
PEOPLE's DEMOCRATIC REPUBLIC OF ALGERIA

MINISTRY OF HIGHER EDUCATION AND
SCIENTIFIC RESEARCH



Mohamed Boudiaf university of Msila

Faculty of Mathematics and computer sciences

Department of Mathematics



1985

Master memory

Field : Mathematics and computer sciences

Branch : Mathematics

Option : Functional analysis

Theme

Notes On Vectors And Tensors

Presented by : Bakhti Loubna

In front of the jury composed of :

Abdelhamid Tallab	MCA,	University of Msila	President.
Ahmed Mazouz	MCB,	University of Msila	Supervisor.
Khaled Haimidi	MCA,	University of Msila	Examiner.

University year 2025/2026

Contents

Introduction	4
1 Multilinear Algebra and Tensor Spaces	7
1.1 Algebraic Duality: Real and Complex Cases.	7
Covectors, Anti-Covectors, and Double Duals: A Unified View of Dual- Type Spaces	7
Canonical Bases for Dual, Conjugate Dual, and Conjugate Spaces . . .	8
1.2 Examples	13
1.3 Multilinear Maps, Tensor Products, and the Universality Principle . . .	13
2 Tensor Algebras and the Abstract Index Formalism	25
2.1 Building Tensor Algebras from a Base Vector Space.	25
2.2 The Algebraic Structure of Tensor Spaces	26
2.3 Symbolic Tensor Calculus via Abstract Indices.	26
2.4 Contracting Tensors: The Universal Reduction	31
2.5 Physical Laws and Tensorial Invariance	34
3 From Abstract Tensors to Physical Applications	36
3.1 Group Representations on Tensor Product Spaces	36
Example	37
3.2 Tensor Products in Composite Quantum Systems	40
3.3 Symmetric and Antisymmetric Tensors under Permutation	42
Examples	44
Example	45

4	Scalar Products and the Geometry of Tensor Spaces	46
4.1	Scalar products	46
	Conclusion	50

Introduction

The structure, definitions, and main results of Chapters 1 to 4 follow closely the lecture notes of Valter Moretti [14] on multilinear algebra and tensor theory. Additional references are cited where supplementary material or alternative expositions have been consulted. We have provided reformulations, examples, and minor adaptations throughout. Tensors furnish the natural mathematical language for formulating physical laws in a manner that remains structurally invariant across changes of reference frame. Unlike coordinate-dependent descriptions, tensorial equations retain their form under arbitrary basis transformations, a property that renders them indispensable in both special and general relativity. [6, 23]. Emerging from the study of multilinear forms and their representations, tensor calculus generalizes the elementary notions of scalar quantities, vector fields, and linear transformations into a unified algebraic structure capable of encoding geometric and physical relationships in a covariant manner [21, 22].

This memoir offers a systematic exposition of tensor spaces generated by a finite-dimensional vector space over the real or complex numbers. Our treatment is organized into four thematic chapters, each building upon the preceding material.

In Chapter 1, we lay the algebraic groundwork by examining the dual space V^* , the conjugate dual space $\overline{V^*}$, and the conjugate space $\overline{V}$ associated with a given vector space V [9, 10]. We establish the existence of natural bases in these spaces and prove the fundamental dimension identities $\dim V = \dim V^* = \dim \overline{V^*} = \dim \overline{V}$. The core of this chapter is the construction of the tensor product via the universality property: we show that for any multilinear mapping $f : V_1 \times \cdots \times V_n \rightarrow W$, there exists a unique linear map $f^\otimes$ on the tensor product space making the canonical diagram commute [2].

This universality theorem underpins the identification of $U_1 \otimes \cdots \otimes U_n$ with the space of multilinear functionals $\mathcal{L}(V_1, \dots, V_n)$, and yields the natural isomorphisms governing associativity of tensor products and distributivity of the duality functor.

In Chapter 2, we assemble the tensor algebra $\mathcal{A}_K(V)$ generated by a vector space V and introduce the abstract index notation [16]. This notation, distinguishing contravariant (upper) and covariant (lower) indices, encodes the transformation laws of tensor components under basis changes while preserving the form of physical equations across reference frames [6, 15, 20]. We develop the operations of linear combination, tensor product, and contraction, and prove the canonical isomorphism between the space of linear operators $\mathcal{L}(S, S')$ and the tensor product $S^* \otimes S'$. The invariance of these operations under coordinate transformations explains their ubiquity in the formulation of relativity principles [23].

In Chapter 3, we turn to applications that illustrate the power of the tensor formalism. We construct the tensor product of group representations and prove that it yields a new representation on the tensor product space, a result of paramount importance in the quantum mechanical description of composite systems [18, 24, 7]. We analyze the action of the symmetric group $\mathcal{P}_n$ on $V^{\otimes n}$, obtaining the decomposition into symmetric and anti-symmetric tensor subspaces [8]. These symmetry properties govern the statistical behavior of quantum particles and find extensive application in quantum field theory [24, 1].

In Chapter 4, we introduce metric structures on vector spaces through the notion of scalar products [11, 19]. Beginning with pseudo scalar products and Hermitian pseudo scalar products, we carefully distinguish the structural requirements of bilinearity, sesquilinearity, symmetry, Hermiticity, and non-degeneracy [10]. We define semi scalar products and establish the Cauchy–Schwarz inequality in this generality, ultimately characterizing genuine (Hermitian) scalar products as those mappings that are simultaneously non-degenerate and positive definite [6, 2]. These metric notions are indispensable for the transition from algebraic tensor theory to the geometric structures of Riemannian and pseudo-Riemannian manifolds [3, 22, 21].

Throughout this work, we have aimed to present a coherent and self-contained treatment that emphasizes both the intrinsic algebraic structure of tensor spaces and their far-reaching applications in modern physics [1].

Chapter 1

Multilinear Algebra and Tensor Spaces

Within this section we introduce basic concepts concerning multi-linear algebra and tensors. The theory of vector spaces and linear mappings is assumed to be well known [4, 14, 12].

1.1 Algebraic Duality: Real and Complex Cases.

As a first step we introduce the dual space and the conjugate space of a given vector space.

Covectors, Anti-Covectors, and Double Duals: A Unified View of Dual-Type Spaces

Definition 1.1.1. *Let V be a vector space on the field either $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ [14, 12].*

- (a) *The **dual space** of V , V^* , is the vector space of linear functionals on V , i.e., the linear mappings $f : V \rightarrow \mathbb{K}$.*
- (b) *If $\mathbb{K} = \mathbb{C}$, the **conjugate dual space** of V , $\overline{V^*}$, is the vector space of anti-linear functionals on V , i.e., the antilinear mappings $g : V \rightarrow \mathbb{C}$. Finally the **conjugate space** of V , $\overline{V}$, is the space $\overline{(V^*)^*}$ [8, 14]:*

Comments 1. (1) If V and V' are vector spaces on $\mathbb{C}$, a mapping $f : V \rightarrow V'$ is called *anti-linear* or *conjugate linear* if it satisfies

$$f(\alpha u + \beta v) = \bar{\alpha}f(u) + \bar{\beta}f(v)$$

for all $u, v \in V$ and $\alpha, \beta \in \mathbb{C}$, $\bar{\alpha}$ denoting the complex conjugate of $\alpha \in \mathbb{C}$. If $V' = \mathbb{C}$ the given definition reduces to the definition of anti-linear functional.

(2) V^* , $\overline{V^*}$ and $\overline{V}$ turn out to be vector spaces on the field $\mathbb{K}$ when the composition rule of vectors and the product with elements of the field are defined in the usual way. For instance, if $f, g \in V^*$ or $\overline{V^*}$, and $\alpha \in \mathbb{K}$ then $f + g$ and αf are functions such that:

$$(f + g)(u) := f(u) + g(u)$$

and

$$(\alpha f)(u) := \alpha f(u)$$

for all $u \in V$.

Canonical Bases for Dual, Conjugate Dual, and Conjugate Spaces

Definition 1.1.2. Let V be a finite-dimensional vector space on either $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\{e_i\}_{i \in I}$ be a vector basis of V . The set $\{e^{*j}\}_{j \in I} \subset V^*$ whose elements are defined by

$$e^{*j}(e_i) := \delta_i^j$$

for all $i, j \in I$ is called the **dual basis** of $\{e_i\}_{i \in I}$.

Similarly, if $\mathbb{K} = \mathbb{C}$, the set $\{\overline{e^{*j}}\}_{j \in I} \subset \overline{V^*}$ defined by:

$$\overline{e^{*j}}(e_i) := \delta_i^j$$

for all $i, j \in I$ is called the **conjugate dual basis** of $\{e_i\}_{i \in I}$.

Finally the set of elements $\{\bar{e}_j\}_{j \in I} \subset \bar{V}$ defined by:

$$\bar{e}_p(e^{*q}) := \delta_q^p$$

for all $p, q \in I$ is called the **conjugate basis** of $\{e_i\}_{i \in I}$.

Remark 1.1.1. We collect several observations that contextualize the definitions above and motivate the restrictions imposed.

- (1) **Why finite dimensions?** The algebraic dual space V^* can be defined for any vector space, regardless of dimension. However, when $\dim V = \infty$, the dual space V^* is strictly larger than V in cardinality, and the natural map $V \rightarrow V^{**}$ fails to be surjective. Recovering the isomorphism $V \cong V^{**}$ then requires topological vector spaces, continuous functionals, and completeness conditions—ingredients foreign to the purely algebraic treatment pursued here. We therefore assume $\dim V < \infty$ throughout, noting that this covers most applications in differential geometry and quantum mechanics on finite-level systems.
- (2) **Functionals are determined by basis images.** Once a basis $\{e_i\}_{i \in I}$ is fixed, a linear (or conjugate-linear) functional f is completely known from the scalars $f_i := f(e_i)$. Indeed, expanding $v = \sum_i v^i e_i$ and applying linearity:

$$f(v) = f\left(\sum_i v^i e_i\right) = \sum_i v^i f(e_i) = \sum_i v^i f_i.$$

The coefficients v^i in the expansion of v are the same scalars that appear in the output of f ; no auxiliary constants enter.

- (3) **Verification .** That the systems $\{e^{*i}\}$, $\{\overline{e^{*i}}\}$, and $\{\bar{e}_i\}$ actually constitute bases—and not merely linearly independent sets—is demonstrated in Theorem 1.1.1 by explicit spanning and independence arguments.
- (4) **Connection to spinor calculus.** When V is a two-dimensional complex vector space, the spaces introduced above appear in Penrose’s two-spinor formalism

with standard names: elements of V are dotted contravariant spinors (upper primed indices); elements of V^* are dotted covariant spinors (lower primed indices); and elements of $\overline{V}^*$ are undotted contravariant spinors (upper unprimed indices). These identifications reflect the $\mathrm{SL}(2, \mathbb{C})$ representation theory underlying relativistic spinors; see [16, 14] for the full geometric development.

Concerning Definition 1.1.2 we have the next

Theorem 1.1.1. [14] *Let V be a finite-dimensional vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let $\{e_i\}_{i \in I}$ be a basis of V . Then:*

- (a) *The dual basis $\{e^{*i}\}_{i \in I}$ is a basis of V^* ;*
- (b) *If $\mathbb{K} = \mathbb{C}$, the conjugate dual basis $\{\overline{e^{*i}}\}_{i \in I}$ is a basis of $\overline{V}^*$;*
- (c) *The conjugate basis $\{\overline{e_i}\}_{i \in I}$ is a basis of $\overline{V}$.*

Consequently,

$$\dim V = \dim V^* = \dim \overline{V}^* = \dim \overline{V}.$$

Proof. We establish the three claims in turn, beginning with the dual basis.

(a) To verify that $\{e^{*i}\}_{i \in I}$ spans V^* , consider an arbitrary functional $f \in V^*$. The action of f on any vector $v = \sum_j v^j e_j$ is determined by the scalars $f_j := f(e_j)$, since linearity gives $f(v) = \sum_j v^j f_j$. Now form the linear combination $g := \sum_i f_i e^{*i}$ and evaluate it at basis elements:

$$g(e_k) = \sum_i f_i e^{*i}(e_k) = \sum_i f_i \delta_k^i = f_k = f(e_k).$$

Because two linear functionals agreeing on a basis must coincide, $g = f$. Thus every $f \in V^*$ is a linear combination of the e^{*i} .

For linear independence, suppose $\sum_k c_k e^{*k} = 0$ in V^* . Testing this functional on e_i yields $c_i = 0$ for each $i \in I$, since $e^{*k}(e_i) = \delta_i^k$. Hence the family is free.

(b) When $\mathbb{K} = \mathbb{C}$, the argument for $\{\overline{e^{*i}}\}$ proceeds identically, replacing the linearity condition with conjugate-linearity at each step.

