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Directeur de thése : Douadi DRIHEM

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To the people who paved our way of science and knowledge.

DEDICATE

The simplest words
are the strongest, I send all my love
*to my **father** and my **mother** thank you for*
making me what I am today.

To whose love flows
in my veins, and my heart always
*remembers them, to my **brothers** and **sisters**To the most*
precious to my heart
Mehdi,Djana,Mohamed
*To the spirit of my **sister** and grandmother*
ZOUINA AND KALTOM

Finally,
to the best person in my life
To the person who taught me the meaning
of patience and respect you're just yours all my life.

وفـــاء
هباش

Contents

Notation

- $\mathbb{R}^n$ is the n-dimensional real Euclidean space.
- $\mathbb{N}_0$ is the collection of all natural numbers.
- $\mathbb{Z}$ is the collection of all integer numbers.
- If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E.
- For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x,r)$ the open ball in $\mathbb{R}^n$ with center x and radius r.
- Q will denote an cube in the space $\mathbb{R}^n$ with sides parallel to the coordinate axes and $l(Q)$ will denote the side length of the cube Q.
- For all cubes rQ and $r > 0$, rQ is the cube concentric with Q having the side length $rl(Q)$.
- $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ stands for some multi-index, whose length is denoted by $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and
$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \ldots \partial x_n^{\alpha_n}}.$$
- The Euclidean scalar product of $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ is given by
$$x \cdot y = x_1 y_1 + \ldots + x_n y_n.$$
- The expression $f \lesssim g$ means that $f < cg$ for some independent constant c (and non-negative functions f and g).
- $f \approx g$ means $f \lesssim g \lesssim f$.
- The notation $X \hookrightarrow Y$ stands for continuous embeddings from X to Y, where X and Y are quasi-normed spaces.
- $f * g\,(x) = \int_{\mathbb{R}^n} f\,(x-y)\, g\,(y)\, dy$ is the product of the convolution of functions f and g.
- $L^1_{\text{loc}}\,(\mathbb{R}^n)$ is the collection of all locally integrable functions on $\mathbb{R}^n$.

- $L_p(\mathbb{R}^n)$ for $0 < p \leq \infty$ stands for the Lebesgue spaces on $\mathbb{R}^n$ for which

$$\|f \mid L_p(\mathbb{R}^n)\| = \|f\|_p = \Big(\int_{\mathbb{R}^n} |f(x)|^p \, dx \Big)^{1/p} < \infty, \quad 0 < p < \infty$$

and

$$\|f \mid L_\infty(\mathbb{R}^n)\| = \|f\|_\infty = \operatorname*{ess\text{-}sup}_{x \in \mathbb{R}^n} |f(x)| < \infty.$$

- If $1 \leq p \leq \infty$, then $p'$is the conjugate exponent of p given by $\frac{1}{p} + \frac{1}{p'} = 1$.

- $c_1, c_2, c_3, \ldots$positive constants, their values may depend on certain parameters and some auxiliary functions, and change from one line to another.

- $\mathcal{D}(\mathbb{R}^n)$ is the space of functions with continuous derivatives of all orders and compact support.

- $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on $\mathbb{R}^n$.

- By supp f we denote the support of the function f.

- The dual $\mathcal{S}'(\mathbb{R}^n)$ is the space of temperate distributions.

- We define the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} f(x) \, dx, \quad \xi \in \mathbb{R}^n.$$

and its inverse Fourier transform by:

$$\mathcal{F}^{-1} f(x) = \check{f}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} f(\xi) \, d\xi, \quad x \in \mathbb{R}^n.$$

- f its spectrum in $B(0, r)$ and $r > 0$; supp$\hat{f} \subset B(0, r)$.

- By $\ell^q, 0 < q \leq \infty$, we denote the space of all (complex) sequences $\{a_k\}_{k \in \mathbb{Z}}$ equipped with the quasi-norm

$$\left\| \{a_k\}_{k \in \mathbb{Z}} | \ell^q \right\| = \left(\sum_{k=-\infty}^{\infty} |a_k|^q \right)^{\frac{1}{q}}$$

(with the usual modification if $q = \infty$).

- If $a : \mathbb{R}^n \times \mathbb{R}^n \to C$ is a function, we denote by $a(x, D)$ the pseudo-differential operator (P.D.O) of symbol a is noted $a(x, D)$ and defined on the class $\mathcal{S}(\mathbb{R}^n)$ by:

$$a(x, D) f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} a(x, D) \hat{f}(\xi) \, d\xi, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

- "i.e." stands simply for "in other words".

- "a.e." stands simply for "almost everywhere".

- χ_E is the characteristic function of $E \subset \mathbb{R}^n$.

- $\eta_{v,m}(x) = 2^{nv}(1 + 2^v |x|)^{-m}$, for any $x \in \mathbb{R}^n$, $v \in \mathbb{N}_0$ and $m > 0$.

Introduction

In recent years there has been a growing interesting in generalizing classical spaces such as Lebesgue and Sobolev spaces to cases with either variable integrability or variable smoothness see [15].

The motivation to study such function spaces also comes from applications to other fields of applied mathematics, but also from applications to image processing and PDE with non-standard growth conditions.

Some example of these spaces can be mentioned such as: variable Lebesgue space, variable Besov and Tiebel-Lizorkin spaces.

Pseudo-differential operators play an import role in Harmonic analysis and in nonlinear partial differential. The boundedness these operators has been extensively addressed in several works. In Lebesgue spaces with symbols in the Hörmander classes can be found in [5-8, 11-12, 21, 23, 37-38] and references therein. In another function spaces, such that Besov spaces, Triebel-Lizorkin spaces, BMO spaces and Hardy spaces, see [26, 32-34, 39, 46-48].

In [34] J. Marschall introduced the class $SB_{\delta}^{m}(r, \mu, v; N, \lambda)$, which is defined by means of vector-valued Besov spaces, and proved the boundedness of the corresponding pseudodifferential operators on Besov spaces and Triebel-Lizorkin spaces.

Boundedness of pseudodifferential operators, with symbols in the Hörmander classes, on weighted variable exponent Lebesgue and Bessel potential spaces was studied by V.S. Rabinovich and S. Samko [37-38] and by A. Yu. Karlovich and I. M. Spitkovsky in [28] (in variable Lebesgue space). Since Besov spaces can be written as a (real) interpolation space between appropriate Bessel potential spaces, Almeida and Hästö [2] extend the results of V.S. Rabinovich and S. Samko to Besov spaces with variable integrability $B_{p(\cdot),q}^{s}$.

The variable Besov spaces $B_{p(\cdot),q(\cdot)}^{s(\cdot)}$, initially appeared in the paper of A. Almeida and P. Hästo [2], serveral bassic properties where established, such as the Fourier analytical characterization when p, q, s are constants they coincide with the usual function spaces $B_{p,q}^{s}$ studied in detail by H. Triebel in [43-45].

Diening, Hästö and Roudenko, [14] introduced and investigated Triebel-Lizorkin spaces with variable smoothness and integrability $F_{p(\cdot),q(\cdot)}^{s(\cdot)}$ with $s(\cdot) \geq 0$, and showed that these spaces behaved nicely with respect to the trace operator.

The aim of the thesis is to study the boundedness of the pseudodifferential operators on variable Besov spaces Triebel-Lizorkin spaces.

Our thesis consists four chapter, in the first chapter, we give some basic properties of variable Lebesgue spaces and mixed variable Lebesgue's sequence space, after this we define variable Besov spaces where all the three parameters are variable and we recall some their basic properties.

In the second chapter, we investigate the $L^{p(\cdot)}$- boundedness of certain class of pseudo-differential operators with non regular symbols. We employ regularisation methods.

In Chapter 3, we present the boundedness properties of the pseudo-differential operators on Besov spaces with variable smoothness and integrability with symbols in $SB_{\delta}^{m}(r, \mu, v; N, \lambda)$, where we use the decomposition of Littelwood-Paley.

In Chapter 4, we present the boundedness properties of the pseudo-differential operators on Triebel-Lizorkin spaces with variable smoothness and integrability with symbols in $SB_{\delta}^{m}(r, \mu, v; N, \lambda)$. Also we give some technical lemmas needed in the proof of the main statement.

The Chapter three is a paper published in Journal of Pseudo-Differential operators and applications "Boundedness of non regular pseudodifferential operators on variable Besov spaces, 8 (2), 167-189 2017", while the Chapter four is papers in preparation with the advisor.

In the future we will try to apply these results to study some problems in harmonic analysis and partial differential equations.

Chapter 1

Variable Besov spaces

In this chapter we present some results, which remain fixed throughout this theses. We recall some conventions and results on modular and variable Lebesgue spaces, we then define the Besov space $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ and give some their basic properties in analogy to the Besov spaces with fixed exponents.

1.1 Variable Lebesgue spaces

The object of this section is to recall some fundamental properties of variable Lebesgue spaces, see [10, Chapters 1–2] and [15, Chapters 1–3] for details and further properties.

1.1.1 Modular space

We will use the modular to define the variable Lebesgue spaces.

Definition 1.1 *Let X be a $\Bbbk$-vector space. A function $\varrho : X \to [0, \infty]$ is called a semimodular on X if the following properties hold.*

(a) $\varrho(0) = 0$.

(b) $\varrho(\lambda x) = \varrho(x)$ *for all* $x \in X, \lambda \in \Bbbk$ *with* $|\lambda| = 1$.

(c) ϱ *is quasi convex.*

(d) ϱ *is left-continuous.*

(e) $\varrho(\lambda x) = 0$ *for all* $\lambda > 0$ *implies* $x = 0$.

A semimodular is called a modular if

(f) $\varrho(x) = 0$ *implies* $x = 0$.

A semimodular is called a continuous if

(g) *The mapping* $\lambda \mapsto \varrho(\lambda x)$ *is continuous on* $[0, \infty)$ *for every* $x \in X$.

The following examples are from [15].

Example 1.2 **(a)** *If* $1 \leq p < \infty$, *then*

$$\varrho_p(f) = \int_{\mathbb{R}^n} |f(x)|^p \, dx$$

defines a continuous modular on $L^0(\mathbb{R}^n)$.

(b) *Let* $w \in L^1_{loc}(\mathbb{R}^n)$ *with* $w > 0$ *almost everywhere and* $1 \leq p < \infty$. *Then*

$$\varrho(f) = \int_{\mathbb{R}^n} |f(x)|^p \, w(x) \, dx$$

defines a continuous modular on $L^0(\mathbb{R}^n)$.

(c) *If* $1 \leq p < \infty$, *then*

$$\varrho_p((x_j)) = \sum_{j=0}^{\infty} |x_j|^p .$$

defines a continuous modular on $\mathbb{R}^{\mathbb{N}}$.

Remark 1.3 *Let* ϱ *be a semimodular on* X. *Then by quasi convexity, non-negative of* ϱ *and* $\varrho(0) = 0$ *it follows that,*

$$\begin{aligned} \varrho(\lambda x) &= \varrho(|\lambda| x) \leq C |\lambda| \varrho(x) \qquad \text{for all } |\lambda| \leq 1, \\ \varrho(\lambda x) &= \varrho(|\lambda| x) \geq \frac{1}{C} |\lambda| \varrho(x) \qquad \text{for all } |\lambda| \geq 1, \end{aligned} \tag{1.1}$$

where $C \geq 1$ *is independent of* λ *and* x.

Definition 1.4 *If* ϱ *be a semimodular or modular on* X, *then*

$$X_\varrho = \left\{ x \in X : \quad \lim_{\lambda \to 0} \varrho(\lambda x) = 0 \right\}$$

is called a semimodular space or modular space, respectively, where the limit $\lambda \to 0$ *takes place in* $\mathbb{k}$.

Since $\varrho(\lambda x) = \varrho(|\lambda| x)$ it is enough to require $\lim_{\lambda \to 0} \varrho(\lambda x)$ with $\lambda \in (0, \infty)$. From (1.1) we can alternatively define X_ϱ by

$$X_\varrho = \{x \in X : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

since for $\lambda' < \lambda$ we have by (1.1) that

$$\varrho(\lambda x) = \varrho\left(\frac{\lambda'}{\lambda}\lambda x\right) \leq C\frac{\lambda'}{\lambda}\varrho(\lambda x) \to 0$$

as $\lambda' \to 0$ and $C \geq 1$.

We equipped this space with the following quasi-norm, for the proof see [15, Theorem 2.1.7].

Theorem 1.5 *Let ϱ be a semimodular on X. Then X_ϱ is a quasi-normed $\Bbbk$-vector space. The quasi-norm, called the Luxemburg quasi-norm, is defined by*

$$\|x\|_\varrho = \inf\left\{\lambda > 0 : \varrho\left(\frac{1}{\lambda}x\right) \leq 1\right\}.$$

Example 1.6 *Let $1 \leq p < \infty$ and ϱ_p be as in Example 1.2. Then*

$$X_{\varrho_p} = L^p(\mathbb{R}^n).$$

For an exposition of these concepts we refer to the monographs [15] and [36].

1.2 Lebesgue spaces $L^{p(\cdot)}$

In this subsection we recall the definition and basic properties of the variable Lebesgue spaces. They differ from classical L^p spaces where the exponent p is not constant but a function from $\mathbb{R}^n$ to $[c, \infty)$, $c > 0$.

The variable exponents that we consider are always measurable functions p on $\mathbb{R}^n$ with range in $[c, \infty[$ for some $c > 0$. We denote the set of such functions by $\mathcal{P}_0(\mathbb{R}^n)$. The subset of variable exponents with range $[1, \infty[$ is denoted by $\mathcal{P}(\mathbb{R}^n)$. We use the standard notation

$$p^- = \operatorname*{ess-inf}_{x \in \mathbb{R}^n} p(x), \quad p^+ := \operatorname*{ess-sup}_{x \in \mathbb{R}^n} p(x).$$

The variable exponent Lebesgue space $L^{p(\cdot)}$ is the class of all measurable functions f on $\mathbb{R}^n$ such that the modular

$$\varrho_{p(\cdot)}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx$$

is finite. This is a quasi-Banach function space equipped with the quasi-norm

$$\|f\|_{p(\cdot)} = \inf\left\{\mu > 0 : \varrho_{p(\cdot)}\left(\frac{1}{\mu}f\right) \leq 1\right\}.$$

If $p(x) = p$ is constant, then $L^{p(\cdot)} = L^p$ is the classical Lebesgue space.

The following classes for the exponents are necessary for us. We say that $g : \mathbb{R}^n \to \mathbb{R}$ is *locally* log-*Hölder continuous*, abbreviated $g \in C^{\log}_{\text{loc}}$, if there exists $c_{\log}(g) > 0$ such that

$$|g(x) - g(y)| \leq \frac{c_{\log}(g)}{\log(e + 1/|x-y|)} \tag{1.2}$$

for all $x, y \in \mathbb{R}^n$. We say that g satisfies the log-*Hölder decay condition*, if there exists $g_\infty \in \mathbb{R}$ and a constant $c_{\log} > 0$ such that

$$|g(x) - g_\infty| \leq \frac{c_{\log}}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$. The constants $c_{\log}(g)$ and $c_{\log}$ are called the *locally* log-*Hölder constant* and the log-*Hölder decay constant*, respectively. We note that all functions $g \in C^{\log}_{\text{loc}}$ always belong to L^∞.

We say that g is *globally*-log-*Hölder continuous*, abbreviated $g \in C^{\log}$, if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition. We define the following class of variable exponents

$$\mathcal{P}^{\log}(\mathbb{R}^n) = \left\{p \in \mathcal{P}(\mathbb{R}^n) : \frac{1}{p} \in C^{\log}\right\},$$

were introduced in [16, Section 2]. Note that $\frac{1}{p}$ although is bounded, the variable exponent p itself can be unbounded.

Example 1.7 *We set*

$$p(x) = \max(1 - e^{3-|x|}, \min(6/5, \max(1/2.3/2 - x^2))), \quad x \in \mathbb{R}$$

then $p \in C^{\log}$, see [37].

A basic tool to study the variable Besov spaces is the following.

Lemma 1.8 *If $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, then the convolution with a radially decreasing L^1-function is bounded on $L^{p(\cdot)}$:*

$$\|\varphi * f\|_{p(\cdot)} \leq c\|\varphi\|_1 \|f\|_{p(\cdot)}.$$

Proof. For the proof, see [15] ■

Lemma 1.9 *If $p \in \mathcal{P}(\mathbb{R}^n)$, then $\|f\|_{p(\cdot)} \leq 1$ and $\varrho_{L^{p(\cdot)}}(f) \leq 1$ are equivalent. For $f \in L^{p(\cdot)}(\mathbb{R}^n)$ we have*

1. *If $\|f\|_{p(\cdot)} \leq 1$, then $\varrho_{L^{p(\cdot)}}(f) \leq \|f\|_{p(\cdot)}$.*

2. *If $1 < \|f\|_{p(\cdot)}$, then $\|f\|_{p(\cdot)} \leq \varrho_{L^{p(\cdot)}}(f)$.*

Lemma 1.10 (***Hölder's inequality***) *Let $p, q, s \in \mathcal{P}(\mathbb{R}^n)$ such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}$$

for every $y \in \mathbb{R}^n$. Then

$$\|fg\|_{s(\cdot)} \leq 2\|f\|_{p(\cdot)}\|g\|_{q(\cdot)},$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{q(\cdot)}(\mathbb{R}^n)$.

For the proof, see [15].

1.3 Besov spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$

We recall in this subsection the definition and some properties of mixed Lebesgue sequence and Besov spaces with variable smoothness and integrability.

Definition 1.11 *Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the modular*

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{f_v\}_v) = \sum_v \inf\Big\{\lambda_v > 0 : \varrho_{p(\cdot)}\Big(\frac{f_v}{\lambda_v^{1/q(\cdot)}}\Big) \leq 1\Big\}.$$

The (quasi)-norm is defined from this as usual:

$$\|\{f_v\}_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf\Big\{\mu > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\Big(\frac{1}{\mu}\{f_v\}_v\Big) \leq 1\Big\}. \tag{1.3}$$

If $q^+ < \infty$, then we can replace (1.3) by the simpler expression

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{f_v\}) = \sum_v \left\||f_v|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Furthermore, if p and q are constants, then $\ell^{q(\cdot)}(L^{p(\cdot)}) = \ell^q(L^p)$. The case $p := \infty$ can be included by replacing the last modular by

$$\varrho_{\ell^{q(\cdot)}(L^\infty)}(\{f_v\}_v) = \sum_v \left\| |f_v|^{q(\cdot)} \right\|_\infty .$$

We define $\frac{1}{p_\infty} = \lim_{|x|\to\infty} \frac{1}{p(x)}$ and we use the convention $\frac{1}{\infty} = 0$.

These function spaces introduced recently in [1].

Now we recall the definition of the spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$, as given in [1]. We first need the concept of a smooth dyadic resolution of unity. Let Ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\Psi(x) = 1$ for $|x| \le 1$ and $\Psi(x) = 0$ for $|x| \ge 2$. We put

$$\mathcal{F}\varphi_0(x) = \Psi(x)$$

$$\mathcal{F}\varphi_1(x) = \Psi(x) - \Psi(2x)$$

and

$$\mathcal{F}\varphi_v(x) = \mathcal{F}\varphi_1(2^{-v}x) \quad for \quad v = 2, 3, ...$$

Then $\{\mathcal{F}\varphi_v\}_{v\in\mathbb{N}_0}$ is a smooth dyadic resolution of unity,

$$\sum_{v=0}^{\infty} \mathcal{F}\varphi_v(x) = 1$$

for all $x \in \mathbb{R}^n$. Thus we obtain the Littlewood-Paley decomposition

$$f = \sum_{v=0}^{\infty} \varphi_v * f$$

of all $f \in \mathcal{S}'(\mathbb{R}^n)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$).

