1, there exists at least one solution $u \in H_0^1(\Omega)$ to the problem (P_0).

2.3 the case: $f \in L^1(\Omega)$

In this case, for every datum $f \in L^1(\Omega)$, the solutions to problem (P_0) belong to $W_0^{1,q}(\Omega)$.

2.3.1 Weak solutions

Definition 2.1. *We say that u is a weak solution to problem (P_0) if $u \in W_0^{1,1}(\Omega)$ and*

$$\int_\Omega a(x)\nabla u\cdot\nabla\varphi dx \;=\; \int_\Omega f\varphi dx, \quad \forall\varphi \in W_0^{1,\infty}(\Omega). \tag{2.4}$$

The result proved in this case is the following Theorem

Theorem 2.3. *Let $f \in L^1(\Omega)$. Then The problem (P_0) has at least one weak solution u which is in $W_0^{1,q}(\Omega)$, for all*

$$q \in \left[1, q_c = \frac{N}{N-1}\right[.$$

2.3.2 Model Example

Let $\Omega = B(0,1)$, the open Euclidean unit ball centered at zero with radius 1. find $u : \overline{\Omega} \to \mathbb{R}$ as a solution of

$$(P_0) \qquad \begin{cases} -\mathrm{div}(\nabla u) &= \delta \text{ in } D'(\Omega);\\ u &= 0 \text{ on } \partial\Omega; \end{cases}$$

where δ is the Dirac distribution at the origin.

the problem (P_0) has a solution (model) u_{mod} is given by (see [**?**]):

$$u_{mod} = \begin{cases} C(|x|^{2-N} - 1) & \text{if } N \neq 2; \\ C\log|x| & \text{if } N = 2; \end{cases}$$

In the case $N \neq 2$, we have

$$|\nabla u_{mod}| = C_1|x|^{1-N},$$

then

$$\begin{aligned} \int_\Omega |\nabla u_{mod}|dx &= C_1 \int_{B(0,1)} |x|^{1-N} dx \\ &= C_2 \int_0^1 r^{1-N} r^{N-1} dr \\ &= C_2 \int_0^1 dr = C_2 \end{aligned}$$

This integral is finite. Thus, this calculation shows that: u_{mod} is in $W_0^{1,1}(B(0,1))$. Now, we are looking for critical value of the exponent $q \geq 1$ so that $u_{mod} \in W_0^{1,q}(B(0,1))$ we have :

$$\int_\Omega |\nabla u_{mod}|^q dx = C_1 \int_0^1 r^{q(1-N)+N-1} dr dx.$$

Our integral will be finite if $q(1-N)+N-1 > -1$, which gives

$$u_{mod} \in W_0^{1,q}(B(0,1)) \quad \text{for all } q \text{ such that} \quad 1 \leq q < \frac{N}{N-1}.$$

Now, the proof of the Theorem 2.3 needs several steps: First, we approximate the problem (P_0) with sequence of problems (P_{0n}) having smooth solutions (u_n). Then, after deriving uniform estimates on u_n , we pass to the limit.

2.4 Approximation of (P)

Consider the following approximated problem:

$$(P_{0n}) \qquad \begin{cases} -\operatorname{div}(a(x)\nabla u_n) = f_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases}$$

with f_n a sequence of bounded functions defined in Ω by:

$$f_n(x) = \frac{f(x)}{1 + \frac{|f(x)|}{n}}. \tag{2.5}$$

such that

$$\begin{cases} \|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}, \\ f_n \to f \text{ strongly in } L^1(\Omega). \end{cases} \tag{2.6}$$

The existence of a solution $u_n \in H_0^1(\Omega)$ of the problem (P_n) is classical (see Theorem 2.2), since $f_n \in H^{-1}(\Omega)$ and the operator $Au_n = -\operatorname{div}(a(x)\nabla u_n)$ is coercive, and **pseudo-monotone**. Thus, there exists a unique function $u_n \in H_0^1(\Omega)$ solution of problem (P_{0n}).

2.5 Uniform Estimates

Since we only have $\|f_n\|_{L^1(\Omega)} \leq C, \quad \forall n \geq 1$ in order to to bound the term $\int_\Omega f_n \varphi$ uniformly with respect to n, we need to choose a test function

$$\varphi_n = \quad \text{a function of } u_n \text{ such that} \quad \|\varphi_n\|_{L^\infty(\Omega)} \leq C, \forall n \geq 1 \quad \text{with} \quad \varphi_n \in H_0^1(\Omega).$$

To choose the test functions, we use the following result due to Stampacchia :

Lemma 2.1 (**Stampacchia**). *Let $T : \mathbb{R} \to \mathbb{R}$ be a globally lipschitz function,i.e.*

$$\exists C > 0 \quad \textit{such that} \quad |T(s) - T(t)| \leq C|s - t|, \quad \forall s, t \in \mathbb{R},$$

