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Commutator estimates for vector fields on variable Triebel–Lizorkin spaces

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Abstract

In this paper we present a bilinear estimate for commutators on Triebel–Lizorkin spaces with variable smoothness and integrability, and under no vanishing assumptions on the divergence of vector fields.

Keywords Commutator · Vector fields · Triebel–Lizorkin space · Variable exponent

Mathematics Subject Classification 46E35 · 42B37

1 Introduction

Function spaces have been a central topic in modern analysis, and are now of increasing applications in many fields of mathematics especially harmonic analysis and partial differential equations. The most known general scales of function spaces are the scales of Sobolev spaces, Besov spaces and Triebel–Lizorkin spaces, and it is known that they cover many well-known classical function spaces such as Hölder–Zygmund spaces, Hardy spaces, Lebesgue spaces and Sobolev spaces. Further details on the classical theory of these spaces can be found in [14–16]. An example where the function spaces play an important role is the Euler equations for the homogeneous incompressible fluid flows:

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla u)u = -\nabla p, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ \mathrm{div} u = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $u = (u_1, \dots, u_n)$ is the velocity of the fluid flows, p is the scalar pressure, and u_0 is the given initial velocity satisfying $\mathrm{div} u_0 = 0$.

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The theory of Euler equation in function spaces has been developed in detail in [3] but has a longer history already including many contributors; we do not want to discuss this here. One of the main tools to study the Euler equation is the estimation of the commutator

$$[V \cdot \nabla, \Delta_j]f = \sum_{k=1}^{n} V_k \partial_k \Delta_j f - \Delta_j (V_k \partial_k f), \tag{1}$$

with $\Delta_j f = \varphi_j * f, j \in \mathbb{N}_0$, $V = (V_1, \dots, V_n)$ is a smooth vector field in $\mathbb{R}^n$ and $(\mathcal{F}\varphi_j)_{j \in \mathbb{N}_0}$ is a smooth dyadic resolution of unity, see Sect. 2. For the Euler equations V in (1) can be taken the velocity of the fluid flows, see for example [3, Lemma 2.100] and [4, p. 663].

Recently in [11] were presented new estimates of (1) in weighted and variable exponent Lebesgue, Triebel–Lizorkin, and Besov spaces. These estimates are obtained under no vanishing assumptions on the divergence of the vector field.

In recent years, there has been growing interest in generalizing classical spaces such as Sobolev spaces, Besov spaces, Triebel–Lizorkin spaces to the case with either variable integrability or variable smoothness. The motivation for the increasing interest in such spaces comes not only from theoretical purposes, but also from applications to fluid dynamics [13], image restoration [5] and PDE with non-standard growth conditions.

Triebel–Lizorkin spaces of variable smoothness and integrability, $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$, initially appeared in the paper [9]. Several basic properties were established. When p, q, s are constants they coincide with the usual function spaces $F^{s}_{p,q}$. Also the atomic and molecular representations for the spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$ were already obtained in [9]. Taking $s \in \mathbb{R}$ and $q \in (0, \infty]$ as constants we derive the spaces $F^{s}_{p(\cdot),q}$ studied by Xu [18]. Some properties of these function spaces such as local means characterizations and characterizations by ball means of differences can be found in [12, 17].

The main aim of this paper is to estimate (1) in variable Triebel–Lizorkin spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$. Allowing s, p and q to vary from point to point will raise extra difficulties which, in general, are overcome by imposing some regularity assumptions on these exponents. When s and q are constants the commutator estimate in $F^{s}_{p(\cdot),q}$ is given in [11], which is based on the boundedness of the maximal function on the space $L^{p(\cdot)}(\ell^{q})$. Since the maximal operator is in general not bounded on $L^{p(\cdot)}(\ell^{q(\cdot)})$, we are forced to introduce some new methods. In the case of constant exponents we recover the results of [11].

The paper is organized as follows. First we give some preliminaries where we fix some notation and recall some basic facts on the variable Triebel–Lizorkin spaces. Also we give the key technical lemmas needed in the proofs of the main statements. The main statements and their proofs are formulated in Sect. 3.

2 Variable function spaces

As usual, we denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The Euclidean scalar product of $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ is given by $x \cdot y = x_1 y_1 + \cdots + x_n y_n$. The expression $f \lesssim g$ means that $f \leq c\, g$ for some independent constant c (and non-negative functions f and g), and $f \approx g$ means $f \lesssim g \lesssim f$.

For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball in $\mathbb{R}^n$ with center x and radius r. By supp f we denote the support of the function f, i.e., the closure of its non-zero set.

The symbol $\mathcal{S}(\mathbb{R}^n)$ is used in place of the set of all Schwartz functions on $\mathbb{R}^n$. We denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. We define the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

The Hardy–Littlewood maximal operator $\mathcal{M}$ is defined on L^1_{loc} by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy.$$

By c we denote generic positive constants, which may have different values at different occurrences. Further notation will be properly introduced whenever needed.

The variable exponents that we consider are always measurable functions p on $\mathbb{R}^n$ with range in $[1,\infty[$. We denote the set of such functions by $\mathcal{P}$. We use the standard notation $p^- = \underset{x\in\mathbb{R}^n}{\text{ess-inf}}\, p(x)$ and $p^+ = \underset{x\in\mathbb{R}^n}{\text{ess-sup}}\, p(x)$.

The variable exponent modular is defined by

$$\varrho_{p(\cdot)}(f) = \int_{\mathbb{R}^n} \rho_{p(x)}(|f(x)|)dx,$$

where $\rho_p(t) = t^p$. The variable exponent Lebesgue space $L^{p(\cdot)}$ consists of measurable functions f on $\mathbb{R}^n$ such that $\varrho_{p(\cdot)}(\lambda f) < \infty$ for some $\lambda > 0$. We define the Luxemburg norm on this space by the formula

$$\|f\|_{p(\cdot)} = \inf\left\{ \lambda > 0 : \varrho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1 \right\}.$$

As is known, the following inequalities hold (see [10], Lemma 3.2.4)

$$\|f\|_{p(\cdot)} \leq 1 \Leftrightarrow \varrho_{p(\cdot)}(f) \leq 1. \tag{2}$$

Let $p, q \in \mathcal{P}$. The space $L^{p(\cdot)}(\ell^{q(\cdot)})$ is defined to be the set of all sequences $\left(f_j\right)_{j\in\mathbb{N}_0}$ of functions such that

$$\|(f_j)_{j\in\mathbb{N}_0}\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} = \big\|\|(f_j)_{j\in\mathbb{N}_0}\|_{\ell^{q(\cdot)}}\big\|_{p(\cdot)} < \infty.$$

It is easy to show that $L^{p(\cdot)}(\ell^{q(\cdot)})$ is always a normed space. If $p \in \mathcal{P}$, then p' denotes the conjugate exponent of p given by $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