Now, we define the Besov spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$.

Definition 1.12 *Let $\{\mathcal{F}\varphi_v\}_{v\in\mathbb{N}_0}$ be as resolution of unity. For $s : \mathbb{R}^n \to \mathbb{R}$ and $p, q \in \mathcal{P}_0(\mathbb{R}^n)$, the Besov space $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ consists of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} = \left\| \left\{ 2^{vs(\cdot)} \varphi_v * f \right\}_v \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty. \tag{1.4}$$

To the Besov space we can also associate the following modular:

$$\varrho^{\varphi}_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}}(f) = \varrho_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)} \left(\left\{ 2^{vs(\cdot)} \varphi_v * f \right\}_v \right).$$

We directly obtain the following simplification in the case when q is constant, see [1]. If q is a constant, then

$$\|f\|^{\varphi}_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} = \left\| \left\| \left\{ 2^{vs(\cdot)} \varphi_v * f \right\}_v \right\|_{p(\cdot)} \right\|_{\ell^q},$$

For any $p, q \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ and $s \in C^{\log}_{\mathrm{loc}}$, the space $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ does not depend on the chosen smooth dyadic resolution of unity $\{\mathcal{F}\varphi_v\}_{v\in\mathbb{N}_0}$ (in the sense of equivalent quasi-norms) and

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^{s(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

If $p, q \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ and $s \in C^{\log}_{\mathrm{loc}}$, then $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ and they are quasi-Banach spaces. Moreover, if p, q, α are constants, we re-obtain the usual Besov spaces $B^s_{p,q}$, studied in detail by H. Triebel in [44], [45] and [46]. See also [20] and [41] for further properties.

The following theorem gives basic embedding generalize the constant exponent versions.

Theorem 1.13 *Let* $\alpha, \alpha_0, \alpha_1 \in L^\infty(\mathbb{R}^n)$ *and* $p, q_0, q_1 \in \mathcal{P}_0(\mathbb{R}^n)$.

(a) *If* $q_0 \leq q_1$, *then*

$$B^{\alpha(\cdot)}_{p(\cdot),q_0(\cdot)}(\mathbb{R}^n) \hookrightarrow B^{\alpha(\cdot)}_{p(\cdot),q_1(\cdot)}(\mathbb{R}^n).$$

(b) *If* $(\alpha_0 - \alpha_1)^- > 0$, *then*

$$B^{\alpha_0(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow B^{\alpha_1(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n).$$

We next consider embeddings of Sobolev-type. For constant exponents it is well-known that

$$B^{\alpha_0}_{p_0,q} \hookrightarrow B^{\alpha_1}_{p_1,q}$$

if $\alpha_0 - \frac{n}{p_0} = \alpha_1 - \frac{n}{p_1}$, where $0 < p_0 \leq p_1 \leq \infty, 0 < q \leq \infty, -\infty < \alpha_1 \leq \alpha_0 < \infty$ (see e.g. [44], Theorem 2.7.1). For variable case we have the following results, see [1].

Theorem 1.14 (***Sobolev inequality***) *Let* $p_0, p_1, q \in \mathcal{P}_0(\mathbb{R}^n)$ *and* $\alpha_0, \alpha_1 \in L^\infty(\mathbb{R}^n)$ *with* $\alpha_0 \geq \alpha_1$. *If* $\frac{1}{q}$ *and*

$$\alpha_0(x) - \frac{n}{p_0(x)} = \alpha_1(x) - \frac{n}{p_1(x)}$$

are locally log-Hölder continuous, then

$$B^{\alpha_0(\cdot)}_{p_0(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow B^{\alpha_1(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n).$$

Corollary 1.15 *Let* $p_0, p_1, q_0, q_1 \in \mathcal{P}_0(\mathbb{R}^n)$ *and* $\alpha_0, \alpha_1 \in L^\infty(\mathbb{R}^n)$ *with* $\alpha_0 \geq \alpha_1$. *If*

$$\alpha_0(x) - \frac{n}{p_0(x)} = \alpha_1(x) - \frac{n}{p_1(x)} + \varepsilon(x)$$

is locally log-Hölder continuous and $\varepsilon^- > 0$, *then*

$$B^{\alpha_0(\cdot)}_{p_0(\cdot),q_0(\cdot)} \hookrightarrow B^{\alpha_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}.$$

The full treatment of the spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ can be found in [1], [17], [29],[30], [31] and [50]. We refer to the papers [50-51], for further results on the variable Besov spaces $B^{s(\cdot)}_{p(\cdot),q}$ (only the case of constant q was considered), see also [3], [4]).

Chapter 2

Boundedness of pseudo-differential operators in the $L^{p(\cdot)}$ spaces

In this chapter we present the boundedness of a certain class of pseudodifferential operators in variable Lebesgue spaces.

2.1 Definitions and basic properties

We now introduce the basic pseudodifferential symbol class S^m. There are many other more general or modified symbol classes, which are used in the literature and research for different purposes. But the following symbol class is the most simple and most common. It is a natural symbol class that contains the symbols of differential operators with smooth coefficients.

Definition 2.1 *Let $m \in (-\infty, +\infty)$. The symbol S^m is used in place of the set of all functions $a(x,\xi)$ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for any two multi-indices α and β, there is positive constant $C_{\alpha,\beta}$, depending on α and β only, such that*

$$\left| D_x^\alpha D_\xi^\beta a(x,\xi) \right| \le C_{\alpha,\beta} (1+|\xi|)^{m-|\beta|}, \quad x,\xi \in \mathbb{R}^n. \tag{2.1}$$

We call any function a in $\cup_{m\in\mathbb{R}} S^m$ a symbol.

Definition 2.2 *Let a be a symbol. The pseudo-differential operator T_a associated to a is defined by*

$$(T_a\varphi)(x) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{ix\xi} a(x,\xi)\, \hat{\varphi}(\xi)\, d\xi, \qquad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

We give some examples.

Example 2.3 *Let* $P(x,D) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha$ *be a linear partial differential operator on* $\mathbb{R}^n$*. If all the coefficients* $a_\alpha(x)$ *are* C^∞ *and have bounded derivatives of all orders, then the polynomial*

$$P(x,D) = \sum_{|\alpha|\le m} a_\alpha(x)\,\xi^\alpha$$

is in S^m *and hence* $P(x,D)$ *is a pseudo-differential operator, see [55].*

Example 2.4 *Let* $a(\xi) = \left(1+|\xi|^2\right)^{\frac{m}{2}}, -\infty < m < \infty$*. Then* $a \in S^m$ *and hence* T_a *is a pseudo-differential operator, see [55].*

The proof of the following results can be found in [55, Chapters, 4 and 9].

Proposition 2.5 *Let* a *be a symbol. Then* T_a *maps the Schwartz space* $\mathcal{S}$ *into itself.*

Proposition 2.6 T_a *is a linear mapping from* $\mathcal{S}'(\mathbb{R}^n)$ *into* $\mathcal{S}'(\mathbb{R}^n)$*.*

Let us recall the begin definition of the Hardy-Littlewood maximal function, which plays a very important role in harmonic analysis.

Definition 2.7 *Suppose that* f *is a locally integrable on* $\mathbb{R}^n$*, i.e.* $f \in L^1_{\text{loc}}(\mathbb{R}^n)$*. The Hardy-Littlewood maximal operator* $\mathcal{M}$ *is defined on* L^1_{loc} *by*

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,dy,$$

where, the supremum is taken over balls B *in* $\mathbb{R}^n$ *which contain the point* x*.*

The Hardy-Littelwood maximal operator $\mathcal{M}$, in general, is not a bounded from $L^1(\mathbb{R}^n)$ to itself. Take $f(x) = \chi_{[0,1]}(x)$, then for any $x \ge 1$, we have

$$\mathcal{M}f(x) \ge \frac{1}{2x}\int_0^{2x} |f(y)|\,dy = \frac{1}{2x}.$$

Hence

$$\int_{\mathbb{R}^n} \mathcal{M}f(x)dx \ge \int_1^\infty \mathcal{M}f(x)dx \ge \frac{1}{2x}dx = \infty.$$

Although $\mathcal{M}$ is not a bounded operator on $L^1(\mathbb{R}^n)$, however, as its a replacement result we shall see that $\mathcal{M}$ is a bounded operator from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, i.e. the weak $L^1(\mathbb{R}^n)$ space.

Remark 2.8 *The maximum function $\mathcal{M}$ is bounded on $L^{\infty}(\mathbb{R}^n)$,*

$$\|\mathcal{M}f\|_{\infty} \leq \|f\|_{\infty}.$$

Theorem 2.9 *Let f be a measurable function on $\mathbb{R}^n$.*

(a) *If $f \in L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$, then $\mathcal{M}f(x) < \infty$ a.e. $x \in \mathbb{R}^n$.*

(b) *There exists a constant $C = C(n,p) > 0$ such that for any $f \in L^p(\mathbb{R}^n)$,*

$$\|\mathcal{M}f\|_p \leq C \|f\|_p, \quad 1 < p \leq \infty.$$

The proof can be found in [23]. For the variable case we have, see [13], [15], [16].

Theorem 2.10 *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p^- > 1$. Then there exists $K > 0$ only depending on the dimension n and $c_{log}(p)$ such that*

$$\|\mathcal{M}f\|_{p(\cdot)} \leq K \left(p^-\right)' \|f\|_{p(\cdot)}.$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

2.2 Fourier multipliers

The purpose of this subsection is to review some known properties of Fourier multipliers.

Definition 2.11 (A_p *weights* $1 \leq p < \infty$) *Let $\omega > 0$ and $\omega \in L^1_{\text{loc}}(\mathbb{R}^n)$. We say that $\omega \in A_p$ for $1 < p < \infty$ if there is a constant $C > 0$ such that*

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}\,dx\right)^{p-1} \leq C, \tag{2.2}$$

where and below, $1/p + 1/p' = 1$. We say that $\omega \in A_1$ if there is a constant $C > 0$ such that

$$\mathcal{M}\omega(x) \leq C\omega(x) \quad \textit{for almost all } x \in \mathbb{R}^n. \tag{2.3}$$

The smallest constant C for which (2.2) or (2.3) hold, denoted by A_p. As an example, we can take

$$\gamma(x) = |x|^{\alpha}, \quad \alpha \in \mathbb{R}.$$

Then $\gamma \in A_p$, $1 < p < \infty$, if and only if $-n < \alpha < n(p-1)$.

Clearly, $\omega \in A_1$ if and only if there is a constant $C > 0$ such that for any cube Q

$$\frac{1}{|Q|}\int \omega\left(x\right)dx \leq C \inf_{x\in Q}\omega\left(x\right).$$

Some properties of A_p weights can be found in [23].

Let m be a bounded function on $\mathbb{R}^n$ and consider the multiplier operator Tf defined initially for functions f in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}(Tf)\left(x\right) = m\left(x\right)\mathcal{F}f\left(x\right).$$

Denote by s a real number greater than or equal to 1 and l a positive integer. We say $m \in M\left(s,l\right)$ if

$$\sup_{R>0}\left(R^{s|\alpha|-n}\int_{R<|\alpha|<2R}\left|D^{\alpha}m\left(x\right)\right|^{s}dx\right)^{\frac{1}{s}} < \infty \qquad \text{for all } |\alpha| \leq l.$$

Definition 2.12 *Let $1 < p < \infty$. Given a measurable set E and a weight γ, we denote the space of all functions $f : \mathbb{R}^n \to \mathbb{C}$ with finite quasi-norm*

$$\|f\|_{L^p(\mathbb{R}^n,\omega)} = \left\|fw^{\frac{1}{p}}|L^p(\mathbb{R}^n)\right\|,$$

by $L^p(\mathbb{R}^n,\omega)$.

We need a weighted version of Hörmender's multiplier theorem, see [32].

Theorem 2.13 *Let $k > \left[\frac{n}{2}\right]$ and $m \in C^k\left(\mathbb{R}^n - \{0\}\right)$. Suppose that $m \in M\left(2,k\right)$,*

$$\sup r^{2|\alpha|-n}\int_{r\leq|x|\leq 2r}\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}m\left(x\right)\right|^2 dx \leq B, \qquad \text{for } |\alpha| \leq k.$$

When $k < n$ and $\frac{n}{k} < p < \infty$, m is a bounded multiplier from $L^p(\mathbb{R}^n,\omega)$ to $L^p(\mathbb{R}^n,\omega)$ if $\omega \in A_{\frac{pn}{k}}$. Finally, if $k > n$, m is a bounded multiplier from $L^p(\mathbb{R}^n,\omega)$ to $L^p(\mathbb{R}^n,\omega)$, $1 < p < \infty$, the norm of the operator depends only on B, p, w, k and n.

Let $\mathcal{B}\left(\mathbb{R}^n\right)$ be the set of $p\left(\cdot\right) \in \mathcal{P}\left(\mathbb{R}^n\right)$ such that $\mathcal{M}$ is bounded on $L^{p(\cdot)}\left(\mathbb{R}^n\right)$. Here F will denote a family of ordered pairs of non-negative, measurable functions (f, g).

Theorem 2.14 *Given a family F. Suppose that for some p_0, $0 < p_0 < \infty$, and for every weight $w \in A_1$*

$$\int_{\mathbb{R}^n} f\left(x\right)^{p_0} w\left(x\right)dx \leq C_0\int_{\mathbb{R}^n} g\left(x\right)^{p_0} w\left(x\right)dx, \qquad (f,g) \in F. \tag{2.4}$$

where C_0 depends only on p_0 and the A_1 constant of w. Let $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ be such that $p_0 < p_-$, and $(p(\cdot)/p_0)' \in \mathcal{B}(\mathbb{R}^n)$, that is the Hardy-Littlewood maximal operator is bounded on $L^{(p(\cdot)/p_0)'}$. Then for all $(f,g) \in F$ such that $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\|f\|_{p(\cdot)} \leq C \|g\|_{p(\cdot)}, \tag{2.5}$$

where the constant C is independent of the pair (f,g).

Proof. See [9]. ■

Observe that if $\mathcal{P}^{\log}(\mathbb{R}^n)$ with $p^- > 1$, then $(p(\cdot)/p_0)' \in \mathcal{B}(\mathbb{R}^n)$ for any $p_0 < p_-$.

2.3 $L^{p(\cdot)}$ boundedness

Let us present some results which are useful for us. The following lemmas are from [55] and [22]. Set

$$|a|_{\alpha,\beta}^{(m)} = \sup_{x,\xi} \langle\xi\rangle^{|\beta|-m} |D_x^\alpha D_\xi^\beta a(x,\xi)|$$

and

$$\langle\xi\rangle = 1 + |\xi|, \quad \xi \in \mathbb{R}^n.$$

Lemma 2.15 *Let Q_0 be the cube with center at the origin and edges of length 1 parallel to the coordinates axes in $\mathbb{R}^n$. Let $\eta \in \mathcal{D}(\mathbb{R}^n)$ be identically 1 on Q_0. Let $a \in S^0$, $a_m(x,\xi) = \eta(x-m)a(x,\xi)$ for $m \in \mathbb{Z}^n$ and*

$$\hat{a}_m(\lambda,\xi) = \int_{\mathbb{R}^n} e^{-i\lambda\varepsilon} a_m(x,\xi)dx.$$

Then, for all $\alpha \in \mathbb{N}^n$ and all $N \in \mathbb{N}$ there exists $C > 0$ depending only on n, η and N such that

$$\left|D_\xi^\alpha \hat{a}_m(\lambda,\xi)\right| \leq C \sup_{|\beta|\leq N} |a|_{\alpha,\beta}^{(0)} \langle\xi\rangle^{-|\alpha|} \langle\lambda\rangle^{-N}$$

for all $(\lambda,\xi) \in \mathbb{R}^{2n}$.

Lemma 2.16 *Let $a \in S^0$ and K_a be the distribution $\mathcal{F}_{\xi\to z}^{-1}(a(x,\xi))$ in $S(\mathbb{R}^{2n})$. Then*

1. *for each $x \in \mathbb{R}^n$, $K_a(x,.)$ is a function defined on $\mathbb{R}^n\backslash\{0\}$,*

2. *for each N sufficiently large there exists a constant c, depending only on N and n such that*

$$\left|K_a\left(x,z\right)\right| \leq c \sup_{|\alpha|\leq N} |a|^{(0)}_{\alpha,0} \left|z\right|^{-N}$$

for all $z \neq 0$,

3. *for each $x \in \mathbb{R}^n$ and $\varphi \in \mathcal{S}\left(\mathbb{R}^n\right)$, vanishing in a neighborhood of x*

$$a(x,D)\varphi\left(x\right) = \int_{\mathbb{R}^n} K_a\left(x,x-z\right)\varphi\left(z\right)dz.$$

To prove the main result of this chapter we need the weight version of [55, Theorem 9.7] and [22].

Theorem 2.17 *Let $a \in S^0$ and $p_0 \in \mathcal{P}^{\log}$. Then, there exists $N \in \mathbb{N}$ and a constant C depending only on n, N and p such that*

$$\|a(x,D)\varphi\|_{L^{p_0}(\mathbb{R}^n,\omega)} \leq C \sup_{|\alpha+\beta|\leq N} |a|^{(0)}_{\alpha,\beta} \|\varphi\|_{L^{p_0}(\mathbb{R}^n,\omega)}$$

for all $\varphi \in \mathcal{S}\left(\mathbb{R}^n\right)$ and any $\omega \in A_1$.

Proof. We write $\mathbb{R}^n$ as a union of cube Q_m, where Q_m is the cube with center $m \in \mathbb{Z}^m$ and edge of length 1. Let $Q^*_m = \frac{3}{2}Q_m$ and $Q^{**}_m = 2Q_m$. It follows that

$$Q_m \subset Q^*_m \subset Q^{**}_m$$

and that for some $\delta > 0$ one has $|x-z| \geq \delta$, for all $x \in Q_m$ and $z \in \mathbb{R}^n \backslash Q^*_m$.