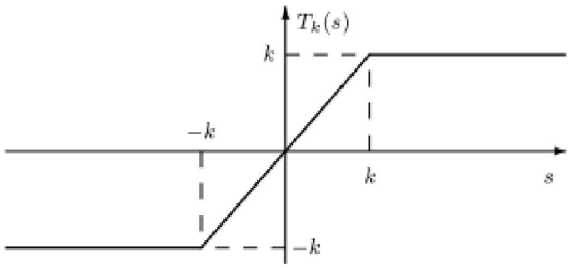
such that $T(0) = 0$. then, $\forall v \in H_0^1(\Omega)$. we have :

$$T(v) \in H_0^1(\Omega). \quad \textit{and} \quad \nabla T(v) = T'(v)\nabla v \quad \textit{in} \quad \mathscr{D}'(\Omega) \quad \textit{and} \quad \textit{almosteverywhere} \quad \textit{in} \quad \Omega$$

Exemple 2.1. *Let $k > 0$. The truncation at levels $-k$ and k is defined by the function T_k of $\mathbb{R}$ from $\mathbb{R}$ given by*

$$T_k(r) = \begin{cases} k, & \textit{if } r \geq k, \\ r, & \textit{if } |r| < k, \\ -k, & \textit{if } r \leq -k. \end{cases}$$

It can be verified that the function T_k is a globally Lipschitz function, The graph of T_k is:



We can verify that the function T_k is a globally Lipschitz function satisfying $|T_k(r)| \le k$ and $|T_k(r)| \le |r|$.

Lemma 2.2. *Let $\delta > 0$, there exists a constant C_δ independent of n such that*

$$\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^{1+\delta}} dx \le C_\delta, \quad \forall n \ge 1.$$

Proof. Let $\delta > 0$. the function ψ_δ given by

$$\psi_\delta(t) = \int_0^t \frac{d\sigma}{(1+|\sigma|)^{1+\delta}} = \frac{1}{\delta}\left(1 - \frac{1}{(1+|t|)^\delta}\right)\text{sign}(t), \quad t \in \mathbb{R}$$

is globally Lipschitz $\psi_\delta(0) = 0$ and $|\psi_\delta(t)| \le \frac{1}{\delta}$, $\forall t \in \mathbb{R}$. Taking $\psi_\delta(u_n)$ as a test function in (P_{0n}), we obtain:

$$\langle Au_n, \psi_\delta(u_n)\rangle = \langle f_n, \psi_\delta(u_n)\rangle, \quad Au_n = -\text{div}(a(x)\nabla u_n)$$

so that

$$\int_\Omega a(x)\nabla u_n \nabla \psi_\delta(u_n) dx = \int_\Omega f_n \psi_\delta(u_n) dx$$

Using the fact that $\nabla \psi_\delta(u_n) = \nabla u_n \psi'_\delta(u_n)$ with $\psi'_\delta(u_n) = \frac{1}{(1+|u_n|)^{1+\delta}}$, we have

$$\begin{aligned}\int_\Omega a(x)\nabla u_n \nabla \psi_\delta(u_n) dx &= \int_\Omega a(x)\nabla u_n \nabla u_n \psi'_\delta(u_n) dx \\ &= \int_\Omega a(x)\nabla u_n \frac{\nabla u_n}{(1+|u_n|)^{1+\delta}} dx,\end{aligned}$$

Therefore,

$$\int_\Omega a(x)\nabla u_n \frac{\nabla u_n}{(1+|u_n|)^{1+\delta}} dx \le \int_\Omega |f_n||\psi_\delta(u_n)| dx \le \frac{1}{\delta}\|f_n\|_{L^1(\Omega)},$$

Since $a_n(x) \geq \alpha > 0$ and $\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$, we obtain

$$\alpha \int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^{1+\delta}}\, dx \leq \frac{1}{\delta}\|f\|_{L^1(\Omega)}.$$

Thus,

$$\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^{1+\delta}}\, dx \leq \frac{1}{\alpha\delta}\|f\|_{L^1(\Omega)} \leq C_\delta.$$

□

Lemma 2.3. *Let q be such that*

$$1 \leq q \leq \frac{N}{N-1}.$$

Then, the sequence (∇u_n) is bounded in $L^q(\Omega)$, i.e., there exists a constant $C = C(q) > 0$ such that:

$$\int_\Omega |\nabla u_n|^q\, dx \leq C, \quad \forall n \geq 1. \tag{2.7}$$

Proof. Note that

$$1 \leq q \leq \frac{N}{N-1} \leq 2 \quad (\text{since } N > 2).$$

Let q^* be the Sobolev exponent associated with q, i.e., $q^* = \frac{Nq}{N-q}$. For δ chosen such that

$$\delta = q^*\frac{2-q}{q} - 1 \quad \Longleftrightarrow \quad q^* = (1+\delta)\frac{q}{2-q}.$$

