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Etude non linéaire de quelques propriétés de sommabilité et de factorisation

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Introduction

In this thesis, we mainly focus on summabilities and factorizations of non linear operators such as multilinear and Lipschitz operators. These different classes of summability play an important role in Banach and metric spaces theory. In the famous works of Grothendieck in 1953 and Pietsch in 1968, a continuous linear operator $T : E \rightarrow F$ between Banach spaces is absolutely p -summing if it sends weakly p -summable sequences (i.e., an element of $\ell_p^w(E)$) to absolutely p -summable sequences (i.e., an element of $\ell_p(F)$), linear p -summing operators an important class of operator ideals. In addition, Cohen in 1973 has also introduced the space of strongly p -summable sequences $\ell_p \langle E \rangle$ in order to define two new classes of summability, the space of strongly p -summing operators as a characterization of the conjugates of p^* -summing operators and the space of p -nuclear operators. Several analogous concepts for operator ideals have been defined, such as absolutely (p, σ) -continuous, strongly (p, σ) -continuous, (p, G) -summing and (\tilde{p}, G) -summing which were introduced and studied by several authors (see [1, 2, 3, 38, 25]). For a detailed approach to the linear theory of absolutely summing operators we refer to the excellent books of Diestel, Jarchow and Tonge [24]. On the other hand, Karn and Sinha have introduced the mid p -summable sequences $\ell_p^{mid}(E)$ such that $\ell_p(F) \subset \ell_p^{mid}(E) \subset \ell_p^w(E)$. A sequence $x = (x_j)_j$ in Banach space E is called mid p -summable if $((x_n^*(x_j))_j)_n \in \ell_p(\ell_p)$

whenever $(x_n^*)_n \in \ell_p^w(E^*)$. equivalently $(S(x_j))_j \in \ell_p(\ell_p)$ for every $S \in \mathcal{L}(E, \ell_p)$. This last space led Karn and Sinha to define and study a new class of operators called weakly mid p -summing operators [30]. For $1 < p \leq \infty$, a linear map T between Banach spaces E and F is said to be weakly mid p -summing if $(T(x_j))_{j=1}^\infty \in \ell_p^{mid}(F)$ whenever $(x_j)_{j=1}^\infty \in \ell_p^w(E)$.

Lately, it has appeared, in most natural way, the interest in extending the concept of p -summing linear operator to the multilinear (polynomials) and Lipschitz mappings. The theory of multilinear summing mappings was first outlined by Pietsch in 1983 [53]. Since then, it has developed intensely, several interesting results being obtained in this field. The main motivations, which led to this development, is finding that concept which extends best the linear summing properties to the multilinear (polynomials) and Lipschitz cases. Thus, several analogous concepts for multilinear mappings have been defined, such as absolutely p -summing, p -dominated Cohen p -summing and strong p -summing multilinear operators, which was introduced and studied by several authors (Saadi, Dahia, Pietsch, Matos, Mezrag, Achour, Botelho, Santos, among others, see [53, 36, 37, 39, 6, 10, 8, 46, 52]).

Farmer and Johnson [26] introduced the notion of Lipschitz p -summing operators and showed that it is a good generalization of the concept of linear p -summing operators. This notion marked the beginning of the theory of Lipschitz summability. Motivated by the importance of this theory, several authors then developed and studied various related concepts [16, 17, 18, 19, 26, 42, 57, 59].

The main goal of this thesis is to introduce some new multilinear and Lipschitz ideals. This work is divided into four chapters.

In Chapter 1, we recall briefly some basic notions and terminologies needed in our thesis.

In Chapter 2, We introduce and study the classes of $(p; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated, $(\tilde{p}; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated multilinear operators defined on the tensor product of Banach spaces for $1 \leq p, p_1, \dots, p_n < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_n}$. We prove a natural analog to “Pietsch domination/factorization theorem” for such classes as the linear case introduced and studied by Maurey and Elezović [38, 25]. We establish also the relationship between the three classes of dominated multilinear operators. Finally, we treat the relationship between a multilinear operator T and its associated operators T^t , \tilde{T} and $T^\#$ for these classes of summability.

In Chapter 3, we introduce the new ideal of weakly mid- (p_1, \dots, p_m) -summing multilinear operators as a multilinear version of weakly mid- p -summing linear operators. Using the space of mid- p -summable sequences, we present a characterization given by summability property. We prove that, an operator $T \in \mathcal{L}(X_1, \dots, X_m; Y)$ is weakly mid- (p_1, \dots, p_m) -summing if and only if $S \circ T : X_1 \times \dots \times X_m \rightarrow \ell_p$ is (p_1, \dots, p_m) -dominated for every $S \in \mathcal{L}(Y; \ell_p)$. Also, we give an analogue of the Pietsch domination theorem.

In Chapter 4, we introduce and study a new ideal of strongly mid p -summing linear operators between Banach spaces. We prove that: an operator is strongly mid p -summing if, and only if, its adjoint is absolutely mid p^* -summing. This result led us to prove that X has p -Dunford-Pettis property ($1 < p < \infty$) if, and only if, X^* has this property. We extend to multilinear operators the concept of absolutely mid p -summing operators, for which the resulting class $\Pi_{p_1, \dots, p_m}^{mid}$ of the absolutely mid (p_1, \dots, p_m) -summing multilinear operators is a Banach ideal of multilinear operators. As application, we prove a Kwapien’s factorization theorem for dominated multilinear operators.

Finally in Chapter 5, we study the set of bounded sublinear operators which is

a positive cone in $\text{Lip}_0(X, Y)$. We shall consider the following situation: let X be a Banach space, Y be a complete Banach lattice and T be a bounded sublinear operator from X into Y (i.e., positively homogeneous and subadditive). We denote by ∇T the set of u in $\mathcal{B}(X, Y)$ (the Banach space of all bounded linear operators from X into Y) such that $u \leq T$. The aim of this is to study the relation between T and its subdifferential ∇T concerning the notion of Lipschitz p -summing and other classes of summability. We look also at the other types of summability. We are interested to the notion of Lipschitz p -dominated operators introduced by Chen and Zheng [19]. We show that if T is Lipschitz p -dominated sublinear operator, then $\nabla T \subset \Pi_p(X, Y)$ and we prove that, if T is Lipschitz strongly p -summing operators for $1 < p \leq \infty$, we have u positive strongly p -summing for all u in ∇T and hence u^* is positive p^* -summing. We also demonstrate that u is positive p -summing for all u in ∇T ($1 < p \leq 2$) whenever T is Lipschitz p -dominated sublinear operator from X into $L_2(\Omega, \mu)$.

PRELIMINARIES

The notation used in the chapter is in general standard. The letters X, Y shall denote Banach spaces over \mathbb{K} ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}). The closed unit ball of X is denoted by B_X and its topological dual by X^* . Let m be in \mathbb{N} and X_1, \dots, X_m, Y be Banach spaces over \mathbb{K} ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}). We will denote by $\mathcal{L}(X_1, \dots, X_m; Y)$ the Banach space of all bounded multilinear operators from $X_1 \times \dots \times X_m$ into Y equipped with the operator norm. If $m = 1$, $\mathcal{L}(X_1, \dots, X_m; Y) = \mathcal{L}(X; Y)$ the Banach space of all bounded linear operators $T : X \rightarrow Y$ endowed with the usual sup norm.

We also denote by $\mathcal{C}(K, G)$ the space of all continuous functions $f : K \rightarrow G$, where K is a compact Hausdorff space. If $G = \mathbb{K}$, we write simply $\mathcal{C}(K)$ instead of $\mathcal{C}(K, \mathbb{K})$.

As usual, $E \otimes G$ denotes the 2-fold tensor product of E and G .

Let $E \otimes_\varepsilon G$ be the injective tensor product, i.e., the algebraic tensor product $E \otimes G$ endowed with the injective crossnorm

$$\varepsilon(u) = \sup \{ |\langle u, x^* \otimes z^* \rangle|, \quad \|x^*\|_{E^*} \leq 1, \|z^*\|_{G^*} \leq 1 \}, u \in E \otimes G,$$

and $E \widehat{\otimes}_\varepsilon G$ its completion.

We say that [22, 55] a norm α on $E \otimes G$ is a crossnorm if it has the following properties:

1. $\alpha(x \otimes z) \leq \|x\| \|z\|$ for every $x \in E$ and $z \in G$.
2. For every $x^* \in E^*$ and $z^* \in G^*$, the linear functional $x^* \otimes z^*$ on $E \otimes F$ is bounded, and $\|x^* \otimes z^*\| \leq \|x^*\| \|z^*\|$.

1.1 Sequences spaces

For a Banach space X , and $1 \leq p < \infty$. Let us recall some important Banach sequence spaces, and will be used in the sequel. (see [24, 14])

The Banach space of absolutely p -summable sequences given by

$$\ell_p(X) := \left\{ (x_j)_{j=1}^\infty \subset X : \|(x_j)_{j=1}^\infty\|_p := \left(\sum_{j=1}^\infty \|x_j\|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

The Banach space of weakly p -summable sequences given by

$$\ell_p^w(X) := \left\{ (x_j)_{j=1}^\infty \subset X : \|(x_j)_{j=1}^\infty\|_{w,p} := \sup_{x^* \in B_{X^*}} \left(\sum_{j=1}^\infty |x^*(x_j)|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

The Banach space of unconditionally p -summable sequences given by

$$\ell_p^u(X) := \left\{ (x_j)_{j=1}^\infty \in \ell_p^w(X) : \lim_k \|(x_j)_{j=k}^\infty\|_{w,p} = 0 \right\}.$$

with the norm inherited from $\ell_p^w(X)$.

1.2 Linear operator ideals

1.2.1 Definition

Definition 1.2.1. [28] An ideal of linear operators is a subclass \mathcal{I} of all continuous linear operators between Banach spaces such that, the components

$$\mathcal{I}(X; Y) := \mathcal{L}(X; Y) \cap \mathcal{I}$$

satisfy.

1) $\mathcal{I}(X; Y)$ is a linear subspace of $\mathcal{L}(X; Y)$ which contains the linear operators of finite type.

2) The ideal property: if $T \in \mathcal{I}(G; F)$, $u \in \mathcal{L}(X, G)$ and $v \in \mathcal{L}(F, Y)$, then $v \circ T \circ u$ is in $\mathcal{I}(X; Y)$.

If $\|\cdot\|_{\mathcal{I}} : \mathcal{I} \rightarrow \mathbb{R}^+$ satisfies:

a) $(\mathcal{I}(X; Y), \|\cdot\|_{\mathcal{I}})$ is a normed (Banach) space,

b) the linear form $T : \mathbb{K} \rightarrow \mathbb{K}$ given by $T(x) = x$ satisfies $\|T\|_{\mathcal{I}} = 1$,

c) if $T \in \mathcal{I}(G; F)$, $u \in \mathcal{L}(X, G)$ and $v \in \mathcal{L}(F, Y)$ then

$$\|v \circ T \circ u\|_{\mathcal{I}} \leq \|v\| \|T\|_{\mathcal{I}} \|u\|.$$

We say that $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a normed (Banach) ideal.

Let $(\mathcal{I}(X; Y), \alpha(\cdot))$ be an operator ideal, we put

$$\mathcal{I}^d(X; Y) = \{T \in \mathcal{L}(X; Y) : T^* \in \mathcal{I}(Y^*; X^*)\}.$$

For an operator $T \in \mathcal{I}^d(X; Y)$, we put $\alpha^d(T) = \alpha(T^*)$. With these notations $(\mathcal{I}^d(X; Y), \alpha^d(\cdot))$ is also a Banach operator ideal and is called the dual ideal of $(\mathcal{I}(X; Y), \alpha(\cdot))$ (see [43, Section 4]).

1.2.2 Examples

The ideal of absolutely p -summing operators

For $1 \leq p < \infty$, we say that a linear operator $T : X \rightarrow Y$ is p -summing if $(T(x_j))_{j=1}^{\infty} \in \ell_p(Y)$ whenever $(x_j)_{j=1}^{\infty} \in \ell_p^w(X)$ (see [54]). The class of linear p -summing operators from X to Y will be represented by $\Pi_p(X; Y)$. An equivalent definition asserts that T is p -summing if there is a positive constant C such that

$$\|(T(x_j))_{j=1}^{\infty}\|_p \leq C \|(x_j)_{j=1}^{\infty}\|_{w,p}$$

for all $(x_j)_{j=1}^\infty \in \ell_p^w(X)$. We denote by $\pi_p(T)$ for the infimum of all $C > 0$ that satisfies the above inequality which defined a norm on $\Pi_p(X; Y)$, denoted by $\pi_p(T)$.

The ideal of strongly p -summing operators

Let us recall that (see Cohen in [20]) an operator $T \in \mathcal{L}(X; Y)$ is strongly p -summing if there is a constant C such that

$$\|(\langle T(x_j), y_j^* \rangle)_{j=1}^\infty\|_1 \leq C \|(x_j)_{j=1}^\infty\|_p \|(y_j^*)_{j=1}^\infty\|_{w,p^*}$$

for all $(x_j)_{j=1}^\infty \in \ell_p(X)$ and $(y_j^*)_{j=1}^\infty \in \ell_{p^*}^w(Y^*)$. The space of these operators from X into Y denoted by $\mathcal{D}_p(X; Y)$, is a Banach space under the norm $d_p(T)$ which is the infimum of all $C > 0$ that satisfy the above inequality.

1.3 Ideals of multilinear mappings

1.3.1 Definition

Definition 1.3.1. [28] An ideal of multilinear operators (or multi-ideal) is a subclass \mathcal{M} of all continuous multilinear operators between Banach spaces such that for all $m \in \mathbb{N}$, the components

$$\mathcal{M}(X_1, \dots, X_m; Y) := \mathcal{L}(X_1, \dots, X_m; Y) \cap \mathcal{M}$$

satisfy.

- 1) $\mathcal{M}(X_1, \dots, X_m; Y)$ is a linear subspace of $\mathcal{L}(X_1, \dots, X_m; Y)$ which contains the multilinear operators of finite type.
- 2) The ideal property: if $T \in \mathcal{M}(G_1, \dots, G_m; F)$, $u_j \in \mathcal{L}(X_j, G_j)$ for $j = 1, \dots, m$ and $v \in \mathcal{L}(F, Y)$, then $v \circ T \circ (u_1, \dots, u_m)$ is in $\mathcal{M}(X_1, \dots, X_m; Y)$.

If $\|\cdot\|_{\mathcal{M}} : \mathcal{M} \rightarrow \mathbb{R}^+$ satisfies:

- a) $(\mathcal{M}(X_1, \dots, X_m; Y), \|\cdot\|_{\mathcal{M}})$ is a normed (Banach) space,

b) the multilinear form $T^m : \mathbb{K}^m \rightarrow \mathbb{K}$ given by $T^m(x^1, \dots, x^m) = x^1 \dots x^m$ satisfies $\|T^m\|_{\mathcal{M}} = 1$ for all m ,

c) if $T \in \mathcal{M}(G_1, \dots, G_m; F)$, $u_j \in \mathcal{L}(X_j, G_j)$ for $j = 1, \dots, m$ and $v \in \mathcal{L}(F, Y)$ then

$$\|v \circ T \circ (u_1, \dots, u_m)\|_{\mathcal{M}} \leq \|v\| \|T\|_{\mathcal{M}} \|u_1\| \dots \|u_m\|.$$

We say that $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ is a normed (Banach) multi-ideal.

1.3.2 Example

Consider $1 \leq p, p_1, \dots, p_m < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$. A multilinear operator T from $X_1 \times \dots \times X_m$ into Y is called (p_1, \dots, p_m) -dominated (see [10, 36, 37, 47, 53, 52]) if $(T(x_j^1, \dots, x_j^m))_{j=1}^{\infty} \in \ell_p(Y)$ whenever $(x_j^i)_{j=1}^{\infty} \in \ell_{p_i}^w(X_i)$. This is equivalent to say that, there is a constant $C > 0$ such that for any $(x_j^i)_{j=1}^{\infty} \in \ell_{p_i}^w(X_i)$, $(1 \leq i \leq m)$ we have

$$\left\| (T(x_j^1, \dots, x_j^m))_{j=1}^{\infty} \right\|_p \leq C \prod_{i=1}^m \left\| (x_j^i)_{j=1}^{\infty} \right\|_{w, p_i}. \quad (1.1)$$

The class of all (p_1, \dots, p_m) -dominated multilinear operators from $X_1 \times \dots \times X_m$ into Y is denoted by $\Pi_{p_1, \dots, p_m}(X_1, \dots, X_m; Y)$, which is a Banach space equipped with the norm $\pi_{p_1, \dots, p_m}(T)$ that is the smallest constant C such that the Inequality (1.1) holds. This class satisfies a Pietsch's domination theorem, so T is (p_1, \dots, p_m) -dominated if and only if there is a constant $C > 0$ and regular Borel probability measures μ_i on $B_{X_i^*}$, $1 \leq i \leq m$, (with the weak star topology) so that for all $(x^1, \dots, x^m) \in X_1 \times \dots \times X_m$ the inequality

$$\|T(x^1, \dots, x^m)\| \leq C \prod_{i=1}^m \left(\int_{B_{X_i^*}} |\langle x^i, x_i^* \rangle|^{p_i} d\mu_i(x_i^*) \right)^{\frac{1}{p_i}} \quad (1.2)$$

is valid. In this case $\pi_{p_1, \dots, p_m}(T)$ is the least of all $C > 0$ such that (1.2) holds.

For the convenience of the reader we also give the following theorem due to Geiss [28].

Theorem 1.3.2. $T \in \Pi_{p_1, \dots, p_m}(X_1, \dots, X_m; Y)$ if and only if, there exists a Radon probability μ_i on $(B_{X_i^*}, w^*)$, a closed subspace Y_i of $L_{p_i}(\mu_i)$ ($1 \leq i \leq m$) and a continuous multilinear operator $S : Y_1 \times \dots \times Y_m \rightarrow Y$ such that the following diagram commutes

$$\begin{array}{ccccccc} X_1 & \times & \cdots & \times & X_m & \rightarrow & Y \\ \downarrow & & & & \downarrow & & \uparrow \\ I_{X_1}(X_1) & \times & \cdots & \times & I_{X_m}(X_m) & \rightarrow & Y_1 \times \cdots \times Y_m \\ \cap & & & & \cap & & \cap \\ \mathcal{C}(B_{X_1^*}) & \times & \cdots & \times & \mathcal{C}(B_{X_m^*}) & \rightarrow & L_{p_1}(\mu_1) \times \cdots \times L_{p_m}(\mu_m) \end{array}$$

$T = S \circ (R_1, \dots, R_m)$, where $R_i = J_{p_i}|_{I_{X_i}(X_i)} \circ I_{X_i}$, $I_{X_i} : X_i \rightarrow I_{X_i}(X_i)$ and $J_{p_i} : \mathcal{C}(B_{X_i^*}) \rightarrow L_{p_i}(\mu_i)$ are the canonical operators, $J_{p_i}|_{I_{X_i}(X_i)} : I_{X_i}(X_i) \rightarrow Y_i$ is the restriction of J_{p_i} to $I_{X_i}(X_i)$ $1 \leq i \leq m$.

We can see the following definition in [45] Definition 4].

Definition 1.3.3 (Factorization method). Let $(\mathcal{I}_i, \alpha_i(\cdot))$ ($i = 1, \dots, m$) be Banach ideals of bounded linear operators. We say that $T \in \mathcal{L}(X_1, \dots, X_m; Y)$ belongs to $\mathcal{L} \circ (\mathcal{I}_1, \dots, \mathcal{I}_m)(X_1, \dots, X_m; Y)$ if and only if there exist a Banach spaces Y_1, \dots, Y_m , $R_i \in \mathcal{I}_i$ ($i = 1, \dots, m$) and $S \in \mathcal{L}(Y_1, \dots, Y_m; Y)$ such that $T = S \circ (R_1, \dots, R_m)$. In this case, we define $\|T\|_{\mathcal{L} \circ (\mathcal{I}_1, \dots, \mathcal{I}_m)} = \inf \{ \|S\| \prod_{i=1}^m \alpha_i(R_i) \}$, where the infimum is taken over all factorizations of T as above.

The next theorem is due to [53](see [8] Theorem 2.2)].

Theorem 1.3.4 (Factorization Theorem). A multilinear operator $T \in \mathcal{L}(X_1, \dots, X_m; Y)$ is (p_1, \dots, p_m) -dominated if and only if there are Banach spaces Y_1, \dots, Y_m , operators $R_i \in \Pi_{p_i}(X_i; Y_i)$, ($i = 1, \dots, m$) and $S \in \mathcal{L}(Y_1, \dots, Y_m; Y)$ such that $T = S \circ (R_1, \dots, R_m)$. Moreover

$$\pi_{p_1, \dots, p_m}(T) = \inf \left\{ \|S\| \prod_{i=1}^m \pi_{p_i}(R_i) : T = S \circ (R_1, \dots, R_m) \right\}.$$

DOMINATED MULTILINEAR OPERATORS DEFINED ON TENSOR PRODUCTS OF BANACH SPACES

In this chapter, we introduce and study new classes of dominated multilinear operators, which we call $(p; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated and $(\tilde{p}; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated multilinear operators defined on the tensor product of Banach spaces. Some characterizations of this type of operators are given and some important coincidence results are proved. As an application, we characterize $(p; p_1, \dots, p_n)$ -dominated multilinear operators on $\mathcal{C}(K, G)$ and $(p; p_1, \dots, p_n)$ -dominated multilinear operators in the sense of Dinculeanu on $\mathcal{C}(K, G)$, where K is a compact Hausdorff space and G a Banach space. We also treat the connection between an operator T and its associated operators T^t, \tilde{T} and $T^\#$ for certain classes.

2.1 Introduction and background

Let K_1, \dots, K_n be compact Hausdorff topological spaces. Every multilinear operator $T \in \mathcal{L}(\mathcal{C}(K_1) \widehat{\otimes}_\varepsilon G_1, \dots, \mathcal{C}(K_n) \widehat{\otimes}_\varepsilon G_n; F)$ induces an associated multilinear operator $T^\# : \mathcal{C}(K_1) \times \dots \times \mathcal{C}(K_n) \rightarrow \mathcal{L}(G_1, \dots, G_n; F)$ defined by

$$T^\#(x_1, \dots, x_n)(z_1, \dots, z_n) = T(x_1 \otimes z_1, \dots, x_n \otimes z_n).$$

The origin of this notion, for the linear case, is due to Dinculeanu [23] ($T^\#$ is denoted by T' in [23, page 377]). If $T : \mathcal{C}(K) \widehat{\otimes}_\varepsilon G \rightarrow F$ is a bounded linear operator, then there is a unique bounded linear operator $T^\# : \mathcal{C}(K) \rightarrow \mathcal{L}(G, F)$ such that $T(f \otimes z) = T^\#(f)(z)$. Swartz proved [56, Theorem 12] that $T : \mathcal{C}(K) \widehat{\otimes}_\varepsilon G \rightarrow F$ is 1-summing if, and only if, $T^\# : \mathcal{C}(K) \rightarrow \Pi_1(G, F)$ is 1-summing and this result was extended by Montgomery-Smith and Saab in [40] to any \mathcal{L}_∞ -space. Popa [49] gave necessary and sufficient conditions for an operator on the space $\mathcal{C}(K, G)$ to be (r, p) -absolutely summing and in [50] he generalized this result to multilinear operators, proving that if $T \in \Pi_{p; p_1, \dots, p_n}(\mathcal{C}(K_1) \widehat{\otimes}_\varepsilon G_1, \dots, \mathcal{C}(K_n) \widehat{\otimes}_\varepsilon G_n; F)$ is $(p; p_1, \dots, p_n)$ -dominated then $T^\# : \mathcal{C}(K_1) \times \dots \times \mathcal{C}(K_n) \rightarrow \Pi_{p; p_1, \dots, p_n}(G_1, \dots, G_n; F)$ is $(p; p_1, \dots, p_n)$ -dominated. He also proved a similar result for multiple summing operators in [51, Theorem 1].

Let us recall the notion of (p, G) -summing linear operators introduced in [38, Exposé I, page 1] and studied in [9, 31], which together with the results in [50] are our main motivations for this chapter. We try to generalize these notions to multilinear operators, replacing $\mathcal{C}(K)$ by any Banach space E and the ε -crossnorm by any α -crossnorm.

