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Groups and symmetries

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List of Symbols

Notation	Name
S_n	Permutation group
$\mathbb{Q}$	Rational numbers
$\mathbb{R}$	Real numbers
$< X >$	Subgroup generated by subset X
$o(g)$	The order of the element g in the groups
$[G : H]$	The number of distinct left or right cosets of H in G
$Kerf$	The kernel of the homomorphism f
$G_1 \cong G_2$	G_1 and G_2 are isomorphic
A_n	Alternating group
G_x	Stabilizer of x
O_x	The orbit of x
$M_{m \times n}$	The set of all $m \times n$ matrices
$GL_n(\mathbb{R})$	The set of all $n \times n$ invertible matrices
C_n	Cyclic group
D_n	Dihedral Group

Introduction

Symmetry is a vast significant subject. For example, in sculpture and certain types of architectural decoration are obtained using symmetry. Crystals are classified according to the symmetry properties they prosses.

In mathematics, the concept of symmetry does not still in geometric thinking and transform many of these ideas into abstract concepts as a group theories(Galois theory is concerned with symmetries at the roots of polynomial...).

In this memory, the Group theory is used to study the notion of symmetry.

In the first chapter, we introduce some definitions and notions about groups, symmetric group, and notions of groups acting on sets. The second chapter, is concerned with the orthogonal transformations. Finally, the last chapter is about some symmetry groups in 2 and 3 dimensions.

Dedication

In the name of Allah, the Most Gracious, the Most Merciful.
This work is wholeheartedly dedicated to my beloved parents who taught me to be unique and determined.

To my brothers and sisters who shared their words of advice and encouragement to help accomplish this study.

To all those who taught me throughout my educational journey. To my lifelong companions,

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Chapter 1

Notions of Groups

1.1 Groups

Definition 1.1.1 *(Group)*

A ***group*** *is a set* G *with a binary operation* $*$*, such that:*

1. $x * y \in G$ *for all* $x, y \in G$*, that is the operation* $*$*, is* ***closed****.*
2. $x * (y * z) = (x * y) * z$ *for all* $x, y, z \in G$*; that is, the operation* $*$ *is* ***associative****.*
3. *There exist an element* $e \in G$*, called the* ***identity element****, such that:*

   $$\text{for all } x \in G\text{: } x * e = e * x = x.$$

4. *For each element* $x \in G$*, there exists an* ***inverse element*** *in* G *, denoted* x^{-1} *such that :*

   $$x^{-1} * x = x * x^{-1} = e$$

The group G *with the operation* $*$ *is written* $(G, *)$*.*
A group $(G, *)$ *is said to be* ***abelian*** *if it also satisfies* $x * y = y * x$ *for all* $x, y \in G$ *that is, if the operation is commutative.*

Example 1.1.2

1. *The integers* $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ *under addition form a groups, in which the identity element is the integer* 0, *and the inverse of* x *is* $-x$.

2. *The nonzero rational numbers* $\mathbb{Q}^*$ *under multiplication form a group that is written as* $(\mathbb{Q}^*, \times)$. *Similarly,* $(\mathbb{R}^*, \times)$ *and* $(\mathbb{C}^*, \times)$ *are also groups. In each of them the identity element is 1, and the inverse element of* x *is* $1/x$.

Example 1.1.3

1. *The groups* $(\mathbb{Z}, +)$, $(\mathbb{Q}^*, \times)$, *are abelian groups.*

2. *The group* $(M_2(\mathbb{R}), +)$, *is abelian group*
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$

   The identity is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, *and the inverse of* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is* $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$

Theorem 1.1.4

Givenv a group $(G, *)$

1. *The identity element of group* $(G, *)$ *is unique.*

2. *An element* $x \in G$, *there is only one element* $y \in G$ *such that:*

$$y * x = x * y = e$$

Proof.

1. Suppose that there are two identities elements of $(G, *)$. Call them e, $e^{'}$ such that $x * e = e * x = x$ and $x * e^{'} = e^{'} * x = x$ for every $x \in G$. Then consider $e^{'} * e$. Using the fact that e is an identity element, $e * e^{'} = e^{'}$. Similarly, using the fact that $e^{'}$ is an identity element, $e * e^{'} = e$. Therefore $e = e^{'}$, so the identity element e of the group $(G, *)$is unique.

2. Suppose that there are two element y and $y^{'}$ in G with the property that $y * x = x * y = e$ and $y^{'} * x = x * y^{'} = e$.

$$\begin{aligned} \text{Then } y' &= y' \cdot e && \text{where } e \text{ is the identity} \\ &= y' \cdot (x \cdot y) && \text{since } x \cdot y = e \\ &= (y' \cdot x) \cdot y && \text{the associativity property} \\ &= e \cdot y && \text{since } y' \cdot x = e \\ &= y && \text{where } e \text{ is the identity} \end{aligned}$$

□

1.1.1 Subgroups

Definition 1.1.5 *(Subgroup)*

Given a group G under a binary operation $*$, a subset H of G is called a ***subgroup*** *of G if H also forms a group under the operation $*$.*

We use the notation $H \leqslant G$ to means that H is a subgroup of G.

Definition 1.1.6

- *The* ***trivial subgroup*** *of any group is the subgroup $\{e\}$.*
- *The* ***proper subgroup*** *of a group G is a subgroup H which is a proper subset of G (that is $H \neq G$).*

Theorem 1.1.7

Let G be a group, and H be a nonempty subset of G, H is a subgroup of G if, and only if

1. *$x * y \in H$ for all $x, y \in H$*
2. *$x^{-1} \in H$ for each $x \in H$*

Proof.