(c) By definition $\bar{V} = (V^*)^*$, so the conjugate basis $\{\bar{e}_i\}$ is precisely the dual basis of V^* constructed from $\{e^{*i}\}$. Applying part (a) to V^* in place of V yields the claim.

The dimension equalities follow because each basis is indexed by the same set I . $\square$

Notation 1.1.1. *Henceforth, we adopt the following notational convention: whenever an index appears exactly twice in a term—once as a superscript and once as a subscript—summation over all admissible values of that index is implied without explicit indication. For instance,*

$$t_{ijkl}f^{rsil}$$

is understood to mean

$$\sum_{i,l} t_{ijkl}f^{rsil}.$$

The admissible range of each index is to be inferred from the context or specified explicitly when ambiguity might arise see [14].

Remark 1.1.2. *Iterating the dual space construction yields a chain $V, V^*, V^{**}, \dots$. The surprise is that for $\dim V < \infty$, the second dual V^{**} contains no new information: evaluation at vectors, $\text{ev} : V \rightarrow V^{**}$, is already an isomorphism.*

This phenomenon is special to finite dimensions. When $\dim V = \infty$, the canonical injection $V \hookrightarrow V^{**}$ is never surjective, and each successive dual introduces genuinely larger spaces. The finite-dimensional case is thus atypical—an algebraic artifact of dimension counting rather than a general property of vector spaces.

The evaluation isomorphism is privileged among isomorphisms $V \rightarrow W$ in that it requires no choices: no basis selection, no inner product, no additional structure on V whatsoever. Such natural isomorphisms are rare; most constructions (including $V \cong V^*$) demand extra data. Their systematic study belongs to category theory; see [12] for a functorial treatment.

Theorem 1.1.2. [12] *If V is a finite-dimensional vector space on the field $K = \mathbb{R}$ or $\mathbb{C}$, there is a natural isomorphism $F : V \rightarrow (V^*)^*$ given by*

$$(F(v))(u) := u(v),$$

for all $u \in V^*$ and $v \in V$.

Proof. Define $F : V \rightarrow (V^*)^*$ by evaluation: $(F(v))(u) := u(v)$ for $v \in V$ and $u \in V^*$.

First, $F(v)$ is indeed a linear functional on V^* , since for $\alpha, \beta \in \mathbb{K}$ and $u, u' \in V^*$:

$$(F(v))(\alpha u + \beta u') = (\alpha u + \beta u')(v) = \alpha u(v) + \beta u'(v) = \alpha(F(v))(u) + \beta(F(v))(u').$$

Linearity of F . For $\alpha, \beta \in \mathbb{K}$ and $v, w \in V$, we must verify $F(\alpha v + \beta w) = \alpha F(v) + \beta F(w)$. Evaluating both sides at arbitrary $u \in V^*$:

$$(F(\alpha v + \beta w))(u) = u(\alpha v + \beta w) = \alpha u(v) + \beta u(w) = \alpha(F(v))(u) + \beta(F(w))(u),$$

where the second equality uses the linearity of $u \in V^*$. Since u was arbitrary, the functionals coincide.

Injectivity. Suppose $F(v) = 0$ in $(V^*)^*$, meaning $u(v) = 0$ for all $u \in V^*$. If $v \neq 0$, extend $\{v\}$ to a basis of V and let u be the dual functional picking out the v -coordinate. Then $u(v) = 1 \neq 0$, a contradiction. Hence $v = 0$.

Surjectivity. Let $\varphi \in (V^*)^*$. Choose a basis $\{e_i\}$ of V with dual basis $\{e^{*i}\}$. Set $v_\varphi := \sum_i \varphi(e^{*i})e_i$. For any $u = \sum_j u_j e^{*j} \in V^*$:

$$(F(v_\varphi))(u) = u(v_\varphi) = \sum_j u_j e^{*j} \left(\sum_i \varphi(e^{*i})e_i \right) = \sum_i u_i \varphi(e^{*i}) = \varphi \left(\sum_i u_i e^{*i} \right) = \varphi(u).$$

Thus $F(v_\varphi) = \varphi$, proving that F is onto. Since $\dim V = \dim V^{**}$ finiteness ensures that injectivity implies surjectivity, but the explicit construction above provides a direct argument. $\square$

Definition 1.1.3. Let V be a finite-dimensional vector space on $K = \mathbb{C}$ or $\mathbb{R}$ with dual space V^* . The bilinear map $\langle \cdot, \cdot \rangle : V \times V^* \rightarrow K$ such that

$$\langle u, v \rangle := v(u)$$

for all $u \in V$, $v \in V^*$, is called **pairing**. [4].

Comment. Because of the theorem proven above, we may indifferently think $\langle u, v \rangle$ as representing either the action of $u \in (V^*)^*$ on $v \in V^*$ or the action of $v \in V^*$ on $u \in V$.

Notation 1.1.2. From now on $V \simeq W$ indicates that the vector spaces V and W are isomorphic under some natural isomorphism.

If the field of V and W is $\mathbb{C}$, $V \cong W$ indicates that there is a natural anti-isomorphism, i.e., an injective, surjective, anti-linear mapping $G : V \rightarrow W$ built up using the definition of V and $\overline{V}$ only.

1.2 Examples

Example 1.2.1. Show that if $v \in V$ then $v = \langle v, e^{*i} \rangle e_i$, where $\{e_i\}_{i \in I}$ is any basis of the finite-dimensional vector space V .

(Hint. Decompose $v = c^i e_i$, compute $\langle v, e^{*k} \rangle$ taking into account the linearity of the left entrance.)

account. Alternatively, show that $v - \langle v, e^{*j} \rangle e_j = 0$ proving that

$$f(v - \langle v, e^{*j} \rangle e_j) = 0$$

for every $f \in V^*$.

1.3 Multilinear Maps, Tensor Products, and the Universality Principle

Let us consider $n \geq 1$ vector spaces $V_1, \dots, V_n$ on the common field $K = \mathbb{R}$ or $\mathbb{C}$ and another vector space W on K . In the following $\mathcal{L}(V_1, \dots, V_n; W)$ denotes the vector space (on K) of multilinear maps from $V_1 \times \dots \times V_n$ to W . We recall that a mapping $f : V_1 \times \dots \times V_n \rightarrow W$ is said to be multilinear if, arbitrarily fixing $n - 1$ vectors $v_i \in V_i$ for $i \neq k$, each mapping

$$v_k \mapsto f(v_1, \dots, v_k, \dots, v_n)$$

is linear for every $k = 1, \dots, n$. We leave as a routine exercise the verification that $\mathcal{L}(V_1, \dots, V_n; W)$ is a vector space over K under the standard operations of pointwise addition and scalar multiplication. When $W = K$, we write $\mathcal{L}(V_1, \dots, V_n)$ in place of $\mathcal{L}(V_1, \dots, V_n; K)$, where K denotes the common underlying field $(V_1, \dots, V_n)$. [8, 14]:

Theorem 1.3.1. [12, 14]. *If $V_1, \dots, V_n$ are finite-dimensional vector spaces on the common field K ($= \mathbb{C}$ or $\mathbb{R}$), and W is another finite-dimensional vector space on K , the vector spaces $\mathcal{L}(V_1, \dots, V_n; W)$ and $\mathcal{L}(V_1, \dots, V_n, W^*)$ are naturally isomorphic by means of the map*

$$\Psi : \mathcal{L}(V_1, \dots, V_n; W) \rightarrow \mathcal{L}(V_1, \dots, V_n, W^*)$$

with $F : f \mapsto \Psi_f$ defined by

$$\Psi_f(v_1, \dots, v_n, w^*) := \langle f(v_1, \dots, v_n), w^* \rangle,$$

for all $(v_1, \dots, v_n) \in V_1 \times \dots \times V_n$ and $w^* \in W^*$.

Proof. Define $\Psi : \mathcal{L}(V_1, \dots, V_n; W) \rightarrow \mathcal{L}(V_1, \dots, V_n, W^*)$ by

$$\Psi_f(v_1, \dots, v_n, w^*) := w^*(f(v_1, \dots, v_n)),$$

where we have written the pairing as evaluation of the functional w^* .

Ψ_f is multilinear. Fixing all but one argument, the linearity in each v_k follows from the multilinearity of f , while linearity in w^* follows from the definition of addition and scalar multiplication in W^* .

Ψ is linear. For $f, g \in \mathcal{L}(V_1, \dots, V_n; W)$ and scalars α, β :

$$\begin{aligned} \Psi_{\alpha f + \beta g}(v_1, \dots, v_n, w^*) &= w^*((\alpha f + \beta g)(v_1, \dots, v_n)) \\ &= \alpha w^*(f(v_1, \dots, v_n)) + \beta w^*(g(v_1, \dots, v_n)) \\ &= (\alpha \Psi_f + \beta \Psi_g)(v_1, \dots, v_n, w^*). \end{aligned}$$

Injectivity. Assume $\Psi_f = \Psi_g$. Then for all arguments, $w^*(f(\mathbf{v})) = w^*(g(\mathbf{v}))$. If $f(\mathbf{v}) \neq g(\mathbf{v})$ for some $\mathbf{v}$, choose w^* separating these two vectors (possible by Hahn-Banach in general, or explicitly by extending to a basis in finite dimensions). This yields a contradiction, so $f = g$.

Surjectivity. Given $\Phi \in \mathcal{L}(V_1, \dots, V_n, W^*)$, we reconstruct f_Φ as follows. Select bases $\{e_{k,i}\}$ for each V_k and $\{e_j\}$ for W with dual basis $\{e^{*j}\}$. Define

$$f_\Phi(v_1, \dots, v_n) := \sum_j \Phi(e_{1,i_1}, \dots, e_{n,i_n}, e^{*j}) v_1^{i_1} \cdots v_n^{i_n} e_j,$$

where $v_k = \sum_{i_k} v_k^{i_k} e_{k,i_k}$. By construction f_Φ is multilinear, and verifying $\Psi_{f_\Phi} = \Phi$ reduces to checking agreement on basis elements, which follows from the duality pairing. $\square$

Definition 1.3.1. Let U and V two finite-dimensional vector space on the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We say that U **linearly acts on V by means of F** , if $F : U \rightarrow V^*$ is a natural isomorphism.

Examples 1.3.1. 1. If V is a finite-dimensional vector space over $K = \mathbb{R}$ or $\mathbb{C}$, then V^* acts linearly on V via $F = \text{id}_{V^*}$.

2. If V is as in (1), then V acts linearly on V^* by means of the natural isomorphism between V and $(V^*)^*$.

In the following, if U acts linearly on V by means of F , and $u \in U$, $v \in V$, we write $u(v)$ instead of $(F(u))(v)$ whenever F is unambiguously determined.

For instance, in the second example above, $v(u)$ means $\langle v, u \rangle$ when $v \in V$ and $u \in V^*$.

Definition 1.3.2. Let $U_1, \dots, U_n$ be $n \geq 1$ finite-dimensional vector spaces on the common field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, suppose each U_i linearly acts on V_i (by means of some F_i), for $i = 1, \dots, n$.

(1) if $(u_1, \dots, u_n) \in U_1 \times \dots \times U_n$, $u_1 \otimes \dots \otimes u_n$ denotes the multi linear mapping in

$\mathcal{L}(V_1, \dots, V_n)$ defined by

$$(u_1 \otimes \dots \otimes u_n)(v_1, \dots, v_n) := u_1(v_1) \cdots u_n(v_n),$$

for all $(v_1, \dots, v_n) \in V_1 \times \dots \times V_n$. $u_1 \otimes \dots \otimes u_n$ is called **tensor product of vectors** $u_1, \dots, u_n$.

(2) The mapping $\otimes : U_1 \times \dots \times U_n \rightarrow \mathcal{L}(V_1, \dots, V_n)$ given by: $\otimes : (u_1, \dots, u_n) \mapsto u_1 \otimes \dots \otimes u_n$, is called **tensor product map**.

(3) The vector subspace of $\mathcal{L}(V_1, \dots, V_n)$ generated by all of $u_1 \otimes \dots \otimes u_n$ for all $(u_1, \dots, u_n) \in U_1 \times \dots \times U_n$ is called **tensor product of spaces** $U_1, \dots, U_n$ and is indicated by $U_1 \otimes \dots \otimes U_n$. The vectors in $U_1 \otimes \dots \otimes U_n$ are called **tensors**.

Remark 1.3.1. We deal several consequences of Definition 1.3.2 that are used repeatedly in what follows.