We say that $g : \mathbb{R}^n \to \mathbb{R}$ is *locally* log *-Hölder continuous*, abbreviated $g \in C^{\log}_{\text{loc}}$, if there exists $c_{\log}(g) > 0$ such that

$$|g(x) - g(y)| \leq \frac{c_{\log}(g)}{\log(e + 1/|x-y|)} \tag{3}$$

for all $x, y \in \mathbb{R}^n$. We say that g satisfies the log-*H ölder decay condition*, if there exists $g_\infty \in \mathbb{R}$ and a constant $c_{\log} > 0$ such that

$$\left|g(x) - g_\infty\right| \leq \frac{c_{\log}}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$. We say that g is *globally*-log *-Hölder continuous*, abbreviated $g \in C^{\log}$, if it is locally log-Hölder continuous and satisfies the log-Hö lder decay condition. The constants $c_{\log}(g)$ and $c_{\log}$ are called the *locally* log*-Hölder constant* and the log*-Hölder decay constant*, respectively. We note that all functions $g \in C^{\log}_{\text{loc}}$ always belong to L^∞.

We define the following class of variable exponents

$$\mathcal{P}^{\log} = \left\{ p \in \mathcal{P} : \frac{1}{p} \text{ is globally-log-Hölder continuous} \right\}.$$

We define $1/p_\infty := \lim_{|x|\to\infty} 1/p(x)$ and we use the convention $\frac{1}{\infty} = 0$. Note that although $\frac{1}{p}$ is bounded, the variable exponent p itself can be unbounded. It was shown in [10], Theorem 4.3.8 that $\mathcal{M} : L^{p(\cdot)} \to L^{p(\cdot)}$ is bounded if $p \in \mathcal{P}^{\log}$ and $p^- > 1$. Note that if $p \in \mathcal{P}$ and $p^+ < \infty$, then $p \in \mathcal{P}^{\log}$ if and only if $p \in C^{\log}_{\text{loc}}$.

We also refer to the papers [6–8], where various results on maximal function in variable Lebesgue spaces were obtained.

Recall that $\eta_{j,m}(x) = 2^{nj}(1 + 2^j|x|)^{-m}$, for any $x \in \mathbb{R}^n$, $j \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Note that $\eta_{j,m} \in L^1$ when $m > n$ and that $\left\|\eta_{j,m}\right\|_1 = c_m$ is independent of j.

We need to recall the definition of variable Triebel–Lizorkin spaces and their basic properties. Let Φ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\Phi(x) = 1$ for $|x| \leq 1$ and $\Phi(x) = 0$ for $|x| \geq 2$. We put $\mathcal{F}\varphi_0(x) = \Phi(x)$, $\mathcal{F}\varphi_1(x) = \Phi(\frac{x}{2}) - \Phi(x)$ and

$$\mathcal{F}\varphi_j(x) = \mathcal{F}\varphi_1(2^{1-j}x) \quad \text{for} \quad j = 2, 3, \ldots .$$

Then $(\mathcal{F}\varphi_j)_{j\in\mathbb{N}_0}$ is a smooth dyadic resolution of unity, $\sum_{j=0}^{\infty} \mathcal{F}\varphi_j(x) = 1$ for all $x \in \mathbb{R}^n$. Thus we obtain the Littlewood–Paley decomposition

$$f = \sum_{j=0}^{\infty} \varphi_j * f$$

of all $f \in \mathcal{S}'(\mathbb{R}^n)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$).

We are now in a position to state the definition of the spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$.

Definition 1 Let $(\mathcal{F}\varphi_j)_{j\in\mathbb{N}_0}$ be a smooth dyadic resolution of unity. For $s : \mathbb{R}^n \to \mathbb{R}$ and $p, q \in \mathcal{P}$, the Triebel–Lizorkin space $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$ consists of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} = \left\|(2^{js(\cdot)}\varphi_j * f)_{j\in\mathbb{N}_0}\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} < \infty.$$

Using the system $(\varphi_j)_{j\in\mathbb{N}_0}$ we can define the quasi-norm

$$\|f\|_{F^s_{p,q}} = \left\|(2^{js}\varphi_j * f)_{j\in\mathbb{N}_0}\right\|_{L^p(\ell^q)}$$

for constants $s \in \mathbb{R}, p \in [1, \infty)$ and $q \in [1, \infty]$. The Triebel–Lizorkin space $F^s_{p,q}$ consists of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{F^s_{p,q}} < \infty$. It is well-known that these spaces do not depend on the choice of the system $(\varphi_j)_{j\in\mathbb{N}_0}$ (up to equivalence of norms). Further details on the classical theory of these spaces can be found in [14, 15], see also [16] for recent developments. One recognizes immediately that if s, p and q are constants, then $F^{s(\cdot)}_{p(\cdot),q(\cdot)} = F^s_{p,q}$.

For any $p, q \in \mathcal{P}^{\log}$ with $1 \leq p^- \leq p^+ < \infty, 1 \leq q^- \leq q^+ < \infty$, and $s \in C^{\log}_{\text{loc}}$, the space $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$ does not depend on the chosen system $(\varphi_j)_{j\in\mathbb{N}_0}$ (in the sense of equivalent norms). They are Banach spaces, and

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow F^{s(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

In particular if $p \in C^{\log}_{\text{loc}}, 1 < p^- \leq p^+ < \infty$, and $m \in \mathbb{N}_0$, then

$$F^m_{p(\cdot),2} = W^{m,p(\cdot)},$$

see [9]. The full treatment of the spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$ can be found in [9, 10]. We refer to the paper [18], for further results on the variable Triebel–Lizorkin spaces $F^{s(\cdot)}_{p(\cdot),q}$ (only the case of constant q was considered). We also mention the papers [1, 2], for further results on the variable Bessel potential spaces and variable Sobolev spaces.

2.1 Key lemmas

In this section we present some results which are useful for us. The following lemma is from [9, Lemma 6.1], see also [12, Lemma 19].

Lemma 1 *Let* $\alpha \in C^{\log}_{\text{loc}}$ *and let* $R \geq c_{\log}(\alpha)$, *where* $c_{\log}(\alpha)$ *is the constant from* (3) *for* α. *Then*

$$2^{j\alpha(x)}\eta_{j,m+R}(x-y) \leq c\, 2^{j\alpha(y)}\eta_{j,m}(x-y)$$

with $c > 0$ *independent of* $x, y \in \mathbb{R}^n$ *and* $j, m \in \mathbb{N}_0$.