- If H is a subgroup of G, the assertions (1) and (2) are clearly verified .
- Reciprocally

The first condition, that $x * y \in H$, makes sure that H is closed under the operations in G.

Suppose that $x, y, z \in H$, then as H is a subset of G, then $x, y, z \in G$.

But $x * (y * z) = (x * y) * z$ since G is a group. Thus associativity in H is inherited from associativity in G.

For all element x of H we have $x^{-1} \in H$ then, after the first condition $x * x^{-1} \in H$. Such $x * x^{-1}$ is identity element of G, we deduce that the identity element of G is the same identity element of H.

Then H under the operation in G is a group . $\square$

Proposition 1.1.8

1. *The conditions (1) and (2) are obviously equivalent to the single condition $H \neq \emptyset$ whenever*

$$x, y \in H \Longrightarrow x * y^{-1} \in H$$

2. *If H is a subgroup of group G, and if K is a subgroup of a subgroup H, then K is a subgroup of G .*

Example 1.1.9

1. *The sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, are subgroups of $\mathbb{C}$ undre the addition.*
2. *The set $\{-1, 1\}$ is a subgroup of $(\mathbb{C}^*, \times)$.*

Theorem 1.1.10

Let $(S_i)_{i \in I}$ be a family of subgroups of the group G, with $I \neq \phi$. Then $\bigcap_{i \in I} S_i$ is a subgroup of G.

Proof.

$x, y \in \bigcap_{i\in I} S_i = T$. Since $e \in S_i$ for all $i \in I$, then $T \neq \phi$.

Then $x, y \in S_i$ for all $i \in I$. Thus $x * y, x^{-1} \in S_i$ for all $i \in I$, and so $x * y, x^{-1} \in T$.

Thus $\bigcap_{i\in I} S_i$ is a subgroup of G. □

1.1.2 Cyclic Groups

Definition 1.1.11

- *Let G be a group, and let X be a subset of G, then the smallest subgroup of G containing X is called* ***subgroup generated*** *by X, and is denoted $< X >$, and G is* ***finitely generated*** *if there is a finite subset X of G such that $G =< X >$.*
- *The group G is* **cyclic** *if there is an element $g \in G$ such that $G =< g >$.*

Proposition 1.1.12

Let H the subgroup generated by an singleton $\{g\}$, $g \in G$, then

$$< g >= \{g^n : n \in \mathbb{Z}\}$$

Proof.

The subgroup $< g >$ contained g, we have $< g >$ is closed under the operation of G, then contained $g.g = g^2$, $g^2.g = g^3$ and by recurrence g^n for all $n \in \mathbb{N}^*$.

But $(g^n)^{-1} = g^{-n} \in < g >$, whence $g^n \in < g >$ for all $n \in \mathbb{Z}$

$< g >$ must be contain all product of such g^n $(g^n \in < g >)$.

It should be clear that the set of such product is a subgroup of G contained g. □

Example 1.1.13

The group of units , $U(9) \subset \mathbb{Z}_9$ is a cyclic group .

$U(9)$ is $\{1, 2, 4, 5, 7, 8\}$. The element 2 is a generator for $U(9)$ since

$$2^1 = 2$$
$$2^2 = 4$$
$$2^3 = 8$$
$$2^4 = 7$$
$$2^5 = 5$$
$$2^6 = 1$$

Definition 1.1.14 *(Order of element, Order of group)*

Let G be a group and $g \in G$. If $< g >$ is finite, then the order of g is the number of elements in $< g >$. If is infinite, then the order of g is infinite. The order of g is denoted $o(g)$, and if $o(g)$ is infinite, we write $o(g) = \infty$.

The number of elements in a finite group G is called the order of G and denoted $o(G)$. If G is infinite, we write $o(G) = \infty$.

Example 1.1.15

1. *The number of bijections of $\{1, 2, 3..., n\}$ in $\{1, 2, 3..., n\}$ equal $n!$. The symmetric group S_n of order $n!$.*
2. *In $(\mathbb{Z}, +)$ all elements not null has infinite order .*
3. *In all groups G, the only element of order 1 is the identity element .*

1.2 Quotient Groups

1.2.1 Coset

Definition 1.2.1 *(Left coset, Right coset)*

Let H be a subgroup of G, and let $g \in G$, then $gH = \{gh : h \in H\}$ is the ***left coset*** *of H determined by g, and $Hg = \{hg : h \in H\}$ is the* ***right coset*** *of H determined by g.*

Example 1.2.2

For the group $\mathbb{Z}$ of integers under addition, the subgroup of all multiples of 3 has just the three cosets:

$$\{... - 9, -6, -3, 0, 3, 6, 9...\}$$
$$\{... - 8, -5, -2, 1, 4, ...\}$$
$$\{... - 7, -4, -1, 2, 5, 8, 11, ...\}$$

Proposition 1.2.3

Let H be a subgroup of group G, and let a and b be elements of G, the following conditions are equivalent:

1. $a \in bH$
2. $b \in aH$
3. $aH = bH$
4. $b^{-1}a \in H$
5. $a^{-1}b \in H$

Theorem 1.2.4 ***(Lagrange)***

Let H be a subgroup of a finite group G. Then the order of H divide the order of G (if $H \leq G$, then $|G| = |H|.[G : H]$ where $[G : H]$ is the number of distinct left cosets of H in G).