Let now $\psi \in \mathcal{D}\left(\mathbb{R}^n\right)$ be such that $0 \leq \psi \leq 1$, $\operatorname{supp}\psi \subseteq Q^{**}_0$ and $\psi\left(x\right) = 1$ on a neighborhood of Q^*_0. It follows that $\psi_m\left(x\right) = \psi\left(x-m\right)$ has support contained in Q^{**}_m and $\psi_m\left(x\right) = 1$ on a neighborhood of Q^*_m. For each $\varphi \in \mathcal{S}\left(\mathbb{R}^n\right)$ we can write

$$\varphi = \varphi_{1,m} + \varphi_{2,m},$$

where $\varphi_{1,m} = \psi_m\varphi$ and $\varphi_{2,m} = \left(1-\psi_m\right)\varphi$. Therefore,

$$a(x,D)\varphi = a(x,D)\varphi_{1,m} + a(x,D)\varphi_{2,m}.$$

It is clear that

$$\begin{aligned} \|a(x,D)\varphi\|^{p_0}_{L^{p_0}(w)} &= \sum_{m\in\mathbb{Z}^m} \int_{Q_m} \left|a(x,D)\varphi\left(x\right)\right|^{p_0} w\left(x\right)dx \qquad (2.6) \\ &\leq \sum_{m\in\mathbb{Z}^m} \int_{Q_m} \left|a(x,D)\varphi_{1,m}\left(x\right)\right|^{p_0} w\left(x\right)dx \\ &\quad + \sum_{m\in\mathbb{Z}^m} \int_{Q_m} \left|a(x,D)\varphi_{2,m}\left(x\right)\right|^{p_0} w\left(x\right)dx. \end{aligned}$$

We will present the proof on three steps:

Step 1. Let us estimate

$$\int_{Q_m} \left|a(x,D)\varphi_{1,m}(x)\right|^{p_0} w(x)\, dx.$$

Let $\eta \in \mathcal{D}(\mathbb{R}^n)$ be identically 1 on Q_0 and $a_m(x,\xi) = \eta(x-m)\, a(x,\xi)$. Hence,

$$\int_{Q_m} \left|a(x,D)\varphi_{1,m}(x)\right|^{p_0} w(x)\, dx \leq \int_{\mathbb{R}^n} \left|a(x,D)\varphi_{1,m}(x)\right|^{p_0} w(x)\, dx \tag{2.7}$$

Since a_m is compactly supported in x we get

$$a(x,D)\varphi_{1,m}(x) = \int_{\mathbb{R}^n} e^{ix\lambda} \int_{\mathbb{R}^n} e^{ix\xi}\hat{a}_m(\lambda,D)\hat{\varphi}_{1,m}(\xi)\, d\xi d\lambda = \int_{\mathbb{R}^n} e^{ix\lambda}\hat{a}_m(\lambda,D)\left(\varphi_{1,m}\right)(x)\, d\lambda.$$

From Lemma 2.15 we have that for all $N \in \mathbb{N}$

$$\left|D_\xi^\alpha \hat{a}_m(\lambda,\xi)\right| \leq C \sup_{|\beta|\leq N} |a|_{\alpha,\beta}^{(0)} \langle\xi\rangle^{-|\alpha|} \langle\lambda\rangle^{-N},$$

where C depends only on n, η and N. We apply Theorem 2.13 to $f(\xi) = \hat{a}_m(\lambda,\xi)$ with

$$B = C \sup_{|\beta|\leq N, |\alpha|\leq[n/2]+1} |a|_{\alpha,\beta}^{(0)} \langle\lambda\rangle^{-N}$$

and obtain that there exists a constant $\acute{C}$, depending on N, n, η and p such that

$$\left\|\hat{a}_m(\lambda,D)\varphi_{1,m}\right\|_{L^{p_0}(w)} \leq \acute{C} \sup_{|\beta|\leq N, |\alpha|\leq[n/2]+1} |a_m|_{\alpha,\beta}^{(0)} \langle\lambda\rangle^{-N} \left\|\varphi_{1,m}\right\|_{L^{p_0}(w)} \tag{2.8}$$

for all $\lambda \in \mathbb{R}^n$ for all $m \in \mathbb{Z}^n$ and $\varphi_{1,m} \in \mathcal{S}(\mathbb{R}^n)$. An application of the Minkowski's inequality in integral from leads from (2.8) to

$$\begin{aligned}
&\left\|a_m(\lambda,D)\varphi_{1,m}\right\|_{L^{p_0}(w)} \\
= &\left\{\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} e^{ix\lambda}\hat{a}_m(\lambda,D)\left(\varphi_{1,m}\right)(x)\, d\lambda\right|^{p_0} w(x)\, dx\right\}^{\frac{1}{p_0}} \\
\leq &\int_{\mathbb{R}^n} \left\{\int_{\mathbb{R}^n} \left|\hat{a}_m(\lambda,D)\left(\varphi_{1,m}\right)(x)\right|^{p_0} w(x)\, dx\right\}^{\frac{1}{p_0}} d\lambda \\
\leq &\int_{\mathbb{R}^n} \left\|\hat{a}_m(\lambda,D)\varphi_{1,m}\right\|_{L^{p_0}(w)} d\lambda \\
\leq &\acute{C} \sup_{|\beta|\leq N, |\alpha|\leq[n/2]+1} |a_m|_{\alpha,\beta}^{(0)} \int_{\mathbb{R}^n} \langle\lambda\rangle^{-N} d\lambda \left\|\varphi_{1,m}\right\|_{L^{p_0}(w)}
\end{aligned}$$

Thus, choosing $N = n+1$ we get

$$\left\|a_m(\lambda,D)\varphi_{1,m}\right\|_{L^{p_0}(w)} \leq \acute{C} \sup_{|\beta|\leq n+1, |\alpha|\leq[n/2]+1} |a_m|_{\alpha,\beta}^{(0)} \left\|\varphi_{1,m}\right\|_{L^{p_0}(w)}, \tag{2.9}$$

valid for all $m \in \mathbb{Z}^m$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Going back to

$$\int_{Q_m} \left|a(x,D)\varphi_{1,m}(x)\right|^{p_0} dx,$$

the estimate (2.9) combined with (2.7) yields:

$$\begin{aligned}\int_{Q_m} \left|a(x,D)\varphi_{1,m}(x)\right|^{p_0} w(x)\,dx &\leq \left\|a_m(\lambda,D)\varphi_{1,m}\right\|_{L^{p_0}(w)}^{p_0} \\ &\leq C_p\Big(\sup_{|\beta|\leq n+1, |\alpha|\leq [n/2]+1} |a_m|_{\alpha,\beta}^{(0)}\Big)^{p_0} \left\|\varphi_{1,m}\right\|_{L^{p_0}(w)}^{p_0},\end{aligned}$$

where C_p does not depend on m.

Step 2. We now estimate

$$\left(\int_{Q_m} \left|a(x,D)\varphi_{2,m}(x)\right|^{p_0} w(x)\,dx\right)^{\frac{1}{p_0}}.$$

Since $\varphi_{2,m}$ is identically 0 on $Q_m \subset Q_m^*$ from Lemma 2.16 we have

$$\begin{aligned}&\left(\int_{Q_m} \left|a(x,D)\varphi_{2,m}(x)\right|^{p_0} w(x)\,dx\right)^{\frac{1}{p_0}} \\ =\ &\left(\int_{Q_m} \left|\int_{\mathbb{R}^n} K_a(x,x-z)\varphi_{2,m}(z)\,dz\right|^{p_0} w(x)\,dx\right)^{\frac{1}{p_0}} \\ \leq\ &\int_{\mathbb{R}^n}\left(\int_{Q_m} \left|K_a(x,x-z)\varphi_{2,m}(z)\right|^{p_0} w(x)\,dx\right)^{\frac{1}{p_0}} dz \\ \leq\ &C_{\lambda,N}\sup_{|\alpha|\leq 2N} |a|_{\alpha,0}^{(0)} \int_{\mathbb{R}^n\backslash Q_m^*}\left(\int_{Q_m} |x-z|^{-2Np_0}\left|\varphi_{2,m}(z)\right|^{p_0} w(x)\,dx\right)^{\frac{1}{p_0}} dz,\end{aligned}$$

valid for $2N > n$. Let us fix $\lambda \geq \sqrt{n}+1$. Since $|x-z| \geq \delta$ for all $x \in Q_m$ and all $z \in \mathbb{R}^n\backslash Q_m^*$, there exists a constant $C_{\lambda,N}$ such that

$$\frac{|x-z|^{-2N}}{(\lambda+|x-z|)^{-2N}} \leq C_{\lambda,N} \tag{2.10}$$

and

$$\lambda+|x-z| \geq \lambda+|m-z|-|x-m| \geq \left(\lambda-\frac{\sqrt{n}}{2}\right)+|m-z| \geq \frac{\sqrt{n}}{2}+1+|m-z| = \mu+|m-z|. \tag{2.11}$$

for all $x \in Q_m$ and all $z \in \mathbb{R}^n\backslash Q_m^*$. By (2.10) and (2.11) we get

$$\left(\int_{Q_m} \left|a(x,D)\varphi_{2,m}(x)\right|^{p_0} w(x)\,dx\right)^{\frac{1}{p_0}}$$

can be estimated by

$$
\begin{aligned}
& c \sup_{|\alpha|\leq 2N} |a|^{(0)}_{\alpha,0} \int_{\mathbb{R}^n\setminus Q_m^*} \left(\int_{Q_m} (\lambda+|x-z|)^{-2Np_0} \left|\varphi_{2,m}(z)\right|^{p_0} w(x)\,dx \right)^{\frac{1}{p_0}} dz \\
\leq\ & c \sup_{|\alpha|\leq 2N} |a|^{(0)}_{\alpha,0} \int_{\mathbb{R}^n\setminus Q_m^*} \left(\int_{Q_m} \frac{(\mu+|x-z|)^{-Np_0}}{(\mu+|m-z|)^{Np_0}} \left|\varphi_{2,m}(z)\right|^{p_0} w(x)\,dx \right)^{\frac{1}{p_0}} dz \\
\leq\ & c \sup_{|\alpha|\leq 2N} |a|^{(0)}_{\alpha,0} \int_{\mathbb{R}^n\setminus Q_m^*} \frac{\left|\varphi_{2,m}(z)\right|}{(\mu+|m-z|)^{N}} \left\{ \int_{Q_m} (\mu+|x-z|)^{-Np_0} w(x)\,dx \right\}^{\frac{1}{p_0}} dz.
\end{aligned}
$$

We prove that

$$
\int_{Q_m} (\mu+|x-z|)^{-Np_0} w(x)\,dx < c\, w(z), \quad z\in\mathbb{R}^n\setminus Q_m^*.
$$

We have

$$
\begin{aligned}
& \int_{Q_m\cap\{|x-z|>\delta\}} (\mu+|x-z|)^{-Np_0} w(x)\,dx \\
\leq\ & \int_{|x-z|>\delta} (\mu+|x-z|)^{-Np_0} w(x)\,dx \\
=\ & \sum_{j=0}^{\infty} \int_{2^j\delta<|x-z|\leq 2^{j+1}\delta} (\mu+|x-z|)^{-Np_0} w(x)\,dx \\
\leq\ & \sum_{j=0}^{\infty} \int_{2^j\delta<|x-z|\leq 2^{j+1}\delta} |x-z|^{-Np_0} w(x)\,dx.
\end{aligned}
$$

Observe that $|x-z|^{-Np_0} \leq (2^j\delta)^{-Np_0}$, then the last term is bounded by

$$
\begin{aligned}
& \sum_{j=0}^{\infty} \left(2^j\delta\right)^{-Np_0} \int_{|x-z|\leq 2^{j+1}\delta} w(x)\,dx, \\
=\ & c\sum_{j=0}^{\infty} \left(2^j\delta\right)^{n-Np_0} \frac{1}{|B(z,2^{j+1}\delta)|} \int_{B(z,2^{j+1}\delta)} w(x)\,dx \\
\leq\ & w(z) \sum_{j=0}^{\infty} \left(2^j\delta\right)^{n-Np_0}, \quad z\in\mathbb{R}^n\setminus Q_m^*,
\end{aligned}
$$

since $w\in A_1$. Assume that $N>\frac{n}{p_0}$, then

$$
\left(\int_{Q_m} (\mu+|x-z|)^{-Np_0} w(x)\,dx \right)^{\frac{1}{p_0}} < cw(z)^{\frac{1}{p_0}}
$$

valid for all $z\in\mathbb{R}^n\setminus Q_m^*$. Hence

$$
\int_{Q_m} \left|a(x,D)\varphi_{2,m}(x)\right|^{p_0} w(x)\,dx \leq C_{\lambda,N,p_0} \big(\sup_{|\alpha|\leq 2N} |a|^{(0)}_{\alpha,0} \big)^{p_0} \int_{\mathbb{R}^n\setminus Q_m^*} \frac{\left|\varphi_{2,m}(z)\right|^{p_0}}{(\mu+|m-z|)^{Np_0/2}} w(z)\,dz \tag{2.12}
$$

for all $l \in \mathbb{Z}^n$.

Step3. A combination of (2.6) with (2.9) and (2.12) yields

$$\begin{aligned}\|a(x,D)\varphi\|_{L^{p_0}(w)} &\leq C\sum_{m\in\mathbb{Z}^n}\sup_{\substack{|\beta|\leq n+1\\|\alpha|\leq[n/2]+1}}|a_m|^{(0)}_{\alpha,\beta}\left\|\varphi_{1,m}\right\|_{L^{p_0}(w)}\\&\quad+C\sup_{|\alpha|\leq 2N}|a|^{(0)}_{\alpha,0}\sum_{m\in\mathbb{Z}^n}\sum_{l\neq m}\int_{Q_l}\frac{\left|\varphi_{2,m}(z)\right|^{p_0}}{(\mu+|m-z|)^{Np_0/2}}w(z)\,dz,\end{aligned}$$

with $\lambda\geq\sqrt{n}+1$ and $Np_0>2n(p_0-1)$. From the definition of a_m and $\varphi_{1,m}$ and $\varphi_{2,m}$ we get

$$\begin{aligned}\|a(x,D)\varphi\|_{L^{p_0}(w)} &\leq C\sum_{m\in\mathbb{Z}^n}\sup_{\substack{|\beta|\leq n+1\\|\alpha|\leq[n/2]+1}}|a_m|^{(0)}_{\alpha,\beta}\left\|\varphi_{1,m}\right\|_{L^{p_0}(w)}\\&\quad+C\sup_{|\alpha|\leq 2N}|a|^{(0)}_{\alpha,0}\sum_{m\in\mathbb{Z}^n}\sum_{l\neq m}\left(\int_{Q_l}\frac{\left|\varphi_{2,m}(z)\right|^{p_0}}{(\mu+|m-z|)^{Np_0/2}}w(z)\,dz\right)^{\frac{1}{p_0}}.\end{aligned}$$

Arguing as in (2.11) we have that $\mu+|m-z|\geq 1+|m-l|$ when with $z\in Q_l$ with $l\neq m$. Hence

$$\begin{aligned}&\sum_{m\in\mathbb{Z}^n}\sum_{l\neq m}\int_{Q_l}\frac{\left|\varphi_{2,m}(z)\right|^{p_0}}{(\mu+|m-z|)^{Np_0/2}}w(z)\,dz\\&\leq\sum_{m\in\mathbb{Z}^n}\sum_{l\neq m}(1+|m-l|)^{-Np_0/2}\int_{Q_l}\left|\varphi_{2,m}(z)\right|^{p_0}w(z)\,dz\\&\leq\sum_{m\in\mathbb{Z}^n}\sum_{l\in\mathbb{Z}^n}(1+|m-l|)^{-Np_0/2}\int_{Q_l}\left|\varphi_{2,m}(z)\right|^{p_0}w(z)\,dz\\&\leq\sum_{m\in\mathbb{Z}^n}(1+|m|)^{-Np_0/2}\sum_{l\in\mathbb{Z}^n}\int_{Q_l}\left|\varphi_{2,m}(z)\right|^{p_0}w(z)\,dz.\end{aligned}$$

Choosing $Np_0\geq\max(2n(p_0-1),2n)$ and we obtain

$$\|a(x,D)\varphi\|_{L^{p_0}(w)}\leq C\Big(\sup_{\substack{|\beta|\leq n+1\\|\alpha|\leq[n/2]+1}}|a_m|^{(0)}_{\alpha,\beta}+\sup_{|\alpha|\leq 2N}|a|^{(0)}_{\alpha,0}\Big)\|\varphi\|_{L^{p_0}(w)}$$

valid for all $\varphi\in\mathcal{S}(\mathbb{R}^n)$. This completes the proof. ■

From this theorem and Theorem 2.14 we obtain.

Theorem 2.18 *Let $a\in S^0$ and $p\in\mathcal{P}^{\log}$. Then, there exists $N\in\mathbb{N}$ and a constant C such that*

$$\|a(x,D)\varphi\|_{p(\cdot)}\leq C\sup_{|\alpha+\beta|\leq N}|a|^{(0)}_{\alpha,\beta}\|\varphi\|_{p(\cdot)}$$

for all $\varphi\in\mathcal{S}(\mathbb{R}^n)$.

We would like to mention that this theorem was proved in [38] but here we present a another method.

Chapter 3

Boundedness of pseudo-differential operators on variable Besov spaces

In this chapter concerns the boundedness properties of the pseudodifferential operators on Besov spaces with variable smoothness and integrability with symbols in $SB_{\delta}^{m}\left(r,\mu,v;N,\lambda\right)$.

The chapter is arranged as follows. In section 1 we give some key technical lemmas needed in the proofs of the main statements, where we recall the definition of the Besov spaces with variable smoothness and integrability. For making the presentation clearer, we give the proof of the main result of this section. With the help of the results of Section 1, we prove the boundedness of non-regular pseudodifferential operators in the space $B_{p(\cdot),q(\cdot)}^{s(\cdot)}$.

Let $\left\{\mathcal{F}\varphi_j\right\}_j$ be a resolution of unity. For a function $a:\mathbb{R}^n\times\mathbb{R}^n\rightarrow\mathbb{C}$, we write

$$a_j\left(x,\xi\right)=\mathcal{F}_{y\rightarrow x}^{-1}\left(\mathcal{F}\varphi_j\left(y\right)\mathcal{F}a\left(y,\xi\right)\right).$$

Let $0<\mu\leq\infty$, $1\leq\lambda\leq\infty$, $r\geq\frac{n}{\mu}$ and $N>\frac{n}{\lambda}$. The space $B_{\mu,v}^{r}(B_{\lambda,\infty}^{N})$ consists of all distributions $a\in\mathcal{S}'(\mathbb{R}^n\times\mathbb{R}^n)$ such that

$$\left\|a\right\|_{B_{\mu,v}^{r}(B_{\lambda,\infty}^{N})}=\left\|\left\{2^{jr}\left\|a_j\left(x,\cdot\right)\right\|_{B_{\lambda,\infty}^{N}}\right\}_j\right\|_{\ell^{v}(L^{\mu})}<\infty.$$

Notice that these spaces are just the spaces $SB_{\bar{p},\bar{q}}^{\bar{r}}$ with $\bar{r}=(N,r)$, $\bar{p}=(\lambda,\mu)$ and $\bar{q}=(\infty,v)$, see [42] for further properties of these function spaces. Let $m,r,N\in\mathbb{R}$, $0\leq\delta\leq1$, $0<\mu\leq\infty$, $r>\frac{n}{\mu}$ and $N>\frac{n}{\lambda}$. We say that a symbol a belongs to $SB_{\delta}^{m}\left(r,\mu,v;N,\lambda\right)$ if

$$\begin{aligned}\sup_k 2^{-km}\left\|\left\|a\left(x,2^k\cdot\right)\mathcal{F}\varphi_k\left(2^k\cdot\right)\right\|_{B_{\lambda,\infty}^{N}}\right\|_{L^\infty(dx)} &< \infty\\ \sup_k 2^{-k(m+\delta r)}\left\|a\left(x,2^k\cdot\right)\mathcal{F}\varphi_k\left(2^k\cdot\right)\right\|_{B_{\mu,v}^{r}(B_{\lambda,\infty}^{N})} &< \infty,\end{aligned}$$

which, introduced by J. Marschall [34] and [35]. Choosing $\mu = v = N = \lambda = \infty$, these symbols include the classical Hörmander classes $S^m_{1,\delta}$. Moreover the class $SB^m_0(r, \mu, v; \infty, 1)$ equal the class $S'(B^{(1,\ldots,1),r}_{\mu,v})^m$ of M. Yamazaki [53]. Notice that

$$SB^m_\delta(r, \mu, v; N, \lambda) \hookrightarrow SB^m_{\delta_1}(r_1, \mu_1, v; N, \lambda), \tag{3.1}$$

if $0 < \mu < \mu_1 \leq \infty$, $0 < v \leq \infty$, $r - \frac{n}{\mu} = r_1 - \frac{n}{\mu_1}$ and $\delta r = \delta_1 r_1$, see [35, Lemma 10].

A pseudo-differential operator with symbol $a \in SB^m_\delta(r, \mu; N, \lambda)$ is defined by

$$a(x, D) f(x) = \frac{1}{(2\pi)^n} \int e^{ix\xi} a(x, \xi) \mathcal{F}f(\xi) d\xi,$$

where $f \in \mathcal{S}(\mathbb{R}^n)$. Besov estimates, with fixed exponents, for such operators were given by J. Marschall [35]

3.1 Auxiliary results

In this section we present some results which are useful for us. The following lemma is from [31, Lemma 19], see also [14, Lemma 6.1].