Consider $1 \leq p < \infty$. An operator T from the injective tensor product $E \otimes_\varepsilon G$ into F is said to be (p, G) -summing if there exists a constant $C > 0$ such that, for

all finite sequence $(u_i)_{1 \leq i \leq n}$ in $E \otimes G$, we have

$$\left(\sum_{i=1}^n \|T(u_i)\|_F^p \right)^{\frac{1}{p}} \leq C \sup_{x^* \in \mathcal{B}_{E^*}} \left(\sum_{i=1}^n \|u_i(x^*)\|_G^p \right)^{\frac{1}{p}} \quad (2.1)$$

where $u_i(x^*) = \sum_{j=1}^{m_i} x^*(x_{ij})z_{ij}$, for $u_i = \sum_{j=1}^{m_i} x_{ij} \otimes z_{ij}$, $x_{ij} \in E$ and $z_{ij} \in G$. By [38], this definition is equivalent to say that: there is a probability measure μ on $(\mathcal{B}_{E^*}, \sigma(E^*, E))$ such that

$$\|T(u)\|_F^p \leq C^p \int_{\mathcal{B}_{E^*}} \|u(x^*)\|_G^p d\mu(x^*)$$

for every $u \in E \otimes G$. The smallest of such constants defines the (p, G) -norm of T , denoted by $\pi_p^G(T)$ and the space $\Pi_p^G(E \widehat{\otimes}_\varepsilon G, F)$ of all (p, G) -summing operators is a Banach space endowed with such norm.

Let $T: E \widehat{\otimes}_\varepsilon G \rightarrow F$ be a bounded linear operator. We denote by \tilde{T} the associated linear operator from G to $\mathcal{L}(E; F)$ defined by $\tilde{T}(z)(x) = T(x \otimes z)$, $x \in E$ and $z \in G$ (in [9] \tilde{T} is denote by $T^\# = \Phi(T)$). It was proved by Blasco and Signes in [9, Proposition 1.4] that, if $T: E \widehat{\otimes}_\varepsilon G \rightarrow F$ is (p, G) -summing then $\tilde{T}(z)$ is p -summing for all $z \in G$.

Later in [25], Elezovic introduced the following notion in the setting of tensor product. Let α be a crossnorm on $E \otimes G$ and $1 \leq p < \infty$. An operator T from the tensor product $E \otimes_\alpha G$ into F is said to be (\tilde{p}, G) -summing if there exists a positive constant C such that, for any finite sequence $(u_i)_{1 \leq i \leq n}$ in $E \otimes G$, we have

$$\left(\sum_{i=1}^n \|T(u_i)\|_F^p \right)^{\frac{1}{p}} \leq C \sup_{x^* \in \mathcal{B}_{E^*}, z^* \in \mathcal{B}_{G^*}} \left(\sum_{i=1}^n |\langle u_i(x^*), z^* \rangle|^p \right)^{\frac{1}{p}}$$

where $\langle u_i(x^*), z^* \rangle = \sum_{j=1}^{m_i} x^*(x_{ij})z^*(z_{ij})$, for $u_i = \sum_{j=1}^{m_i} x_{ij} \otimes z_{ij}$, $x_{ij} \in E$ and $z_{ij} \in G$. The smallest of such constants defines the (\tilde{p}, G) -norm of T , denoted by $\pi_{\tilde{p}}^G(T)$ and the space $\Pi_{\tilde{p}}^G(E \widehat{\otimes}_\alpha G, F)$ of all (\tilde{p}, G) -summing operators is a Banach space endowed with such norm.

Our main focus in this chapter is to develop new classes of multilinear operators on tensor products of Banach spaces, similar to those of (p, G) -summing and (\tilde{p}, G) -summing linear operators.

The chapter is organized as follows. In Section 2, we introduce the notions of $(p; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated and $(\tilde{p}; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated multilinear operators for $1 \leq p, p_1, \dots, p_n < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_n}$.

We prove a natural analog to “Pietsch domination/factorization theorem” for such classes as the linear case introduced and studied by Maurey in [38]. In Section 3 we establish the relationship between the three classes of dominated multilinear operators. Finally, in Section 4 we treat the relationship between a multilinear operator T and its associated operators T^t , \tilde{T} and $T^\#$ for the classes of summability cited above. As a consequence, some interesting properties are given concerning certain generalizations of the linear case.

2.2 Characterization

Let α_i be a crossnorm on the space $E_i \otimes G_i$, $1 \leq i \leq n$. We introduce the following definition for a multilinear operator $T : E_1 \otimes_{\alpha_1} G_1 \times \dots \times E_n \otimes_{\alpha_n} G_n \rightarrow F$.

Definition 2.2.1. Consider $1 \leq p, p_1, \dots, p_n < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_n}$. The multilinear operator T is said to be

1) $(p; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated if there exists a constant $C > 0$ such that

$$\left(\sum_{k=1}^l \|T(u_1^k, \dots, u_n^k)\|_F^p \right)^{\frac{1}{p}} \leq C \prod_{i=1}^n \sup_{\|x_i^*\|_{E_i^*} \leq 1} \left(\sum_{k=1}^l \|u_i^k(x_i^*)\|_{G_i}^{p_i} \right)^{\frac{1}{p_i}} \quad (2.2)$$

for any finite sequences $u_i^k \in E_i \otimes G_i$, $1 \leq i \leq n$, $1 \leq k \leq l$.

2) $(\tilde{p}; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated if there exists a constant $C > 0$ such

that

$$\left(\sum_{k=1}^l \|T(u_1^k, \dots, u_n^k)\|_F^p \right)^{\frac{1}{p}} \leq C \prod_{i=1}^n \sup_{\|z_i^*\|_{G_i^*}, \|x_i^*\|_{E_i^*} \leq 1} \left(\sum_{k=1}^l |\langle u_i^k(x_i^*), z_i^* \rangle|^{p_i} \right)^{\frac{1}{p_i}} \quad (2.3)$$

for every choice of elements $u_i^k \in E_i \otimes G_i$, $1 \leq i \leq n$, $1 \leq k \leq l$.

We denote by $\pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(T)$ and $\pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(T)$, respectively, the infimum of all constants C in (2.2) and (2.3) and by $\Pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(E_1 \otimes_{\alpha_1} G_1, \dots, E_n \otimes_{\alpha_n} G_n; F)$ and $\Pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(E_1 \otimes_{\alpha_1} G_1, \dots, E_n \otimes_{\alpha_n} G_n; F)$ the corresponding spaces of such multilinear operators.

Example 2.2.2. Consider $1 \leq i \leq n$. Then, for every $x_i^* \in E_i^*$, $z_i^* \in G_i^*$ and $y \in F$, the operator $T = x_1^* \otimes z_1^* \otimes \dots \otimes x_n^* \otimes z_n^* \otimes y$ defined by

$$(x_1^* \otimes z_1^* \otimes \dots \otimes x_n^* \otimes z_n^* \otimes y)(u_1, \dots, u_n) = \langle u_1, x_1^* \otimes z_1^* \rangle \cdots \langle u_n, x_n^* \otimes z_n^* \rangle y$$

is $(\tilde{p}; p_1, \dots, p_n, G_1, \dots, G_n)$ -dominated and

$$\pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(x_1^* \otimes z_1^* \otimes \dots \otimes x_n^* \otimes z_n^* \otimes y) = \|y\| \prod_{i=1}^n \|x_i^*\| \|z_i^*\|.$$

Suppose that the α_i ($1 \leq i \leq n$) are uniform crossnorms. Let X_i, Z_i, E_i, G_i be Banach spaces and let $A_i : X_i \rightarrow E_i$, $B_i : Z_i \rightarrow G_i$ be bounded linear operators. Let $T_i = A_i \otimes_{\alpha_i} B_i : X_i \otimes_{\alpha_i} Z_i \rightarrow E_i \otimes_{\alpha_i} G_i$ be the unique operator defined by $T_i(a \otimes c) = A_i(a) \otimes B_i(c)$ such that $\|T_i\| \leq \|A_i\| \|B_i\|$ (see [22, 55]).

Proposition 2.2.3. Let $T \in \Pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(E_1 \otimes_{\alpha_1} G_1, \dots, E_n \otimes_{\alpha_n} G_n; F)$ and h be a bounded linear operator from F to Y . Then

$$h \circ T \circ (T_1, \dots, T_n) \in \Pi_{\tilde{p}; p_1, \dots, p_n}^{Z_1, \dots, Z_n}(X_1 \otimes_{\alpha_1} Z_1, \dots, X_n \otimes_{\alpha_n} Z_n; Y)$$

and

$$\pi_{\tilde{p}; p_1, \dots, p_n}^{Z_1, \dots, Z_n}(h \circ T \circ (T_1, \dots, T_n)) \leq \|h\| \pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \prod_{i=1}^n \|A_i\| \|B_i\|.$$

Remark 2.2.4. The same holds for $\Pi_{\vec{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(E_1 \otimes_{\alpha_1} G_1, \dots, E_n \otimes_{\alpha_n} G_n; F)$.

Now, we characterize the $(p; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated multilinear operators by giving the Pietsch domination theorem. For the proof we use the full general Pietsch domination theorem presented by Pellegrino et al. in [46].

Let $X_1, \dots, X_m, Y, E_1, \dots, E_k$ be (arbitrary) non-void sets and \mathcal{H} be a family of mappings from $X_1 \times \dots \times X_m$ to Y . Let K_1, \dots, K_t be compact Hausdorff topological spaces and G_1, \dots, G_t be Banach spaces. Suppose that the maps

$$\begin{cases} R_j : K_j \times E_1 \times \dots \times E_k \times G_j \rightarrow [0, +\infty), & j = 1, \dots, t \\ S : \mathcal{H} \times E_1 \times \dots \times E_k \times G_1 \times \dots \times G_t \rightarrow [0, +\infty) \end{cases}$$

satisfy:

(1) For each $x^l \in E_l$ and $b \in G_j$, with $(j, l) \in \{1, \dots, t\} \times \{1, \dots, k\}$, the mapping $(R_j)_{x^1, \dots, x^k, b} : K_j \rightarrow [0, \infty)$, defined by $(R_j)_{x^1, \dots, x^k, b}(\varphi_j) = R_j(\varphi_j, x^1, \dots, x^k, b)$, is continuous.

(2) The following inequalities hold:

$$\begin{cases} R_j(\varphi_j, x^1, \dots, x^k, \eta_j b^j) \leq \eta_j R_j(\varphi_j, x^1, \dots, x^k, b^j) \\ S(f, x^1, \dots, x^k, \alpha_1 b^1, \dots, \alpha_t b^t) \geq \alpha_1 \dots \alpha_t S(f, x^1, \dots, x^k, b^1, \dots, b^t). \end{cases}$$

For R_1, \dots, R_t and S as above and $1 \leq p, p_1, \dots, p_t < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_t}$, a mapping $f \in \mathcal{H}$ is said to be R_1, \dots, R_t - S -abstract (p_1, \dots, p_t) -summing if there is a constant $C > 0$ such that

$$\left(\sum_{i=1}^n S(f, x_i^1, \dots, x_i^k, b_i^1, \dots, b_i^t)^p \right)^{\frac{1}{p}} \leq C \prod_{j=1}^t \sup_{\varphi_j \in K_j} \left(\sum_{i=1}^n R_j(\varphi_j, x_i^1, \dots, x_i^k, b_i^j)^{p_j} \right)^{\frac{1}{p_j}}$$

for all $x_1^s, \dots, x_n^s \in E_s, b_1^j, \dots, b_n^j \in G_j$ and $(s, j) \in \{1, \dots, k\} \times \{1, \dots, t\}$.

Now, we can present the abstract domination theorem.

Theorem 2.2.5. [46] Theorem 4.6] A map $f \in \mathcal{H}$ is R_1, \dots, R_t - S -abstract (p_1, \dots, p_t) -summing if, and only if, there are a constant $C > 0$ and Borel probability measures μ_j on K_j such that

$$S(f, x^1, \dots, x^k, b^1, \dots, b^t) \leq C \prod_{j=1}^t \left(\int_{K_j} R_j(\varphi_j, x^1, \dots, x^k, b^j)^{p_j} d\mu_j \right)^{\frac{1}{p_j}}$$

for all $x^l \in E_l$ and $b \in G_j$, with $(j, l) \in \{1, \dots, t\} \times \{1, \dots, k\}$.

From the above theorem, we present the domination theorem concerning the class of $(p; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated multilinear operators

Theorem 2.2.6. *Let $1 \leq p, p_1, \dots, p_n < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_n}$. A multilinear operator $T : E_1 \otimes_{\alpha_1} G_1 \times \dots \times E_n \otimes_{\alpha_n} G_n \rightarrow F$ is $(p; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated if, and only if, there are a constant $C > 0$ and probability measures μ_i on $(\mathcal{B}_{E_i^*}, \sigma(E_i^*, E_i))$ such that*

$$\|T(u_1, \dots, u_n)\|_F \leq C \prod_{i=1}^n \left(\int_{\mathcal{B}_{E_i^*}} \|u_i(x_i^*)\|_{G_i}^{p_i} d\mu_i(x_i^*) \right)^{1/p_i} \quad (2.4)$$

for all $u_i \in E_i \otimes G_i$. Moreover, $\pi_{p; p_1, \dots, p_n}^{G_1, \dots, G_n}(T)$ is the smallest of the constants verifying the inequality (2.4).

Proof. Choosing the parameters

$$\left\{ \begin{array}{l} t = n \\ E_i = \mathbb{K}, i = 1, \dots, n \\ G_i = E_i \otimes_{\alpha_i} G_i, i = 1, \dots, n \\ K_i = \mathcal{B}_{E_i^*}, i = 1, \dots, n \\ \mathcal{H} = \mathcal{L}(E_1 \otimes_{\alpha_1} G_1, \dots, E_n \otimes_{\alpha_n} G_n; F) \\ S(T, \lambda_1, \dots, \lambda_n, u_1, \dots, u_n) = \|T(u_1, \dots, u_n)\| \\ R_i(x_i^*, \lambda_1, \dots, \lambda_n, u_i) = \|u_i(x_i^*)\|, i = 1, \dots, n \end{array} \right.$$

we can easily conclude that $T : E_1 \otimes_{\alpha_1} G_1 \times \dots \times E_n \otimes_{\alpha_n} G_n \rightarrow F$ is $(p; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated if, and only if, it is R_1, \dots, R_n - S -abstract (p_1, \dots, p_n) -summing. Theorem 2.2.5 tells us that T is R_1, \dots, R_n - S -abstract (p_1, \dots, p_n) -summing if, and only if, there exist a positive constant C and probability measures μ_i on $K_i, i = 1, \dots, n$, such that

$$S(T, \lambda_1, \dots, \lambda_n, u_1, \dots, u_n) \leq C \prod_{i=1}^n \left(\int_{K_i} R_i(x_i^*, \lambda_1, \dots, \lambda_n, u_i)^{p_i} d\mu_i(x_i^*) \right)^{\frac{1}{p_i}}.$$

Consequently,

$$\|T(u_1, \dots, u_n)\|_F \leq C \prod_{i=1}^n \left(\int_{K_i} \|u_i(x_i^*)\|^{p_i} d\mu_i(x_i^*) \right)^{1/p_i}$$

and this concludes the proof. □

Theorem 2.2.7 (Factorization Theorem). *Consider $1 \leq p, p_1, \dots, p_n < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_n}$. Then, the following properties are equivalent:*

- 1) *A multilinear operator T is in $\Pi_{p; p_1, \dots, p_n}^{G_1, \dots, G_n}(E_1 \otimes_{\alpha_1} G_1, \dots, E_n \otimes_{\alpha_n} G_n; F)$.*
- 2) *There exist Banach spaces F_1, \dots, F_n , (p_i, G_i) -summing linear operators $T_i \in \Pi_{p_i}^{G_i}(E_i \otimes_{\alpha_i} G_i; F_i)$ ($1 \leq i \leq n$) and an n -linear mapping $S \in \mathcal{L}(F_1, \dots, F_n; F)$ such that the following diagram commutes*

$$\begin{array}{ccccccc} E_1 \otimes_{\alpha_1} G_1 & \times & \dots & \times & E_n \otimes_{\alpha_n} G_n & \xrightarrow{T} & F \\ T_1 \downarrow & & & & T_n \downarrow & S \nearrow & \\ F_1 & \times & \dots & \times & F_n & & \end{array}$$

Proof. First we prove the converse. Let $u_i \in E_i \otimes G_i$ ($1 \leq i \leq n$). If T has such a factorization, we have

$$\begin{aligned} \|T(u_1, \dots, u_n)\| &= \|S(T_1(u_1), \dots, T_n(u_n))\| \\ &\leq \|S\| \prod_{i=1}^n \|T_i(u_i)\|. \end{aligned}$$

Since T_i is (p_i, G_i) -summing then, by Theorem [2.2.6](#) for $n = 1$, there is a probability measure μ_i on $(\mathcal{B}_{E_i^*}, \sigma(E_i^*, E_i))$ such that, for all $u_i \in E_i \otimes G_i$,

$$\|T_i(u_i)\| \leq \pi_{p_i}^{G_i}(T_i) \left(\int_{\mathcal{B}_{E_i^*}} \|u_i(x_i^*)\|^{p_i} d\mu_i(x_i^*) \right)^{\frac{1}{p_i}}.$$

Consequently,

$$\|T(u_1, \dots, u_n)\| \leq \|S\| \prod_{i=1}^n \pi_{p_i}^{G_i}(T_i) \left(\int_{\mathcal{B}_{E_i^*}} \|u_i(x_i^*)\|^{p_i} d\mu_i(x_i^*) \right)^{\frac{1}{p_i}}.$$

Therefore, by Theorem [2.2.6](#), T is $(p; p_1, \dots, p_n, G_1, \dots, G_n)$ -dominated and $\pi_{p; p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \leq \|S\| \prod_{i=1}^n \pi_{p_i}^{G_i}(T_i)$.

To prove the first implication, take $T \in \Pi_{p; p_1, \dots, p_n}^{G_1, \dots, G_n}(E_1 \otimes_{\alpha_1} G_1, \dots, E_n \otimes_{\alpha_n} G_n; F)$. So, there are, by Theorem [2.2.6](#), probability measures μ_i on $(\mathcal{B}_{E_i^*}, \sigma(E_i^*, E_i))$ such that, for all $u_i \in E_i \otimes G_i$, we have

$$\|T(u_1, \dots, u_n)\| \leq \pi_{p; p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \prod_{i=1}^n \left(\int_{\mathcal{B}_{E_i^*}} \|u_i(x_i^*)\|_{G_i}^{p_i} d\mu_i(x_i^*) \right)^{1/p_i}.$$

Now we consider the operator $T_i^0 : E_i \otimes_{\alpha_i} G_i \rightarrow L_{p_i}(\mathcal{B}_{E_i^*}, \mu_i, G_i)$ given by $T_i^0(u_i) = u_i(\cdot)$ (where $L_{p_i}(\mathcal{B}_{E_i^*}, \mu_i, G_i)$ is the space of the p_i -Bochner integrable functions).

Notice that

$$\begin{aligned} \|T_i^0(u_i)\| &= \left(\int_{\mathcal{B}_{E_i^*}} \|u_i(x_i^*)\|^{p_i} d\mu_i(x_i^*) \right)^{\frac{1}{p_i}} \\ &\leq \sup_{\|x_i^*\| \leq 1} \|u_i(x_i^*)\| \\ &\leq \sup_{\|x_i^*\| \leq 1, \|z_i^*\| \leq 1} |\langle u_i(x_i^*), z_i^* \rangle| \\ &\leq \alpha_i(u_i), \end{aligned}$$

for all $u_i \in E_i \otimes G_i$. Let F_i be the closure in $L_{p_i}(\mathcal{B}_{E_i^*}, \mu_i, G_i)$ of the range of T_i^0 , i.e., $F_i = \overline{T_i^0(E_i \otimes_{\alpha_i} G_i)}^{L_{p_i}(\mathcal{B}_{E_i^*}, \mu_i, G_i)}$ and $T_i : E_i \otimes_{\alpha_i} G_i \rightarrow F_i$ be the extension operator. Note that T_i is (p_i, G_i) -summing with $\pi_{p_i}^{G_i}(T_i) \leq 1$. Let S_0 be the multilinear operator defined on $T_1^0(E_1 \otimes_{\alpha_1} G_1) \times \cdots \times T_n^0(E_n \otimes_{\alpha_n} G_n)$ by

$$S_0(T_1^0(u_1), \dots, T_n^0(u_n)) = T(u_1, \dots, u_n).$$

This definition makes sense because

$$\|S_0(T_1^0(u_1), \dots, T_n^0(u_n))\| \leq \prod_{i=1}^n \|T_i^0(u_i)\|.$$

It follows that S_0 is continuous on $T_1^0(E_1 \otimes_{\alpha_1} G_1) \times \cdots \times T_n^0(E_n \otimes_{\alpha_n} G_n)$ and has a unique bounded multilinear extension S to

$$F_1 \times \cdots \times F_n = \overline{T_1^0(E_1 \otimes_{\alpha_1} G_1)}^{L_{p_1}(\mathcal{B}_{E_1^*}, \mu_1, G_1)} \times \cdots \times \overline{T_n^0(E_n \otimes_{\alpha_n} G_n)}^{L_{p_n}(\mathcal{B}_{E_n^*}, \mu_n, G_n)}.$$

Finally, note also that $T = S \circ (T_1, \dots, T_n)$, where $T_i \in \Pi_{p_i}^{G_i}(E_i \otimes_{\alpha_i} G_i, F_i)$ ($1 \leq i \leq n$), and this ends the proof. \square

Corollary 2.2.8. *If $T : E_1 \otimes_{\alpha_1} G_1 \times \cdots \times E_n \otimes_{\alpha_n} G_n \rightarrow F$ is $(p; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated, then T admits a multilinear extension \hat{T} from $E_1 \hat{\otimes}_{\alpha_1} G_1 \times \cdots \times E_n \hat{\otimes}_{\alpha_n} G_n$ into F which is $(p; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated and $\pi_{p; p_1, \dots, p_n}^{G_1, \dots, G_n}(T) = \pi_{p; p_1, \dots, p_n}^{G_1, \dots, G_n}(\hat{T})$.*

Remark 2.2.9. By Corollary [2.2.8](#), we can replace $E_i \otimes_{\alpha_i} G_i$ in Definition [2.2.1](#) ([2.2.1](#)) and Theorem [2.2.6](#) with $E_i \hat{\otimes}_{\alpha_i} G_i$.

Now, we characterize the $(\tilde{p}; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated multilinear operators by giving the Pietsch domination theorem. For the proof we use the full general Pietsch domination theorem as in Theorem [2.2.6](#).

Theorem 2.2.10. *Let $1 \leq p, p_1, \dots, p_n < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_n}$. A multilinear operator $T : E_1 \otimes_{\alpha_1} G_1 \times \dots \times E_n \otimes_{\alpha_n} G_n \rightarrow F$ is $(\tilde{p}; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated if, and only if, there exist Radon probability measures μ_i on $K_i = \mathcal{B}_{E_i^*} \times \mathcal{B}_{G_i^*}$ and a constant $C > 0$ such that, for all $u_i \in E_i \otimes G_i$, one has*

$$\|T(u_1, \dots, u_n)\|_F \leq C \prod_{i=1}^n \left(\int_{K_i} |\langle u_i, x_i^* \otimes z_i^* \rangle|^{p_i} d\mu_i(x_i^*, z_i^*) \right)^{1/p_i}. \quad (2.5)$$

Moreover, $\pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(T)$ is the smallest of the constants verifying the estimate [\(2.5\)](#).

Now, we give the factorization theorem for the class of $(\tilde{p}; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated multilinear operators. For the proof, we use Theorem 1 in [\[25\]](#) and the same idea in Theorem [2.2.7](#). The proof will be omitted.

Theorem 2.2.11. *Let $1 \leq p, p_1, \dots, p_n < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_n}$. Then, the following assertions are equivalent:*

- 1) An operator $T \in \Pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(E_1 \otimes_{\alpha_1} G_1, \dots, E_n \otimes_{\alpha_n} G_n; F)$.
- 2) There exist Banach spaces F_1, \dots, F_n , (\tilde{p}_i, G_i) -summing linear operators $T_i \in \Pi_{\tilde{p}_i}^{G_i}(E_i \otimes_{\alpha_i} G_i; F_i)$ ($1 \leq i \leq n$) and an n -linear mapping $S \in \mathcal{L}(F_1, \dots, F_n; F)$ such that T admits the factorization $T = S \circ (T_1, \dots, T_n)$.

Remark 2.2.12. As in Remark [2.2.9](#), we can replace in Definition [\(2.2.1\)](#) [\(2.2.1\)](#), Theorem [2.2.10](#) and Theorem [2.2.11](#), $E_i \otimes_{\alpha_i} G_i$ with $E_i \hat{\otimes}_{\alpha_i} G_i$.