Proof.

Let G be a finite group and H a subgroup of G

1. The left cosets of H in G form a partition of G, so G can be written as a disjoint union :

   $G = g_1H \cup g_2H \cup ... \cup g_kH$ for a finite set of elements $g_1, g_2, ..., g_k \in G$.

2. define $\varphi : H \longrightarrow gH$. Then φ is clearly surjective . Now suppose that

   $\varphi(h_1) = \varphi(h_2)$, so $gh_1 = gh_2$ multiplying each side by g^{-1} on the left, we obtain $h_1 = h_2$,φ injective, then φ is a bijection. Hence |gH|=|H|.

   Finally, if G is finite, then we have $|G| = k.|H|$, and hence $|H|$ divides $|G|$, then $|G| = [G : H].|H|$

□

1.2.2 Normal Subgroups

Definition 1.2.5 *(Normal subgroup)*

A subgroup H of a group $(G, .)$ is called a ***normal subgroup*** *of G if $g^{-1}hg \in H$ for all $g \in G$ and $h \in H$.*

Proposition 1.2.6

$Hg = gH$, for all $g \in G$ if, and only if H is a normal subgroup of G.

Proof.

Suppose that $Hg = gH$. Then $hg \in Hg = gH$, for any $h \in H$. Hence $hg = gh_1$ for some $h_1 \in H$ and $g^{-1}hg = g^{-1}gh_1 = h_1 \in H$. Therefore, H is a normal subgroup.

Conversely, if H is normal, let $hg \in Hg$ and $g^{-1}hg = h_1 \in H$. Then $hg = gh_1 \in gH$ and $Hg \subseteq gH$. Also $ghg^{-1} = (g^{-1})^{-1}hg^{-1} = h_2 \in H$, since H is normal, so $gh = h_2g \in Hg$. Hence $gH \subseteq Hg$, and so $Hg = gH$. □

Example 1.2.7

Any subgroup of an abelian group is normal.

1.2.3 Quotient group

Definition 1.2.8 *(Quotient group)*

if H is a normal subgroup of $(G, .)$, the set of cosets $G/H = \{gH : g \in G\}$ forms a group $(G/H, .)$, where the operation is defined by $(g_1H).(g_2H) = (g_1.g_2).H$. This group is called the ***quotient group*** *or* ***factor group*** *of G by H.*

Example 1.2.9

Consider the normal subgroup $3\mathbb{Z}$ *of* $\mathbb{Z}$. *The cosets of* $3\mathbb{Z}$ *in* $\mathbb{Z}$ *are*

$$0 + 3\mathbb{Z} = \{..., -3, 0, 3, 6...\}$$

$$1 + 3\mathbb{Z} = \{..., -2, 1, 4, 7...\}$$

$$2 + 3\mathbb{Z} = \{..., -1, 2, 5, 8...\}$$

The group $\mathbb{Z}/3\mathbb{Z}$ *is given by the multiplication table below .*

$+$	$0+3\mathbb{Z}$	$1+3\mathbb{Z}$	$2+3\mathbb{Z}$
$0+3\mathbb{Z}$	$0+3\mathbb{Z}$	$1+3\mathbb{Z}$	$2+3\mathbb{Z}$
$1+3\mathbb{Z}$	$1+3\mathbb{Z}$	$2+3\mathbb{Z}$	$0+3\mathbb{Z}$
$2+3\mathbb{Z}$	$2+3\mathbb{Z}$	$0+3\mathbb{Z}$	$1+3\mathbb{Z}$

1.3 Group Homomorphisms

1.3.1 Group Homomorphism

Definition 1.3.1 *(Group Morphism)*

If $(G_1, .)$ *and* $(G_2, *)$ *are two groups, the function* $f : G_1 \to G_2$ *is called group* ***morphism*** *if*

$$f(a.b) = f(a) * f(b), \text{ for all } a, b \in G_1$$

Example 1.3.2

Let G *be a group and* $g \in G$. *Define a map* $h : \mathbb{Z} \to G$ *by* $h(n) = g^n$. *Then* h *is a group morphism, since* $h(m+n) = g^{m+n} = g^m g^n = h(m)h(n)$, *for all* $n, m \in \mathbb{Z}$

Example 1.3.3 *Let* $G = GL_2(\mathbb{R})$ *if* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is in* G , *then the determinant is non zero; that is,* $det(A) = ad - bc \neq 0$. *Also for any two elements* A *and* B *in* G $det(A \times B) = det(A)det(B)$. *Using the determinant, we can define a homomorphism* $det : GL_2 \to \mathbb{R}^*$ *by* $A \longmapsto det(A)$.

Definition 1.3.4 *(Kernel of a homomorphism)*

The ***Kernel*** *of a homomorphism* f *from a group* G_1 *to a group* G_2 *with identity* e *is the set* $\{x \in G_1 : f(x) = e\}$. *The kernel of* f *is denoted by* $Kerf$.