Lemma 3.1 *Let $\alpha \in C^{\log}_{\mathrm{loc}}$ and let $R \geq c_{\log}(\alpha)$, where $c_{\log}(\alpha)$ is the constant from (1.2) for α. Then*

$$2^{v\alpha(x)} \eta_{v,m+R}(x - y) \leq c\, 2^{v\alpha(y)} \eta_{v,m}(x - y)$$

with $c > 0$ independent of $x, y \in \mathbb{R}^n$ and $v, m \in \mathbb{N}_0$.

The next lemma often allows us to deal with exponents which are smaller than 1.

Lemma 3.2 *Let $r > 0$, $v \in \mathbb{N}_0$ and $m > n$. Then there exists $c = c(r, m, n) > 0$ such that for all $g \in \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{supp}\mathcal{F}g \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2^{v+1}\}$, we have*

$$|g(x)| \leq c(\eta_{v,m} * |g|^r(x))^{1/r}, \quad x \in \mathbb{R}^n.$$

The following lemma is from A. Almeida and P. Hästö [1, Lemma 4.7] (we use it, since the maximal operator is in general not bounded on $\ell^{q(\cdot)}(L^{p(\cdot)})$, see [1, Example 4.1]).

Lemma 3.3 *Let $p \in \mathcal{P}^{\log}$, $q \in \mathcal{P}^{\log}_0$ with $0 < q^- \leq q^+ < \infty$ and $p^- > 1$. For $m > n + c_{\log}(1/q)$, there exists $c > 0$ such that*

$$\left\| \{\eta_{v,m} * f_v\}_v \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq c \left\| \{f_v\}_v \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

We will also make use of the following statement were proved by Franke [21, Theorem 2.4.1] in the case of constant p, see [18] for variable case.

Lemma 3.4 *Let $p \in \mathcal{P}_0^{\log}$, $l, k \in \mathbb{N}_0$ with $k \leq l$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then for all $\{f_l\}_{l \in \mathbb{N}_0} \subset \mathcal{S}'(\mathbb{R}^n) \cap L^{p(\cdot)}$ with $\operatorname{supp} \mathcal{F} f_l \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2^l\}$, we have*

$$\|\varphi_k * f_l\|_{p(\cdot)} \leq c\, 2^{(k-l)n\left(1-1/\min(1,p^-)\right)} \|f_l\|_{p(\cdot)},$$

where $\varphi_k = 2^{kn}\varphi(2^k \cdot)$ and $c > 0$ is independent of k and l.

The next lemma is a Hardy-type inequality which is easy to prove.

Lemma 3.5 *Let $0 < a < 1$ and $0 < q \leq \infty$. Let $\{\varepsilon_k\}_{k \in \mathbb{N}_0}$ be a sequence of positive real numbers, such that $\|(\varepsilon_k)_k\|_{\ell^q} = I < \infty$. The sequence*

$$\{\delta_k : \delta_k = \sum_{j=0}^{\infty} a^{|k-j|} \varepsilon_j\}_{k \in \mathbb{N}_0}$$

is in ℓ^q with

$$\|\{\delta_k\}_k\|_{\ell^q} \leq c\, I.$$

c depends only on a and q.

The following proposition plays a fundamental role in this section.

Proposition 3.6 *Let $s \in C_{\mathrm{loc}}^{\log}$, p_1, p_2, $q \in \mathcal{P}_0^{\log}$, $0 < \mu \leq \infty$ and $1 \leq \lambda \leq \infty$ with $\frac{1}{p_1(\cdot)} = \frac{1}{p_2(\cdot)} + \frac{1}{\mu}$. Let $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a bounded and measurable symbol such that*

$$\operatorname{supp} a(x, \cdot) \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq c 2^k\}.$$

If $p_1^- \geq 1$ or if $0 < p_1^- < 1$ and

$$\operatorname{supp} \mathcal{F} f \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq c 2^k\},$$

and if $N > n \max\left\{\frac{1}{2}, \frac{1}{\lambda}, \frac{1}{p_2^-}\right\} + c_{\log}(s) + c_{\log}(\frac{1}{q})$, then

$$\left\|2^{ks(\cdot)} \delta^{-\frac{1}{q(\cdot)}} a(x, D) f\right\|_{p_1(\cdot)} \lesssim \left\| \|a(\cdot, 2^k \cdot)\|_{B_{\lambda,\infty}^N} \right\|_{\mu} \left\|2^{ks(\cdot)} \delta^{-\frac{1}{q(\cdot)}} f\right\|_{p_2(\cdot)}$$

for any $k \in \mathbb{N}_0$ and any $\delta \in \left[2^{-k}, 1 + 2^{-k}\right]$, with the implicit constant not depending on k.

Proof. The proof follows the ideas in [35, Proposition 4], see also [33, Lemma 3]. We will do the proof in two steps.

Step 1. Let us begin by the case $p_1^- \geq 1$. Let

$$K(x, x-y) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, \xi)\, d\xi = \mathcal{F}_\xi^{-1} a(x, \cdot)(x-y),$$

be the kernel of $a(x, D)$. Set $\frac{1}{\tau} = \max\left\{\frac{1}{2}, \frac{1}{\lambda}, \frac{1}{p_2^-}\right\}$. It follows from the Hölder inequality

$$\begin{aligned}
&\left|2^{ks(x)} \delta^{-\frac{1}{q(x)}} a(x, D) f(x)\right| \\
= &\left|\int 2^{ks(x)} \delta^{-\frac{1}{q(x)}} K(x, x-y) f(y)\, dy\right| \\
\leq &\sum_{v=-\infty}^{\infty} \int \left|2^{ks(x)} \delta^{-\frac{1}{q(x)}} K(x, x-y) \mathcal{F}\varphi_v(y-x) f(y)\right| dy \\
\leq &\sum_{v=-\infty}^{\infty} \left(\int |K(x, x-y) \mathcal{F}\varphi_v(y-x)|^{\tau'} dy\right)^{\frac{1}{\tau'}} \left(\int_{|x-y| \leq 2^{v+1}} \left|2^{ks(x)} \delta^{-\frac{1}{q(x)}} f(y)\right|^{\tau} dy\right)^{\frac{1}{\tau}}.
\end{aligned}$$

Observe that

$$\begin{aligned}
K(x, x-y) \mathcal{F}\varphi_v(y-x) &= \mathcal{F}_\xi^{-1}\left(\mathcal{F}_\xi\left(K(x, \cdot) \mathcal{F}\varphi_v(-\cdot)\right)\right)(x-y) \\
&= \mathcal{F}_\xi^{-1}\left(\mathcal{F}_\xi^{-1}\left(\mathcal{F}\varphi_v \mathcal{F}_\xi a(x, \cdot)\right)\right)(x-y).
\end{aligned}$$

Let us recall the Hausdorff-Young inequality. If $f \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq 2$ then

$$\|\mathcal{F}f\|_{p'} \leq c\, \|f\|_p\,.$$

Hence by this inequality, since $1 \leq \tau \leq 2$,

$$\begin{aligned}
&\left|2^{ks(x)} \delta^{-\frac{1}{q(x)}} a(x, D) f(x)\right| \\
\leq\ & c \sum_{v=-\infty}^{\infty} \left\|\mathcal{F}_\xi^{-1}\left(\mathcal{F}\varphi_v \mathcal{F}_\xi a(x, \cdot)\right)\right\|_\tau \left(\int_{|x-y| \leq 2^{v+1}} \left|2^{ks(x)} \delta^{-\frac{1}{q(x)}} f(y)\right|^\tau dy\right)^{\frac{1}{\tau}}.
\end{aligned}$$

Since s and $\frac{1}{q}$ are log-Hölder continuous and $\delta \in \left[2^{-k}, 1+2^{-k}\right]$ we get

$$2^{ks(x)} \leq 2^{-kn} \eta_{k, c_{\log}(s)}(x-y) 2^{ks(y)} \leq \left(1 + 2^{k+v+1}\right)^{c_{\log}(s)} 2^{ks(y)}$$

and

$$\delta^{-\frac{1}{q(x)}} \leq 2^{-kn} \eta_{k, c_{\log}\left(\frac{1}{q}\right)}(x-y) \delta^{-\frac{1}{q(y)}} \leq \left(1 + 2^{k+v+1}\right)^{c_{\log}\left(\frac{1}{q}\right)} \delta^{-\frac{1}{q(y)}}$$

for any $x, y \in \mathbb{R}^n$ such that $|x-y| \leq 2^{v+1}$. Therefore,

$$\left|2^{ks(x)} \delta^{-\frac{1}{q(x)}} a(x, D) f(x)\right|$$

is bounded by

$$c\sum_{v=-k}^{\infty}2^{(v+k)(\frac{n}{\tau}+c_{\log}(s)+c_{\log}(\frac{1}{q}))}H_{v,k}(x)+c\sum_{v=-\infty}^{-k-1}2^{(v+k)\frac{n}{\tau}}H_{v,k}(x),$$

where

$$H_{v,k}(x)=\left\|\mathcal{F}_{\xi}^{-1}\left(\mathcal{F}\varphi_{v+k}\mathcal{F}_{\xi}a\left(x,2^{k}\cdot\right)\right)\right\|_{\tau}\mathcal{M}_{\tau}\big(2^{ks(\cdot)}\delta^{-\frac{1}{q(\cdot)}}f\big)\left(x\right).$$

The first sum clearly is bounded by

$$c\left\|a\left(x,2^{k}\cdot\right)\right\|_{B^{N}_{\tau,\infty}}\mathcal{M}_{\tau}\big(2^{ks(\cdot)}\delta^{-\frac{1}{q(\cdot)}}f\big)\left(x\right).$$

The second sum is bounded by

$$\begin{aligned}&c\sup_{i\leq 0}\left\|\mathcal{F}_{\xi}^{-1}\left(\varphi_{i}\mathcal{F}_{\xi}a\left(x,2^{k}\cdot\right)\right)\right\|_{\tau}\mathcal{M}_{\tau}\big(2^{ks(\cdot)}\delta^{-\frac{1}{q(\cdot)}}f\big)\left(x\right)\\ \leq\;&c\left\|a\left(x,2^{k}\cdot\right)\right\|_{\tau}\mathcal{M}_{\tau}\big(2^{ks(\cdot)}\delta^{-\frac{1}{q(\cdot)}}f\big)\left(x\right),\end{aligned}$$

where we have used the fact that

$$\left\|\mathcal{F}_{\xi}^{-1}\left(\varphi_{i}\mathcal{F}_{\xi}a\left(x,2^{k}\cdot\right)\right)\right\|_{\tau}\leq\left\|\mathcal{F}^{-1}\varphi_{i}\right\|_{1}\left\|a\left(x,2^{k}\cdot\right)\right\|_{\tau}\lesssim\left\|a\left(x,2^{k}\cdot\right)\right\|_{\tau}.$$

Therefore,

$$\begin{aligned}&\left\|2^{ks(\cdot)}\delta^{-\frac{1}{q(\cdot)}}a\left(x,D\right)f\right\|_{p_{1}(\cdot)}\\ \lesssim\;&\left(\left\|\left\|a(\cdot,2^{k}\cdot)\right\|_{\tau}+\left\|a(\cdot,2^{k}\cdot)\right\|_{B^{N}_{\tau,\infty}}\right\|_{u}\right)\left\|2^{ks(\cdot)}\delta^{-\frac{1}{q(\cdot)}}f\right\|_{p_{2}(\cdot)},\end{aligned}$$

since the the maximal function is bounded in $L^{p_{2}(\cdot)/\tau}$. Our estimate follows by the fact that

$$\left\|a\left(x,2^{k}\cdot\right)\right\|_{\tau}+\left\|a\left(x,2^{k}\cdot\right)\right\|_{B^{N}_{\tau,\infty}}\lesssim\left\|a\left(x,2^{k}\cdot\right)\right\|_{B^{N}_{\lambda,\infty}}.$$

Step 2. Let now $0<p_{1}^{-}<1$. Here we have in addition that $\mathrm{supp}\mathcal{F}f\subseteq\{\xi\in\mathbb{R}^{n}:|\xi|\leq c2^{k}\}$. Let us recall the Plancherel-Polya-Nikol'skij inequality in a form stated in [44, Sect. 1.3.2]. Let $0<p\leq q\leq\infty$, $R>0$ and $f\in L^{p}(\mathbb{R}^{n})\cap\mathcal{S}'(\mathbb{R}^{n})$ with

$$\mathrm{supp}\mathcal{F}f\subset\{\xi\in\mathbb{R}^{n}:|\xi|\leq R\}.$$

Then $\left\|f\right\|_{q}$ can be estimated by

$$c\;R^{n(1/p-1/q)}\left\|f\right\|_{p}.$$

Applying this inequality, we get

$$\begin{aligned}
&\left|2^{ks(x)}\delta^{-\frac{1}{q(x)}}a\left(x,D\right)f\left(x\right)\right| \\
\lesssim\;& 2^{kn\left(\frac{1}{\tau}-1\right)}\left(\int 2^{ks(x)\tau}\delta^{-\frac{\tau}{q(x)}}\left|K\left(x,x-y\right)\right|^{\tau}\left|f\left(y\right)\right|^{\tau}dy\right)^{1/\tau}.
\end{aligned}$$

This expression with power τ is bounded by

$$\begin{aligned}
& c2^{kn(1-\tau)}\sum_{v=-\infty}^{\infty}\sup_{y}\left|K\left(x,x-y\right)\mathcal{F}\varphi_{v}\left(x-y\right)\right|^{\tau}\int_{|x-y|<2^{v+1}}\left|2^{ks(x)}\delta^{-\frac{1}{q(x)}}f\left(y\right)\right|^{\tau}dy \\
\lesssim\;& 2^{kn(1-\tau)}\sum_{v=-\infty}^{\infty}2^{\left(c_{\log}(s)+c_{\log}\left(\frac{1}{q}\right)\right)\max(0,k+v)\tau+vn}\sup_{y}\left|K\left(x,x-y\right)\mathcal{F}\varphi_{v}\left(x-y\right)\right|^{\tau}\times \\
& \left(\mathcal{M}_{\tau}\left(2^{ks(\cdot)}\delta^{-\frac{1}{q(\cdot)}}f\right)\left(x\right)\right)^{\tau}.
\end{aligned}$$

As before,

$$\begin{aligned}
\left\|K\left(x,x-\cdot\right)\mathcal{F}\varphi_{v}\left(x-\cdot\right)\right\|_{\infty} &\leq \left\|\mathcal{F}_{\xi}^{-1}\left(\mathcal{F}\varphi_{v}\mathcal{F}_{\xi}a\left(x,\cdot\right)\right)\right\|_{1} \\
&= 2^{kn}\left\|\mathcal{F}_{\xi}^{-1}\left(\mathcal{F}\varphi_{v+k}\mathcal{F}_{\xi}a\left(x,2^{k}\cdot\right)\right)\right\|_{1}.
\end{aligned}$$

Hence

$$\left|2^{ks(x)}\delta^{-\frac{1}{q(x)}}a\left(x,D\right)f\left(x\right)\right|\lesssim\left\|a\left(x,2^{k}\cdot\right)\right\|_{B_{1,\tau}^{\frac{n}{\tau}+c_{\log}(s)+c_{\log}\left(\frac{1}{q}\right)}}\mathcal{M}_{\tau}\left(2^{ks(\cdot)}\delta^{-\frac{1}{q(\cdot)}}f\right)\left(x\right).$$

Now our estimate follows by Hölder's inequality and the boundedness of the maximal function in $L^{p_2(\cdot)/\tau}$. ■

The next three lemmas are used in the proof of our result, see [35] for the constant exponents.

Lemma 3.7 *Let $A,B>0$, p, $q\in\mathcal{P}_0^{\log}$ and $s\in C_{\mathrm{loc}}^{\log}$ such that $q^{+}<\infty$. Let $\{f_k\}_{k\in\mathbb{N}_0}$ be a sequence of functions such that*

$$\mathrm{supp}\mathcal{F}f_0\subseteq\{\xi\in\mathbb{R}^n:|\xi|\leq A\}$$

and

$$\mathrm{supp}\mathcal{F}f_k\subseteq\left\{\xi\in\mathbb{R}^n:B2^{k+1}\leq|\xi|\leq A2^{k+1}\right\}.$$

Then it holds that:

$$\left\|\sum_{k=0}^{\infty}f_k\right\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot)}}\lesssim\left\|\left\{2^{ks(\cdot)}f_k\right\}_k\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

Proof. Let $\{\mathcal{F}\varphi_j\}_{j\in\mathbb{N}_0}$ be as resolution of unity. Using the support properties of $\mathcal{F}f_k$ and $\mathcal{F}\varphi_j$, the sum $\varphi_j * \sum_{k=0}^{\infty} f_k$ become $\sum_{l=-\kappa_1}^{\kappa_2} \varphi_j * f_{j+l}$ for some natural numbers $\kappa_1, \kappa_2 \in \mathbb{N}_0$. Observe that

$$\left|\varphi_j * f_{j+l}\right| \lesssim \left(\eta_{j,m} * \left|f_{j+l}\right|^t\right)^{1/t}$$

for any $m > n + c_{\log}(s) + c_{\log}(1/q)$ and any $t > 0$. Therefore, with $0 < t < p^-$,

$$\begin{aligned}\Big\|\sum_{k=0}^{\infty} f_k\Big\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} &\lesssim \sum_{l=-\kappa_1}^{\kappa_2} \Big\|\Big\{\Big(\eta_{j,m-c_{\log}(\alpha)} * 2^{js(\cdot)t}\left|f_{j+l}\right|^t\Big)^{1/t}\Big\}_j\Big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ &\lesssim \sum_{l=-\kappa_1}^{\kappa_2} \Big\|\left\{2^{js(\cdot)} f_{j+l}\right\}_j\Big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})},\end{aligned}$$

by Lemmas 3.1 and 3.3. ■

Lemma 3.8 *Let $A > 0$, $p, q \in \mathcal{P}_0^{\log}$ and $s \in C_{\mathrm{loc}}^{\log}$ such that $0 < q^+ < \infty$. Let $s^- > n(\max\{1, 1/p^-\} - 1)$. Let $\{f_k\}_{k\in\mathbb{N}_0}$ be a sequence of functions such that*

$$\mathrm{supp}\mathcal{F}f_k \subseteq \left\{\xi \in \mathbb{R}^n : |\xi| \leq A2^{k+1}\right\}.$$

Then it holds that

$$\Big\|\sum_{k=0}^{\infty} f_k\Big\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \Big\|\left\{2^{ks(\cdot)} f_k\right\}_k\Big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

Proof. By the scaling argument, we see that it suffices to consider the case

$$\big\|\left\{2^{ks(\cdot)} f_k\right\}_k\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = 1$$

and show that the modular of the function on the left-hand side is bounded. In particular, we will show that

$$\sum_{j=0}^{\infty} \Big\|\Big|2^{js(\cdot)}\varphi_j * \sum_{k=0}^{\infty} f_k\Big|^{q(\cdot)}\Big\|_{\frac{p(\cdot)}{q(\cdot)}} \lesssim 1.$$