Lemma 2.2 implies that there exists a constant $C = C(\delta) > 0$ such that:

$$\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^{1+\delta}}\, dx \leq C, \quad \forall n \geq 1. \tag{2.8}$$

Let $y_n = \int_\Omega |\nabla u_n|^q\, dx$. We can write:

$$y_n = \int_\Omega \frac{|\nabla u_n|^q}{(1+|u_n|)^{(1+\delta)q/2}}(1+|u_n|)^{(1+\delta)q/2}\, dx.$$

Using Hölder's inequality with exponents $\frac{2}{q}$ and $\frac{2}{2-q}$, we obtain:

$$\begin{aligned} y_n &\leq \left(\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^{1+\delta}}\, dx\right)^{q/2} \left(\int_\Omega (1+|u_n|)^{(1+\delta)\frac{q}{2-q}}\, dx\right)^{1-q/2} \\ &\leq C^{q/2}\left(\int_\Omega (1+|u_n|)^{q^*}\, dx\right)^{1-q/2}. \end{aligned} \tag{2.9}$$

By the Sobolev inequality, we have:

$$\int_\Omega (1+|u_n|)^{q^*}\, dx \leq C\left(1+\int_\Omega |\nabla u_n|^q\, dx\right)^{q^*/q} = C\left(1+y_n^{q^*/q}\right).$$

Thus,

$$y_n \leq C^{q/2}\left(1 + y_n^{q^*/q}\right)^{1-q/2}.$$

Using the inequality $(a+b)^p \leq 2^p(a^p + b^p)$ (for $a, b \geq 0$ and $p > 0$), we obtain:

$$y_n \leq a + b y_n^{\frac{q^*}{q}(1-q/2)},$$

where a and b are constants depending on q, N and $|\Omega|$.

Let $\theta = \frac{q^*}{q}(1-q/2)$. For $q < \frac{N}{N-1}$, we have $\theta < 1$. Indeed:

$$\theta - 1 = \frac{N}{N-q}\left(1 - \frac{q}{2}\right) - 1 = \frac{N(2-q) - 2(N-q)}{2(N-q)} = \frac{q(2-N)}{2(N-q)} < 0,$$

since $N > 2$. Since $\theta < 1$, the inequality $y_n \leq a + b y_n^{\theta}$ implies that the sequence y_n is bounded.

For the case $q = \frac{N}{N-1}$, we repeat the argument by choosing α such that $\max\{1, \frac{q}{N-q}\} < \alpha < q^*$ and setting $\delta = \alpha\frac{N-q}{q} - 1$. Using the continuity of the embedding, we similarly obtain the boundedness.

□

2.6 Passing to the Limit

Since the operator $Au = -\text{div}(a(x)\nabla u)$ depends on the gradient ∇u. We need at least convergence almost everywhere of the sequence (∇u_n).

Lemma 2.4. *The sequence (∇u_n) converges almost everywhere to ∇u (possibly passing to a subsequence).*

Proof. We aim to show that (∇u_n) is **Cauchy in measure**, which implies that $\nabla u_n \longrightarrow \nabla u$ almost everywhere for a subsequence. This requires proving that:

$$\forall \delta > 0, \forall \varepsilon > 0, \exists n_0 \text{ such that } \forall p, q \geq n_0, \quad \text{meas}\{x \in \Omega : |(\nabla u_p - \nabla u_q)(x)| \geq \delta\} \leq \varepsilon.$$

Let us fix $\delta > 0$ and $\varepsilon > 0$. For any $\lambda > 0$ and $\eta > 0$, we observe the following inclusion:

$$\{x \in \Omega : |(\nabla u_p - \nabla u_q)(x)| \geq \delta\} \subset E_1 \cup E_2 \cup E_3 \cup E_4,$$

where

$$E_1 = \{x \in \Omega : |\nabla u_p| \geq \lambda\},$$

$$E_2 = \{x \in \Omega : |\nabla u_q| \geq \lambda\},$$

$$E_3 = \{x \in \Omega : |u_p - u_q| \geq \eta\},$$

$$E_4 = \{x \in \Omega : |\nabla u_p - \nabla u_q| \geq \delta, |\nabla u_p| \leq \lambda, |\nabla u_q| \leq \lambda, |u_p - u_q| \leq \eta\}.$$

Since the sequence (∇u_n) is bounded in $L^q(\Omega)$ for all $q \in \left[1, \frac{N}{N-1}\right[$ (see Lemma 2.3), choosing a sufficiently large λ allows us to make meas(E_1) and meas(E_2) arbitrarily small. For instance:

$$\text{meas}(E_1) = \int_{E_1} 1 \, dx = \frac{1}{\lambda} \int_{E_1} \lambda \, dx \leq \frac{1}{\lambda} \int_{\Omega} |\nabla u_p| \, dx \leq \frac{C}{\lambda} \leq \varepsilon.$$

Thus, meas$(E_1) \to 0$ as $\lambda \to +\infty$.