Example 1.3.5

The mapping f from $\mathbb{R}^*$ to $\mathbb{R}^*$, defined by $f(x) = |x|$, is a homomorphism with

$$Kerf = \{-1, 1\}$$

Proposition 1.3.6

Let $f : G_1 \to G_2$ be a homomorphism of groups. Then

1. *If e is the identity of G_1, then $f(e)$ is the identity of G_2.*

2. *For any element $g \in G_1$, $f(g^{-1}) = [f(g)]^{-1}$.*

3. *If H_1 is a subgroup of G_1, then $f(H_1)$ is a subgroup of G_2.*

4. *If H_2 is a subgroup of G_2, then $f^{-1}(H_2) = \{g \in G_1 : f(g) \in H_2\}$ is a subgroup of G_1.*

Proof.

1. Suppose that e and e' are identities of G_1 and G_2 respectively, then

$$e'f(e) = f(e) = f(ee) = f(e)f(e)$$

   Then, $f(e) = e'$.

2. This statement follows from the fact that

$$f(g^{-1})f(g) = f(g^{-1}g) = f(e) = e'$$

3. The set $f(H_1)$ is nonempty since the identity of G_2 is in $f(H_1)$. Suppose that H_1 is a subgroup of G_1 and let x and y be in $f(H_1)$. There exist elements $a, b \in H_1$ such that $f(a) = x$ and $f(b) = y$. Since $xy^{-1} = f(a)[f(b)]^{-1} = f(ab^{-1}) \in f(H_1)$, $f(H_1)$ is a subgroup of G_2.

4. Let H_2 be a subgroup of G_2 and define H_1 to be $f^{-1}(H_2)$; that is, H_1 is the set of all $g \in G_1$ such that $f(g) \in H_2$. The identity is in H_1 since $f(e) = e'$. If a and b are in H_1, then $f(ab^{-1})$=$f(a)[f(b)]^{-1}$ is in H_2 since H_2 is a subgroup of G_2. Therefore, $ab^{-1} \in H_1$ and H_1 is a subgroup of G_1 .

□

1.3.2 Group Isomorphisms

Definition 1.3.7 *(Group isomorphism)*

*An **isomorphism** is a group homomorphism that is bijective. For such a isomorphism $G_1 \to G_2$, we say that G_1 and G_2 are **isomorphic** and write $G_1 \cong G_2$.*

Example 1.3.8

Let G_1 be the real numbers under addition and let G_2 be the positive real numbers under multiplication. Then G_1 and G_2 are isomorphic under the mapping $\phi(x) = 2^x$. Certainly, ϕ is a function from G_1 to G_2. To prove that it is one-to-one, suppose that $2^x = 2^y$. Then $log_2 2^x = log_2 2^y$, and therefore $x = y$. For onto, we must find for any positive real number y some real number x such the $\phi(x) = y$, that is , $2^x = y$. Well, solving for x gives log_2 y. Finally,

$$\phi(x + y) = 2^{x+y} = 2^x.2^y = \phi(x)\phi(y)$$

For all x and y on G_1, so that ϕ is operation-preserving as well.

Theorem 1.3.9 ***(Fundamental Theorem on Homomorphisms)***

Given a homomorphism $f : G \to H$, the factor group $G/Kerf$ is isomorphic to $f(G)$.

Proof.

let K denoted $Kerf$ of the homomorphism f. We define a map
$\theta : G/K \to f(G)$ by $\theta(gK) = f(g)$ for all $g \in G$. To verify that $\theta(gK)$ is determined uniquely, we need to show that $g'K = gK \Rightarrow f(g') = f(g)$. Indeed, if the cosets $g'K$ and gK are the same then $g' = gk$ for some $k \in K$. Hence

$$f(g') = f(gk) = f(g)f(k) = f(g)e_H = f(g)$$

The fact that θ is a homomorphism of groups will follow from the definition of the factor group. For any cosets g_1K and g_2K of the subgroup K, we have

$$\theta((g_1K)(g_2K)) = \theta(g_1g_2K) = f(g_1g_2) = f(g_1)f(g_2) = \theta((g_1K)\theta(g_2K))$$

By construction, θ is surjective. To prove injectivity, we need to show that $f(g') = f(g) \Rightarrow g'K = gK$. Let $a = g^{-1}g'$. If $f(g') = f(g)$, then

$$f(a) = f(g^{-1})f(g') = (f(g))^{-1}f(g') = (f(g))^{-1}f(g) = e_H. \text{ hence } a \in K.$$

Consequently, $g' = ga \in gK$ so that $g'K = gK$. Thus θ is bijective. □

1.4 Symmetric Group

1.4.1 Permutation Group

Definition 1.4.1 *(Symmetric Group)*

*The **symmetric group**, defined over any set is the group whose elements are all the bijections from the set to it self, and whose group operation is the composition of functions. In particular, the finite symmetric group S_n defined over a finite set $X = \{1, 2, 3..., n\}$ consists of the permutations that can be performed on the n symbols.*

Proposition 1.4.2

$(S_n, \circ)$ *is a group. This group called the symmetric group .*

Proof.

- Let $f, g \in S_n$, then the composition $f \circ g$ is an application of X in X and also is a bijection. Then $f \circ g$ is a permutation of X (S_n intern with composition). Composition of maps functions is associative, which makes the operation also associative.

- The identity element of S_n is the identity application $f \circ Id_X(x) = f(x) = Id_X \circ f(x)$ then $Id_X \circ f = f \circ Id_X$.

- Like f is bijective then the inverse application exist and is a permutation.

Then $(S_n, \circ)$ is a group. $\square$

Proposition 1.4.3

The order of S_n is $n!$.