Let $0 < r < \min(\frac{1}{q^+}, \frac{p^-}{q^+})$. Using the support properties of $\mathcal{F}f_k$ and $\mathcal{F}\varphi_j$, the sum $\varphi_j * \sum_{k=0}^{\infty} f_k$ become $\sum_{k=j-\kappa}^{\infty} \varphi_j * f_k$ for some natural number $\kappa \in \mathbb{N}_0$. The left-hand side of the above expression can be rewritten us

$$c\sum_{j=0}^{\infty} \Big\|\Big|\sum_{k=j}^{\infty} 2^{js(\cdot)}\varphi_j * f_k\Big|^{rq(\cdot)}\Big\|_{\frac{p(\cdot)}{rq(\cdot)}}^{\frac{1}{r}}. \tag{3.2}$$

Using the fact that $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we obtain

$$2^{js(\cdot)}\left|\varphi_j * f_k\right| \lesssim 2^{js(\cdot)}\left(\eta_{j,m} * |f_k|\right), \quad m > n.$$

Since s is log-Hölder continuous, we move $2^{js(\cdot)}$ inside the convolution by Lemma 3.1,

$$2^{js(\cdot)}\left(\eta_{j,m} * |f_k|\right) \lesssim \eta_{j,m_1} * 2^{js(\cdot)}\,|f_k|\,,$$

where $m_1 = m - c_{\log}(s)$. Therefore, (3.2) is bounded by

$$\begin{aligned}
& c\sum_{j=0}^{\infty}\Big\|\Big(\sum_{k=j}^{\infty}\eta_{j,m_1} * 2^{js(\cdot)}\,|f_k|\Big)^{rq(\cdot)}\Big\|_{\frac{p(\cdot)}{rq(\cdot)}}^{\frac{1}{r}}. \\
\lesssim\ & \sum_{j=0}^{\infty}\Big(\sum_{k=j}^{\infty}2^{(j-k)(nd+s^-)q^-r}\Big\|\big|2^{n(k-j)d}\eta_{j,m_1} * 2^{ks(\cdot)}\,|f_k|\big|^{q(\cdot)}\Big\|_{\frac{p(\cdot)}{q(\cdot)}}^{r}\Big)^{\frac{1}{r}}.
\end{aligned} \tag{3.3}$$

Let us prove that

$$\begin{aligned}
\Big\|\big|2^{n(k-j)d}\eta_{j,m_1} * 2^{ks(\cdot)}\,|f_k|\big|^{q(\cdot)}\Big\|_{\frac{p(\cdot)}{q(\cdot)}} &\le \Big\|\big|2^{ks(\cdot)}f_k\big|^{q(\cdot)}\Big\|_{\frac{p(\cdot)}{q(\cdot)}} + 2^{-j} \\
&= \delta, \quad k \ge j,
\end{aligned}$$

where $d = 1 - \max\{1, 1/p^-\}$. This is equivalent to

$$\Big\|\delta^{-\frac{1}{q(\cdot)}}2^{n(k-j)d}\eta_{j,m_1} * 2^{ks(\cdot)}\,|f_k|\Big\|_{p(\cdot)} \le 1.$$

Since $\frac{1}{q}$ is log-Hölder continuous and $\delta \in [2^{-j}, 1+2^{-j}]$, we can move $\delta^{-\frac{1}{q(\cdot)}}$ inside the convolution by Lemma 3.1,

$$\begin{aligned}
& c\ 2^{n(k-j)d}\Big\|\delta^{-\frac{1}{q(\cdot)}}\eta_{j,m_1} * 2^{ks(\cdot)}\,|f_k|\Big\|_{p(\cdot)} \\
\lesssim\ & 2^{n(k-j)d}\Big\|\eta_{j,h} * \delta^{-\frac{1}{q(\cdot)}}2^{ks(\cdot)}\,|f_k|\Big\|_{p(\cdot)},
\end{aligned}$$

with $h = m - c_{\log}(s) - c_{\log}(1/q)$. We claim that this expression is bounded by

$$c\Big\|2^{ks(\cdot)}\delta^{-\frac{1}{q(\cdot)}}f_k\Big\|_{p(\cdot)}.$$

The last quasi-norm is less than or equal to one if and only if:

$$\Big\|\,\Big|2^{ks(\cdot)}\delta^{-\frac{1}{q(\cdot)}}f_k\Big|^{q(\cdot)}\,\Big\|_{\frac{p(\cdot)}{q(\cdot)}} \le 1,$$

which follows immediately from the definition of δ. Therefore, the right-hand side of (3.3) is bounded by

$$\begin{aligned}
& c\sum_{j=0}^{\infty}\Big(\sum_{k=j}^{\infty}2^{n(j-k)(nd+s^-)q^-r}\Big\|\big|2^{ks(\cdot)}f_k\big|^{q(\cdot)}\Big\|_{\frac{p(\cdot)}{q(\cdot)}}^{r}\Big)^{\frac{1}{r}} + c \\
\lesssim\ & \sum_{k=0}^{\infty}\Big\|\big|2^{ks(\cdot)}f_k\big|^{q(\cdot)}\Big\|_{\frac{p(\cdot)}{q(\cdot)}} + c \\
\lesssim\ & 1,
\end{aligned}$$

where we have used Lemma 3.5 (since $s^- > -nd$). Now we prove our claim. Here we use the same arguments of [18]. If $p^- \geq 1$, then convolution with a radially decreasing L^1-function is bounded on $L^{p(\cdot)}$:

$$\begin{aligned}\left\|\eta_{j,h} * \delta^{-\frac{1}{q(\cdot)}} 2^{ks(\cdot)} |f_k|\right\|_{p(\cdot)} &\lesssim \|\eta_{j,h}\|_1 \left\|\delta^{-\frac{1}{q(\cdot)}} 2^{ks(\cdot)} f_k\right\|_{p(\cdot)} \\ &\lesssim \left\|\delta^{-\frac{1}{q(\cdot)}} 2^{ks(\cdot)} f_k\right\|_{p(\cdot)},\end{aligned}$$

by taking $m > n + c_{\log}(s) + c_{\log}(1/q)$. Let $0 < p^- < 1$. By Lemma 3.2, we have

$$|f_k(y)| \lesssim \left(\eta_{k,L} * |f_k|^{p^-}(y)\right)^{1/p^-}, \quad L > n, y \in \mathbb{R}^n.$$

Since $\frac{1}{q} \in C^{\log}_{\text{loc}}$, then

$$\delta^{-\frac{1}{q(y)}} \leq c\,\delta^{-\frac{1}{q(x)}}(1+2^j|y-x|)^{c_{\log}(1/q)} \leq c\,\delta^{-\frac{1}{q(x)}}(1+2^k|y-x|)^{c_{\log}(1/q)}$$

for any $j \leq k$ and any $x, y \in \mathbb{R}^n$. Hence, with $L_1 = L - c_{\log}(\alpha) - c_{\log}(1/q)$,

$$\begin{aligned}&\eta_{j,h} * \delta^{-\frac{1}{q(\cdot)}} 2^{ks(\cdot)} |f_k|(x) \\ \lesssim\ &\eta_{j,h} * \left(\eta_{k,L_1} * \delta^{-\frac{p^-}{q(\cdot)}} 2^{ks(\cdot)p^-} |f_k|^{p^-}\right)^{1/p^-}(x) \\ =\ &c \int_{\mathbb{R}^n} \eta_{j,h}(x-y)\left(\eta_{k,L_1} * \delta^{-\frac{p^-}{q(\cdot)}} 2^{ks(\cdot)p^-} |f_k|^{p^-}(y)\right)^{1/p^-} dy,\end{aligned}$$

where $c > 0$ is independent of j and k. By the inequalities

$$\begin{aligned}\left(1+2^j|x-y|\right)^{-h} &\leq \left(1+2^j|x-z|\right)^{-h}\left(1+2^j|y-z|\right)^{h} \\ &\leq \left(1+2^j|x-z|\right)^{-h}\left(1+2^k|y-z|\right)^{h}, \quad x, y, z \in \mathbb{R}^n, j \leq k,\end{aligned}$$

the last expression can be estimated by

$$\begin{aligned}&c\,2^{j(n-n/p^-)} \\ &\times \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \eta_{j,hp^-}(x-z)\,\eta_{k,L_1-hp^-}(y-z)\,\delta^{-\frac{p^-}{q(z)}} 2^{ks(z)p^-}|f_k(z)|^{p^-} dz\right)^{1/p^-} dy.\end{aligned}$$

Therefore, Minkowiski's inequality gives

$$\begin{aligned}&\eta_{j,h} * \delta^{-\frac{1}{q(\cdot)}} 2^{ks(\cdot)} |f_k|(x) \\ \lesssim\ &2^{(j-k)(n-n/p^-)} \left\|\eta_{k,L_1/p^- - h}\right\|_1 \left(\eta_{j,hp^-} * \delta^{-\frac{p^-}{q(\cdot)}} 2^{ks(\cdot)p^-} |f_k|^{p^-}(x)\right)^{1/p^-} \\ \lesssim\ &2^{(j-k)(n-n/p^-)} \left(\eta_{j,hp^-} * \delta^{-\frac{p^-}{q(\cdot)}} 2^{ks(\cdot)p^-} |f_k|^{p^-}(x)\right)^{1/p^-}\end{aligned}$$

for any $L > (n+h)p^-$ and any $x \in \mathbb{R}^n$. Since $\frac{p(\cdot)}{p^-} \in \mathcal{P}^{\log}$, then convolution with a radially decreasing L^1-function is bounded on $L^{\frac{p(\cdot)}{p^-}}$:

$$\begin{aligned}
&\left\| \eta_{j,h} * \delta^{-\frac{1}{q(\cdot)}} 2^{js(\cdot)} |f_k| \right\|_{p(\cdot)} \\
\lesssim\ & 2^{(j-k)(n-n/p^-)} \left\| \eta_{j,hp^-} * \delta^{-\frac{p^-}{q(\cdot)}} 2^{ks(\cdot)p^-} |f_k|^{p^-} \right\|_{\frac{p(\cdot)}{p^-}}^{1/p^-} \\
\lesssim\ & 2^{(j-k)(n-n/p^-)} \left\| \eta_{j,hp^-} \right\|_1^{1/p^-} \left\| \delta^{-\frac{p^-}{q(\cdot)}} 2^{ks(\cdot)p^-} |f_k|^{p^-} \right\|_{\frac{p(\cdot)}{p^-}}^{1/p^-} \\
=\ & c 2^{(j-k)(n-n/p^-)} \left\| \delta^{-\frac{1}{q(\cdot)}} 2^{ks(\cdot)} f_k \right\|_{p(\cdot)}.
\end{aligned}$$

The proof is completed. ■

Lemma 3.9 *Let $A > 0, p, q \in \mathcal{P}_0^{\log}$ such that $0 < q^+ < \infty$. Let $\{f_k\}_{k \in \mathbb{N}_0}$ be a sequence of functions such that*

$$\operatorname{supp} \mathcal{F} f_k \subseteq \left\{ \xi \in \mathbb{R}^n : |\xi| \leq A 2^{k+1} \right\}.$$

Let $\alpha = n(\max\{1, 1/p^-\} - 1)$. Then it holds that, for some constant $c > 0$,

$$\left\| \sum_{k=0}^{\infty} f_k \right\|_{B^{\alpha}_{p(\cdot),\infty}} \lesssim \left\| \left\{ 2^{k\alpha} f_k \right\}_k \right\|_{\ell^{\min(1,p^-)}(L^{p(\cdot)})}. \tag{3.4}$$

Moreover if the right-hand side inequality in (3.4) is finite, then $\left\{ \sum_{k=0}^{N} f_k \right\}_N$ converges in $\mathcal{S}'(\mathbb{R}^n)$ to a distribution $\sum_{k=0}^{\infty} f_k$ satisfying this inequality.

Proof. The proof follows the ideas in [35, Lemma 3]. First we assume the convergence and putting $\beta = \min(1, p^-)$. Using the support properties of $\mathcal{F} f_k$ and $\mathcal{F}\varphi_j$, the sum $\varphi_j * \sum_{k=0}^{\infty} f_k$ become $\sum_{k=j-\kappa}^{\infty} \varphi_j * f_k$ for some natural number $\kappa \in \mathbb{N}_0$. Using Lemma 3.4, we get

$$\begin{aligned}
\left\| \sum_{k=0}^{\infty} f_k \right\|_{B^{\alpha}_{p(\cdot),\infty}} &= \sup_{j \geq 0} \left\| \sum_{k=j-\kappa}^{\infty} 2^{\alpha j} \varphi_j * f_k \right\|_{p(\cdot)} \\
&\leq \sup_{j \geq 0} \left(\sum_{k=j-\kappa}^{\infty} 2^{\alpha\beta j} \left\| \varphi_j * f_k \right\|_{p(\cdot)}^{\beta} \right)^{1/\beta} \\
&\lesssim \sup_{j \geq 0} 2^{\alpha j} \left(\sum_{k=j-\kappa}^{\infty} 2^{(k-j)\alpha\beta} \left\| f_k \right\|_{p(\cdot)}^{\beta} \right)^{1/\beta} \\
&\lesssim \left\| \left\{ 2^{k\alpha} f_k \right\}_k \right\|_{\ell^{\beta}(L^{p(\cdot)})}.
\end{aligned}$$

Let us now prove the convergence of $\left\{ \sum_{k=0}^{N} f_k \right\}_N$ in $\mathcal{S}'(\mathbb{R}^n)$. It follows from the result we just proved above,

$$\left\| \sum_{k=-N}^{M} f_k \right\|_{B^{\alpha}_{p(\cdot),\infty}} \lesssim \left\| \left\{ 2^{k\alpha} f_k \right\}_{k=-N}^{k=M} \right\|_{\ell^{\beta}(L^{p(\cdot)})}$$

which tend to zero if $N, M \to \infty$ and then $\left\{\sum_{k=0}^{N} f_k\right\}_N$ is a Cauchy sequence in $B^{\alpha}_{p(\cdot),\infty}$ and hence a convergent sequence in $\mathcal{S}'(\mathbb{R}^n)$. ■

We remark that Lemmas 3.7 and 3.8 are true in $B^{s(\cdot)}_{p(\cdot),\infty}$ spaces with the same assumptions on p and s.

3.2 Main results

The following theorem concerning the continuity of pseudo-differential operator on variable Besov spaces.

Theorem 3.10 *Let $s \in C^{\log}_{\mathrm{loc}}$, p, $q \in \mathcal{P}^{\log}_0$ with $0 < q^+ < \infty$. Let $a \in SB^m_\delta(r, \mu, v; N, \lambda)$ be such that $0 < \mu, v \le \infty$, $r > 0$, $(1-\delta) r \ge \frac{n}{\mu}$ and $1 \le \lambda \le \infty$. Let $N > n \max\{\frac{1}{2}, \frac{1}{\lambda}, \frac{1}{p^-}\} + c_{\log}(s) + c_{\log}(\frac{1}{q})$.*

(i) *If*

$$n \max\left\{1, \frac{1}{\mu} + \frac{1}{p^-}\right\} - n - (1-\delta) r < s^- \le s^+ < r - n \max\left\{\frac{1}{\mu} - \frac{1}{p^+}, 0\right\},$$

then $a(x, D)$ is a continuous linear mapping from $B^{s(\cdot)+m}_{p(\cdot),q(\cdot)}$ to $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$.

(ii) *If $(1-\delta) r > \frac{n}{\mu}$, $v \le q^- < \infty$ and*

$$s := r - n \max\left\{\frac{1}{\mu^-} - \frac{1}{p^+}, 0\right\},$$

then $a(x, D)$ is a continuous linear mapping from $B^{s+m}_{p(\cdot),q(\cdot)}$ to $B^{s}_{p(\cdot),q(\cdot)}$.

(iii) *We suppose that $\frac{1}{\mu} + \frac{1}{p^-} \le 1$ or $0 < p^+ \le 1$ and $\frac{1}{\mu} + \frac{1}{p^-} > 1$. If $(1-\delta) r > \frac{n}{\mu}$, $0 < q^+ \le \min\{1, p^-\}$ and*

$$s := n \max\left\{1, \frac{1}{\mu} + \frac{1}{p^-}\right\} - n - (1-\delta) r,$$

then $a(x, D)$ is a continuous linear mapping from $B^{s+m}_{p(\cdot),q(\cdot)}$ to $B^{s}_{p(\cdot),q(\cdot)}$.

Proof. Let $\{\mathcal{F}\varphi_k\}_k$ be a resolution of unity. We set

$$a_{j,k}(x, \xi) = \mathcal{F}^{-1}_{\eta \to x}\left(\mathcal{F}\varphi_j(\eta) \mathcal{F}_{x \to \eta} a(\cdot, \xi)\right) \mathcal{F}\varphi_k(\xi).$$

We decompose the symbol into three parts

$$a(x, \xi) = a^{(1)}(x, \xi) + a^{(2)}(x, \xi) + a^{(3)}(x, \xi),$$

where

$$
\begin{aligned}
a^{(1)}(x,\xi) &= \sum_{k=4}^{\infty}\sum_{j=0}^{k-4} a_{j,k}(x,\xi) \\
a^{(2)}(x,\xi) &= \sum_{k=0}^{\infty}\sum_{j=k-3}^{k+3} a_{j,k}(x,\xi) \\
a^{(3)}(x,\xi) &= \sum_{k=0}^{\infty}\sum_{j=k+4}^{\infty} a_{j,k}(x,\xi).
\end{aligned}
$$

As in Marschall [35], we need only to estimate $a^{(i)}$, $i=1,2,3$ in $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ spaces.

Step 1. **Proof of (i).**

Substep 1.1. We will prove in this step that there is a constant $c>0$ such that for every $f\in B^{s(\cdot)+m}_{p(\cdot),q(\cdot)}$

$$
\left\|a^{(1)}(x,D)f\right\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} \leq c\left\|f\right\|_{B^{s(\cdot)+m}_{p(\cdot),q(\cdot)}}. \tag{3.5}
$$

Observe that $\sum_{j=0}^{k-4} a_{j,k}(x,D)f_k$ has its spectrum in $\left\{\xi\in\mathbb{R}^n : c_1 2^k \leq |\xi| \leq c_2 2^k\right\}$. Then we can apply Lemma 3.7 to obtain

$$
\left\|a^{(1)}(x,D)f\right\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} \leq c\left\|\left\{2^{ks(\cdot)}\sum_{j=0}^{k-4} a_{j,k}(x,D)f_k\right\}_k\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})},
$$

where $f_k := \varphi_k * f$. Let us show that the last quasi-norm is bounded by

$$
c\left\|f\right\|_{B^{s(\cdot)+m}_{p(\cdot),q(\cdot)}}.
$$

By the scaling argument, we see that it suffices to consider the case

$$
\sum_{k=0}^{\infty}\left\|\left|2^{k(s(\cdot)+m)}f_k\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} = 1
$$

and show that the modular of a constant times the function on the left-hand side is bounded. In particular, we will show that

$$
\sum_{k=4}^{\infty}\left\|\left|2^{ks(\cdot)}\sum_{j=0}^{k-4} a_{j,k}(x,D)f_k\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq C.
$$

This clearly follows from the inequality

$$
\begin{aligned}
\left\|\left|2^{ks(\cdot)}\sum_{j=0}^{k-4} a_{j,k}(x,D)f_k\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} &\lesssim \left\|\left|2^{k(s(\cdot)+m)}f_k\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} + 2^{-k} \\
&= \delta,
\end{aligned}
$$

which we proceed to prove. The claim can be reformulated as showing that

$$\left\| \delta^{-1} \left| 2^{ks(\cdot)} \sum_{j=0}^{k-4} a_{j,k}\left(x,D\right) f_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \lesssim 1,$$

which is equivalent to

$$\left\| \delta^{-\frac{1}{q(\cdot)}} 2^{ks(\cdot)} \sum_{j=0}^{k-4} a_{j,k}\left(x,D\right) f_k \right\|_{p(\cdot)} \lesssim 1.$$

By Proposition 3.6, the left-hand side is bounded by

$$\begin{aligned} & c \left\| \left\| \sum_{j=0}^{k-4} a_{j,k}\left(\cdot, 2^k \cdot\right) \right\|_{B^N_{\lambda,\infty}} \right\|_{\infty} \left\| 2^{ks(\cdot)} \delta^{-\frac{1}{q(\cdot)}} f_k \right\|_{p(\cdot)} \\ \lesssim & \left\| 2^{k(s(\cdot)+m)} \delta^{-\frac{1}{q(\cdot)}} f_k \right\|_{p(\cdot)}, \end{aligned}$$

where the implicit constants not depending on k and δ. Now the right-hand side is less than or equal to one if and only if

$$\left\| \left| 2^{k(s(\cdot)+m)} \delta^{-\frac{1}{q(\cdot)}} f_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1,$$

which follows immediately from the definition of δ. This finish the proof of (3.5).