The previous lemma allows us to treat the variable smoothness in many cases as if it were not variable at all, namely we can move the term inside the convolution as follows:

$$2^{j\alpha(x)}\eta_{j,m+R} * f(x) \leq c\, \eta_{j,m} * (2^{j\alpha(\cdot)}f)(x).$$

Since the maximal operator is in general not bounded on $L^{p(\cdot)}(\ell^{q(\cdot)})$, see [9, Section 5], we will make use of the following statement.

Lemma 2 *Let* $p, q \in \mathcal{P}^{\log}$ *with* $1 < p^- \leq p^+ < \infty$ *and* $1 < q^- \leq q^+ < \infty$. *For* $m > n$, *there exists* $c > 0$ *such that*

$$\left\|(\eta_{j,m} * f_j)_{j\in\mathbb{N}_0}\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \leq c\left\|(f_j)_{j\in\mathbb{N}_0}\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}.$$

The proof of this lemma is given in [9, Section 5]. The next lemma is a Hardy-type inequality which is easy to prove.

Lemma 3 *Let* $0 < a < 1$ *and* $1 \leq q \leq \infty$. *Let* $(\varepsilon_k)_{k\in\mathbb{N}_0}$ *be a sequence of positive real numbers and denote*

$$\delta_k = \sum_{j=0}^{k} a^{k-j}\varepsilon_j, \quad \text{and} \quad \eta_k = \sum_{j=k}^{\infty} a^{j-k}\varepsilon_j, \quad k \in \mathbb{N}_0.$$

Then there exists a constant $c > 0$ *depending only on a and q such that*

$$\left(\sum_{k=0}^{\infty} \delta_k^q\right)^{\frac{1}{q}} + \left(\sum_{k=0}^{\infty} \eta_k^q\right)^{\frac{1}{q}} \leq c \left(\sum_{k=0}^{\infty} \varepsilon_k^q\right)^{\frac{1}{q}}.$$

Lemma 4 *Let* $p, q \in \mathcal{P}$ *with* $1 < p^- \leq p^+ < \infty, 1 < q^- \leq q^+ < \infty$ *and* $\left(f_j\right)_{j\in\mathbb{N}_0} \in L^{p(\cdot)}(\ell^{q(\cdot)})$. *Then*

$$\|(f_j)_{j\in\mathbb{N}_0}\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \approx \sup \int_{\mathbb{R}^n} \sum_{j=0}^{\infty} |f_j(x)||g_j(x)|dx,$$

where the supremum is taken over all sequences of functions $\left(g_j\right)_{j\in\mathbb{N}_0} \in L^{p'(\cdot)}(\ell^{q'(\cdot)})$ *such that*

$$\|(g_j)_{j\in\mathbb{N}_0}\|_{L^{p'(\cdot)}(\ell^{q'(\cdot)})} \leq 1.$$

Proof Let

$$\|(f_j)_{j\in\mathbb{N}_0}\|^{\bullet}_{L^{p(\cdot)}(\ell^{q(\cdot)})} = \sup \int_{\mathbb{R}^n} \sum_{j=0}^{\infty} |f_j(x)||g_j(x)|dx.$$

Since $\frac{1}{q(\cdot)} + \frac{1}{q'(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$, by Hölder's inequality,

$$\|(f_j)_{j\in\mathbb{N}_0}\|^{\bullet}_{L^{p(\cdot)}(\ell^{q(\cdot)})} \lesssim \|(f_j)_{j\in\mathbb{N}_0}\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}.$$

Now let us prove the converse inequality. By the scaling argument, it suffices to consider the case $\|(f_j)_{j\in\mathbb{N}_0}\|^{\bullet}_{L^{p(\cdot)}(\ell^{q(\cdot)})} \leq 1$ and prove that $\|(f_j)_{j\in\mathbb{N}_0}\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \leq 1$. Since $(f_j)_{j\in\mathbb{N}_0} \in L^{p(\cdot)}(\ell^{q(\cdot)})$ we have

$$\int_{\mathbb{R}^n} \|(f_j(x))_{j\in\mathbb{N}_0}\|^{p(x)}_{\ell^{q(x)}} dx < \infty.$$

Assume, for the sake of contradiction, that

$$\int_{\mathbb{R}^n} \|\left(f_j(x)\right)_{j\in\mathbb{N}_0}\|^{p(x)}_{\ell^{q(x)}} dx > 1.$$

Then by the continuity of the modular there exists $d > 1$ such that

$$\int_{\mathbb{R}^n} \|(d^{-1}f_j(x))_{j\in\mathbb{N}_0}\|^{p(x)}_{\ell^{q(x)}} dx = 1.$$

Define

$$g_j(x) = \frac{|\frac{f_j(x)}{d}|^{q(x)-1}}{\left(\sum_{v=0}^{\infty} |\frac{f_v(x)}{d}|^{q(x)}\right)^{1-\frac{p(x)}{q(x)}}}, \quad j \in \mathbb{N}_0, x \in \mathbb{R}^n,$$

which leads to

$$\left\|(f_j)_{j\in\mathbb{N}_0}\right\|^{\bullet}_{L^{p(\cdot)}(\ell^{q(\cdot)})} \geq \int_{\mathbb{R}^n}\sum_{j=0}^{\infty}|f_j(x)||g_j(x)|dx = d\int_{\mathbb{R}^n}\left\|(d^{-1}f_j(x))_{j\in\mathbb{N}_0}\right\|^{p(x)}_{\ell^{q(x)}}dx > 1,$$

which contradicts our assumption. The proof is completed by applying the unit ball property (2). □

Let $(\mathcal{F}\varphi_j)_{j\in\mathbb{N}_0}$ be a smooth dyadic resolution of unity. Let $\Psi \in \mathcal{S}(\mathbb{R}^n)$ and

$$\Lambda_{j,m}(f,g)(x) = \int_{\mathbb{R}^{2n}} \varphi_j(x-y)(\Psi_m(x-z)-\Psi_m(y-z))f(y)\varphi_m * g(z)dydz,$$

where $j,m\in\mathbb{N}_0$ and $\Psi_m = 2^m\Psi(2^m\cdot)$.

