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Theme

Eigenvalue and Nonlinear Fractional Problems with Riemann Liouville Operator

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الحمد لله اولا واخرا وبه استعين وعليه اتوكل اهدي هذا العمل المتواضع الى من كان لي السند والداعم ومن علمني معنى الصبر والعطاء الى والدي العزيز " مخلوفي " اطال الله في عمره وادامه تاجا فوق راسي والى القلب الكبير ونبع الحنان الى امي الغالية " نورة " التي كانت دعواتها سر نجاحي وقوتي الى اخوتي الاحباء : " بلال واشرف " والى اختاي الغاليتين " كريمة وسميحة " رفقاء دربي وشركاء ايامي، الى جدتي وجدي وخالي وخالتي وعمي وعمتي وكل افراد عائلتي الكريمة الذين احاطوني بالمحبة والدعم والتشجيع.

كما اتقدم بخالص الشكر والتقدير لاستاذي المشرف " بوغرارة ابراهيم " الذي انار لي طريق البحث بعلمه وخبرته وكانت لتوجيهاته السديدة الفضل الكبير في اتمام هذا العمل فجزاه الله عني خير الجزاء.

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مزوزي كنوز

Contents

Notation

$L^p(a,b)$: The Lebesgue space of p-integrable function on (a,b).

$\|\cdot\|_{L^p}$: The norm in $L^p(a,b)$.

$L^\infty(a,b)$: The Lebesgue space of essentially bounded functions on (a,b).

$\|\cdot\|_{L^\infty}$: The norm in $L^\infty(a,b)$.

$\Gamma(\cdot)$: The Gamma function.

$B(\cdot)$: The Beta function.

$AC(a,b)$: Space of absolute continuous function.

$AC^n(a,b)$: The Space of n-times absolutely functions.

$C^\alpha(a,b)$: The Holder Space.

$\mathcal{L}(H)$: The space of continuous operator.

$I^\alpha_{a+}f$: Riemann-Liouville left-sided fractional integral.

$I^\alpha_{b-}f$: Riemann-Liouville right-sided fractional integral.

$P\alpha^\star$: The critical Sobolev exponent.

$D^\alpha_{a+}f$: Riemann-Liouville left-sided fractional derivative.

$D^\alpha_{b-}f$: Riemann-Liouville right-sided fractional derivative.

$AC^\alpha_{a+}(a,b)$: The left-sided fractional absolutely continuous space of order α.

$AC^\alpha_{b-}(a,b)$: The right-sided fractional absolutely continuous space of order α.

$\mathcal{D}(a,b)$: The space of test functions (infinitely differentiable functions with compact support).

$E^{\alpha,p}_{a+}(a,b)$: The left-sided fractional integral space of order α.

$E^{\alpha,p}_{b-}(a,b)$: The right-sided fractional integral space of order α.

Introduction

Many phenomena in nature and engineering change continuously, such as heat diffusion in materials, fluid motion, or population growth. **Mathematics** provides precise framework to understand and analyze these changes by describing the relationships between variables. In this context, **differential equations** are essential tools to study how variables evolve over time and space.

As scientific research advanced, it became clear that some systems are influenced by past states or have memory, which classical differential equations cannot fully capture. Therefore, **fractional differential equations**, involving derivatives of non-integer order, were developed to provide more flexible and accurate models for complex systems.The scientific references from the foundation for understanding fractional differential equations. Among the most notable are the works of [10], which provide a detailed explanation of fractional integrals and derivatives, and the book by [8], which establishes the theoretical foundations of these equations and discuses their practical applications.

Fractional differential equations can be classified as linear or nonlinear, with nonlinear equations presenting greater challenges that require advanced mathematical techniques. This research is devoted exclusively to the study of nonlinear fractional differential equations. We rely on several fundamental references, such as [2], which provide important results in nonlinear functional analysis and Sobolev spaces. The study begins by establishing the existence of a weak solution in an appropriate functional space.

Accordingly, the thesis is organized into four chapters:

- **In chapter 1:** We present the key definitions and theorems that, We use throughout the thesis as a foundation for understanding and analyzing the core mathematical concepts of the study.

- **In chapter 2:** We introduce fractional calculus by presenting fractional integrals and the Riemann-Liouville derivative. We also provide simple examples to help clarify the main ideas. In addition, a set of definitions related to fractional Sobolev spaces was presented.

- **In chapter 3:** This chapter is devoted to the study of a fractional eigenvalue problem involving the fractional p-Laplacian, where we prove the existence of a weak solution using variational methods.

- **In chapter 4:** This chapter investigates a nonlinear fractional problem using variational methods. The existence of weak solutions is proved in the sublinear case by means of a minimization method, and in the superlinear case via the Nehari manifold.

This work provides clear evidence of the fundamental role played by fractional calculus.

Chapter 1

Preliminaries

This chapter is devoted to outlining the main mathematical tools and fundamental results that will be employed throughout the thesis. These elements play a vital role in the study of differential equations.

1.1 Functional Spaces

1.1.1 The Lebesgue Space

Definition 1.1. *[2] Let $(a,b) \subset \mathbb{R}$ open set and let $p \in \mathbb{R}$ with $1 \leq p < +\infty$, the lebesgue space is defined by:*

$$L^p(a,b) = \left\{ f :]a,b[\to \mathbb{R} \,|\, f \textit{ is measurable and } \quad \int_a^b |f(x)|^p \, dx < \infty \right\}$$

equipped with the norm:

$$\|f\|_{L^p(a,b)} = \|f\|_p = \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

Definition 1.2. *[2] The lebesgue space $L^\infty(a,b)$ is defined by :*

$$L^\infty(a,b) = \{ f :]a,b[\to \mathbb{R} \,|\, f \, is \, measurable \, and \, \exists \, c > 0 : |f(x)| \leq c \quad a.e \ x \ on \ (a,b) \}$$

equipped with the norm:

$$\|f\|_{L^\infty(a,b)} = \|f\|_\infty = \operatorname*{ess\,sup}_{x \in (a,b)} |f(x)|$$

Properities 1.1. *[2]*

- *$L^p(a,b)$ is Banach space: for every $1 \leq p \leq \infty$*
- *$L^p(a,b)$ is reflexive: for every $1 < p < \infty$*

- *$L^p(a,b)$ is separable: for every $1 \le p < \infty$*

Remark 1.1. *For $p = 2$, the space $L^2(a,b)$ is a Hilbert space with the inner product*

$$(f,g) = \int_a^b f g \, dx$$

Notation 1.1. *Let $1 \le p \le \infty$ we denote by p' the conjugate exponent of p, where*

$$\frac{1}{p} + \frac{1}{p'} = 1$$

Theorem 1.1. ***(The Hölder Inequality):****[10][2]*

Let$f \in L^p$ and $g \in L^{p'}$ with $1 \le p \le \infty$ then $f, g \in L^1$

$$\int_a^b |f(x)\, g(x)|\, dx \le \|f\|_{L^p(a,b)} \|g\|_{L^{p'}(a,b)} \tag{1.1}$$

we have also the generalized Hölder inequality

$$\int_a^b |f_1(x) \ldots f_m(x)|\, dx \le \|f_1\|_{L^{p_1}(a,b)} \cdots \|f_m\|_{L^{p_m}(a,b)} \tag{1.2}$$

where $f_k(x) \in L^{p_k}$, $\quad k = 1, 2, \ldots, m, \quad \sum_{k=1}^m \frac{1}{p_k} = 1$

Remark 1.2. *If $p = p' = 2$ we obtain*

$$\int_a^b |f(x)\, g(x)| \le \|f\|_{L^2(a,b)} \|g\|_{L^2(a,b)} \tag{1.3}$$

which called Cauchy-Schwartz inequality

Theorem 1.2. ***(The Young Inequality):****[2]*

For every $a, b > 0$

$$a\, b \le \frac{1}{p} |a|^p + \frac{1}{p'} |b|^{p'} \tag{1.4}$$

with $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$

Moreover: $\forall \epsilon > 0, \exists c > 0$ such that:

$$ab \le \epsilon\, a^p + c\, b^{p'} \qquad c = \epsilon^{-\frac{1}{p-1}} \tag{1.5}$$

Theorem 1.3. ***(The Minkowsky Inequality):****[10]*

Let $f \in L^p(a,b)$ *and* $g \in L^p(a,b)$ *then* $f+g \in L^p(a,b)$

$$\|f+g\|_{L^p(a,b)} \leq \|f\|_{L^p(a,b)} + \|g\|_{L^p(a,b)} \tag{1.6}$$

In case of $p=\infty$

$$\|f+g\|_{L^\infty(a,b)} \leq \|f\|_{L^\infty(a,b)} + \|g\|_{L^\infty(a,b)} \tag{1.7}$$

Theorem 1.4. ***(Interpolation Inequality):****[2]*

Let $0<\alpha<1$, *If* $f \in L^p(a,b) \cap L^q(a,b)$ *with* $1 \leq p \leq q \leq \infty$, *then* $f \in L^r(a,b)$ *for all* $p \leq r \leq q$

$$\|f\|_{L^r(a,b)} \leq \|f\|^{\alpha}_{L^p(a,b)} \|f\|^{1-\alpha}_{L^{p'}(a,b)} \qquad where \quad \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{p'} \tag{1.8}$$

1.2 Special functions

Definition 1.3. *[9][8] the gamma function* Γ *is defined as the function given by:*

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx \qquad (z \in \mathbb{C}, Re(z) > 0) \tag{1.9}$$

with $x^{z-1} = e^{(z-1)\ln(x)}$

1. *For all* $z>0 \in \mathbb{C}$ *we have* , $\Gamma(z+1) = z\Gamma(z)$
2. *If* $n \in \mathbb{N}^*$ *then* $\Gamma(n+1) = n!$

Definition 1.4. *[9][8] the beta function is a type of euler integral defined by :*

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx \qquad p,q \in \mathbb{C} \quad Re(p)>0, Re(q)>0$$

1. *for all* $p,q \in \mathbb{C}$, $Re(p)>0$, $Re(q)>0$, *we have*

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{1.10}$$

2. if $n, m \in \mathbb{N}^*$ then:

$$B(n,m) = \frac{(n-1)!\,(m-1)!}{(n+m-1)!}$$

It should also be mentioned that the Beta function is symmetric, i.e

$$\beta(p,q) = \beta(q,p)$$

1.2.1 Space of absolutely continuous functions

Definition 1.5. *[8] Let $[a,b]$, $-\infty < a < b < +\infty$ a finite interval of $\mathbb{R}$ and $AC[a,b]$ is the space of primitive function f*

$$AC([a,b]) = \left\{ f : \exists\varphi \in L^1(a,b) : f(x) = c + \int_a^x \varphi(t)\,dt \right\}$$

Definition 1.6. *[8] Let $[a,b] \subset \mathbb{R}$ and let $n \in \mathbb{N}^\star$ the space AC^n is defined inductively as follows*

$$AC^n([a,b]) = \left\{ f \in C^{n-1}(a,b) \,/\, f^{(n-1)} \in AC([a,b]) \right\}$$

Remark 1.3. *If $n = 1$: $AC^n = AC$*

Definition 1.7. *(**Hölder Space**) For $0 < \alpha < 1$, the Holder space is defined as follow*

$$C^\alpha(a,b) = \left\{ f \in C(a,b) : \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty \right\}$$

equipped with the norm

$$\|f\|_{C^\alpha(a,b)} = \|f\|_{L^\infty(a,b)} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

1.3 Some useful theorem

Theorem 1.5. ***(Tonelli):****[2]*

Let Ω_1 and Ω_2 two open sets of $\mathbb{R}^n$ assume that $F(x,y) : \Omega_1 \star \Omega_2 \to \mathbb{R}$ be a measurable function satisfying

1. $\int_{\Omega_2} |F(x,y)|\, dy < \infty$ *for a.e* $x \in \Omega_1$
2. $\int_{\Omega_1} dx \int_{\Omega_2} |F(x,y)|\, dy < \infty$

Then:

$$F \in L^1(\Omega_1 \star \Omega_2)$$

Theorem 1.6. ***(Fubini):****[2]*
We assume that $F \in L^1(\Omega_1 \star \Omega_2)$ *then for all* $x \in \Omega_1$

$$F(x,y) \in L^1_y(\Omega_2) \qquad and \qquad \int_{\Omega_2} F(x,y)\, dy \in L^1_x(\Omega_1)$$

likewise, for all $y \in \Omega_2$

$$F(x,y) \in L^1_x(\Omega_1) \qquad and \qquad \int_{\Omega_1} F(x,y)\, dx \in L^1_y(\Omega_2)$$

Furthermore, we have

$$\int_{\Omega_1} dx \int_{\Omega_2} F(x,y)\, dy = \int_{\Omega_2} dy \int_{\Omega_1} F(x,y)\, dx = \int_{\Omega_2}\int_{\Omega_1} F(x,y)\, dx\, dy$$

Theorem 1.7. *[2]****(Lebesgue dominated convergence theorem)*** *Let* f_n *be sequence of functions in* L^1 *that satisfy:*

1. $f_n(x) \to f(x)$ *a.e. on* Ω.
2. *There is a function* $g \in L^1$ *such that for all* n, $|f_n(x)| \leq g(x)$ *a.e. on* Ω

Then $f \in L^1$ *and* $\|f_n - f\|_{L^1} \to 0$

Theorem 1.8. *[2]****(Inverse Lebesgue dominated convergence theorem)*** *Let* (f_n) *be a sequence in* L^p *and let* $f \in L^p$ *be such that* $\|f_n - f\|_{L^p} \to 0$. *then, there exist a subsequence* (f_{n_k}) *and a function* $h \in L^p$ *such that*

1. $f_{n_k}(x) \to f(x)$ *a.e. on* Ω
2. $|f_{n_k}(x)| \leq h(x) \quad \forall\ k,$ *a.e. on* Ω

1.3.1 Compactness

Definition 1.8. ***(Compact set)***

Let H be Hilbert space. a subset $K \subset H$ is called compact set if for every sequence (u_n) in K has a subsequence (u_{n_k}) that converges to a point in K.

Definition 1.9. ***(Precompact set)*** *Let E be a normed space. A subset $A \subset E$ is called precompact (or relatively compact) in E if:*
For every $\varepsilon > 0$, the set A can be covered by a finite number of subsets of E, each having diameter less than or equal to ε.

Theorem 1.9. *A normed space is compact if and only if it is precompact and complete.*

Definition 1.10. ***(compact operator)***

Let E, F two Hilbert spaces assume that $T : E \to F$ we say that T is compact if for every bounded sequence $(u_n) \subset E$ there exists a subsequence (u_{n_k}) such that Tu_{n_k} converges to y in F

Definition 1.11. *Let H be Hilbert space. we denote by $\mathcal{L}(H, H) = \mathcal{L}(H)$ the space of continuous linear operator defined by:*

$$\mathcal{L}(H) = \{T : H \to H;\, linear\, and\, continuous\}$$

equipped with the norm

$$\|T\|_{\mathcal{L}(H)} = \sup_{\|x\| \leq 1} \|Tx\|$$

Definition 1.12. ***(Uniformly equicontinuous)***

Let E be a subset of $C([a, b])$ we say that E s uniformly equicontinuous if:

$$\forall \varepsilon > 0, \exists \delta > 0, such\, that\, d(x, y) < \delta \implies |f(x) - f(y)| < \varepsilon \quad \forall f \in E$$

Theorem 1.10. *[4] Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ and let $1 \leq p \leq \infty$, a bounded subset $k \subset L^p(\Omega)$ is precompact in $L^p(\Omega)$ if and only if:*

$\forall \varepsilon > 0$, $\exists \delta > 0$, $\exists G \subset \Omega$ open such that $\forall u \in k$, $\forall h \in \mathbb{R}^{\mathbb{N}}$ satisfying: $|h| < \delta$, $|h| < d(G, \partial\Omega)$, we have

$$\int_G |u(x+h) - u(x)|^p \, dx < \varepsilon^p \quad \text{and} \quad \int_{\Omega \setminus \overline{G}} |u(x)|^p \, dx < \varepsilon^p$$

Theorem 1.11. *[4]**(Ascoli–Arzelà)***
Let E be a bounded subset of $C([a,b])$, the space of continuous functions on $[a,b]$ with the uniform norm. Assume that E is uniformly equicontinuous. Then the closure of E in $C([a,b])$ is compact.

1.4 Some important result

Lemma 1.1. *Let $a, b > 0$ such that $b \leq a$ we have:*

$$If \mu \geq 1 \qquad then \qquad (a-b)^\mu \leq a^\mu - b^\mu \tag{1.11}$$

$$If 0 < \mu < 1 \qquad then \qquad b^\mu - a^\mu \leq (a-b)^\mu \tag{1.12}$$

Lemma 1.2. *The following function is strictly increasing*

$$\begin{aligned} f : [a, a+h] &\longrightarrow \mathbb{R} \\ s &\longmapsto f(s) = (b-h-s)^\alpha - (b-s)^\alpha + (a+2h-s)^\alpha - (a+h-s)^\alpha \end{aligned}$$

Lemma 1.3. *For all $a, b \geq 0$ we have*

$$If \ p \geq 1 \quad then \quad (a+b)^p \leq 2^{p-1}(a^p + b^p) \tag{1.13}$$

$$If \ p < 1 \quad then \quad (a+b)^p \leq a^p + b^p \tag{1.14}$$

Lemma 1.4. *For all $a, b \geq 0$ we have*

$$If\ 1 < p < 2 \quad then \quad \left| |a|^{p-2} a - |b|^{p-2} b \right| \leq c_p |a-b|^{p-1} \tag{1.15}$$

$$If\ p \geq 2 \quad then \quad \left| |a|^{p-2} a - |b|^{p-2} b \right| \leq c_p \left(|a|^{p-2} + |b|^{p-2} \right) |a-b| \tag{1.16}$$

1.5 Gateaux and Frechet differentiability

Definition 1.13. ***(Gateaux differentiable)*** *Let E be a Banach space, $\Omega \subset E$ an open set, and let $I : \Omega \to \mathbb{R}$ be a functional, we say that I is Gateaux differentiable at $u \in \Omega$, if there exists $A \in E'$ (A linear and continuous), denoted by I' such that, for all $v \in E$,*

$$\lim_{t \to 0} \frac{I(u+tv) - I(u)}{t} = \langle A, v \rangle \tag{1.17}$$

If I is Gateaux differentiable at u, there exists only one linear functional $A \in E'$ satisfying (1.17), It is called the Gateaux differential of I at u and is denoted by $I'(u)$

Definition 1.14. ***(Frechet differentiable)***

Let E be a Banach space, $\Omega \subset E$ an open set, and let $I : \Omega \to \mathbb{R}$ be a functional, we say that I is Frechet differentiable at $u \in \Omega$, if there exists $A_u \in E'$ such that

$$I(u+v) - I(u) = \langle A_u, v \rangle + o(v) \tag{1.18}$$

With: $\lim\limits_{\|v\| \to 0} \frac{o(v)}{\|v\|} = 0$
we denote by $I'(u) = A_u$

Proposition 1.1. ***(Relationship between G-differentiable and F-differentiable)*** *Let $I : E \to \mathbb{R}$ be a Gateaux differentiable and $I' : E \to E'$ is continuous.*

Then, I is of class $C^1(E)$ and in particular I is Frechet differentiable.