Substep 1.2. We will prove in this step that there is a constant $c > 0$ such that for every $f \in B^{s(\cdot)+m}_{p(\cdot),q(\cdot)}$

$$\left\| a^{(2)}(x,D) f \right\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} \leq c \left\| f \right\|_{B^{s(\cdot)+m}_{p(\cdot),q(\cdot)}} . \tag{3.6}$$

We have $\sum_{j=k-3}^{k+3} a_{j,k}(x,D) f_k$ has its spectrum in $\left\{\xi \in \mathbb{R}^n : |\xi| \leq c_2 2^k\right\}$. Let $\frac{1}{p_1(\cdot)} = \frac{1}{\mu} + \frac{1}{p(\cdot)}$. Since $(1-\delta)\, r \geq \frac{n}{\mu}$, we have the Sobolev embedding

$$B^{s(\cdot)+(1-\delta)r}_{p_1(\cdot),q(\cdot)} \hookrightarrow B^{s(\cdot)+(1-\delta)r-\frac{n}{\mu}}_{p(\cdot),q(\cdot)} \hookrightarrow B^{s(\cdot)}_{p(\cdot),q(\cdot)}.$$

By Lemma 3.8,

$$\left\| a^{(2)}(x,D) f \right\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \left\| \left\{ 2^{k(s(\cdot)+(1-\delta)r)} \sum_{j=k-3}^{k+3} a_{j,k}(x,D) f_k \right\}_k \right\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})}.$$

Let us show that the last quasi-norm is bounded by

$$c \left\| f \right\|_{B^{s(\cdot)+m}_{p(\cdot),q(\cdot)}} .$$

Again by the scaling argument, we see that it suffices to consider the case

$$\sum_{k=0}^{\infty} \left\| \left| 2^{k(s(\cdot)+m)} f_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} = 1$$

and show that the modular of a constant times the function on the left-hand side is bounded. In particular, we will show that

$$\sum_{k=0}^{\infty}\left\|\left|2^{k(s(\cdot)+(1-\delta)r)}\sum_{j=k-3}^{k+3}a_{j,k}(x,D)f_k\right|^{q(\cdot)}\right\|_{\frac{p_1(\cdot)}{q(\cdot)}}\leq C.$$

This clearly follows from the inequality:

$$\left\|\left|2^{k(s(\cdot)+(1-\delta)r)}\sum_{j=k-3}^{k+3}a_{j,k}(x,D)f_k\right|^{q(\cdot)}\right\|_{\frac{p_1(\cdot)}{q(\cdot)}}\lesssim\left\|\left|2^{k(s(\cdot)+m)}f_k\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}+2^{-k}=\delta.$$

The claim can be reformulated as showing that

$$H=\left\|\delta^{-\frac{1}{q(\cdot)}}2^{k(s(\cdot)+(1-\delta)r)}\sum_{j=k-3}^{k+3}a_{j,k}(x,D)f_k\right\|_{p_1(\cdot)}\lesssim 1.$$

By Proposition 3.6,

$$\begin{aligned}
H &\lesssim 2^{k(1-\delta)r}\left\|\left\|\sum_{j=k-3}^{k+3}a_{j,k}\left(\cdot,2^k\cdot\right)\right\|_{B^N_{\lambda,\infty}}\right\|_{\mu}\left\|\delta^{-\frac{1}{q(\cdot)}}2^{ks(\cdot)}f_k\right\|_{p(\cdot)}\\
&\lesssim \left\|\delta^{-\frac{1}{q(\cdot)}}2^{k(s(\cdot)+m)}f_k\right\|_{p(\cdot)}\\
&\lesssim 1.
\end{aligned}$$

Substep 1.3. We will prove in this step that there is a constant $c>0$ such that for every $f\in B^{s(\cdot)+m}_{p(\cdot),q(\cdot)}$

$$\left\|a^{(3)}(x,D)f\right\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}}\leq c\left\|f\right\|_{B^{s(\cdot)+m}_{p(\cdot),q(\cdot)}}. \tag{3.7}$$

We can apply Lemma 3.7 to obtain

$$\left\|a^{(3)}(x,D)f\right\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}}\leq c\left\|\left\{2^{js(\cdot)}\sum_{k=0}^{j-4}a_{j,k}(x,D)f_k\right\}_j\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

Let us show that the last quasi-norm is bounded by

$$c\left\|f\right\|_{B^{s(\cdot)+m}_{p(\cdot),q(\cdot)}}.$$

By the scaling argument, it suffices to suppose that $\|f\|_{B^{s(\cdot)+m}_{p(\cdot),q(\cdot)}}\leq 1$ and we will show that

$$\sum_{j=4}^{\infty}\left\|\left|2^{js(\cdot)}\sum_{k=0}^{j-4}a_{j,k}\left(x,D\right)f_k\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\lesssim 1. \tag{3.8}$$

We set $\mu_1 = \max(\mu, p^+)$. Let $r_1 > 0$ and $\delta_1 > 0$ be such that $r - \frac{n}{\mu} = r_1 - \frac{n}{\mu_1}$ and $\delta r = \delta_1 r_1$. Let $0 < \sigma < \min(\frac{1}{q^+}, \frac{p^-}{q^+})$. The quasi-norm on the right-hand side with power σ is bounded by

$$\sum_{k=0}^{j-4} \left\| \left| 2^{js(\cdot)} a_{j,k}(x,D) f_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}^{\sigma}.$$

Let us prove that

$$\begin{aligned} \left\| \left| 2^{ks(\cdot)+(j-k)r_1} a_{j,k}(x,D) f_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} &\lesssim \left\| \left| 2^{k(s(\cdot)+m+(1-\delta_1)r_1)} f_k \right|^{q(\cdot)} \right\|_{\frac{p_2(\cdot)}{q(\cdot)}} + 2^{-k} \\ &= \delta, \end{aligned}$$

where $\frac{1}{p(\cdot)} = \frac{1}{\mu_1} + \frac{1}{p_2(\cdot)}$. The claim can be reformulated as showing that

$$\left\| \delta^{-1} \left| 2^{ks(\cdot)+(j-k)r_1} a_{j,k}(x,D) f_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \lesssim 1,$$

which is equivalent to

$$\left\| \delta^{-\frac{1}{q(\cdot)}} 2^{ks(\cdot)+(j-k)r_1} a_{j,k}(x,D) f_k \right\|_{p(\cdot)} \lesssim 1.$$

Observe that f_k and $a_{j,k}(x,D)$ have its spectrum in $\left\{\xi \in \mathbb{R}^n : |\xi| \leq c_2 2^k\right\}$, by Proposition 3.6 and the Sobolev embedding (3.1), the left-hand side is bounded by

$$\begin{aligned} & c \left\| \left\| 2^{jr_1} a_{j,k}(\cdot, 2^k \cdot) \right\|_{B^N_{\lambda,\infty}} \right\|_{\mu_1} \left\| \delta^{-\frac{1}{q(\cdot)}} 2^{ks(\cdot)-kr_1} f_k \right\|_{p_2(\cdot)} \\ \lesssim & \left\| \delta^{-\frac{1}{q(\cdot)}} 2^{k(s(\cdot)+m-(1-\delta_1)r_1)} f_k \right\|_{p_2(\cdot)}, \end{aligned}$$

where the implicit constants not depending on k and δ. Now the right-hand side is less than or equal to one if and only if

$$\left\| \left| 2^{k(s(\cdot)+m-(1-\delta_1)r_1)} \delta^{-\frac{1}{q(\cdot)}} f_k \right|^{q(\cdot)} \right\|_{\frac{p_2(\cdot)}{q(\cdot)}} \leq 1,$$

which follows immediately from the definition of δ. Our estimate (3.8) follows by Lemma 3.5 and the embeddings

$$B^{s(\cdot)+m}_{p(\cdot),q(\cdot)} \hookrightarrow B^{s(\cdot)+m-(1-\delta)r+\frac{n}{\mu}}_{p(\cdot),q(\cdot)} \hookrightarrow B^{s(\cdot)+m-(1-\delta_1)r_1}_{p_2(\cdot),q(\cdot)}.$$

Step 2. **Proof of (ii).** We need only to estimate $a^{(3)}(x,D)$. Let μ_1, r_1 and δ_1 be as in Substep 1.3. Let $\varrho = \min(1, p^-, q^-)$. If $v \leq q^- < \infty$, then

$$\begin{aligned} \left\| a^{(3)}(x,D) f \right\|_{B^s_{p(\cdot),q^-}} &= \left\| \sum_{k=0}^{\infty} \sum_{j=k+4}^{\infty} a_{j,k}(x,D) f_k \right\|_{B^s_{p(\cdot),q^-}} \\ &\lesssim \left(\sum_{k=0}^{\infty} \left\| \sum_{j=k+4}^{\infty} a_{j,k}(x,D) f_k \right\|_{B^s_{p(\cdot),q^-}}^{\varrho} \right)^{\frac{1}{\varrho}}. \end{aligned}$$

Using Lemma 3.7 and Proposition 3.6 we get

$$\begin{aligned}\Big\|\sum_{j=k+4}^{\infty} a_{j,k}(x,D)f_k\Big\|_{B^s_{p(\cdot),q^-}}^{q^-} &\lesssim \sum_{j=k+4}^{\infty}\left\|2^{js}a_{j,k}(x,D)f_k\right\|_{p(\cdot)}^{q^-}\\ &\lesssim \|f_k\|_{p_2(\cdot)}^{q^-}\sum_{j=k+4}^{\infty}2^{jsq^-}\Big\|\left\|a_{j,k}\left(\cdot,2^k\cdot\right)\right\|_{B^N_{\lambda,\infty}}\Big\|_{\mu_1}^{q^-}.\end{aligned}$$

Since $a\in SB^m_\delta\left(r_1,\mu_1,v;N,\lambda\right)$, $v\le q^-<\infty$ and $s=r_1$,

$$\sum_{j=k+4}^{\infty}2^{jsq^-}\Big\|\left\|a_{j,k}\left(\cdot,2^k\cdot\right)\right\|_{B^N_{\lambda,\infty}}\Big\|_{\mu_1}^{q^-}\le 2^{k(m+\delta_1 r_1)q^-}.$$

Therefore,

$$\begin{aligned}\left\|a^{(3)}(x,D)f\right\|_{B^s_{p(\cdot),q^-}} &\lesssim \|f\|_{B^{m+\delta_1 r_1}_{p_2(\cdot),\varrho}}\\ &= \|f\|_{B^{s+m+\frac{n}{\mu}-(1-\delta)r-\frac{n}{\mu_1}}_{p_2(\cdot),\varrho}}\\ &\lesssim \|f\|_{B^{s+m+\frac{n}{\mu}-(1-\delta)r}_{p(\cdot),\varrho}}\\ &\le \|f\|_{B^{s+m}_{p(\cdot),q(\cdot)}},\end{aligned}$$

since $(1-\delta)\,r>\frac{n}{\mu}$.

Step 3. **Proof of (iii).** We need only to estimate $a^{(2)}(x,D)$. Let us consider two cases.

Case 1. $\frac{1}{\mu}+\frac{1}{p^-}\le 1$. Let $\frac{1}{p_1(\cdot)}=\frac{1}{\mu}+\frac{1}{p(\cdot)}$. Then we have

$$B^0_{p_1(\cdot),\infty}\hookrightarrow B^{s+(1-\delta)r-\frac{n}{\mu}}_{p(\cdot),\infty}\hookrightarrow B^s_{p(\cdot),q(\cdot)}.$$

Applying Lemma 3.9, we obtain

$$\begin{aligned}\left\|a^{(2)}(x,D)f\right\|_{B^s_{p(\cdot),q(\cdot)}} &\lesssim \left\|a^{(2)}(x,D)f\right\|_{B^0_{p_1(\cdot),\infty}}\\ &\lesssim \sum_{k=0}^{\infty}\sum_{j=k-3}^{k+3}\left\|a_{j,k}(x,D)f_k\right\|_{p_1(\cdot)}.\end{aligned}$$

By Proposition 3.6 we get

$$\begin{aligned}\sum_{j=k-3}^{k+3}\left\|a_{j,k}(x,D)f_k\right\|_{p_1(\cdot)} &\lesssim \sum_{j=k-3}^{k+3}\Big\|\left\|a_{j,k}\left(\cdot,2^k\cdot\right)\right\|_{B^N_{\lambda,\infty}}\Big\|_u\|f_k\|_{p(\cdot)}\\ &\lesssim 2^{k(m-(1-\delta)r)}\|f_k\|_{p(\cdot)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|a^{(2)}(x,D)f\right\|_{B^s_{p(\cdot),q(\cdot)}} &\lesssim \|f\|_{B^{m-(1-\delta)r}_{p(\cdot),1}}\\ &\lesssim \|f\|_{B^{s+m}_{p(\cdot),q(\cdot)}},\end{aligned}$$

since $s=-(1-\delta)r$.

Case 2. $\frac{1}{\mu}+\frac{1}{p^-}>1$ and $0<p^+\leq 1$. Let

$$\mu_1=\begin{cases}\mu & \text{if} \quad p^+\leq\mu\\ \max(1,\mu) & \text{if} \quad p^+>\mu.\end{cases}$$

Obviously

$$B^{\frac{n}{p^-}-n}_{p(\cdot),\infty}=B^{s+(1-\delta)r-\frac{n}{\mu}}_{p(\cdot),\infty}\hookrightarrow B^{s}_{p(\cdot),q(\cdot)}.$$

Let $r_1>0$ and $\delta_1>0$ be such that $r-\frac{n}{\mu}=r_1-\frac{n}{\mu_1}$ and $\delta r=\delta_1 r_1$. Let $\frac{1}{p(\cdot)}=\frac{1}{\mu_1}+\frac{1}{p_2(\cdot)}$. Then, $a\in SB^m_\delta\left(r_1,\mu_1,v;N,\lambda\right)$, $p_2(\cdot)\geq p(\cdot)$ and

$$s-\frac{n}{p(\cdot)}=\frac{n}{p^-}-n-(1-\delta_1)r_1-\frac{n}{p_2(\cdot)}.$$

Applying again Lemma 3.9, we obtain

$$\begin{aligned}\left\|a^{(2)}(x,D)f\right\|_{B^{s}_{p(\cdot),q(\cdot)}} &\leq \left\|a^{(2)}(x,D)f\right\|_{B^{\frac{n}{p^-}-n}_{p(\cdot),\infty}}\\ &\lesssim \Big(\sum_{k=0}^{\infty}2^{kn(1-p^-)}\sum_{j=k-3}^{k+3}\left\|a_{j,k}\left(x,D\right)f_k\right\|^{p^-}_{p(\cdot)}\Big)^{1/p^-}\\ &\lesssim \left\|f\right\|_{B^{m+\frac{n}{p^-}-n-(1-\delta_1)r_1}_{p_2(\cdot),p^-}},\end{aligned}$$

by Proposition 3.6. Our estimate follows by the Sobolev embedding

$$B^{s+m}_{p(\cdot),q(\cdot)}\hookrightarrow B^{m+\frac{n}{p^-}-n-(1-\delta_1)r_1}_{p_2(\cdot),p^-}.$$

This finish the proof. ■

Chapter 4

Boundedness of pseudo-differential operators on variable Triebel-Lizorkin spaces

This chapter concerns the boundedness properties of the pseudodifferential operators on Triebel-Lizorkin spaces with variable smoothness and integrability with symbols in the class $SB_{\delta}^{m}(r,\mu,v;N,\lambda)$.

Let $\{\mathcal{F}\varphi_j\}_{j\in\mathbb{N}_0}$ be as resolution of unity. For a function $a:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$, we write

$$a_j(x,\xi)=\mathcal{F}_{y\to x}^{-1}\left(\mathcal{F}\varphi_j(y)\,\mathcal{F}a(y,\xi)\right).$$

Let $0<\mu<\infty, 0<v\leq\infty$, $1\leq\lambda\leq\infty$, $0\leq\delta\leq1$, $(1-\delta)r\geq\frac{n}{\mu}$ and $N>\frac{n}{\lambda}$. The space $F_{\mu,v}^{r}(B_{\lambda,\infty}^{N})$ consists of all distributions $a\in\mathcal{S}'(\mathbb{R}^n\times\mathbb{R}^n)$ such that

$$\|a\|_{F_{\mu,v}^{r}(B_{\lambda,\infty}^{N})}=\left\|\left\{2^{jr}\left\|a_j(x,\cdot)\right\|_{B_{\lambda,\infty}^{N}}\right\}_j\right\|_{L^{\mu}(\ell^{v})}<\infty.$$

Let $m,r,N\in\mathbb{R}$, $0\leq\delta\leq1$, $0<\mu,v\leq\infty$, $1\leq\lambda\leq\infty$, $(1-\delta)r\geq\frac{n}{\mu}$ and $N>\frac{n}{\lambda}$. We say that a symbol a belongs to $SF_{\delta}^{m}(r,\mu,v;N,\lambda)$ if

$$\begin{aligned}\sup_k 2^{-km}\left\|\left\|a\left(x,2^k\cdot\right)\mathcal{F}\varphi_k\left(2^k\cdot\right)\right\|_{B_{\lambda,\infty}^{N}}\right\|_{L^{\infty}(dx)} &< \infty\\ \sup_k 2^{-k(m+\delta r)}\left\|a\left(x,2^k\cdot\right)\mathcal{F}\varphi_k\left(2^k\cdot\right)\right\|_{F_{\mu,v}^{r}(B_{\lambda,\infty}^{N})} &< \infty,\end{aligned}$$

which, introduced by J. Marschall [34]. Notice that

$$SB_{\delta}^{m}(r,\mu,p;N,\lambda)\hookrightarrow SF_{\delta_1}^{m}(r_1,p,q;N,\lambda), \tag{4.1}$$

if $0<\mu<p<\infty$, $0<v\leq\infty$, $r-\frac{n}{\mu}=r_1-\frac{n}{p}$ and $\delta r=\delta_1 r_1$.

Let $p,q\in\mathcal{P}_0$. The space $L^{p(\cdot)}(\ell^{q(\cdot)})$ is defined to be the set of all sequences $(f_v)_{v\geq 0}$ of functions such that

$$\left\|(f_v)_{v\geq 0}\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}:=\left\|\left\|(f_v(x))_{v\geq 0}\right\|_{\ell^{q(x)}}\right\|_{L^{p(\cdot)}}<\infty.$$

It is easy to show that $L^{p(\cdot)}(\ell^{q(\cdot)})$ is always a quasi-normed space and it is a normed space, if $\min(p(x),q(x))\geq 1$ holds point-wise.

We recall the definition and some properties of Triebel-Lizorkin spaces with variable smoothness and integrability.