2.3 Coincidence and inclusion properties

In this section, we investigate the relationships between various classes of dominated multilinear operators and some coincidence results. Firstly, since

$$\sup_{\|z_i^*\|_{G_i^*}, \|x_i^*\|_{E_i^*} \leq 1} \left(\sum_{k=1}^l \left| \langle u_i^k(x_i^*), z_i^* \rangle \right|^{p_i} \right)^{\frac{1}{p_i}} \leq \sup_{\|x_i^*\|_{E_i^*} \leq 1} \left(\sum_{k=1}^l \|u_i^k(x_i^*)\|_{G_i}^{p_i} \right)^{\frac{1}{p_i}}$$

holds for every $u_j^i \in E_i \otimes G_i$, $1 \leq i \leq n$, $1 \leq k \leq l$, we obtain the following proposition.

Proposition 2.3.1. *If T is $(\tilde{p}; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated multilinear operator then T is $(p; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated and $\pi_{p; p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \leq \pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(T)$.*

The next theorem is one of our main results. Mimicking the same idea of [9], we have the following result.

Theorem 2.3.2. *Let α_i be an arbitrary crossnorm on $E_i \otimes G_i$, $1 \leq i \leq n$. If*

$$\Pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(E_1 \hat{\otimes}_{\alpha_1} G_1, \dots, E_n \hat{\otimes}_{\alpha_n} G_n; F) = \Pi_{p; p_1, \dots, p_n}^{G_1, \dots, G_n}(E_1 \hat{\otimes}_{\alpha_1} G_1, \dots, E_n \hat{\otimes}_{\alpha_n} G_n; F)$$

then

$$\mathcal{L}(G_1, \dots, G_n; F) = \Pi_{p; p_1, \dots, p_n}(G_1, \dots, G_n; F).$$

Moreover $\Pi_{\tilde{p}; p_1, \dots, p_n}^{G, \dots, G}(E_1 \hat{\otimes}_{\alpha_1} G, \dots, E_n \hat{\otimes}_{\alpha_n} G; G) = \Pi_{p; p_1, \dots, p_n}^{G, \dots, G}(E_1 \hat{\otimes}_{\alpha_1} G, \dots, E_n \hat{\otimes}_{\alpha_n} G; G)$

if, and only if, G is finite-dimensional.

Proof. Suppose that

$$\Pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(E_1 \hat{\otimes}_{\alpha_1} G_1, \dots, E_n \hat{\otimes}_{\alpha_n} G_n; F) = \Pi_{p; p_1, \dots, p_n}^{G_1, \dots, G_n}(E_1 \hat{\otimes}_{\alpha_1} G_1, \dots, E_n \hat{\otimes}_{\alpha_n} G_n; F).$$

Let $S \in \mathcal{L}(G_1, \dots, G_n; F)$. Fix $x_{i,0}^* \in \mathcal{B}_{E_i^*}$ ($1 \leq i \leq n$) and consider $T: E_1 \hat{\otimes}_{\alpha_1} G_1 \times \dots \times E_n \hat{\otimes}_{\alpha_n} G_n \rightarrow F$ given by

$$T(u_1, \dots, u_n) = S(u_1(x_{1,0}^*), \dots, u_n(x_{n,0}^*)).$$

So,

$$\begin{aligned} \left(\sum_{k=1}^l \|T(u_1^k, \dots, u_n^k)\|_F^p \right)^{\frac{1}{p}} &= \left(\sum_{k=1}^l \|S(u_1^k(x_{1,0}^*), \dots, u_n^k(x_{n,0}^*))\|_F^p \right)^{\frac{1}{p}} \\ &\leq \|S\| \prod_{i=1}^n \left(\sum_{k=1}^l \|u_i^k(x_{i,0}^*)\|^{p_i} \right)^{1/p_i} \\ &\leq \|S\| \prod_{i=1}^n \sup_{\|x_i^*\| \leq 1} \left(\sum_{k=1}^l \|u_i^k(x_i^*)\|^{p_i} \right)^{1/p_i} \end{aligned}$$

and T is $(p; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated multilinear operator. By hypothesis T is $(\tilde{p}; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated, i.e., there is a positive constant C such that for all $l \in \mathbb{N}$ and $u_i^k \in E_i \otimes G_i$ ($1 \leq k \leq l, 1 \leq i \leq n$) we have

$$\left(\sum_{k=1}^l \|T(u_1^k, \dots, u_n^k)\|_F^p \right)^{\frac{1}{p}} \leq C \prod_{i=1}^n \sup_{\|z_i^*\|_{G_i^*}, \|x_i^*\|_{E_i^*} \leq 1} \left(\sum_{k=1}^l |\langle u_i^k(x_i^*), z_i^* \rangle|^{p_i} \right)^{\frac{1}{p_i}}. \quad (2.6)$$

In particular, if we take $u_i^k = x_{i,0} \otimes z_i^k$ where $x_{i,0} \in \mathcal{B}_{E_i}$, $\langle x_{i,0}^*, x_{i,0} \rangle = 1$ ($1 \leq i \leq n$) and use (2.6), we get

$$\begin{aligned} \left(\sum_{k=1}^l \|S(z_1^k, \dots, z_n^k)\|_F^p \right)^{\frac{1}{p}} &= \left(\sum_{k=1}^l \|T(x_{1,0} \otimes z_1^k, \dots, x_{n,0} \otimes z_n^k)\|_F^p \right)^{\frac{1}{p}} \\ &\leq C \prod_{i=1}^n \sup_{\|z_i^*\|_{G_i^*}, \|x_i^*\|_{E_i^*} \leq 1} \left(\sum_{k=1}^l |\langle x_{i,0} \otimes z_i^k, x_i^* \otimes z_i^* \rangle|^{p_i} \right)^{\frac{1}{p_i}} \\ &\leq C \prod_{i=1}^n \sup_{\|z_i^*\|_{G_i^*}, \|x_i^*\|_{E_i^*} \leq 1} \left(\sum_{k=1}^l |\langle x_{i,0}, x_i^* \rangle \langle z_i^k, z_i^* \rangle|^{p_i} \right)^{\frac{1}{p_i}} \\ &\leq C \prod_{i=1}^n \sup_{\|z_i^*\|_{G_i^*} \leq 1} \left(\sum_{k=1}^l |\langle z_i^k, z_i^* \rangle|^{p_i} \right)^{\frac{1}{p_i}}. \end{aligned}$$

This shows that S is $(p; p_1, \dots, p_n)$ -dominated and completes the proof of the first part.

For the second part, if $\dim(G)$ is finite the implication is obvious. The second implication is given by the Dvoretzky-Rogers theorem. \square

Lemma 2.3.3. [48] *Lemma 1] For $u^1, \dots, u^l \in E \otimes_\varepsilon G$, we have*

$$\left\| (u^k)_{1 \leq k \leq l} \right\|_{p,w} = \sup_{\|x^*\|, \|z^*\| \leq 1} \left(\sum_{k=1}^l |\langle u^k, x^* \otimes z^* \rangle|^p \right)^{\frac{1}{p}}. \quad (2.7)$$

Proposition 2.3.4. *Let α_i be an arbitrary crossnorm on $E_i \otimes G_i$, $1 \leq i \leq n$. Then,*

$$1) \Pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n} (E_1 \widehat{\otimes}_{\alpha_1} G_1, \dots, E_n \widehat{\otimes}_{\alpha_n} G_n; F) \subset \Pi_{p; p_1, \dots, p_n} (E_1 \widehat{\otimes}_{\alpha_1} G_1, \dots, E_n \widehat{\otimes}_{\alpha_n} G_n; F).$$

$$2) \Pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n} (E_1 \widehat{\otimes}_{\varepsilon} G_1, \dots, E_n \widehat{\otimes}_{\varepsilon} G_n; F) = \Pi_{p; p_1, \dots, p_n} (E_1 \widehat{\otimes}_{\varepsilon} G_1, \dots, E_n \widehat{\otimes}_{\varepsilon} G_n; F).$$

Proof. 1) Let $T \in \Pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n} (E_1 \otimes_{\alpha_1} G_1, \dots, E_n \otimes_{\alpha_n} G_n; F)$ and take a finite sequences $(u_i^k)_{1 \leq k \leq l}$ in $E_i \otimes G_i$. One has

$$\begin{aligned} \left(\sum_{k=1}^l \|T(u_1^k, \dots, u_n^k)\|^p \right)^{\frac{1}{p}} &\leq \pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n} (T) \prod_{i=1}^n \sup_{\|x_i^*\|, \|z_i^*\| \leq 1} \left(\sum_{k=1}^l |\langle u_i^k(x_i^*), z_i^* \rangle|^{p_i} \right)^{\frac{1}{p_i}} \\ &= \pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n} (T) \prod_{i=1}^n \sup_{\|x_i^*\|, \|z_i^*\| \leq 1} \left(\sum_{k=1}^l |\langle u_i^k, x_i^* \otimes z_i^* \rangle|^{p_i} \right)^{\frac{1}{p_i}} \\ &\leq \pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n} (T) \prod_{i=1}^n \sup_{\|u_i^*\|_{(E_i \otimes_{\alpha_i} G_i)^*} \leq 1} \left(\sum_{k=1}^l |\langle u_i^k, u_i^* \rangle|^{p_i} \right)^{\frac{1}{p_i}}. \end{aligned}$$

Consequently T is $(p; p_1, \dots, p_n)$ -dominated and $\pi_{p; p_1, \dots, p_n} (T) \leq \pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n} (T)$.

2) We can deduce this coincidence from [\(2.7\)](#).

We conclude by extending T to the entire space by density and the factorization theorem. □

Corollary 2.3.5. *If an operator $T \in \Pi_{p; p_1, \dots, p_n} (E_1 \widehat{\otimes}_{\varepsilon} G_1, \dots, E_n \widehat{\otimes}_{\varepsilon} G_n; F)$, then $T \in \Pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n} (E_1 \widehat{\otimes}_{\varepsilon} G_1, \dots, E_n \widehat{\otimes}_{\varepsilon} G_n; F)$.*

The following corollary is an extension of [\[9 Proposition 1.2\]](#).

Corollary 2.3.6. *If*

$$\Pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n} (E_1 \widehat{\otimes}_{\varepsilon} G_1, \dots, E_n \widehat{\otimes}_{\varepsilon} G_n; F) = \Pi_{p; p_1, \dots, p_n} (E_1 \widehat{\otimes}_{\varepsilon} G_1, \dots, E_n \widehat{\otimes}_{\varepsilon} G_n; F),$$

then

$$\mathcal{L}(G_1, \dots, G_n; F) = \Pi_{p; p_1, \dots, p_n} (G_1, \dots, G_n; F)$$

and $\Pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n} (E_1 \widehat{\otimes}_{\varepsilon} G, \dots, E_n \widehat{\otimes}_{\varepsilon} G; G) = \Pi_{p; p_1, \dots, p_n} (E_1 \widehat{\otimes}_{\varepsilon} G, \dots, E_n \widehat{\otimes}_{\varepsilon} G; G)$, if and only if, G is finite-dimensional.

Remark 2.3.7. Since $\mathcal{C}(K_i, G_i) = \mathcal{C}(K_i) \widehat{\otimes}_\varepsilon G_i$ $1 \leq i \leq n$ (see [55]), then the space of $(p; p_1, \dots, p_n)$ -dominated in the sense of Dinculeanu (see [50]) actually coincides by (2.4) with $\Pi_{p; p_1, \dots, p_n}^{G_1, \dots, G_n}(\mathcal{C}(K_1) \widehat{\otimes}_\varepsilon G_1, \dots, \mathcal{C}(K_n) \widehat{\otimes}_\varepsilon G_n; F)$ and the class of $(p; p_1, \dots, p_n)$ -dominated in the sense of Matos coincides by Proposition 2.3.4 with

$$\Pi_{\bar{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(\mathcal{C}(K_1) \widehat{\otimes}_\varepsilon G_1, \dots, \mathcal{C}(K_n) \widehat{\otimes}_\varepsilon G_n; F).$$

The following proposition is an extension of [38, Proposition 1].

Proposition 2.3.8. *If $T \in \Pi_{p; p_1, \dots, p_n}(E_1, \dots, E_n; F)$, then $\bar{T}: E_1 \widehat{\otimes}_{\alpha_1} G_1 \times \dots \times E_n \widehat{\otimes}_{\alpha_n} G_n \rightarrow F \widehat{\otimes}_\varepsilon G_1 \widehat{\otimes}_\varepsilon \dots \widehat{\otimes}_\varepsilon G_n$ defined by $\bar{T}(x_1 \otimes z_1, \dots, x_n \otimes z_n) = T(x_1, \dots, x_n) \otimes z_1 \otimes \dots \otimes z_n$ is $(p; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated.*

Proof. Let $u_i = \sum_{j=1}^{m_i} x_{ij} \otimes z_{ij} \in E_i \otimes G_i$. For $z_i^* \in G_i^*$, $1 \leq i \leq n$, we have $u_i(z_i^*) = \sum_{j=1}^{m_i} x_{ij} \langle z_{ij}, z_i^* \rangle$ and

$$\begin{aligned} \bar{T}(u_1, \dots, u_n) &= \bar{T}\left(\sum_{j=1}^{m_1} x_{1j} \otimes z_{1j}, \dots, \sum_{j=1}^{m_n} x_{nj} \otimes z_{nj}\right) \\ &= \sum_{j=1}^{m_1} \dots \sum_{j=1}^{m_n} \bar{T}(x_{1j} \otimes z_{1j}, \dots, x_{nj} \otimes z_{nj}) \\ &= \sum_{j=1}^{m_1} \dots \sum_{j=1}^{m_n} T(x_{1j}, \dots, x_{nj}) \otimes z_{1j} \otimes \dots \otimes z_{nj}. \end{aligned}$$

Consider (z_1^*, \dots, z_n^*) in $(G_1^* \times \dots \times G_n^*)$. It follows that

$$\begin{aligned} \bar{T}(u_1, \dots, u_n)(z_1^*, \dots, z_n^*) &= \sum_{j=1}^{m_1} \dots \sum_{j=1}^{m_n} T(x_{1j}, \dots, x_{nj}) \langle z_{1j}, z_1^* \rangle \dots \langle z_{nj}, z_n^* \rangle \\ &= \sum_{j=1}^{m_1} \dots \sum_{j=1}^{m_n} T(x_{1j} \langle z_{1j}, z_1^* \rangle, \dots, \langle z_{nj}, z_n^* \rangle x_{nj}) \\ &= T\left(\sum_{j=1}^{m_1} x_{1j} \langle z_{1j}, z_1^* \rangle, \dots, \sum_{j=1}^{m_n} \langle z_{nj}, z_n^* \rangle x_{nj}\right) \\ &= T(u_1(z_1^*), \dots, u_n(z_n^*)). \end{aligned}$$

Consider now T in $\Pi_{p; p_1, \dots, p_n}(E_1, \dots, E_n; F)$. By [37, Theorem 3.2], there exist a constant $C > 0$ and probability measures μ_i on $\mathcal{B}_{E_i^*}$, $1 \leq i \leq n$, such that

$$\begin{aligned}
\|\bar{T}(u_1, \dots, u_n)\| &= \sup_{\|z_1^*\|, \dots, \|z_n^*\| \leq 1} \|\bar{T}(u_1, \dots, u_n)(z_1^*, \dots, z_n^*)\| \\
&= \sup_{\|z_1^*\|, \dots, \|z_n^*\| \leq 1} \|T(u_1(z_1^*), \dots, u_n(z_n^*))\| \\
&\leq C \prod_{i=1}^n \sup_{\|z_1^*\|, \dots, \|z_n^*\| \leq 1} \left(\int_{\mathcal{B}_{E_i^*}} |\langle u_i(x_i^*), z_i^* \rangle|^{p_i} d\mu_i(x_i^*) \right)^{\frac{1}{p_i}} \\
&\leq C \prod_{i=1}^n \left(\int_{\mathcal{B}_{E_i^*}} \|u_i(x_i^*)\|^{p_i} d\mu_i(x_i^*) \right)^{\frac{1}{p_i}}.
\end{aligned}$$

This ends the proof by using Corollary 2.2.8. □

The next result generalizes [25] Proposition 4].

Proposition 2.3.9. *Let $T_i: G_i \rightarrow F_i$ be p_i -summing operators and $\bar{T}_i = id_{E_i} \otimes T_i$, $1 \leq i \leq n$. If $T \in \Pi_{\tilde{p}; p_1, \dots, p_n}^{F_1, \dots, F_n}(E_1 \hat{\otimes}_{\alpha_1} F_1, \dots, E_n \hat{\otimes}_{\alpha_n} F_n; F)$, then $T \circ (\bar{T}_1, \dots, \bar{T}_n) \in \Pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(E_1 \hat{\otimes}_{\alpha_1} G_1, \dots, E_n \hat{\otimes}_{\alpha_n} G_n; F)$ and $\pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(T \circ (\bar{T}_1, \dots, \bar{T}_n)) \leq \pi_{\tilde{p}; p_1, \dots, p_n}^{F_1, \dots, F_n}(T) \prod_{i=1}^n \pi_{p_i}(T_i)$.*

Proof. For $1 \leq i \leq n$, $1 \leq k \leq l$, let $u_i^k = \sum_{j=1}^{m_{ik}} x_{ij}^k \otimes z_{ij}^k \in E_i \otimes G_i$. We have for $x_i^* \in E_i^*$, the equality

$$\bar{T}_i(u_i^k)(x_i^*) = \sum_{j=1}^{m_{ik}} \langle x_i^*, x_{ij}^k \rangle T_i(z_{ij}^k) = T_i \left(\sum_{j=1}^{m_{ik}} \langle x_i^*, x_{ij}^k \rangle z_{ij}^k \right) = T_i(u_i^k(x_i^*)).$$

From this equality, we obtain

$$\begin{aligned}
&\left(\sum_{k=1}^l \|T(\bar{T}_1(u_1^k), \dots, \bar{T}_n(u_n^k))\|^p \right)^{\frac{1}{p}} \\
&\leq \pi_{\tilde{p}; p_1, \dots, p_n}^{F_1, \dots, F_n}(T) \prod_{i=1}^n \sup_{\|x_i^*\|_{E_i^*} \leq 1} \left(\sum_{k=1}^l \|\bar{T}_i(u_i^k)(x_i^*)\|_{F_i}^{p_i} \right)^{\frac{1}{p_i}} \\
&= \pi_{\tilde{p}; p_1, \dots, p_n}^{F_1, \dots, F_n}(T) \prod_{i=1}^n \sup_{\|x_i^*\|_{E_i^*} \leq 1} \left(\sum_{k=1}^l \|T_i(u_i^k(x_i^*))\|_{F_i}^{p_i} \right)^{\frac{1}{p_i}} \\
&\leq \pi_{\tilde{p}; p_1, \dots, p_n}^{F_1, \dots, F_n}(T) \prod_{i=1}^n \pi_{p_i}(T_i) \sup_{\|x_i^*\|_{E_i^*}, \|z_i^*\|_{G_i^*} \leq 1} \left(\sum_{k=1}^l |\langle u_i^k(x_i^*), z_i^* \rangle|^{p_i} \right)^{\frac{1}{p_i}}.
\end{aligned}$$

This shows that $T \circ (\bar{T}_1, \dots, \bar{T}_n)$ is $(\tilde{p}; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated and $\pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(T \circ (\bar{T}_1, \dots, \bar{T}_n)) \leq \pi_{\tilde{p}; p_1, \dots, p_n}^{F_1, \dots, F_n}(T) \prod_{i=1}^n \pi_{p_i}(T_i)$, which proves the proposition. □

2.4 Relations with associated operators

For every multilinear operator T from $E_1 \otimes G_1 \times \cdots \times E_n \otimes G_n$ into F we denote by $T^\#$, \tilde{T} and T^t the associated operators from $E_1 \times \cdots \times E_n$ into $\mathcal{L}(G_1, \dots, G_n; F)$, from $G_1 \times \cdots \times G_n$ into $\mathcal{L}(E_1, \dots, E_n; F)$ and from $G_1 \otimes E_1 \times \cdots \times G_n \otimes E_n$ into F , respectively, defined by $T^\#(x_1, \dots, x_n)(z_1, \dots, z_n) = T(x_1 \otimes z_1, \dots, x_n \otimes z_n)$, $\tilde{T}(z_1, \dots, z_n)(x_1, \dots, x_n) = T(x_1 \otimes z_1, \dots, x_n \otimes z_n)$ and $T^t(u_1^t, \dots, u_n^t) = T(u_1, \dots, u_n)$, where $u_i = \sum_{j=1}^{m_i} x_{ij} \otimes z_{ij} \in E_i \otimes G_i$, and $u_i^t = \sum_{j=1}^{m_i} z_{ij} \otimes x_{ij} \in G_i \otimes E_i$. Clearly $T^\#$, \tilde{T} and T^t are multilinear operators. In this section, we try to find some relationships between the operator T and their associated operators $T^\#$, \tilde{T} and T^t relative to the different variants of summabilities.

We recall (see [6] and [39, Remark 2.2]) that $T \in \mathcal{D}_p^n(E_1, \dots, E_n; F)$, the class of all Cohen strongly p -summing n -linear operators from $E_1 \times \cdots \times E_n$ into F , if there is a positive constant C such that for all $m \in \mathbb{N}$, $x_i^1, \dots, x_i^m \in E_i$ and $y_1^*, \dots, y_m^* \in F^*$, we have

$$\|(\langle T(x_1^j, \dots, x_n^j), y_j^* \rangle)\|_1 \leq C \prod_{i=1}^n \left(\sum_{j=1}^m \|x_i^j\|_{E_i}^{p_i} \right)^{\frac{1}{p_i}} \sup_{\|y\|_F \leq 1} \|(y_j^*(y))\|_{p^*}. \quad (2.8)$$

We equip $\mathcal{D}_p^n(E_1, \dots, E_n; F)$ with the norm $d_p^n(\cdot)$, which is the smallest of the constants C verifying (2.8) and with which it becomes a Banach space.

The Chevet-Saphar crossnorms are defined, for $1 \leq p \leq \infty$, as follows (see [22, 55]):

$$d_p(u) = \inf \left\{ \|(x_i)_{i=1}^n\|_{p^*, w} \|(y_i)_{i=1}^n\|_p : u = \sum_{i=1}^n x_i \otimes y_i \in E \otimes F \right\}$$

and

$$g_p(u) = \inf \left\{ \|(x_i)_{i=1}^n\|_p \|(y_i)_{i=1}^n\|_{p^*, w} : u = \sum_{i=1}^n x_i \otimes y_i \in E \otimes F \right\}.$$

We have $g_p = d_p^t$, for every p , in the sense that $g_p(u) = d_p(u^t)$.

Proposition 2.4.1. *Let α_i be an arbitrary crossnorm on $E_i \otimes G_i$, $1 \leq i \leq n$. Then*

1) *A multilinear operator $T \in \Pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(E_1 \widehat{\otimes}_{\alpha_1} G_1, \dots, E_n \widehat{\otimes}_{\alpha_n} G_n; F)$ if, and only if, $T^t \in \Pi_{\tilde{p}; p_1, \dots, p_n}^{E_1, \dots, E_n}(G_1 \widehat{\otimes}_{\alpha_1} E_1, \dots, G_n \widehat{\otimes}_{\alpha_n} E_n; F)$.*

2) *A multilinear operator $T \in \mathcal{D}_p^n(E_1 \widehat{\otimes}_{d_{p_1}} G_1, \dots, E_n \widehat{\otimes}_{d_{p_n}} G_n; F)$ if, and only if, $T^t \in \mathcal{D}_p^n(E_1 \widehat{\otimes}_{g_{p_1}} G_1, \dots, E_n \widehat{\otimes}_{g_{p_n}} G_n; F)$.*

Proof. 1) We can deduce this from the equality $\langle u_i^k(x_i^*), z_i^* \rangle = \langle (u_i^k)^t(x_i^*), z_i^* \rangle$ and we deduce $\pi_{\tilde{p}; p_1, \dots, p_n}^{E_1, \dots, E_n}(T^t) = \pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(T)$.

2) Follows immediately from the definitions of the Chevet-Saphar norms. \square

Remark 2.4.2. Using Proposition [2.3.4](#), we have $T \in \Pi_{p; p_1, \dots, p_n}(E_1 \widehat{\otimes}_\varepsilon G_1, \dots, E_n \widehat{\otimes}_\varepsilon G_n; F)$ if, and only if, $T^t \in \Pi_{p; p_1, \dots, p_n}(G_1 \widehat{\otimes}_\varepsilon E_1, \dots, G_n \widehat{\otimes}_\varepsilon E_n; F)$.