- (1) The space $U_1 \otimes \dots \otimes U_n$ consists of finite sums $\sum_{j=1}^N \alpha_j u_{1,j} \otimes \dots \otimes u_{n,j}$ with $\alpha_j \in \mathbb{K}$ and $u_{k,j} \in U_k$. The integer N is not fixed; it varies with the tensor. Simple tensors correspond to $N = 1$, but most elements require $N > 1$.
- (2) The map $\otimes : U_1 \times \dots \times U_n \rightarrow \mathcal{L}(V_1, \dots, V_n)$ sending $(u_1, \dots, u_n)$ to $u_1 \otimes \dots \otimes u_n$ is multilinear. To verify this, fix $k \in \{1, \dots, n\}$ and vectors $u_k, v_k \in U_k$ with scalars $\alpha, \beta \in \mathbb{K}$. For any $(v_1, \dots, v_n) \in V_1 \times \dots \times V_n$:

$$\begin{aligned} & (u_1 \otimes \dots \otimes (\alpha u_k + \beta v_k) \otimes \dots \otimes u_n)(v_1, \dots, v_n) \\ &= u_1(v_1) \cdots (\alpha u_k + \beta v_k)(v_k) \cdots u_n(v_n) \\ &= \alpha u_1(v_1) \cdots u_k(v_k) \cdots u_n(v_n) + \beta u_1(v_1) \cdots v_k(v_k) \cdots u_n(v_n) \\ &= [\alpha(u_1 \otimes \dots \otimes u_k \otimes \dots \otimes u_n) + \beta(u_1 \otimes \dots \otimes v_k \otimes \dots \otimes u_n)](v_1, \dots, v_n). \end{aligned}$$

Since the arguments were arbitrary, the tensors agree. A useful special case is the balanced identity $(\alpha u) \otimes v = u \otimes (\alpha v) = \alpha(u \otimes v)$, valid for any two-factor product and extendable to any position in a longer product.

- (3) *Mixed tensor products such as $V_1^* \otimes V_2 \otimes V_3$ or $V_1 \otimes V_2^* \otimes V_3^*$ are unambiguous once the canonical actions from Definition 1.3.1 are understood: V acts on V^* via the natural isomorphism $V \cong (V^*)^*$, and V^* acts on V by evaluation. No additional choices are required.*

A foundational question remains: does $U_1 \otimes \cdots \otimes U_n$ exhaust $\mathcal{L}(V_1, \dots, V_n)$, or is it merely a proper subspace?

Referring to Definition 1.3.1, the next theorem holds

Theorem 1.3.2. *The following tensorial product yields*

$$U_1 \otimes \cdots \otimes U_n = \mathcal{L}(V_1, \dots, V_n).$$

Proof. We must show $U_1 \otimes \cdots \otimes U_n \supseteq \mathcal{L}(V_1, \dots, V_n)$, the reverse inclusion being immediate from the definition. Take $f \in \mathcal{L}(V_1, \dots, V_n)$.

Choose bases $\{e_{k,i}\}$ for V_k and let $\{e_k^{*i}\}$ be the dual bases. The scalars $f_{i_1, \dots, i_n} := f(e_{1,i_1}, \dots, e_{n,i_n})$ completely determine f by multilinearity: for $v_k = \sum_{i_k} v_k^{i_k} e_{k,i_k}$,

$$f(v_1, \dots, v_n) = \sum_{i_1, \dots, i_n} v_1^{i_1} \cdots v_n^{i_n} f_{i_1, \dots, i_n}.$$

Via the isomorphisms $F_k : U_k \rightarrow V_k^*$ from Definition 1.3.1, pull back the dual basis vectors to $\tilde{e}_k^i := F_k^{-1}(e_k^{*i}) \in U_k$. Consider the tensor

$$\mathcal{T}_f := \sum_{i_1, \dots, i_n} f_{i_1, \dots, i_n} \tilde{e}_1^{i_1} \otimes \cdots \otimes \tilde{e}_n^{i_n} \in U_1 \otimes \cdots \otimes U_n.$$

Evaluating $\mathcal{T}_f$ on $(v_1, \dots, v_n)$:

$$\begin{aligned}\mathcal{T}_f(v_1, \dots, v_n) &= \sum_{i_1, \dots, i_n} f_{i_1, \dots, i_n} \tilde{e}_1^{i_1}(v_1) \cdots \tilde{e}_n^{i_n}(v_n) \\ &= \sum_{i_1, \dots, i_n} f_{i_1, \dots, i_n} e_1^{*i_1}(v_1) \cdots e_n^{*i_n}(v_n) \\ &= \sum_{i_1, \dots, i_n} f_{i_1, \dots, i_n} v_1^{i_1} \cdots v_n^{i_n} \\ &= f(v_1, \dots, v_n).\end{aligned}$$

Thus $\mathcal{T}_f$ and f agree as multilinear maps, proving $f \in U_1 \otimes \cdots \otimes U_n$. $\square$

Another relevant result is stated by the theorem below from Definition 1.3.1

Theorem 1.3.3. *The following statements hold. (a) The dimension of $U_1 \otimes \cdots \otimes U_n$ is:*

$$\dim(U_1 \otimes \cdots \otimes U_n) = \prod_{k=1}^n \dim U_k = \prod_{k=1}^n \dim V_k.$$

(b) *If $\{e_{k,i_k}\}_{i_k \in I_k}$ is a basis of U_k , $k = 1, \dots, n$, then:*

$$\{e_{1,i_1} \otimes \cdots \otimes e_{n,i_n}\}_{(i_1, \dots, i_n) \in I_1 \times \cdots \times I_n}$$

is a vector basis of $U_1 \otimes \cdots \otimes U_n$.

(c) *If $\mathcal{T} \in U_1 \otimes \cdots \otimes U_n$, the components of $\mathcal{T}$ with respect to a basis $\{e_{1,i_1} \otimes \cdots \otimes e_{n,i_n}\}$ are given by:*

$$\mathcal{T}^{i_1 \cdots i_n} := t(e_1^{*i_1}, \dots, e_n^{*i_n}),$$

and thus it holds

$$\mathcal{T} = \mathcal{T}^{i_1 \cdots i_n} e_{1,i_1} \otimes \cdots \otimes e_{n,i_n}.$$

Proof. We prove (b) directly; statement (a) follows by counting, and (c) is a component computation.

Linear independence. Suppose $\sum_{i_1, \dots, i_n} C^{i_1 \cdots i_n} e_{1,i_1} \otimes \cdots \otimes e_{n,i_n} = 0$ in $U_1 \otimes \cdots \otimes U_n$. This tensor annihilates all arguments in $V_1 \times \cdots \times V_n$. In particular, evaluating on

$(e_1^{*j_1}, \dots, e_n^{*j_n})$ where e_k^{*j} is the dual basis element satisfying $e_k^{*j}(e_{k,i}) = \delta_i^j$:

$$0 = \sum_{i_1, \dots, i_n} C^{i_1 \dots i_n} e_{1,i_1}(e_1^{*j_1}) \cdots e_{n,i_n}(e_n^{*j_n}) = C^{j_1 \dots j_n}.$$

Since the indices $(j_1, \dots, j_n)$ were arbitrary, all coefficients vanish.

Spanning. Any tensor $\mathcal{T} \in U_1 \otimes \cdots \otimes U_n$ is a finite linear combination of simple tensors. Each simple tensor $u_1 \otimes \cdots \otimes u_n$ expands in the respective bases as $u_k = \sum_{i_k} u_k^{i_k} e_{k,i_k}$. By multilinearity of the tensor product map:

$$u_1 \otimes \cdots \otimes u_n = \sum_{i_1, \dots, i_n} u_1^{i_1} \cdots u_n^{i_n} e_{1,i_1} \otimes \cdots \otimes e_{n,i_n}.$$

Thus the basis elements span the entire tensor product space.

For (c), the components $\mathcal{T}^{i_1 \dots i_n} := \mathcal{T}(e_1^{*i_1}, \dots, e_n^{*i_n})$ satisfy

$$\mathcal{T}(v_1, \dots, v_n) = \sum_{i_1, \dots, i_n} \mathcal{T}^{i_1 \dots i_n} v_1^{i_1} \cdots v_n^{i_n},$$

which is precisely the evaluation of $\sum_{i_1, \dots, i_n} \mathcal{T}^{i_1 \dots i_n} e_{1,i_1} \otimes \cdots \otimes e_{n,i_n}$. Uniqueness follows from the linear independence established above. $\square$

There are two important theorems which, together with the identification of V and $(V^*)^*$, imply that all of the spaces which one may build up coherently using the symbols $\otimes$, V_k , * and $(\cdot)$ are naturally isomorphic to spaces which are of the form

$$V_1^{(*)} \otimes \cdots \otimes V_n^{(*)}.$$

The rule to produce spaces naturally isomorphic to a given initial space is that one has to: (1) ignore parentheses, (2) assume that * is distributive with respect to $\otimes$, and (3) assume that * is involutive (i.e. $(X^*)^* \simeq X$).

For instance, one has:

$$((V_1^* \otimes V_2) \otimes (V_3 \otimes V_4^*))^* \simeq V_1 \otimes V_2^* \otimes V_3^* \otimes V_4.$$

Let us state the theorems corresponding to the rules (2) and (1) respectively (the rule (3) being nothing but Theorem 1.1.2)

Theorem 1.3.4. *The $*$ is **distributive** with respect to $\otimes$ by means of **natural isomorphisms**. In other words, if $V_1, \dots, V_n$ are finite dimensional vector spaces on the same field $K = \mathbb{R}$ or $\mathbb{C}$, it holds:*

$$(V_1 \otimes \cdots \otimes V_n)^* \simeq V_1^* \otimes \cdots \otimes V_n^*.$$

Theorem 1.3.5. *The tensor product is **associative** up to **natural isomorphisms**. That is, for finite-dimensional vector spaces over the same field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), any parenthesization of a multiple tensor product yields a space naturally isomorphic to every other. In particular,*

$$(V_1 \otimes V_2) \otimes V_3 \simeq V_1 \otimes V_2 \otimes V_3$$

and

$$V_1 \otimes (V_2 \otimes V_3) \simeq V_1 \otimes V_2 \otimes V_3,$$

where $\simeq$ denotes a natural isomorphism.

Sketch of proof. The natural isomorphisms required in the two theorems above can be built up as follows.

In the former case, consider a linear mapping

$$F : V_1^* \otimes \cdots \otimes V_n^* \longrightarrow (V_1 \otimes \cdots \otimes V_n)^*$$

which satisfies

$$F : v_1^* \otimes \cdots \otimes v_n^* \mapsto [v_1 \otimes \cdots \otimes v_n]^* \in (V_1 \otimes \cdots \otimes V_n)^*,$$

where, for all $u_1 \otimes \cdots \otimes u_n \in V_1 \otimes \cdots \otimes V_n$,

$$[v_1 \otimes \cdots \otimes v_n]^*(u_1 \otimes \cdots \otimes u_n) := \langle u_1, v_1^* \rangle \cdots \langle u_n, v_n^* \rangle.$$

In the latter case, consider the first proposed example. The required natural isomorphism can be built up as a linear mapping

$$F_1 : V_1 \otimes V_2 \otimes V_3 \longrightarrow (V_1 \otimes V_2) \otimes V_3$$

such that

$$F_1 : v_1 \otimes v_2 \otimes v_3 \longmapsto (v_1 \otimes v_2) \otimes v_3,$$

for all $v_1 \otimes v_2 \otimes v_3 \in V_1 \otimes V_2 \otimes V_3$.

$$\dim((V_1 \otimes \cdots \otimes V_n)^*) = \dim(V_1 \otimes \cdots \otimes V_n) = \dim(V_1^* \otimes \cdots \otimes V_n^*),$$

and

$$\dim((V_1 \otimes V_2) \otimes V_3) = \dim(V_1 \otimes V_2) \cdot \dim(V_3) = \dim(V_1 \otimes V_2 \otimes V_3).$$

Theorem 1.3.6. (*Universality Theorem*) *Given $n > 1$ finite-dimensional vector spaces $U_1, \dots, U_n$ on the same field $K = \mathbb{R}$ or $\mathbb{C}$, the following statements hold.*

(a) *For each finite-dimensional vector space W and each multi-linear mapping*

$$f : U_1 \times \cdots \times U_n \longrightarrow W,$$

there is a unique linear mapping

$$f^\otimes : U_1 \otimes \cdots \otimes U_n \longrightarrow W$$

such that

$$f^\otimes(u_1 \otimes \cdots \otimes u_n) = f(u_1, \dots, u_n).$$

$$\begin{array}{ccc}
 U_1 \times \cdots \times U_n & \xrightarrow{\otimes} & U_1 \otimes \cdots \otimes U_n \\
 & \searrow f & \downarrow f^\otimes \\
 & & W
 \end{array}$$

(b) For $U_1, \dots, U_n$ fixed as above, suppose there is another pair (U_T, T) such that, for each W and each $f : U_1 \times \cdots \times U_n \rightarrow W$ multilinear, there is a unique $f^T : U_T \rightarrow W$ such that

$$f^T \otimes T = f.$$

$$\begin{array}{ccc}
 U_1 \times \cdots \times U_n & \xrightarrow{T} & U_T \\
 & \searrow f & \downarrow f^T \\
 & & W
 \end{array}$$

Then there is a vector space natural isomorphism

$$\phi : U_1 \otimes \cdots \otimes U_n \longrightarrow U_T$$

such that

$$f^T \otimes \phi = f^\otimes.$$

In other words, the pair $(\otimes, U_1 \otimes \cdots \otimes U_n)$ is determined up to vector space isomorphisms by the diagram property above.