Regarding E_3, we have:

$$\text{meas}(E_3) \leq \frac{1}{\eta} \int_{\Omega} |u_p - u_q| \, dx.$$

Since (u_n) is a Cauchy sequence in $L^1(\Omega)$ [see (3.15)], for a fixed $\eta > 0$, we have meas$(E_3) \to 0$ as $p, q \to +\infty$.

It remains to control meas(E_4). By the monotonicity of the mapping $a(x,\xi) = M(x)\xi$, we have $(a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2) > 0$. On the compact set defined by $\{(\xi_1,\xi_2) : |\xi_1| \leq \lambda, |\xi_2| \leq \lambda, |\xi_1 - \xi_2| \geq \delta\}$, the continuous function $(a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2)$ attains a minimum $\mu(x) > 0$.

Consequently, for any $\varepsilon > 0$, there exists $\varepsilon' > 0$ such that:

$$\int_{E_4} \mu(x) \, dx < \varepsilon' \implies \text{meas}(E_4) < \varepsilon. \tag{2.10}$$

To satisfy this, we use the definition of $\mu(x)$ and E_4:

$$\int_{E_4} \mu(x) \, dx \leq \int_{\Omega} (a(x)\nabla u_p - a(x)\nabla u_q)\nabla T_\eta(u_p - u_q) \, dx, \tag{2.11}$$

where T_η is the truncation function. By using $T_\eta(u_p - u_q)$ as a test function in the equations for u_p and u_q and subtracting the results, we obtain:

$$\int_{\Omega} (a(x)\nabla u_p - a(x)\nabla u_q)\nabla T_\eta(u_p - u_q) \, dx \leq \text{small terms related to } \eta. \tag{2.12}$$

As $\eta \to 0$, the integral vanishes uniformly in p and q. Thus, for sufficiently small η, we obtain meas$(E_4) \leq \varepsilon$. Combining these estimates, we conclude that:

$$\text{meas}\{x \in \Omega : |(\nabla u_p - \nabla u_q)(x)| \geq \delta\} \leq 4\varepsilon.$$

This establishes the convergence of ∇u_n to ∇u in measure, and consequently, $\nabla u_n \to \nabla u$ almost everywhere in Ω (up to a subsequence). □

Proposition 2.1. *since $(u_n)_n$ is bounded in the space $W_0^{1,q}(\Omega)$ for all $q \in [1, q_c = \frac{N}{N-1}[$, we can extract from the sequence (u_n) a subsequence, still denoted by (u_n), such that*

$$u_n \rightharpoonup u \quad \text{weakly in} \quad W_0^{1,q}(\Omega), \tag{2.13}$$

$$u_n \longrightarrow u \quad \text{strongly in} \quad L^q(\Omega). \tag{2.14}$$

Therefore,

$$u_n \longrightarrow u \quad \text{a.e in} \quad \Omega. \tag{2.15}$$

For $v \in W_0^{1,\infty}(\Omega)$, we have

$$\int_\Omega a(x)\nabla u_n \nabla v \, dx = \int_\Omega f_n v \, dx. \tag{2.16}$$

1) Passage to the limit in $\int_\Omega f_n v \, dx$. Using that $f_n \to f$ strongly in $L^1(\Omega)$, we have for every $v \in L^\infty(\Omega)$

$$\lim_{n\to\infty} \int_\Omega f_n v \, dx = \int_\Omega f v \, dx.$$

2) Passage to the limit in $\int_\Omega a(x)\nabla u_n \nabla v \, dx$. As $\nabla u_n \rightharpoonup \nabla u$ almost everywhere in Ω, we have:

$$a(x)\nabla u_n \to a(x)\nabla u \quad \text{almost everywhere in} \quad \Omega. \tag{2.17}$$

For any measurable subset for $E \subset \Omega$ we have :

$$\begin{aligned} \int_E |a(x)\nabla u \nabla \varphi| \, dx &\leq \beta \|\nabla \varphi\|_{L^\infty(E)} \int_E |\nabla u_n| dx \\ &\leq \beta \|\nabla \varphi\|_{L^\infty(\Omega)} \int_E |\nabla u_n| dx \\ &\leq C \int_E |\nabla u_n| dx \end{aligned}$$

Since $q > 1$ and according to Hölder inequality, we have :

$$\begin{aligned} \int_E |a(x)\nabla u_n \nabla \varphi| \, dx &\leq C \left(\int_E |\nabla u_n|^q dx \right)^{1/q} . \left(\int_E 1 dx \right)^{1/q'} \\ &\leq C \left(\int_E |\nabla u_n|^q dx \right)^{1/q} . |E|^{1/q'} \end{aligned}$$