Definition 4.1 *Let $\{\mathcal{F}\varphi_v\}_{v\in\mathbb{N}_0}$ be as resolution of unity. For $s:\mathbb{R}^n\to\mathbb{R}$ and $p,q\in\mathcal{P}_0(\mathbb{R}^n)$, with $0<p^+,q^+<\infty$, the Triebel-Lizorkin space $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$ consists of all distributions $f\in\mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}}=\left\|\left\{2^{vs(\cdot)}\varphi_v*f\right\}_v\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}<\infty.$$

We directly obtain the following simplification in the case when q is constant. If q is a constant, then

$$\|f\|^{\varphi}_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}=\left\|\left\|\left\{2^{v\alpha(\cdot)}\varphi_v*f\right\}_v\right\|_{\ell^q}\right\|_{p(\cdot)},$$

For any $p,q\in\mathcal{P}^{\log}_0(\mathbb{R}^n)$, with $0<p^+,q^+<\infty$ and $s\in C^{\log}_{\rm loc}$, the space $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$ does not depend on the chosen smooth dyadic resolution of unity $\{\mathcal{F}\varphi_v\}_{v\in\mathbb{N}_0}$ (in the sense of equivalent quasi-norms) and

$$\mathcal{S}(\mathbb{R}^n)\hookrightarrow F^{s(\cdot)}_{p(\cdot),q(\cdot)}\hookrightarrow\mathcal{S}'(\mathbb{R}^n).$$

We next consider embeddings of Sobolev-type. For constant exponents it is well-known that

$$F^{\alpha_0}_{p_0,\infty}\hookrightarrow F^{\alpha_1}_{p_1,q}$$

if $\alpha_0-\frac{n}{p_0}=\alpha_1-\frac{n}{p_1}$, where $0<p_0\leq p_1<\infty, 0<q\leq\infty, -\infty<\alpha_1\leq\alpha_0<\infty$ (see e.g. [44], Theorem 2.7.1). For variable case we have the following results, see [50].

Theorem 4.2 *Let $p_0,p_1,q_0,q_1\in\mathcal{P}^{\rm loc}_0(\mathbb{R}^n)$, with $0<p_0^+,p_1^+,q_0^+,q_1^+<\infty$ and $\alpha_0,\alpha_1\in C^{\log}_{\rm loc}$ with $\alpha_0\geq\alpha_1$. If*

$$\alpha_0(x)-\frac{n}{p_0(x)}=\alpha_1(x)-\frac{n}{p_1(x)}$$

and

$$\inf(\alpha_0 - \alpha_1) > 0,$$

then

$$F^{\alpha_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}(\mathbb{R}^n) \hookrightarrow F^{\alpha_1(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n).$$

More information about these function spaces can be found in [14],[29],[30],[31] and reference theiren.

4.1 Basic tools

In this section we present some results which are useful for us. The following lemma is from [14, Theorem 3.2] (we use it, since the maximal operator is in general not bounded on $L^{p(\cdot)}(\ell^{q(\cdot)})$).

Lemma 4.3 *Let $p, q \in \mathcal{P}^{\log}$ with $1 < p^- \leq p^+ < \infty$ and $1 < q^- \leq q^+ < \infty$. For $m > n$, there exists $c > 0$ such that*

$$\left\|\left\{\eta_{v,m} * f_v\right\}_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq c \left\|\left\{f_v\right\}_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$$

for every sequence $\{f_v\}_v$ of L^1_{loc}-functions.

Lemma 4.4 *Let $A, B > 0$, p, $q \in \mathcal{P}^{\log}_0$ and $s \in C^{\log}_{\text{loc}}$ such that $p^+, q^+ < \infty$. Let $\{f_k\}_{k\in\mathbb{N}_0}$ be a sequence of functions such that*

$$\text{supp}\mathcal{F}f_0 \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq A\}$$

and

$$\text{supp}\mathcal{F}f_k \subseteq \left\{\xi \in \mathbb{R}^n : B2^{k+1} \leq |\xi| \leq A2^{k+1}\right\}.$$

Then it holds that:

$$\Big\|\sum_{k=0}^{\infty} f_k\Big\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} \leq c \left\|\left\{2^{ks(\cdot)} f_k\right\}_k\right\|_{L^{p(\cdot)}\left(\ell^{q(\cdot)}\right)}.$$

Proof. Let $\left\{\mathcal{F}\varphi_j\right\}_{j\in\mathbb{N}_0}$ be as resolution of unity. Using the support properties of $\mathcal{F}f_k$ and $\mathcal{F}\varphi_j$, the sum $\varphi_j * \sum_{k=0}^{\infty} f_k$ become

$$\sum_{l=-\kappa_1}^{\kappa_2} \varphi_j * f_{j+l}$$

for some natural numbers $\kappa_1, \kappa_2 \in \mathbb{N}_0$. Observe that

$$\left|\varphi_j * f_{j+l}\right| \lesssim \left(\eta_{j,m} * \left|f_{j+l}\right|^t\right)^{1/t}, \quad l = -\kappa_1, ..., \kappa_2$$

for any $m > n + c_{\log}(\alpha)$ and any $t > 0$. Therefore, with $t \in (0, \min\{1, p^-, q^-\})$,

$$\begin{aligned}\Big\|\sum_{k=0}^{\infty} f_k\Big\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} &\lesssim \Big\|\Big\{\left(\eta_{j,m-c_{\log}(\alpha)} * 2^{js(\cdot)t}\left|f_{j+l}\right|^t\right)^{1/t}\Big\}_k\Big\|_{L^{p(\cdot)}\left(\ell^{q(\cdot)}\right)} \\ &\lesssim \left\|\left\{2^{js(\cdot)} f_{j+l}\right\}_k\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})},\end{aligned}$$

by Lemmas 3.1 and 4.3. ■

The following proposition plays a fundamental role in this section.

Proposition 4.5 *Let $s \in C^{\log}_{\mathrm{loc}}$, $m > 0$, $1 \leq \lambda \leq \infty$, $\frac{1}{\tau} > \max(\frac{1}{2}, \frac{1}{\lambda})$ and $N > \frac{m}{\tau} + c_{\log}(s)$. Let $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a bounded and measurable symbol such that*

$$\mathrm{supp} a(x, \cdot) \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq c2^k\}.$$

(i) *If $\tau \geq 1$, then*

$$\left|2^{ks(x)} a(x, D) f(x)\right| \lesssim C_N \left\|a\left(x, 2^k \cdot\right)\right\|_{B^N_{\lambda,\infty}} \left(\eta_{k,m} * 2^{ks(\cdot)\tau} |f|^\tau(x)\right)^{\frac{1}{\tau}}, \quad x \in \mathbb{R}^n \tag{4.2}$$

for any $k \in \mathbb{N}_0$, with the implicit constant not depending on k.

(ii) *If $0 < \tau < 1$ and*

$$\mathrm{supp}\mathcal{F} f \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq c2^k\},$$

then we have (4.2).

Proof. The proof follows the ideas in Proposition 3.6 and [35, Proposition 4]. We will do the proof in two steps.

Step 1. *Proof of* (i). Let

$$K(x, x-y) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, \xi)\, d\xi = \mathcal{F}_\xi a(x, \cdot)(x-y),$$

be the kernel of $a(x, D)$. It follows from the Hölder inequality

$$\begin{aligned}&\left|2^{ks(x)} a(x, D) f(x)\right| \\ \leq\ & \sum_{v=-\infty}^{\infty} \int \left|2^{ks(x)} K(x, x-y) \mathcal{F}\varphi_v(y-x) f(y)\right| dy \\ \leq\ & \sum_{v=-\infty}^{\infty} \left(\int |K(x, x-y) \mathcal{F}\varphi_v(y-x)|^{\tau'} dy\right)^{\frac{1}{\tau'}} \left(\int_{|x-y| \leq 2^{v+1}} \left|2^{ks(x)} f(y)\right|^\tau dy\right)^{\frac{1}{\tau}}.\end{aligned}$$

Observe that

$$K\left(x,x-y\right)\mathcal{F}\varphi_{v}\left(y-x\right)=\mathcal{F}_{\xi}^{-1}\left(\mathcal{F}_{\xi}^{-1}\left(\mathcal{F}\varphi_{v}\mathcal{F}_{\xi}a\left(x,\cdot\right)\right)\right)\left(x-y\right).$$

Hence by the Hausdorff-Young inequalities, since $1\leq\tau\leq 2$,

$$\left|2^{ks(x)}a\left(x,D\right)f\left(x\right)\right|\leq c\sum_{v=-\infty}^{\infty}\left\|\mathcal{F}_{\xi}^{-1}\left(\mathcal{F}\varphi_{v}\mathcal{F}_{\xi}a\left(x,\cdot\right)\right)\right\|_{\tau}\Big(\int_{|x-y|\leq 2^{v+1}}\left|2^{ks(x)}f\left(y\right)\right|^{\tau}dy\Big)^{\frac{1}{\tau}}.$$

Since s is log-Hölder continuous we get

$$2^{ks(x)}\leq 2^{-kn}\eta_{k,-c_{\log}(s)}(x-y)2^{ks(y)}\leq\left(1+2^{k+v+1}\right)^{c_{\log}(s)}2^{ks(y)}$$

and

$$2^{-kn}\eta_{k,-m}(x-y)\leq\left(1+2^{k+v+1}\right)^{m}$$

for any $x,y\in\mathbb{R}^{n}$ such that $|x-y|\leq 2^{v+1}$. Therefore

$$\left|2^{ks(x)}a\left(x,D\right)f\left(x\right)\right|\leq c\sum_{v=-k}^{\infty}2^{(v+k)(\frac{m}{\tau}+c_{\log}(s))}H_{v,k}(x)+c\sum_{v=-\infty}^{-k-1}2^{(v+k)\frac{m}{\tau}}H_{v,k}(x),$$

where

$$H_{v,k}(x)=\left\|\mathcal{F}_{\xi}^{-1}\left(\mathcal{F}\varphi_{v+k}\mathcal{F}_{\xi}a\left(x,2^{k}\cdot\right)\right)\right\|_{\tau}\left(\eta_{k,m}*2^{ks(\cdot)\tau}\left|f\right|^{\tau}(x)\right)^{\frac{1}{\tau}}.$$

The first sum clearly is bounded by

$$c\left\|a\left(x,2^{k}\cdot\right)\right\|_{B^{N}_{\tau,\infty}}\left(\eta_{k,m}*2^{ks(\cdot)\tau}\left|f\right|^{\tau}(x)\right)^{\frac{1}{\tau}},\quad x\in\mathbb{R}^{n}.$$

The second sum is bounded by

$$\begin{aligned}&c\sup_{i\leq 0}\left\|\mathcal{F}_{\xi}^{-1}\left(\varphi_{i}\mathcal{F}_{\xi}a\left(x,2^{k}\cdot\right)\right)\right\|_{\tau}\left(\eta_{k,m}*2^{ks(\cdot)\tau}\left|f\right|^{\tau}(x)\right)^{\frac{1}{\tau}}\\ \leq\ &c\left\|a\left(x,2^{k}\cdot\right)\right\|_{\tau}\left(\eta_{k,m}*2^{ks(\cdot)\tau}\left|f\right|^{\tau}(x)\right)^{\frac{1}{\tau}},\quad x\in\mathbb{R}^{n},\end{aligned}$$

where we have used the fact that

$$\begin{aligned}\left\|\mathcal{F}_{\xi}^{-1}\left(\varphi_{i}\mathcal{F}_{\xi}a\left(x,2^{k}\cdot\right)\right)\right\|_{\tau}&\leq\left\|\mathcal{F}^{-1}\varphi_{i}\right\|_{1}\left\|a\left(x,2^{k}\cdot\right)\right\|_{\tau}\\&\lesssim\left\|a\left(x,2^{k}\cdot\right)\right\|_{\tau}.\end{aligned}$$

Our estimate follows by the fact that

$$\left\|a\left(x,2^{k}\cdot\right)\right\|_{\tau}+\left\|a\left(x,2^{k}\cdot\right)\right\|_{B^{N}_{\tau,\infty}}\lesssim\left\|a\left(x,2^{k}\cdot\right)\right\|_{B^{N}_{\lambda,\infty}}.$$

Step 2. *Proof of* (ii). Applying Plancherel-Polya-Nikol'skij, we get

$$\left|2^{ks(x)}a\left(x,D\right)f\left(x\right)\right|\lesssim 2^{kn\left(\frac{1}{\tau}-1\right)}\left(\int 2^{ks(x)\tau}\left|K\left(x,x-y\right)\right|^{\tau}\left|f\left(y\right)\right|^{\tau}dy\right)^{1/\tau},\quad x\in\mathbb{R}^{n}.$$

This expression with power τ is bounded by

$$\begin{aligned} & c2^{kn(1-\tau)}\sum_{v\in\mathbb{Z}}\sup_{y}\left|K\left(x,x-y\right)\mathcal{F}\varphi_{v}\left(x-y\right)\right|^{\tau}\int_{|x-y|<2^{v}}\left|2^{ks(x)}f\left(y\right)\right|^{\tau}dy \\ \lesssim\ & c\sum_{v\in\mathbb{Z}}2^{(c_{\log}(s)\tau+m)\max(0,k+v)-kn\tau}\sup_{y}\left|K\left(x,x-y\right)\mathcal{F}\varphi_{v}\left(x-y\right)\right|^{\tau} \\ & \times\int_{|x-y|<2^{v}}2^{ks(y)\tau}\eta_{k,m}(x-y)\left|f\left(y\right)\right|^{\tau}dy. \end{aligned}$$

As before,

$$\begin{aligned} \left\|K\left(x,x-\cdot\right)\varphi_{v}\left(x-\cdot\right)\right\|_{\infty} &\leq \left\|\mathcal{F}_{\xi}^{-1}\left(\mathcal{F}\varphi_{v}\mathcal{F}_{\xi}a\left(x,\cdot\right)\right)\right\|_{1} \\ &= 2^{kn}\left\|\mathcal{F}_{\xi}^{-1}\left(\varphi_{v+k}\mathcal{F}_{\xi}a\left(x,2^{k}\cdot\right)\right)\right\|_{1}. \end{aligned}$$

Hence

$$\begin{aligned} \left|2^{ks(x)}a\left(x,D\right)f\left(x\right)\right|^{\tau} &\lesssim \left\|a\left(\cdot,2^{k}\cdot\right)\right\|_{\dot{B}_{1,\tau}^{\frac{m}{\tau}+c_{\log}(s))}}^{\tau}\eta_{k,m}*2^{ks(\cdot)\tau}\left|f\right|^{\tau}(x) \\ &\lesssim \left\|a\left(\cdot,2^{k}\cdot\right)\right\|_{B_{1,\tau}^{\frac{m}{\tau}+c_{\log}(s))}}^{\tau}\eta_{k,m}*2^{ks(\cdot)}\left|f\right|^{\tau}(x),\quad x\in\mathbb{R}^{n}. \end{aligned}$$

The proof is completed. ■

We need the Marschall's inequalities, see [34, Proposition 1.3] and [54, Proposition 6.1] for the case of constant exponent.

Lemma 4.6 *Let* $A>0, R\geq 1$ *and* $s\in C_{\text{loc}}^{\log}$. *Let* $b\in\mathcal{D}(\mathbb{R}^{n})$ *and a function* $f\in C^{\infty}(\mathbb{R}^{n})$ *such that*

$$\operatorname{supp}\mathcal{F}f\subseteq\left\{\xi\in\mathbb{R}^{n}:\left|\xi\right|\leq AR\right\}\quad\textit{and}\quad\operatorname{supp}b\subseteq\left\{\xi\in\mathbb{R}^{n}:\left|\xi\right|\leq A\right\}.$$

Then

$$2^{js(x)}\left|\mathcal{F}^{-1}b*f(x)\right|\leq c2^{(1-\frac{1}{t})jn}(AR)^{\frac{n}{t}-n}\left\|b(2^{j}\cdot)\right\|_{B_{1,t}^{\frac{n}{t}+c_{\log}(s)}}\left(\mathcal{M}(2^{js(\cdot)}\left|f\right|^{t})(x)\right)^{\frac{1}{t}}$$

for any $0<t\leq 1$ *and any* $x\in\mathbb{R}^{n}$, *where* c *is independant of* A, R, b, j *and* f.

Proof. With x fixed, $y \longmapsto \mathcal{F}^{-1}b(x-y)f(y)$ has its spectrum in $B(0,(R+1)A)$. Applying Plancherel-Polya-Nikol'skij, we get

$$\begin{aligned}
\left|\mathcal{F}^{-1}b * f(x)\right| &\leq \int |\mathcal{F}^{-1}b(x-y)f(y)|dy \\
&= \left\|\mathcal{F}^{-1}b(x-\cdot)f\right\|_1 \\
&\leq c(AR)^{\frac{n}{t}-n} \left\|\mathcal{F}b(x-\cdot)f\right\|_t .
\end{aligned}$$

Let $\phi \in \mathcal{D}(\mathbb{R}^n)$ be supported in $\{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$ and such that

$$\sum_{k=-\infty}^{\infty} \phi(2^{-k}\xi) = 1, \quad \xi \neq 0.$$

We have

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}b(x-\cdot)f\right\|_t^t \\
= &\int |\mathcal{F}^{-1}b(x-y)f(y)|^t dy \leq \sum_{k=-\infty}^{\infty} \int |\mathcal{F}^{-1}b(x-y)\phi(2^{-k}(x-y))f(y)|^t dy \\
\leq &\sum_{k=-\infty}^{\infty} \sup_y |\mathcal{F}^{-1}b(x-y)\phi(2^{-k}(x-y))|^t \int_{\overline{B(x,2^{k+1})}} |f(y)|^t dy \\
\leq &\sum_{k=-\infty}^{\infty} \sup_z |\mathcal{F}^{-1}b(z)\phi(2^{-k}(z))|^t \int_{\overline{B(x,2^{k+1})}} |f(y)|^t dy.
\end{aligned}$$

Since,

$$\mathcal{F}^{-1}b(z)\phi(2^{-k}z) = \mathcal{F}^{-1}\mathcal{F}(\mathcal{F}^{-1}b\phi(2^{-k}\cdot))(z),$$

we obtain

$$\begin{aligned}
|\mathcal{F}^{-1}b(z)\phi(2^{-k}(z))| &\leq \int |\mathcal{F}(\mathcal{F}^{-1}b\phi(2^{-k}\cdot))(y)|dy \\
&= \left\|\mathcal{F}(\phi(2^{-k}\cdot)) * b\right\|_1 .
\end{aligned}$$

From the fact that

$$2^{js(x)} \leq (1+2^j|x-y|)^{c_{\log}(s)} 2^{js(y)}, \quad x,y \in \mathbb{R}^n,$$

we get,

$$\begin{aligned}
&2^{js(x)t} \left\|\mathcal{F}^{-1}b(x-\cdot)f\right\|_t^t \\
\leq &\sum_{k=-\infty}^{\infty} 2^{\max(0,j+k)c_{\log}(s)t} \left\|\mathcal{F}(\phi(2^{-k}\cdot)) * b\right\|_1^t \int_{\overline{B(x,2^{k+1})}} 2^{js(y)t}|f(y)|^t dy \\
\leq &\sum_{k=-\infty}^{\infty} 2^{\max(0,j+k)c_{\log}(s)t+kn} \left\|\mathcal{F}(\phi(2^{-k}\cdot)) * b\right\|_1^t \mathcal{M}(2^{js(\cdot)t}\,|f|^t)(x), \quad x \in \mathbb{R}^n.
\end{aligned}$$