Remark 1.3.2. The property of the pair $(\otimes, U_1 \otimes \cdots \otimes U_n)$ stated in the second part of the theorem is called **universality property**.

Before proving the theorem, let us explain how it gives a precise answer to the stated question.

The theorem says that a linear extension of any function f with values in W , initially defined on simple tensor products only,

$$f(v_1 \otimes \cdots \otimes v_n),$$

does exist on the whole domain space $U_1 \otimes \cdots \otimes U_n$ and it is uniquely determined

provided

$$f(v_1 \otimes \cdots \otimes v_n) = g(v_1, \dots, v_n),$$

where

$$g : U_1 \times \cdots \times U_n \longrightarrow W$$

is some multilinear function.

Concerning the mappings F and F_1 introduced above, we may profitably use the universality theorem to show that they are well-defined on the whole domain made of linear combinations of simple tensor products.

In fact, consider F for example. We can define the multi-linear mapping

$$G : V_1^* \times \cdots \times V_n^* \longrightarrow (V_1 \otimes \cdots \otimes V_n)^*$$

such that

$$G(v_1^*, \dots, v_n^*) := [v_1 \otimes \cdots \otimes v_n]^*.$$

Then the universality theorem with $U_k = V_k^*$ and $W = (V_1 \otimes \cdots \otimes V_n)^*$ assures the existence of a linear mapping

$$F : V_1^* \otimes \cdots \otimes V_n^* \longrightarrow (V_1 \otimes \cdots \otimes V_n)^*$$

with the required properties because $F = G^\otimes$ such that

$$F(v_1^* \otimes \cdots \otimes v_n^*) = (G^\otimes \otimes \otimes)(v_1^*, \dots, v_n^*) = G(v_1^*, \dots, v_n^*) = [v_1 \otimes \cdots \otimes v_n]^*.$$

A similar multi-linear mapping G_1 can be found for F_1 :

$$G_1 : (v_1, v_2, v_3) \longmapsto (v_1 \otimes v_2) \otimes v_3.$$

$$F_1 := G_1^\otimes$$

can be used in the proof of Theorem 1.3.5

Proof. (a) **Construction of $f^\otimes$.** Select bases $\{e_{k,i}\}$ for U_k and $\{e_j\}$ for W . Every multilinear $f : U_1 \times \cdots \times U_n \rightarrow W$ is uniquely specified by the coefficients

$$f_{i_1, \dots, i_n}^j := e^{*j}(f(e_{1,i_1}, \dots, e_{n,i_n})) \in \mathbb{K}.$$

Conversely, any array of scalars $\{g_{i_1, \dots, i_n}^j\}$ determines a unique linear map $g : U_1 \otimes \cdots \otimes U_n \rightarrow W$ via

$$g(e_{1,i_1} \otimes \cdots \otimes e_{n,i_n}) := \sum_j g_{i_1, \dots, i_n}^j e_j.$$

Define $f^\otimes$ to be the linear map corresponding to the coefficients $f_{i_1, \dots, i_n}^j$. Then on simple tensors:

$$f^\otimes(v_1 \otimes \cdots \otimes v_n) = \sum_{i_1, \dots, i_n} v_1^{i_1} \cdots v_n^{i_n} f^\otimes(e_{1,i_1} \otimes \cdots \otimes e_{n,i_n}) = f(v_1, \dots, v_n),$$

verifying the commutative diagram.

Uniqueness. If $g^\otimes$ also satisfies $g^\otimes \circ \otimes = f$, then $(f^\otimes - g^\otimes)$ vanishes on all simple tensors. Since simple tensors span the tensor product and the difference is linear, $f^\otimes - g^\otimes = 0$.

(b) **Uniqueness up to isomorphism.** Given two pairs $(U_1 \otimes \cdots \otimes U_n, \otimes)$ and (U_T, T) both satisfying the universal property, apply part (a) with $f = T$ to obtain $T^\otimes : U_1 \otimes \cdots \otimes U_n \rightarrow U_T$ with $T^\otimes \circ \otimes = T$. Similarly, swapping the roles yields $\otimes^T : U_T \rightarrow U_1 \otimes \cdots \otimes U_n$ with $\otimes^T \circ T = \otimes$.

Composing these:

$$(T^\otimes \circ \otimes^T) \circ T = T^\otimes \circ \otimes = T.$$

Since T maps into U_T and its image spans U_T (otherwise the universal property would fail for maps factoring through the span), the endomorphism $T^\otimes \circ \otimes^T$ fixes a spanning set, hence equals Id_{U_T} . The reverse composition is analogous, proving $\phi := T^\otimes$ is an isomorphism with inverse $\otimes^T$.

Finally, $f^T \circ \phi = f^\otimes$ follows from the uniqueness in the universal property applied to the map $f : U_1 \times \cdots \times U_n \rightarrow W$. □

Chapter 2

Tensor Algebras and the Abstract Index Formalism

Having established the foundational machinery of multilinear maps and tensor products through the universality theorem, we now turn to the systematic organization of these spaces. In this chapter, we assemble the tensor algebra generated by a vector space and introduce the abstract index notation, a powerful symbolic framework that encodes covariance and contravariance while preserving the invariance of physical laws under coordinate transformations [14].

2.1 Building Tensor Algebras from a Base Vector Space.

Let V be a finite-dimensional vector space on the field $K = \mathbb{R}$ or $\mathbb{C}$. Equipped with this algebraic structure, in principle we may build up several different tensor spaces.[14] Notice that we have to consider also K , K^* and V^* as admissible tensor factors [12].

First consider K^* when we consider K as a vector space on the field K itself. Each $c^* \in K^*$ has the form $c^*(k) = c \cdot k$ for all $k \in K$. Thus the mapping $c \mapsto c^*$ is a natural vector space isomorphism: $K \simeq K^*$.

Then we consider products $K \otimes \cdots \otimes K \simeq \mathcal{L}(K, \dots, K)$. Each multilinear mapping

is determined by $f(1, \dots, 1)$, hence $K \otimes \dots \otimes K \simeq K$.

We pass now to $K \otimes V \simeq \mathcal{L}(K, V^*) \simeq \mathcal{L}(K, V)$. Each multilinear functional f is determined by $f(1, \cdot) : V^* \rightarrow K$, thus we obtain a natural isomorphism: $K \otimes V \simeq V$.

Notice that this isomorphism satisfies $k \otimes v \mapsto kv$.

Concluding, only the spaces K, V, V^* and their tensor products may be essentially different.

2.2 The Algebraic Structure of Tensor Spaces

Definition 2.2.1. *Let V be a finite-dimensional vector space with field K .*

(1) *The tensor algebra $\mathcal{A}_K(V)$ generated by V is the class whose elements are the vector spaces K, V, V^* and all tensor products of factors V and V^* in any order.*[4]

(2) *Tensors of K are called **scalars**, tensors of V are called **contravariant vectors**, tensors of V^* are called **covariant vectors**.*

Tensors of spaces $V^{\otimes n} := V \otimes \dots \otimes V$ (n times) are called contravariant tensors of order n . Tensors of spaces $(V^*)^{\otimes n} := V^* \otimes \dots \otimes V^*$ are called covariant tensors of order n .

More generally tensors containing n factors V and m factors V^* are called tensors of order (n, m) .

Remark. Pairs of tensor spaces with the same number of factors V and V^* in different order are naturally isomorphic (see exercise 1.3.1), but we treat them as different spaces and use the identification when and if necessary.

2.3 Symbolic Tensor Calculus via Abstract Indices.

Let us introduce the abstract index notation.[14, 6] Consider a finite-dimensional vector space V with field $K = \mathbb{R}$ or $\mathbb{C}$. After specification of a basis $\{e_i\}_{i \in I} \subset V$ and

the corresponding basis in V^* , each tensor is completely determined by giving its components with respect to the induced basis in the corresponding tensor space in $A_k(V)$. [?] We are interested in the transformation rule of these components under change of the base in V . Suppose to fix another basis $\{e'_i\}_{i \in I} \subset V$ with

$$e_i = A_i^j e'_j.$$

The coefficients A_i^j determine a matrix $[A] := [A_i^j]$ in the matrix group $GL(\dim V, K)$.

First consider a contravariant vector $\mathcal{T} = \mathcal{T}^i e_i$, passing to the other basis, we have

$$\mathcal{T} = \mathcal{T}^i e_i = \mathcal{T}^i A_i^j e'_j = \mathcal{T}'^j e'_j,$$

and thus

$$\mathcal{T}'^j = A_i^j \mathcal{T}^i.$$

Similarly, if we specify a set of components $\{\mathcal{T}^i\}_{i \in I}$ for each basis $\{e_i\}_{i \in I} \subset V$ and these components, changing the basis to $\{e'_i\}_{i \in I}$, transform as

$$\mathcal{T}'^j = A_i^j \mathcal{T}^i,$$

then a contravariant tensor $\mathcal{T}$ is defined.

Concerning covariant vectors, a similar result holds. Indeed a covariant vector $u \in V^*$ is completely determined by the specification of a set of components $\{u_i\}_{i \in I}$ for each basis $\{e_i\}_{i \in I} \subset V$ and dual basis $\{e^i\}_{i \in I} \subset V^*$. When these components, changing the basis to $\{e'_i\}_{i \in I}$, transform as

$$u'_j = B_j^i u_i,$$

where the coefficients B_j^i are defined by

$$e'^i = B_j^i e^j.$$

What is the relation between the matrix $[A] = [A_i^j]$ and the matrix $[B] = [B_j^i]$? The

answer is clear:

$$\delta_j^i = \langle e_j, e^i \rangle = A_j^k B_k^i,$$

so that

$$I = A^T B,$$

which is equivalent to

$$B = (A^{-1})^T.$$

Proper tensors have components which transform similarly to the vector case. For instance, consider $\mathcal{T} \in V \otimes V^*$, fix a basis $\{e_i\}_{i \in I} \subset V$, the dual one $\{e^i\}_{i \in I} \subset V^*$ and consider that induced in $V \otimes V^*$, $\{e_i \otimes e^j\}_{(i,j) \in I \times I}$. Then

$$\mathcal{T} = \mathcal{T}_j^i e_i \otimes e^j.$$

Passing to another basis, the components transform as

$$\mathcal{T}_l^{ik} = A_i^k B_l^j \mathcal{T}_j^i.$$

It is clear that the specification of a tensor of $V \otimes V^*$ is completely equivalent to the specification of a set of components for each basis of $V \otimes V^*$, provided these components transform as specified above under change of basis.

Definition 2.3.1. [14] *Let $A_k(V)$ be the tensor algebra generated by the finite-dimensional vector space V on the field $K (= \mathbb{C} \text{ or } \mathbb{R})$. If $B = \{e_i\}_{i \in I}$ is a in V with dual basis $B^* = \{e^i\}_{i \in I} \subset V^*$, the canonical bases associated to the former are the bases in the tensor spaces of $A_k(V)$ obtained by tensor products of elements of B and B^* .*

Remark 2.3.1. *Notice that also $\{e_i \otimes e_j\}_{i,j \in I}$ is a basis of $V \otimes V$ if $\{e_i\}_{i \in I}$ and $\{e_j\}_{j \in I}$ are bases of V . However, $\{e_i \otimes e_j\}_{i,j \in I}$ is not canonical unless $e_i = e_j$ for all $i \in I$.*

Theorem 2.3.1. [14] *Consider the tensor algebra $A_k(V)$ generated by a finite-dimensional vector space V with field $K = \mathbb{R}$ or $\mathbb{C}$ and take a tensor space $V^{\otimes n} \otimes V^{*\otimes m} \subset A_k(V)$. The specification of a tensor $\mathcal{T} \in V^{\otimes n} \otimes V^{*\otimes m}$ is completely equivalent to the speci-*

cation of a set of components

$$\mathcal{T}_{j_1 \dots j_m}^{i_1 \dots i_n}$$

with respect to each canonical basis

$$\{e_{i_1} \otimes \dots \otimes e_{i_n} \otimes e^{j_1} \otimes \dots \otimes e^{j_m}\}.$$

which, under change of basis:

$$\{e'_{i_1} \otimes \dots \otimes e'_{i_n} \otimes e^{*j_1} \otimes \dots \otimes e^{*j_m}\}_{i_1, \dots, i_n, j_1, \dots, j_m \in I}$$

transform as:

$$\mathcal{T}_{j_1 \dots j_m}^{i_1 \dots i_n} = A_{k_1}^{i_1} \dots A_{k_n}^{i_n} B_{j_1}^{l_1} \dots B_{j_m}^{l_m} \mathcal{T}_{l_1 \dots l_m}^{k_1 \dots k_n}$$

where

$$e_i = A_i^j e'_j,$$

and the coefficients B_j^i are those of the matrix:

$$[B] = [A]^{-1t},$$

with $[A] := [A_i^j]$. The associated tensor t is represented by

$$\mathcal{T} = \mathcal{T}_{j_1 \dots j_m}^{i_1 \dots i_n} e_{i_1} \otimes \dots \otimes e_{i_n} \otimes e^{*j_1} \otimes \dots \otimes e^{*j_m}$$

for each considered canonical basis. Analogous results hold for tensor spaces whose factors V and V^* take different positions[8].