Remark 2.4.3. Let α_i be an arbitrary crossnorm on $E_i \otimes G_i$, $1 \leq i \leq n$. If $T \in \Pi_{p; p_1, \dots, p_n}^{G_1, \dots, G_n}(E_1 \widehat{\otimes}_{\alpha_1} G_1, \dots, E_n \widehat{\otimes}_{\alpha_n} G_n; F)$ in general $T^t \notin \Pi_{\tilde{p}; p_1, \dots, p_n}^{E_1, \dots, E_n}(G_1 \widehat{\otimes}_{\alpha_1} E_1, \dots, G_n \widehat{\otimes}_{\alpha_n} E_n; F)$. Indeed, consider $T \in \mathcal{L}(G, \dots, G; G) = \Pi_{p; p_1, \dots, p_n}^{G, \dots, G}(\mathbb{R} \widehat{\otimes}_{\alpha_1} G, \dots, \mathbb{R} \widehat{\otimes}_{\alpha_n} G; G)$ but $T^t \notin \Pi_{p; p_1, \dots, p_n}^{\mathbb{R}, \dots, \mathbb{R}}(G \widehat{\otimes}_{\alpha_1} \mathbb{R}, \dots, G \widehat{\otimes}_{\alpha_n} \mathbb{R}; G) = \Pi_{p; p_1, \dots, p_n}(G, \dots, G; G)$, if $\dim(G)$ is infinite.

Proposition 2.4.4. *If $T \in \Pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(E_1 \widehat{\otimes}_{\alpha_1} G_1, \dots, E_n \widehat{\otimes}_{\alpha_n} G_n; F)$, then:*

1) $T^\#(x_1, \dots, x_n)$ is $(p; p_1, \dots, p_n)$ -dominated for all $(x_1, \dots, x_n) \in E_1 \times \dots \times E_n$.

Moreover, there exist Radon probability measures μ_i on $K_i = \mathcal{B}_{E_i^*} \times \mathcal{B}_{G_i^*}$ such that for all $u_i \in E_i \otimes G_i$, $1 \leq i \leq n$, the following inequality holds:

$$\pi_{p; p_1, \dots, p_n}(T^\#(x_1, \dots, x_n)) \leq C \prod_{i=1}^n \left(\int_{K_i} |\langle x_i, x_i^* \rangle|^{p_i} d\mu_i(x_i^*, z_i^*) \right)^{1/p_i}.$$

2) $T^\#: E_1 \times \dots \times E_n \rightarrow \Pi_{p; p_1, \dots, p_n}(G_1, \dots, G_n; F)$ is $(p; p_1, \dots, p_n)$ -dominated.

Proof. Since T is $(\tilde{p}; p_1, \dots, p_n; G_1, \dots, G_n)$ -dominated, by Theorem [2.2.10](#), there exist Radon probability measures μ_i on $K_i = \mathcal{B}_{E_i^*} \times \mathcal{B}_{G_i^*}$ such that for all $u_i \in E_i \otimes G_i$, $1 \leq i \leq n$, the following inequality holds:

$$\|T(u_1, \dots, u_n)\|_F \leq \pi_{\tilde{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \prod_{i=1}^n \left(\int_{K_i} |\langle u_i(x_i^*), z_i^* \rangle|^{p_i} d\mu_i(x_i^*, z_i^*) \right)^{1/p_i}.$$

1) We would like to show that $T^\#(x_1, \dots, x_n)$ is $(p; p_1, \dots, p_n)$ -dominated for all $x_i \in E_i$, $1 \leq i \leq n$. Let $u_i^k = x_i \otimes z_i^k \in E_i \otimes_{\alpha_i} G_i$ for $1 \leq k \leq l$. Using Hölder's inequality, we obtain

$$\begin{aligned}
 & \left(\sum_{k=1}^l \left\| T^\#(x_1, \dots, x_n)(z_1^k, \dots, z_n^k) \right\|_F^p \right)^{\frac{1}{p}} \\
 &= \left(\sum_{k=1}^l \left\| T(x_1 \otimes z_1^k, \dots, x_n \otimes z_n^k) \right\|_F^p \right)^{\frac{1}{p}} \\
 &\leq \pi_{\bar{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \left(\sum_{k=1}^l \left[\prod_{i=1}^n \left(\int_{K_i} |\langle z_i^k, z_i^* \rangle \langle x_i, x_i^* \rangle|^{p_i} d\mu_i(x_i^*, z_i^*) \right)^{\frac{1}{p_i}} \right]^p \right)^{\frac{1}{p}} \\
 &\leq \pi_{\bar{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \prod_{i=1}^n \left(\sum_{k=1}^l \int_{K_i} |\langle z_i^k, z_i^* \rangle \langle x_i, x_i^* \rangle|^{p_i} d\mu_i(x_i^*, z_i^*) \right)^{\frac{1}{p_i}} \\
 &\leq \pi_{\bar{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \prod_{i=1}^n \left(\int_{K_i} \left(\sum_{k=1}^l |\langle z_i^k, z_i^* \rangle|^{p_i} \right) |\langle x_i, x_i^* \rangle|^{p_i} d\mu_i(x_i^*, z_i^*) \right)^{\frac{1}{p_i}} \\
 &\leq \pi_{\bar{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \left\| (z_i^k)_{1 \leq k \leq l} \right\|_{p_i, w} \prod_{i=1}^n \left(\int_{K_i} |\langle x_i, x_i^* \rangle|^{p_i} d\mu_i(x_i^*, z_i^*) \right)^{\frac{1}{p_i}}.
 \end{aligned}$$

This yields that $T^\#(x_1, \dots, x_n)$ is $(p; p_1, \dots, p_n)$ -dominated for all $x_i \in E_i$, $1 \leq i \leq n$, and

$$\pi_{p; p_1, \dots, p_n}(T^\#(x_1, \dots, x_n)) \leq \pi_{\bar{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \prod_{i=1}^n \left(\int_{K_i} |\langle x_i, x_i^* \rangle|^{p_i} d\mu_i(x_i^*, z_i^*) \right)^{1/p_i}.$$

2) For the second property, we use the last inequality and Hölder's inequality again. We deduce, for all $(x_1^k, \dots, x_n^k) \in E_1 \times \dots \times E_n$, $1 \leq k \leq l$, that

$$\begin{aligned}
 & \left(\sum_{k=1}^l \left[\pi_{p; p_1, \dots, p_n}(T^\#(x_1^k, \dots, x_n^k)) \right]^p \right)^{\frac{1}{p}} \\
 &\leq \pi_{\bar{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \left(\sum_{k=1}^l \left[\prod_{i=1}^n \left(\int_{K_i} |\langle x_i^k, x_i^* \rangle|^{p_i} d\mu_i(x_i^*, z_i^*) \right)^{1/p_i} \right]^p \right)^{\frac{1}{p}} \\
 &\leq \pi_{\bar{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \prod_{i=1}^n \left(\sum_{k=1}^l \int_{K_i} |\langle x_i^k, x_i^* \rangle|^{p_i} d\mu_i(x_i^*, z_i^*) \right)^{\frac{1}{p_i}} \\
 &\leq \pi_{\bar{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \prod_{i=1}^n \left(\int_{K_i} \sum_{k=1}^l |\langle x_i^k, x_i^* \rangle|^{p_i} d\mu_i(x_i^*, z_i^*) \right)^{\frac{1}{p_i}} \\
 &\leq \pi_{\bar{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \prod_{i=1}^n \left\| (x_i^k)_{1 \leq k \leq l} \right\|_{p_i, w}.
 \end{aligned}$$

Thus $T^\#: E_1 \times \dots \times E_n \rightarrow \Pi_{p; p_1, \dots, p_n}(G_1, \dots, G_n; F)$ is $(p; p_1, \dots, p_n)$ -dominated and $\pi_{p; p_1, \dots, p_n}(T^\#) \leq \pi_{\bar{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(T)$. \square

The following proposition can be proved with the same arguments as the previous result.

Proposition 2.4.5. *If $T \in \Pi_{\vec{p}; p_1, \dots, p_n}^{G_1, \dots, G_n}(E_1 \widehat{\otimes}_{\alpha_1} G_1, \dots, E_n \widehat{\otimes}_{\alpha_n} G_n; F)$, then*

- 1) $\tilde{T}(z_1, \dots, z_n)$ is $(p; p_1, \dots, p_n)$ -dominated for all $(z_1, \dots, z_n) \in G_1 \times \dots \times G_n$.
- 2) $\tilde{T}: G_1 \times \dots \times G_n \rightarrow \Pi_{p; p_1, \dots, p_n}(E_1, \dots, E_n; F)$ is $(p; p_1, \dots, p_n)$ -dominated.

Using Proposition 2.3.4, Proposition 2.4.4 and Proposition 2.4.5, we have the next

Corollary 2.4.6. *If $T \in \Pi_{p; p_1, \dots, p_n}(E_1 \widehat{\otimes}_{\varepsilon} G_1, \dots, E_n \widehat{\otimes}_{\varepsilon} G_n; F)$, then*

- 1) \tilde{T} is $(p; p_1, \dots, p_n)$ -dominated and $\tilde{T}(z_1, \dots, z_n)$ is $(p; p_1, \dots, p_n)$ -dominated.
- 2) $T^\#$ is $(p; p_1, \dots, p_n)$ -dominated and $T^\#(x_1, \dots, x_n)$ is $(p; p_1, \dots, p_n)$ -dominated.

We end this chapter by the following proposition.

Proposition 2.4.7. *If $T \in \Pi_{p; p_1, \dots, p_n}^{G_1, \dots, G_n}(E_1 \widehat{\otimes}_{\alpha_1} G_1, \dots, E_n \widehat{\otimes}_{\alpha_n} G_n; F)$, then $\tilde{T}(z^1, \dots, z^n)$ is $(p; p_1, \dots, p_n)$ -dominated.*

Proof. Let T be in $\Pi_{p; p_1, \dots, p_n}^{G_1, \dots, G_n}(E_1 \widehat{\otimes}_{\alpha_1} G_1, \dots, E_n \widehat{\otimes}_{\alpha_n} G_n; F)$, then by Theorem 2.2.6, there exist probability measures μ_i on $K_i = \mathcal{B}_{E_i^*}$ such that, for every $u_i \in E_i \otimes G_i$, we have

$$\|T(u_1, \dots, u_n)\|_F \leq \pi_{p; p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \prod_{i=1}^n \left(\int_{K_i} \|u_i(x_i^*)\|_{G_i}^{p_i} d\mu_i(x_i^*) \right)^{1/p_i}.$$

For $u_i = x_i \otimes z_i \in E_i \otimes G_i$, by the previous inequality, we obtain

$$\begin{aligned} \|\tilde{T}(z_1, \dots, z_n)(x_1, \dots, x_n)\| &= \|T(x_1 \otimes z_1, \dots, x_n \otimes z_n)\| \\ &\leq \pi_{p; p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \prod_{i=1}^n \left(\int_{K_i} \|x_i \otimes z_i(x_i^*)\|_{G_i}^{p_i} d\mu_i(x_i^*) \right)^{1/p_i} \\ &\leq \pi_{p; p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \prod_{i=1}^n \|z_i\| \prod_{i=1}^n \left(\int_{K_i} |x_i^*(x_i)|^{p_i} d\mu_i(x_i^*) \right)^{1/p_i}. \end{aligned}$$

Using Hölder's inequality, we have

$$\begin{aligned}
 & \left(\sum_{k=1}^l \|\tilde{T}(z_1, \dots, z_n)(x_1^k, \dots, x_n^k)\|^p \right)^{\frac{1}{p}} \\
 & \leq \pi_{p;p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \prod_{i=1}^n \|z_i\| \left(\sum_{k=1}^l \left[\prod_{i=1}^n \left(\int_{K_i} |\langle x_i^k, x_i^* \rangle|^{p_i} d\mu_i(x_i^*) \right)^{1/p_i} \right]^p \right)^{\frac{1}{p}} \\
 & \leq \pi_{p;p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \prod_{i=1}^n \|z_i\| \prod_{i=1}^n \left(\sum_{k=1}^l \int_{K_i} |\langle x_i^k, x_i^* \rangle|^{p_i} d\mu_i(x_i^*) \right)^{\frac{1}{p_i}} \\
 & \leq \pi_{p;p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \prod_{i=1}^n \|z_i\| \prod_{i=1}^n \left(\int_{K_i} \sum_{k=1}^l |\langle x_i^k, x_i^* \rangle|^{p_i} d\mu_i(x_i^*) \right)^{\frac{1}{p_i}} \\
 & \leq \pi_{p;p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \prod_{i=1}^n \|z_i\| \prod_{i=1}^n \left\| (x_i^k)_{1 \leq k \leq l} \right\|_{p_i, w},
 \end{aligned}$$

and this shows $\tilde{T}(z_1, \dots, z_n)$ is $(p; p_1, \dots, p_n)$ -dominated and $\pi_{p;p_1, \dots, p_n}(\tilde{T}(z_1, \dots, z_n)) \leq$

$$\pi_{p;p_1, \dots, p_n}^{G_1, \dots, G_n}(T) \prod_{i=1}^n \|z_i\|.$$

□

WEAKLY MID SUMMING MULTILINEAR OPERATORS

This chapter is divided into four sections. In the first section, we recall some basic definitions and notations concerning sequences Banach spaces. We give in the second section, the definition and properties concerning the linear weakly mid p -summing operators. We introduce in the next section (section 3), a multilinear version of weakly mid- p -summing operators for which the resulting vector space is a Banach ideal of multilinear operators. Finally, in last section we prove a natural analogue of Pietsch domination theorem of this new class.

Karn and Sinha in [30] by using the notion of mid p -summable (operator p -summability), they introduced the class of weakly mid- p -summing (sequentially p -limited) operators. There are many attempts available in the literature to study the ideal of weakly mid p -summing operators in different textures, see [14, 30, 27, 21]. Recently, Baweja and Philip in [12] studied some aspects of the operator ideal of weakly mid p -summing operators using the theory of tensor products. This concept led them to a new space of summability between p -summable and weakly p -summable sequences. A bounded linear operator $T : X \rightarrow Y$ between Banach spaces is weakly mid p -summing if it sends weakly p -summable sequences to mid p -summable sequences. Equivalently by Theorem 4.4 [30], $S \circ T : X \rightarrow \ell_p$ is p -summing for all bounded linear operator S from Y to ℓ_p . The aim of this chapter is to study the multilinear version of weakly mid- p -summing operators by considering the (p_1, \dots, p_m) -dominated multilinear operators $S \circ T$.

3.1 Notation and preliminaries

Definition 3.1.1. [30, 14] A sequence $(x_j)_{j=1}^\infty$ in a Banach space X is said to be mid- p -summable, $1 \leq p < \infty$, if $((x_n^*(x_j))_{j=1}^\infty)_{n=1}^\infty \in \ell_p(\ell_p)$ whenever $(x_n^*)_{n=1}^\infty \in \ell_p^w(X^*)$. The space of all such sequences shall be denoted by $\ell_p^{mid}(X)$ which is a Banach space under the norm

$$\|(x_j)_{j=1}^\infty\|_{mid,p} := \sup_{(x_n^*)_{n=1}^\infty \in B_{\ell_p^w(X^*)}} \left(\sum_{j=1}^\infty \sum_{n=1}^\infty |x_n^*(x_j)|^p \right)^{\frac{1}{p}}.$$

The relationships between the various sequence spaces are given by [14]

$\ell_p(X) \subsetneq \ell_p^{mid}(X) \subsetneq \ell_p^w(X)$, with $\|(x_j)_{j=1}^\infty\|_{w,p} \leq \|(x_j)_{j=1}^\infty\|_{mid,p} \leq \|(x_j)_{j=1}^\infty\|_p$, for all $(x_j)_{j=1}^\infty \in \ell_p(X)$. If $p \leq q$, then for every Banach space X , $\ell_p^{mid}(X) \subset \ell_q^{mid}(X)$ and

$$\|(x_j)_{j=1}^\infty\|_{mid,q} \leq \|(x_j)_{j=1}^\infty\|_{mid,p}.$$

Given Banach spaces E and F , the symbol $E \xrightarrow{1} F$ means that E is a linear subspace of F and $\|x\|_F \leq \|x\|_E$, for every $x \in E$. A class of vector-valued sequences \mathcal{S} , or simply a sequence class \mathcal{S} , is a rule that assigns to each $E \in \mathcal{BAN}$ a Banach space $\mathcal{S}(E)$ of E -valued sequences such that $c_{00}(E) \subset \mathcal{S}(E) \xrightarrow{1} \ell_\infty(E)$ and $\|e_n\|_{\mathcal{S}(\mathbb{K})} = 1$, for any n .

The sequence class $\mathcal{S}(E)$ is said to be: *finitely determined* if for any sequence $(x_j)_{j=1}^\infty \in E^\mathbb{N}$, $(x_j)_{j=1}^\infty \in \mathcal{S}(E)$ if and only if $\sup_k \|(x_j)_{j=1}^k\|_{\mathcal{S}(E)} < \infty$ and in this case, $\|(x_j)_{j=1}^\infty\|_{\mathcal{S}(E)} = \sup_k \|(x_j)_{j=1}^k\|_{\mathcal{S}(E)}$. *Linearly stable* if for every $u \in \mathcal{L}(E; F)$ it holds that $(u(x_j))_{j=1}^\infty \in \mathcal{S}(F)$, whenever $(x_j)_{j=1}^\infty \in \mathcal{S}(E)$ and $\|\hat{u} : \mathcal{S}(E) \rightarrow \mathcal{S}(F)\| = \|u\|$, where \hat{u} is the induced linear operator given by $\hat{u}((x_j)_{j=1}^\infty) = (u(x_j))_{j=1}^\infty$.

Example 3.1.2. The sequence classes $\ell_\infty(\cdot) = \ell_\infty^w(\cdot) = \ell_\infty^{mid}(\cdot)$, $\ell_p(\cdot)$, $\ell_p^w(\cdot)$, $\ell_p(\cdot)$ and $\ell_p^{mid}(\cdot)$ ($1 \leq p < \infty$) are linearly stable and finitely determined (see [13, 14]).

Definition 3.1.3. [13, Definition 3.1] Let $m \in \mathbb{N}$ and $X_1, \dots, X_m; Y$ be sequence classes. A multilinear operator $T \in \mathcal{L}(E_1, \dots, E_m; F)$ is $(X_1, \dots, X_m; Y)$ -summing if $(T(x_j^1, \dots, x_j^m))_{j=1}^\infty \in Y(F)$ whenever $(x_j^i)_{j=1}^\infty \in X_i(E_i)$, $i = 1, \dots, m$. In this case we write $T \in \mathcal{L}_{X_1, \dots, X_m; Y}(E_1, \dots, E_m; F)$.

The following proposition describes how to works the transformation of vector-valued sequences by multilinear operators.

Proposition 3.1.4. [13, Proposition 2.4] Let $m \in \mathbb{N}$ and $X_1, \dots, X_m; Y$ be sequence classes. If the sequence classes X_1, \dots, X_m and Y are finitely determined, then the following condition are equivalent for a given $T \in \mathcal{L}(E_1, \dots, E_m; F)$

- (i) $T \in \mathcal{L}_{X_1, \dots, X_m; Y}(E_1, \dots, E_m; F)$
- (ii) The induced map $\hat{T} : X_1(E_1) \times \dots \times X_m(E_m) \rightarrow Y(F)$ given by,

$$\hat{T}((x_j^1)_{j=1}^\infty, \dots, (x_j^m)_{j=1}^\infty) = (T(x_j^1, \dots, x_j^m))_{j=1}^\infty,$$

is a well-defined continuous multilinear operator.

(iii) There is a constant $C > 0$ such that

$$\|(T(x_j^1, \dots, x_j^m))_{j=1}^k\|_{Y(F)} \leq C \prod_{i=1}^m \|(x_j^i)_{j=1}^k\|_{X_i(E_i)}. \quad (3.1)$$

for every $k \in \mathbb{N}$ and for all finite sequences $x_j^i \in E_i, j = 1, \dots, k, i = 1, \dots, m$.

In this case,

$$\|T\|_{X_1, \dots, X_m; Y} = \|\hat{T}\|_{\mathcal{L}(X_1(E_1), \dots, X_m(E_m); Y(F))} = \inf \{C : (3.1) \text{ holds}\}.$$

Given sequence classes $X_1, \dots, X_m; Y$, we say that $X_1(\mathbb{K}) \cdots X_m(\mathbb{K}) \xrightarrow{1} Y(\mathbb{K})$ if $(\lambda_j^1 \cdots \lambda_j^m)_{j=1}^\infty \in Y(\mathbb{K})$ and

$$\|(\lambda_j^1 \cdots \lambda_j^m)_{j=1}^\infty\|_{Y(\mathbb{K})} \leq \prod_{i=1}^m \|(\lambda_j^i)_{j=1}^\infty\|_{X_i(\mathbb{K})},$$

whenever $(\lambda_j^i)_{j=1}^\infty \in X_i(\mathbb{K}), i = 1, \dots, m$.

Theorem 3.1.5. [13, Theorem 3.6] Let $m \in \mathbb{N}$ and X_1, \dots, X_m, Y be linearly stable sequence classes such that $X_1(\mathbb{K}) \cdots X_m(\mathbb{K}) \xrightarrow{1} Y(\mathbb{K})$. Then $(\mathcal{L}_{X_1, \dots, X_m; Y}, \|\cdot\|_{X_1, \dots, X_m; Y})$ is a Banach ideal of multilinear operators.

3.2 Linear weakly mid p -summing operators

The following definition was given by Karn and Sinha in [30, Definition 4.1] and [14, Definition 2.1].

Definition 3.2.1. Let T be in $\mathcal{L}(X; Y)$. For $1 \leq p < \infty$, we say that T is weakly mid- p -summing, if $(T(x_j))_{j=1}^\infty \in \ell_p^{mid}(Y)$ whenever $(x_j)_{j=1}^\infty \in \ell_p^w(X)$. The set of all weakly mid- p -summing operators from X into Y will be denoted by $\mathcal{W}_p^{mid}(X, Y)$.

The weakly mid p -summing operators characterized by the following theorem [14, Theorem 2.3].

Theorem 3.2.2. *The following conditions are equivalent*

1) T is weakly mid p -summing.

2) The induced map $\widehat{T} : \ell_p^w(X) \rightarrow \ell_p^{\text{mid}}(Y)$ given by $\widehat{T}((x_j)_{j=1}^\infty) = (T(x_j))_{j=1}^\infty$ is a well-define and continuous linear operator.

3) The induced map $\widetilde{T} : \ell_p^u(X) \rightarrow \ell_p^{\text{mid}}(Y)$ given by $\widetilde{T}((x_j)_{j=1}^\infty) = (T(x_j))_{j=1}^\infty$ is a well-define and continuous linear operator.

4) $(T(x_j))_{j=1}^\infty \in \ell_p^{\text{mid}}(Y)$ whenever $(x_j)_{j=1}^\infty \in \ell_p^u(X)$

5) There is a constant $B > 0$ such that

$$\|(T(x_j))_{j=1}^\infty\|_{\text{mid},p} \leq B \|(x_j)_{j=1}^\infty\|_{w,p}, \quad \text{for any } (x_j)_{j=1}^\infty \in \ell_p^w(X).$$

6) There is a constant $C > 0$ such that

$$\|(T(x_j))_{j=1}^k\|_{\text{mid},p} \leq C \|(x_j)_{j=1}^k\|_{w,p}, \quad \text{for any } (x_j)_{j=1}^k \subset X.$$

7) There is a constant $C' > 0$ such that

$$\|(\langle T(x_j), y_n^* \rangle)_{n=1}^\infty)_{j=1}^\infty\|_{\ell_p(\ell_p)} \leq C' \|(x_j)_{j=1}^\infty\|_{w,p} \|(y_n^*)_{n=1}^\infty\|_{w,p},$$

for any $(x_j)_{j=1}^\infty \in \ell_p^w(X)$, and every $(y_n^*)_{n=1}^\infty \in \ell_p^w(Y^*)$.

Moreover,

$$w_p^{\text{mid}}(T) := \|\widehat{T}\| = \|\widetilde{T}\| = \inf\{B : 5 \text{ holds}\} = \inf\{C : 6 \text{ holds}\} = \inf\{C' : 7 \text{ holds}\}.$$

Theorem 3.2.3. *The class $(\mathcal{W}_p^{\text{mid}}, w_p^{\text{mid}})$ is a Banach operator ideal*

The weakly mid p -summing operators characterized by the following theorem see [30, Theorem 4.4].

Theorem 3.2.4. *Let T be in $\mathcal{L}(X; Y)$. For $1 \leq p < \infty$, the following are equivalent:*

1) T is weakly mid p -summing.

2) $S \circ T \in \Pi_p(X; \ell_p)$ for every $S \in \mathcal{L}(Y; \ell_p)$.

3) $T \circ U \in \Pi_p^d(\ell_{p^*}; Y)$ for every $U \in \mathcal{L}(\ell_{p^*}; X)$, $1/p + 1/p^* = 1$ (where Π_p^d is the dual ideal of Π_p).