Observe that

$$\begin{aligned}
\mathcal{F}(\phi(2^{-k}\cdot)) * b(x) &= \int \mathcal{F}(\phi(2^{-k}\cdot))(z) b(x-z)dz \\
&= 2^{-jn} \int \mathcal{F}(\phi(2^{-k-j}\cdot))(2^{-j}z) b(x-z)dz \\
&= \mathcal{F}(\phi(2^{-k-j}\cdot)) * b(2^{j}\cdot)(2^{-j}x),
\end{aligned}$$

then

$$\left\|\mathcal{F}(\phi(2^{-k}\cdot)) * b\right\|_1 \leq 2^{jn} \left\|\mathcal{F}(\phi(2^{-k-j}\cdot)) * b(2^{j}\cdot)\right\|_1 .$$

Therefore,

$$\sum_{k=-\infty}^{\infty} 2^{\max(0,j+k)c_{\log}(s)t+(j+\frac{k}{t})nt} \left\|\mathcal{F}(\phi(2^{-k-j}\cdot)) * b(2^{j}\cdot)\right\|_1^t = \sum_{k+j\geq 0} \cdots + \sum_{k+j<0} \cdots .$$

The first sum is bounded by

$$2^{(1-\frac{1}{t})jnt} \left\|b(2^{j}\cdot)\right\|^t_{B_{1,t}^{\frac{n}{t}+c_{\log}(s)}}$$

and since

$$\left\|\mathcal{F}(\phi(2^{-k-j}\cdot)) * b(2^{j}\cdot)\right\|_1 \lesssim \left\|b(2^{j}\cdot)\right\|_1 \leq \left\|b(2^{j}\cdot)\right\|_{B_{1,t}^{\frac{n}{t}+c_{\log}(s)}} ,$$

we obtain the desired estimate. ■

Lemma 4.7 *Let $A > 0, p \in \mathcal{P}_0^{\log}$ and $s \in C_{\mathrm{loc}}^{\log}$ such that $p^+ < \infty$ and $s^- > n(\max\{1, 1/p^-\}-1)$. Let $\{f_k\}_{k\in\mathbb{N}_0}$ be a sequence of functions such that* $\mathrm{supp}\mathcal{F} f_k \subseteq \left\{\xi \in \mathbb{R}^n : |\xi| \leq A2^{k+1}\right\}$*. Then it holds that*

$$\Big\| \sum_{k=0}^{\infty} f_k \Big\|_{F^{s(\cdot)}_{p(\cdot),\infty}} \leq c \left\|\left\{2^{ks(\cdot)} f_k\right\}_k\right\|_{L^{p(\cdot)}(\ell^\infty)} .$$

Proof. Let $\left\{\mathcal{F}\varphi_j\right\}_{j\in\mathbb{N}_0}$ be as resolution of unity. Using the support properties of $\{f_k\}_{k\in\mathbb{N}_0}$, we obtain

$$\sum_{k=0}^{\infty} \varphi_j * f_k = \sum_{k=j+\sigma}^{\infty} \varphi_j * f_k = \sum_{i=\sigma}^{\infty} \varphi_j * f_{j+i}, \quad \sigma \in \mathbb{R}.$$

Let $\beta = \min\{1, p^-\}$. Then we have

$$\begin{aligned}
\Big\| \sum_{k=0}^{\infty} f_k \Big\|^{\beta}_{F^{s(\cdot)}_{p(\cdot),\infty}} &\leq \left\|\left\{2^{js(\cdot)} \sum_{i=\sigma}^{\infty} |\varphi_j * f_{j+i}|\right\}_j\right\|^{\beta}_{L^{p(\cdot)}(\ell^\infty)} \\
&= \left\|\left\{\left(2^{js(\cdot)} \sum_{i=\sigma}^{\infty} |\varphi_j * f_{j+i}|\right)^{\beta}\right\}_j\right\|_{L^{\frac{p(\cdot)}{\beta}}(\ell^\infty)} \\
&\leq \sum_{i=\sigma}^{\infty} \left\|\left\{2^{js(\cdot)} |\varphi_j * f_{j+i}|\right\}_j\right\|^{\beta}_{L^{p(\cdot)}(\ell^\infty)} .
\end{aligned}$$

Observe that

$$\varphi_j = \mathcal{F}^{-1}\mathcal{F}\varphi_j$$

and $\operatorname{supp}\mathcal{F}\varphi_j \subset \{\xi : |\xi| \leq 2^{j+1}\}$ then we can use Lemma 4.6;

$$\begin{aligned} & 2^{js(x)}|\varphi_j * f_{j+i}(x)| \\ \lesssim\ & 2^{(n-\frac{n}{t})j}2^{(j+i)(\frac{n}{t}-n)} \left\|\mathcal{F}\varphi_j(2^j\cdot)\right\|_{B_{1,t}^{\frac{n}{t}+c_{\log}(s)}} \left(\mathcal{M}(2^{js(\cdot)}\left|f_{j+i}\right|^t)(x)\right)^{\frac{1}{t}} \\ \lesssim\ & 2^{i(\frac{n}{t}-n)} \left\|\varphi\right\|_{B_{1,t}^{\frac{n}{t}+c_{\log}(s)}} \left(\mathcal{M}(2^{js(\cdot)}\left|f_{j+i}\right|^t)(x)\right)^{\frac{1}{t}}, \quad x \in \mathbb{R}^n, 0 < t \leq 1. \end{aligned}$$

The proof is completed in view of the fact that $s^- > n(\max\{1, 1/p^-\} - 1)$. ■

4.2 Main results

The following theorem concerning the continuity of pseudo-differential operator on the spaces $F_{p(\cdot),q(\cdot)}^{s(\cdot)}$.

Theorem 4.8 *Let $s \in C_{\text{loc}}^{\log}$, p, $q \in \mathcal{P}_0^{\log}$ with $0 < p^+, q^+ < \infty$. Let $a \in SB_\delta^m\left(r, \mu, v; N, \lambda\right)$ be such that $0 < \mu < \infty$, $0 < v \leq \infty$, $r > 0$, $\left(1-\delta\right)r \geq \frac{n}{\mu}$ and $1 \leq \lambda \leq \infty$. Let $N > n\max\left\{\frac{1}{2}, \frac{1}{\lambda}, \frac{1}{p^-}, \frac{1}{q^-}\right\}$ and*

$$n\max\left\{1, \frac{1}{\mu} + \frac{1}{p^-}\right\} - n - \left(1-\delta\right)r < s^- \leq s^+ < r - n\max\left\{\frac{1}{\mu} - \frac{1}{p^+}, 0\right\},$$

then $a\left(x, D\right)$ is a continuous linear mapping from $F_{p(\cdot),q(\cdot)}^{s(\cdot)+m}$ to $F_{p(\cdot),q(\cdot)}^{s(\cdot)}$.

Proof. Let $\left\{\mathcal{F}\varphi_k\right\}_k$ be a resolution of unity. We set

$$a_{j,k}\left(x, \xi\right) = \mathcal{F}^{-1}\left(\mathcal{F}\varphi_j\left(\eta\right)\mathcal{F}_x a\left(\cdot, \xi\right)\right)\mathcal{F}\varphi_k\left(\xi\right).$$

We decompose the symbol into three parts

$$a\left(x, \xi\right) = a^{(1)}\left(x, \xi\right) + a^{(2)}\left(x, \xi\right) + a^{(3)}\left(x, \xi\right),$$

where

$$\begin{aligned} a^{(1)}\left(x, \xi\right) &= \sum_{k=4}^{\infty}\sum_{j=0}^{k-4} a_{j,k}\left(x, \xi\right) \\ a^{(2)}\left(x, \xi\right) &= \sum_{k=0}^{\infty}\sum_{j=k-3}^{k+3} a_{j,k}\left(x, \xi\right) \\ a^{(3)}\left(x, \xi\right) &= \sum_{k=0}^{\infty}\sum_{j=k+4}^{\infty} a_{j,k}\left(x, \xi\right). \end{aligned}$$

Step 1. We will prove in this step that there is a constant $c>0$ such that for every $f\in F^{s(\cdot)+m}_{p(\cdot),q(\cdot)}$

$$\left\|a^{(1)}(x,D)f\right\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}}\leq c\left\|f\right\|_{F^{s(\cdot)+m}_{p(\cdot),q(\cdot)}}. \tag{4.3}$$

Observe that

$$\sum_{j=0}^{k-4}a_{j,k}(x,D)f_k,$$

has its spectrum in $\left\{\xi\in\mathbb{R}^n:c_12^k\leq|\xi|\leq c_22^k\right\}$. Then we can apply Lemma 4.4 to obtain

$$\left\|a^{(1)}(x,D)f\right\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}}\lesssim\left\|\left\{2^{ks(\cdot)}\sum_{j=0}^{k-4}a_{j,k}(x,D)f_k\right\}_k\right\|_{L^{p(\cdot)}\left(\ell^{q(\cdot)}\right)},$$

where $f_k:=\varphi_k*f$. Let us show that the last quasi-norm is bounded by

$$c\left\|f\right\|_{F^{s(\cdot)+m}_{p(\cdot),q(\cdot)}}.$$

By Proposition 4.5, the left-hand side is bounded by

$$\begin{aligned}\left\|\left\{\left(\eta_{k,\sigma}*2^{k(s(\cdot)+m)\tau}\left|f_k\right|^{\tau}\right)^{\frac{1}{\tau}}\right\}_k\right\|_{L^{p(\cdot)}\left(\ell^{q(\cdot)}\right)}&\lesssim\left\|\left\{2^{k(s(\cdot)+m)}f_k\right\}_k\right\|_{L^{p(\cdot)}\left(\ell^{q(\cdot)}\right)}\\&\lesssim\left\|f\right\|_{F^{s(\cdot)+m}_{p(\cdot),q(\cdot)}},\end{aligned}$$

by Lemma 4.3, with $\frac{1}{\tau}=\max\left(\frac{1}{2},\frac{1}{\lambda},\frac{1}{p^-},\frac{1}{q^-}\right)$ and $\sigma>n$. This finish the proof of (4.3).

Step 2. We will prove in this step that there is a constant $c>0$ such that for every $f\in F^{s(\cdot)+m}_{p(\cdot),q(\cdot)}$

$$\left\|a^{(2)}(x,D)f\right\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}}\leq c\left\|f\right\|_{F^{s(\cdot)+m}_{p(\cdot),q(\cdot)}}. \tag{4.4}$$

We have $\sum_{j=k-3}^{k+3}a_{j,k}(x,D)f_k$ has its spectrum in $\left\{\xi\in\mathbb{R}^n:|\xi|\leq c_22^k\right\}$. Let $\frac{1}{p_1(\cdot)}=\frac{1}{\mu}+\frac{1}{p(\cdot)}$. Since $(1-\delta)\,r\geq\frac{n}{\mu}$, we have the Sobolev embedding

$$F^{s(\cdot)+(1-\delta)r}_{p_1(\cdot),\infty}\hookrightarrow F^{s(\cdot)+(1-\delta)r-\frac{n}{\mu}}_{p(\cdot),q(\cdot)}\hookrightarrow F^{s(\cdot)}_{p(\cdot),q(\cdot)},$$

see Theorem 4.2. By Lemma 4.7,

$$\left\|a^{(2)}(x,D)f\right\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}}\lesssim\left\|\left\{2^{k(s(\cdot)+(1-\delta)r)}\sum_{j=k-3}^{k+3}a_{j,k}(x,D)f_k\right\}_k\right\|_{L^{p_1(\cdot)}(\ell^{\infty})}.$$

We have

$$\begin{aligned}&\left|2^{k(s(x)+(1-\delta)r)}\sum_{j=k-3}^{k+3}a_{j,k}(x,D)f_k(x)\right|\\&\leq\left(\sup_k 2^{((1-\delta)r-m)}\Big\|\sum_{j=k-3}^{k+3}a_{j,k}\left(x,2^k\cdot\right)\Big\|_{B^N_{\lambda,\infty}}\right)\left(\eta_{k,m}*2^{k(s(\cdot)+m)\tau}\left|f_k\right|^{\tau}(x)\right)^{\frac{1}{\tau}},\end{aligned}$$

by Proposition 4.5. Taking the $\ell^{q(\cdot)}$-norm and then the $L^{p_1(\cdot)}$-norm we obtain by Hölder's inequality that

$$\begin{aligned}\left\|a^{(2)}(x,D)f\right\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} &\lesssim \left\|\left\{\left(\eta_{k,\sigma} * 2^{k(s(\cdot)+m)\tau}\left|f_k\right|^{\tau}\right)^{\frac{1}{\tau}}\right\}_k\right\|_{L^{p(\cdot)}(\ell^{\infty})}\\ &\lesssim \left\|\left\{\left(2^{k(s(\cdot)+m)\tau}\left|f_k\right|^{\tau}\right)^{\frac{1}{\tau}}\right\}_k\right\|_{L^{p(\cdot)}(\ell^{\infty})}\\ &\lesssim \|f\|_{F^{s(\cdot)+m}_{p(\cdot),q(\cdot)}},\end{aligned}$$

by Lemma 4.3, with $\frac{1}{\tau}=\max\left(\frac{1}{2},\frac{1}{\lambda},\frac{1}{p^-},\frac{1}{q^-}\right)$ and $\sigma>n$.

Step 1.3. We will prove in this step that there is a constant $c>0$ such that for every $f\in F^{s(\cdot)+m}_{p(\cdot),q(\cdot)}$

$$\left\|a^{(3)}(x,D)f\right\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} \le c\|f\|_{F^{s(\cdot)+m}_{p(\cdot),q(\cdot)}}.$$

We can apply Lemma 4.4 to obtain

$$\left\|a^{(3)}(x,D)f\right\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} \le c\left\|\left\{2^{js(\cdot)}\sum_{k=0}^{j-4}a_{j,k}(x,D)f_k\right\}_j\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}.$$

We set $\mu_1=\max(\mu,p^+)$. Let $r_1>0$ and $\delta_1>0$ be such that $r-\frac{n}{\mu}=r_1-\frac{n}{\mu_1}$ and $\delta r=\delta_1 r_1$. Then

$$r_1=r-n\max\left\{\frac{1}{\mu}-\frac{1}{p^+},0\right\}$$

and

$$SB^m_{\delta}(r,\mu,v;N,\lambda)\hookrightarrow SF^m_{\delta_1}(r_1,\mu_1,v;N,\lambda),$$

see (4.1). By Proposition 4.5

$$\begin{aligned}\left|2^{ks(x)}a_{j,k}(x,D)f_k(x)\right| &\le C_N\left\|a_{j,k}\left(x,2^k\cdot\right)\right\|_{B^N_{\lambda,\infty}}\left(\eta_{k,\sigma}*2^{ks(\cdot)\tau}\left|f_k\right|^{\tau}(x)\right)^{\frac{1}{\tau}}\\ &\le C_N 2^{-r_1 j}\sup_i\left\|2^{r_1 i}a_{i,k}\left(x,2^k\cdot\right)\right\|_{B^N_{\lambda,\infty}}\left(\eta_{k,\sigma}*2^{ks(\cdot)\tau}\left|f_k\right|^{\tau}(x)\right)^{\frac{1}{\tau}}\end{aligned}$$

for any $x\in\mathbb{R}^n$. Applying Lemma 3.5 we get

$$\begin{aligned}&\left\|a^{(3)}(x,D)f\right\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}}\\ \le\ & c\left\|\left\{\sup_i\left\|2^{r_1 i}a_{i,k}\left(x,2^k\cdot\right)\right\|_{B^N_{\lambda,\infty}}\left(\eta_{k,\sigma}*2^{k(s(\cdot)-r_1)\tau}\left|f_k\right|^{\tau}\right)^{\frac{1}{\tau}}\right\}_k\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}\\ \le\ & c\left\|\left\{2^{-k(m+\delta_1 r_1)}\sup_i\left\|2^{r_1 i}a_{i,k}\left(x,2^k\cdot\right)\right\|_{B^N_{\lambda,\infty}}\left(\eta_{k,\sigma}*2^{k(s(\cdot)+m-(1-\delta_1)r_1)\tau}\left|f_k\right|^{\tau}\right)^{\frac{1}{\tau}}\right\}_k\right\|_{L^{p_2(\cdot)}(\ell^{q(\cdot)})}.\end{aligned}$$

Puting $\frac{1}{p(\cdot)} = \frac{1}{\mu_1} + \frac{1}{p_2(\cdot)}$ and applying Holder's inequality we estimate the last term by

$$\begin{aligned} & c\Big\| \Big\{ \big(\eta_{k,\sigma} * 2^{k(s(\cdot)+m-(1-\delta_1)r_1)\tau} \left| f_k \right|^{\tau} \big)^{\frac{1}{\tau}} \Big\}_k \Big\|_{L^{p_2(\cdot)}(\ell^{q(\cdot)})} \\ \lesssim & \Big\| \Big\{ \big(2^{k(s(\cdot)+m-(1-\delta_1)r_1)\tau} \left| f_k \right|^{\tau} \big)^{\frac{1}{\tau}} \Big\}_k \Big\|_{L^{p_2(\cdot)}(\ell^{q(\cdot)})} \\ \lesssim & \|f\|_{F^{s(\cdot)+m-(1-\delta_1)r_1}_{p_2(\cdot),q(\cdot)}} , \end{aligned}$$

where we have used Lemma 4.3 with $\sigma > n$. The proof is completed. ■

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ملخص المذكرة:

نتطرق في هذه الرسالة إلى دراسة استمرارية المؤثرات شبه التفاضلية في فضاء $L^{p(x)}$ التي تنتمي الى صنف هرموندار S^0 ، حيث نستخدم أساليب التنظيم.

استنادا إلى الفئة $SB_\delta^m(r,\mu,v;N,\lambda)$ لـ مارشال، التي تم تعريفها بواسطة فضاء بيزوف قمنا ببرهان استمرارية المؤثرات شبه التفاضلية على الفضاءات الدالية لزوركان تريبال وفضاء بيزوف مع الاسس المتغيرة.

نتائجنا هي عبارة عن تغطية النتائج الكلاسيكية في فضاءات لوباغ، بيزوف و تريبال–ليزوركان .

الكلمات المفتاحية :

المؤثرات شبه التفاضلية، فضاء لزوركان تريبال، فضاء بيزوف، الدالة العظمى ، تحليل ليتلوود و بايلي، الأسس المتغيرة .

Résumé de thèses

Dans cette thèse, nous étudions la continuité des opérateurs pseudo-différentiels dans l'espaces $L^{p(\cdot)}$ avec des symboles appartenant aux classes de Hörmander S^0, où nous employons les méthodes de régularisation.

Basé sur la classe $SB_\delta^m(r,\mu,v;N,\lambda)$ de J. Marschall, nous prouvons la continuité des opérateurs pseudo-différentiels sur les espaces de Besov et les espaces de Triebel-Lizorkin avec des exposants variables, où ces symboles incluent dans la classes de Hörmander classiques. Nos résultats couvrent les résultats des espaces classiques de Lebesgue, Besov et Triebel-Lizorkin.

Mot-clés: Opérateurs Pseudo-différentiels, espace de Triebel-Lizorkin, espace de Besov, la fonction maximale, décomposition de Littlewood-Paley, exposant variable.

Abstract of thesis

In this thesis we present the $L^{p(\cdot)}$ boundedness of pseudo-differential operators with symbols belonging to Hörmander classes S^0, where we employ the regularisation methods.

Based on the class $SB_\delta^m(\mathrm{r},\mu,\mathrm{v};\mathrm{N},\lambda)$ of J. Marschall, which is defined by means of vector-valued Besov spaces, we prove the boundedness of the corresponding pseudodifferential operators on Besov spaces and Triebel-Lizorkin spaces with variable smoothness and integrability, where these symbols include the classical Hörmander classes. Our results cover the results on classical Lebesgue, Besov and Triebel-Lizorkin spaces.

Key words: Pseudo-differential operators, Triebel-Lizorkin spaces, Besov spaces, maximal function, Littlewood-Paley decomposition, variable exponent.