Notation 2.3.1. In the **abstract index notation** a tensor is indicated by writing its generic component in a non-specified basis. E.g. $t \in V^* \otimes V$ is indicated by $\mathcal{T}_i^j$.

Somewhere in the following we adopt a cumulative index notation, i.e., letters $A, B, C, \dots$ denote set of covariant, contravariant or mixed indices. For instance $\mathcal{T}_{lm}^{ijk}$ can be written as $\mathcal{T}^A$ with $A = {}^{ijk}_{lm}$. Similarly e_A denotes the element of a canonical

basis

$$e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes e^{*j_1} \otimes \cdots \otimes e^{*j_m}.$$

Moreover, if A and B are cumulative indices, the indices of the cumulative index AB are those of A immediately followed by those of B , e.g. if $A = {}^{ijk}_{lm}$ and $B = {}^{pq}_u$, then $AB = {}^{ijkpq}_{lm u}$.

Let us specify the allowed mathematical rules to produce tensors from given tensors. To this end we shall use both the synthetic and the index notation.

Linear combinations of tensors of a fixed tensor space. Take a tensor space $S \in A_k(V)$. This is a vector space by definition, and thus picking out $\mathcal{S}, \mathcal{T} \in S$, and $\alpha, \beta \in K$, linear combinations can be formed which still belong to S . In other words we may define the tensor $\mathcal{U} := \alpha\mathcal{S} + \beta\mathcal{T}$ or, in the abstract index notation $\mathcal{U}^A = \alpha\mathcal{S}^A + \beta\mathcal{T}^A$.

The definition of u above given by the abstract index notation means that the components of u are related to the components of s and t by a linear combination which has the same form in whatever canonical basis of the space S and the coefficients α, β do not depend on the basis.

Products of tensors of generally different tensor spaces. Take two tensor spaces $S, S' \in A_k(V)$ and pick out $\mathcal{T} \in S, \mathcal{T}' \in S'$. Then is well defined the tensor $\mathcal{T} \otimes \mathcal{T}' \in S \otimes S'$.

By the associativity of the tensor product up to natural isomorphisms, we find a unique tensor space $S'' \in \mathcal{A}_K(V)$ which is naturally isomorphic to $S \otimes S'$, and thus we may identify $\mathcal{T} \otimes \mathcal{T}'$ with a tensor in S'' , which we shall again denote by $\mathcal{T} \otimes \mathcal{T}'$ with a slight abuse of notation. The resulting tensor $\mathcal{T} \otimes \mathcal{T}'$ is called the **product of tensors** $\mathcal{T}$ and $\mathcal{T}'$. Therefore, the product of tensors coincides with the usual tensor product up to a natural isomorphism, and in the abstract index notation:

$$(\mathcal{T} \otimes \mathcal{T}')^{AB} = \mathcal{T}^A \mathcal{T}'^B,$$

so that, for instance, if $S = V \otimes V^*$ and $S' = V^*$, the tensors $\mathcal{T}$ and $\mathcal{S}$ are respectively indicated by $\mathcal{T}^i_j$ and $\mathcal{S}_k$, and thus $(\mathcal{T} \otimes \mathcal{S})^i_{jk} = \mathcal{T}^i_j \mathcal{S}_k$.

2.4 Contracting Tensors: The Universal Reduction

Consider a tensor space $A_k(V)$ of the form

$$U_1 \otimes \cdots \otimes U_k \otimes V \otimes U_{k+1} \otimes \cdots \otimes U_l \otimes V^* \otimes U_{l+1} \otimes \cdots \otimes U_n$$

where U_i denotes either V or V^* . Everything we are going to say can be re-stated for the analogous space

$$U_1 \otimes \cdots \otimes U_k \otimes V^* \otimes U_{k+1} \otimes \cdots \otimes U_l \otimes V \otimes U_{l+1} \otimes \cdots \otimes U_n.$$

Then consider the multi-linear mapping C with domain

$$U_1 \times \cdots \times U_k \times V \times U_{k+1} \times \cdots \times U_l \times V^* \times U_{l+1} \times \cdots \times U_n,$$

and values in $U_1 \otimes \cdots \otimes U_k \otimes U_{k+1} \otimes \cdots \otimes U_l \otimes U_{l+1} \otimes \cdots \otimes U_n$ defined by:

$$\begin{aligned} & (u_1, \dots, u_k, v, u_{k+1}, \dots, u_l, v^*, u_{l+1}, \dots, u_n) \\ & \mapsto \langle v, v^* \rangle u_1 \otimes \cdots \otimes u_k \otimes u_{k+1} \otimes \cdots \otimes u_l \otimes u_{l+1} \otimes \cdots \otimes u_n. \end{aligned}$$

By the universality theorem there is a linear mapping $C^\otimes$, called contraction of V and V^* , defined on the whole tensor space

$$U_1 \otimes \cdots \otimes U_k \otimes V \otimes U_{k+1} \otimes \cdots \otimes U_l \otimes V^* \otimes U_{l+1} \otimes \cdots \otimes U_n$$

taking values in $U_1 \otimes \cdots \otimes U_k \otimes U_{k+1} \otimes \cdots \otimes U_l \otimes U_{l+1} \otimes \cdots \otimes U_n$ such that, on simple products of vectors reduces to

$$\begin{aligned} & u_1 \otimes \cdots \otimes u_k \otimes v \otimes u_{k+1} \otimes \cdots \otimes u_l \otimes v^* \otimes u_{l+1} \otimes \cdots \otimes u_n \\ & \mapsto \langle v, v^* \rangle u_1 \otimes \cdots \otimes u_k \otimes u_{k+1} \otimes \cdots \otimes u_l \otimes u_{l+1} \otimes \cdots \otimes u_n. \end{aligned}$$

This linear mapping takes tensors in a tensor product space with $n + 2$ factors and

produces tensors in a space with n factors. The simplest case arises for $n = 0$. In that case $C : V \times V^* \rightarrow \mathbb{K}$ is nothing but the bilinear pairing

$$C : (v, v^*) \mapsto \langle v, v^* \rangle$$

and $C^\otimes$ is the linear associated mapping by the universality theorem.

Finally, let us represent the contraction mapping within the abstract index picture. It is quite simple to show that $C^\otimes$ takes a tensor $\mathcal{T}^{AiB}_j C$ where A, B and C are arbitrary cumulative indices, and produces the tensor $(C^\otimes \mathcal{T})^{ABC} := \mathcal{T}^{AkB}_k C$ where we remark the convention of summation of the double repeated index k . To show that the abstract-index representation of contractions is that above notice that the contractions are linear and thus

$$\begin{aligned} C^\otimes(\mathcal{T}^{AiB}_j C e_A \otimes e_i \otimes e_B \otimes e^{*j} \otimes e_C) &= \mathcal{T}^{AiB}_j C^\otimes(e_A \otimes e_i \otimes e_B \otimes e^{*j} \otimes e_C) \\ &= \mathcal{T}^{AiB}_j C^j_i e_A \otimes e_B \otimes e_C, \end{aligned}$$

$$\text{and thus } C^\otimes(\mathcal{T}^{AiB}_j C e_A \otimes e_i \otimes e_B \otimes e^{*j} \otimes e_C) = \mathcal{T}^{AkB}_k C e_A \otimes e_B \otimes e_C.$$

Which is : $(C^\otimes \mathcal{T})^{ABC} := \mathcal{T}^{AkB}_k C$. To conclude we pass to consider a final theorem which shows that there is a one-to-one correspondence between linear mappings on tensors and tensors themselves; see [14, 4]

Theorem 2.4.1. [14] *Let S, S' be a pair of tensor spaces of a tensor algebra $A_k(V)$. The vector space of linear mappings from S to S' is naturally isomorphic to*

$$S^* \otimes S'$$

(which it is naturally isomorphic to the corresponding tensor space of $A_k(V)$). The isomorphism $F : \mathcal{L}(S, S') \rightarrow S^* \otimes S'$ is given by $f \mapsto \mathcal{T}(f)$, where the tensor $\mathcal{T}(f) \in S^* \otimes S'$ is that (uniquely) determined by the requirement

$$\mathcal{T}(f)(s, s^*) = s^*(f(s)), \quad \text{for all } s \in S, s^* \in S^*.$$

Moreover, fixing a basis $\{e_i\}_{i \in I}$ in V , let $\{e_A\}$ denote the canonical basis induced in S , $\{e'_B\}$ that induced in S' and $\{e_C^*\}$ that induced in S^* . With those definitions

$$(\mathcal{T}(f))^C_A = \langle f(e_A), e^{*C}_C \rangle$$

and, if $s = s^A e_A \in S$, $f(s)^C = s^A (\mathcal{T}(f))^C_A$.

Proof. The identification proceeds in two stages. First, Theorem 1.3.1 provides a natural isomorphism $\mathcal{L}(S, S') \cong \mathcal{L}(S, S'^*)$. Second, the tensor product construction identifies $\mathcal{L}(S, S'^*) \cong S^* \otimes S'$. Composing these yields $F : \mathcal{L}(S, S') \rightarrow S^* \otimes S'$.

Explicitly, for $f \in \mathcal{L}(S, S')$, define $\mathcal{T}(f) \in S^* \otimes S'$ by requiring

$$\mathcal{T}(f)(s, s^*) = s^*(f(s)) \quad \text{for all } s \in S, s^* \in S'^*.$$

This condition uniquely determines $\mathcal{T}(f)$ because the pairing $(s, s^*) \mapsto s^*(f(s))$ is bilinear, and the universal property guarantees a unique linear functional on $S \otimes S'^* \cong (S^* \otimes S')^*$.

In components: let $\{e_A\}$ be a basis of S with dual $\{e^A\}$, and $\{e'_B\}$ a basis of S' with dual $\{e'^B\}$. Then

$$\mathcal{T}(f) = \sum_{A,B} \langle f(e_A), e'^B \rangle e^A \otimes e'_B,$$

so the components are $(\mathcal{T}(f))^B_A = e'^B(f(e_A))$. For $s = \sum_A s^A e_A$:

$$f(s) = \sum_A s^A f(e_A) = \sum_{A,B} s^A (\mathcal{T}(f))^B_A e'_B,$$

confirming that $f(s)^B = \sum_A s^A (\mathcal{T}(f))^B_A$. □

Let us illustrate how one may use that theorem. For instance consider a linear mapping

$$f : V \rightarrow V$$

where V is finite-dimensional with field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Then f defines a tensor of

$$V^* \otimes V.$$

In fact, fixing a basis $\{e_i\}_{i \in I}$ in V and considering those canonically associated, by the linearity of f and the pairing:

$$f(v) = \langle f(v), e^{*k} \rangle e_k = v^i \langle f(e_i), e^{*k} \rangle e_k.$$

We may define the tensor $\mathcal{T}(f) \in V^* \otimes V$ such that $\mathcal{T}(f) := \langle f(e_i), e^{*k} \rangle e^i \otimes e_k$. Employing Theorem 2.3.1 one can trivially show that the given definition is well posed. we omit the proof that the obtained maps which associate linear functions with tensors is the isomorphism of the theorem above.

The action of f on v can be represented in terms of $\mathcal{T}(f)$ and v using the rules presented above. Indeed, by the abstract index notation, one has $(f(v))^k = v^i \mathcal{T}(f)^k_i$. In other words the action of f on v reduces to:

1. A product of the involved tensors: $v^i \mathcal{T}(f)^j_i$
2. Followed by a convenient contraction: $(f(v))^k = v^i \mathcal{T}(f)^k_i$.