Notation 1.4.4

- *δ is the permutation of the set $X = \{1, 2, ..., n\}$, then*

$$\delta = \begin{pmatrix} 1 & 2 & \cdots & n \\ \delta(1) & \delta(2) & \cdots & \delta(n) \end{pmatrix}$$

- *The identity element Id_X represent by*

$$Id_X = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$$

- *The inverse element of δ represent by*

$$\delta^{-1} = \begin{pmatrix} \delta(1) & \delta(2) & \cdots & \delta(n) \\ 1 & 2 & \cdots & n \end{pmatrix}$$

Example 1.4.5

Let the permutation $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$

The inverse of δ is $\delta^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$

Example 1.4.6

1. *For* $n = 2$ *then* $|S_2| = 2$

$$S_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$$

2. *For n=3 , then* $|S_3| = 6$

$$S_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$$

1.4.2 Composition of permutations

If $\pi, \delta \in S_n$ are two permutations, their product $\pi \circ \delta$ is the permutation obtained by applying δ first and then π. This agrees with our notation of composition of functions because $(\pi \circ \delta)(x) = \pi(\delta(x))$. (However, the reader should be aware that some others use the opposite convention in which π is applied first and then δ).

Example 1.4.7
for $\pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ *and* $\delta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ *are two elements of* S_3
calculate $\pi \circ \delta$, *we see that under* δ, *1 is mapping to 3 , and under* π, *3 is mapped to 2; thus under* $\pi \circ \delta$, *1 is mapped to 2. by the same way we find that under* $\pi \circ \delta$, *2 is mapped to 1, and under* $\pi \circ \delta$, *3 is mapped to 3 we see that :*

$$\pi \circ \delta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

In a similar way we can show that

$$\delta \circ \pi = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Theorem 1.4.8

If $n \geq 3$, *then* S_n *is non-Abelian*

Proof.

Let δ and π, are defined by

$\delta = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 3 & 2 & \cdots & n \end{pmatrix}$ and $\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 3 & 2 & 1 & \cdots & n \end{pmatrix}$

with each number after 3 mapped to itself in each case, then

$\pi \circ \delta = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 3 & 1 & 2 & \cdot & n \end{pmatrix}$ but $\delta \circ \pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 3 & 1 & \cdots & n \end{pmatrix}$

Thus $\pi \circ \delta \neq \delta \circ \pi$, and the group is non-Abelian .

□

1.4.3 Cycles and Transpositions

Definition 1.4.9 *(r-cycle)*

*If $a_1, a_2, \ldots, a_r$ are distinct elements of $\{1, 2, 3, \ldots, n\}$ the permutation $\delta \in S_n$; defined by : $\delta(a_1) = a_2$; $\delta(a_2) = a_3$; $\ldots$; $\delta(a_{r-1}) = a_r$; $\delta(a_r) = a_1$ and $\delta(x) = x$ if $x \notin \{a_1, a_2, ...a_r\}$ is called a cycle of length r or an r-**cycle** we denoted by $(a_1 a_2 \ldots a_r)$*

Example 1.4.10

- $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} = (1342)$ *, is an 4-cycle in S_4.*
- $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix} = (1243)(56)$ *,contenants a 2-cycle and 4-cycle in S_6.*
- *If $X = \{1, 2, 3, 4, 5\}$ and $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 1 & 4 \end{pmatrix}$ in S_5 .*
  Remarque that $\delta = (13254) = (32541) = (25413) = (41325) = (54132)$.

Proposition 1.4.11

An r-cycle in S_n has order r .

Proof. If $\delta = (a_1 a_2 \ldots a_r)$ is an r-cycle in S_n , then

$\delta(a_1) = a_2$

$\delta^2(a_1) = a_3$

$\delta^3(a_1) = a_4$

$\vdots$

$\delta^r(a_1) = a_1$

Similarly , $\delta^r(a_i) = a_i$ for $i = 1, 2, ..., r$. since δ^r fixes all the other elements , it is the identity permutations. But none of the permutations $\delta, \delta^2, ..., \delta^r$ equal the identity permutation because they all move the element a_i. Hence the order of δ is r.

□

Definition 1.4.12 *(disjoint cycles)*

Two cycles in S_n, $\delta = (a_1 a_2 \dots a_k)$ and $\tau = (b_1 b_2 \dots b_r)$ are ***disjoint*** *if $a_i \neq b_j$ for all i and j .*

Example 1.4.13

The cycles (135) *and* (274) *are disjoint; however, the cycles* (135) *and* (347) *are not.*

Proposition 1.4.14

Let δ and τ be two disjoint cycles in S_n. Then $\delta \circ \tau = \tau \circ \delta$.

Proof.

Let $\delta = (a_1 \cdots a_k)$ and $\tau = (b_1 \cdots b_l)$. We must show that $\delta \circ \tau(x) = \tau \circ \delta(x)$ for all $x \in \{1, 2, ..., n\}$. If x neither in $\{a_1, ..., a_k\}$ nor $\{b_1, ..., b_l\}$, then both δ and τ fix x. That is, $\delta(x) = x$ and $\tau(x) = x$. hence,

$$\delta\tau(x) = \delta(\tau(x)) = \delta(x) = x = \tau(x) = \tau(\delta(x)) = \tau\delta(x)$$

Suppose that $x \in \{a_1, \dots, a_k\}$. Then $\delta(a_i) = a_{(imodk)+1}$.