Using lemma 2.3 we have (∇u_n) is bounded in $L^q(\Omega)$, we have :

$$\int_E |a(x)\nabla u_n \nabla \varphi| \, dx \leq C|E|^{1/q'} = C'$$

This proves the equi-integrability of the sequence $(a(x)\nabla u_n \nabla \varphi)_n$. It follows from Vitali's convergence theorem 1.9 that

$$\lim_{n\to+\infty} \int_\Omega a(x)\nabla u_n \nabla \varphi \, dx = \int_\Omega a(x)\nabla u \nabla \varphi \, dx$$

Finally, passing to the limit in the formulation (2.16) , we obtain

$$\int_\Omega a(x)\nabla u \nabla v \, dx = \int_\Omega f v \, dx, \tag{2.18}$$

for every $v \in W_0^{1,\infty}(\Omega)$. Therefore, u is a solution to problem (P_0).

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Chapter 3

Elliptic problem with degenerate coercivity

This chapter is devoted to the analysis of a class of degenerate elliptic equations featuring $L^m(\Omega)$ data with $m > 1$. We investigate the existence of weak solutions by leveraging suitable analytical tools and appropriate structural assumptions on the coefficients.

3.1 The problem (P)

The problem we are considering is as follows:

$$(P) \qquad \begin{cases} -\operatorname{div}\left(\frac{a(x)\nabla u}{(1+|u|)^\theta}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here Ω is a bounded, open subset of $\mathbb{R}^N$, with $N > 2$, $f \in L^m(\Omega)$ with $m > 1$ and $a : \Omega \times \mathbb{R} \to \mathbb{R}$ is a mesurable function, satisfying the following conditions:

$$\alpha \le a(x) \le \beta \tag{3.1}$$

where α and β are positive constants and

$$0 \le \theta < 1 \tag{3.2}$$

3.2 Weak solutions

Definition 3.1. *We say that u is a weak solution to problem (P), if $u \in W_0^{1,1}(\Omega)$ and*

$$\int_\Omega a(x)\frac{\nabla u}{(1+|u|)^\theta} \cdot \nabla\varphi \, dx = \int_\Omega f\varphi \, dx, \quad \forall \varphi \in W_0^{1,\infty}(\Omega). \tag{3.3}$$

In the following result too, we will prove the existence of a $W_0^{1,1}(\Omega)$ solution.

Theorem 3.1. *Let f be a function in $L^m(\Omega)$, with $m = \frac{N}{N+1-\theta(N-1)}$ and $\frac{1}{N-1} < \theta < 1$. Assume (3.1). Then there exists a $W_0^{1,1}(\Omega)$ weak solution of (P).*

We are going to prove Theorem 3.1, that is, the existence of a solution to problem (P). Note that $m > 1$ if and only if $\theta > \frac{1}{N-1}$. In the sequel, C will denote a constant depending on $\alpha, N, \text{meas}(\Omega), \theta$ and the norm of f.

3.3 Approximation of (P)

Consider the following approximated problem:

$$(P_n)\quad \begin{cases} -\operatorname{div}\left(a(x)\dfrac{\nabla u_n}{(1+|T_n(u_n)|)^\theta}\right) = T_n(f) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

i.e., satisfying

$$\int_\Omega a(x)\frac{\nabla u_n}{(1+|T_n(u_n)|)^\theta}\cdot\nabla\varphi\,dx = \int_\Omega T_n(f)\varphi\,dx, \quad \forall\varphi\in W_0^{1,\infty}(\Omega), \tag{3.4}$$

with $T_n(f)$ a sequence of bounded functions defined in Ω by

$$T_n(f)(x) = \frac{f(x)}{1+\frac{|f(x)|}{n}}. \tag{3.5}$$

such that

$$\begin{cases} \|T_n(f)\|_{L^m(\Omega)} \le \|f\|_{L^m(\Omega)}, \\ T_n(f)\to f \text{ strongly in } L^m(\Omega), \quad m>1. \end{cases} \tag{3.6}$$

since $T_n(f)\in L^\infty(\Omega)\subset W^{-1,2}(\Omega)$ and the operator

$$A_n(u) = -\operatorname{div}\left(a(x)\frac{\nabla u_n}{(1+|T_n(u_n)|)^\theta}\right)$$

is bounded, coercive, and pseudo-monotone on $W_0^{1,2}(\Omega)$ (see Theorem2.1). Thus, there exists a unique function $u_n\in W_0^{1,2}(\Omega)$ solution of problem (P_n).