Chapter 2

Fractional operator and Sobolev spaces

In this chapter, we first introduce the concept of fractional integrals, followed by an explanation of the Riemann–Liouville derivative. We also present simple examples to illustrate these concepts and make them easier to understand, along with a discussion of some important related properties. Furthermore, we have provided some definitions specific to fractional Sobolev spaces.Moreover, we present several specific definitions related to fractional sobolev spaces.

2.1 Riemann-Liouville Fractional Integrals

Definition 2.1. *Let $0 < \alpha < 1$ and $f \in L^1(a,b)$. the left Riemann-Liouville fractional integral I^{α}_{a+} and right Riemann-Liouville fractional integral I^{α}_{b-} of f are defined as follow:*

$$\left(I^{\alpha}_{a+}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}dt \qquad (a < x \le b) \tag{2.1}$$

$$\left(I^{\alpha}_{b-}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)}{(t-x)^{1-\alpha}}dt \qquad (a \le x < b) \tag{2.2}$$

Example 2.1. *1. for $\alpha > 0$, $\beta > 0$ we put $f(x) = (x-a)^{\beta-1}$*

$$\left(I^{\alpha}_{a+}(x-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1} \tag{2.3}$$

And

$$\left(I^{\alpha}_{b-}(b-x)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(b-x)^{\alpha+\beta-1} \tag{2.4}$$

2. Let $0 < \alpha < 1$ and let $f(x) = c$ where c is a constant

$$\left(I^{\alpha}_{a+}c\right)(x) = \frac{c}{\Gamma(\alpha+1)}(x-a)^{\alpha} \tag{2.5}$$

Now we collect some fundamental properties of Riemann-Liouville fractional integral.

Proposition 2.1. *Let $\alpha > 0, \beta > 0$ et $f \in L^1[a,b]$, then*

$$I_{a+}^{\alpha} I_{a+}^{\beta} f = I_{a+}^{\beta} I_{a+}^{\alpha} f = I_{a+}^{\alpha+\beta} f, \quad I_{b-}^{\alpha} I_{b-}^{\beta} f = I_{b-}^{\beta} I_{b-}^{\alpha} f = I_{b-}^{\alpha+\beta} f \tag{2.6}$$

Theorem 2.1. *[1] The Riemann-Liouville integral $I_{a+}^{\alpha}, I_{b-}^{\alpha} : L^p \to L^p$ are well defined and continuous. Moreover, we have*

$$\|I_{a+}^{\alpha} f\|_{L^p} \leq \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \|f\|_{L^p} \tag{2.7}$$

$$\|I_{b-}^{\alpha} f\|_{L^p} \leq \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \|f\|_{L^p} \tag{2.8}$$

Proof. We have $|I_{a+}^{\alpha} f| = \left|\frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt\right|$

$$\begin{aligned}
|I_{a+}^{\alpha} f| &\leq \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} |f(t)|\, dt \\
&= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\frac{\alpha-1}{p}} (x-t)^{\frac{\alpha-1}{p'}} |f(t)|\, dt
\end{aligned}$$

By Holder Inequality

$$\begin{aligned}
|I_{a+}^{\alpha} f| &\leq \frac{1}{\Gamma(\alpha)} \left(\int_a^x (x-t)^{\alpha-1} |f(t)|^p\, dt\right)^{\frac{1}{p}} \left(\int_a^x (x-t)^{\alpha-1}\, dt\right)^{\frac{1}{p'}} \\
&= \frac{1}{\Gamma(\alpha)} \frac{(x-a)^{\frac{\alpha(p-1)}{p}}}{\alpha^{\frac{p-1}{p}}} \left(\int_a^x |f(t)|^p (x-t)^{\alpha-1}\right)^{\frac{1}{p}} \\
|I_{a+}^{\alpha} f|^p &\leq \frac{1}{\Gamma(\alpha)^p} \frac{(x-a)^{\alpha(p-1)}}{\alpha^{(p-1)}} \int_a^x |f(t)|^p (x-t)^{\alpha-1}\, dt \\
\|I_{a+}^{\alpha} f\|_{L^p}^p &\leq \frac{1}{\Gamma(\alpha)^p} \frac{(x-a)^{\alpha(p-1)}}{\alpha^{(p-1)}} \int_a^b \int_a^x |f(t)|^p (x-t)^{\alpha-1}\, dt\, dx
\end{aligned}$$

by Fubini Theorem

$$\begin{aligned}
\|I_{a+}^{\alpha} f\|_{L^p}^p &\leq \frac{1}{\Gamma(\alpha)^p} \frac{(x-a)^{\alpha(p-1)}}{\alpha^{(p-1)}} \int_a^b |f(t)|^p \int_t^b (x-t)^{\alpha-1}\, dx\, dt \\
&\leq \frac{1}{\Gamma(\alpha)^p} \frac{(b-a)^{\alpha(p-1)} (b-a)^{\alpha}}{\alpha^{p-1}\alpha} \|f\|_{L^p}^p
\end{aligned}$$

Hence

$$\left\| I_{a+}^{\alpha} f \right\|_{L^p} \leq \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \left\| f \right\|_{L^p}$$

The same idea for the right Riemann-Liouville integral □

2.2 Riemann-Liouvill Fractional Derivatives

Definition 2.2. *Let $0 < \alpha < 1$ we define the left Riemann-Liouville fractional derivative D_{a+}^{α} and right Riemann-Liouville fractional derivative D_{b-}^{α} of f as follows:*

$$\left(D_{a+}^{\alpha} f\right)(x) = \frac{d}{dx}\left(I_{a+}^{1-\alpha} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f(t)\, dt \tag{2.9}$$

Where $\frac{d}{dx}$ in the weak sense

And

$$\left(D_{b-}^{\alpha} f\right)(x) = \frac{-d}{dx}\left(I_{b-}^{1-\alpha} f\right)(x) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (t-x)^{-\alpha} f(t)\, dt \tag{2.10}$$

Example 2.2. *for $0 < \alpha < 1$, $\beta > 0$. we put $f(x) = (x-a)^{\beta-1}$*

$$\left(D_{a+}^{\alpha} f\right)(x) = \frac{d}{dx}\left(I_{a+}^{1-\alpha} f\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (x-a)^{\beta-\alpha-1}$$

And

$$\left(D_{b-}^{\alpha} f\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (b-x)^{\beta-\alpha-1}$$

Example 2.3. *Let $0 < \alpha < 1$ If we take $\beta = 1$ we obtain*

$$D_{a+}^{\alpha} (x-a)^0 = D_{a+}^{\alpha} 1 = \frac{\Gamma(1)}{\Gamma(1-\alpha)} (x-a)^{-\alpha}$$

And

$$D_{b-}^{\alpha} 1 = \frac{1}{\Gamma(1-\alpha)} (b-x)^{-\alpha}$$

Also, we have

$$\left(D_{a+}^{\alpha} (t-a)^{\alpha-1}\right)(x) = 0 \quad \textit{and} \quad \left(D_{b-}^{\alpha} (b-t)^{\alpha-1}\right)(x) = 0 \tag{2.11}$$

2.2.1 Some properties

Proposition 2.2. *For $0 < \alpha < 1$, $0 < \beta < 1$ the following statement hold*

1. *If $f(x) \in L^1([a,b])$ then*

$$\left(D_{a+}^{\alpha} I_{a+}^{\alpha} f\right)(x) = f(x) \quad and \quad \left(D_{b-}^{\alpha} I_{b-}^{\alpha} f\right)(x) = f(x) \tag{2.12}$$

2. *If $\alpha > \beta$ and $f \in L^p([a,b])$, $1 \leq p \leq \infty$ then*

$$\left(D_{a+}^{\beta} I_{a+}^{\alpha} f\right)(x) = \left(I_{a+}^{\alpha-\beta} f\right)(x) \qquad and \qquad \left(D_{a+}^{\beta} I_{a+}^{\alpha} f\right)(x) = \left(I_{a+}^{\alpha-\beta} f\right)(x) \tag{2.13}$$

3. *If $f \in L^1([a,b])$, and $\left(I_{a+}^{1-\alpha} f\right) \in AC([a,b])$, then*

$$\left(I_{a+}^{\alpha} D_{a+}^{\alpha} f\right)(x) = f(x) - \frac{I_{a+}^{1-\alpha} f(a)}{\Gamma(\alpha)} (x-a)^{\alpha-1} \tag{2.14}$$

and

$$\left(I_{b-}^{\alpha} D_{b-}^{\alpha} f\right)(x) = f(x) - \frac{I_{b-}^{1-\alpha} f(b)}{\Gamma(\alpha)} (b-x)^{\alpha-1} \tag{2.15}$$

Lemma 2.1. *[8] If $f(x) \in AC([a,b])$, then $I_{a+}^{1-\alpha} f(x) \in AC([a,b])$ and*

$$I_{a+}^{1-\alpha} f(x) = \frac{1}{\Gamma(2-\alpha)} \left[f(a)(x-a)^{1-\alpha} + \int_a^x f'(t)(x-t)^{1-\alpha} dt \right] \tag{2.16}$$

Proof. We have

$$f \in AC([a,b]) \quad then \quad f(x) = f(a) + \int_a^x f'(s)\, ds \tag{2.17}$$

we apply $I_{a+}^{1-\alpha}$ to both sided we obtain

$$\begin{aligned} I_{a+}^{1-\alpha} f(x) &= I_{a+}^{1-\alpha} f(a) + \int_a^x I_{a+}^{1-\alpha} f'(s)\, ds \\ &= \frac{1}{\Gamma(1-\alpha)} \left[\int_a^x (x-a)^{-\alpha} f(a)\, dt + \int_a^x \int_a^s (s-t)^{-\alpha} f'(t)\, dt\, ds \right] \end{aligned}$$

by Fubini theorem

$$= \frac{1}{\Gamma(1-\alpha)}\left[\frac{f(a)}{(1-\alpha)}(x-a)^{1-\alpha} + \int_a^x f'(t)\int_t^x (s-t)^{-\alpha}\,ds\,dt\right]$$

$$= \frac{1}{\Gamma(1-\alpha)}\left[\frac{f(a)}{(1-\alpha)}(x-a)^{1-\alpha} + \frac{1}{(1-\alpha)}\int_a^x f'(t)(x-t)^{1-\alpha}\,dt\right]$$

then

$$I_{a+}^{1-\alpha} f(x) = \frac{1}{\Gamma(2-\alpha)}\left[f(a)(x-a)^{1-\alpha} + \int_a^x f'(t)(x-t)^{1-\alpha}\,dt\right]$$

□

Lemma 2.2. *[8] If $0 < \alpha < 1$ and $f(x) \in AC[a,b]$ then*

$$\left(D_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{f(a)}{(x-a)^{\alpha}} + \int_a^x \frac{f'(t)}{(x-t)^{\alpha}}dt\right] \tag{2.18}$$

$$\left(D_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{f(b)}{(b-x)^{\alpha}} + \int_x^b \frac{f'(t)}{(t-x)^{\alpha}}dt\right] \tag{2.19}$$

Definition 2.3. ***(Weak fractional derivatives):*** *Let $f \in L^1_{Loc}(a,b)$ and let $0 < \alpha < 1$*

1. *If there exists $g \in L^1_{Loc}(a,b)$ such that*

$$\int_a^b f(x)\, D_{b-}^{\alpha}\varphi(x)\,dx = \int_a^b g(x)\,\varphi(x)\,dx, \quad \forall \varphi \in \mathcal{D} \tag{2.20}$$

   Then g is called the weak left fractional derivative of f, and we write

$$\mathcal{D}_{a+}^{\alpha} f(x) = g(x)$$

2. *If there exists $h \in L^1_{Loc}(a,b)$ such that*

$$\int_a^b f(x)\, D_{a+}^{\alpha}\varphi(x)\,dx = \int_a^b h(x)\,\varphi(x)\,dx, \quad \forall \varphi \in \mathcal{D} \tag{2.21}$$

   Then h is called the weak right fractional derivative of f, and we write

$$\mathcal{D}_{b-}^{\alpha} f(x) = h(x)$$

Remark 2.1. *If $I^{1-\alpha} f \in AC(a,b)$ then the derivative of $I^{1-\alpha} f$ in the weak sense coincide a.e with the derivative in the usual sense.*

2.3 Fractional Sobolev Spaces

2.3.1 Spaces $E_{a+}^{\alpha,p}(a,b)$ and $E_{b-}^{\alpha,p}(a,b)$

Definition 2.4. *Let $0<\alpha<1$ and $1\le p<\infty$ The spaces $E_{a+}^{\alpha,p}$, $E_{b-}^{\alpha,p}$ are defined by*

$$E_{a+}^{\alpha,p}(a,b)=\left\{f\in L^{p}(a,b): D_{a+}^{\alpha}f\in L^{p}(a,b)\right\} \tag{2.22}$$

and

$$E_{b-}^{\alpha,p}(a,b)=\left\{f\in L^{p}(a,b): D_{b-}^{\alpha}f\in L^{p}(a,b)\right\} \tag{2.23}$$

these spaces are equipped with the norms

$$\|f\|_{E_{a+}^{\alpha,p}(a,b)}=\left(\int_a^b |f(x)|^p\,dx+\int_a^b |D_{a+}^{\alpha}f(x)|^p\,dx\right)^{\frac{1}{p}}$$

and

$$\|f\|_{E_{b-}^{\alpha,p}(a,b)}=\left(\int_a^b |f(x)|^p\,dx+\int_a^b |D_{b-}^{\alpha}f(x)|^p\,dx\right)^{\frac{1}{p}}$$

Theorem 2.2. ***(Integral representation)*** *Let $1\le p<\infty$*

1. *Let $\alpha>\frac{1}{p'}$ then $f\in E_{b-}^{\alpha,p}(a,b)$ if and only if*

$$f(x)=\frac{d}{\Gamma(\alpha)}(b-x)^{\alpha-1}+I_{b-}^{\alpha}\varphi(x)\qquad \text{with } \varphi\in L^{P} \tag{2.24}$$

with $d=I_{b-}^{1-\alpha}f(b)$

and $g\in E_{a+}^{\alpha,p}$ if and only if

$$g(x)=\frac{c}{\Gamma(\alpha)}(x-a)^{\alpha-1}+I_{a+}^{\alpha}\psi(x) \tag{2.25}$$

With $c=I_{a+}^{1-\alpha}g(a)$

In particular for $p=1$ and $0<\alpha<1$ then

$$f(x)=\frac{d}{\Gamma(\alpha)}(b-x)^{\alpha-1}+I_{b-}^{\alpha}\varphi(x)$$

and

$$g(x)=\frac{c}{\Gamma(\alpha)}(x-a)^{\alpha-1}+I_{a+}^{\alpha}\psi(x)$$

2. *If* $\alpha \leq \frac{1}{p'}$ *then* $f \in E_{b-}^{\alpha,p}(a,b)$ *if and only if* $f = I_{b-}^{\alpha}\varphi$ *and* $g \in E_{a+}^{\alpha,p}$ *if and only if* $g = I_{a+}^{\alpha}\psi$

Proof. 1. ($\Rightarrow$) For $\alpha > \frac{1}{p'}$ we show that if $f \in E_{b-}^{\alpha,p}$ then

$$f(x) = \frac{d}{\Gamma(\alpha)}(b-x)^{\alpha-1} + I_{b-}^{\alpha}\varphi(x)$$

we have $f \in E_{b-}^{\alpha,p}$ which implies $D_{b-}^{\alpha}f \in L^{P}$ then $I_{b-}^{1-\alpha}f(x) \in AC$ there exists $d \in \mathbb{R}$ and $\varphi \in L^{p}$ such that

$$I_{b-}^{1-\alpha}f(x) = I_{b-}^{1-\alpha}f(b) + I_{b-}^{1}D_{b-}^{\alpha}f(x)$$

With: $I_{b-}^{1-\alpha}f(b) = d$ and $D_{b-}^{\alpha}f(x) = \frac{d}{dx}\left(I_{b-}^{1-\alpha}f\right)(x) = \varphi(x)$ we obtain

$$I_{b-}^{1-\alpha}f(x) = d + I_{b-}^{1}\varphi(x)$$

By applying I_{b-}^{α} we obtain

$$I_{b-}^{1}f(x) = I_{b-}^{\alpha}d + I_{b-}^{1}I_{b-}^{\alpha}\varphi(x)$$

By using property (2.6) we obtain

$$I_{b-}^{1}f(x) = I_{b-}^{\alpha}d + I_{b-}^{1+\alpha}\varphi(x)$$

Then $I_{b-}^{1}f(x) = \frac{d}{\alpha\Gamma(\alpha)}(b-x)^{\alpha} + I_{b-}^{1+\alpha}\varphi(x)$
Differentiating both sides we obtain:

$$f(x) = \frac{d}{\Gamma(\alpha)}(b-x)^{\alpha-1} + I_{b-}^{\alpha}\varphi(x) \tag{2.26}$$

For $\alpha \leq \frac{1}{p'}$ in this case we have $E_{b-}^{\alpha,p} = I_{b-}^{\alpha}(L^{p})$
Since,

$$f \in L^{p} \implies d(b-x)^{\alpha-1} \in L^{p}$$

but we have $(b-a)^{\alpha-1} \notin L^{p}$
Hence,

$$d = 0$$

Then

$$f = I_{b-}^{\alpha}\varphi$$

The same idea for g

2. ($\Leftarrow$) For $\alpha > \frac{1}{p'}$, we show that if $f(x) = \frac{d}{\Gamma(\alpha)}(b-x)^{\alpha-1} + I_{b-}^{\alpha}\varphi(x)$ then $f \in E_{b-}^{\alpha,p}(a,b)$

   we have

$$f(x) = \frac{d}{\Gamma(\alpha)}(b-x)^{\alpha-1} + I_{b-}^{\alpha}\varphi(x)$$

   We apply the left Riemann-Liouville fractional derivative of order α to both sides of the equation

$$D_{b-}^{\alpha}f(x) = D_{b-}^{\alpha}\left(\frac{d}{\Gamma(\alpha)}(b-x)^{\alpha-1}\right) + D_{b-}^{\alpha}\left(I_{b-}^{\alpha}\varphi(x)\right)$$

   By the definition of the Riemann-Liouville derivative (see definition (2.2)) and the semi group property $D_{a+}^{\alpha}I_{a+}^{\alpha}\varphi = \varphi$ (see equation (2.12)) and by equation (2.11) we obtain:

$$D_{b-}^{\alpha}f(x) = \varphi(x)$$

   Then,

$$D_{b-}^{\alpha}f(x) \in L^{p} \tag{2.27}$$

   For $\alpha \leq \frac{1}{p'}$ evident

□

Lemma 2.3. *[6]*

1. *The operator $I_{a+}^{\alpha} : L^{p}(a,b) \to E_{a+}^{\alpha,p}(a,b)$ is linear and bounded.*
2. *The operator $D_{a+}^{\alpha} : E_{a+}^{\alpha,p}(a,b) \to L^{p}(a,b)$ is linear and bounded.*

Proof. 1. From the definition of the fractional integral I_{a+}^{α} (*see* (2.1)), we know that I_{a+}^{α} is a linear operator. Let $f \in L^{p}(a,b)$ and define $g = I_{a+}^{\alpha}f$ then, by lemma (2.3), we have $g \in E_{a+}^{\alpha,p}(a,b)$.