Lemma 5 *Let* $s\in C^{\log}_{\mathrm{loc}}, a\in\mathbb{R}, p, p_1, p_2, q\in\mathcal{P}^{\log}$ *with* $1<p^-\leq p^+<\infty, 1<p_1^-\leq p_1^+\leq\infty, 1<p_2^-\leq p_2^+<\infty$ *and* $1<q^-\leq q^+<\infty$. *Assume that* $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ *and* $(s+a)^->0$. *Then*

$$\sum_{j=0}^{\infty}\sum_{m=j}^{\infty}\int_{\mathbb{R}^n}2^{ja}|\Lambda_{j,m}(f,g)(x)||h_j(x)|dx \lesssim \|f\|_{p_1(\cdot)}\|g\|_{F^{s(\cdot)+a}_{p_2(\cdot),q(\cdot)}}$$

holds for any $(h_j)_{j\in\mathbb{N}_0}\subset\mathcal{S}(\mathbb{R}^n)$ *such that*

$$\left\|\left\|(2^{-js(\cdot)}h_j)_{j\in\mathbb{N}_0}\right\|_{\ell^{q'(\cdot)}}\right\|_{p'(\cdot)}\leq 1.$$

Proof We use some decomposition techniques of [11]. Since $\varphi, \Psi\in\mathcal{S}(\mathbb{R}^n)$, we have

$$|\varphi_j|\leq c\eta_{j,N} \quad \text{and} \quad |\Psi_m|\leq c\eta_{m,N}, \quad j,m\in\mathbb{N}_0, N>n,$$

where the positive constant c is independent of j and m. Therefore $|\Lambda_{j,m}(f,g)(x)|$ can be estimated by

$$\begin{aligned} c&\int_{\mathbb{R}^{2n}}\eta_{j,N}(x-y)\eta_{m,N}(x-z)|f(y)||\varphi_m * g(z)|dydz \\ &\quad + c\int_{\mathbb{R}^{2n}}\eta_{j,N}(x-y)\eta_{m,N}(y-z)|f(y)||\varphi_m * g(z)|dydz \\ &\leq c(\eta_{j,N} * |f|(x))(\eta_{m,N} * |\varphi_m * g|(x)) + c\,I_{j,m}(x) \\ &= H_{j,m}(x) + c\,I_{j,m}(x), \end{aligned}$$

with

$$I_{j,m}(x) = \eta_{j,N} * (|f|\eta_{m,N} * |\varphi_m * g|)(x), \quad x\in\mathbb{R}^n, j,m\in\mathbb{N}_0, N>n. \tag{4}$$

Let us estimate each term separately. Let $(h_j)_{j\in\mathbb{N}_0}$ be a sequence of Schwartz functions with

$$\left\|\left\|(2^{-s(\cdot)j}h_j)_{j\in\mathbb{N}_0}\right\|_{\ell^{q'(\cdot)}}\right\|_{p'(\cdot)}\leq 1.$$

Using the well-known estimate

$$\eta_{j,N} * |f| \lesssim \mathcal{M}(f) \tag{5}$$

and Lemma 1 we can estimate

$$\sum_{j=0}^{\infty}\sum_{m=j}^{\infty}\int_{\mathbb{R}^n} 2^{ja} H_{j,m}(x)|h_j(x)|dx \tag{6}$$

by

$$\begin{aligned}
&c\int_{\mathbb{R}^n}\mathcal{M}(f)(x)\sum_{j=0}^{\infty}\sum_{m=j}^{\infty} 2^{j(a+s(x))}\eta_{m,N} * |\varphi_m * g|(x)2^{-js(x)}|h_j(x)|dx \\
&\quad\lesssim \int_{\mathbb{R}^n}\mathcal{M}(f)(x)\sum_{j=0}^{\infty}\sum_{m=j}^{\infty} 2^{(j-m)(a+s(x))}\eta_{m,N_1} * (2^{m(a+s(\cdot))}|\varphi_m * g|)(x)2^{-js(x)}|h_j(x)|dx
\end{aligned}$$

for some $N_1 > n$. Now

$$\sum_{j=0}^{\infty}\sum_{m=j}^{\infty} 2^{(j-m)(a+s(x))}\eta_{m,N_1} * (2^{m(a+s(\cdot))}|\varphi_m * g|)(x)2^{-js(x)}|h_j(x)|$$

can be estimated by

$$\begin{aligned}
&\left\|\left(\sum_{m=j}^{\infty} 2^{(j-m)(a+s(x))}\eta_{m,N_1} * (2^{m(a+s(\cdot))}|\varphi_m * g|)(x)\right)_{j\in\mathbb{N}_0}\right\|_{\ell^{q(x)}} \\
&\qquad\times \left\|\left(2^{-js(x)}h_j(x)\right)_{j\in\mathbb{N}_0}\right\|_{\ell^{q'(x)}} \\
&\quad\lesssim \left\|\left(\eta_{j,N_1} * (2^{j(a+s(\cdot))}|\varphi_j * g|)(x)\right)_{j\in\mathbb{N}_0}\right\|_{\ell^{q(x)}}\left\|\left(2^{-js(x)}h_j(x)\right)_{j\in\mathbb{N}_0}\right\|_{\ell^{q'(x)}},
\end{aligned}$$

where we have used Hölder's inequality and Lemma 3. Hence (6) does not exceed

$$\begin{aligned}
&c\int_{\mathbb{R}^n}\mathcal{M}(f)(x)\|(\eta_{j,N_1} * (2^{j(a+s(\cdot))}|\varphi_j * g|)(x))_{j\in\mathbb{N}_0}\|_{\ell^{q(x)}}\|(2^{-s(\cdot)j}h_j(x))_{j\in\mathbb{N}_0}\|_{\ell^{q'(x)}}dx \\
&\quad\lesssim \|\|\mathcal{M}(f)(\eta_{j,N_1} * 2^{j(a+s(\cdot))}|\varphi_j * g|)_{j\in\mathbb{N}_0}\|_{\ell^{q(\cdot)}}\|_{p(\cdot)}\|\|(2^{-s(\cdot)j}h_j)_{j\in\mathbb{N}_0}\|_{\ell^{q'(\cdot)}}\|_{p'(\cdot)} \\
&\quad\lesssim \|\mathcal{M}(f)\|_{p_1(\cdot)}\|\|(\eta_{j,N_1} * 2^{j(a+s(\cdot))}|\varphi_j * g|)_{j\in\mathbb{N}_0}\|_{\ell^{q(\cdot)}}\|_{p_2(\cdot)} \\
&\quad\lesssim \|f\|_{p_1(\cdot)}\|\|(2^{j(a+s(\cdot))}\varphi_j * g)_{j\in\mathbb{N}_0}\|_{\ell^{q(\cdot)}}\|_{p_2(\cdot)} \\
&\quad\lesssim \|f\|_{p_1(\cdot)}\|g\|_{F^{s(\cdot)+a}_{p_2(\cdot),q(\cdot)}}.
\end{aligned}$$

In the first estimate we used Hölder's inequality, while the second follows by Lemma 2 and the boundedness of maximal function on variable Lebesgue spaces. We estimate the term (4). Observe that

$$\begin{aligned}
&\sum_{j=0}^{\infty}\sum_{m=j}^{\infty}\int_{\mathbb{R}^n} 2^{ja} I_{j,m}(x)|h_j(x)|dx \\
&\quad= \int_{\mathbb{R}^n}\sum_{j=0}^{\infty} 2^{-js(x)}|h_j(x)|2^{j(a+s(x))}\eta_{j,N} * \sum_{m=j}^{\infty}(|f|\eta_{m,N} * |\varphi_m * g|)(x)dx \\
&\quad= \int_{\mathbb{R}^n}\sum_{j=0}^{\infty} 2^{-js(x)}|h_j(x)|2^{j(a+s(x))}\eta_{j,N} * \kappa_j(x)dx,
\end{aligned}$$

where

$$\kappa_j=\sum_{m=j}^{\infty}(|f|\eta_{m,N}*|\varphi_m*g|),\quad j\in\mathbb{N}_0.$$