3.4 Uniform Estimates

Lemma 3.1. *Let $f\in L^m(\Omega)$,with $m=\frac{N}{N+1-\theta(N-1)}$ and $\frac{1}{N-1}<\theta<1$. then*

$$\int_\Omega |\nabla T_k(u_n)|^2 \le \frac{k(1+k)^\theta\|f\|_{L^1(\Omega)}}{\alpha}$$

Proof. We consider $T_k(u_n)$ as a test function in (P_n). Then

$$\int_\Omega a(x)\frac{\nabla u_n}{(1+|T_n(u_n)|)^\theta}\nabla(T_k(u_n))\,dx = \int_\Omega T_n(f)T_k(u_n)\,dx. \tag{3.7}$$

Since (3.1) and the fact that $|T_n(f)|\le|f|$, we have

$$\alpha\int_\Omega \frac{|\nabla T_k(u_n)|^2}{(1+|T_n(u_n)|)^\theta}\,dx \le \int_\Omega |f||T_k(u_n)|\,dx. \tag{3.8}$$

Using that $|T_k(u_n)| \le k$ and for all $n > k > 0$, we have

$$\frac{\alpha}{(1+k)^\theta}\int_\Omega |\nabla T_k(u_n)|^2 dx \le k\int_\Omega |f| dx = k\|f\|_{L^1(\Omega)}. \tag{3.9}$$

So, we find

$$\int_\Omega |\nabla T_k(u_n)|^2\, dx \le \frac{k(1+k)^\theta}{\alpha}\,\|f\|_{L^1(\Omega)}. \tag{3.10}$$

Moreover, by Fatous Lemma, we have

$$\int_\Omega |\nabla T_k(u)|^2 dx \le \liminf_{n\to+\infty}\int_\Omega |\nabla T_k(u_n)|^2 dx \le C.$$

□

Lemma 3.2. *Let $f \in L^m(\Omega)$, with $m = \frac{N}{N+1-\theta(N-1)}$ and $\frac{1}{N-1} < \theta < 1$. then there exist $C > 0$ such that*

$$\int_\Omega |u_n|^{\frac{N}{N-1}}\, dx \le C.$$

Moreover, the sequence $\{u_n\}$ is bounded in the space $W_0^{1,1}(\Omega)$.

Proof. Choosing $[(1+|u_n|)^p - 1]\,\mathrm{sign}(u_n)$, with $p = \theta - \frac{1}{N-1}$, as a test function in (P_n), we obtain

$$\int_\Omega a(x)\frac{\nabla u_n}{(1+|T_n(u_n)|)^\theta}\nabla([(1+|u_n|)^p - 1])dx = \int_\Omega T_n(f)[(1+|u_n|)^p - 1]dx. \tag{3.11}$$

By Hölder's inequality on the right hand side and assumption (3.1) on the left one

$$\alpha p\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^\theta}(1+|u_n|)^{p-1}dx \le \|f\|_{L^m(\Omega)}\left[\int_\Omega ((1+|u_n|)^p - 1)^{m'}dx\right]^{\frac{1}{m'}}. \tag{9}$$

The fact that $\theta - p + 1 = \frac{N}{N-1}$ and $\|f\|_{L^m(\Omega)} \le C$, we have

$$\alpha p\int_\Omega \left(\frac{|\nabla u_n|}{(1+|u_n|)^{\frac{N}{2(N-1)}}}\right)^2 dx \le C\left[\int_\Omega ((1+|u_n|)^p - 1)^{m'}dx\right]^{\frac{1}{m'}}. \tag{9}$$

The Sobolev embedding used on the left hand side implies

$$\left[\int_\Omega \left((1+|u_n|)^{\frac{N-2}{2(N-1)}} - 1\right)^{\frac{2N}{N-2}} dx\right]^{\frac{2}{2^*}} \le C\left[\int_\Omega ((1+|u_n|)^p - 1)^{m'}dx\right]^{\frac{1}{m'}}. \tag{3.12}$$

We observe that $\frac{N-2}{2(N-1)}\frac{2N}{N-2} = pm'$, moreover $\frac{2}{2^*} > \frac{1}{m'}$, since $m < \frac{N}{2}$. Therefore the above inequality implies that

$$\int_\Omega |u_n|^{\frac{N}{N-1}}\, dx \le C. \tag{10}$$

One deduces that

$$\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^{\frac{N}{N-1}}}\,dx \le C. \tag{3.13}$$

Höldr's inequality with exponent 2 applied to

$$\int_\Omega |\nabla u_n| = \int_\Omega \frac{|\nabla u_n|}{(1+|u_n|)^{\frac{N}{2(N-1)}}}(1+|u_n|)^{\frac{N}{2(N-1)}}$$

gives

$$\int_\Omega |\nabla u_n| \le C.$$

Thus, $\{u_n\}$ is bounded in $W_0^{1,1}(\Omega)$. □

3.5 Passage to the limite

By Lemma 3.2, the sequence of approximated solutions u_n is bounded in $W_0^{1,1}(\Omega)$. Therefore, there exists a function u belongs to $W_0^{1,1}(\Omega)$ such that, up to subsequences,

$$u_n \rightharpoonup u \quad \text{weakly in} \quad W_0^{1,1}(\Omega), \tag{3.14}$$

$$u_n \longrightarrow u \quad \text{strongly in} \quad L^1(\Omega). \tag{3.15}$$

Therefore

$$u_n \longrightarrow u \quad \text{a.e in} \quad \Omega, \tag{3.16}$$

Now, we are going to prove the almost everywhere convergence of ∇u_n to ∇u in Ω.