Moreover:

$$I_{a+}^{1-\alpha} g = I_{a+}^{1-\alpha} I_{a+}^{\alpha} f = I_{a+}^{1} f \qquad (By\, proposition\, 2.1)$$

Hence, the fractional derivative $D_{a+}^{\alpha} g$ exists, and we have

$$D_{a+}^{\alpha} g = f$$

Consequently,

$$\|D_{a+}^{\alpha} g\|_{L^p(a,b)} = \|f\|_{L^p(a,b)} \tag{2.28}$$

and

$$I_{a+}^{\alpha} f = g \in E_{a+}^{\alpha,p}(a,b)$$

Combining (2.22) and (2.28), we get

$$\|I_{a+}^{\alpha} f\|_{E_{a+}^{\alpha,p}(a,b)} = \left(\|I_{a+}^{\alpha} f\|_{L^p(a,b)}^p + \|D_{a+}^{\alpha} I_{a+}^{\alpha} f\|_{L^p(a,b)}^p\right)^{\frac{1}{p}}$$

Then, by (2.7), it follows that

$$\|I_{a+}^{\alpha} f\|_{E_{a+}^{\alpha,p}(a,b)} \leq \left(\frac{(b-a)^{p\alpha}}{\Gamma^p(\alpha+1)} \|f\|_{L^p(a,b)}^p + \|f\|_{L^p(a,b)}^p\right)^{\frac{1}{p}} \leq \left(\frac{(b-a)^{p\alpha}}{\Gamma^p(\alpha+1)} + 1\right)^{\frac{1}{p}} \|f\|_{L^p(a,b)}$$

This shows that the operator $I_{a+}^{\alpha} : L^p(a,b) \to E_{a+}^{\alpha,p}(a,b)$ is bounded.

2. From lemma (2.3) the operator D_{a+}^{α} is linear. Moreover, for $f \in E_{a+}^{\alpha,p}(a,b)$, we have

$$\|D_{a+}^{\alpha} f\|_{L^p(a,b)} \leq \left(\|f\|_{L^p(a,b)}^p + \|D_{a+}^{\alpha} f\|_{L^p(a,b)}^p\right)^{\frac{1}{p}} = \|f\|_{E_{a+}^{\alpha,p}(a,b)}$$

which implies that $D_{a+}^{\alpha} : E_{a+}^{\alpha,p}(a,b) \to L^p(a,b)$ is also bounded operator

□

2.3.2 Embedding for Riemann-Liouvill integral

Theorem 2.3. *[11] Let* $0 < \alpha < 1$, $1 \leq p < \infty$

1. *If* $0 < \alpha \leq \frac{1}{p} < 1$ *then the operator* I_{a+}^{α}, I_{b-}^{α}: $L^p(a,b) \to L^q(a,b)$ *are continuous*

   - *If* $\alpha < \frac{1}{p}$, *then for every* $1 \leq q \leq p_{\alpha}^{\star}$ *with* $p_{\alpha}^{\star} = \frac{p}{1-\alpha p}$ *which called (the critical Sobolev exponent)*
   - *If* $\alpha = \frac{1}{p}$, *then for every* $1 \leq q < +\infty$

2. *If* $0 < \frac{1}{p} < \alpha < 1$ *then the operator* I_{a+}^{α}, I_{b-}^{α}: $L^p(a,b) \to L^{\infty}(a,b)$ *are continuous*

Theorem 2.4. *[11] Let* $0 < \alpha < 1$ *and* $p \geq 1$, *then the operator* $I_{a+}^{\alpha}, I_{b-}^{\alpha}$: $L^p(a,b) \to L^p(a,b)$ *are compact.*

Theorem 2.5. *[11] If* $0 < \alpha \leq \frac{1}{p}$ *the operator* $I_{a+}^{\alpha}, I_{b-}^{\alpha} : L^p(a,b) \to L^q(a,b)$ *are compact*

- *If* $\alpha < \frac{1}{p}$, *then for every* $q \in [1, p_{\alpha}^{\star}[$.
- $\alpha = \frac{1}{p}$, *then for every* $1 \leq q < +\infty$

2.3.3 Spaces $E_{a+,0}^{\alpha,p}(a,b)$ and $E_{b-,0}^{\alpha,p}(a,b)$

Definition 2.5. *Let* $0 < \alpha < 1$ *and let* $(a,b) \subset \mathbb{R}$. *We define the fractional space* $E_{a+,0}^{\alpha,p}$ *by:*

$$E_{a+,0}^{\alpha,p}(a,b) = \overline{\mathcal{D}(a,b)}^{\|\cdot\|_{E_{a+}^{\alpha,p}(a,b)}}. \tag{2.29}$$

Lemma 2.4. *[6] Let* $f \in E_{a+,0}^{\alpha,p}(a,b)$ *then*

$$I_{a+}^{\alpha} D_{a+}^{\alpha} f(x) = f(x) \ \ a.e \ in \, (a,b)$$

Proof. From the definition of the space $E_{a+,0}^{\alpha,p}(a,b)$ there exists a sequence $(\varphi_n) \subset \mathcal{D}(a,b)$ such that:

$$\|\varphi_n - f\|_{E_{a+,0}^{\alpha,p}(a,b)} \to 0 \quad as \quad n \to \infty$$

Which together with (2.14) we get

$$I_{a+}^{\alpha} D_{a+}^{\alpha} \varphi_n = \varphi_n - \frac{I^{1-\alpha}\varphi_n(a)}{\Gamma(\alpha)}(x-t)^{\alpha-1}$$

Since $I^{1-\alpha}\varphi_n(a) = 0$, we have

$$I_{a+}^{\alpha} D_{a+}^{\alpha} \varphi_n = \varphi_n$$

By lemma (2.3) we add subtract the term $I_{a+}^{\alpha} D_{a+}^{\alpha} \varphi_n$

$$\begin{aligned}
\|I_{a+}^{\alpha} D_{a+}^{\alpha} f - f\|_{E_{a+}^{\alpha,p}} &\leq \|(I_{a+}^{\alpha} D_{a+}^{\alpha} f - I_{a+}^{\alpha} D_{a+}^{\alpha} \varphi_n) + (I_{a+}^{\alpha} D_{a+}^{\alpha} \varphi_n - f)\|_{E_{a+}^{\alpha,p}(a,b)} \\
&\leq \|I_{a+}^{\alpha} D_{a+}^{\alpha} f - I_{a+}^{\alpha} D_{a+}^{\alpha} \varphi_n\|_{E_{a+}^{\alpha,p}(a,b)} + \|I_{a+}^{\alpha} D_{a+}^{\alpha} \varphi_n - f\|_{E_{a+}^{\alpha,p}(a,b)} \\
&\leq \|I_{a+}^{\alpha} D_{a+}^{\alpha}(f - \varphi_n)\|_{E_{a+}^{\alpha,p}(a,b)} + \|I_{a+}^{\alpha} D_{a+}^{\alpha} \varphi_n - f\|_{E_{a+}^{\alpha,p}(a,b)}
\end{aligned}$$

By the continuity of I_{a+}^{α} and D_{a+}^{α} it follows that:

$$\begin{aligned}
\|I_{a+}^{\alpha} D_{a+}^{\alpha} f - f\|_{E_{a+}^{\alpha,p}} &\leq c\,\|\varphi_n - f\|_{E_{a+}^{\alpha,p}(a,b)} + \|f - \varphi_n\|_{E_{a+}^{\alpha,p}(a,b)} \\
&= (c+1)\,\|\varphi_n - f\| \to 0 \quad n \to \infty
\end{aligned}$$

Where $c > 0$ is constant

Hence:

$$I_{a+}^{\alpha} D_{a+}^{\alpha} f(x) = f(x) \quad a.e\ x \in [a,b]$$

□

Corollarry 2.1. *[6]**(Fractional Poincare inequality)** Let $0 < \alpha < 1$ for every $f \in E_{a+,0}^{\alpha,p}(a,b)$. Then there exists $C > 0$ such that*

$$\|f\|_{L^p(a,b)} \leq C\,\|D_{a+}^{\alpha} f\|_{L^p(a,b)} \qquad where \quad C = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \tag{2.30}$$

Proof. By lemma(2.4) and expression (2.7) we get

$$\begin{aligned}\|f\|_{L^p(a,b)} &= \|I^{\alpha}_{a+}D^{\alpha}_{a+}f\|_{L^p(a,b)} \\ &\leq C\,\|D^{\alpha}_{a+}f\|_{L^p(a,b)} \qquad \textit{with } C=\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\end{aligned}$$

□

Remark 2.2. *By Corollary (2.1), we can endowed $E^{\alpha,p}_{a+,0}(a,b)$ with the norm:*

$$\|f\|_{E^{\alpha,p}_{a+,0}(a,b)} = \left(\int_a^b |D^{\alpha}_{a+}f(x)|^p\right)^{\frac{1}{p}}$$

which is equivalent to$\|\cdot\|_{E^{\alpha,p}_{a+}(a,b)}$

2.4 Embedding results

2.4.1 Embedding for $E^{\alpha,p}_{a+,0}$

Theorem 2.6. *[1][11]**(Continuous embedding)***

1. *If $0<\alpha\leq\frac{1}{p}$, then the embedding $E^{\alpha,p}_{a+,0}(a,b)\hookrightarrow L^q(a,b)$ is continuous*
   - *If $\alpha<\frac{1}{p}$, then for every $1\leq q\leq p^{\star}_{\alpha}$ with $p^{\star}_{\alpha}=\frac{p}{1-\alpha p}$ which called (the critical Sobolev exponent)*
   - *If $\alpha=\frac{1}{p}$, then for every $1\leq q<+\infty$*
2. *If $\frac{1}{p}<\alpha<1$, then the embedding $E^{\alpha,p}_{a+,0}(a,b)\hookrightarrow C^{\alpha-\frac{1}{p}}$ is continuous, Moreover:*

$$\|f\|_{L^{\infty}(a,b)} \leq \frac{(b-a)^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)\left(p'(\alpha-1)+1\right)^{\frac{1}{p'}}}\|D^{\alpha}_{a+}f\|_{L^p(a,b)} \tag{2.31}$$

Proof. 1. By lemma (2.4), for every $f\in E^{\alpha,p}_{a+,0}(a,b)$, we know that

$$f(x) = I^{\alpha}_{a+}D^{\alpha}_{a+}f(x)$$

$$\|f\|_{L^{p^{\star}}(a,b)} = \|I^{\alpha}_{a+}D^{\alpha}_{a+}f\|_{L^{p^{\star}}(a,b)}$$

Applying theorem (2.3) yields the estimate

$$\|f\|_{L^{p^\star}(a,b)} \leq c_1 \|D^\alpha_{a+} f\|_{L^p(a,b)} \qquad c_1 = \frac{(b-a)^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)\left(p'(\alpha-1)+1\right)^{\frac{1}{p'}}}$$

Moreover, for any $q \in [1, p^\star]$, the inclusion $L^{p^\star}(a,b) \subset L^q(a,b)$ ensures that

$$\|f\|_{L^q(a,b)} \leq c_2 \|f\|_{L^{p^\star}(a,b)}$$

Combining the above inequalities, we obtain

$$\|f\|_{L^q(a,b)} \leq c_1 c_2 \|D^\alpha_{a+} f\|_{L^p(a,b)}$$

Since,

$$\|f\|_{L^q(a,b)} \leq c \|D^\alpha_{a+} f\|_{L^p(a,b)}$$

2. Let $f \in E^{\alpha,p}_{a+,0}[a,b]$. Then, by lemma (2.4) and Holder inequality, we have:

$$\begin{aligned}
|f(x)| = |I^\alpha_{a+} D^\alpha_{a+} f(x)| &= \frac{1}{\Gamma(\alpha)} \left| \int_a^x (x-t)^{\alpha-1} D^\alpha_{a+} f(t)\, dt \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\int_a^x (x-t)^{p'(\alpha-1)}\, dt \right)^{\frac{1}{p'}} \left(\int_a^x |D^\alpha_{a+} f(t)|^p\, dt \right)^{\frac{1}{p}} \\
&\leq \frac{\|D^\alpha_{a+} f\|_{L^p(a,b)}}{\left(p'(\alpha-1)+1\right)^{\frac{1}{p'}} \Gamma(\alpha)} (b-a)^{\alpha-\frac{1}{p}}
\end{aligned}$$

Hence,

$$\|f\|_{L^\infty(a,b)} \leq \frac{(b-a)^{\alpha-\frac{1}{p}}}{\left(p'(\alpha-1)+1\right)^{\frac{1}{p'}} \Gamma(\alpha)} \|f\|_{E^{\alpha,p}_{a+,0}(a,b)} \tag{2.32}$$

On the other hand, let $a \leq x \leq z \leq b$, then:

$$\begin{aligned}
|f(z) - f(x)| &= |I^\alpha_{a+} D^\alpha_{a+} f(z) - I^\alpha_{a+} D^\alpha_{a+} f(x)| \\
&= \frac{1}{\Gamma(\alpha)} \left| \int_a^x \left((z-t)^{\alpha-1} - (x-t)^{\alpha-1}\right) D^\alpha_{a+} f(t)\, dt \right. \\
&\left. + \int_x^z (z-t)^{\alpha-1} D^\alpha_{a+} f(t)\, dt \right|
\end{aligned}$$

Using the Holder inequality :

$$\begin{aligned}
|f(z) - f(x)| \leq & \frac{1}{\Gamma(\alpha)} \left(\int_a^x \left|(z-t)^{\alpha-1} - (x-t)^{\alpha-1}\right|^{p'} dt \right)^{\frac{1}{p'}} \|D^\alpha_{a+} f\|_{L^p(a,b)} \\
& + \frac{(z-x)^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)\left(p'(\alpha-1)+1\right)^{\frac{1}{p'}}} \|D^\alpha_{a+} f\|_{L^p(a,b)}
\end{aligned}$$

Using estimates (1.11) we obtain:

$$
\begin{aligned}
|f(z)-f(x)| \leq & \frac{\|D_{a+}^{\alpha} f\|_{L^p(a,b)}}{\Gamma(\alpha)} \left(\int_a^x \left((x-t)^{p'(\alpha-1)} - (z-t)^{p'(\alpha-1)} \right) \right)^{\frac{1}{p'}} \\
& + \frac{(z-x)^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)\left(p'(\alpha-1)+1\right)^{\frac{1}{p'}}} \|D_{a+}^{\alpha} f\|_{L^p(a,b)} \\
\leq & \frac{\|D_{a+}^{\alpha} f\|_{L^p(a,b)}}{\Gamma(\alpha)\left(p'(\alpha-1)+1\right)^{\frac{1}{p'}}} \left((z-x)^{\alpha-\frac{1}{p}} + (x-a)^{\alpha-\frac{1}{p}} - (z-a)^{\alpha-\frac{1}{p}} \right) \\
& + \frac{(z-x)^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)\left(p'(\alpha-1)+1\right)^{\frac{1}{p'}}} \|D_{a+}^{\alpha} f\|_{L^p(a,b)}
\end{aligned}
$$

using (1.12) we obtain:

$$
|f(z)-f(x)| \leq \frac{\|D_{a+}^{\alpha} f\|_{L^p(a,b)}}{\Gamma(\alpha)\left(p'(\alpha-1)+1\right)^{\frac{1}{p'}}} \left((z-x)^{\alpha-\frac{1}{p}} + (z-x)^{\alpha-\frac{1}{p}} + (z-x)^{\alpha-\frac{1}{p}} \right)
$$

Hence

$$
|f(z)-f(x)| \leq \frac{3(z-x)^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)\left(p'(\alpha-1)+1\right)^{\frac{1}{p'}}} \|D_{a+}^{\alpha} f\|_{L^p(a,b)} \tag{2.33}
$$

Therefore,

$$
\begin{aligned}
\|f\|_{C^{\alpha-\frac{1}{p}}(a,b)} &= \|f\|_{L^{\infty}(a,b)} + \sup_{(x,z)\in(a,b)} \frac{|f(z)-f(x)|}{|z-x|^{\alpha-\frac{1}{p}}} \\
&\leq \frac{(b-a)^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)\left(p'(\alpha-1)+1\right)^{\frac{1}{p'}}} \|D_{a+}^{\alpha} f\|_{L^p(a,b)} + \frac{3}{\Gamma(\alpha)\left(p'(\alpha-1)+1\right)^{\frac{1}{p'}}} \|D_{a+}^{\alpha} f\|_{L^p} \\
&= \left(\frac{(b-a)^{\alpha-\frac{1}{p}}+3}{\Gamma(\alpha)\left(p'(\alpha-1)+1\right)^{\frac{1}{p'}}} \right) \|D_{a+}^{\alpha} f\|_{L^p(a,b)}
\end{aligned}
$$

Hence

$$
\|f\|_{C^{\alpha-\frac{1}{p}}(a,b)} \leq \left(\frac{(b-a)^{\alpha-\frac{1}{p}}+3}{\Gamma(\alpha)\left(p'(\alpha-1)+1\right)^{\frac{1}{p'}}} \right) \|D_{a+}^{\alpha} f\|_{L^p(a,b)}
$$

□

Theorem 2.7. *[1][11]**(Compact embedding)***

1. *If $0<\alpha\leq\frac{1}{p}$, then the embedding $E_{a+,0}^{\alpha,p}(a,b) \hookrightarrow L^q(a,b)$ is compact*

- *If $\alpha < \frac{1}{p}$, then for every $1 \leq q \leq p_{\alpha}^{\star}$*
- *If $\alpha = \frac{1}{p}$, then for every $1 \leq q < +\infty$*

2. *If $\frac{1}{p} < \alpha < 1$, then the embedding $E_{a+,0}^{\alpha,p}(a,b) \hookrightarrow C(a,b)$ is compact.*

Proof. 1. By theorem (2.6) the embedding

$$i : E_{a+,0}^{\alpha,p}(a,b) \to L^{q}(a,b)$$

for all $q \in [1, p_{\alpha}^{\star}]$ is continuous, That is, there exists a constant $C > 0$ such that

$$\|f_n\|_{L^q(a,b)} \leq C \|f_n\|_{E_{a+,0}^{\alpha,p}(a,b)}$$

Let $(f_n) \subset E_{a+,0}^{\alpha,p}(a,b)$ such that $f_n \rightharpoonup f$ in $E_{a+,0}^{\alpha,p}$, then (f_n) is bounded. hence f_n is bounded in $L^p(a,b)$

By definition of the space $D_{a+}^{\alpha} f_n$ is bounded in $L^p(a,b)$, by theorem (2.5) $I_{a+}^{\alpha}(D_{a+}^{\alpha} f_n)$ is precompact in $L^q(a,b)$, for all $q \in [1, p_{\alpha}^{\star}]$, Ihis means that there exist a subsequence (f_{n_k}), still denoted $(f_n)_{n \in \mathbb{N}}$, and a function $f \in L^q(a,b)$ such that

$$\|I_{a+}^{\alpha} D_{a+}^{\alpha} f_n - f\|_{L^q(a,b)} \to 0 \quad as \quad n \to \infty$$

Moreover, by lemma (2.4) we have

$$f_n(x) = I_{a+}^{\alpha} D_{a+}^{\alpha} f_n(x)$$

Then

$$i(f_n(x)) = i(I_{a+}^{\alpha} D_{a+}^{\alpha} f_n(x))$$

By the continuity of i the composition is compact.