To continue we apply Hölder's inequality and obtain

$$\sum_{j=0}^{\infty}2^{-js(x)}|h_j(x)|2^{j(a+s(\cdot))}\eta_{j,N}*\kappa_j(x)$$

is bounded by

$$\big\|(2^{j(a+s(x))}\eta_{j,N}*\kappa_j(x))_{j\in\mathbb{N}_0}\big\|_{\ell^{q(x)}}\big\|(2^{-s(x)j}h_j(x))_{j\in\mathbb{N}_0}\big\|_{\ell^{q'(x)}}.$$

Taking the L^1-norm and using the Hölder inequality, we get

$$\begin{aligned}
&\sum_{j=0}^{\infty}\sum_{m=j}^{\infty}\int_{\mathbb{R}^n}2^{ja}I_{j,m}(x)|h_j(x)|dx\\
&\quad\lesssim\big\|\big\|(2^{j(a+s(\cdot))}\eta_{j,N}*\kappa_j)_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\big\|_{p(\cdot)}\big\|\big\|(2^{-s(\cdot)j}h_j)_{j\in\mathbb{N}_0}\big\|_{\ell^{q'(\cdot)}}\big\|_{p'(\cdot)}\\
&\quad\lesssim\big\|\big\|(\eta_{j,N_1}*2^{j(a+s(\cdot))}\kappa_j)_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\big\|_{p(\cdot)}\\
&\quad\lesssim\big\|\big\|(2^{j(a+s(\cdot))}\kappa_j)_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\big\|_{p(\cdot)}\\
&\quad\lesssim\big\|\big\||f(2^{j(a+s(\cdot))}\eta_{j,N}*|\varphi_j*g|)_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\big\|_{p(\cdot)},
\end{aligned}$$

by Lemmas 1 and 2, and Lemma 3, where $N_1=N-c_{\log}(s)>n$. Using again Hölder's inequality and Lemmas 1–2 we estimate the last expression by

$$\begin{aligned}
&c\|f\|_{p_1(\cdot)}\big\|\big\|(2^{j(a+s(\cdot))}\eta_{j,N}*|\varphi_j*g|)_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\big\|_{p_2(\cdot)}\\
&\quad\lesssim\|f\|_{p_1(\cdot)}\big\|\big\|(2^{j(a+s(\cdot))}\varphi_j*g)_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\big\|_{p_2(\cdot)}\\
&\quad\lesssim\|f\|_{p_1(\cdot)}\|g\|_{F^{s(\cdot)+a}_{p_2(\cdot),q(\cdot)}}.
\end{aligned}$$

The proof is completed. □

For $0\le m\le j,j,m\in\mathbb{N}_0$, $x\in\mathbb{R}^n$ and $K\in\mathbb{N}$, we set

$$\begin{aligned}
&E_{j,m,K}(f,g)(x)\\
&\quad=2^{(m-j)K}\int_{\mathbb{R}^{2n}}\eta_{j,N}(x-y)\eta_{m,N}(x-z)|f(y)||\varphi_m*g(z)|dydz\\
&\qquad+2^{(m-j)K}\int_{\mathbb{R}^{2n}}\eta_{j,N}(x-y)\eta_{m,N}(y-z)|f(y)||\varphi_m*g(z)|dydz,
\end{aligned}$$

where $N>n$ large enough.

Lemma 6 *Let* $s\in C^{\log}_{\mathrm{loc}},a\in\mathbb{R},K\in\mathbb{N},p,p_1,p_2,q\in\mathcal{P}^{\log}$ *with* $1<p^-\le p^+<\infty,1<p_1^-\le p_1^+\le\infty,1<p_2^-\le p_2^+<\infty$ *and* $1<q^-\le q^+<\infty$. *Assume that* $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ *and* $(s+a)^+<K$. *Then*

$$\sum_{j=0}^{\infty}\sum_{m=0}^{j}\int_{\mathbb{R}^n} 2^{ja}E_{j,m,K}(f,g)(x)|h_j(x)|dx \lesssim \|f\|_{p_1(\cdot)}\|g\|_{F^{s(\cdot)+a}_{p_2(\cdot),q(\cdot)}}$$

holds for any $(h_j)_{j\in\mathbb{N}_0}\subset\mathcal{S}(\mathbb{R}^n)$ *such that*

$$\big\|\|(2^{-js(\cdot)}h_j)_{j\in\mathbb{N}_0}\|_{\ell^{q'(\cdot)}}\big\|_{p'(\cdot)} \le 1. \tag{7}$$

Proof We employ the same notation as in Lemma 5. Let $0\le m\le j$. We have

$$\begin{aligned} 2^{ja}E_{j,m,K}(f,g)(x) \lesssim& 2^{(m-j)K+aj}\big(\eta_{j,N} * |f|(x)\eta_{m,N} * |\varphi_m * g|(x) + I_{j,m}(x)\big)\\ =& 2^{(m-j)K+aj}\big(H_{j,m}(x)+I_{j,m}(x)\big). \end{aligned}$$

where N large enough and $x\in\mathbb{R}^n$. The estimate (5) and Lemma 1, yield that

$$\sum_{j=0}^{\infty}\sum_{m=0}^{j}2^{(m-j)K+aj}H_{j,m}(x)$$

is bounded by

$$\begin{aligned} &c\mathcal{M}(f)(x)\sum_{j=0}^{\infty}2^{-js(x)}\sum_{m=0}^{j}2^{(m-j)(K-s(x)-a)}\eta_{m,N_1} * (2^{m(s(\cdot)+a)}|\varphi_m * g|)(x)\\ &= c\mathcal{M}(f)(x)\sum_{j=0}^{\infty}2^{-js(x)}\vartheta_j(x), \end{aligned}$$

where N_1 large enough and the constant c is independent of x. Observe that

$$\sum_{m=0}^{j}2^{(m-j)K+ja}I_{j,m}(x) = \eta_{j,N} * \omega_j,$$

with

$$\omega_j = \sum_{m=0}^{j}2^{(m-j)K+ja}(|f|\eta_{m,N} * |\varphi_m * g|), \quad j\in\mathbb{N}_0.$$