That is

$$\begin{gathered} a_1 \to a_2 \\ a_2 \to a_3 \\ \vdots \\ a_k \to a_1 \end{gathered}$$

However $\tau(a_i) = a_i$ since δ and τ are dijoint therefore

$$\begin{aligned}\delta\tau(a_i) &= \delta(a_i) \\ &= a_{(imod(k))+1} \\ &= \tau(a_{(imod(k)+1}) \\ &= \tau(\delta(a_i)) \\ &= \tau\delta(a_i)\end{aligned}$$

Similarly if $x \in \{b_1...., b_l\}$, then τ and δ also commute. □

Theorem 1.4.15

Every permutation in S_n can be written as cycle or as a product of disjoint cycles.

Example 1.4.16
$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 1 & 5 & 2 \end{pmatrix}$ *and* $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 5 & 6 & 4 \end{pmatrix}$
$\pi = (1624)$ *and* $\delta = (13)(456)$

$\pi\delta = (136)(245)$

$\delta\pi = (143)(256)$

Definition 1.4.17 *(Transpositions)*

A permutation δ which interchange two letters i and j and leaves all the others letters unchanged is called a ***transposition****. And we denoted by (ij) or $T_{(i;j)}$.*

Example 1.4.18

$\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix} = (24)$ *is transposition* $T_{(2;4)}$

Definition 1.4.19 *(Even and Odd permutation)*

- *A permutation is called an* ***even permutation*** *if it is the product of an even number of transpositions.*
- *A permutation is called an* ***odd permutation*** *if it is the product of an odd number of transpositions .*

Example 1.4.20

1. $(123) = (12)(23)$*, then* (123) *is an even permutation.*
2. $(1352) = (13)(35)(52)$*, then* (1352) *is an odd permutation.*

1.4.4 The Alternating Groups

Definition 1.4.21 *(The Alternating Groups)*

The set of all even permutations in S_n *is denoted by* A_n *and is called* ***alternating group*** *and is a subgroup of* S_n *.*

Example 1.4.22

The alternating group A_4 *consist of the following elements* $\{(1), (12)(34), (13)(24), (14)(23), (13)(12), (14)(12), (14)(13), (24)(23), (31)(32), (42)(41), (41)(43), (42)(43)\}$

Theorem 1.4.23

$|A_n| = \frac{n!}{2}$ *for every* $n \geq 2$*.*

Proof.

Let O_n denoted the set of odd permutations in S_n, so that $S_n = O_n \cup A_n$ and $O_n \cap A_n = \phi$. Hence $|A_n| + |O_n| = n!$. If we can show that there a bijection $f : A_n \to O_n$ they must contain the same number of elements. Fix a transposition $\sigma = (12) \in S_n$. Since $n \geq 2$ and define f by $f(\tau) = \sigma \circ \tau$ for all $\tau \in A_n$ ($\sigma \circ \tau$ is odd because τ is even and σ is odd). Then f is injective because $f(\tau) = f(\tau_1)$ implies that $\sigma \circ \tau = \sigma \circ \tau_1$, so $\tau = \tau_1$ by cancellations in S_n. To see that f is surjective, let $\lambda \in O_n$. Then $\sigma \circ \lambda \in A_n$ and $f(\sigma \circ \lambda) = \sigma \circ (\sigma \circ \lambda) = \lambda$ because $\sigma \circ \sigma = e$. Thus f is surjective, then $n! = 2|A_n|$. $\square$

Example 1.4.24

$|A_4| = \frac{4!}{2} = 12.$

Theorem 1.4.25 ***(Cayley's Theorem)***

Every group $(G, .)$ is isomorphic to a subgroup of its symmetric group $(S(G), \circ)$.

Proof.

For each element $g \in G$, define $\pi_g : G \to G$ by $\pi_g(x) = g.x$, we show that π_g is a bijection. It is surjective because, for any $y \in G$, $\pi_g(g^{-1}.y) = g.(g^{-1}.y) = y$. It is injective because $\pi_g(x) = \pi_g(y)$ implies that $gy = gx$, and so $y = x$ .Hence $\pi_g \in S(G)$.

Let $H = \{\pi_g \in S(G) : g \in G\}$. We show that $(H, \circ)$ is a subgroup of $(S(G), \circ)$ isomorphic to $(G, .)$. In fact, we show that the function $\psi : G \to H$ by $\psi(g) = \pi_g$ is a group isomorphism. This is clearly surjective. It is also injective because $\psi(g) = \psi(h)$ implies that $\pi_g = \pi_h$, and $\pi_g(e) = \pi_h(e)$ implies that $g = h$.

We show that ψ preserves the group operation. If $g, h \in G$,

$$\pi_{g.h}(x) = (g.h)(x) = g.(h.x) = \pi_g(h.x) = (\pi_g \circ \pi_h)(x)$$

and $\pi_{g.h} = \pi_g \circ \pi_h$. Also $\pi_{h^{-1}} \circ \pi_h = \pi_{h^{-1}.h} = \pi_e$; thus $(\pi_h)^{-1} = \pi_{h^{-1}} \in H$. Hence H is a subgroup of $S(G)$ and $\psi(g.h) = \psi(g) \circ \psi(h)$. $\square$

1.5 Actions of Groups

1.5.1 Definition and Examples

Definition 1.5.1

The group $(G, .)$ left acts on the set X if there is a map:

$$G \times X \to X$$

$$(g, x) \mapsto g.x$$

Such that :

1. $(g_1 g_2).x = g_1.(g_2.x)$ *for all* $g_1, g_2 \in G, x \in X$.