Moreover $w_p^{mid}(T) = lt_p(T)$, where

$$lt_p(T) = \sup\{\pi_p(S \circ T) : S \in \mathcal{L}(Y, \ell_p), \|S\| \leq 1\}.$$

Theorem 3.2.5. *The class $(\mathcal{W}_p^{mid}, lt_p)$ is a Banach operator ideal*

Theorem 3.2.6. *Let $T \in \mathcal{L}(X; Y)$*

1) *If $T \in \Pi_p^d(X, Y)$ ($\Leftrightarrow T^* \in \Pi_p(Y^*, X^*)$), then $T \in \mathcal{W}_p^{mid}(X; Y)$.*

2) *If $T \in \Pi_p(X, Y)$, then $T \in \mathcal{W}_p^{mid}(X; Y)$.*

3.3 Weakly mid summing multilinear operators

Now, we extend the definition of weakly mid p -summing linear operators to the multilinear operators.

Definition 3.3.1. Let $1 \leq p, p_1, \dots, p_m < \infty$ with $1/p = 1/p_1 + \dots + 1/p_m$. A multilinear operator $T : X_1 \times \dots \times X_m \rightarrow Y$ is weakly mid- (p_1, \dots, p_m) -summing if $S \circ T \in \Pi_{p_1, \dots, p_m}(X_1, \dots, X_m; \ell_p)$ for every $S \in \mathcal{L}(Y; \ell_p)$. The set of all weakly mid- (p_1, \dots, p_m) -summing operators from $X_1 \times \dots \times X_m$ into Y will be denoted by $\mathcal{W}_{p_1, \dots, p_m}^{mid}(X_1, \dots, X_m; Y)$ we define

$$lt_{p_1, \dots, p_m}(T) = \sup\{\pi_{p_1, \dots, p_m}(S \circ T) : S \in \mathcal{L}(Y; \ell_p), \|S\| \leq 1\}.$$

Question 3.3.2. *The class of $(\mathcal{W}_{p_1, \dots, p_m}^{mid}, lt_{p_1, \dots, p_m})$ is Banach ideal of multilinear operators?*

We will answer on this question in the last of this section.

Replacing Y in [27] Definition 2.1] by ℓ_p we have the following lemma.

Lemma 3.3.3. *Let $(x_j)_{j=1}^\infty \in \ell_p^w(X)$. The following are equivalent.*

- 1- $(x_j)_{j=1}^\infty \in \ell_p^{\text{mid}}(X)$.
- 2- $(S(x_j))_{j=1}^\infty \in \ell_p(\ell_p)$ for all $S \in \mathcal{L}(X; \ell_p)$. Moreover

$$\|(x_j)_{j=1}^\infty\|_{\text{mid},p} = \sup\{\|(S((x_j)_{j=1}^\infty))_{j=1}^\infty\| : S \in \mathcal{L}(X; \ell_p), \|S\| \leq 1\}.$$

The following proposition characterize the weakly mid- (p_1, \dots, p_m) -summing operators.

Proposition 3.3.4. *For $T \in \mathcal{L}(X_1, \dots, X_m; Y)$, the following statements are equivalent*

- 1- T is weakly mid- (p_1, \dots, p_m) -summing operator.
- 2- $(T(x_j^1, \dots, x_j^m))_{j=1}^\infty \in \ell_p^{\text{mid}}(Y)$ whenever $(x_j^i)_{j=1}^\infty \in \ell_{p_i}^w(X_i)$ $1 \leq i \leq m$.

Proof. We take $T \in \mathcal{W}_{p_1, \dots, p_m}(X_1, \dots, X_m; Y)$, then by the definition of weakly mid- (p_1, \dots, p_m) -summing we have $S \circ T \in \Pi_{p_1, \dots, p_m}(X_1, \dots, X_m, \ell_p)$ for every $S \in \mathcal{L}(Y; \ell_p)$, this equivalent by definition of (p_1, \dots, p_m) -dominated $(S \circ T(x_j^1, \dots, x_j^m))_{j=1}^\infty \in \ell_p(\ell_p)$ for every $(x_j^i)_{j=1}^\infty \in \ell_{p_i}^w(X_i)$, $1 \leq i \leq n$ and for every $S \in \mathcal{L}(Y; \ell_p)$ equivalently $(S \circ T(x_j^1, \dots, x_j^m))_{j=1}^\infty \in \ell_p(\ell_p)$ for every $S \in \mathcal{L}(Y; \ell_p)$ and for every $(x_j^i)_{j=1}^\infty \in \ell_{p_i}^w(X_i)$, $1 \leq i \leq n$ hence $(T(x_j^1, \dots, x_j^m))_{j=1}^\infty \in \ell_p^{\text{mid}}(Y)$ for every $(x_j^i)_{j=1}^\infty \in \ell_{p_i}^w(X_i)$, $1 \leq i \leq n$. This complete the proof. \square

As $\ell_p^{\text{mid}}(\cdot)$ and $\ell_p^w(\cdot)$ are finitely determined (Example 3.1.2), then the next theorem is immediate consequences of Proposition 3.1.4.

Theorem 3.3.5. *For $T \in \mathcal{L}(X_1, \dots, X_m; Y)$, the following statements are equivalent*

- 1- $T \in \mathcal{W}_{p_1, \dots, p_m}^{\text{mid}}(X_1, \dots, X_m; Y)$.
- 2- The induced map $\tilde{T} : \ell_{p_1}^w(X_1) \times \dots \times \ell_{p_m}^w(X_m) \rightarrow \ell_p^{\text{mid}}(Y)$ given by $\tilde{T}((x_j^1)_{j=1}^\infty, \dots, (x_j^m)_{j=1}^\infty) = (T(x_j^1, \dots, x_j^m))_{j=1}^\infty$ is a well-defined continuous multilinear operator.
- 3- There is a constant $C > 0$ such that for any $x_1^i, \dots, x_k^i \in X_i$, $(1 \leq i \leq m)$, we have

$$\|(T(x_j^1, \dots, x_j^m))_{j=1}^k\|_{\text{mid},p} \leq C \prod_{i=1}^m \|(x_j^i)_{j=1}^k\|_{w,p_i}. \quad (3.2)$$

3'- there is a constant $C > 0$ such that for any $(x_j^i)_{j=1}^\infty \in \ell_{p_i}^w(X_i)$, $(1 \leq i \leq m)$ we have

$$\|(T(x_j^1, \dots, x_j^m))_{j=1}^\infty\|_{mid,p} \leq C \prod_{i=1}^m \|(x_j^i)_{j=1}^\infty\|_{w,p_i}. \quad (3.3)$$

Moreover, $w_{p_1, \dots, p_m}^{mid}(T) = \|\tilde{T}\| = \|\widehat{T}\| = \inf\{C : 3 \text{ holds}\} = \inf\{C : 3' \text{ holds}\}$.

Theorem 3.3.6. *The class $(\mathcal{W}_{p_1, \dots, p_m}^{mid}, w_{p_1, \dots, p_m}^{mid}(\cdot))$ is a Banach ideal of multilinear mappings.*

Proof. As $\ell_{p_i}^w(\mathbb{K}) = \ell_{p_i}(\mathbb{K})$ and $\ell_p^{mid}(\mathbb{K}) = \ell_p(\mathbb{K})$ are linearly stable sequence classes and since $1/p = 1/p_1 + \dots + 1/p_m$ we obtain

$$\ell_{p_1}^w(\mathbb{K}) \cdots \ell_{p_m}^w(\mathbb{K}) = \ell_{p_1}(\mathbb{K}) \cdots \ell_{p_m}(\mathbb{K}) \xrightarrow{1} \ell_p^{mid}(\mathbb{K}) = \ell_p(\mathbb{K}).$$

So, from Theorem 3.1.5 it follows that $(\mathcal{W}_{p_1, \dots, p_m}^{mid}, w_{p_1, \dots, p_m}^{mid}(\cdot))$ is a Banach ideal of multilinear operators. \square

We finish this section solving a problem left open in the first section.

Proposition 3.3.7. *For $T \in \mathcal{W}_{p_1, \dots, p_m}^{mid}(X_1, \dots, X_m; Y)$, we have $lt_{p_1, \dots, p_m}(T) = w_{p_1, \dots, p_m}^{mid}(T)$.*

Proof. Let $T \in \mathcal{W}_{p_1, \dots, p_m}^{mid}(X_1, \dots, X_m; Y)$ and $S \in \mathcal{L}(Y; \ell_p)$ with $\|S\| \leq 1$. Here we use that the spaces $\ell_p^w(Y^*)$ and $\mathcal{L}(Y; \ell_p)$ are canonically isometrically isomorphic via the correspondence $y^* = (y_n^*)_{n=1}^\infty \in \ell_p^w(Y^*) \mapsto S_{y^*} \in \mathcal{L}(Y; \ell_p)$, $S_{y^*}(y) = (y_n^*(y))_{n=1}^\infty$ [22, Proposition 8.2(2)]. So there exists $(y_n^*)_{n=1}^\infty \in B_{\ell_p^w(Y^*)}$ such that $S(y) = (y_n^*(y))_{n=1}^\infty$ for every $y \in Y$. Thus

$$\left(\sum_{j=1}^\infty \|ST(x_j^1, \dots, x_j^m)\|_p^p \right)^{1/p} = \left(\sum_{j=1}^\infty \sum_{n=1}^\infty |\langle y_n^*, T(x_j^1, \dots, x_j^m) \rangle|^p \right)^{1/p} \leq lt_{p_1, \dots, p_m}(T) \prod_{i=1}^m \|(x_j^i)_{j=1}^\infty\|_{w,p_i}$$

for every $(x_j^i)_{j=1}^\infty \in \ell_{p_i}^w(X_i)$. Therefore $ST \in \Pi_{p_1, \dots, p_m}(X_1, \dots, X_m; \ell_p)$ and $w_{p_1, \dots, p_m}(T) \leq lt_{p_1, \dots, p_m}(T)$. From

$$\left(\sum_{j=1}^\infty \sum_{n=1}^\infty |\langle y_n^*, T(x_j^1, \dots, x_j^m) \rangle|^p \right)^{1/p} = \left(\sum_{j=1}^\infty \|ST(x_j^1, \dots, x_j^m)\|_p^p \right)^{1/p} \leq \pi_{p_1, \dots, p_m}(S \circ T) \prod_{i=1}^m \|(x_j^i)_{j=1}^\infty\|_{w,p_i}$$

we obtain $lt_{p_1, \dots, p_m}(T) \leq w_{p_1, \dots, p_m}(T)$, proving that $lt_{p_1, \dots, p_m}(T) = w_{p_1, \dots, p_m}(T)$. \square

Now we solve it in the affirmative: Since $lt_{p_1, \dots, p_m}(T) = w_{p_1, \dots, p_m}^{mid}(T)$ for every $T \in \mathcal{W}_{p_1, \dots, p_m}^{mid}(X_1, \dots, X_m; Y)$, we obtain the following Theorem.

Theorem 3.3.8. *The class $(\mathcal{W}_{p_1, \dots, p_m}^{mid}, lt_{p_1, \dots, p_m}(\cdot))$ is a Banach ideal of multilinear mappings.*

3.4 Pietsch domination theorems

Theorem 3.4.1. [21, Theorem 4.8] *Let $T \in \mathcal{L}(X; Y)$. The following statements are equivalent:*

(a) *T is weakly mid- p -summing.*

(b) *There are a constant $C > 0$ and a regular Borel probability measure μ on Y^{**} such that*

$$\left(\sum_{j=1}^{\infty} |y^*(T(x_j))|^p \right)^{1/p} \leq C \left(\int_{B_{Y^{**}}} |y^{**}(y^*)|^p d\mu(y^{**}) \right)^{1/p} \quad (3.4)$$

for all $y^ \in B_{Y^*}$ and all $(x_j)_{j=1}^{\infty} \in \ell_p^w(X)$.*

(c) *There are a constant $C > 0$ and a regular Borel probability measure μ on $B_{Y^{**}}$ such that*

$$\left(\sum_{j=1}^k |y^*(T(x_j))|^p \right)^{1/p} \leq C \left(\int_{B_{Y^{**}}} |y^{**}(y^*)|^p d\mu(y^{**}) \right)^{1/p}$$

for all $y^ \in Y^*$, $x_1, \dots, x_k \in X$.*

Theorem 3.4.2. *Let $T \in \mathcal{L}(X_1, \dots, X_m; Y)$. The following statements are equivalent.*

(a) *T is weakly mid- (p_1, \dots, p_m) -summing*

(b) *There are a constant $C > 0$ and a regular Borel probability measures μ_i on $B_{X_i^*}$ such that,*

$$\|S \circ T(x^1, \dots, x^m)\| \leq C \prod_{i=1}^m \left(\int_{B_{X_i^*}} |x^i(x_i^*)|^{p_i} d\mu(x_i^*) \right)^{1/p_i}$$

for all $S \in \mathcal{L}(Y; \ell_p)$, $x^i \in X_i$.

(c) *There are a constant $C > 0$ and a regular Borel probability measures μ_i on $B_{X_i^*}$ such*

that,

$$\left(\sum_{n=1}^{\infty} |\langle y_n^*, T(x^1, \dots, x^m) \rangle|^p \right)^{1/p} \leq C \prod_{i=1}^m \left(\int_{B_{X_i^*}} |x^i(x_i^*)|^{p_i} d\mu(x_i^*) \right)^{1/p_i} \quad (3.5)$$

for all $(y_n^*)_{n=1}^{\infty} \in \ell_p^w(Y^*)$, $x^i \in X_i$.

Proof. (a) \Rightarrow (b) If $T \in Lt_{p_1, \dots, p_m}(X_1, \dots, X_m; Y)$, we have $S \circ T : X_1 \times \dots \times X_m \rightarrow \ell_p$ is (p_1, \dots, p_m) -dominated. Therefore, there are a constant $C > 0$ and a regular Borel probability measures μ_i on $B_{X_i^*}$ such that,

$$\|S \circ T(x^1, \dots, x^m)\| \leq C \prod_{i=1}^m \left(\int_{B_{X_i^*}} |x^i(x_i^*)|^{p_i} d\mu(x_i^*) \right)^{1/p_i}$$

for all $x^i \in X_i$ and for all $S \in \mathcal{L}(Y; \ell_p)$. (b) \Rightarrow (c) For $S \in \mathcal{L}(Y; \ell_p)$. So there exists $(y_n^*)_{n=1}^{\infty} \in \ell_p^w(Y^*)$ such that $S(y) = (y_n^*(y))_{n=1}^{\infty}$ for every $y \in Y$. Thus

$$\|S \circ T(x^1, \dots, x^m)\| = \left(\sum_{n=1}^{\infty} |\langle y_n^*, T(x^1, \dots, x^m) \rangle|^p \right)^{1/p}$$

for all $x_i \in X_i$, i.e.,

$$\left(\sum_{n=1}^{\infty} |\langle y_n^*, T(x^1, \dots, x^m) \rangle|^p \right)^{1/p} \leq C \prod_{i=1}^m \left(\int_{B_{X_i^*}} |x^i(x_i^*)|^{p_i} d\mu(x_i^*) \right)^{1/p_i}$$

for all $(y_n^*)_{n=1}^{\infty} \in \ell_p^w(Y^*)$, $x^i \in X_i$. (c) \Rightarrow (a) Let $(x_j^1, \dots, x_j^m) \in (X_1 \times \dots \times X_m)$ ($1 \leq j \leq k$) we have from (3.5)

$$\sum_{n=1}^{\infty} |\langle y_n^*, T(x_j^1, \dots, x_j^m) \rangle|^p \leq C^p \prod_{i=1}^m \left(\int_{B_{X_i^*}} |x_j^i(x_i^*)|^{p_i} d\mu(x_i^*) \right)^{p/p_i}.$$

By Hölder's inequality $((\sum_{j=1}^k \prod_{i=1}^m (\lambda_j^i)^p)^{1/p} \leq \prod_{i=1}^m (\sum_{j=1}^k (\lambda_j^i)^{p_i})^{1/p_i}$, $\lambda_j^i \geq 0$) we

obtain

$$\begin{aligned}
 \left(\sum_{j=1}^k \sum_{n=1}^{\infty} |\langle y_n^*, T(x_j^1, \dots, x_j^m) \rangle|^p \right)^{1/p} &\leq C \left(\sum_{j=1}^k \prod_{i=1}^m \left(\int_{B_{X_i^*}} |x_j^i(x_i^*)|^{p_i} d\mu(x_i^*) \right)^{p/p_i} \right)^{1/p} \\
 &= C \left(\sum_{j=1}^k \left[\prod_{i=1}^m \left(\int_{B_{X_i^*}} |x_j^i(x_i^*)|^{p_i} d\mu(x_i^*) \right)^{1/p_i} \right]^p \right)^{1/p} \\
 &\leq C \prod_{i=1}^m \left(\sum_{j=1}^k \int_{B_{X_i^*}} |x_j^i(x_i^*)|^{p_i} d\mu(x_i^*) \right)^{1/p_i} \\
 &\leq C \prod_{i=1}^m \left(\int_{B_{X_i^*}} \sum_{j=1}^k |x_j^i(x_i^*)|^{p_i} d\mu(x_i^*) \right)^{1/p_i} \\
 &\leq C \prod_{i=1}^m \| (x_j^i)_{j=1}^k \|_{w, p_i}.
 \end{aligned}$$

We deduce $((\langle T(x_j^1, \dots, x_j^m), y_n^* \rangle)_{n=1}^{\infty})_{j=1}^k \in \ell_p(\ell_p)$ whenever $(y_n^*)_{n=1}^{\infty} \in \ell_p^w(Y^*)$, $(x_j^i)_{j=1}^k \in \ell_{p_i}^w(X_i)$, i.e., $(T(x_j^1, \dots, x_j^m))_{j=1}^k \in \ell_p^{mid}(Y)$ whenever $(x_j^i)_{j=1}^k \in \ell_{p_i}^w(X_i)$.

□

Corollary 3.4.3. *Let $T \in \mathcal{L}(X; Y)$. The following statements are equivalent.*

(a) *T is weakly mid- p -summing*

(b) *There are a constant $C > 0$ and a regular Borel probability measure μ on $B_{Y^{**}}$ such that*

$$\left(\sum_{j=1}^{\infty} |y^*(T(x_j))|^p \right)^{1/p} \leq C \left(\int_{B_{Y^{**}}} |y^{**}(y^*)|^p d\mu(y^{**}) \right)^{1/p}$$

for all $y^ \in Y^*$ and all $(x_j)_{j=1}^{\infty} \in \ell_p^w(X)$.*

(c) *There are a constant $C > 0$ and a regular Borel probability measures μ on B_{X^*} such that,*

$$\left(\sum_{n=1}^{\infty} |\langle y_n^*, T(x) \rangle|^p \right)^{1/p} \leq C \left(\int_{B_{X^*}} |x(x^*)|^p d\mu(x^*) \right)^{1/p}$$

for all $(y_n^)_{n=1}^{\infty} \in \ell_p^w(Y^*)$, $x \in X$.*

MULTILINEAR MID SUMMING OPERATORS AND APPLICATIONS

Our results in this chapter are presented as follows. In first section , we recall important results and definitions to be used later. In section 2, we introduce and investigate a new ideal of strongly mid p -summing operators. We present a characterization given by a summability property. We also prove the related dual result. We study Banach space X for which $id_X \in \Pi_p^{mid}$, $id_X \in \mathcal{D}_p^{mid}$ and $\mathcal{D}_p(\ell_p; X) = \mathcal{L}(\ell_p; X)$. We also prove, X has the p -Dunford-Pettis property if and only if X^* has the p -Dunford-Pettis property. In section 3, we introduce the notion of absolutely mid (p_1, \dots, p_m) -summing multilinear operators. The original motivation for our section is to give a multilinear version of Kwapień's factorization theorem for (p_1, \dots, p_m) -dominated multilinear operators. $\Pi_{p_1, \dots, p_m} = \Pi_{p_1, \dots, p_m}^{mid} \circ (\mathcal{W}_{p_1}^{mid}, \dots, \mathcal{W}_{p_m}^{mid})$.

4.1 Strongly mid summing linear operators

Botelho, Campos and Santos [14] introduced the concept of absolutely mid p -summing operators. For $1 \leq p < \infty$, an operator $T \in \mathcal{L}(X; Y)$ is absolutely mid p -summing if $(T(x_j))_{j=1}^{\infty} \in \ell_p(Y)$ whenever $(x_j)_{j=1}^{\infty} \in \ell_p^{mid}(X)$. Equivalently, if there is a positive constant C such that

$$\|(T(x_j))_{j=1}^{\infty}\|_p \leq C \|(x_j)_{j=1}^{\infty}\|_{mid,p}$$

for all $(x_j)_{j=1}^{\infty} \in \ell_p^{mid}(X)$. The space of these operators from X into Y denoted by $\Pi_p^{mid}(X; Y)$, is a Banach space under the norm $\pi_p^{mid}(T)$ which is the infimum of all $C > 0$ that satisfy the above inequality.

Now, we introduce the concept of strongly mid p -summing linear operators as a characterization of the conjugates of absolutely mid p^* -summing linear operators.

Definition 4.1.1. Let $1 < p \leq \infty$. A mapping $T \in \mathcal{L}(X; Y)$ is strongly mid p -summing if there exist a constant $C > 0$ such that

$$\|(\langle T(x_j), y_j^* \rangle)_{j=1}^k\|_1 \leq C \|(x_j)_{j=1}^k\|_p \|(y_j^*)_{j=1}^k\|_{mid,p^*} \quad (4.1)$$

for any $(x_j)_{j=1}^k \subset X$ and $(y_j^*)_{j=1}^k \subset Y^*$. We denote by $\mathcal{D}_p^{mid}(X; Y)$ the space of all strongly mid p -summing operators from X to Y . The least C for which (4.1) holds will be denoted by $d_p^{mid}(T)$. From this definition it is clear to see that $\mathcal{D}_p(X; Y) \subset \mathcal{D}_p^{mid}(X; Y)$ and $d_p^{mid}(T) \leq d_p(T)$ for all $T \in \mathcal{D}_p(X; Y)$.

For $T \in \mathcal{L}(X; Y)$, the induced map $\varphi_T : X \times Y^* \rightarrow \mathbb{K}$ given by $\varphi_T(x, y^*) = \langle T(x), y^* \rangle$ for all $x \in X$ and $y^* \in Y^*$ is a continuous bilinear operator. Using the abstract approach of [13] and [14, Proposition 1.9], we can see that the next proposition is an immediate consequences of [13, Proposition 2.4].

Proposition 4.1.2. *Let $T \in \mathcal{L}(X; Y)$, the following assertions are equivalent.*

- 1) $T \in \mathcal{D}_p^{mid}(X; Y)$.
- 2) $(\varphi_T(x_j, y_j^*))_{j=1}^\infty \in \ell_1$ whenever $(x_j)_{j=1}^\infty \in \ell_p(X)$ and $(y_j^*)_{j=1}^\infty \in \ell_{p^*}^{mid}(Y^*)$.
- 3) The induced map $\widehat{\varphi}_T : \ell_p(X) \times \ell_{p^*}^{mid}(Y^*) \rightarrow \ell_1$ given by $\widehat{\varphi}_T((x_j)_{j=1}^\infty, (y_j^*)_{j=1}^\infty) = (\varphi_T(x_j, y_j^*))_{j=1}^\infty$ is a well-defined and continuous bilinear operator.
- 4) There exists a constant $C > 0$ such that

$$\|(\varphi_T(x_j, y_j^*))_{j=1}^\infty\|_1 \leq C \|(x_j)_{j=1}^\infty\|_p \|(y_j^*)_{j=1}^\infty\|_{mid, p^*}. \quad (4.2)$$

for any $(x_j)_{j=1}^\infty \in \ell_p(X)$ and $(y_j^*)_{j=1}^\infty \in \ell_{p^*}^{mid}(Y^*)$.

- 5) There exists a constant $C > 0$ such that

$$\|(\varphi_T(x_j, y_j^*))_{j=1}^k\|_1 \leq C \|(x_j)_{j=1}^k\|_p \|(y_j^*)_{j=1}^k\|_{mid, p^*}. \quad (4.3)$$

for any $k \in \mathbb{N}$ and any $(x_j)_{j=1}^k \subset X$, $(y_j^*)_{j=1}^k \subset Y^*$. Moreover, $d_p^{mid}(T) = \|\widehat{\varphi}_T\| = \inf\{C : \text{\textcircled{4.2}} \text{ holds}\} = \inf\{C : \text{\textcircled{4.3}} \text{ holds}\}$.