More complicated cases can be treated similarly. For example, linear mappings f from $V^* \otimes V \rightarrow V^* \otimes V \otimes V$ are determined by tensors $\mathcal{T}(f)^i{}_{jk}{}^{lm}$ of $V \otimes V^* \otimes V^* \otimes V \otimes V$ and their action on tensors u_q^p of $V^* \otimes V$ is $f(u)_k^{lm} = u_j^i \mathcal{T}(f)^j{}_{ik}{}^{lm}$, i.e., a product of tensors and two contractions. Obviously $\mathcal{T}(f)^i{}_{jk}{}^{lm} = (f(e^i \otimes e_j))(e_k, e^{*l}, e^{*m})$, where we have used the multi-linear action of elements of $V^* \otimes V \otimes V$ on elements of $V \times V^* \times V^*$.

2.5 Physical Laws and Tensorial Invariance

A key physical consequence of the tensor formalism developed above is the *basis independence* of tensor operations: linear combinations, tensor products, and contractions retain their symbolic structure regardless of the chosen basis. This invariance is precisely what renders the abstract index notation powerful. In physical terms, selecting

a basis corresponds to selecting an observer's reference frame, and the foundational postulates of relativity demand that the mathematical expressions of physical laws remain structurally identical across all admissible frames . Consequently, the tensor calculus provides the natural language for encoding such laws: by representing physical quantities as tensors and combining them through the invariant operations established herein, one automatically guarantees that the resulting equations are *covariant*—that is, unchanged in form under transformations between reference frames [6, 14]. This is why the theories of Special and General Relativity are inherently tensorial theories.

Chapter 3

From Abstract Tensors to Physical Applications

In this section we present a few applications of the theory previously developed we refer to [12, 4].

3.1 Group Representations on Tensor Product Spaces

As is known a **group** is an algebraic structure, $(G, \otimes)$, where G is a set and $\otimes : G \times G \rightarrow G$ is a mapping called the **composition rule** of the group. Moreover the following three conditions have to be satisfied.[4, 12]

1. $\otimes$ is **associative**, i.e., $g_1 \otimes (g_2 \otimes g_3) = (g_1 \otimes g_2) \otimes g_3$, for all $g_1, g_2, g_3 \in G$.
2. There is a **group unit**, i.e., there is $e \in G : e \otimes g = g \otimes e = g$, for all $g \in G$.
3. Each element $g \in G$ admits an **inverse element**, i.e., for each $g \in G$ there is $g^{-1} \in G$ with $g \otimes g^{-1} = g^{-1} \otimes g = e$.

We remind the unit element turns out to be unique and so does the inverse element for each element of the group. A group $(G, \otimes)$ is said to be **commutative** or **Abelian** if $g \otimes g' = g' \otimes g$ for each pair of elements, $g, g' \in G$; otherwise it is said to be **non-commutative** or **non-Abelian**. A subset $G' \subset G$ of a group is called **subgroup** if it is a group with respect to the restriction to $G' \times G'$ of the composition rule of G .

If $(G_1, \otimes_1)$ and $(G_2, \otimes_2)$ are groups, a (group) **homomorphism** from G_1 to G_2 is a mapping $h : G_1 \rightarrow G_2$ which **preserves the group structure**, i.e., the following requirement has to be fulfilled:

$$h(g \otimes_1 g') = h(g) \otimes_2 h(g') \quad \text{for all } g, g' \in G_1.$$

As a consequence of the uniqueness of the unit element and inverse element, it also holds with obvious notations: $h(e_1) = e_2$, and $h(g^{-1}) = (h(g))^{-1}$ for each $g \in G_1$.

A group **isomorphism** is a bijective group homomorphism.

A simple example of group is $GL(n, \mathbb{K})$ which is the set of the $n \times n$ matrices A with components in the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ and $\det A \neq 0$. The group composition rule is the usual product of matrices. If $n > 0$ the group is non-commutative. An important subgroup of $GL(n, \mathbb{K})$ is $SL(n, \mathbb{K})$, i.e., the set of matrices B in $GL(n, \mathbb{K})$ with $\det B = 1$ is a subgroup of $GL(n, \mathbb{K})$.) However there are groups which are not defined as group of matrices, e.g., $(\mathbb{Z}, +)$. A non completely trivial example is given by the **group of permutations** of n elements which we shall consider in the next subsection.

Example

Example 3.1.1. [4] Show that if $h : G_1 \rightarrow G_2$ is a group homomorphism, then $h(e_1) = e_2$ and $h(g^{-1}) = (h(g))^{-1}$ for each $g \in G_1$.

Definition 3.1.1. Let V a vector space, $GL(V)$ denotes the group of linear mappings $f : V \rightarrow V$ such that f is injective and surjective, with group composition rule given by the usual mappings composition. $GL(V)$ is called the **linear group on V** .

Remarks

- (a) If $V := K^n$, then $GL(V) = GL(n, K)$.
- (b) If $V \neq V'$, it is not possible to define the analogue of $GL(V)$ considering some subset of $\mathfrak{L}(V, V')$.

Definition 3.1.2. [8] Let $(G, \otimes)$ be a group and V a vector space. A (linear group) representation of G on V is a homomorphism $\xi : G \rightarrow GL(V)$. Moreover a representation $\xi : G \rightarrow GL(V)$ is called:

1. faithful if ξ is injective,
2. free if for each $v \in V$, there is no $g \in G$ with $g \neq e$ (the unit element of G) and $\xi(g)v = v$,
3. transitive if for each pair $v, v' \in V$ there is $g \in G$ with $v' = \xi(g)v$,
4. irreducible if there is no proper vector subspace $S \subset V$ which is invariant under the action of $\xi(G)$, i.e. which satisfies $\xi(g)S \subset S$ for all $g \in G$.

Equipped with the above-given definitions we are able to study the simplest interplay of tensors and group representations. We want to show that the notion of tensor product allows the definitions of tensor products of representations. That mathematical object is of fundamental importance in applications to Quantum Mechanics, in particular as far as systems with many components are concerned. Consider a group G (from now on we omit to specify the symbol of the composition rule whenever it does not produce misunderstandings) and several representations of the group $\xi_i : G \rightarrow GL(V_i)$, where $V_1, \dots, V_n$ are finite-dimensional vector spaces on the same field $K = \mathbb{C}$ or $\mathbb{R}$. For each $g \in G$, we define a multi-linear mapping $[\xi_1(g), \dots, \xi_n(g)] \in \mathcal{L}(V_1, \dots, V_n; V_1 \otimes \dots \otimes V_n)$ given by, for all $(v_1, \dots, v_n) \in V_1 \times \dots \times V_n$,

$$[\xi_1(g), \dots, \xi_n(g)] : (v_1, \dots, v_n) \mapsto (\xi_1(g)v_1) \otimes \dots \otimes (\xi_n(g)v_n).$$

That mapping is multi linear because of the multi linearity of the tensor-product mapping and the linearity of the operators $\xi_i(g)$. Using the universality theorem, we uniquely find a linear mapping which we indicate by

$$\xi_1(g) \otimes \dots \otimes \xi_n(g) : V_1 \otimes \dots \otimes V_n \rightarrow V_1 \otimes \dots \otimes V_n$$

such that

$$\xi_1(g) \otimes \cdots \otimes \xi_n(g)(v_1 \otimes \cdots \otimes v_n) = (\xi_1(g)v_1) \otimes \cdots \otimes (\xi_n(g)v_n).$$

.

Definition 3.1.3. Let $V_1, \dots, V_n$ be finite-dimensional vector spaces on the same field $K = \mathbb{C}$ or $\mathbb{R}$ and suppose there are n representations $\xi_i : G \rightarrow GL(V_i)$ of the same group G on the given vector spaces. The set of linear maps

$$\{\xi_1(g) \otimes \cdots \otimes \xi_n(g) : V_1 \otimes \cdots \otimes V_n \rightarrow V_1 \otimes \cdots \otimes V_n \mid g \in G\},$$

defined above is called **tensor product of representations** $\xi_1, \dots, \xi_n$.

The relevance of the definition above is evident because of the following theorem [8].

Referring to the definition above, the elements of a tensor product of representations is given in the next theorem

Theorem 3.1.1. The representations $\xi_1, \dots, \xi_n$ of the group G on the spaces $V_1, \dots, V_n$ define $\xi_1, \dots, \xi_n$ of the group G on spaces $V_1, \dots, V_n$ define a linear group representation of G on the tensor-product space $V_1 \otimes \cdots \otimes V_n$.

Proof. We verify the homomorphism property directly. For $g, h \in G$ and simple tensors $v_1 \otimes \cdots \otimes v_n$:

$$\begin{aligned} & (\xi_1(h) \otimes \cdots \otimes \xi_n(h)) \circ (\xi_1(g) \otimes \cdots \otimes \xi_n(g))(v_1 \otimes \cdots \otimes v_n) \\ &= (\xi_1(h) \otimes \cdots \otimes \xi_n(h))(\xi_1(g)v_1 \otimes \cdots \otimes \xi_n(g)v_n) \\ &= \xi_1(h)\xi_1(g)v_1 \otimes \cdots \otimes \xi_n(h)\xi_n(g)v_n \\ &= \xi_1(hg)v_1 \otimes \cdots \otimes \xi_n(hg)v_n \\ &= (\xi_1(hg) \otimes \cdots \otimes \xi_n(hg))(v_1 \otimes \cdots \otimes v_n), \end{aligned}$$

where the third equality uses that each ξ_k is a group representation. By linearity, this identity extends to all of $V_1 \otimes \cdots \otimes V_n$.

Each $\xi_1(g) \otimes \cdots \otimes \xi_n(g)$ is invertible with inverse $\xi_1(g^{-1}) \otimes \cdots \otimes \xi_n(g^{-1})$, as verified by the composition above with $h = g^{-1}$. Hence the image lies in $GL(V_1 \otimes \cdots \otimes V_n)$. $\square$

More generally, if $A_k : V_k \rightarrow U_k$ are n linear mappings (operators), and all involved vector spaces are finite dimensional and with the same field $K = \mathbb{C}$ or $\mathbb{R}$, it is defined the tensor product of operators.

Definition 3.1.4. Let $A_k : V_k \rightarrow U_k$, $k = 1, \dots, n$ are n linear mappings (operators), and all the vector spaces U_i, V_i are finite dimensional with the same field $K = \mathbb{C}$ or $\mathbb{R}$, the tensor product of $A_1, \dots, A_n$ is the linear mapping $A_1 \otimes \cdots \otimes A_n : V_1 \otimes \cdots \otimes V_n \rightarrow U_1 \otimes \cdots \otimes U_n$ uniquely determined by the universality theorem and the requirement:

$$(A_1 \otimes \cdots \otimes A_n) \otimes \otimes = A_1 \times \cdots \times A_n,$$

where

$$A_1 \times \cdots \times A_n : V_1 \times \cdots \times V_n \rightarrow U_1 \otimes \cdots \otimes U_n$$

is the multi-linear mapping such that, for all $(v_1, \dots, v_n) \in V_1 \times \cdots \times V_n$,

$$A_1 \times \cdots \times A_n : (v_1, \dots, v_n) \mapsto (A_1 v_1) \otimes \cdots \otimes (A_n v_n).$$

Remark 3.1.1. Employing the given definition and the same proof used for the relevant part of the proof of Theorem 3.1, it is simply shown that if

$$A_i : V_i \rightarrow U_i, \quad B_i : U_i \rightarrow W_i, \quad i = 1, \dots, n$$

are $2n$ linear maps and all involved spaces V_i, U_i, W_i are finite dimensional with the same field $K = \mathbb{C}$ or $\mathbb{R}$, then $(B_1 \otimes \cdots \otimes B_n)(A_1 \otimes \cdots \otimes A_n) = (B_1 A_1) \otimes \cdots \otimes (B_n A_n)$.

3.2 Tensor Products in Composite Quantum Systems

Physicists are involved with Hilbert spaces whenever they handle quantum mechanics. A Hilbert space is nothing but a complex vector space equipped with a Hermitean

scalar product[6]. (see the next chapter) such that it is complete with respect to the norm topology induced by that scalar product.

As far as we are concerned we need only the structure of vector space. Physically speaking, the vectors of the Hilbert space represent the states of the considered physical system (actually things are more complicated but we do not mind that). To consider the simplest case we assume that the vector space which describes the states of the system is finite-dimensional (that is the case for the spin part of a quantum particle). [6]

Moreover, see [6, 14], physics implies that the space $\mathcal{H}$ of the states of a composite system S made of two systems S_1 and S_2 is associated with Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, is the tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$.