Therefore,

$$i : E_{a+,0}^{\alpha,p} \to L^{q}(a,b)$$

is compact. for all $q \in [1, p_{\alpha}^{\star}]$

2. Let $f_n \rightharpoonup f$ in $E^{\alpha,p}_{a+,0}$. then the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded in $E^{\alpha,p}_{a+,0}(a,b)$. by theorem (2.6) f_n is bounded in $C([a,b])$, there exist $K>0$

   By equation (2.31) we obtain:

$$\|f_n\|_{L^\infty(a,b)} \leq \frac{K(b-a)^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(p'(\alpha-1)+1)^{\frac{1}{p'}}}$$

   Therefore, the sequence (f_n) is uniformly bounded in $C([a,b])$ by equation (2.33) we obtain:

$$|f_n(z)-f_n(x)| \leq \frac{K}{\Gamma(\alpha)(p'(\alpha-1)+1)^{\frac{1}{p'}}}(z-x)^{\alpha-\frac{1}{p}}$$

   Let $\varepsilon>0$. choose $\delta>0$ such that

$$\varepsilon = \frac{K}{\Gamma(\alpha)(p'(\alpha-1)+1)^{\frac{1}{p'}}}\delta^{\alpha-\frac{1}{p}}$$

   That is,

$$\delta = \left(\frac{\Gamma(\alpha)(p'(\alpha-1)+1)^{\frac{1}{p'}}}{K}\varepsilon\right)^{\frac{1}{\alpha-\frac{1}{p}}}$$

   Then, for all $x,z\in[a,b]$ such that $|z-x|<\delta$, we obtain $|f_n(z)-f_n(x)|<\varepsilon$ Hence, the sequence (f_n) is uniformly equicontinuous.

   Since, (f_n) is uniformly bounded and equicontinuous on the compact interval $[a,b]$, by the **Arzela-Ascoli** theorem (1.11) the sequence is relatively compact in $C([a,b])$

   Therefore, there exist a subsequence (still denoted (f_n)) and a function $f\in C([a,b])$, such that

$$\|f_n-f\|_{L^\infty(a,b)} \to 0 \quad as \quad n\to\infty$$

□

Proposition 2.3. *1. If $\alpha\in\left]0,\frac{1}{p}\right]$, then*

$$E^{\alpha,p}_{a+,0}(a,b) \subset \left\{f\in L^p(a,b), D^\alpha_{a+}f\in L^p(a,b) \;\; and \;\; \lim_{x\to a+} I^{1-\alpha}_{a+}f(x)=0\right\} \tag{2.34}$$

2. *And, if* $\alpha \in \left]\frac{1}{p}, 1\right[$, *then:*

$$E^{\alpha,p}_{a+,0}(a,b) \subset \{f \in L^p(a,b), D^{\alpha}_{a+}f \in L^p(a,b) \ \text{and} \ f(a) = f(b) = 0\} \quad (2.35)$$

Proof. • For $\alpha \in \left]0, \frac{1}{p}\right]$ by definition (2.5) we have

$$E^{\alpha,p}_{a+,0}(a,b) = \overline{\mathcal{D}(a,b)}^{\|\cdot\|_{E^{\alpha,p}_{a+}(a,b)}}$$

So for every $f \in E^{\alpha,p}_{a+,0}(a,b)$ there is $(\varphi_n) \subset \mathcal{D}(a,b)$ such that

$$\lim_{n\to+\infty} \|\varphi_n - f\|_{E^{\alpha,p}_{a+,0}(a,b)} = 0$$

Then by theorem (2.3) we have

$$\lim_{x\to a^+} I^{1-\alpha}_{a+} f(x) = I^{1-\alpha}_{a+} f(a) = \lim_{n\to+\infty} I^{1-\alpha}_{a+}\varphi_n(a) = 0 \qquad \forall \varphi \in \mathcal{D}(a,b)$$

Hence,

$$\lim_{x\to a^+} I^{1-\alpha}_{a+} f(x) = 0 \qquad \forall x > a$$

Because $1-\alpha > \frac{1}{p'}$, so

$$E^{\alpha,p}_{a+,0}(a,b) \subset \left\{f \in L^p(a,b), D^{\alpha}_{a+}f \in L^p(a,b) \ \text{and} \ \lim_{x\to a^+} I^{1-\alpha}_{a+} f(x) = 0\right\}$$

• Now, for $\alpha \in \left]\frac{1}{p}, 1\right[$ we recall from the definition (2.5) thet, for any $f \in E^{\alpha,p}_{a+,0}(a,b)$ there exists a sequence $(f_n) \subset \mathcal{D}(a,b)$ such that

$$f_n \to f \ in \ E^{\alpha,p}_{a+,0}(a,b)$$

By the compact embedding (2.7), we have

$$\|f\|_{L^\infty(a,b)} \leq \frac{(b-a)^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)\left(p'(\alpha-1)+1\right)^{\frac{1}{p'}}} \|f\|_{E^{\alpha,p}_{a+,0}(a,b)}$$

That is,

$$\max_{x\in[a,b]} |f(x) - f_n(x)| \leq c\,\|f - f_n\|_{E^{\alpha,p}_{a+,0}(a,b)} \to 0$$

Where

$$c = \frac{(b-a)^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)\left(p'(\alpha-1)+1\right)^{\frac{1}{p'}}}$$

Therefore, we obtain

$$0 \leq |f(a)| = |f(a) - f_n(a)| < \|f - f_n\|_{L^\infty} \to 0 \ as \ n \to +\infty$$

Which implies that $f(a) = 0$, In the same way, we also have $f(b) = 0$ so

$$E^{\alpha,p}_{a+,0}(a,b) \subset \left\{f \in L^p(a,b), D^\alpha_{a+}f \in L^p(a,b) \ \ and \ f(a) = f(b) = 0\right\}$$

□

2.5 Integration by parts

2.5.1 Integration by parts for fractional integrals

Theorem 2.8. *Let $f \in L^p(a,b)$ and $g \in L^q(a,b)$ such that $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$, then we have*

$$\int_a^b f(x)\, I^\alpha_{b-}g(x)\, dx = \int_a^b g(x)\, I^\alpha_{a+}f(x)\, dx \tag{2.36}$$

Proof. By Holder Inequality and theorem (2.3) the integral$\int_a^b f(x)\, I^\alpha_{b-}g(x)\, dx$ is well defined, and we have:

$$\begin{aligned}\int_a^b f(x)\, I^\alpha_{b-}g(x)\, dx &= \frac{1}{\Gamma(\alpha)}\int_a^b f(x)\int_x^b (t-x)^{\alpha-1} g(t)\, dt\, dx \\ &= \frac{1}{\Gamma(\alpha)}\int_a^b\int_x^b (t-x)^{\alpha-1} f(x)\, g(t)\, dt\, dx\end{aligned}$$

Using Fubini's theorem we obtain:

$$\begin{aligned}\int_a^b f(x)\, I_{b-}^{\alpha} g(x)\, dx &= \frac{1}{\Gamma(\alpha)} \int_a^b \int_a^t (t-x)^{\alpha-1} f(x)\, g(t)\, dt\, dx \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b g(t) \int_a^t (t-x)^{\alpha-1} f(x)\, dx\, dt \\ \\ &= \int_a^b g(t) \left(\frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x)\, dx \right) dt \\ &= \int_a^b g(t)\, I_{a+}^{\alpha} f(t)\, dt\end{aligned}$$

Hence

$$\int_a^b f(x)\, I_{b-}^{\alpha} g(x)\, dx = \int_a^b g(t)\, I_{a+}^{\alpha} f(t)\, dt$$

- By the same way for the right fractional integral

□

2.5.2 Integration by parts for fractional derivatives

Proposition 2.4. *Let* $0 < \alpha < 1$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ *and* $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$

$$\int_a^b D_{b-}^{\alpha} f(x)\, g(x)\, dx = \int_a^b f(x)\, D_{a+}^{\alpha} g(x)\, dx \tag{2.37}$$

for all $f \in I_{b-}^{\alpha}(L^p)$ *and* $g \in I_{a+}^{\alpha}(L^q)$

Proof. We put $f = I_{b-}^{\alpha}\varphi$ and $g = I_{a+}^{\alpha}\psi$

$$\begin{aligned}\int_a^b D_{b-}^{\alpha} f(x)\, g(x)\, dx &= \int_a^b \varphi(x)\, I_{a+}^{\alpha}\psi(x)\, dx \\ &= \int_a^b I_{b-}^{\alpha}\varphi(x)\, \psi(x)\, dx \qquad by (2.36) \\ &= \int_a^b f(x)\, D_{a+}^{\alpha} g(x)\, dx\end{aligned}$$

Hence

$$\int_a^b D_{b-}^{\alpha} f(x)\, g(x)\, dx = \int_a^b f(x)\, D_{a+}^{\alpha} g(x)\, dx \tag{2.38}$$

□

Theorem 2.9. *Let* $1 \leq p < \infty$ $1 \leq p' < \infty$. *if* $\alpha \leq \min\left\{\frac{1}{p}, \frac{1}{p'}\right\}$,*then*

$$\int_a^b D_{b-}^{\alpha} f(x)\, g(x)\ dx = \int_a^b f(x)\, D_{a+}^{\alpha} g(x)\ dx \tag{2.39}$$

for all $f \in E_{b-}^{\alpha,p}(a,b)$ *and* $g \in E_{a+}^{\alpha,p'}(a,b)$

Proof. To simplify the analysis, we restrict ourselves to the case where $\alpha < \frac{1}{p'}$ and $a < \frac{1}{p}$. Under these assumptions, the conditions required to apply (theorem 2.2) are fulfilled.

$$f = I_{b-}^{\alpha}\varphi \in I_{b-}^{\alpha}(L^p) = E_{b-}^{\alpha,p}$$

and

$$g = I_{a+}^{\alpha,p'}\psi \in I_{a+}^{\alpha}\left(L^{p'}\right) = E_{a+}^{\alpha,p'}$$

By proposition (2.4) we obtain

$$\int_a^b D_{b-}^{\alpha} f(x)\, g(x)\ dx = \int_a^b f(x)\, D_{a+}^{\alpha} g(x)\ dx \tag{2.40}$$

□

Theorem 2.10. *Let* $1 \leq p < \infty$ *and* $1 \leq p' < \infty$, *if* $\alpha > \max\left\{\frac{1}{p}, \frac{1}{p'}\right\}$ *then*

$$\int_a^b D_{b-}^{\alpha} f(x)\, g(x)\, dx = \int_a^b f(x)\, D_{a+}^{\alpha} g(x)\, dx + I_{a+}^{1-\alpha} g(a)\, f(a) - I_{b-}^{1-\alpha} f(b)\, g(b) \tag{2.41}$$

for $f \in E_{b-}^{\alpha,p}(a,b)$ *and* $g \in E_{a+}^{\alpha,p'}(a,b)$

Proof. By (2.24) and (2.25)

$$f(x) = \frac{d}{\Gamma(\alpha)}(b-x)^{\alpha-1} + I_{b-}^{\alpha}\varphi(x) \qquad where \quad d = I_{b-}^{1-\alpha} f(b)$$

and

$$g(x) = \frac{c}{\Gamma(\alpha)}(x-a)^{\alpha-1} + I_{a+}^{\alpha}\psi(x) \qquad where \quad c = I_{a+}^{1-\alpha} g(a)$$

We have

$$\int_a^b D_{b-}^{\alpha} f(x)\, g(x)\ dx = \int_a^b \varphi(x) \left[\frac{c}{\Gamma(\alpha)} (x-a)^{\alpha-1} + I_{a+}^{\alpha} \psi(x)\right] dx$$
$$= \frac{c}{\Gamma(\alpha)} \int_a^b \varphi(x)\,(x-a)^{\alpha-1}\, dx + \int_a^b \varphi(x)\, I_{a+}^{\alpha} \psi(x)\, dx \qquad by(2.36)$$
$$= c\, I_{b-}^{\alpha} \varphi(a) + \int_a^b I_{b-}^{\alpha} \varphi(x)\, \psi(x)\, dx$$

$$\int_a^b D_{b-}^{\alpha} f(x)\, g(x)\ dx = I_{a+}^{1-\alpha} g(a)\, I_{b-}^{\alpha} \varphi(a) + \int_a^b I_{b-}^{\alpha} \varphi(x)\, \psi(x)\, dx \tag{2.42}$$

on the other hand we have

$$\int_a^b f(x)\, D_{a+}^{\alpha} g(x)\, dx = \int_a^b \left[\frac{d}{\Gamma(\alpha)} (b-x)^{\alpha-1} + I_{b-}^{\alpha} \varphi(x)\right] \psi(x)\, dx$$
$$= \frac{d}{\Gamma(\alpha)} \int_a^b \psi(x)\,(b-x)^{\alpha-1}\, dx + \int_a^b I_{b-}^{\alpha} \varphi(x)\, \psi(x)\, dx$$
$$= \ d I_{a+}^{\alpha} \psi(b) + \int_a^b I_{b-}^{\alpha} \varphi(x)\, \psi(x)\, dx$$

$$\int_a^b f(x)\, D_{a+}^{\alpha} g(x)\, dx = I_{b-}^{1-\alpha} f(b)\, I_{a+}^{\alpha} \psi(b) + \int_a^b I_{b-}^{\alpha} \varphi(x)\, \psi(x)\, dx \tag{2.43}$$

By substracting equation(2.42) and (2.43) , we obtain

$$\int_a^b D_{b-}^{\alpha} f(x)\, g(x)\ dx - \int_a^b f(x)\, D_{a+}^{\alpha} g(x)\, dx = I_{a+}^{1-\alpha} g(a)\ I_{b-}^{\alpha} \varphi(a) - I_{b-}^{1-\alpha} f(b)\ I_{a+}^{\alpha} \psi(b)$$

we have

$$I_{b-}^{\alpha} \varphi(a) = f(a) - \frac{I_{b-}^{1-\alpha} f(b)}{\Gamma(\alpha)} (b-a)^{\alpha-1}$$

and

$$I_{a+}^{\alpha} \psi(b) = g(b) - \frac{I_{a+}^{1-\alpha} g(a)}{\Gamma(\alpha)} (b-a)^{\alpha-1}$$

Hence

$$\int_a^b D_{b-}^{\alpha} f(x)\, g(x)\, dx = \int_a^b f(x)\, D_{a+}^{\alpha} g(x)\, dx + I_{a+}^{1-\alpha} g(a)\, f(a) - I_{b-}^{1-\alpha} f(b)\, g(b)$$

□

Theorem 2.11. *Let $1 \leq p < \infty$ and $1 \leq p' < \infty$, if $\min\left\{\frac{1}{p}, \frac{1}{p'}\right\} < \alpha \leq \max\left\{\frac{1}{p}, \frac{1}{p'}\right\}$ then*

$$\int_a^b D_{b-}^{\alpha} f(x)\, g(x)\, dx = \int_a^b f(x)\, D_{a+}^{\alpha} g(x)\, dx - I_{b-}^{1-\alpha} f(b)\, g(b) \tag{2.44}$$

for $f \in E_{b-}^{\alpha,p}(a,b)$ and $g \in E_{a+}^{\alpha,p'}(a,b)$

Proof. Without loss of generality, suppose that $\frac{1}{p'} < \frac{1}{p}$ then $\frac{1}{p'} < \alpha \leq \frac{1}{p}$. In this case we have

$$g(x) = I_{a+}^{\alpha} \psi(x)$$

and

$$f(x) = \frac{I_{b-}^{1-\alpha} f(b)}{\Gamma(\alpha)} (b-x)^{\alpha-1} + I_{b-}^{\alpha} \varphi(x)$$

$$\begin{aligned} \int_a^b D_{b-}^{\alpha} f(x)\, g(x)\, dx &= \int_a^b \varphi(x)\, g(x)\, dx \\ &= \int_a^b \varphi(x)\, I_{a+}^{\alpha} \psi(x)\, dx \qquad by (2.36) \\ &= \int_a^b I_{b-}^{\alpha} \varphi(x)\, \psi(x)\, dx \end{aligned} \tag{2.45}$$