Theorem 4.1.3. $(\mathcal{D}_p^{mid}(X; Y), d_p^{mid}(\cdot))$ is a Banach operator ideal.

Proof. Using the abstract framework, notation and language [13], we find that a linear operator T is strongly p -summing if and only if φ_T is $(\ell_p(\cdot), \ell_{p^*}^{mid}(\cdot); \ell_1)$ -summing. Since $1/p + 1/p^* = 1$ we obtain

$$\|(\lambda_j^1, \lambda_j^2)_{j=1}^\infty\|_1 \leq \|(\lambda_j^1)_{j=1}^\infty\|_{\ell_p(\mathbb{K})} \|(\lambda_j^2)_{j=1}^\infty\|_{\ell_{p^*}(\mathbb{K})} = \|(\lambda_j^1)_{j=1}^\infty\|_{\ell_p(\mathbb{K})} \|(\lambda_j^2)_{j=1}^\infty\|_{\ell_{p^*}^{mid}(\mathbb{K})}.$$

Therefore, $\ell_p(\mathbb{K}) \ell_{p^*}^{mid}(\mathbb{K}) = \ell_p(\mathbb{K}) \ell_{p^*}(\mathbb{K}) \xrightarrow{1} \ell_1$. In addition, all the sequence classes involved are linearly stable (see [14, Proposition 1.10]). So, from [13, Theorem 3.6] it follows that $(\mathcal{D}_p^{mid}, d_p^{mid}(\cdot))$ is a Banach operators ideal. \square

In the next theorem, we will show that the absolutely mid p -summing linear operators is the adjoint of the strongly mid p -summing linear operators. It will be useful throughout this section.

Theorem 4.1.4. *Let $1 < p \leq \infty$, $T \in \mathcal{L}(X; Y)$. Then $T \in \mathcal{D}_p^{mid}(X; Y)$ if and only if $T^* \in \Pi_{p^*}^{mid}(Y^*; X^*)$. Moreover, $d_p^{mid}(T) = \pi_{p^*}^{mid}(T^*)$.*

Proof. Assume that $T \in \mathcal{D}_p^{mid}(X; Y)$. For $(x_j)_{j=1}^\infty \in \ell_p(X)$ and $(y_j^*)_{j=1}^\infty \in \ell_{p^*}^{mid}(Y^*)$.

We have

$$\|(\langle T(x_j), y_j^* \rangle)_{j=1}^\infty\|_1 = \|(\langle T^*(y_j^*), x_j \rangle)_{j=1}^\infty\|_1 \leq d_p^{mid}(T) \| (x_j)_{j=1}^\infty \|_p \| (y_j^*)_{j=1}^\infty \|_{mid, p^*}$$

by taking the supremum over the unit ball in $\ell_p(X)$ we obtain

$$\| (T^*(y_j^*))_{j=1}^\infty \|_{p^*} \leq C \| (y_j^*)_{j=1}^\infty \|_{mid, p^*}.$$

Therefore, $T^* : Y^* \rightarrow X^*$ is absolutely mid p^* -summing and $\pi_{p^*}^{mid}(T^*) \leq d_p^{mid}(T)$.

Conversely, let $T^* \in \Pi_{p^*}^{mid}(Y^*; X^*)$. By Hölder's inequality we have, for $(x_j)_{j=1}^\infty \in \ell_p(X)$ and $(y_j^*)_{j=1}^\infty \in \ell_{p^*}^{mid}(Y^*)$,

$$\sum_{j=1}^\infty |\langle T(x_j), y_j^* \rangle| \leq \| (x_j)_{j=1}^\infty \|_p \| (T^*(y_j^*))_{j=1}^\infty \|_{p^*} \leq \pi_{p^*}^{mid}(T^*) \| (x_j)_{j=1}^\infty \|_p \| (y_j^*)_{j=1}^\infty \|_{mid, p^*}.$$

Thus T is strongly mid p -summing and $d_p^{mid}(T) = \pi_{p^*}^{mid}(T^*)$. □

The following corollary which is a straightforward consequence of the preceding theorem and Theorem 2.8 in [14].

Corollary 4.1.5. *Let $1 \leq p < \infty$, $T \in \mathcal{L}(X; Y)$. If $T^* \in \mathcal{D}_{p^*}^{mid}(Y^*; X^*)$ then $T \in \Pi_p^{mid}(X; Y)$.*

Recall that an operator ideal $\mathcal{I}(X; Y)$ is surjective if $T \in \mathcal{I}(X; Y)$ whenever $T \circ Q \in \mathcal{I}(X_0; Y)$, where $Q : X_0 \rightarrow X$ is a quotient map. As consequence of Theorem 4.1.4 [14, Proposition 2.8] and [43, Section 4], we obtain the following proposition.

Proposition 4.1.6. *The operator ideal $(\mathcal{D}_p^{mid}, d_p^{mid})$ is surjective.*

The Dvoretzky-Rogers theorem (see [24]) states that the Banach space X is of finite dimensional if, and only if, id_X is p -summing. Now we will give a result in this context.

We say that a Banach space X is a weak mid p -space (or has p -Dunford-Pettis property see [30]) if $\ell_p^{mid}(X) = \ell_p^w(X)$ and it is a strong mid p -space if $\ell_p^{mid}(X) = \ell_p(X)$.

The next result is a reformulation of [14, Theorem 2.7].

Theorem 4.1.7. *Let $1 \leq p < \infty$. Then the following are equivalent.*

- 1) X is a strong mid p -space.
- 2) $id_X \in \Pi_p^{mid}(X; X)$.
- 3) $\mathcal{L}(X; Y) = \Pi_p^{mid}(X; Y)$ for every Banach space Y .
- 4) $\mathcal{L}(Y; X) = \Pi_p^{mid}(Y; X)$ for every Banach space Y .
- 5) X is a subspace of $L_p(\mu)$ for some Borel measure μ .

Corollary 4.1.8. *If X^{**} is a strong mid p -space then X is a strong mid p -space.*

Proof. Is a direct consequence of Theorem 4.1.7 and [14, Proposition 2.8]. □

If we replace X by X^* in the precedent theorem, we have by Theorem 4.1.4 and Theorem 4.1.7 the next dual theorem.

Theorem 4.1.9. *The following properties are equivalent:*

- 1) X^* is a strong mid p^* -space.
- 2) $id_X \in \mathcal{D}_p^{mid}(X; X)$.
- 3) $\mathcal{L}(X; Y) = \mathcal{D}_p^{mid}(X; Y)$ for every Banach space Y .
- 4) $\mathcal{L}(Y; X) = \mathcal{D}_p^{mid}(Y; X)$ for every Banach space Y .
- 5) $id_{X^*} \in \Pi_{p^*}^{mid}(X^*; X^*)$.
- 6) $\mathcal{L}(X^*; Y) = \Pi_{p^*}^{mid}(X^*; Y)$ for every Banach space Y .

- 7) $\mathcal{L}(Y; X^*) = \Pi_{p^*}^{mid}(Y; X^*)$ for every Banach space Y .
 8) X^* is a subspace of $L_{p^*}(\mu)$ for some Borel measure μ .

From [14, Theorem 2.6] and [30, Theorem 3.7], we have the following equivalences for $1 \leq p < \infty$.

Theorem 4.1.10. *The following assertions are equivalent.*

- 1) X has the p -Dunford-Pettis property.
- 2) $\Pi_p^{mid}(X; Y) = \Pi_p(X; Y)$ for every Banach space Y .
- 3) $\Pi_p^{mid}(X; \ell_p) = \Pi_p(X; \ell_p) = \mathcal{L}(X; \ell_p)$.
- 4) $\mathcal{D}_{p^*}(\ell_{p^*}; X^*) = \mathcal{L}(\ell_{p^*}; X^*)$.

As consequence of Theorem 4.1.10 and Theorem 4.1.4, we have.

Theorem 4.1.11. *The following properties are equivalent.*

- 1) X^* has the p -Dunford-Pettis property.
- 2) $\mathcal{D}_{p^*}^{mid}(Y; X) = \mathcal{D}_{p^*}(Y; X)$ for every Banach space Y .
- 3) $\Pi_p^{mid}(X^*; Y^*) = \Pi_p(X^*; Y^*)$ for every Banach space Y .
- 4) $\Pi_p^{mid}(X^*; \ell_p) = \Pi_p(X^*; \ell_p) = \mathcal{L}(X^*; \ell_p)$.
- 5) $\mathcal{D}_{p^*}^{mid}(\ell_{p^*}; X) = \mathcal{D}_{p^*}(\ell_{p^*}; X) = \mathcal{L}(\ell_{p^*}; X)$.
- 6) $\mathcal{D}_{p^*}(\ell_{p^*}; X) = \mathcal{L}(\ell_{p^*}; X)$.

Corollary 4.1.12. *X^{**} has the p -Dunford-Pettis property if and only if X has the p -Dunford-Pettis property.*

Theorem 4.1.13. *X^* has the p -Dunford-Pettis property if and only if X has the p -Dunford-Pettis property.*

Proof. By [30, Corollary 3.9], if X^{**} has the p -Dunford-Pettis property, then X^* is too, and consequently X has the p -Dunford-Pettis property. On the other hand, by a Corollary 4.1.12, if X has the p -Dunford-Pettis property, then X^{**} it is also.

□

4.2 Kwapień's factorization theorem for dominated multilinear operators

In this section we extend to multilinear operators the concept of absolutely mid p -summing linear operators, for which the resulting vector space $\Pi_{p_1, \dots, p_m}^{mid}$ of the absolutely mid (p_1, \dots, p_m) -summing is a Banach ideal of multilinear operators. The original motivation for our section is to give a multilinear version of Kwapień's factorization theorem for (p_1, \dots, p_m) -dominated multilinear operators such that

$$\Pi_{p_1, \dots, p_m} = \Pi_{p_1, \dots, p_m}^{mid} \circ (\mathcal{W}_{p_1}^{mid}, \dots, \mathcal{W}_{p_m}^{mid}).$$

Now, we state our definition.

Definition 4.2.1. Let $1 \leq p, p_1, \dots, p_m < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$. A mapping $T \in \mathcal{L}(X_1, \dots, X_m; Y)$ is absolutely mid (p_1, \dots, p_m) -summing if $(T(x_j^1, \dots, x_j^m))_j \in \ell_p(Y)$ whenever $(x_j^i)_j \in \ell_{p_i}^{mid}(X_i)$, $i = 1, \dots, m$.

Let $\Pi_{p_1, \dots, p_m}^{mid}(X_1, \dots, X_m; Y)$ be the space of all absolutely mid (p_1, \dots, p_m) -summing operators from $X_1 \times \dots \times X_m$ to Y .

As $\ell_p^{mid}(\cdot)$ and $\ell_p(\cdot)$ are finitely determined (Example [3.1.2](#)), then the next theorem is immediate consequences of Proposition [3.1.4](#).

Proposition 4.2.2. For $T \in \mathcal{L}(X_1, \dots, X_m; Y)$ the following conditions are equivalent.

- (a) T is absolutely mid (p_1, \dots, p_m) -summing.
- (b) The induced map $\hat{T} : \ell_{p_1}^{mid}(X_1) \times \dots \times \ell_{p_m}^{mid}(X_m) \rightarrow \ell_p(Y)$ given by $\hat{T}((x_j^1)_{j=1}^\infty, \dots, (x_j^m)_{j=1}^\infty) = (T(x_j^1, \dots, x_j^m))_{j=1}^\infty$ is a well-defined continuous multilinear operator.
- (c) There is a constant $C > 0$ such that for any $x_1^i, \dots, x_k^i \in X_i$, $(1 \leq i \leq m)$, we have

$$\|(T(x_j^1, \dots, x_j^m))_{j=1}^k\|_p \leq C \prod_{i=1}^m \|(x_j^i)_{j=1}^k\|_{mid, p_i}. \quad (4.4)$$

(d) There is a constant $C > 0$ such that for any $(x_j^i)_{j=1}^\infty \in \ell_{p_i}^{mid}(X_i)$, $(1 \leq i \leq m)$ we have

$$\|(T(x_j^1, \dots, x_j^m))_{j=1}^\infty\|_p \leq C \prod_{i=1}^m \|(x_j^i)_{j=1}^\infty\|_{mid, p_i}. \quad (4.5)$$

In this case $\pi_{p_1, \dots, p_m}^{mid}(T) = \|\widehat{T}\| = \inf\{C : \boxed{4.4} \text{ holds}\} = \inf\{C : \text{verifying } \boxed{4.5}\}$.

As a direct consequence of [\[14\]](#), Proposition 1.4], we have the next proposition.

Proposition 4.2.3. For $T \in \mathcal{L}(X_1, \dots, X_m; Y)$ and $1/p = 1/p_1 + \dots + 1/p_m$. If $T \in \Pi_{p_1, \dots, p_m}(X_1, \dots, X_m; Y)$ then $T \in \Pi_{p_1, \dots, p_m}^{mid}(X_1, \dots, X_m; Y)$ and $\pi_{p_1, \dots, p_m}^{mid}(T) \leq \pi_{p_1, \dots, p_m}(T)$. In particular if X_i has the p_i -Dunford-Pettis property for all $(1 \leq i \leq m)$, we have $\Pi_{p_1, \dots, p_m}(X_1, \dots, X_m; Y) = \Pi_{p_1, \dots, p_m}^{mid}(X_1, \dots, X_m; Y)$.

Theorem 4.2.4. The class $(\Pi_{p_1, \dots, p_m}^{mid}, \pi_{p_1, \dots, p_m}^{mid}(\cdot))$ is a Banach ideal of multilinear mappings.

Proof. As $\ell_{p_i}^{mid}(\mathbb{K}) = \ell_{p_i}(\mathbb{K})$ are linearly stable sequence classes and since $1/p = 1/p_1 + \dots + 1/p_m$ we obtain

$$\ell_{p_1}^{mid}(\mathbb{K}) \cdots \ell_{p_m}^{mid}(\mathbb{K}) = \ell_{p_1}(\mathbb{K}) \cdots \ell_{p_m}(\mathbb{K}) \xrightarrow{1} \ell_p(\mathbb{K}).$$

So, from Theorem [3.1.5](#) it follows that $(\Pi_{p_1, \dots, p_m}^{mid}, \pi_{p_1, \dots, p_m}^{mid}(\cdot))$ is a Banach ideals of multilinear operators.

□

Definition 4.2.5. [\[44\]](#), Definition 5] Let $m \in \mathbb{N}$ and let $(\mathcal{I}_i, \alpha_i(\cdot))$ $(i = 1, \dots, m)$ be Banach ideals of bounded linear operators and $(\mathcal{M}, \beta(\cdot))$ be a Banach ideal of continuous multilinear operators. We say that $T \in \mathcal{L}(X_1, \dots, X_m; Y)$ belongs to $\mathcal{M} \circ (\mathcal{I}_1, \dots, \mathcal{I}_m)(X_1, \dots, X_m; Y)$ if there exist Banach spaces Y_1, \dots, Y_m , $R_i \in \mathcal{I}_i$ $(i = 1, \dots, m)$ and $S \in \mathcal{M}(Y_1, \dots, Y_m; Y)$ such that $T = S \circ (R_1, \dots, R_m)$. In this case, we define $\|T\|_{\mathcal{M} \circ (\mathcal{I}_1, \dots, \mathcal{I}_m)} = \inf\{\beta(S) \prod_{i=1}^m \alpha_i(R_i)\}$, where the infimum is taken over all factorizations of T as above.

We continue by recalling the notion of weakly mid p -summing. An operator $T \in \mathcal{L}(X; Y)$ is weakly mid p -summing if $(T(x_j))_{j=1}^\infty \in \ell_p^{\text{mid}}(Y)$ whenever $(x_j)_{j=1}^\infty \in \ell_p^w(X)$ (see [30, 14] for more details).

Theorem 4.2.6. *Every (p_1, \dots, p_m) -dominated multilinear operator factors through absolutely mid (p_1, \dots, p_m) -summing multilinear operator and weakly mid p_i -summing linear operators ($1 \leq i \leq m$), that is,*

$$\Pi_{p_1, \dots, p_m} = \Pi_{p_1, \dots, p_m}^{\text{mid}} \circ (\mathcal{W}_{p_1}^{\text{mid}}, \dots, \mathcal{W}_{p_m}^{\text{mid}}).$$

Proof. Firstly, we get to prove that $\Pi_{p_1, \dots, p_m}^{\text{mid}} \circ (\mathcal{W}_{p_1}^{\text{mid}}, \dots, \mathcal{W}_{p_m}^{\text{mid}}) \subset \Pi_{p_1, \dots, p_m}$.

Let $S : Y_1 \times \dots \times Y_m \rightarrow Y$ be an absolutely mid (p_1, \dots, p_m) -summing and let $R_i : X_i \rightarrow Y_i$ be a weakly mid p_i -summing operators. We prove that $T = S \circ (R_1, \dots, R_m) : X_1 \times \dots \times X_m \rightarrow Y$ is (p_1, \dots, p_m) -dominated multilinear operators. Let $(x_j^i)_{j=1}^k \subset X_i$. We have

$$\begin{aligned} \|(T(x_j^1, \dots, x_j^m))_{j=1}^k\|_p &= \|(S \circ (R_1(x_j^1), \dots, R_m(x_j^m)))_{j=1}^k\|_p \\ &\leq \pi_{p_1, \dots, p_m}^{\text{mid}}(S) \prod_{i=1}^m \|(S(x_j^i))_{j=1}^k\|_{p_i, \text{mid}} \\ &\leq \pi_{p_1, \dots, p_m}^{\text{mid}}(S) w_{p_1}^{\text{mid}}(R_1) \dots w_{p_m}^{\text{mid}}(R_m) \prod_{i=1}^m \|(x_j^i)_{j=1}^k\|_{p_i, w}. \end{aligned}$$

Then T is (p_1, \dots, p_m) -dominated and $\pi_{p_1, \dots, p_m}(T) \leq \pi_{p_1, \dots, p_m}^{\text{mid}}(S) w_{p_1}^{\text{mid}}(R_1) \dots w_{p_m}^{\text{mid}}(R_m)$.

Secondly, we prove that $\Pi_{p_1, \dots, p_m} \subset \Pi_{p_1, \dots, p_m}^{\text{mid}} \circ (\mathcal{W}_{p_1}^{\text{mid}}, \dots, \mathcal{W}_{p_m}^{\text{mid}})$. Let $T \in \Pi_{p_1, \dots, p_m}(X_1, \dots, X_m; Y)$. We have by Theorem [1.3.2] $T = S \circ (R_1, \dots, R_m)$. Now we show that S is absolutely mid (p_1, \dots, p_m) -summing. Let $(y_j^i)_{j=1}^\infty \in \ell_{p_i}^{\text{mid}}(Y_i)$. As Y_i is closed subspace of $L_{p_i}(\mu_i)$, from [14, Theorem 1.2] we have $(y_j^i)_{j=1}^\infty \in \ell_{p_i}(Y_i)$. Thus, as S is continuous multilinear and $\mathcal{L}_{\ell_{p_1}(\cdot), \dots, \ell_{p_m}(\cdot); \ell_p(\cdot)} = \mathcal{L}^m$ (see [13, equation 5]), then $(S(y_j^1, \dots, y_j^m))_{j=1}^\infty \in \ell_p(Y)$. This proves that $S \in \Pi_{p_1, \dots, p_m}^{\text{mid}}(Y_1, \dots, Y_m; Y)$. We now prove that $R_i = J_{p_i}|_{I_{X_i}(X_i)} \circ I_{X_i}$ is weakly mid p_i -summing operators. Let

$(x_j^i)_{j=1}^\infty \in \ell_{p_i}^w(X_i)$, since J_{p_i} is absolutely p_i -summing we have

$$\begin{aligned} \|(R_i(x_j^i))_{j=1}^\infty\|_{mid, p_i} &= \|(J_{p_i}|_{I_{X_i}(X_i)}(I_{X_i}(x_j^i)))_{j=1}^\infty\|_{mid, p_i} \\ &\leq \|(J_{p_i}|_{I_{X_i}(X_i)}(I_{X_i}(x_j^i)))_{j=1}^\infty\|_{p_i} \\ &\leq \|(I_{X_i}(x_j^i))_{j=1}^\infty\|_{w, p_i} \\ &\leq \|(x_j^i)_{j=1}^\infty\|_{w, p_i}. \end{aligned}$$

Then R_i is weakly mid p_i -summing and $w_{p_i}^{mid}(R_i) \leq 1$. □

Corollary 4.2.7. *Let $T \in \mathcal{L}(X_1, \dots, X_m; Y)$. Then T is (p_1, \dots, p_m) -dominated if and only if there exist Banach spaces Y_1, \dots, Y_m , weakly mid p_i -summing operators $R_i \in \mathcal{L}(X_i; Y_i)$ and a absolutely mid (p_1, \dots, p_m) -summing multilinear operator $S \in \mathcal{L}(Y_1, \dots, Y_m; Y)$ such that $T = S \circ (R_1, \dots, R_m)$. Moreover $\pi_{p_1, \dots, p_m}^{mid}(T) = \inf\{\pi_{p_1, \dots, p_m}^{mid}(S) \prod_{i=1}^m w_{p_i}^{mid}(R_i) : T = S \circ (R_1, \dots, R_m)\}$.*

RELATIONSHIPS BETWEEN T AND ∇T FOR CERTAIN SOMMABILITIES

Let $SB(X, Y)$ be the set of all bounded sublinear operators from a Banach space X into a complete Banach lattice Y ; which is a pointed convex cone not salient in $\text{Lip}_0(X, Y)$. In this chapter, we are interested in studying the relationship between T and its subdifferential ∇T (the set of all bounded linear operators $u : X \rightarrow Y$ such that $u(x) \leq T(x)$ for all x in X); concerning certain notions of Lipschitz summability. We also answer negatively a question posed previously concerning this type of relation in the sublinear case. For this, we introduce and study a new concept of summability in the category of Lipschitz operators, which we call "*super Lipschitz p -summing operators*". We prove some characterizations in terms of a domination theorem and some properties of this notion.

Introduction

The notion of p -summing linear operators goes back to Grothendieck in the 1950s, but just in 1967 and 1968, the classical works of Pietsch [54] and Lindenstrauss-Pełczyński [34] clarified Grothendieck's precious ideas and contributed clearly to the vigorous development of this notion. Recently, Lipschitz versions of different types of summing linear operators were investigated by several authors such as [16], [18], [19], [26], [57], [42] and [59] among others. The first paper is due to Farmer and Johnson [26]. They introduced the notion of Lipschitz p -summing operators and showed that is really a good generalization of the concept of linear p -summing operators [26, Theorem 2]. This notion marked the beginning of the theory of nonlinear summability. Motivated by the importance of this theory, several authors have developed and studied different concepts of summability. Chen and Zheng introduced in [19] (strongly) Lipschitz p -integral and p -nuclear operators. In [16] Chávez-Domínguez introduced the notion of Lipschitz (r, p, q) -summing operators and Lipschitz (q, p) -mixing in [17]. Independently, Yahi, Achour and Rueda [59] and Saadi [57] introduced and studied the class of Lipschitz strongly p -summing operators. The first authors introduced the notion of summing Lipschitz conjugates and (p, σ) -summability with an appropriate factorization. They characterized also those Lipschitz operators whose Lipschitz conjugates are absolutely p -summing. The chapter is organized as follows. In section 1, we recall some basic definitions and properties concerning Banach lattices and sublinear operators. In section 2, we study the Lipschitz p -summing sublinear operators and we introduce the class of Lipschitz super p -summing sublinear operators. We characterize this type of operators by giving a domination theorem. Also, we give some properties concerning this class. We study the relationship between T and u in ∇T concerning this type of summability and we answer negatively a

question posed in [5] relative to the linear case, which is: T is p -summing if, and only if, u is p -summing for all u in ∇T ? The first implication was shown in [5] and the reciprocal remained open. Here we answer this negatively. Finally, we look in section 3 at the other types of summability. We are interested to the notion of Lipschitz p -dominated operators introduced in [19]. We show that if T is Lipschitz p -dominated sublinear operator, then $\nabla T \subset \Pi_p(X, Y)$. We prove in Proposition 5.3.6 below, if T is in $\mathcal{D}_p^+(X, Y)$ (the space of positive strongly p -summing sublinear operators) for $1 < p \leq \infty$, we have u positive strongly p -summing for all u in ∇T and hence u^* is positive p^* -summing with $\pi_{p^*}^+(u^*) \leq 2d_p^+(T)$. We also demonstrate that u is positive p -summing for all u in ∇T ($1 < p \leq 2$) whenever T is in $\mathcal{D}_p^+(X, L_2(\Omega, \mu))$ with $\pi_p^+(u) \leq Cd_p^+(T)$.