Let $k>0$ and let us take

$$w_n = T_{2k}(T_k(u_n) - T_k(u)),$$

as test function in (P_n), we have

$$\int_\Omega \frac{a(x)}{(1+|T_n(u_n)|)^\theta}\nabla u_n \nabla w_n dx = \int_\Omega f_n w_n dx. \tag{3.17}$$

For the first integral in (3.17) and the fact that $\frac{1}{1+|T_n(u_n)|} \ge \frac{1}{1+k}$ on the set $\{|u_n| \le k\}$, we can write it as

$$\begin{aligned}\int_\Omega \frac{a(x)}{(1+|T_n(u_n)|)^\theta}\nabla u_n \nabla w_n dx \ge{}& \frac{\alpha}{(1+k)^\theta}\int_{\{|u_n|\le k\}} \nabla T_k(u_n)\nabla(T_k(u_n)-T_k(u))dx \\ &- \int_{\{|u_n|>k\}} \frac{a(x)}{(1+|T_n(u_n)|)^\theta}\nabla T_k(u_n)\nabla T_k(u)dx\end{aligned}$$

Using that $\frac{1}{1+|T_n(u_n)|} \leq 1$ and the assumption (3.1), we have

$$-\int_{\{|u_n|>k\}} \frac{a(x)}{(1+|T_n(u_n)|)^\theta} \nabla T_k(u_n) \nabla T_k(u) dx \geq -\beta \int_{\{|u_n|>k\}} \nabla T_k(u_n) \nabla T_k(u) dx$$

So,

$$\begin{aligned}\int_\Omega \frac{a(x)}{(1+|T_n(u_n)|)^\theta} \nabla u_n \nabla w_n dx \geq \frac{\alpha}{(1+k)^\theta} \int_{\{|u_n|\leq k\}} \nabla T_k(u_n) \nabla (T_k(u_n) - T_k(u)) dx \\ -\beta \int_{\{|u_n|>k\}} \nabla T_k(u_n) \nabla T_k(u) dx.\end{aligned}$$

It follows from the above inequality that

$$\frac{\alpha}{(1+k)^\theta} \int_\Omega \nabla T_k(u_n) \nabla (T_k(u_n) - T_k(u)) dx \leq \beta \int_{\{|u_n|>k\}} \nabla T_k(u_n) \nabla T_k(u) dx + \int_\Omega f_n w_n dx.$$

Furthermore, we have

$$\begin{aligned}\frac{\alpha}{(1+k)^\theta} \int_\Omega [\nabla T_k(u_n) - \nabla T_k(u)] \nabla (T_k(u_n) - T_k(u)) dx \leq \beta \int_{\{|u_n|>k\}} |\nabla T_k(u_n)||\nabla T_k(u)| dx \\ -\frac{\alpha}{(1+k)^\theta} \int_\Omega \nabla T_k(u) \nabla (T_k(u_n) - T_k(u)) dx + \int_\Omega f_n w_n dx \\ = I_1 + I_2 + I_3. \end{aligned} \tag{3.18}$$

Limit of I_1.The fact that $|\nabla T_k(u_n)|$ is bounded in $L^2(\Omega)$ and by the dominated convergence theorem

$$\chi_{\{|u_n|>k\}} |\nabla T_k(u)| \longrightarrow \chi_{\{|u|>k\}} |\nabla T_k(u)| \quad \text{strongly in} \quad L^2(\Omega),$$

which is zero, as $n \to +\infty$. Thus, we obtain

$$\lim_{n \longrightarrow +\infty} I_1 = 0. \tag{3.19}$$

Limit of I_2. Since Lemma 3.1, we obtain $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ in $L^2(\Omega)$ and the bounded of $\nabla T_k(u)$ in $L^2(\Omega)$, we have

$$\lim_{n \longrightarrow +\infty} I_2 = 0. \tag{3.20}$$

Limit of I_3. Notice that

$$\int_\Omega |f_n - g||w_n| dx + \int_\Omega |f w_n| dx \leq 2k \int_\Omega |f_n - f| dx dt + \int_\Omega |f T_{2k}(T_k(u_n) - T_k(u)| dx.$$