On the other hand we obtain

$$\begin{aligned} \int_a^b f(x)\, D_{a+}^{\alpha} g(x)\, dx &= \frac{I_{b-}^{1-\alpha} f(b)}{\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1} \psi(x)\, dx + \int_a^b I_{b-}^{\alpha} \varphi(x)\, \psi(x)\, dx \\ &= I_{b-}^{1-\alpha} f(b)\, I_{a+}^{\alpha} \psi(b) + \int_a^b I_{b-}^{\alpha} \varphi(x)\, \psi(x)\, dx \end{aligned} \tag{2.46}$$

By substracting equation (2.45) and (2.46), we obtain

$$\int_a^b D_{b-}^{\alpha} f(x)\, g(x)\, dx = \int_a^b f(x)\, D_{a+}^{\alpha} g(x)\, dx - I_{b-}^{1-\alpha} f(b)\, g(b) \tag{2.47}$$

□

Theorem 2.12. *Let $1 \leq p < \infty$ and $1 \leq p' < \infty$, if $\alpha > \max\left\{\frac{1}{p}, \frac{1}{p'}\right\}$*

$$\begin{aligned} \int_a^b D_{b-}^{\alpha} \left(\left| D_{a+}^{\alpha} f(x) \right|^{p-2} D_{a+}^{\alpha} f(x) \right) g(x)\, dx &= \int_a^b \left| D_{a+}^{\alpha} f(x) \right|^{p-2} D_{a+}^{\alpha} f(x)\, D_{a+}^{\alpha} g(x)\, dx \\ &+ I_{a+}^{1-\alpha} g(a) \left(\left| D_{a+}^{\alpha} f \right|^{p-2} D_{a+}^{\alpha} f \right)(a) - I_{b-}^{1-\alpha} \left(\left| D_{a+}^{\alpha} f \right|^{p-2} D_{a+}^{\alpha} f \right)(b)\, g(b) \end{aligned} \tag{2.48}$$

for $\left| D_{a+}^{\alpha} f \right|^{p-2} D_{a+}^{\alpha} f \in E_{b-}^{\alpha,p'}(a,b)$ and $g \in E_{a+}^{\alpha,p}(a,b)$

Proof. This is result deduced from theorem (2.10) □

Corollarry 2.2. *Let* $0 < \alpha < 1$, $1 \leq p < \infty$ *and* $1 \leq p' < \infty$ *then*

$$\int_a^b D_{b-}^{\alpha} f(x) g(x) dx = \int_a^b f(x) D_{a+}^{\alpha} g(x) dx \tag{2.49}$$

for all $f \in E_{b-,0}^{\alpha,p}(a,b)$ *and* $g \in E_{a+,0}^{\alpha,p'}(a,b)$

Proof. 1. By theorem (2.10) we have

$$\int_a^b D_{b-}^{\alpha} f(x) g(x) dx = \int_a^b f(x) D_{a+}^{\alpha} g(x) dx + I_{a+}^{1-\alpha} g(a) f(a) - I_{b-}^{1-\alpha} f(b) g(b)$$

Also, by proposition (2.3) we have $f(a) = f(b) = 0$ and $g(a) = g(b) = 0$

Then,

$$\int_a^b D_{b-}^{\alpha} f(x) g(x) dx = \int_a^b f(x) D_{a+}^{\alpha} g(x) dx \tag{2.50}$$

2. By theorem (2.11we have)

$$\int_a^b D_{b-}^{\alpha} f(x) g(x) dx = \int_a^b f(x) D_{a+}^{\alpha} g(x) dx - I_{b-}^{1-\alpha} f(b) g(b)$$

Also, by proposition (2.3) we have $I_{b-}^{1-\alpha} f(b) = 0$

Then,

$$\int_a^b D_{b-}^{\alpha} f(x) g(x) dx = \int_a^b f(x) D_{a+}^{\alpha} g(x) dx \tag{2.51}$$

By this results and theorem (2.9) we conclude

$$\int_a^b D_{b-}^{\alpha} f(x) g(x) dx = \int_a^b f(x) D_{a+}^{\alpha} g(x) dx \tag{2.52}$$

□

Chapter 3

Nonlinear Fractional Eigenvalue Problem

This chapter is devoted to the study of a fractional eigenvalue problem involving the fractional p-Laplacian operator. we are interested in establishing the existence of solution to this eigenvalue problem. The analysis is based on a variational characterization of the first eigenvalue. By introducing an appropriate functional frame work, we prove the existence of a nontrivial weak solution, obtained as a minimizer of the associated Rayleigh-type quotient. Furthermore, we derive some fundamental properties of the first eigenvalue.

3.1 Preliminaries

3.1.1 Position of problem

Let (a, b) be a bounded interval, $\lambda \in \mathbb{R}$ we consider the following non linear fractional eigenvalue problem with homogeneous boundary condition:

$$(P_\lambda)\begin{cases} D_{b-}^{\alpha}\left(\left|D_{a+}^{\alpha}u\left(x\right)\right|^{p-2}D_{a+}^{\alpha}u\left(x\right)\right)=\lambda\left|u\right|^{p-2}u & in\ ]a,b[\\ \gamma_\alpha\left(u\right)=0 & on\,\partial\,]a,b[\end{cases}$$

Where $\gamma_\alpha(u)$ denotes the boundary condition associated with problem (P_λ), which depends on the value of $\alpha \in (0, 1)$, and is defined as follows:

$$\gamma_\alpha\left(u\right)=\begin{cases}\lim\limits_{x\to a}I_{a+}^{1-\alpha}u\left(x\right) & if\ \alpha\in\left]0,\frac{1}{p}\right]\\ u\left(a\right)=u\left(b\right) & if\ \alpha\in\left]\frac{1}{p},1\right[\end{cases}$$

Definition 3.1. ***(Weak solution)****A function $u \in E_{a+,0}^{\alpha,p}$ is called weak solution* *of problem (P_λ) if it satisfies the following variational formulation.*

$$\int_a^b |D_{a+}^\alpha u(x)|^{p-2} D_{a+}^\alpha u(x) D_{a+}^\alpha v(x) \, dx = \lambda \int_a^b |u(x)|^{p-2} u(x) v(x) \, dx \qquad \forall v \in \mathcal{D}(a,b) \tag{3.1}$$

Definition 3.2. • *The first eigenvalue of operator $D_{b-}^\alpha \left(|D_{a+}^\alpha u|^{p-2} D_{a+}^\alpha u \right)$ is defined as follow*

$$\lambda_1 = \inf_{u \in E_{a+,0}^{\alpha,p} \setminus \{0\}} \frac{\|D_{a+}^\alpha u\|_{L^p(a,b)}^p}{\|u\|_{L^p(a,b)}^p} \tag{3.2}$$

- *The first eigen function associated to λ_1 is a non trivial weak solution of (P_{λ_1}).*

Definition 3.3. ***(Energy functional)*** *The Rayleigh-type quotient is the functional φ defined as follows:*

$$\varphi : E_{a+,0}^{\alpha,p}(a,b) \setminus \{0\} \longrightarrow \mathbb{R}$$

$$u \longmapsto \varphi(u) = \frac{\displaystyle\int_a^b |D_{a+}^\alpha u(x)|^p \, dx}{\displaystyle\int_a^b |u(x)|^p \, dx} = \frac{\|D_{a+}^\alpha u\|_{L^p(a,b)}^p}{\|u\|_{L^p(a,b)}^p} \tag{3.3}$$

we set,

$$I(u) = \|D_{a+}^\alpha u\|_{L^p(a,b)}^p$$

and

$$J(u) = \|u\|_{L^p(a,b)}^p$$

3.2 Existence of weak solution

We now present our result concerning the existence of the first eigenvalue and the corresponding eigenfunction.

Theorem 3.1. *Let $p > 1$, $0 < \alpha < 1$. Then the first eigenvalue λ_1 is positive, and problem (P_{λ_1}) admits a nontrivial weak solution.*

To prove this theorem, we adopt a variational approach. More precisely, we establish the existence of a solution as aminimizer of the associated Rayleigh-type *quotient. We begin wiyh the following result concerning the functional φ defined above.*

Proposition 3.1. *The functional φ is well defined and differentiable on $E^{\alpha,p}_{a+,0}(a,b)\backslash\{0\}$*
Moreover, its derivative is given by:

$$\langle\varphi'(u),v\rangle=\frac{1}{\|u\|^p_{L^p}}\left(\int_a^b|D^\alpha_{a+}u|^{p-2}\,D^\alpha_{a+}u\,D^\alpha_{a+}v\,dx-\frac{\|D^\alpha_{a+}\|^p_{L^P}}{\|u\|^p_{L^P}}\int_a^b|u|^{p-2}\,u\,v\,dx\right)$$
$$\forall v\in E^{\alpha,p}_{a+,0}(a,b)$$

Proof.

- **Firstly**, we show that the functional defined in (3.3) is well defined for any $u\in E^{\alpha,p}_{a+,0}(a,b)\backslash\{0\}$

  1. For the first term:
  $$|I(u)|=\int_a^b|D^\alpha_{a+}u(x)|^p\,dx\ =\|D^\alpha_{a+}u\|^p_{L^p(a,b)}<\infty \tag{3.4}$$
  2. For the second term:
  $$|J(u)|=\int_a^b|u(x)|^p\,dx\ =\|u\|^p_{L^p(a,b)}<\infty \tag{3.5}$$

  Hence, by (3.4) and (3.5) the functional φ is well defined.

- **Secondly**, we prove that φ is Gateaux differentiable and $\varphi'(u)$ is continuous that is:
  $$\lim_{t\to0}\frac{\varphi(u+tv)-\varphi(u)}{t}=\langle\varphi'(u),v\rangle$$
  We set,
  $$I(u)=\int_a^b|D^\alpha_{a+}u(x)|^p\,dx$$
  Thus,
  $$\lim_{t\to0}\frac{I(u+tv)-I(u)}{t}=\lim_{t\to0}\int_a^b\frac{|D^\alpha_{a+}u(x)+tD^\alpha_{a+}v(x)|^p-|D^\alpha_{a+}u(x)|^p}{t}dx$$

By the Dominated Convergence Theorem (1.7)on this expression:

$$\frac{\left|D_{a+}^{\alpha}u\left(x\right)+tD_{a+}^{\alpha}v\left(x\right)\right|^{p}-\left|D_{a+}^{\alpha}u\left(x\right)\right|^{P}}{t}$$

We define

$$g:s\longrightarrow g\left(s\right)=\left|D_{a+}^{\alpha}u\left(x\right)+sD_{a+}^{\alpha}v\left(x\right)\right|^{p}$$

Since $p>1$, g is differentiable for s and we have:

$$g'\left(s\right)=p\left|D_{a+}^{\alpha}u\left(x\right)+sD_{a+}^{\alpha}v\left(x\right)\right|^{p-2}\left(D_{a+}^{\alpha}u\left(x\right)+sD_{a+}^{\alpha}v\left(x\right)\right)D_{a+}^{\alpha}v\left(x\right)$$

Then

$$\lim_{t\to0}\frac{\left|D_{a+}^{\alpha}u\left(x\right)+tD_{a+}^{\alpha}v\left(x\right)\right|^{p}-\left|D_{a+}^{\alpha}u\left(x\right)\right|^{P}}{t}=g'\left(0\right)=p\left|D_{a+}^{\alpha}u\left(x\right)\right|^{p-2}D_{a+}^{\alpha}u\left(x\right)D_{a+}^{\alpha}v\left(x\right)$$

by the means value theorem, $\exists c_t$ between 0 and t such that:

$$g\left(t\right)-g\left(0\right)=g'\left(c_t\right)t$$

Thus we obtain

$$\begin{aligned}\frac{\left|D_{a+}^{\alpha}u\left(x\right)+tD_{a+}^{\alpha}v\left(x\right)\right|^{p}-\left|D_{a+}^{\alpha}u\left(x\right)\right|^{p}}{t}&=p\left|D_{a+}^{\alpha}u\left(x\right)+c_tD_{a+}^{\alpha}v\left(x\right)\right|^{p-2}\\&\quad\left(D_{a+}^{\alpha}u\left(x\right)+c_tD_{a+}^{\alpha}v\left(x\right)\right)D_{a+}^{\alpha}v\left(x\right)\end{aligned}$$

Hence,

$$\left|\frac{\left|D_{a+}^{\alpha}u\left(x\right)+tD_{a+}^{\alpha}v\left(x\right)\right|^{p}-\left|D_{a+}^{\alpha}u\left(x\right)\right|^{p}}{t}\right|=p\left|D_{a+}^{\alpha}u\left(x\right)+c_tD_{a+}^{\alpha}v\left(x\right)\right|^{p-1}\left|D_{a+}^{\alpha}v\left(x\right)\right|$$

We apply the inequality (1.13) we obtain:

$$\begin{aligned}\left|\frac{\left|D_{a+}^{\alpha}u\left(x\right)+tD_{a+}^{\alpha}v\left(x\right)\right|^{p}-\left|D_{a+}^{\alpha}u\left(x\right)\right|^{p}}{t}\right|&\leq2^{p-2}p\left(\left|D_{a+}^{\alpha}u\right|^{p-1}+\left|D_{a+}^{\alpha}v\right|^{p-1}\right)\left|D_{a+}^{\alpha}v\right|\\&\leq2^{p-2}p\left(\left|D_{a+}^{\alpha}u\right|^{p-1}\left|D_{a+}^{\alpha}v\right|+\left|D_{a+}^{\alpha}v\right|^{p}\right)\end{aligned}$$

Then, by Holder inequality $\left|D_{a+}^{\alpha}u\left(x\right)\right|^{p-1}\left|D_{a+}^{\alpha}v\left(x\right)\right|\in L^1\left(a,b\right)$, and $\left|D_{a+}^{\alpha}v\left(x\right)\right|^{p}\in L^1\left(a,b\right)$

Hence, $2^{p-2}p\left(\left|D_{a+}^{\alpha}u\left(x\right)\right|^{p-1}\left|D_{a+}^{\alpha}v\left(x\right)\right|+\left|D_{a+}^{\alpha}v\left(x\right)\right|^{p}\right)\in L^1\left(a,b\right)$

By the Dominated Convergence Theorem (1.7) we conclude that

$$\lim_{t\to 0}\int_a^b \frac{\left|D_{a+}^{\alpha}u\,(x)+t\,D_{a+}^{\alpha}v\,(x)\right|^P-\left|D_{a+}^{\alpha}u\,(x)\right|^p}{t}dx = p\int_a^b \left|D_{a+}^{\alpha}u\right|^{p-2}D_{a+}^{\alpha}u\,D_{a+}^{\alpha}v\,dx$$

Hence, the functional I is Gateaux differentiable, and its derivative is given by:

$$\langle I'\,(u)\,,v\rangle = p\int_a^b \left|D_{a+}^{\alpha}u\,(x)\right|^{p-2}D_{a+}^{\alpha}u\,(x)\,D_{a+}^{\alpha}v\,(x)\,dx \tag{3.6}$$

Let $u\in E_{a+,0}^{\alpha,p}\,(a,b)$ we show that $I'\,(u)$ is continuous with respect to v

$$\begin{aligned}\left|\langle I'\,(u)\,,v\rangle\right| &= p\left|\int_a^b \left|D_{a+}^{\alpha}u\,(x)\right|^{p-2}D_{a+}^{\alpha}u\,(x)\,D_{a+}^{\alpha}v\,(x)\,dx\right|\\ &\le \int_a^b \left|D_{a+}^{\alpha}u\,(x)\right|^{p-1}\left|D_{a+}^{\alpha}v\,(x)\right|dx\\ &\le C\left\|D_{a+}^{\alpha}u\right\|_{L^p(a,b)}^{p-1}\left\|D_{a+}^{\alpha}v\right\|_{L^p(a,b)}^{p} \qquad (by\ Holder\ inequality)\end{aligned}$$

Therefore, we show that $I'\,(u)$ is linear and continuous functional.

By the same way, we show that the functional J is G-differentiable and we have:

$$\langle J'\,(u)\,,v\rangle = p\int_a^b \left|u\,(x)\right|^{p-2}u\,(x)\,v\,(x)\,dx \tag{3.7}$$

By (3.6) and (3.7) the functional φ is Gateaux differentiable on $E_{a+,0}^{\alpha,p}\,(a,b)\setminus\{0\}$

$$\langle\varphi'(u),v\rangle = \frac{1}{\|u\|_{L^p}^p}\left(\int_a^b \left|D_{a+}^{\alpha}u\right|^{p-2}D_{a+}^{\alpha}u\,D_{a+}^{\alpha}v\,dx-\frac{\|D_{a+}^{\alpha}\|_{L^P}^p}{\|u\|_{L^P}^p}\int_a^b |u|^{p-2}\,u\,v\,dx\right)$$

We now establish the differentiability of φ. By a standard result in functional analysis (proposition (1.1)), it suffices to show that $\varphi'\,(u)$ is continuous.