5.1 Definitions, notations and properties

Unless otherwise stated X, Y, Z will always real Banach spaces. As customary, \mathcal{B}_X denotes the closed unit ball of X and X^* its topological linear dual. Now, we are going to introduce some terminology concerning the Banach lattices. For more details, the interested reader can consult the references [35, 41].

We recall the abstract definition of Banach lattice. Let X be a Banach space. If X is a vector lattice and $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$ ($|x| = \sup\{x, -x\}$) we say that X is a Banach lattice. If the lattice is complete, we say that X is a complete Banach lattice. Note that this implies obviously that for any $x \in X$ the elements x and $|x|$ have the same norm. We denote by $X_+ = \{x \in X : x \geq 0\}$. An element x of X is positive if $x \in X_+$.

The dual X^* of a Banach lattice X is a complete Banach lattice endowed with the natural order

$$x_1^* \leq x_2^* \iff \langle x_1^*, x \rangle \leq \langle x_2^*, x \rangle, \quad \forall x \in X_+ \quad (5.1)$$

where $\langle \cdot, \cdot \rangle$ denotes the bracket of duality.

By a sublattice of a Banach lattice X we mean a linear subspace E of X so that $\sup \{x, y\}$ belongs to E whenever $x, y \in E$. The canonical embedding $i : X \rightarrow X^{**}$ such that $\langle i(x), x^* \rangle = \langle x^*, x \rangle$ of X into its second dual X^{**} is an order isometry from X onto a sublattice of X^{**} , see [35, Proposition 1.a.2]. If we consider X as a sublattice of X^{**} we have for $x_1, x_2 \in X$

$$x_1 \leq x_2 \iff \langle x_1, x^* \rangle \leq \langle x_2, x^* \rangle, \quad \forall x^* \in X_+^*. \quad (5.2)$$

Let K be a compact space. The space $C(K)$ is a Banach lattice, it is a complete Banach lattice if K is a Stonian compact space. The L_p ($1 \leq p \leq \infty$) are complete Banach lattices and any reflexive Banach lattice is a complete Banach lattice.

For the convenience of the reader, we give in this section some elementary definitions and fundamental properties relative to sublinear operators. For more details see [5, 33].

Definition 5.1.1. A mapping T from a Banach space X into a Banach lattice Y is said to be sublinear if for all x, y in X and $\lambda \in \mathbb{R}_+$, we have

$$\begin{aligned} (i) \quad T(\lambda x) &= \lambda T(x) && \text{(i.e., positively homogeneous),} \\ (ii) \quad T(x + y) &\leq T(x) + T(y) && \text{(i.e., subadditive).} \end{aligned}$$

The operator T is said to be superlinear if T is positively homogeneous and superadditive (i.e., $T(x + y) \geq T(x) + T(y)$ for all x, y in X (i.e., $-T$ is sublinear). Note that the sum of two sublinear operators is a sublinear operator and the multiplication by a positive number is also a sublinear operator. Let us denote by

$$\mathcal{SL}(X, Y) = \{ \text{sublinear mappings } T : X \rightarrow Y \}$$

and we equip it with the natural order induced by Y

$$T_1 \leq T_2 \iff T_1(x) \leq T_2(x), \quad \forall x \in X. \quad (5.3)$$

We will note by ∇T the subdifferential of T , which is the set of all linear operators $u : X \rightarrow Y$ such that $u(x) \leq T(x)$ for all x in X . We know (see [5]) that ∇T is not empty if Y is a complete Banach lattice and $T(x) = \sup \{u(x) : u \in \nabla T\}$, moreover, the supremum is attained (i.e., for all x in X there is $u_x \in \nabla T$ such that, $T(x) = u_x(x)$). If Y is simply a Banach lattice then ∇T is empty in general (see [33]). As a consequence

$$u \in \nabla T \iff -T(-x) \leq u(x) \leq T(x), \quad \forall x \in X. \quad (5.4)$$

Proposition 5.1.2. *Let T be a sublinear from a Banach space X into a Banach lattice Y .*

Then, the following properties are equivalent.

- (1) *The operator T is continuous on X .*
- (2) *The operator T is continuous in 0.*
- (3) *There is a constant $C > 0$ such that for all $x \in X$, $\|T(x)\| \leq C \|x\|$.*
- (4) *The operator T is Lipschitz i.e., $\|T(x) - T(y)\| \leq K \|x - y\|$, for some positive*

constant K .

Proof. The proof is in general like the linear case. For (3) implies (4), we have

$$|T(x) - T(y)| \leq |T(x - y)| + |T(y - x)|. \quad (5.5)$$

and this concludes the proof. □

In both cases, we say that T is bounded and we put

$$\|T\| = \sup\{\|T(x)\| : \|x\|_X = 1\}.$$

Immediately, we have for all $x \in X$

$$\|T(x)\| \leq \sup_{u \in \nabla T} \|u(x)\| \leq \|T(x)\| + \|T(-x)\| \quad (5.6)$$

and consequently

$$\|T\| \leq \sup_{u \in \nabla T} \|u\| \leq 2 \|T\|. \quad (5.7)$$

This gives the following result: let C be a positive constant, then the operator T is bounded and $\|T\| \leq C$ if, and only if, for all $u \in \nabla T$, $\|u\| \leq C$.

We will denote by $\mathcal{SB}(X, Y)$ the set of all bounded sublinear operators from X into Y . The set $\mathcal{SB}(X, Y)$ is a pointed positive convex cone of $\text{Lip}_0(X, Y)$ but not salient and the great vector space in $\mathcal{SB}(X, Y)$ is

$$\mathcal{SB}(X, Y) \cap (-\mathcal{SB}(X, Y)) = \mathcal{B}(X, Y).$$

We denote also by $\text{Lip}_0(X, Y)$ the Banach space of Lipschitz functions $T : X \rightarrow Y$ such that $T(0) = 0$ with pointwise addition and the Lipschitz norm $\text{Lip}(\cdot)$ is given by

$$\text{Lip}(T) = \sup_{x \neq y} \frac{\|T(x) - T(y)\|}{\|x - y\|}.$$

We use the shorthand $X^\# := \text{Lip}_0(X) := \text{Lip}_0(X, \mathbb{R})$. The closed unit ball $\mathcal{B}_{X^\#}$ of $X^\#$ is a compact Hausdorff space for the topology of pointwise convergence on X . We can define a preorder on $\text{Lip}_0(X, Y)$ by setting

$$T \leq S \text{ if } S - T \in \mathcal{SB}(X, Y).$$

Remark 5.1.3. We have by Proposition 5.1.2 for $T \in \mathcal{SB}(X, Y)$

$$\|T\| \leq \text{Lip}(T) \leq 2 \|T\|.$$

In addition if T is symmetric (i.e., $T(x) = T(-x)$ for all x in X), then

$$\|T\| = \text{Lip}(T).$$

In the sequel, we can see $\mathcal{SB}(X, Y)$ as a cone in $(\text{Lip}_0(X, Y), \text{Lip}(\cdot))$. We denote by

$$\Delta \mathcal{SB}(X, Y) = \mathcal{SB}(X, Y) - \mathcal{SB}(X, Y)$$

the subspace of $\text{Lip}_0(X, Y)$ spanned by $\mathcal{SB}(X, Y)$, i.e.

$$\Delta\mathcal{SB}(X, Y) = \{T_1 - T_2 : T_1, T_2 \in \mathcal{SB}(X, Y)\}.$$

Remark 5.1.4. For all T in $\Delta\mathcal{SB}(X, Y)$ there is $\varphi_T \in \mathcal{SB}(X, Y)$ and ψ_T super linear such that $\varphi_T \leq T \leq \psi_T$ and $\varphi_T(-x) = \varphi_{-T}(x)$ (resp. $\psi_T(-x) = \psi_{-T}(x)$) for all x in X . We define φ_T, ψ_T by

$$\psi_T(x) = T_1(x) + T_2(-x) \quad \varphi_T(x) = -T_1(-x) - T_2(x).$$

where $T = T_1 - T_2$.

5.2 Lipschitz p -summing sublinear operators

We start by giving some standard notations. We denote by $\|\cdot\|_p$ the norm on l_p of a sequence of real numbers. For a sequence of vectors $(x_i)_i$ in a Banach space X , its strong p -norm is the l_p -norm of the sequence $(\|x_i\|)_i$ and we denote its weak p -norm (cf. [22]) by

$$\omega_p((x_i)_i) = \sup_{x^* \in \mathcal{B}_{X^*}} \|(x^*(x_i))_i\|_p.$$

We denote respectively these spaces by $l_p(X)$ and $l_p^\omega(X)$ ($l_p^n(X)$ and $l_p^{n\omega}(X)$ if we take finite sequences $(x_i)_{1 \leq i \leq n} \subset X$). We know (see [24]) that $l_p(X) \equiv l_p^\omega(X)$ (the symbol \equiv indicates that two Banach spaces are isometrically isomorphic) for some $1 \leq p < \infty$ if, and only if, $\dim(X)$ is finite. If $p = \infty$, we have $l_\infty(X) \equiv l_\infty^\omega(X)$. We have also if $1 < p \leq \infty$, $l_p^\omega(X) \equiv \mathcal{B}(l_{p^*}, X)$ isometrically (where p^* is the conjugate of p i.e. $\frac{1}{p} + \frac{1}{p^*} = 1$). In other words, let $v : l_{p^*} \rightarrow X$ be a linear operator such that $v(e_i) = x_i$ (namely $v = \sum_1^\infty e_j \otimes x_j$, e_j denotes the unit vector basis of l_{p^*}) then $\|v\| = \|(x_n)\|_{l_p^\omega(X)}$.

Analogously for a sequence $(\lambda_i)_i$ of real numbers and $(x_i)_i, (x'_i)_i$ of points in X , we denote their weak Lipschitz p -norm (not really a norm because there is no linear structure) by

$$\omega_p^L((\lambda_i), (x_i)_i, (x'_i)_i) = \sup_{f \in \mathcal{B}_{X^\#}} \left\| \left(|\lambda_i|^{\frac{1}{p}} (f(x_i) - f(x'_i)) \right)_i \right\|_p. \quad (5.8)$$

Inspired by the useful concept of absolutely summing operators, J. Farmer and W.-B. Johnson introduced in [26] the following definition. This is a good generalization of the concept of linear p -summing operators, since it is shown in [26] that the Lipschitz p -summing norm of a linear operator is the same as its p -summing norm.

Definition 5.2.1. A Lipschitz operator T between X, Y is called Lipschitz p -summing ($1 \leq p < \infty$), if there is a positive constant C such that for all n in \mathbb{N} , $(x_i)_{1 \leq i \leq n}, (y_i)_{1 \leq i \leq n}$ in X and $(a_i)_{1 \leq i \leq n}$ in \mathbb{R}^+ , we have

$$\left\| \left(a_i^{\frac{1}{p}} (T(x_i) - T(y_i)) \right)_i \right\|_p \leq C \omega_p^L((a_i), (x_i)_i, (y_i)_i). \quad (5.9)$$

We denote by $\Pi_p^L(X, Y)$ the Banach space of all Lipschitz p -summing operators from X into Y ($1 \leq p < \infty$). The Lipschitz p -summing ($1 \leq p < \infty$) norm, $\pi_p^L(T)$ of T is the smallest constant C verifying (5.9).

Notice that for any embedding $j : Y \rightarrow Z$, we have $\pi_p^L(T) = \pi_p^L(jT)$ and

$$\pi_p^L(T) = \sup_{X_0 \subset X} \left\{ \pi_p^L(T|_{X_0}) : X_0 \text{ finite dimensional subspace of } X \right\}.$$

Also, we can omit in the definition the sequence $(a_i)_{1 \leq i \leq n}$ in \mathbb{R}_+ and we restrict to $a_i = 1$ (see [26] for an implicit proof).

Now, we proceed to generalize (5.7) to the notion of Lipschitz p -summing operators, i.e., we study the following question: let C be a positive constant, then the operator $T \in \mathcal{SB}(X, Y)$ is Lipschitz p -summing and $\pi_p^L(T) \leq C$ if, and only if, for all $u \in \nabla T$, $\|\pi_p(u)\| \leq C$? This question is difficult for us, but we can resolve it

partially by introducing a new intermediary notion of summability which we call "Lipschitz super p -summing operators". Also, we answer negatively a question posed in [5] in the linear case, i.e., if T is " p -summing" $1 \leq p < \infty$ in the sense of definition below (Definition 5.2.2), then for all u in ∇T , u is p -summing and the reciprocal is false.

Definition 5.2.2. A map T in $\Delta\mathcal{SB}(X, Y)$ is called Lipschitz super p -summing ($1 \leq p < \infty$), if there is a positive constant C such that for all n in \mathbb{N} , $(x_i)_{1 \leq i \leq n}$, $(y_i)_{1 \leq i \leq n}$ in X and $(a_i)_{1 \leq i \leq n}$ in \mathbb{R}^+ , we have

$$\left\| \left(a_i^{\frac{1}{p}} T(x_i - y_i) \right) \right\|_p \leq C \omega_p^L((a_i), (x_i)_i, (y_i)_i). \quad (5.10)$$

We denote by $\Pi_p^{Ls}(X, Y)$ the space of the Lipschitz super p -summing ($1 \leq p < \infty$) in $\Delta\mathcal{SB}(X, Y)$ and by $\pi_p^{Ls}(T)$, the Lipschitz super p -summing norm of T ; which is the smallest constant C verifying (5.10).

Notice that for any embedding $j : Y \rightarrow Z$, we have $\pi_p^{Ls}(T) = \pi_p^{Ls}(jT)$ and

$$\pi_p^{Ls}(T) = \sup_{X_0 \subset X} \left\{ \Pi_p^{Ls}(T/X_0) : X_0 \text{ finite dimensional space of } X \right\}.$$

Also, we can omit in the definition the sequence $(a_i)_{1 \leq i \leq n}$ in \mathbb{R}_+ and we restrict to $a_i = 1$.

Remark 5.2.3. By (5.5). Let $T \in \mathcal{SB}(X, Y)$ if T is in $\Pi_p^{Ls}(X, Y)$ then T is in $\Pi_p^L(X, Y)$ and $\pi_p^L(T) \leq 2\pi_p^{Ls}(T)$. The converse is false see Example 1 below.

Now we are ready to give the Pietsch domination theorem. The proof follows in an analogous way of [26, Theorem 1].

Theorem 5.2.4. Let $1 \leq p < \infty$. Let X be a Banach space and Y be a Banach lattice. The following properties are equivalent for T in $\Delta\mathcal{SB}(X, Y)$ and a positive constant C .

- (a) The mapping T is Lipschitz super p -summing and $\pi_p^{Ls}(T) \leq C$.

(b) There is a probability μ on $\mathcal{B}_{X^\#}$ such that

$$\|T(x - y)\| \leq C \left(\int_{\mathcal{B}_{X^\#}} |f(x) - f(y)|^p d\mu(f) \right)^{\frac{1}{p}} \quad (5.11)$$

for all x, y in X .

Proof. The proof is the same than that used in the Lipschitz p -summing case. \square

As an immediate consequence, we have an inclusion result and some composition properties of our class.

Proposition 5.2.5. *Let $1 \leq p < q < \infty$. If $T : X \rightarrow Y$ is Lipschitz super p -summing, then T is Lipschitz super q -summing and $\pi_q^{Ls}(T) \leq \pi_p^{Ls}(T)$.*

Proposition 5.2.6. *Let $S : E \rightarrow X$ be a bounded linear function and T in $\Pi_p^{Ls}(X, Y)$. Then, $T \circ S$ is in $\Pi_p^{Ls}(E, Y)$ and $\pi_p^{Ls}(T \circ S) \leq \pi_p^{Ls}(T) \|S\|$.*

Proof. Let x, y be in X . Then

$$\begin{aligned} & \| (T \circ S)(x - y) \| \\ &= \| T(S(x) - S(y)) \| \\ &\leq \pi_p^{Ls}(T) \left(\int_{\mathcal{B}_{X^\#}} |f(S(x)) - f(S(y))|^p d\mu(f) \right)^{\frac{1}{p}} \\ &\leq \pi_p^{Ls}(T) \|S\| \left(\int_{\mathcal{B}_{E^\#}} |g(x) - g(y)|^p d\mu(g) \right)^{\frac{1}{p}}. \end{aligned}$$

Where $g(x) = \frac{f(S(x))}{\|S\|}$. Therefore, $T \circ S$ is Lipschitz super p -summing and

$$\pi_p^{Ls}(T \circ S) \leq \pi_p^{Ls}(T) \|S\|.$$

This ends the proof. \square

Proposition 5.2.7. *Consider S in $\Pi_p^{Ls}(X, Y)$ and let u be in $\mathcal{B}^+(Y, Z)$ (Z is a Banach lattice). Then, $u \circ S$ is in $\Pi_p^{Ls}(Y, Z)$ and $\pi_p^{Ls}(u \circ S) \leq \|u\| \pi_p^{Ls}(S)$.*

Proof. Let x, y be in X . Then

$$\begin{aligned} \|(u \circ S)(x - y)\| &\leq \|u\| \|S(x - y)\| \\ &\leq \|u\| \pi_p^{Ls}(S) \left(\int_{B_{X^\#}} |f(x) - f(y)|^p d\mu(f) \right)^{\frac{1}{p}}. \end{aligned}$$

This implies that, $u \circ S$ is Lipschitz super p -summing and $\pi_p^{Ls}(u \circ S) \leq \|u\| \pi_p^{Ls}(S)$. □

Proposition 5.2.8. *Let X be a Banach and Y be a complete Banach lattice. Let T be a bounded sublinear operator from X into Y . Suppose that T is Lipschitz super p -summing ($1 \leq p < \infty$). Then for all $S \in \mathcal{SB}(X, Y)$ such that $S \leq T$, S is Lipschitz super p -summing.*

Proof. By (5.3), we have for all x, y in X , $S(x - y) \leq T(x - y)$ and $-S(x - y) \leq T(y - x)$. Thus $|S(x - y)| \leq |T(x - y)| + |T(y - x)|$. Using (5.11), we get $\pi_p^{Ls}(S) \leq 2\pi_p^{Ls}(T)$. □

Corollary 5.2.9. *If T is Lipschitz super p -summing ($1 \leq p < \infty$), then for all $u \in \nabla T$, u is p -summing.*

Let now $T : X \rightarrow Y$ be a sublinear operator between a Banach space X and a Banach lattice Y .

Definition 5.2.10. We will say that T is “ p -summing” ($1 \leq p < \infty$, we write $T \in \Pi_p(X, Y)$), if there exists a positive constant C such that for every n in \mathbb{N} the mappings

$$\begin{aligned} T_n : l_p^n \omega(X) &\longrightarrow l_p^n(Y) \\ (x_i) &\longmapsto l_p^n(Y) \end{aligned}$$

are uniformly bounded by C . We put in this case

$$\pi_p(T) = \sup_n \|T_n\|.$$

If T is linear, then from the closed graph theorem, it is p -summing if, and only if, it satisfies that for every infinite sequence $(x_n)_{n \in \mathbb{N}}$ in X , we have

$$\left(\sum_{n \in \mathbb{N}} |\langle \xi, x_n \rangle|^p < \infty, \quad \forall \xi \in X^* \right) \implies \sum_{n \in \mathbb{N}} \|T(x_n)\|^p < \infty. \quad (5.12)$$

Proposition 5.2.11. *Let u be a bounded linear operator from a Banach space X into a Banach lattice Y and $1 \leq p < \infty$. Then, the notions of Lipschitz super p -summing, Lipschitz p -summing and p -summing coincide and $\pi_p(u) = \pi_p^L(u) = \pi_p^{Ls}(u)$.*

Proof. We have $\pi_p(u) = \pi_p^L(u)$ by [26, Theorem 2] and $\pi_p^L(u) = \pi_p^{Ls}(u)$ because u is linear. □

By the weak Dvoretzky-Rogers Theorem which state that: every infinite dimensional Banach space X contains a weakly p -summable ($1 \leq p < \infty$) sequence which fails to be strongly p -summable (see [24, Theorem 2.18]). We then have the following corollary.

Corollary 5.2.12. *Let X be a Banach lattice. Then, the following properties are equivalent.*

- (1) *The identity is in $\Pi_p^{Ls}(X, X)$*
- (2) *The identity is in $\Pi_p^L(X, X)$.*
- (3) *The identity is in $\Pi_p(X, X)$.*
- (4) *The space X is of finite dimensional.*

Proof. The assertions (1) (2) and (3) are equivalent by Proposition 5.2.11. The assertion (2) is equivalent to (4) by the weak Dvoretzky-Rogers Theorem. □

We study in the Lipschitz case, the following question posed in [5]: Does u is Lipschitz p -summing (i.e., p -summing) for every u in ∇T if, and only if, T is p -summing sublinear operator (this question was inspired by [eqn6])? (For the

linear case, we have shown that, if T is p -summing then u is p -summing for every u in ∇T . The converse has remained open. In this paper, we ask this question and we prove that the reciprocal is false by the following example.

Example 5.2.13. Let X be a Banach space of infinite dimensional and consider the map $T: X \rightarrow \mathbb{R}$ defined by $T(x) = \|x\|$. The operator T is a bounded sublinear operator. The set

$$\nabla T = \{u \in \mathcal{B}(X, \mathbb{R}) \subset \mathcal{B}_{X^*}, u \leq T\}$$

is in $\Pi_p(X, \mathbb{R})$ and $\pi_p(u) \leq \|T\|$. But T is not p -summing. Indeed, suppose the contrary, then there is an absolute positive constant C such that for all n in \mathbb{N} and for all $(x_i)_{1 \leq i \leq n}$ in X , we have

$$\sum_{i=1}^n \|x_i\|^p \leq C^p \sup_{\xi \in \mathcal{B}_{X^*}} \sum_{i=1}^n |\xi(x_i)|^p.$$

This implies that the Id_X is p -summing and by (5.12), we have for every infinite sequence (x_i) in X

$$\left(\sum_{n \in \mathbb{N}} |\langle \xi, x_n \rangle|^p < \infty, \quad \forall \xi \in X^* \right) \implies \sum_{n \in \mathbb{N}} \|x_n\|^p < \infty.$$

This contradicts the weak Dvoretzky-Rogers Theorem.

Remark 5.2.14. Let X be a Banach space and Y be a Banach lattice. The notions of p -summing and Lipschitz p -summing do not coincide on $\mathcal{SB}(X, Y)$. The opposite makes our problem trivial.

Let X be a Banach space of infinite dimensional. The sublinear operator T defined on X by $T(x) = \|x\|$ is not p -summing by Example 5.2.13 but is Lipschitz p -summing. Indeed, for $(x_i)_{1 \leq i \leq n}, (y_i)_{1 \leq i \leq n}$ in X , we have

$$\begin{aligned}
 & \left(\sum_{i=1}^n |T(x_i) - T(y_i)|^p \right)^{\frac{1}{p}} \\
 = & \left(\sum_{i=1}^n \left| \|x_i\| - \|y_i\| \right|^p \right)^{\frac{1}{p}} \\
 = & \left(\sum_{i=1}^n |f_0(x_i) - f_0(y_i)|^p \right)^{\frac{1}{p}} \\
 & \text{(where } f_0(\cdot) = \|\cdot\|; \text{ which is a Lipschitz function and } \text{Lip}(f_0) \leq 1) \\
 \leq & \sup_{f \in \mathcal{B}_{X\#}} \left(\sum_{i=1}^n |f(x_i) - f(y_i)|^p \right)^{\frac{1}{p}}.
 \end{aligned}$$

We can add that the sublinear operator T is not super Lipschitz p -summing. In fact, suppose that there exists $C > 0$ such that for every $\{x_i\}_{1 \leq i \leq n}, \{y_i\}_{1 \leq i \leq n}$ in X , we have

$$\begin{aligned}
 & \left(\sum_{i=1}^n |T(x_i - y_i)|^p \right)^{\frac{1}{p}} \\
 = & \left(\sum_{i=1}^n \|x_i - y_i\|^p \right)^{\frac{1}{p}} \\
 \leq & C \sup_{f \in \mathcal{B}_{X\#}} \left(\sum_{i=1}^n |f(x_i) - f(y_i)|^p \right)^{\frac{1}{p}}.
 \end{aligned}$$

This implies that Id_X is Lipschitz p -summing and consequently by Corollary [5.2.12](#) is p -summing. This contradicts the weak Dvoretzky-Rogers.

The following proposition studies the link between $u \in \nabla T$ and T , for T Lipschitz super p -summing.