Since $f_n \longrightarrow f$ strongly in $L^m(\Omega)$, using (3.16) and the Lebesgue dominated convergence theorem, we have

$$\lim_{n \longrightarrow +\infty} I_3 \leq \lim_{n \longrightarrow +\infty} |f T_{2k}(T_k(u_n) - T_k(u))| dx = 0. \tag{3.21}$$

Now, passing to the limits in (3.18) as n tend to infinity, and by (3.19)-(3.21), we deduce that

$$\lim_{n\longrightarrow+\infty} I_n^k = 0,$$

where

$$I_n^k = \int_\Omega [\nabla T_k(u_n) - \nabla T_k(u)]^2 dx.$$

Since $I_n^k \longrightarrow 0$ as $n \longrightarrow +\infty$ and $(T_k(u_n))_n$ is bounded in $W_0^{1,2}(\Omega)$ (see lemma 3.1), we conclude that for every $k > 0$,

$$\nabla T_k(u_n) \longrightarrow \nabla T_k(u) \text{ strongly in } L^2(\Omega).$$

From this result, we deduce that (up to subsequences)

$$\nabla u_n \longrightarrow \nabla u \quad \text{almost everywhere in} \quad \Omega. \tag{3.22}$$

3.6 The end of the proof of Theorem 3.1

For $\varphi \in W_0^{1,\infty}(\Omega)$ (See (3.4)), we have

$$\int_\Omega a(x)\frac{\nabla u_n}{(1+|T_n(u_n)|)^\theta}\nabla\varphi dx = \int_\Omega f_n\varphi dx,$$

1)Passage to the limit in $\int_\Omega f_n\varphi dx$

Using that $f_n \to f$ strongly in $L^m(\Omega)$, we have $\forall\varphi \in W_0^{1,\infty}(\Omega)$

$$\lim_{n\to+\infty}\int_\Omega f_n\varphi dx = \int_\Omega f\varphi dx.$$

2)Passage to the limit in $\int_\Omega a(x)\frac{\nabla u_n}{(1+|T_n(u_n)|)^\theta}\nabla\varphi dx$.

Thanks to the boundedness of the sequence $(u_n)_n$ in $W_0^{1,1}(\Omega)$, we have that

$$\nabla u_n \rightharpoonup \nabla u \quad \text{in} \quad L^1(\Omega). \tag{3.23}$$

By the dominated convergence Theorem, we get

$$a(x)\nabla u_n \to a(x)\nabla u \quad \text{in} \quad L^1(\Omega).$$

and using (3.16), we have that

$$\begin{cases} \frac{1}{(1+|T_n(u_n)|)^\theta}\nabla\varphi \to \frac{1}{(1+|u|)^\theta}\nabla\varphi, \text{a.e in}\,\Omega, \\ |\frac{1}{(1+|T_n(u_n)|)^\theta}\nabla\varphi| \le |\nabla\varphi| \in L^\infty(\Omega), \end{cases} .$$

So,

$$\lim_{n\to+\infty}\int_\Omega a(x)\frac{\nabla u_n}{(1+|T_n(u_n)|)^\theta}\nabla\varphi dx=\int_\Omega a(x)\frac{\nabla u}{1+|u|}\nabla\varphi dx.$$

Finally, we pass to the limit in the formulation (3.4), we obtain

$$\int_\Omega a(x)\frac{\nabla u}{(1+|u|)^\theta}\nabla\varphi dx=\int_\Omega f\varphi dx,$$

for every $\varphi\in W_0^{1,\infty}(\Omega)$. Therefore. we have to prove that u is a solution to problem (P). This finishes the proof of theorem 3.1.

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Conclusion

In this thesis, we have focused our attention on a class of nonlinear elliptic equations with degenerate coercivity of the form:

$$(P) \qquad \begin{cases} Au &= f \text{ in } \Omega; \\ u &= 0 \text{ on } \partial\Omega, \end{cases}$$

where the differential operator:

$$Au = -\operatorname{div}\left(\frac{a(x)\nabla u}{(1+|u|)^{\theta}}\right)$$

and the right-hand f in $L^m(\Omega)$. The method used to prove the existence of solutions $u \in W_0^{1,1}(\Omega)$ for the problem (P) is the same as used in [2]. We are going to prove the existence of the weak solution of the problem (P) in the space $W_0^{1,1}(\Omega)$ in the case $m = \frac{N}{N+1-\theta(N-1)}$ and $\frac{1}{N-1} < \theta < 1$. To do this, we approximate the problem (P) by a sequence of approximate problems (P_n) given in L^∞ whose existence of the solution approximate is guaranteed (See Chapter 2 or [7]). Then we will prove some estimates uniform on the sequence of solutions of these problems (P_n) and their partial derivatives.

Once this is achieved, the linearity of the operator with respect to the gradient (since $a(x)$ is multiplied by ∇u), together with the boundedness and continuity of the coefficient $a(x)$, will allow us to pass to the limit and thus obtain the desired weak solution.

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