Let u_n be a sequence in $E_{a+,0}^{\alpha,p}\,(a,b)$ such that $u_n\to u$ in $E_{a+,0}^{\alpha,p}$ we show that $\varphi'\,(u_n)\to\varphi'\,(u)$ in $\left(E_{a+,0}^{\alpha,p}\,(a,b)\right)'$

We aim to show that:

$$u_n\longrightarrow u\ in\ E_{a+,0}^{\alpha,p}\,(a,b)\implies \left\|I'\,(u_n)-I'\,(u)\right\|_{\left(E_{a+,0}^{\alpha,p}(a,b)\right)'}\longrightarrow 0$$

Where

$$\left\|I'\left(u_n\right)-I'\left(u\right)\right\|_{\left(E^{\alpha,p}_{a+,0}(a,b)\right)'}=\sup_{\|v\|=1}\left|\left\langle I'\left(u_n\right)-I'\left(u\right),v\right\rangle\right|$$

Let $v\in E^{\alpha,p}_{a+,0}\left(a,b\right)$ such that $\|v\|=1$, we have

$$\begin{aligned}\left\langle I'\left(u_n\right)-I'\left(u\right),v\right\rangle&=\left\langle I'\left(u_n\right),v\right\rangle-\left\langle I'\left(u\right),v\right\rangle\\&=p\left|\int_a^b\left|D^\alpha_{a+}u_n\right|^{p-2}D^\alpha_{a+}u_nD^\alpha_{a+}v\,dx-\int_a^b\left|D^\alpha_{a+}u\right|^{p-2}D^\alpha_{a+}uD^\alpha_{a+}v\,dx\right|\\&\leq p\int_a^b\left|\left|D^\alpha_{a+}u_n\right|^{p-2}D^\alpha_{a+}u_n-\left|D^\alpha_{a+}u\right|^{p-2}D^\alpha_{a+}u\right|\left|D^\alpha_{a+}v\right|dx\end{aligned}$$

By Holder inequality we obtain:

$$\left\langle I'\left(u_n\right)-I'\left(u\right),v\right\rangle\leq p\left(\int_a^b\left|\left|D^\alpha_{a+}u_n\right|^{p-2}D^\alpha_{a+}u_n-\left|D^\alpha_{a+}u\right|^{p-2}D^\alpha_{a+}u\right|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\left\|D^\alpha_{a+}v\right\|^p_{L^p}$$

By lemma (1.4) we have

Firstly, if $1<p<2$ by inequality (1.15) we obtain:

$$\begin{aligned}\left\langle I'\left(u_n\right)-I'\left(u\right),v\right\rangle&\leq p\,c_p\left(\int_a^b\left(\left|D^\alpha_{a+}u_n-D^\alpha_{a+}u\right|^{\frac{p}{p-1}}\right)^{p-1}dx\right)^{\frac{p-1}{p}}\|v\|_{E^{\alpha,p}_{a+,0}(a,b)}\\&=p\,c_p\left(\int_a^b\left|D^\alpha_{a+}u_n-D^\alpha_{a+}u\right|^p dx\right)^{\frac{p-1}{p}}\|v\|_{E^{\alpha,p}_{a+,0}(a,b)}\\&=\|u_n-u\|^{p-1}_{E^{\alpha,p}_{a+,0}}\|v\|_{E^{\alpha,p}_{a+,0}(a,b)}\end{aligned}$$

then

$$\sup_{\|v\|_{E^{\alpha,p}_{a+,0}}=1}\left\langle I'\left(u_n\right)-I'\left(u\right)\right\rangle\leq p\,c_p\|u_n-u\|^{p-1}_{E^{\alpha,p}_{a+,0}}\to 0$$

Hence

$$\left\|I'\left(u_n\right)-I'\left(u\right)\right\|_{\left(E^{\alpha,p}_{a+,0}(a,b)\right)'}\to 0 \tag{3.8}$$

Secondly, if $p\geq 2$ by inequality (1.16) we obtain :

$$\begin{aligned}\left\langle I'\left(u_n\right)-I'\left(u\right),v\right\rangle&\leq p\,c_p\left(\int_a^b\left(\left(\left|D^\alpha_{a+}u_n\right|^{p-2}+\left|D^\alpha_{a+}u\right|^{p-2}\right)\left|D^\alpha_{a+}u_n-D^\alpha_{a+}u\right|\right)^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\\&\quad\|v\|_{E^{\alpha,p}_{a+,0}(a,b)}\\&=p\,c_p\left(\int_a^b\left(\left|D^\alpha_{a+}u_n\right|^{p-2}+\left|D^\alpha_{a+}u\right|^{p-2}\right)^{\frac{p}{p-1}}\left|D^\alpha_{a+}u_n-D^\alpha_{a+}u\right|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\\&\quad\|v\|_{E^{\alpha,p}_{a+,0}(a,b)}\end{aligned}$$

By Holder inequality for the composant $(p-1, \frac{p-1}{p-2})$ we obtain

$$\begin{aligned}\langle I'(u_n) - I'(u), v\rangle \leq c &\left[\left(\int_a^b \left(|D_{a+}^\alpha u_n|^{p-2} + |D_{a+}^\alpha u|^{p-2}\right)^{\frac{p}{p-1}\frac{p-1}{p-2}} dx\right)^{\frac{p-2}{p-1}}\right. \\ &\left.\cdot \left(\int_a^b |D_{a+}^\alpha u_n - D_{a+}^\alpha u|^{\frac{p}{p-1}p-1} dx\right)^{\frac{1}{p-1}}\right]^{\frac{p-1}{p}} \|v\|_{E_{a+,0}^{\alpha,p}(a,b)} \\ &= c\left[\left(\int_a^b \left(|D_{a+}^\alpha u_n|^{p-2} + |D_{a+}^\alpha u|^{p-2}\right)^{\frac{p}{p-2}} dx\right)^{\frac{p-2}{p-1}}\right. \\ &\left.\left(\int_a^b |D_{a+}^\alpha u_n - D_{a+}^\alpha u|^{p} dx\right)^{\frac{1}{p-1}}\right]^{\frac{p-1}{p}} \|v\|_{E_{a+,0}^{\alpha,p}(a,b)} \\ &= c\left(\int_a^b \left(|D_{a+}^\alpha u_n|^{p-2} + |D_{a+}^\alpha u|^{p-2}\right)^{\frac{p}{p-2}} dx\right)^{\frac{p-2}{p}} \|u_n - u\|_{E_{a+,0}^{\alpha,p}(a,b)} \\ &\|v\|_{E_{a+,0}^{\alpha,p}(a,b)}\end{aligned}$$

By lemma (1.3) we have

$$\sup_{\|v\|_{E_{a+,0}^{\alpha,p}}=1} \langle I'(u_n) - I'(u), v\rangle \leq 2^{\frac{2}{p}} c\left(\int_a^b \left(|D_{a+}^\alpha u_n|^{p} + |D_{a+}^\alpha u|^{p}\right) dx\right)^{\frac{p-2}{p}} \|u_n - u\|_{E_{a+,0}^{\alpha,p}(a,b)}$$

we have

$$\|u_n - u\|_{E_{a+,0}^{\alpha,p}(a,b)} \to 0$$

Hence

$$\|I'(u_n) - I'(u)\|_{\left(E_{a+,0}^{\alpha,p}(a,b)\right)'} \to 0 \tag{3.9}$$

From (3.8) and (3.9) I' is continuous

$$\|I'(u_n) - I'(u)\|_{\left(E_{a+,0}^{\alpha,p}(a,b)\right)'} \to 0 \tag{3.10}$$

By the same way for the functional J we have:

$$\|J'(u_n) - J'(u)\|_{\left(E_{a+,0}^{\alpha,p}(a,b)\right)'} \to 0 \tag{3.11}$$

Finally, by (3.10) and (3.11)we conclude that φ' is continuous

$$\|\varphi'(u_n) - \varphi'(u)\|_{\left(E_{a+,0}^{\alpha,p}(a,b)\right)'} \to 0 \tag{3.12}$$

Consequently, the functional φ is F-differentiable on $E_{a+,0}^{\alpha,p}(a,b)\backslash\{0\}$.

Proof the Theorem (3.1). By Poincare inequality (see corollary (2.1)) and (equation (2.7)) we have for all $u \in E_{a+,0}^{\alpha,p}(a,b)$, for some constant $c > 0$

$$\|u\|_{L^p}^p \leq c\left\|D_{a+}^{\alpha}u\right\|_{L^p}^p$$

Hence

$$\frac{\left\|D_{a+}^{\alpha}u\right\|_{L^p}^p}{\|u\|_{L^p}^p} \geq \frac{1}{c} > 0$$

Since λ_1 realize the minimum of the above Rayleigh quotient, we deduce that

$$\lambda_1 \geq \frac{1}{c} > 0$$

We have $\lambda_1 = \inf\limits_{w \in E_{a+,0}^{\alpha,p}\backslash\{0\}} \varphi(w)$, then there exist $(w_n)_n \subset E_{a+,0}^{\alpha,p}$ be a minimezer sequence i.e

$$\lim_{n\to+\infty}\varphi(w_n) = \inf_{w\in E_{a+,0}^{\alpha,p}}\varphi(w) = \lambda_1 > 0$$

Now we define the normalized sequence:

$$u_n = \frac{w_n}{\|w_n\|_{L^p(a,b)}} \implies \|u_n\|_{L^p(a,b)} = 1 \quad and\ D_{a+}^{\alpha}u_n = \frac{D_{a+}^{\alpha}w_n}{\|w_n\|_{L^p(a,b)}}$$

Then,

$$\varphi(w_n) = \left\|D_{a+}^{\alpha}u_n\right\|_{L^p(a,b)}^p = \varphi(u_n) \longrightarrow \lambda_1$$

The sequence $(u_n)_n$ is bounded in $E_{a+,0}^{\alpha,p}(a,b)$ then there exists a subsequence (still denoted $(u_n)_n$) and $u \in E_{a+,0}^{\alpha,p}(a,b)$ that

$$u_n \rightharpoonup u\ in\ E_{a+,0}^{\alpha,p}(a,b)$$

By the compact embedding $E_{a+,0}^{\alpha,p}(a,b) \hookrightarrow L^p(a,b)$ (see theorem (2.6)) we obtain

$$u_n \longrightarrow u\ in\ L^p(a,b)\,, \qquad \|u_n\|_{L^p(a,b)} = 1 \longrightarrow \|u\|_{L^p(a,b)} = 1$$

Also, since the norm is weakly lower semi continuous and $\varphi(u_n) \longrightarrow \lambda_1$ we have:

$$\begin{aligned}\varphi(u) = \|D^\alpha_{a+}u\|^p_{L^p(a,b)} &\leq \liminf_{n\to+\infty} \|D^\alpha_{a+}u_n\|^p_{L^p(a,b)} \\ &= \liminf_{n\to+\infty} \varphi(u_n) = \lambda_1\end{aligned}$$

Then,

$$\varphi(u) \leq \lambda_1 \tag{3.13}$$

By the definition of λ_1, we have

$$\varphi(u) \geq \lambda_1 \tag{3.14}$$

Hence, by (3.13) and (3.14) we obtain:

$$\lambda_1 = \varphi(u)$$

Since u is a minimizer of φ we have $\varphi'(u) = 0$, hence

$$\langle \varphi'(u), v\rangle = 0 \qquad \forall v \in E^{\alpha,p}_{a+,0}(a,b)$$

Which is equivalent to

$$\frac{p}{\|u\|^p_{L^p(a,b)}} \left(\langle |D^\alpha_{a+}u|^{p-2} D^\alpha_{a+}u, D^\alpha_{a+}v\rangle - \lambda_1 \langle |u|^{p-2} u, v\rangle\right) = 0$$

Hence,

$$\langle |D^\alpha_{a+}u|^{p-2} D^\alpha_{a+}u, D^\alpha_{a+}v\rangle - \lambda_1 \langle |u|^{p-2} u, v\rangle = 0$$

Hence, u is weak solution of problem (P_{λ_1}), since $\|u\|_{L^p(a,b)} = 1$, then u is nontrivial solution. $\square$

Proposition 3.2. *For $0 < \alpha < 1$ the first eigenvalue λ_1 associated with the operator $D^\alpha_{b-} |D^\alpha_{a+}u|^{p-2} D^\alpha_{a+}u$*

$$\lambda_1 \geq \left(\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\right)^p \tag{3.15}$$

Proof. We have

$$\lambda_1 = \inf_{u \in E_{a+,0}^{\alpha;p} \setminus \{0\}} \frac{\|D_{a+}^{\alpha} u\|_{L^p(a,b)}^p}{\|u\|_{L^p(a,b)}^p}$$

Which to gather by fractional Poincare inequality

$$\inf_{u \in E_{a+,0}^{\alpha;p} \setminus \{0\}} \frac{\|D_{a+}^{\alpha} u\|_{L^p}^p}{\|u\|_{L^p}^p} \geq \left(\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} \right)^p$$

Hence,

$$\lambda_1 \geq \left(\frac{\Gamma(\alpha+1)}{(b-a)} \right)^p$$

$\square$

Chapter 4

Nonlinear Fractional Problem

This chapter deals with a nonlinear fractional problem in different growth cases. We first establish the variational framework and prove that the energy functional is well defined and differentiable. Then, using the variational method, we study the existence of weak solutions. In the sublinear case, existence is obtained via coercivity and weak lower semicontinuity, while in the superlinear case, the Nehari manifold approach is used.

4.1 Position of problem

Let $0 < \alpha < 1$ and $(a, b) \subset \mathbb{R}$

Definition 4.1. *The non linear problem is given as follow*

$$(P)\begin{cases} D^{\alpha}_{b-}\left(\left|D^{\alpha}_{a+}u\left(x\right)\right|^{p-2}D^{\alpha}_{a+}u\left(x\right)\right)=f\left(x,u\right) & in\ ]a,b[\\ \gamma_{\alpha}\left(u\right)=0 \qquad on\,\partial\,]a,b[\end{cases}$$

$\gamma_{\alpha}(u)$ denotes the boundary condition associated with problem (p), which depends on the value of $\alpha \in (0,1)$, and is defined as follows:

$$\gamma_{\alpha}\left(u\right)=\begin{cases} \lim\limits_{x\to a}I^{1-\alpha}_{a+}u\left(x\right) & if\ \alpha\in\left]0,\frac{1}{p}\right] \\ u\left(a\right)=u\left(b\right) & if\ \alpha\in\left]\frac{1}{p},1\right[\end{cases}$$

- *(H_1 The function $f : (a,b) \star \mathbb{R} \to \mathbb{R}$ is called a **Caratheodory function** if:*
  - *for every $u \in \mathbb{R}$, the function $x \mapsto f(x,u)$ is measurable on (a,b).*
  - *for almost every $x \in (a,b)$ the function $u \mapsto f(x,u)$ is continuous.*
- *(H_2) We suppose that, f satisfies the **growth condition**:*

– $\exists q \in (1, p^{\star}_{\alpha}) : |f(x,u)| \leq b_0 + b_1 |u|^{q-1}$

We define the function F as a primitive of f, that is $F(x,u) = \int_0^u f(x,s)\,ds$

Definition 4.2. ***(Weak solution)*** *A function $u \in E^{\alpha,p}_{a+,0}(a,b)$ is called weak solution of problem (P) if it satisfies the following variational formulation.*

$$\int_a^b |D^{\alpha}_{a+}u(x)|^{p-2} D^{\alpha}_{a+}u(x)\, D^{\alpha}_{a+}v(x)\; dx = \int_a^b f(x,u)\, v(x)\, dx \qquad \forall v \in E^{\alpha,p}_{a+,0}(a,b) \tag{4.1}$$

Definition 4.3. ***(Energy functional)*** *The functional associated to the problem (P) is defined as follows*

$$\begin{aligned} h : E^{\alpha,p}_{a+,0}(a,b) &\longrightarrow \mathbb{R} \\ u &\longmapsto h(u) = \frac{1}{p}\int_a^b |D^{\alpha}_{a+}u(x)|^p\, dx - \int_a^b F(x,u)\, dx \end{aligned} \tag{4.2}$$

Proposition 4.1. *The functional h is well defined and differentiable on $E^{\alpha,p}_{a+,0}(a,b)$.*
Moreover, its derivative its given by:

$$\langle h'(u), v\rangle = \int_a^b |D^{\alpha}_{a+}u(x)|^{p-2} D^{\alpha}_{a+}u(x)\, D^{\alpha}_{a+}v(x)\, dx - \int_a^b f(x,u)\, v(x)\, dx \tag{4.3}$$

Proof. 1. **Firstly**, we show that the functional defined in (4.2) is well defined for any $u \in E^{\alpha,p}_{a+,0}(a,b)$

For the first term

$$I(u) = \int_a^b |D^{\alpha}_{a+}u(x)|^p\, dx < \infty \tag{4.4}$$

For the second term

$$\begin{aligned} |J(u)| = \left|\int_a^b F(x,u)\, dx\right| &\leq \int_a^b |F(x,u)|\, dx \\ &\leq b_0 \int_a^b |u(x)|\, dx + \frac{b_1}{q}\int_a^b |u(x)|^q\, dx \\ &\leq b_0 \|u\|_{L^1(a,b)} + \frac{b_1}{q}\|u\|_{L^q(a,b)} < \infty \quad \left(E^{\alpha,p}_{a+,0}(a,b) \hookrightarrow L^q(a,b)\right) \end{aligned} \tag{4.5}$$

Hence, by (4.4) and (4.5)the functional h is well defined.