There should be several remarks in the infinite dimensional case since our definition of tensor product works for finite-dimensional spaces only, however suitable and well-behaved generalizations are, in fact, possible.

Let the system S_1 be described by a state $\psi \in \mathcal{H}_1$ and suppose to transform the system by the action of an element R of some physical group of transformations G (e.g. $SO(3)$).

The transformed state ψ' is given by $U_R^{(1)}\psi$ where $G \ni R \mapsto U_R^{(1)}$ is a representation of G in terms of linear transformations $U_R^{(1)} : \mathcal{H}_1 \rightarrow \mathcal{H}_1$.

Actually, physics and the celebrated Wigner's theorem in particular, requires that every $U_R^{(1)}$ be a unitary (or anti unitary) transformation but this is not relevant for our case.

The natural question concerning the representation of the action of G on the composite system S is:

If we know the representations $G \ni R \mapsto U_R^{(1)}$ and $G \ni R \mapsto U_R^{(2)}$, what about the representation of the action of G on S in terms of linear transformations in the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$? The answer given by physics, at least when the systems S_1 and S_2 do not interact, is that $U_g = U_g^{(2)} \otimes U_g^{(1)}$, for more details see [12, 4].

3.3 Symmetric and Antisymmetric Tensors under Permutation

We remind the definition of the group of permutations of n objects and give some known results of basic group theory whose proofs may be found in any group-theory textbook. [4, 12].

Definition 3.3.1. (*Group of permutations*) Consider the set $I_n := \{1, \dots, n\}$, the group of permutations of n objects, P_n is the set of the bijective mappings $\sigma : I_n \rightarrow I_n$ equipped with the composition rule given by the usual composition rule of functions [12].

Moreover,

- (a) the elements of P_n are called permutations (of n objects);
- (b) a permutation of P_n with $n \geq 2$ is said to be a transposition if it differs from the identity mapping and reduces to the identity mapping when restricted to some subset of I_n containing $n - 2$ elements.[8, ?].

Theorem 3.3.1. see [8]. The above-defined linear $\sigma \mapsto \sigma^\otimes$ with $\sigma \in P_n$, is a group representation of P_n on $V^{\otimes n}$.

Proof. For $\sigma, \tau \in P_n$, we check compatibility on basis elements $e_{i_1} \otimes \dots \otimes e_{i_n}$:

$$\begin{aligned} \sigma^\otimes(\tau^\otimes(e_{i_1} \otimes \dots \otimes e_{i_n})) &= \sigma^\otimes(e_{i_{\tau(1)}} \otimes \dots \otimes e_{i_{\tau(n)}}) \\ &= e_{i_{\tau(\sigma(1))}} \otimes \dots \otimes e_{i_{\tau(\sigma(n))}} \\ &= (\tau \circ \sigma)^\otimes(e_{i_1} \otimes \dots \otimes e_{i_n}), \end{aligned}$$

where the indices are permuted according to the composition $\tau \circ \sigma$. Note that the action on the position indices yields the opposite composition order compared to the abstract group operation; this is the standard right-action convention.

By linear extension, $\sigma^\otimes \circ \tau^\otimes = (\tau \circ \sigma)^\otimes$ on all of $V^{\otimes n}$. Taking $\tau = \sigma^{-1}$ shows each $\sigma^\otimes$ is invertible, hence $\sigma^\otimes \in GL(V^{\otimes n})$. □

Theorem 3.3.2 ([14]). *Let V be a finite-dimensional vector space over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let $V^{\otimes n}$ denote its n -th tensor power. If $\mathcal{T} \in V^{\otimes n}$ with $n \geq 2$ and $\sigma \in \mathcal{P}_n$, then the components of $\sigma^{\otimes} \mathcal{T}$ with respect to any canonical basis of $V^{\otimes n}$ are given by*

$$(\sigma^{\otimes} \mathcal{T})^{i_1 \dots i_n} = \mathcal{T}^{i_{\sigma^{-1}(1)} \dots i_{\sigma^{-1}(n)}}.$$

Proof. Write $\mathcal{T}$ in components relative to the canonical basis: $\mathcal{T} = \sum_{j_1, \dots, j_n} \mathcal{T}^{j_1 \dots j_n} e_{j_1} \otimes \dots \otimes e_{j_n}$. Applying $\sigma^{\otimes}$:

$$\sigma^{\otimes} \mathcal{T} = \sum_{j_1, \dots, j_n} \mathcal{T}^{j_1 \dots j_n} e_{j_{\sigma(1)}} \otimes \dots \otimes e_{j_{\sigma(n)}}.$$

Re-index the sum by setting $i_k := j_{\sigma(k)}$, which implies $j_k = i_{\sigma^{-1}(k)}$ since σ is bijective. Substituting:

$$\sigma^{\otimes} \mathcal{T} = \sum_{i_1, \dots, i_n} \mathcal{T}^{i_{\sigma^{-1}(1)} \dots i_{\sigma^{-1}(n)}} e_{i_1} \otimes \dots \otimes e_{i_n}.$$

Reading off the coefficient of $e_{i_1} \otimes \dots \otimes e_{i_n}$ yields the claimed formula. $\square$

To conclude we introduce the concept of symmetric or anti-symmetric tensor.

Definition 3.3.2. *Let V be a finite-dimensional vector space over a field $K = \mathbb{C}$ or $\mathbb{R}$. Consider the space $V^{\otimes n} \in A_X^n(V)$ of tensors of order $(n, 0)$ with $n \geq 2$.*

(a) $\mathcal{T} \in V^{\otimes n}$ is said to be symmetric if

$$\sigma^{\otimes} \mathcal{T} = \mathcal{T},$$

for all $\sigma \in \mathcal{P}_n$, or equivalently,

$$\mathcal{T}^{i_1 \dots i_n} = \mathcal{T}^{i_{\sigma(1)} \dots i_{\sigma(n)}}.$$

(b) $\mathcal{T} \in V^{\otimes n}$ is said to be anti-symmetric if

$$\sigma^{\otimes} \mathcal{T} = \epsilon_{\sigma} \mathcal{T},$$

for all $\sigma \in \mathcal{P}_n$, or equivalently,

$$\mathcal{T}^{i_1 \dots i_n} = \epsilon_\sigma \mathcal{T}^{i_{\sigma(1)} \dots i_{\sigma(n)}}.$$

Remark 3.3.1. Concerning the definition in (b) notice that $\epsilon_\sigma = \epsilon_{\sigma^{-1}}$.

Examples

Example 3.3.1 ([?]). Suppose $n = 2$, then a symmetric tensor $s \in V \otimes V$ satisfies $s^{ij} = s^{ji}$ and an antisymmetric tensor $a \in V \otimes V$ satisfies $a^{ij} = -a^{ji}$.

Example 3.3.2 ([?]). Suppose $n = 3$, then it is trivially shown that $\sigma \in \mathcal{P}_3$ has parity 1 if and only if σ is a cyclic permutation, i.e. $(\sigma(1), \sigma(2), \sigma(3)) = (1, 2, 3)$ or $(2, 3, 1)$ or $(3, 1, 2)$.

Hence, a tensor $e \in V \otimes V \otimes V$ is antisymmetric if and only if $e^{ijk} = 0$ if (i, j, k) is not a permutation of $(1, 2, 3)$. and otherwise, $e^{ijk} = \pm e^{123}$. where the sign $+$ takes place if the permutation

$$(\sigma(1), \sigma(2), \sigma(3)) = (i, j, k)$$

is cyclic and the sign $-$ takes place otherwise. That relation between parity of a permutation and cyclicity does not hold true for $n > 3$.

Remark 3.3.2. Let $S \in \mathcal{A}_{\mathbb{K}}(V)$ be a tensor space containing $n \geq 2$ factors equal to V . For simplicity, write

$$S = S_1 \otimes V^{\otimes n} \otimes S_2,$$

where $S_1 = U_1 \otimes \dots \otimes U_k$ and $S_2 = U_{k+1} \otimes \dots \otimes U_m$ with each $U_i \in \{V, V^*\}$. The same construction applies when the n copies of V are not contiguous.

Define the multilinear map

$$\hat{\sigma} : U_1 \times \dots \times U_k \times V^n \times U_{k+1} \times \dots \times U_m \longrightarrow S$$

by

$$\hat{\sigma}(u_1, \dots, u_k, v_1, \dots, v_n, u_{k+1}, \dots, u_m) := u_1 \otimes \dots \otimes u_k \otimes v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)} \otimes u_{k+1} \otimes \dots \otimes u_m.$$

By the universality theorem, $\hat{\sigma}$ induces a linear map $\sigma^{\otimes} \in GL(S)$ acting trivially on S_1 and S_2 . In abstract index notation:

$$\sigma^{\otimes} : \mathcal{T}^{A i_1 \dots i_n B} \longmapsto \mathcal{T}^{A i_{\sigma^{-1}(1)} \dots i_{\sigma^{-1}(n)} B}.$$

This permits the definition of partial symmetry: a tensor $\mathcal{T}^{ij}_k{}^r$ may be symmetric or anti-symmetric in any subset of indices of the same variance, e.g. in $\{i, j\}$, $\{j, r\}$, or $\{i, j, r\}$.

Remark 3.3.3. All constructions above apply equally to purely covariant tensors $\mathcal{T}_{i_1 \dots i_n} \in (V^*)^{\otimes n}$. In this case, anti-symmetric tensors of order $(0, n)$ are called n -forms.

Remark 3.3.4. No symmetry condition can be imposed on a pair of indices of different variance (one upper, one lower), as the permutation group $\mathcal{P}_n$ acts only within tensor powers of a single space V or V^* .

Example

Example 3.3.3. Consider a tensor $\mathcal{T}^{i_1 \dots i_n}$, show that the tensor is symmetric if and only if it is symmetric with respect to each arbitrary chosen pair of indices, i.e. $\mathcal{T}^{\dots i_k \dots i_p \dots} = \mathcal{T}^{\dots i_p \dots i_k \dots}$, for all $p, k \in \{1, \dots, n\}$, $p \neq k$.

Chapter 4

Scalar Products and the Geometry of Tensor Spaces

This section concerns the introduction of the notion of scalar product and several applications on tensors we refer to [6, 13]

4.1 Scalar products

First of all we give the definition of a *pseudo scalar product* and *semi scalar products* which differ from the notion of *scalar product* for the positivity and the non-degenerateness requirement see [6, 13]. In fact, a pseudo scalar product is a generalization of the usual definition of scalar product which has many applications in mathematical physics, relativistic theories in particular as in [6]. Semi scalar products are used in several applications of quantum field theory also in [?]

Definition 4.1.1. [6] Let V be a vector space on the field either $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

(a) A pseudo scalar product is a mapping $\langle . | . \rangle : V \times V \rightarrow \mathbb{K}$ which is:

(i) bi linear, i.e., for all $u \in V$ both $\langle u | . \rangle : v \mapsto \langle u | v \rangle$ and $\langle . | u \rangle : v \mapsto \langle v | u \rangle$ are linear functionals on V ;

(ii) symmetric, i.e., $\langle u | v \rangle = \langle v | u \rangle$ for all $u, v \in V$;

(iii) non-degenerate, i.e., $\langle u|v \rangle = 0$ for all $v \in V$ implies $u = 0$.

(b) If $\mathbb{K} = \mathbb{C}$, a Hermitian pseudo scalar product is a mapping $\langle .|. \rangle : V \times V \rightarrow \mathbb{K}$ which is:

(i) sesquilinear, i.e., for all $u \in V$, $\langle u|. \rangle$ and $\langle .|u \rangle$ are a linear functional and an anti-linear functional on V respectively;

(ii) Hermitian, i.e., $\langle u|v \rangle = \overline{\langle v|u \rangle}$ for all $u, v \in V$;

(iii) non-degenerate.

Definition 4.1.2. [6, 13]. Let V be a vector space on the field either $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

(a) If $\mathbb{K} = \mathbb{R}$, a semi scalar product is a mapping $\langle .|. \rangle : V \times V \rightarrow \mathbb{R}$ which satisfies (ai), (aii) above and is

(iv) semi-defined positive, i.e., $\langle u|u \rangle \geq 0$ for all $u \in V$.

(b) If $\mathbb{K} = \mathbb{C}$, a Hermitian semi scalar product is a mapping $\langle .|. \rangle : V \times V \rightarrow \mathbb{K}$ which satisfies (bi), (bii) above and is

(iv) semi-defined positive.

Finally we give the definition of scalar product.

Definition 4.1.3. Let V be a vector space on the field $\mathbb{K} = \mathbb{R}$ (resp. $\mathbb{C}$) endowed with a pseudo scalar product (resp. Hermitian pseudo scalar product) $\langle .|. \rangle$. Then $\langle .|. \rangle$ is called **scalar product** (resp. **Hermitian scalar product**) if $\langle .|. \rangle$ is also a semi scalar product, i.e., if it is semi-defined positive[6].