2. $e.x = x$ *if* e *is the identity of* G *and* $x \in X$.

If G acting on a set X, we say that X is a G-set.

Remark 1.5.2

A right action of a group G on a set X is a map $X \times G \to X$ such that :

1. $e.x = x$ *for all* $x \in X$.

2. $(x.g_1).g_2 = x.(g_1 g_2)$ *for all* $g_1, g_2 \in G, x \in X$.

For thereafter, we will only consider a groups acts on the left of set

Example 1.5.3

Let G a group and let X be the set of elements of G. we define the group action by:

$$\text{for } g, x \in G \ g.x = gxg^{-1} \ .$$

*The action in this case is usually called **conjugation***

for each $x \in G$

$$e.x = exe^{-1} = exe = x.$$

Now suppose $g_1, g_2 \in G$,$x \in G$, then

$$\begin{aligned} g_1.(g_2.x) = g_1(g_2.x)g_1^{-1} &= g_1(g_2xg_2^{-1})g_1^{-1} \\ &= (g_1g_2)x(g_2^{-1}g_1^{-1}) \\ &= (g_1g_2)x(g_2g_1)^{-1} \\ &= (g_1g_2).x \end{aligned}$$

then G acting on itself.

Example 1.5.4

Matrix groups acting on $\mathbb{R}^2$

For $GL_2(\mathbb{R})$ acting on $\mathbb{R}^2$ as follows : $(A, v) \mapsto Av$ for $A \in GL_2(\mathbb{R})$ and $v \in \mathbb{R}^2$

If $v \in \mathbb{R}^2$ and I is the identity matrix, then $Iv = v$. If A and B are 2×2 invertible matrices, then $(AB)v = A(Bv)$ since matrix multiplication is associative .

Example 1.5.5

G acts on the left cosets of a subgroup H

by :

$$G \times (G/H)_L \to (G/H)_L$$
$$(g, xH) \mapsto gxH$$

- $e.(xH) = exH = xH$
- *For g_1, g_2 inG*

$$\begin{aligned} g_1.(g_2.xH) = g_1(g_2xH) &= g_1g_2xH \\ &= (g_1g_2)xH \end{aligned}$$

1.5.2 Orbits and Stabilizers

Definition 1.5.6 *(Stabilizer)*

*Let X be a G-set. Let $x \in X$, then $G_x = Stabx = \{g \in G : gx = x\}$, called **Stabilizer** of x or **Isotropy** subgroup of x .*

Definition 1.5.7 *(Fixed point)*

Let X be a finite G-set and X_g the ***fixed point*** *set of g in X .*

$$X_g = \{x \in X : gx = x\}$$

Example 1.5.8

Let $X = \{1,2,3,4,5,6\}$ and suppose that G is the permutation group given by the permutations : $\{(1), (12)(3456), (35)(46), (12)(3654)\}$

$$G_1 = G_2 = \{(1), (35)(46)\}$$

$$G_3 = G_4 = G_5 = G_6 = \{(1)\}$$

$$X_{(1)} = \{1,2,3,4,5,6\}$$

$$X_{(12)(3456)} = \phi$$

Proposition 1.5.9

Let G be a group acting on a set X and $x \in X$. The stabilizer group G_x , of X is a subgroup of G.

Proof.

1. $e \in G_x$ since the identity fixes every element, in the set X.

2. Let $g, h \in G_x$. Then $gx = x$ and $hx = x$ so $(gh)x = g(hx) = gx = x$. Then

$$gh \in G_x$$

3. If $g \in G_x$. Then $x = ex = (g^{-1}g)x = g^{-1}(gx) = g^{-1}x$ so $g^{-1} \in G_x$.

Then $G_x \leq G$. □

Definition 1.5.10 *(Orbits)*

The set of all images of an element $x \in X$ under the acting group G is called the ***Orbit*** *of x under G, denoted O_x such: $O_x = \{gx : g \in G\}$*

Example 1.5.11

Let G be the permutation group defined by

$G = \{(1), (123), (132), (45), (123)(45), (132)(45)\}$, *and* $X = \{1, 2, 3, 4, 5\}$.

Then X is G-set.

$O_1 = O_2 = O_3 = \{1, 2, 3\}$

$O_4 = O_5 = \{4, 5\}$

Theorem 1.5.12

Let G be a finite group and X a finite G-set. If $x \in X$, then

$$|O_x| = [G : G_x]$$

Proof.

We know that $|\frac{G}{G_x}|$ is the number of left cosets of G_x in G by lagrange's theorem. We define a bijective map θ between the orbit O_x of X and the set of left coset L_{G_x} of G_x in G. Let $y \in O_x$. Then there exists a $g \in G$ such that : $y = gx$.

$$\theta : O_x \to L_{G_x}$$

$$y \mapsto gG_x$$

- We show that θ is well- defined and does not depend on our selection of g. Suppose that $h \in G$ such that $hx = y$. Then $gx = hx$ or $x = g^{-1}hx$; hence, $g^{-1}h$ is in the stabilizer subgroup of x. Therefore, $h \in gG_x$ or $gG_x = hG_x$. Thus, y gets mapped to the same coset regardless of the choice of the representative from that coset .