2. **Secondly**, we prove that h is Gateaux differentiable and $h'(u)$ is continuous

$$\lim_{t\to 0}\frac{h(u+tv)-h(u)}{t}=\langle h'(u),v\rangle$$

We set:

$$I(u)=\|D_{a+}^{\alpha}u\|_{L^p(a,b)}^p$$

is Gateaux differentiable and continuous with respect to v (see proof of proposition (3.1)) the derivative is given by

$$\langle I'(u),v\rangle=p\int_a^b |D_{a+}^{\alpha}u(x)|^{p-2}D_{a+}^{\alpha}u(x)\,D_{a+}^{\alpha}v(x)\,dx \tag{4.6}$$

Now, for the second term

$$J(u)=\int_a^b F(x,u)\,dx$$

Then,

$$\lim_{t\to 0}\frac{J(u+tv)-J(u)}{t}=\lim_{t\to 0}\int_a^b\frac{F(x,u+tv)-F(x,u)}{t}dx$$

By the Dominated Convergence Theorem (1.7) on this expression

$$\frac{F(x,u+tv)-F(x,u)}{t}$$

We define

$$g:(x,s)\longmapsto g(x,s)=F(x,u+sv)$$

By definition of F, for almost every x, $g(x,.)$ is diff and we have:

$$g'(s)=vf(x,u+sv)$$

Then

$$\lim_{t\to 0}\frac{F(x,u+tv)-F(x,u)}{t}=g'(0)=vf(x,u)$$

By the means value theorem, $\exists c_t$ between 0 and t such that

$$g(x,t)-g(x,0)=g'(x,c_t)\,t$$

Hence,

$$F(x,u+tv)-F(x,u)=vf(x,u+c_tv)\,t$$

Which implies

$$\left|\frac{F(x,u+tv)-F(x,u)}{t}\right|=|v|\,|f(x,u+tv)|$$

Using the growth condition (H_2) we obtain

$$\left|\frac{F(x,u+tv)-F(x,u)}{t}\right|\leq|v|\left(a+b\,|u+c_tv|^{q-1}\right)$$

by (1.13) we get:

$$\begin{aligned}\left|\frac{F(x,u+tv)-F(x,u)}{t}\right| &\leq|v|\left(b_0+2^{p-1}b_1\left(|u|+|c_tv|^{q-1}\right)\right)\\ &\leq b_0\,|v|+2^{p-1}b_1\,|u|^{q-1}\,|v|+|v|^q\\ &\leq\max\left\{b_0,2^{p-1}b_1,1\right\}\left(|v|+|u|^{q-1}\,|v|+|v|^q\right)\\ &\leq C\left(|v|+|u|^{q-1}\,|u|+|v|^q\right)\end{aligned}$$

We have $v\in E^{\alpha,p}_{a+,0}(a,b)\hookrightarrow L^q(a,b)$ with $1\leq q\leq p^{\star}_{\alpha}$, by Holder inequality $|u|^{q-1}\,|v|\in L^1(a,b)$ and we have also $|v|^q\in L^1(a,b)$.Hence $C\left(|v|+|u|^{q-1}\,|v|+|v|^q\right)\in L^1(a,b)$

By the dominated convergence theorem, we conclude that:

$$\lim_{t\to0}\int_a^b\frac{F(x,u+tv)-F(x,u)}{t}dx=\int_a^b f(x,u)\,v(x)\,dx$$

Hence, the functional J is Gateaux differentiable and its derivative is given by :

$$\langle J'(u),v\rangle=\int_a^b f(x,v)\,v(x)\,dx \tag{4.7}$$

Let $u \in E^{\alpha,p}_{a+,0}(a,b)$ we show that $J'(u)$ is continuous with respect to v

$$\begin{aligned}
|\langle J'(u), v\rangle| &= \left|\int_a^b f(x,u)\, v(x)\, dx\right| \\
&\leq \int_a^b |f(x,u)|\, |v(x)|\, dx \\
&\leq \int_a^b \left(b_0 + b_1 |u|^{q-1}\right) |v|\, dx \quad (\textit{by growth condition}) \\
&\leq b_0 \int_a^b |v|\, dx + b_1 \int_a^b |u|^{q-1} |v|\, dx
\end{aligned}$$

For the first term, by the continuous embedding $E^{\alpha,p}_{a+,0}(a,b) \hookrightarrow L^1(a,b)$ we have

$$\int_a^b |v|\, dx \leq c \|v\|_{E^{\alpha,p}_{a+,0}}$$

For the second term, applying Holder inequality, we obtain

$$\int_a^b |u|^{q-1} |v|\, dx \leq c \|u\|^{q-1}_{E^{\alpha,p}_{a+,0}} \|v\|_{E^{\alpha,p}_{a+,0}}$$

combining the above estimates, we conclude that

$$|\langle J'(u), v\rangle| \leq c\left(1 + \|u\|^{q-1}_{E^{\alpha,p}_{a+,0}}\right) \|v\|_{E^{\alpha,p}_{a+,0}}$$

Hence,by (4.6) and (4.7)the functional h is Gateaux differentiable

$$\langle h'(u), v\rangle = \int_a^b |D^{\alpha}_{a+}u(x)|^{p-2} D^{\alpha}_{a+}u(x)\, D^{\alpha}_{a+}v(x)\, dx - \int_a^b f(x,u)\, v(x)\, dx$$

We now establish the differentiability of h. By astandard result in functional analysis (proposition (1.1)), it suffices to show that h' is continuous.

Let $(u_n)_n$ be a sequence in $E^{\alpha,p}_{a+,0}(a,b)$ such that $u_n \to u$ in $E^{\alpha,p}_{a+,0}(a,b)$ we show that $h'(u_n) \to h'(u)$ in $\left(E^{\alpha,p}_{a+,0}(a,b)\right)'$

Since $h'(u) = I'(u) - J'(u)$, it is sufficient to prove the continuity of $I'(u)$ and $J'(u)$.

From the previous results, we already know that I' is continuous (see proof of proposition (3.1))

$$\|I'(u_n) - I'(u)\|_{\left(E^{\alpha,p}_{a+,0}(a,b)\right)'} \to 0 \tag{4.8}$$

It remains to prove the continuity of J'. For any $v \in E^{\alpha,p}_{a+,0}(a,b)$ we have $u_n \to u$ in $E^{\alpha,p}_{a+,0}(a,b)$ by the compact embedding $E^{\alpha,p}_{a+,0}(a,b) \hookrightarrow L^q(a,b)$ Which means that $u_n \to u$ in $L^q(a,b)$ by using converse dominated convergence theorem there exists a subsequence (u_{n_k})such that

- $u_{n_k}(x) \to u(x)$ a.e $x \in (a,b)$
- $\exists w \in L^q(a,b) : |u_{n_k}(x)| \leq w$ a.e $x \in (a,b)$

we have f is caratheodory function then $f(x,u_{n_k}) \to f(x,u)$
Which implies

$$|f(x,u_{n_k}) - f(x,u)| \underset{n\longrightarrow\infty}{\longrightarrow 0} \ a.e$$

Using the growth condition we obtain

$$\begin{aligned} |f(x,u_{n_k})| &\leq a\,|u_{n_k}|^{q-1} + b \\ &\leq aw^{q-1} + b \end{aligned}$$

we have $w \in L^q \implies w^{q-1} \in L^{q'} \implies g = aw^{q-1} + b \in L^{q'}$
Hence, by the dominated convergence theorem and Holder inequality we have

$$|\langle J'(u_{n_k}) - J'(u), v\rangle| \leq \left(\int_a^b |f(x,u_{n_k}) - f(x,u)|^q\right)^{\frac{1}{q}} \|v\|_{L^q(a,b)}$$

by compact embedding we obtain

$$|\langle J'(u_{n_k}) - J'(u), v\rangle| \leq c\left(\int_a^b |f(x,u_{n_k}) - f(x,u)|^q\right)^{\frac{1}{q}} \|v\|_{E^{\alpha,p}_{a+,0}(a,b)}$$

Then,

$$\sup_{\|v\|_{E^{\alpha,p}_{a+,0}}} |\langle J'(u_{n_k}) - J'(u)\rangle| \leq c\,\|f(x,u_{n_k}) - f(x,u)\|_{L^{q'}} \to 0$$

Hence,

$$\|J'(u_{n_k}) - J'(u)\|_{\left(E^{\alpha,p}_{a+,0}(a,b)'\right)} \to 0$$

by contadiction we try to prove that $\|J'(u_n) - J'(u)\|_{\left(E^{\alpha,p}_{a+,0}(a,b)'\right)} \to 0$
$\exists \varepsilon > 0 : \forall n \in \mathbb{N}, \exists k_n \geq n : \|J'(u_{n_k}) - J'(u)\|_{\left(E^{\alpha,p}_{a+,0}(a,b)'\right)} \geq \varepsilon$ such that (u_{n_k})

has a subsequence still denoted (u_{n_k})

finally,

$$\|J'(u_n) - J'(u)\|_{\left(E^{\alpha,p}_{a+,0}(a,b)'\right)} \to 0 \tag{4.9}$$

Combining by (4.8) and (4.9) we conclude that

$$\|h'(u_n) - h'(u)\|_{\left(E^{\alpha,p}_{a+,0}(a,b)\right)'} \to 0 \tag{4.10}$$

□

4.2 Existence of the weak solution for the sublinear case $1 \leq q < p$

In this section, we establish the existence of a weak solution of problem (P) in the sublinear case $1 \leq q < p$. Our approach is based on variational methodes by showing that the associated energy functional is coercive, weakly lower semi continuous, and bounded from below.

We have the following existence result

Theorem 4.1. *Let $\alpha \in (0,1)$ and $1 \leq q < p$, the problem (P) admits at least a weak solution in $E^{\alpha,p}_{a+,0}(a,b)$.*

To prove this theorem, we use the following fundamental result.

Theorem 4.2. ***(Minimization Principle)*** *Let E be a reflexive Banach space, and let $h : E \to \mathbb{R}$ be a functional that is coercive, bounded from below and weakly lower semi continuous. then h attains minimum at some point $u \in E$*

Lemma 4.1. ***(Coercivity)*** *The functional h is coercive in $E^{\alpha,p}_{a+,0}(a,b)$*

$$\lim_{\|u\| \to +\infty} h(u) = +\infty$$

Proof. By the definition of h

$$\begin{aligned} h(u) &= \frac{1}{p}\int_a^b |D^{\alpha}_{a+}u(x)|^p \, dx - \int_a^b F(x,u)\, dx \\ &= \frac{1}{p}\|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)} - \int_a^b F(x,u)\, dx \end{aligned}$$

using the growth condition we obtain

$$h(u) \geq \frac{1}{p} \|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)} - b_0 \|u\|_{L^1(a,b)} - \frac{b_1}{q} \|u\|^q_{L^q(a,b)}$$

by the continuous embedding $E^{\alpha,p}_{a+,0}(a,b) \hookrightarrow L^q(a,b)$ with $1 \leq q < p$ we get

$$h(u) \geq \frac{1}{p} \|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)} - c_1 b_0 \|u\|_{E^{\alpha,p}_{a+,0}(a,b)} - c_2 \frac{b_1}{q} \|u\|^q_{E^{\alpha,p}_{a+,0}(a,b)}$$

Hence,

$$\lim_{\|u\| \to +\infty} h(u) \geq \frac{1}{p} \lim_{\|u\| \to +\infty} \|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)} = +\infty$$

□

Lemma 4.2. ***(Weakly lower semi continuous)*** *The functional h is weakly lower semi continuous in $E^{\alpha,p}_{a+,0}(a,b)$ i.e for any sequence $(u_n) \subset E^{\alpha,p}_{a+,0}(a,b)$ if*

$$u_n \rightharpoonup u \text{ in } E^{\alpha,p}_{a+,0}(a,b) \quad \text{then} \quad \liminf_{n \to +\infty} h(u_n) \geq h(u)$$

Proof. By the functional (4.2) we set:

$$I(u) = \frac{1}{p} \|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)}$$

we have $(u_n) \subset E^{\alpha,p}_{a+,0}(a,b)$ such that $u_n \rightharpoonup u$ in $E^{\alpha,p}_{a+,0}(a,b)$ and we have $\|\cdot\|$ is weakly lower semi continuous. then

$$\|u\|_{E^{\alpha,p}_{a+,0}(a,b)} \leq \liminf_{n \to +\infty} \|u_n\|_{E^{\alpha,p}_{a+,0}(a,b)}$$

$$I(u) \leq \liminf_{n \to +\infty} I(u_n) \tag{4.11}$$

We try to prove that

$$\lim_{n \to +\infty} \int_a^b F(x, u_n)\, dx = \int_a^b F(x, u)\, dx$$

by the compact embedding $u_n \to u$ in $L^q(a,b)$ there exist a subsequence (u_{n_k}) such that

- $u_{n_k}(x) \to u(x)$ a.e $x \in (a,b)$

- $\exists g \in L^q(a,b) : |u_{n_k}(x)| \leq g$ a.e $x \in (a,b)$

by the continuity of F implies that

$$F(x, u_{n_k}) \to F(x,u) \, a.e \; x \in (a,b)$$

$$\begin{aligned} |F(x,u_{n_k})| &\leq b_0 |u_{n_k}| + \frac{b_1}{q} |u_{n_k}|^q \\ &\leq b_0 g + \frac{b_1}{q} g^q \end{aligned}$$

we have $g \in L^q(a,b)$ which implies that $|g|^q \in L^1(a,b)$ with $1 \leq q < p$.then $g \in L^1(a,b)$. Hence $b_0 g + \frac{b_1}{q} g^q \in L^1(a,b)$
By dominated convergence theorem we obtain that

$$\lim_{n \to +\infty} \int_a^b F(x, u_{n_k}) \, dx = \int_a^b F(x,u) \, dx$$

$$\lim_{n \to +\infty} J(u_{n_k}) = J(u)$$

By the same argument in (proof of proposition (4.1)) we obtain

$$\lim_{n \to +\infty} \int_a^b F(x, u_n) = \int_a^b F(x,u) \tag{4.12}$$

Conclusion, from (4.11) and (4.12) we conclude that

$$h(u) \leq \liminf_{n \to +\infty} h(u_n) \tag{4.13}$$

Hence, h is weakly lower semi continuous. □

Lemma 4.3. ***(Boundedness from below)*** *The functional h is bounded from below in $E^{\alpha,p}_{a+,0}(a,b)$ i.e, there exists $M \in \mathbb{R}$ such that*

$$h(u) \geq M \qquad \forall u \in E^{\alpha,p}_{a+,0}(a,b)$$

Proof. From the definition of functional h, we have

$$h(u) = \frac{1}{p} \|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)} - \int_a^b F(x,u) \, dx$$

using the growth condition on F

$$\begin{aligned} h(u) &\geq \frac{1}{p}\|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)} - b_0\int_a^b |u|\,dx - \frac{b_1}{q}\int_a^b |u|^q\,dx \\ &\geq \frac{1}{p}\|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)} - b_0\|u\|_{L^1(a,b)} - \frac{b_1}{q}\|u\|^q_{L^q(a,b)} \end{aligned}$$

by the continuous embedding we obtain

$$\begin{aligned} h(u) &\geq \frac{1}{p}\|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)} - b_0c_1\|u\|_{E^{\alpha,p}_{a+,0}(a,b)} - \frac{b_1c_2}{q}\|u\|^q_{E^{\alpha,p}_{a+,0}(a,b)} \\ &\geq \frac{1}{p}\|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)} - c_3\|u\|_{E^{\alpha,p}_{a+,0}(a,b)} - c_4\|u\|^q_{E^{\alpha,p}_{a+,0}(a,b)} \end{aligned}$$

By Young's inequality

$$\begin{aligned} h(u) &\geq \frac{1}{p}\|u\|^P_{E^{\alpha,p}_{a+,0}(a,b)} - c_3\epsilon_1\|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)} - c - c_4\epsilon_2\|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)} - c' \\ &\geq \frac{1}{p}\|u\|^P_{E^{\alpha,p}_{a+,0}(a,b)} - (c_3\epsilon_1 + c_4\epsilon_2)\|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)} - (c + c') \\ &\geq \frac{1}{p}\|u\|^P_{E^{\alpha,p}_{a+,0}(a,b)} - \epsilon\|u\|^P_{E^{\alpha,p}_{a+,0}(a,b)} - c'' \\ &\geq \left(\frac{1}{p} - \epsilon\right)\|u\|^P_{E^{\alpha,p}_{a+,0}(a,b)} - c'' \end{aligned}$$

For $\epsilon < \frac{1}{p}$ then $\frac{1}{p} - \epsilon > 0$

Hence,

$$h(u) \geq -c''$$

$\square$

Proof of theorem (4.1): According to theorem (4.2) the functional h attains its minimum at some $u \in E^{\alpha,p}_{a+,0}(a,b)$. Since h is differentaible (see proposition (4.1)), u is critical point of h.

4.3 Existence of the weak solution for the superlinear case $p < q < p_\alpha^\star$

In this situation, the energy functional associated with problem (P) is not bounded from below on the whole space $E^{\alpha,p}_{a+,0}(a,b)$. However, this property can be recovered by restricting the functional to a suitable subset of the space. A natural choice is the Nehari manifold, which arises directly from the variational structure of the problem.

4.3.1 Preliminaries

In this part, we consider the following nonlinear problem

$$(P)\begin{cases} D_{b-}^{\alpha}\left(\left|D_{a+}^{\alpha}u(x)\right|^{p-2}D_{a+}^{\alpha}u(x)\right)=\left|u(x)\right|^{q-2}u(x) & in\ ]a,b[\\ \gamma_\alpha(u)=0 \qquad on\ \partial\,]a,b[\end{cases}$$

Where the nonlinearity is assumed to be superlinear, that is $(q > p)$.

Definition 4.4. ***(Weak solution)*** *A function $u \in E_{a+,0}^{\alpha,p}(a,b)$ is called weak solution of problem (P) if it satisfies the following variational formulation.*

$$\int_a^b \left|D_{a+}^{\alpha}u(x)\right|^{p-2}D_{a+}^{\alpha}u(x)\,D_{a+}^{\alpha}v(x)\;dx=\int_a^b \left|u(x)\right|^{q-2}u(x)\,v(x)\,dx \qquad \forall v\in\mathcal{D}(a,b) \tag{4.14}$$

Definition 4.5. ***(Energy functional)*** *The functional associated to the problem (P) is defined as follows*

$$\begin{aligned} h : E_{a+,0}^{\alpha,p}(a,b) &\longrightarrow \mathbb{R} \\ u &\longmapsto h(u)=\frac{1}{p}\int_a^b \left|D_{a+}^{\alpha}u(x)\right|^{p}dx-\frac{1}{q}\int_a^b \left|u(x)\right|^{q}dx \end{aligned} \tag{4.15}$$

Proposition 4.2. *The functional h is well defined and differentiable on $E_{a+,0}^{\alpha,p}(a,b)$.*

Moreover, its derivative its given by:

$$\left\langle h'(u),v\right\rangle=\int_a^b \left|D_{a+}^{\alpha}u(x)\right|^{p-2}D_{a+}^{\alpha}u(x)\,D_{a+}^{\alpha}v(x)\,dx-\int_a^b \left|u(x)\right|^{q-2}u(x)\,v(x)\,dx \tag{4.16}$$

Proof. 1. **Firstly**, we show that the functional defined in (4.15) is well defined for any

$u \in E_{a+,0}^{\alpha,p}(a,b)$

For the first term

$$I(u) = \int_a^b |D_{a+}^\alpha u(x)|^p \, dx < \infty \tag{4.17}$$

For the second term

$$\begin{aligned} J(u) = \int_a^b |u(x)|^q \, dx &= \|u\|_{L^q}^q \\ &\leq c\,\|u\|_{E_{a+,0}^{\alpha,p}}^q < \infty \qquad \left(E_{a+,0}^{\alpha,p} \hookrightarrow L^q\right) \end{aligned} \tag{4.18}$$

Hence, by (4.17) and (4.18)the functional h is well defined.