Example 4.1.1. Clearly we have $\langle .|. \rangle$ is a (Hermitian) semi scalar product on V with field $\mathbb{R}$ ($\mathbb{C}$) then the mapping on V , $v \mapsto \|v\| := \sqrt{\langle v|v \rangle}$, satisfies $\|u+v\| \leq \|u\| + \|v\|$ as a consequence of Cauchy-Schwarz' inequality $|\langle u|v \rangle|^2 \leq \langle u|u \rangle \langle v|v \rangle$, which holds true by all (Hermitian) semi scalar product.

Theorem 4.1.1. *Let V be a vector space with field $\mathbb{R}$ ($\mathbb{C}$) equipped with a (Hermitian) semi scalar product $\langle \cdot | \cdot \rangle$. Then, for all $u, v \in V$, Cauchy-Schwarz' inequality holds:*

$$|\langle u | v \rangle|^2 \leq \langle u | u \rangle \langle v | v \rangle.$$

Proof. We treat the complex Hermitian case; the real case follows by omitting conjugations.

Fix $u, v \in V$. For arbitrary $z \in \mathbb{C}$, semi-definiteness requires:

$$0 \leq \langle zu + v, zu + v \rangle.$$

Expanding via sesquilinearity (linear in the second argument, conjugate-linear in the first):

$$\begin{aligned} \langle zu + v, zu + v \rangle &= \langle zu, zu \rangle + \langle zu, v \rangle + \langle v, zu \rangle + \langle v, v \rangle \\ &= |z|^2 \langle u, u \rangle + \bar{z} \langle u, v \rangle + z \langle v, u \rangle + \langle v, v \rangle \\ &= |z|^2 \langle u, u \rangle + 2 \operatorname{Re}(\bar{z} \langle u, v \rangle) + \langle v, v \rangle, \end{aligned}$$

where we used $\langle v, u \rangle = \overline{\langle u, v \rangle}$.

If $\langle u, u \rangle = 0$, then $2 \operatorname{Re}(\bar{z} \langle u, v \rangle) + \langle v, v \rangle \geq 0$ for all z . Varying z over $\mathbb{C}$ forces $\langle u, v \rangle = 0$, and the inequality holds trivially.

Assume $\langle u, u \rangle > 0$. Write $z = re^{i\theta}$ and $\langle u, v \rangle = |\langle u, v \rangle| e^{i\phi}$. Choose $\theta = \phi$ to align the phases, maximizing the real part:

$$0 \leq r^2 \langle u, u \rangle + 2r |\langle u, v \rangle| + \langle v, v \rangle \quad \text{for all } r \in \mathbb{R}.$$

This quadratic in r is non-negative everywhere, so its discriminant must be non-positive:

$$(2|\langle u, v \rangle|)^2 - 4\langle u, u \rangle \langle v, v \rangle \leq 0.$$

Dividing by 4 yields the desired inequality. Equality holds precisely when the quadratic has a real root, i.e., when $zu + v = 0$ for some $z \in \mathbb{C}$. $\square$

Then we pass to the polar representation of z , $z = re^{i\alpha}$ with $r, \alpha \in \mathbb{R}$ arbitrarily and independently fixed. Decompose also $\langle u|v \rangle$, $\langle u|v \rangle = |\langle u|v \rangle| e^{i \arg \langle u|v \rangle}$. we get: $F(r, \alpha) := r^2 \langle u|u \rangle + 2r |\langle u|v \rangle| \operatorname{Re}(e^{i(\arg \langle u|v \rangle - \alpha)}) + \langle u|u \rangle \geq 0$, for all $r \in \mathbb{R}$ when $\alpha \in \mathbb{R}$ is fixed arbitrarily. Since the right-hand side above is a second-order polynomial in r , the inequality implies that for all $\alpha \in \mathbb{R}$,

$$\{2|\langle u|v \rangle| \operatorname{Re}[e^{i(\arg \langle u|v \rangle - \alpha)}]\}^2 - 4 \langle u|u \rangle \langle v|v \rangle \leq 0,$$

which is equivalent to $|\langle u|v \rangle|^2 \cos^2(\arg \langle u|v \rangle - \alpha) - \langle u|u \rangle \langle v|v \rangle \leq 0$.

For all $\alpha \in \mathbb{R}$. Choosing $\alpha = \arg \langle u|v \rangle$, we get Cauchy-Schwarz' inequality:

$$|\langle u|v \rangle|^2 \leq \langle u|u \rangle \langle v|v \rangle.$$

The real case can be treated similarly, replacing $z \in \mathbb{C}$ with $x \in \mathbb{R}$ and the proof is essentially the same. $\square$

Corollary 4.1.1. *A bilinear symmetric (resp. sesquilinear Hermitian) mapping $\langle \cdot | \cdot \rangle : V \times V \rightarrow \mathbb{R}$ (resp. $\mathbb{C}$) is a scalar product (resp. Hermitian scalar product) if and only if it is positive defined, that is $\langle u|u \rangle > 0$ for all $u \in V \setminus \{0\}$.*

Proof. ($\Rightarrow$) Suppose $\langle \cdot, \cdot \rangle$ is a scalar product. By definition it is a semi scalar product, so $\langle u, u \rangle \geq 0$ for all u . If $\langle u, u \rangle = 0$ but $u \neq 0$, then by non-degeneracy there exists v with $\langle u, v \rangle \neq 0$. Cauchy-Schwarz would then give $|\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle = 0$, a contradiction. Hence $\langle u, u \rangle = 0$ implies $u = 0$, which is positive definiteness.

($\Leftarrow$) Suppose $\langle \cdot, \cdot \rangle$ is positive-definite, bilinear symmetric (or sesquilinear Hermitian). It is automatically a semi scalar product since positive definiteness implies semi-definiteness. For non-degeneracy: if $\langle u, v \rangle = 0$ for all v , then in particular $\langle u, u \rangle = 0$, whence $u = 0$ by positive definiteness. $\square$

Conclusion

This work has developed the foundational structures of multilinear algebra and tensor theory, from the duality of vector spaces through the universal construction of tensor products to the geometric framework of metric notions [9, 4, 8]. The abstract index notation provides a powerful invariant calculus for physical applications, while the representation-theoretic perspective reveals the deep significance of tensor symmetries in quantum theory [16, 18, 24]. The systematic treatment presented herein emphasizes the intrinsic algebraic character of tensor spaces alongside their far-reaching consequences in modern mathematical physics [6, 23, 21]. Several directions remain open for future research, notably the extension to infinite-dimensional topological tensor products essential for quantum field theory [11, 19], and the refined study of Lie group actions on tensor algebras [4, 8]. Tensor theory stands as an essential bridge between pure algebraic structures and the conceptual foundations of contemporary physics, and its mastery remains indispensable for advanced study in either discipline.

Bibliography

- [1] Arfken, George B., and Hans J. Weber. *Mathematical Methods for Physicists*. 6th ed. Amsterdam; Boston: Elsevier, 2005.
- [2] Axler, Sheldon. *Linear Algebra Done Right*. 4th ed. Springer Nature, 2024.
- [3] Boothby, William M. *An Introduction to Differentiable Manifolds and Riemannian Geometry*. Rev. 2nd ed. Vol. 120. Pure and Applied Mathematics. Gulf Professional Publishing, 2003.
- [4] Bourbaki, Nicolas. *Algebra I: Chapters 1–3*. Elements of Mathematics. Springer-Verlag, 1989.
- [5] Bourbaki, Nicolas. *Algebra II: Chapters 4–7*. Elements of Mathematics. Springer-Verlag, 2003.
- [6] Carroll, Sean M. *Spacetime and Geometry: An Introduction to General Relativity*. San Francisco: Addison Wesley, 2004.
- [7] Dirac, Paul A. M. *The Principles of Quantum Mechanics*. 4th ed. International Series of Monographs on Physics 27. Oxford: Clarendon Press, 1981.
- [8] Greub, Werner. *Multilinear Algebra*. 2nd ed. Universitext. New York: Springer-Verlag, 1978.
- [9] Halmos, Paul R. *Finite-Dimensional Vector Spaces*. 2nd ed. The University Series in Higher Mathematics. Princeton: D. Van Nostrand, 1958.

- [10] Hoffman, Kenneth, and Ray Kunze. *Linear Algebra*. 2nd ed. Englewood Cliffs, NJ: Prentice-Hall, 1971.
- [11] Kreyszig, Erwin. *Introductory Functional Analysis with Applications*. New York: John Wiley & Sons, 1989.
- [12] Lang, Serge. *Algebra*. Rev. 3rd ed. Graduate Texts in Mathematics 211. New York: Springer, 2002.
- [13] Lee, John M. *Introduction to Smooth Manifolds*. 2nd ed. Graduate Texts in Mathematics 218. New York: Springer, 2012.
- [14] Moretti, Valter. "Geometric Methods in Mathematical Physics I: Multi-Linear Algebra, Tensors, a little Spinors and Special Relativity." Lecture Notes of the University of Trento, <https://moretti.maths.unitn.it/tensori.pdf>
- [15] McConnell, A. J. *Applications of Tensor Analysis*. New York: Dover, 1957.
- [16] Penrose, Roger, and Wolfgang Rindler. *Spinors and Space-Time. Vol. 1: Two-Spinor Calculus and Relativistic Fields*. Cambridge Monographs on Mathematical Physics. Cambridge: Cambridge University Press, 1984.
- [17] Sahraee, Shahab, and Peter Wriggers. "Algebra of Vectors." In *Tensor Calculus and Differential Geometry for Engineers: With Solved Exercises*, 1–33. Cham: Springer Nature Switzerland, 2023.
- [18] Shankar, R. *Principles of Quantum Mechanics*. 2nd ed. New York: Plenum Press, 1994.
- [19] Simmons, George F. *Introduction to Topology and Modern Analysis*. International Student Edition. New York: McGraw-Hill, 1963.
- [20] Sokolnikoff, Ivan S. *Tensor Analysis: Theory and Applications to Geometry and Mechanics of Continua*. New York: John Wiley & Sons, 1964.

- [21] Spivak, Michael. *Calculus on Manifolds: A Modern Approach to Classical Theorems of Advanced Calculus*. Mathematics Monograph Series. New York: W. A. Benjamin, 1965; repr., Boca Raton: CRC Press, 2018.
- [22] Tu, Loring W. *An Introduction to Manifolds*. 2nd ed. Universitext. New York: Springer, 2011.
- [23] Wald, Robert M. *General Relativity*. Chicago: University of Chicago Press, 1984.
- [24] Weinberg, Steven. *The Quantum Theory of Fields. Vol. 1: Foundations*. Cambridge: Cambridge University Press, 1995.
- [25] Wrede, Robert C. *Introduction to Vector and Tensor Analysis*. New York: John Wiley & Sons, 1963; repr., Mineola, NY: Dover Publications, 2011.
- [26] Yokonuma, Takeo. *Tensor Spaces and Exterior Algebra*. Translations of Mathematical Monographs 108. Providence, RI: American Mathematical Society, 1992.

Abstract

Tensor calculus stands as a fundamental pillar of modern mathematical analysis, with decisive applications in both theoretical physics and differential geometry. The objective of this work is to provide a rigorous and self-contained exposition of fundamental tensor concepts, emphasizing:

Keywords: Tensors, tensor analysis, vector spaces, dual spaces, differential geometry, theoretical physics, general relativity, tensor invariance, tensor algebra, tensor tables.

المخلص

التحليل التنسوري الحديث، وله تطبيقات جوهرية في كل من الفيزياء النظرية والهندسة التفاضلية. يهدف هذا العمل إلى تقديم عرض دقيق ومستقل للمفاهيم الأساسية للتنسورات، مع التركيز على:

الكلمات المفتاحية: تنسورات، التحليل التنسوري، الفضاءات المتجهية، الفضاءات المزدوجة، الهندسة التفاضلية، الفيزياء النظرية، النسبية العامة، الثبات التنسوري، الجبر التنسوري، الجداول التنسورية.

Résumé

Le calcul tensoriel constitue un pilier fondamental de l'analyse mathématique moderne avec des applications décisives en physique théorique et en géométrie différentielle. L'objectif de ce travail est de fournir une exposition rigoureuse et autonome des concepts fondamentaux des tenseurs, en mettant l'accent sur sous changement de référentiel.

Mots-clés : Tenseurs, analyse tensorielle, espaces vectoriels, espaces duaux, géométrie différentielle, physique théorique, relativité générale, invariance tensorielle, algèbre tensorielle, tableaux tensoriels.