From Lemma 1 we derive

$$2^{js(x)}\eta_{j,N} * \omega_j \lesssim \eta_{j,N_2} * 2^{js(\cdot)}\omega_j \lesssim \eta_{j,N_2} * \mu_j,$$

where the implicit constant is independent of x, N_2 large enough and

$$\mu_j = \sum_{m=0}^{j}2^{(m-j)(K-s(\cdot)-a)}(|f|\eta_{m,N_3} * 2^{m(s(\cdot)+a)}|\varphi_m * g|), \quad N_3>n.$$

Consequently

$$\begin{aligned}
&\int_{\mathbb{R}^n}\sum_{j=0}^{\infty}\sum_{m=0}^{j}2^{(m-j)K+aj}H_{j,m}(x)|h_j(x)|dx\\
&\quad\lesssim\int_{\mathbb{R}^n}\mathcal{M}(f)(x)\sum_{j=0}^{\infty}|h_j(x)|2^{-js(x)}\vartheta_j(x)dx\\
&\quad\lesssim\|f\|_{p_1(\cdot)}\big\|\big\|\big(\vartheta_j\big)_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\big\|_{p_2(\cdot)},
\end{aligned}$$

where we used Hölder's inequality, (7) and the boundedness of maximal function on variable Lebesgue spaces. Lemmas 3 and 2 yield

$$\begin{aligned}
\big\|\big\|\big(\vartheta_j\big)_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\big\|_{p_2(\cdot)}&\lesssim\big\|\big\|(\eta_{j,N_1}*2^{j(s(\cdot)+a)}|\varphi_j*g|)_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\big\|_{p_2(\cdot)}\\
&\lesssim\|g\|_{F^{s(\cdot)+a}_{p_2(\cdot),q(\cdot)}}.
\end{aligned}$$

Again by Hölder's inequality and (7) we derive

$$\begin{aligned}
\int_{\mathbb{R}^n}\sum_{j=0}^{\infty}\sum_{m=0}^{j}2^{(m-j)K+aj}I_{j,m}(x)|h_j(x)|dx&\lesssim\int_{\mathbb{R}^n}\sum_{j=0}^{\infty}2^{-js(x)}|h_j(x)|\eta_{j,N_2}*\mu_j(x)dx\\
&\lesssim\big\|\big\|(\eta_{j,N_2}*\mu_j)_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\big\|_{p(\cdot)}\\
&\lesssim\big\|\big\|\big(\mu_j\big)_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\big\|_{p(\cdot)},
\end{aligned}$$

where the last term follows by Lemma 2. Applying Lemmas 3, 1 and 2 we obtain

$$\begin{aligned}
\big\|\big\|\big(\mu_j\big)_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\big\|_{p(\cdot)}&\lesssim\big\|\big\|((|f|\eta_{j,N_3}*2^{j(s(\cdot)+a)}|\varphi_j*g|))_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\big\|_{p(\cdot)}\\
&\lesssim\|f\|_{p_1(\cdot)}\big\|\big\|((\eta_{j,N_3}*2^{j(s(\cdot)+a)}|\varphi_j*g|))_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\big\|_{p_2(\cdot)}\\
&\lesssim\|f\|_{p_1(\cdot)}\big\|\big\|(2^{j(s(\cdot)+a)}(\varphi_j*g))_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\big\|_{p_2(\cdot)}\\
&\lesssim\|f\|_{p_1(\cdot)}\|g\|_{F^{s(\cdot)+a}_{p_2(\cdot),q(\cdot)}},
\end{aligned}$$

which is the desired estimate. □

3 The results and their proofs

In this section we present our results of this paper. Let $\mathbf{f}=(f_1,\dots,f_n)\in X^n$ for some normed space X. Then we put

$$\|\mathbf{f}\|_X=\sum_{i=1}^{n}\|f_i\|_X.$$

We have the following result:

Theorem 1 *Let* $s \in C^{\log}_{\mathrm{loc}}, s^- > 0, p, p_1, p_2, q \in \mathcal{P}^{\log}$ *with* $1 < p^- \leq p^+ < \infty, 1 < p_1^- \leq p_1^+ \leq \infty, 1 < p_2^- \leq p_2^+ < \infty$, *and* $1 < q^- \leq q^+ < \infty$. *Assume that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. *Let* $V = (V_1, \dots, V_n) \in (\mathcal{S}(\mathbb{R}^n))^n$ *be vector field. Then for any* $f \in \mathcal{S}(\mathbb{R}^n)$

$$\big\| \big\| (2^{js(\cdot)}[V \cdot \nabla, \Delta_j]f)_{j \in \mathbb{N}_0} \big\|_{\ell^{q(\cdot)}} \big\|_{p(\cdot)} \lesssim \|\nabla f\|_{p_1(\cdot)} \|V\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}} + A \tag{8}$$

and

$$\begin{aligned} &\big\| \big\| (2^{js(\cdot)}[V \cdot \nabla, \Delta_j]f)_{j \in \mathbb{N}_0} \big\|_{\ell^{q(\cdot)}} \big\|_{p(\cdot)} \\ &\quad \lesssim \|f \operatorname{div}(V)\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} + \|\nabla V\|_{p_1(\cdot)} \|f\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}} + \|f\|_{p_1(\cdot)} \|V\|_{F^{s(\cdot)+1}_{p_2(\cdot),q(\cdot)}}, \end{aligned} \tag{9}$$

where

$$A = \|\nabla V\|_{p_1(\cdot)} \|f\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}} \quad \text{or} \quad A = \|V\|_{p_1(\cdot)} \|\nabla f\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}}.$$

Proof The proof follows the ideas in [11, p. 1217].

Step 1 *Preparation.* Let $V = (V_1, \dots, V_n) \in (\mathcal{S}(\mathbb{R}^n))^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$. From Lemma 4 we need to estimate

$$\int_{\mathbb{R}^n} \sum_{j=0}^{\infty} |[V \cdot \nabla, \Delta_j]f(x) h_j(x)| dx$$

for any $(h_j)_{j \in \mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$ such that

$$\big\| \big\| (2^{-s(\cdot)j} h_j)_{j \in \mathbb{N}_0} \big\|_{\ell^{q'(\cdot)}} \big\|_{p'(\cdot)} \leq 1. \tag{10}$$

We have

$$\begin{aligned} [V \cdot \nabla, \Delta_j]f(x) &= \sum_{k=1}^{n} V_k(x) \partial_k \Delta_j f(x) - \Delta_j (V_k \partial_k f)(x) \\ &= \sum_{k=1}^{n} \int_{\mathbb{R}^n} \varphi_j(x-y)(V_k(x) - V_k(y)) \partial_k f(y) dy. \end{aligned}$$