2. **Secondly**, we prove that h is Gateaux differentiable and $h'(u)$ is continuous

$$\lim_{t\to 0} \frac{h(u+tv) - h(u)}{t} = \langle h'(u), v\rangle$$

For the first term,

$$I(u) = \|D_{a+}^\alpha u\|_{L^p(a,b)}^p$$

is Gateaux differentiable and continuous with respect to v (see proof of proposition (3.1)) the derivative is given by

$$\langle I'(u), v\rangle = p\int_a^b |D_{a+}^\alpha u(x)|^{p-2}\, D_{a+}^\alpha u(x)\, D_{a+}^\alpha v(x)\, dx \tag{4.19}$$

For the second term,

$$J(u) = \|u\|_{L^q(a,b)}^q$$

is Gateaux differentiable the proof follows the same steps in proposition (3.1) then the derivative is given by

$$\langle J'(u), v\rangle = q\int_a^b |u(x)|^{q-2}\, u(x)\, v(x)\, dx \tag{4.20}$$

Let $u \in E_{a+,0}^{\alpha,p}(a,b)$ we show that $J'(u)$ is continuous with respect to v

$$\begin{aligned} |\langle J'(u), v\rangle| &= \left|\int_a^b |u(x)|^{q-2}\, u(x)\, v(x)\, dx\right| \\ &\leq \int_a^b |u(x)|^{q-1}\, |v(x)|\, dx \end{aligned}$$

By Holder inequality and the compact embedding we obtain:

$$
\begin{aligned}
\left|\left\langle J'(u), v\right\rangle\right| &\leq \left(\int_a^b |u(x)|^q \, dx\right)^{\frac{q-1}{q}} \|v\|_{L^q(a,b)} \\
&\leq c \, \|u\|_{E_{a+,0}^{\alpha,p}(a,b)}^{q-1} \|v\|_{E_{a+,0}^{\alpha,p}(a,b)}
\end{aligned}
$$

Hence,by (4.19) and (4.20)the functional h is Gateaux differentiable

$$
\left\langle h'(u), v\right\rangle = \int_a^b \left|D_{a+}^\alpha u(x)\right|^{p-2} D_{a+}^\alpha u(x) \, D_{a+}^\alpha v(x) \, dx - \int_a^b |u(x)|^{q-2} u(x) \, v(x) \, dx
$$

We now establish the differentiability of h. By astandard result in functional analysis (proposition (1.1)), it suffices to show that h' is continuous.

Let $(u_n)_n$ be a sequence in $E_{a+,0}^{\alpha,p}(a,b)$ such that $u_n \to u$ in $E_{a+,0}^{\alpha,p}(a,b)$ we show that $h'(u_n) \to h'(u)$ in $\left(E_{a+,0}^{\alpha,p}(a,b)\right)'$

Since $h'(u) = I'(u) - J'(u)$, it is sufficient to prove the continuity of I' and J'.

From the previous results, we already know that I' is continuous (see proof of proposition (3.1))

$$
\left\|I'(u_n) - I'(u)\right\|_{\left(E_{a+,0}^{\alpha,p}(a,b)\right)'} \to 0 \tag{4.21}
$$

Also J' is continuous the proof folllows the same steps (of proposition (3.1))

$$
\left\|J'(u_n) - J'(u)\right\|_{(} E_{a+,0}^{\alpha,p}(a,b))' \to 0 \tag{4.22}
$$

Combining by (4.21) and (4.22) we conclude that

$$
\left\|h'(u_n) - h'(u)\right\|_{\left(E_{a+,0}^{\alpha,p}(a,b)\right)'} \to 0 \tag{4.23}
$$

□

Definition 4.6. ***(Nehari manifold)*** *we define the Nehari manifold associated with problem (P) as follows*

$$
\mathcal{M} = \left\{u \in E_{a+,0}^{\alpha,p}(a,b) : \left\langle h'(u), u\right\rangle = 0\right\} \tag{4.24}
$$

Which is equivalent to:

$$\mathcal{M} = \left\{ u \in E_{a+,0}^{\alpha,p}(a,b) \setminus \{0\} : \|u\|^p_{E_{a+,0}^{\alpha,p}(a,b)} = \|u\|^q_{L^q(a,b)} \right\} \tag{4.25}$$

Remark 4.1. *If $u \in \mathcal{M}$ we have*

$$h(u) = \frac{1}{p}\|u\|^p_{E_{a+,0}^{\alpha,p}(a,b)} - \frac{1}{q}\|u\|^q_{L^q(a,b)} = \left(\frac{1}{p} - \frac{1}{q}\right)\|u\|^p_{E_{a+,0}^{\alpha,p}(a,b)} \tag{4.26}$$

In this case, the functional h is given by (4.26) is coercive on $\mathcal{M}$. Indeed, since $q \in]p, p_\alpha^\star[$ we have

$$\frac{1}{p} - \frac{1}{q} > 0$$

Hence,

$$\lim_{\|u\| \to +\infty} h(u) = \lim_{\|u\| \to +\infty} \|u\|^p_{E_{a+,0}^{\alpha,p}(a,b)} = +\infty$$

4.3.2 Existence of non trivial weak solution

We now state the main theorem.

Theorem 4.3. *Let $q \in]p, p_\alpha^\star[$. The problem (P) admits a non trivial weak solution in $\mathcal{M}$*

The proof of this theorem is based on the proofs of the following lemmas.

Lemma 4.4. *The Nehari manifold defined by (4.25) is not empty.*

Proof. Let $u \in E_{a+,0}^{\alpha,p}(a,b)$, with $u \neq 0$. We try to prove that there exists a real $t > 0$ such that $tu \in \mathcal{M}$ that is:

$$\|tu\|^p_{E_{a+,0}^{\alpha,p}(a,b)} = \|tu\|^q_{L^q(a,b)}$$

$$t^p\|u\|^p_{E_{a+,0}^{\alpha,p}(a,b)} = t^q\|u\|^q_{L^q(a,b)}$$

$$\frac{\|u\|^p_{E_{a+,0}^{\alpha,p}(a,b)}}{\|u\|^q_{L^q(a,b)}} = t^{q-p}$$

Then,

$$t = \left(\frac{\|u\|^p_{E_{a+,0}^{\alpha,p}(a,b)}}{\|u\|^q_{L^q(a,b)}}\right)^{\frac{1}{q-p}} > 0 \tag{4.27}$$

Consequently, there exists $t > 0$ such that $tu \in \mathcal{M}$, which ensures that the Nehari manifold $\mathcal{M}$ is nonempty. $\square$

Lemma 4.5. *We define* $m = \inf\limits_{u \in \mathcal{M}} h(u)$ *and we have* $m > 0$

Proof. Let $u \in \mathcal{M}$ we have

$$\|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)} = \|u\|^q_{L^q(a,b)}$$

by the compact embedding we obtain:

$$\begin{aligned} \|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)} &\leq c\,\|u\|^q_{E^{\alpha,p}_{a+,0}(a,b)} \\ \frac{1}{c} &\leq \|u\|^{q-p}_{E^{\alpha,p}_{a+,0}(a,b)} \end{aligned}$$

Then, for some $c > 0$

$$\|u\|_{E^{\alpha,p}_{a+,0}(a,b)} \geq \frac{1}{c^{q-p}} \tag{4.28}$$

We have

$$\begin{aligned} m = \inf_{u \in \mathcal{M}} h(u) &= \left(\frac{1}{p} - \frac{1}{q}\right) \inf_{u \in \mathcal{M}} \|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)} \\ &\geq \left(\frac{1}{p} - \frac{1}{q}\right) \frac{1}{c^{q-p}} > 0 \end{aligned}$$

Hence,

$$m = \inf_{u \in \mathcal{M}} h(u) > 0$$

$\square$

Lemma 4.6. *The energy functional h attains its minimum over the Nehari manifold $\mathcal{M}$ at some $u \neq 0$. That is*
$\exists u \in \mathcal{M}$*:* $m = \inf\limits_{u \in \mathcal{M}} h(u) = h(u)$

Proof. Let $(u_n) \subset \mathcal{M}$ be a minimizing sequence that is

$$\lim_{n \to +\infty} h(u_n) = \inf_{u \in \mathcal{M}} h(u) = m$$

From the definition of the Nehari manifold $\mathcal{M}$, we have

$$(u_n)_{n\in\mathbb{N}} \in \mathcal{M} \Leftrightarrow \begin{cases} u_n \in E_{a+,0}^{\alpha,p}(a,b), u_n \neq 0 \\ \|u_n\|_{E_{a+,0}^{\alpha,p}(a,b)}^p = \|u_n\|_{L^q(a,b)}^q \end{cases}$$

And

$$\lim_{n\to+\infty} h(u_n) = \left(\frac{1}{p} - \frac{1}{q}\right) \|u_n\|_{E_{a+,0}^{\alpha,p}(a,b)}^p = m$$

Since, h is coercive, the sequence (u_n) is bounded in $E_{a+,0}^{\alpha,p}(a,b)$. Moreover, by the compact embedding $E_{a+,0}^{\alpha,p}(a,b) \hookrightarrow L^q(a,b)$. Therefore, up to a subsequence, there exists $u \in E_{a+,0}^{\alpha,p}(a,b)$ such that

$$u_n \rightharpoonup u \ in \ E_{a+,0}^{\alpha,p}(a,b), \ u_n \to u \ in \ L^q(a,b)$$

Then, by the converse dominated convergence theorem we obtain

$$u_n(x) \to u(x) \ a.e\, x$$

we have

$$h(u) = \frac{1}{p} \|u\|_{E_{a+,0}^{\alpha,p}(a,b)}^p - \frac{1}{q} \|u\|_{L^q(a,b)}^q$$

Since, $u_n \rightharpoonup u$ in $E_{a+,0}^{\alpha,p}(a,b) \implies \|u\| \leq \liminf_{n\to+\infty} \|u_n\|$

$$\begin{aligned} h(u) &\leq \frac{1}{p}\liminf_{n\to+\infty} \|u_n\|_{E_{a+,0}^{\alpha,p}(a,b)}^P - \frac{1}{q}\liminf_{n\to+\infty} \|u_n\|_{L^q(a,b)}^q \\ &= \liminf_{n\to+\infty} \left(\frac{1}{p} \|u_n\|_{E_{a+,0}^{\alpha,p}(a,b)}^p - \frac{1}{q} \|u_n\|_{L^q(a,b)}^q\right) \\ &= \liminf_{n\to+\infty} h(u_n) \end{aligned}$$

Hence,

$$h(u) \leq m \tag{4.29}$$

On the other hand, if $u \in \mathcal{M}$ then by definition of m

$$m = \inf_{u\in\mathcal{M}} h(u) \leq h(u) \tag{4.30}$$

Since $(u_n) \in \mathcal{M}$, we have

$$\|u_n\|_{E_{a+,0}^{\alpha,p}(a,b)}^p = \|u_n\|_{L^q(a,b)}^q$$

Assume by contradiction that

$$\|u_n\|_{E^{\alpha,p}_{a+,0}(a,b)} \to 0 \, as \, n \to +\infty$$

Then, since $E^{\alpha,p}_{a+,0}(a,b)$ is reflexive, up to a subsequence (still denoted (u_n))

$$u_n \rightharpoonup u \, in \, E^{\alpha,p}_{a+,0}(a,b)$$

Using the weak lower semi continuity, we get

$$\|u\|_{E^{\alpha,p}_{a+,0}(a,b)} \leq \liminf_{n\to+\infty} \|u_n\|_{E^{\alpha,p}_{a+,0}(a,b)} = 0 \implies u = 0$$

This contradicts inequality (4.28) which ensures that there exists $c > 0$ such that

$$\|u\|_{E^{\alpha,p}_{a+,0}(a,b)} \geq \frac{1}{c^{q-p}} > 0 \;\; forall \;\; u \in \mathcal{M}$$

Hence, the assumption $\|u_n\|_{E^{\alpha,p}_{a+,0}(a,b)} \to 0$ must be false, then $u \neq 0$

It remains to show that

$$\|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)} = \|u\|^q_{L^q(a,b)}$$

By the lower semi continuity of the norm we have

$$\begin{aligned}\|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)} &\leq \liminf_{n\to+\infty} \|u_n\|^p_{E^{\alpha,p}_{a+,0}(a,b)} \\ &= \liminf_{n\to+\infty} \|u_n\|^q_{L^q(a,b)} = \|u\|^q_{L^q(a,b)}\end{aligned}$$

Assume by contradiction that

$$\|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)} < \|u\|^q_{L^q(a,b)}$$

$$\frac{\|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)}}{\|u\|^q_{L^q(a,b)}} < 1$$

We put $t = \left(\frac{\|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)}}{\|u\|^q_{L^q(a,b)}}\right)^{\frac{1}{q-p}} < 1$

then $tu \in \mathcal{M}$

$$\begin{aligned} h(tu) &= \left(\frac{1}{p}-\frac{1}{q}\right)\|tu\|^p_{E^{\alpha,p}_{a+,0}(a,b)} \\ &= t^P\left(\frac{1}{p}-\frac{1}{q}\right)\|u\|^P_{E^{\alpha,p}_{a+,0}(a,b)} \\ &\leq t^P \liminf_{n\to+\infty}\left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|^p_{E^{\alpha,p}_{a+,0}(a,b)} \\ &= t^p \liminf_{n\to+\infty} h(u_n) \\ &\leq t^p m \end{aligned}$$

Hence by (4.30) we get

$$m < h(u_n) \leq t^p m \implies t \geq 1$$

This leads to a contradiction, thus our assumption was false, and we conclude that

$$\|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)} = \|u\|^q_{L^q(a,b)} \implies u \in \mathcal{M}$$

Finally by (4.29) and (4.30) it follows that

$$h(u) = m$$

□

Lemma 4.7. *Let $u \in \mathcal{M}$ be a minimizer of the energy functional h on the Nehari manifold $\mathcal{M} \subset E^{\alpha,p}_{a+,0}(a,b)$ then u is critical point of h that is:*

$$h'(u) = 0 \qquad in\ E^{\alpha,p}_{a+,0}(a,b)$$

Proof. Let $v \in E^{\alpha,p}_{a+,0}(a,b)$ and we define the functions as follows

$$\phi(s) = \left(\frac{\|u+sv\|^p_{E^{\alpha,p}_{a+,0}(a,b)}}{\|u+sv\|^q_{L^q(a,b)}}\right)^{\frac{1}{q-p}}$$

Where $s \in (-\epsilon,\epsilon)$ is small enough and $\phi(s) > 0$ such that $\phi(s)(u+sv) \in \mathcal{M}$. And we have

$$\phi(0) = \left(\frac{\|u\|^p_{E^{\alpha,p}_{a+,0}(a,b)}}{\|u\|^q_{L^q(a,b)}}\right)^{\frac{1}{q-p}} = 1$$

Thus, we define γ as

$$\gamma(s) = h(\phi(s)(u + sv))$$

Since u is a minimizer of h on $\mathcal{M}$ the function $\gamma(s)$ has a minimum at $s = 0$

$$\gamma(0) = h(u) \leq \gamma(s)$$

Therefore,

$$\gamma'(0) = 0$$

Differentiating γ with respect to s we obtain:

$$\gamma'(s) = \langle h'(\phi(s)(u + sv)), \phi'(s)(u + sv) + \phi(s)v\rangle$$

Since, $\phi(0) = 1$ and $u \in \mathcal{M}$ this yields:

$$\gamma'(0) = \langle h'(\phi(0)u), \phi'(0)u + v\rangle$$

Using the fact $\langle h'(u), u\rangle = 0$ we conclude

$$\gamma'(0) = \langle h'(u), v\rangle = 0$$

Proof of theorem (4.3): By above lemma the functional h admits a critical point whish is the non-trivial weak solution in $\mathcal{M}$. $\square$

General Conclusion

In this thesis, we investigate nonlinear fractional differential equations involving the fractional p-Laplacian operator. The study is carried out within a variational framework based on tools from fractional calculus and functional analysis.

We first develop the necessary mathematical background, including Lebesgue spaces, fractional Sobolev spaces, and fractional derivatives in the sense of Riemann–Liouville. These tools are then applied to the analysis of this type of problem.

In particular, we establish the existence of weak solutions for a nonlinear fractional problem. The sublinear case is treated using a minimization method, while the superlinear case is addressed via the Nehari manifold technique.

Overall, this work highlights the effectiveness of variational methods in the study of nonlinear fractional differential equations and demonstrates their importance in the analysis of complex nonlocal models.

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Abstract:

This thesis investigates nonlinear fractional differential equations involving Riemann–Liouville derivatives. After introducing the appropriate functional framework, we study two nonlinear fractional problems within a variational setting. The first problem, which is a nonlinear eigenvalue problem, is studied using the minimization method to establish the existence of weak solutions. The second problem is a general nonlinear fractional equation, where two cases are considered: the sublinear case, treated via a minimization method, and the superlinear case, analyzed using the Nehari manifold technique.

Key-words: Riemann-Liouville fractional integral - Riemann-Liouville fractional derivative - Fractional Sobolev spaces - Eigenvalue - Eigenfunction.

Resume:

Cette thèse étudie les équations différentielles fractionnaires non linéaires impliquant les dérivées de Riemann–Liouville. Après avoir introduit le cadre fonctionnel approprié, nous analysons deux problèmes fractionnaires non linéaires dans un cadre variationnel. Le premier problème, qui est un problème de valeurs propres non linéaire, est étudié à l'aide de la méthode de minimisation afin d'établir l'existence de solutions faibles. Le second problème est une équation fractionnaire non linéaire générale, dans laquelle deux cas sont considérés : le cas sous-linéaire traité par la méthode de minimisation, et le cas super-linéaire étudié à l'aide de la variété de Nehari (Nehari manifold).

Mots-cles: Intégrale fractionnaire de Riemann-Liouville - Dérivée fractionnaire de Riemann-Liouville - Espaces de Sobolev fractionnaire - valeur propre - fonction proper.

ملخص:

تتناول هذه المذكرة دراسة المعادلات التفاضلية الكسرية غير الخطية التي تتضمن مشتقات ريمان–ليوفيل. بعد تقديم الإطار الدالي المناسب، نقوم بدراسة مسألتين كسريتين غير خطيتين ضمن إطار تغايري. المسألة الأولى، وهي مسألة قيم ذاتية غير خطية، يتم فيها إثبات وجود حلول ضعيفة باستخدام طريقة التصغير. أما المسألة الثانية فهي معادلة كسرية غير خطية عامة، حيث نميز حالتين: الحالة دون الخطية التي تُعالج بطريقة التصغير، والحالة فوق الخطية التي تُدرس باستعمال متشعب نيهاري.

الكلمات مفتاحية: التكامل الكسري لريمان ليوفييل - المشتق الكسري لريمان ليوفييل - فضاءات سوبوليف الكسرية - قيمة ذاتية - دالة ذاتية .