Proposition 5.2.15. *Let X be an arbitrary Banach space and Y be a complete Banach lattice. Consider T in $\mathcal{SB}(X, Y)$. For $1 \leq p < \infty$, if T is in $\Pi_p^{Ls}(X, Y)$, then $\nabla T \subset \Pi_p^{Ls}(X, Y)$. The converse is false in general.*

Proof. Let T be in $\Pi_p^{Ls}(X, Y)$. We have

$$\begin{aligned}
 \|u(x - y)\| & \leq \|T(x - y)\| + \|T(y - x)\| \\
 & \leq 2\pi_p^{Ls}(T) \left(\int_{\mathcal{B}_{X\#}} |f(x) - f(y)|^p d\mu(f) \right)^{\frac{1}{p}}.
 \end{aligned}$$

Hence u is $\Pi_p^{Ls}(X, Y) = \Pi_p(X, Y)$ and $\pi_p(u) \leq 2\pi_p^{Ls}(T)$. According to the previous result the opposite is false.

Consider $T: X \rightarrow \mathbb{R}$ defined $T(x) = \|x\|$. Then T is a bounded sublinear operators. We have

$$\nabla T = \left\{ \begin{array}{l} u \in \mathcal{B}(X, \mathbb{R}) \subset \\ X^*, \quad u \leq T \end{array} \right\} \subset \Pi_p(X, \mathbb{R}) \subset \Pi_p^{Ls}(X, \mathbb{R}).$$

Thus $\nabla T \subset \Pi_p^{Ls}(X, \mathbb{R})$ but T is not Lipschitz super p -summing. Suppose that T is Lipschitz super p -summing, then there is a positive constant C such that for every n in \mathbb{N} , $(x_i)_{1 \leq i \leq n}, (y_i)_{1 \leq i \leq n}$ in X ; we have

$$\begin{aligned} \left(\sum_{i=1}^n |T(x_i - y_i)|^p \right)^{\frac{1}{p}} &= \left(\sum_{i=1}^n \|x_i - y_i\|^p \right)^{\frac{1}{p}} \\ \text{(by Corollary 5.2.12)} &\leq \sup_{f \in \mathcal{B}_{X^\#}} \|(f(x_i) - f(y_i))_i\|_p. \end{aligned}$$

This implies that $\dim(X)$ is finite and hence contradiction. □

problem. For which spaces X, Y we have $\Pi_p^{Ls}(X, Y) = \Pi_p^L(X, Y)$?

problem Let X be a Banach lattice (for example $X = L_q$). Is the Lipschitz operator

$$\begin{aligned} T: X &\longrightarrow X \\ x &\longmapsto T(x) = |x| \end{aligned}$$

Lipschitz p -summing for some p ?

5.3 Relationship between T and $u \in \nabla T$ for other types of summability

We give now the notion of Lipschitz p -dominated operators introduced by D. Chen and B. Zheng in [17]. A Lipschitz mapping $T: X \rightarrow Y$ between Banach spaces X and Y is Lipschitz p -dominated ($1 \leq p < \infty$) if there exist a Banach

space Z and a linear operator L in $\Pi_p(X, Z)$ such that

$$\| T(x) - T(y) \| \leq \| L(x - y) \|, \quad \forall x, y \in X. \quad (5.13)$$

The class of all Lipschitz p -dominated operators between X and Y is denoted by $\mathcal{D}_p^L(X, Y)$. For T in $\mathcal{D}_p^L(X, Y)$, we set $d_p^L(T)$ to be the infimum of $\pi_p(L)$, the infimum being taken over all the above Z and L . Let T sublinear operator, if $T(0) = 0$ we have the following reformulation

$$\| T(x - y) \| \leq \| L(x - y) \|, \quad \forall x, y \in X.$$

Indeed, if we replace in (5.13) x by $x - y$ and y by 0, we will have $\| T(x - y) \| \leq \| L(x - y) \|$, $\forall x, y \in X$. For the converse, by (5.5) we have $\| T(x) - T(y) \| \leq \| 2L(x - y) \|$, $\forall x, y \in X$.

Proposition 5.3.1. *Let X be an arbitrary Banach space and Y be a complete Banach lattice. Consider T in $\mathcal{SB}(X, Y)$. If T is in $\mathcal{D}_p^L(X, Y)$ for $1 \leq p < \infty$ then, T is p -summing in the sense of Definition 5.2.10 and hence T is Lipschitz super p -summing, which implies that T is Lipschitz p -summing.*

Proof. From [17, Theorem 3.2], there is a regular Borel probability measure μ on \mathcal{B}_{X^*} such that for all $x, y \in X$, we have

$$\| T(x) - T(y) \| \leq d_p^L(T) \left(\int_{\mathcal{B}_{X^*}} |\xi(x) - \xi(y)|^p d\mu(\xi) \right)^{\frac{1}{p}}. \quad (5.14)$$

If we take $y = 0$, we obtain $\| T(x) \| \leq d_p^L(T) \left(\int_{\mathcal{B}_{X^*}} |\xi(x)|^p d\mu(\xi) \right)^{\frac{1}{p}}$. Replace in the precedent inequality x by $x - y$, we get

$$\begin{aligned} \| T(x - y) \| &\leq d_p^L(T) \left(\int_{\mathcal{B}_{X^*}} |\xi(x - y)|^p d\mu(\xi) \right)^{\frac{1}{p}}, \\ &\leq d_p^L(T) \left(\int_{\mathcal{B}_{X^*}} |\xi(x) - \xi(y)|^p d\mu(\xi) \right)^{\frac{1}{p}}. \end{aligned}$$

By using Remark 5.2.3, we finish the last implication. □

Proposition 5.3.2. *Let X be an arbitrary Banach space and Y be a complete Banach lattice. Consider T in $\mathcal{SB}(X, Y)$. Then, if T is in $\mathcal{D}_p^L(X, Y)$ for $1 \leq p < \infty$, we have $\nabla T \subset \mathcal{D}_p^L(X, Y)$ and consequently $\nabla T \subset \Pi_p^L(X, Y)$.*

Proof. Using (5.5) and (5.6), we have for all u in ∇T

$$\|u(x - y)\| \leq \|T(x - y)\| + \|T(y - x)\|$$

and hence by (5.14)

$$\begin{aligned} \|u(x - y)\| &\leq 2d_p^L(T) \left(\int_{\mathcal{B}_{X^*}} |\xi(x - y)|^p d\mu(\xi) \right)^{\frac{1}{p}}, \\ (\text{because } T(0) = 0) &\leq 2d_p^L(T) \left(\int_{\mathcal{B}_{X^*}} |\xi(x) - \xi(y)|^p d\mu(\xi) \right)^{\frac{1}{p}}. \end{aligned}$$

This implies that u is p -summing and consequently is Lipschitz p -summing. \square

The definition of "Lipschitz strongly p -summing operators" was introduced independently by [57] and [59]. We deduce in the same spirit from that used in, [32] the following definition.

Definition 5.3.3. A Lipschitz map $T : X \rightarrow Y$ between a Banach space X and a Banach lattice Y is *lattice Lipschitz strongly p -summing* ($1 < p \leq \infty$) if there is a constant $C > 0$, such that for all $n \in \mathbb{N}$, $(x_i)_{1 \leq i \leq n}$, $(x'_i)_{1 \leq i \leq n}$ in X , $(y_i^*)_{1 \leq i \leq n}$ in Y_+^* and $(\lambda_i)_{1 \leq i \leq n}$ in \mathbb{R}_+ , we have Y^{**}

$$\sum_{i=1}^n \lambda_i |\langle T(x_i) - T(x'_i), y_i^* \rangle| \leq C \left(\sum_{i=1}^n \lambda_i \|x_i - x'_i\|^p \right)^{\frac{1}{p}} \sup_{y^{**} \in \mathcal{B}_{Y^{**}}^+} \|(\langle y_i^*, y^{**} \rangle)\|_{l_p^n} \quad (5.15)$$

If T sublinear, then this definition is equivalent by (5.5) to

$$\sum_{i=1}^n \lambda_i |\langle T(x_i - x'_i), y_i^* \rangle| \leq C \left(\sum_{i=1}^n \lambda_i \|x_i - x'_i\|^p \right)^{\frac{1}{p}} \sup_{y^{**} \in \mathcal{B}_{Y^{**}}^+} \|(\langle y_i^*, y^{**} \rangle)\|_{l_p^n}$$

We denote by $\mathcal{D}_{st,p}^{+L}(X, E)$ the class of all latticially Lipschitz strongly p -summing operators from X into E and $d_{st,p}^{+L}(T)$ the smallest C such that (5.15) holds. This generalizes the definition introduced by [20] in the linear case. If T is linear, then we have $\mathcal{D}_{st,p}^L(X, E) = \mathcal{D}_p(X, E)$ because $\mathcal{B}_{X^\#}$ is not involved in the definition.

Remark 5.3.4. Blasco in [11] introduces this definition under the name positive strongly p -summing but the sup of second member is taken on all the ball $\mathcal{B}_{Y^{**}}$. This two definitions are the same.

Indeed, we have for a Banach lattice X and $(x_i)_{1 \leq i \leq n} \subset X^+$

$$\begin{aligned} \sup_{x^* \in \mathcal{B}_{X^*}} \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}} &= \sup_{x^* \in \mathcal{B}_{X^*}} \sup_{(\alpha_i) \in \mathcal{B}_{l_p^*}} \left(\left| \sum_{i=1}^n \alpha_i \langle x_i, x^* \rangle \right| \right) \\ &= \sup_{x^* \in \mathcal{B}_{X^*}} \sup_{(\alpha_i) \in \mathcal{B}_{l_p^*}} \left(\left\langle \sum_{i=1}^n \alpha_i x_i, x^* \right\rangle \right) \\ &= \sup_{(\alpha_i) \in \mathcal{B}_{l_p^+}} \sup_{x^* \in \mathcal{B}_{X^*}} \left(\left\langle \sum_{i=1}^n \alpha_i x_i, x^* \right\rangle \right) \\ &= \sup_{(\alpha_i) \in \mathcal{B}_{l_p^*}} \sup_{x^* \in \mathcal{B}_{X^*_+}} \left(\left\langle \sum_{i=1}^n \alpha_i x_i, x^* \right\rangle \right) \\ &= \sup_{x^* \in \mathcal{B}_{X^*_+}} \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Let $T \in \text{Lip}_0(X; E)$ and $v : l_p^n \rightarrow Y^*$ be a linear operator (\implies bounded). The Lipschitz operator T is strongly Lipschitz p -summing if, and only if,

$$\sum_{i=1}^n \lambda_i |\langle T(x_i) - T(x'_i), v(e_i) \rangle| \leq C \left(\sum_{i=1}^n \lambda_i \|x_i - x'_i\|^p \right)^{\frac{1}{p}} \|v\|.$$

This is equivalent (see [57] and [59]) to Pietsch's domination theorem; which is: there exist a constant $C > 0$ and a Radon probability μ on $\mathcal{B}_{Y^{**}}$ such that for all $x, x' \in X$ and $y^* \in Y^*$, we have

$$|\langle T(x) - T(x'), y^* \rangle| \leq C \|x - x'\| \|y^*\|_{L_p(\mathcal{B}_{Y^{**}}, \mu)} \quad (5.16)$$

Moreover, in this case

$$d_{st,p}^L(T) = \inf \{C > 0 : \text{for all } C \text{ verifying (5.16)}\}.$$

We give the following definition introduced in [4] and its domination theorem.

Definition 5.3.5. Let $1 \leq p \leq +\infty$. A linear operator $u : X \rightarrow Y$ between a Banach space X and a Banach lattice Y is positive strongly p -summing if there exists a constant $C > 0$, such that for all finite sets $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y_+^*$, we have

$$\sum_{i=1}^n |\langle u(x_i), y_i^* \rangle| \leq C \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} \sup_{y^{**} \in \mathcal{B}_{Y^{**}}^+} \left(\sum_{i=1}^n \langle y_i^*, y^{**} \rangle^{p^*} \right)^{\frac{1}{p^*}}. \quad (5.17)$$

Where $\mathcal{B}_{Y^{**}}^+ = \{y^{**} \in \mathcal{B}_{Y^{**}} : y^{**} \geq 0\} = \mathcal{B}_{Y^{**}} \cap Y_+^{**}$.

The class of all positive strongly p -summing operators between X and Y is denoted by $\mathcal{D}_p^+(X, Y)$. The infimum of all the constant C in (5.17) defines the norm d_p^+ on $\mathcal{D}_p^+(X, Y)$. We have $\mathcal{D}_1^+(X, Y) = \mathcal{B}(X, Y)$.

The reformulation in continuous term by [4, Theorem 4.13] is : there exists a positive constant $C > 0$ and Radon probability measure μ on $\mathcal{B}_{Y_+^{**}}$ such that for all $x \in X$ and $y^* \in Y^*$, we have

$$|\langle u(x), y^* \rangle| \leq C \|x\| \left(\int_{\mathcal{B}_{Y^{**}}^+} \|y^*\| (y^{**})^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}}. \quad (5.18)$$

Moreover, in this case

$$d_p^+(u) = \inf \{C, \text{satisfying (5.18)}\}.$$

Proposition 5.3.6. Let X be an arbitrary Banach space and Y be a complete Banach lattice. Consider T in $\mathcal{SB}(X, Y)$. Then, if T is in $\mathcal{D}_p^+(X, Y)$ for $1 < p \leq \infty$, we have u positive strongly p -summing for all u in ∇T and hence u^* is positive p^* -summing and $\pi_{p^*}^+(u^*) \leq 2d_p^+(T)$.

Proof. We have by (5.2) $\langle u(x), y^* \rangle \leq \langle T(x), y^* \rangle$ and $\langle u(-x), y^* \rangle \leq \langle T(-x), y^* \rangle$ for all y^* in Y_+^* . This implies that $|\langle u(x), y^* \rangle| \leq \langle T(x), y^* \rangle + \langle T(-x), y^* \rangle$. And hence by using (5.18)

$$|\langle u(x), y^* \rangle| \leq 2d_p^+(T) \|x\| \left(\int_{\mathcal{B}_{Y^{**}}} |y^*(y^{**})|^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}}$$

for all y^* in Y_+^* . Consider $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y^*$. By using Hölder inequality, we have

$$\begin{aligned} & \sum_{i=1}^n |\langle u(x_i), y_i^* \rangle| \\ & \leq 2d_p^+(T) \sum_{i=1}^n \|x_i\| \left(\int_{\mathcal{B}_{Y^{**}}} |y_i^*(y^{**})|^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}} \\ & \leq 2d_p^+(T) \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n \int_{\mathcal{B}_{Y^{**}}} |y_i^*(y^{**})|^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}} \\ & \leq 2d_p^+(T) \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} \left(\sup_{y^{**} \in \mathcal{B}_{Y^{**}}} \sum_{i=1}^n \int_{\mathcal{B}_{Y^{**}}} |y_i^*(y^{**})|^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}} \\ & \leq 2d_p^+(T) \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} \left(\sup_{y^{**} \in \mathcal{B}_{Y^{**}}} \sum_{i=1}^n |y_i^*(y^{**})|^{p^*} \right)^{\frac{1}{p^*}} \\ & \leq 2d_p^+(T) \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} \left(\sup_{y^{**} \in \mathcal{B}_{Y^{**}}} \sum_{i=1}^n (y_i^*(y^{**}))^{p^*} \right)^{\frac{1}{p^*}}. \end{aligned}$$

We immediately have u positive strongly p -summing from (5.17) and therefore $d_p^+(u) \leq 2d_p^+(T)$. The characterization (5.18) of positive strongly p -summing linear operators yields for all $x \in X$ and $y^* \in Y^*$ that

$$|\langle u(x), y^* \rangle| \leq 2d_p^+(T) \|x\| \left(\int_{\mathcal{B}_{Y^{**}}^+} (|y^*(y^{**})|)^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}}$$

and thus

$$|\langle x, u^*(y^*) \rangle| \leq 2d_p^+(T) \|x\| \left(\int_{\mathcal{B}_{Y^{**}}^+} (|y^*(y^{**})|)^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}}.$$

Hence

$$\|u^*(y^*)\| \leq 2d_p^+(T) \left(\int_{\mathcal{B}_{Y^{**}}^+} (|y^*|(y^{**}))^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}}.$$

This implies by [4, Proposition 3.4] that u^* is positive p^* -summing and $\pi_{p^*}^+(u^*) \leq 2d_p^+(T)$. \square

For the convenience of the reader, we recall the classical Khintchine inequality. Let $D = \{-1, +1\}^{\mathbb{N}}$ equipped with its uniform probability measure μ and its Borel σ -algebras \mathcal{B} (μ is the normalized Haar measure on the compact group D). Let $(\varepsilon_i, i \geq 1)$ be a sequence of independent ± 1 -valued random variables on (D, \mathcal{B}, μ) with $\mu\{\omega : \varepsilon_i(\omega) = +1\} = \mu\{\omega : \varepsilon_i(\omega) = -1\} = \frac{1}{2}$ ($\varepsilon_i : D \rightarrow \{-1, +1\}$ is the i -th coordinate). These are Bernoulli random variables. Let now $0 < p < \infty$ and let $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. Then, the Khintchine inequality is

$$A_p \left(\sum_{i=1}^n |\alpha_i|^2 \right)^{\frac{1}{2}} \leq \left(\int_D \left| \sum_{i=1}^n \alpha_i \varepsilon_i(\omega) \right|^p d\mu(\omega) \right)^{\frac{1}{p}} \leq B_p \left(\sum_{i=1}^n |\alpha_i|^2 \right)^{\frac{1}{2}} \quad (5.19)$$

for some constants $A_p, B_p > 0$ depending only on p . It is clear to see that $A_2 = B_2 = 1$ and $A_p = 1$ when $p \geq 2$, $B_p = 1$ when $p \leq 2$.

Theorem 5.3.7. *Let X be an arbitrary Banach space. Consider T in $\mathcal{SB}(X, L_2(\Omega, \mu))$. Then, if T is in $\mathcal{D}_p^+(X, L_2(\Omega, \mu))$ for $1 < p \leq 2$, we have u positive p -summing for all u in ∇T and $\pi_p^+(u) \leq 2A_1^{-1}B_p d_p^+(T)$. Where A_1^{-1}, B_p are the constants of Khintchine inequality.*

Proof. Start first by $L_2(\Omega, \mu) = l_2^N$ ($N \in \mathbb{N}$) and consider $u : X \rightarrow l_2^N$ in ∇T ,

where $T \in \mathcal{SB}(X, l_2^N)$. Given $(x_i)_{i=1}^n \subset X_+$. We have

$$\begin{aligned}
 & \sum_{i=1}^n \|u(x_i)\| \\
 & ((e_j) \text{ is the canonical basis of } l_2^N) \\
 &= \sum_{i=1}^n \left(\sum_{j=1}^N |\langle u(x_i), e_j \rangle|^2 \right)^{\frac{1}{2}} \\
 &= \sum_{i=1}^n \left(\int_D \left| \sum_{j=1}^N \langle u(x_i), e_j \rangle \varepsilon_j(\omega) \right|^2 d\mu(\omega) \right)^{\frac{1}{2}} \\
 &\leq A_1^{-1} \int_D \left(\sum_{i=1}^n \left| \left\langle x_i, \sum_{j=1}^N u^*(e_j) \varepsilon_j(\omega) \right\rangle \right| \right) d\mu(\omega) \\
 &\leq A_1^{-1} \int_D \left(\sum_{i=1}^n \left\| \sum_{j=1}^N u^*(e_j) \varepsilon_j(\omega) \right\| \left| \left\langle x_i, \frac{\sum_{j=1}^N u^*(e_j) \varepsilon_j(\omega)}{\left\| \sum_{j=1}^N u^*(e_j) \varepsilon_j(\omega) \right\|} \right\rangle \right| \right) d\mu(\omega)
 \end{aligned}$$

We have by Proposition [5.3.6](#) that u^* is positive p^* -summing and $2d_{st,p}^L(T)$. Thus, by [\[60\]](#) and [\[4, Proposition 3.4\]](#) there is a regular Borel-Radon probability ν measure on $\mathcal{B}_{l_2^N}^+$ such that

$$\begin{aligned}
 & \left(\int_D \left\| \sum_{j=1}^N u^*(e_j) \varepsilon_j(\omega) \right\|^{p^*} d\mu(\omega) \right)^{\frac{1}{p^*}} \\
 &= \left(\int_D \left\| u^* \left(\sum_{j=1}^N e_j \varepsilon_j(\omega) \right) \right\|^{p^*} d\mu(\omega) \right)^{\frac{1}{p^*}} \\
 &\leq \pi_{p^*}^+(u^*) \left(\int_D \int_{\mathcal{B}_{l_2^N}^+} \left\langle \left| \sum_{j=1}^N e_j \varepsilon_j(\omega) \right|, \alpha \right\rangle^{p^*} d\mu(\omega) d\mu(\alpha) \right)^{\frac{1}{p^*}} \\
 &\leq 2d_p^+(T) \left(\int_D \int_{\mathcal{B}_{l_2^N}^+} \left\langle \left| \sum_{j=1}^N e_j \varepsilon_j(\omega) \right|, \alpha \right\rangle^{p^*} d\mu(\omega) d\mu(\alpha) \right)^{\frac{1}{p^*}}
 \end{aligned}$$

Using Fubini's theorem and Khintchine inequality [5.19](#), we get

$$\begin{aligned}
 & \left(\int_D \int_{\mathcal{B}_{l_2^N}^+} \left\langle \left| \sum_{j=1}^N e_j \varepsilon_j(\omega) \right|, \alpha \right\rangle^{p^*} d\mu(\omega) d\nu(\alpha) \right)^{\frac{1}{p^*}} \\
 \leq & \left(\int_{\mathcal{B}_{l_2^N}^+} \int_D \left\langle \left| \sum_{j=1}^N e_j \varepsilon_j(\omega) \right|, \alpha \right\rangle^{p^*} d\mu(\omega) d\nu(\alpha) \right)^{\frac{1}{p^*}} \\
 \leq & \left(\int_{\mathcal{B}_{l_2^N}^+} \int_D \left| \left\langle \sum_{j=1}^N e_j \varepsilon_j(\omega), \alpha \right\rangle \right|^{p^*} d\mu(\omega) d\nu(\alpha) \right)^{\frac{1}{p^*}} \\
 \leq & B_{p^*} \left(\int_{\mathcal{B}_{l_2^N}^+} \left(\sum_{j=1}^N |\langle e_j, \alpha \rangle|^2 \right)^{\frac{p^*}{2}} d\nu(\alpha) \right)^{\frac{1}{p^*}} \\
 \leq & B_{p^*}.
 \end{aligned}$$

In conclusion, the preceding inequalities yield $\pi_1^+(u) \leq 2A_1^{-1}Bd_p^+(T)$ and hence $\pi_p^+(u) \leq 2A_1^{-1}Bd_p^+(T)$. This ends the proof. \square

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الملخص. في هذه الأطروحة، قدمنا ودرسنا المؤثرات متعددة الخطية (p, G) -جمعية، (\tilde{p}, G) -جمعية، المطلق ميد p -جمعية، ضعيف ميد p -جمعية. كذلك قمنا بدراسة بعض الخصائص المتعلقة بين المؤثرات تحت الخطية الليبشيتزية p -جمعية وتحت تفاضلاتها. العديد من الخصائص تم إعطاؤها.

الكلمات المفتاحية: المؤثرات (p, G) -جمعية، المؤثرات (\tilde{p}, G) -جمعية، المؤثرات ميد p -جمعية مطلقا، المؤثرات ضعيف ميد p -جمعية، مؤثر تحت خطي، المؤثرات متعددة الخطية، مؤثر ليبشيتزي، نظرية الهيمنة والتفكيك

Résumé : Dans cette thèse, nous avons introduit et étudié les opérateurs multilinéaires (p, G) -sommants, (\tilde{p}, G) -sommants, absolument mid p -sommants et faiblement mid p -sommants. Nous avons aussi étudié certaines propriétés relationnelles entre les opérateurs sous linéaires Lipschitz p -sommant et leurs sous différentiels. Plusieurs propriétés de caractérisation ont été données.

Les mots clés: opérateurs (p, G) -sommants, opérateurs (\tilde{p}, G) -sommants, opérateurs absolument mid- p -sommants, opérateurs faiblement mid- p -sommant, opérateur sous linéaire, opérateur multi-linéaire, opérateur Lipschitzien, théorème de domination-factorisation.

Abstract : In this thesis, we have introduced and studied the multilinear operators (p, G) -summing, (\tilde{p}, G) -summing, absolutely mid- p -summing and weakly mid- p -summing. We have also studied some connexion properties between the sublinear Lipschitz p -summing operators and their subdifferentials. Several characterization properties have been given.

Key words: (p, G) -summing operators, (\tilde{p}, G) -summing operators, absolutely mid- p -summing operators, weakly mid- p -summing operators, sublinear operator, multilinear operator, Lipschitz operator, domination-factorization