Let $(\mathcal{F}\varphi_j)_{j \in \mathbb{N}_0}$ be a smooth dyadic resolution of unity. Then

$$\begin{aligned} [V \cdot \nabla, \Delta_j]f(x) &= \sum_{m=0}^{\infty} \sum_{k=1}^{n} \int_{\mathbb{R}^n} \varphi_j(x-y)(\varphi_m * V_k(x) - \varphi_m * V_k(y)) \partial_k f(y) dy \\ &= \sum_{m=0}^{\infty} \sum_{k=1}^{n} \Pi_{j,m,k}(\partial_k f, V_k)(x) \\ &= \sum_{m=0}^{j} \cdots + \sum_{m=j+1}^{\infty} \cdots \end{aligned}$$

for any $x \in \mathbb{R}^n$ and any $j \in \mathbb{N}_0$. Therefore we need only to estimate

$$\left(\sum_{m=0}^{j} \sum_{k=1}^{n} \Pi_{j,m,k}(\partial_k f, V_k) \right)_{j \in \mathbb{N}_0} \quad \text{and} \quad \left(\sum_{m=j+1}^{\infty} \sum_{k=1}^{n} \Pi_{j,m,k}(\partial_k f, V_k) \right)_{j \in \mathbb{N}_0}, \tag{11}$$

in $L^{p(\cdot)}(\ell^{q(\cdot)})$-norm.

Step 2 In this step we prove (8). From the support properties of $(\mathcal{F}\varphi_j)_{j\in\mathbb{N}_0}$ we have

$$\Pi_{j,m,k}(\partial_k f, V_k)(x) = \Lambda_{j,m}(\partial_k f, V_k)(x), \quad x \in \mathbb{R}^n, j, m \in \mathbb{N}_0, k \in \{1, \ldots, n\},$$

with $\Psi_m = \sum_{i=-2}^{2} \varphi_{m+i}, m \in \mathbb{N}$ and $\Psi_0 = \Phi + \varphi_1$. For simplicity, we use φ_m instead of $\Psi_m, m \in \mathbb{N}_0$. Applying Lemmas 4 and 5, with the help of (10), we estimate the second term of (11) in $L^{p(\cdot)}(\ell^{q(\cdot)})$-norm by

$$\|\nabla f\|_{p_1(\cdot)} \|V\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}}.$$

Let $K \in \mathbb{N}$ be such that $0 < s^- \le s^+ < K$. From [11, Lemma 3.1] we derive

$$\begin{aligned}\Pi_{j,m,k}(\partial_k f, V_k) &= \sum_{1\le|\alpha|<K} 2^{|\alpha|(m-j)}(\theta_{j,\alpha} * \partial_k f)(\partial^\alpha \varphi)_m * \varphi_m * V_k + \Upsilon_{j,m,K,k}(\partial_k f, V_k), \\ &= \sum_{1\le|\alpha|<K} I_{1,j,m,|\alpha|,k} + \Upsilon_{j,m,K,k}(\partial_k f, V_k),\end{aligned} \tag{12}$$

where

$$\Upsilon_{j,m,K,k}(\partial_k f, V_k)(x) = \int_{\mathbb{R}^{2n}} \varphi_j(x-y) \left(\sum_{|\alpha|=K} \frac{1}{\alpha!} (\partial^\alpha \varphi_m)(\xi_\alpha)(y-x)^\alpha \right) \partial_k f(y) \varphi_m * V_k(z) dy dz,$$

ξ_α is on the line segment joining $y - z$ and $x - z$ and

$$\theta_{j,\alpha}(x) = \frac{(-1)^{|\alpha|}}{\alpha!} (2^j x)^\alpha \varphi_j(x), \quad x \in \mathbb{R}^n, j \in \mathbb{N}_0.$$

When $K = 1$, the sum on the right-hand side of (12) is interpreted as zero. Again from [11, Lemma 3.1],

$$|\Upsilon_{j,m,K,k}(\partial_k f, V_k)| \lesssim E_{j,m,K}(\partial_k f, V_k), \quad 0 \le m \le j,$$

which yields by Lemma 6 with $a = 0$ that

$$\sum_{j=0}^{\infty} \sum_{m=0}^{j} \int_{\mathbb{R}^n} |\Upsilon_{j,m,K,k}(\partial_k f, V_k)(x)| |h_j(x)| dx \lesssim \|\partial_k f\|_{p_1(\cdot)} \|V_k\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}}$$

for any $k \in \{1, \ldots, n\}$. From the support properties of $(\mathcal{F}\varphi_j)_{j\in\mathbb{N}_0}$, we have $\theta_{j,\alpha} * \partial_k f = \theta_{j,\alpha} * \tilde{\varphi}_j * \partial_k f$, where $\tilde{\varphi}_j = \sum_{l=-2}^{l=2} \varphi_{j+l}$. Here we put $\varphi_v = 0$ if $v < 0$. Hence

$$\begin{aligned}|I_{1,j,m,1,k}| &\lesssim 2^{m-j}(\eta_{j,N} * |\tilde{\varphi}_j * \partial_k f|)(\eta_{m,N} * |\varphi_m * V_k|) \\ &\lesssim 2^{m-j}(\eta_{j,N} * |\tilde{\varphi}_j * \partial_k f|)\mathcal{M}(V_k)\end{aligned}$$

and then

$$\sum_{j=0}^{\infty} \sum_{m=0}^{j} \int_{\mathbb{R}^n} |I_{1,j,m,1,k}(x)| |h_j(x)| dx \tag{13}$$

does not exceed

$$
\begin{aligned}
&c\int_{\mathbb{R}^n}\mathcal{M}(V_k)(x)\sum_{j=0}^{\infty}(\eta_{j,N}*|\tilde{\varphi}_j*\partial_k f|)|h_j(x)|dx\\
&\quad\lesssim\int_{\mathbb{R}^n}\mathcal{M}(V_k)(x)\big\|(2^{-js(x)}h_j(x))_{j\in\mathbb{N}_0}\big\|_{\ell^{q'(x)}}\big\|(2^{js(x)}\eta_{j,N}*\tilde{\varphi}_j*\partial_k f(x))_{j\in\mathbb{N}_0}\big\|_{\ell^{q(x)}}dx,
\end{aligned}
$$

which is bounded by

$$
\begin{aligned}
&c\big\|\mathcal{M}(V_k)\big\|(2^{js(\cdot)}\eta_{j,N}*\tilde{\varphi}_j*\partial_k f(x))_{j\in\mathbb{N}_0}\big\|_{\ell^{q(\cdot)}}\big\|_{p(\cdot)}\\
&\quad\lesssim\big\|\mathcal{M}(V_k)\big\|_{p_1(\cdot)}\Big\|\Big\|\big(2^{js(\cdot)}\eta_{j,N}*\tilde{\varphi}_j*\partial_k f\big)_{j\in\mathbb{N}_0}\Big\|_{\ell^{q(\cdot)}}\Big\|_{p_2(\cdot)}.
\end{aligned}
$$

Finally the boundedness of maximal function on variable Lebesgue spaces and Lemmas 1 and 2 guarantee that the last term can be estimated by

$$
c\|V_k\|_{p_1(\cdot)}\|\nabla f\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}}.
$$

We have

$$
\sum_{m=0}^{j}\sum_{1\le|\alpha|<K}I_{1,j,m,|\alpha|,k}=\sum_{m=0}^{j}\sum_{1\le|\alpha|<K}2^{|\alpha|(m-j)}(\theta_{j,\alpha}*\tilde{\varphi}_j*\partial_k f)(\partial^{\alpha}\varphi)_m*\varphi_m*V_k.
$$

Recall that

$$
\theta_{j,\alpha}(x)=2^{j|\alpha|}\frac{(-1)^{|\alpha|}}{\alpha!}x^{\alpha}\varphi_j(x),
$$

which yields that

$$
|\theta_{j,\alpha}*\tilde{\varphi}_j*\partial_k f|\lesssim 2^j\eta_{j,L}*|\tilde{\varphi}_j*f|,\quad L>n.
$$

The arguments of [11, p. 1218] yield that

$$
\begin{aligned}
&\left|\sum_{m=0}^{j}\sum_{1\le|\alpha|<K}2^{|\alpha|(m-j)}I_{1,j,m,|\alpha|,k}\right|\\
&\quad\lesssim\eta_{j,L}*|\tilde{\varphi}_j*f|\sum_{m=0}^{j}\sum_{1<|\alpha|<K}2^{(|\alpha|-1)(m-j)}\eta_{m,L}*\\
&\qquad|\nabla V_k|+(\eta_{j,L}*|\tilde{\varphi}_j*f|)(\eta_{j,L}*|\nabla V_k|)\\
&\quad\lesssim\mathcal{M}(\nabla V_k)(\eta_{j,L}*|\tilde{\varphi}_j*f|).
\end{aligned}
$$

Therefore (13) with $I_{1,j,m,|\alpha|,k}$ in place of $I_{1,j,m,1,k}$ can be estimated by

$$
c\|\nabla V\|_{p_1(\cdot)}\|f\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}}.
$$

Step 3 In this step we prove (9). As in [11, p. 1219]

$$
\Pi_{j,m,k}(\partial_k f,V_k)(x)=J^1_{j,m,k}(\partial_k f,V_k)(x)+J^2_{j,m,k}(\partial_k f,V_k)(x),
$$

where

$$J^1_{j,m,k}(\partial_k f, V_k)(x) = \int_{\mathbb{R}^{2n}} \varphi_j(x-y)(\partial_k \Psi_m)(y-z) f(y) \varphi_m * V_k(z) dydz$$

and

$$J^2_{j,m,k}(\partial_k f, V_k)(x) = \int_{\mathbb{R}^{2n}} 2^j (\partial_k \varphi)_j (x-y)(\Psi_m(x-z) - \Psi_m(y-z)) f(y) \varphi_m * V_k(z) dydz.$$

Similarly as in [11, p. 1219] we obtain

$$\left\| \left\| \left(2^{js(\cdot)} \sum_{m=0}^{\infty} \sum_{k=1}^{n} J^1_{j,m,k}(\partial_k f, V_k) \right)_{j\in\mathbb{N}_0} \right\|_{\ell^{q(\cdot)}} \right\|_{p(\cdot)} = \| f \operatorname{div}(V) \|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Observe that $J^2_{j,m,k}(\partial_k f, V_k)$ is just $2^j \Lambda_{j,m}(f, V_k)$ but with $(\partial_k \varphi)_j$ in place of φ_j. Using the same type of arguments as in Step 2 it is easy to see that

$$\begin{aligned} &\left\| \left\| \left(2^{js(\cdot)} \sum_{m=0}^{\infty} \sum_{k=1}^{n} J^2_{j,m,k}(\partial_k f, V_k) \right)_{j\in\mathbb{N}_0} \right\|_{\ell^{q(\cdot)}} \right\|_{p(\cdot)} \\ &\lesssim \|\nabla V\|_{p_1(\cdot)} \|f\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}} + \|f\|_{p_1(\cdot)} \|V\|_{F^{s(\cdot)+1}_{p_2(\cdot),q(\cdot)}}. \end{aligned}$$

The proof is completed. □

Remark 1 Corresponding statements to Theorem 1 were proved in [11, Theorem 1.1] under the assumption s and q are constants and based on the fact that the maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\ell^q)$, which in general is not bounded on $L^{p(\cdot)}(\ell^{q(\cdot)})$. Optimal inequality in (8) is much more complicated even in the case of constant exponents. Probably (8) can be improved in some particular case, see Theorem 2 below.

The last statement of this paper is the following theorem; the case where s and q are constants it is given in [11, Theorem 1.2].

Theorem 2 *Let* $s \in C^{\log}_{\text{loc}}, p, p_1, p_2, q \in \mathcal{P}^{\log}$ *with* $1 < p^- \le p^+ < \infty, 1 < p_1^- \le p_1^+ \le \infty$, $1 < p_2^- \le p_2^+ < \infty$, *and* $1 < q^- \le q^+ < \infty$. *Assume that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. *Let* $V = (V_1, \dots, V_n)$ $\in (\mathcal{S}(\mathbb{R}^n))^n$ *be vector field. Then for any* $f \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned} &\| \| (2^{j(s(\cdot)}[V\cdot\nabla, \Delta_j]f)_{j\in\mathbb{N}_0} \|_{\ell^{q(\cdot)}} \|_{p(\cdot)} \\ &\lesssim \|\nabla f\|_{p_1(\cdot)} \|V\|_{F^{s(\cdot)}_{p_2(\cdot),q(\cdot)}}, \quad 0 < s^- \le s^+ < 1 \end{aligned}$$

and

$$\begin{aligned} \| \| (2^{j(s(\cdot)}[V\cdot\nabla, \Delta_j]f)_{j\in\mathbb{N}_0} \|_{\ell^{q(\cdot)}} \|_{p(\cdot)} \lesssim &\| f \operatorname{div}(V) \|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} \\ &+ \|f\|_{p_1(\cdot)} \|V\|_{F^{s(\cdot)+1}_{p_2(\cdot),q(\cdot)}}, \end{aligned}$$

with $-1 < s^- \le s^+ < 0$.

Proof The first estimate follows by Steps 1–2 of Theorem 1, with $K = 1$ and $a = 0$, while the second one follows by the same arguments of Step 3 in Theorem 1. ☐

Remark 2 In Theorem 2 we present an improvement of (8) with $0 < s^- \le s^+ < 1$ and of (9) with $-1 < s^- \le s^+ < 0$. An extension of Theorem 2 to general case $s \in C^{\log}_{\mathrm{loc}}$ is still open.

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