- assume that $\theta(x_1) = \theta(x_2)$. Then there exist $g_1, g_2 \in G$ such that $x_1 = g_1x$ and $x_2 = g_2x$, since there exist a $g \in G_x$ such that:$g_2 = g_1g$

$$x_2 = g_2x = g_1gx = g_1x = x_1$$

  then θ is injective.

- Let g G_x be the left coset. If $gx = y$, then $\theta(y) = gGx$. Finally θ is surjective, so $|O_x| = [G : G_x] = \frac{|G|}{|G_x|}$

□

Chapter 2

Matrices Groups and Transformations

2.1 Vector space , Inner product

2.1.1 Vector space

Definition 2.1.1 *(Vector space)*

*A **vector space** over a field F is a set V of objects, called vectors, on which two operations called addition and scalar multiplication have been be defined satisfying that :*

1. *V is an Abelian group with respect to addition .*
2. *(ab)v=a(bv)*
3. *(a+b)v=av+bv*
4. *ev=v*

For all $a, b \in F$ and all $v \in V$, e the unity of F.

Example 2.1.2

Let $V = M_{m\times n}$ be the set of all $m \times n$ matrices under the usual operations of addition of matrices and scalar multiplication.

Given matrices $A, B \in M_{m\times n}$ and a scalar α, we defined the sum $A+B$ by adding entry-by-entry, and αA by multiplying each entry of A by α. It is clear that the space $M_{m\times n}$ is closed under these two operations. The 0 vector in $M_{m\times n}$ is the matrix of size $m \times n$ having all entries equal to 0. It can be verified that all others properties of the definition of a vector space are also hold. Thus the set $M_{m\times n}$ is a vector space .

Definition 2.1.3 *(subspaces of vectors spaces)*

Let V be a vector space. A subset W of V is called a ***subspace*** *of V if it satisfies the following properties :*

1. *The zero vectors of V is also in W.*
2. *W is closed under addition, that is, if u and v are in W then $u + v$ is in W.*
3. *W is closed under scalar multiplication, that is, if u is in W and α is a scalar then αu is in W.*

Example 2.1.4

Let W be the graph of the function $f(x) = 2x$

$$W = \{(x, y) \in \mathbb{R}^2 : y = 2x\}$$

- *If $x = 0$ then $y = 0$ and therefor $0 = (0, 0)$ is in W .*
- *Let $u = (a, 2a)$ and $v = (b, 2b)$ be elements of w. Then*

$$\begin{aligned} u + v &= (a, 2a) + (b, 2b) \\ &= (a + b, 2a + 2b) \end{aligned}$$

  Then $u + v$ in W .

- *Let α be any scalar and let $u(a, 2a)$ be element of W. Then $\alpha u = (\alpha a, 2\alpha a)$, then αu in W.*

Then W is a subspace of $\mathbb{R}^2$.

2.1.2 Inner Product

Definition 2.1.5 *(Inner Product)*

Let $u = (u_1, u_2, \cdots, u_n)^t$ and let $v = (v_1, v_2, \cdots, v_n)^t$ be vectors in $\mathbb{R}^n$ ***the inner product*** *of u and v is*

$$< u, v >= uv^t = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

properties Let u, v, w be vectors in $\mathbb{R}^n$ and let α be a scalar. Then:

1. $< u, v > = < v, u >$.

2. $< u + v, w > = < u, w > + < v, w >$.

3. $< \alpha u, v > = < u, \alpha v >$.

4. $< u, u > \geq 0$ and if $< u, u > = 0$ if and only if $u = 0$.

Definition 2.1.6

The length *or* ***norm*** *of a vector* $u = (u_1, u_2, \cdots, u_n)^t \in \mathbb{R}^n$ *is defined as :*

$$\| u \| = \sqrt{< u, u >} = \sqrt{u_1^2 + u_2^2 + \cdots u_n^2}$$

Example 2.1.7

$\| (1, 0, 0, 0, 0, 0)^t \| = 1$

Definition 2.1.8

Let u *and* v *be vectors in* $\mathbb{R}^n$*. The distance between* u *and* v *is the length of the vector* $u - v$*. We will denote the* ***distance*** *between* u *and* v *by* $d(u, v)$ *.*

In other words $d(u, v) = \| u - v \|$

Definition 2.1.9 *(Orthogonality)*

Two vectors u *and* v *in* $\mathbb{R}^n$ *are said to be* ***orthogonal*** *if* $< u, v > = 0$.

Definition 2.1.10 *(Orthogonal Set)*

A set of vectors $\{u_1, u_2, \cdots, u_p\}$ *is said to be an orthogonal set if any pair of distinct vectors* u_i*,*u_j *are orthogonal, that is* $< u_i, u_j > = 0$ *when ever* $i \neq j$.

Definition 2.1.11 *(Orthonormal Set)*

A set of vectors $\{u_1, u_2, \cdots, u_p\}$ *is said* ***orthonormal set*** *if it is an orthogonal set and if each vector* u_i *in the set is a unit vector.*

Example 2.1.12

the set $S = \{u_1, u_2, u_3\} = \left(\begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \right)$ *is orthonormal set*

Definition 2.1.13 *(Euclidean Space)*

*A**Euclidean space** is a finite dimensional vector space over the reals* $\mathbb{R}$ *with an inner product.*

2.2 Matrices Groups

2.2.1 The General and the Special linear groups

Definition 2.2.1 *(The General linear group)*

The set of all $n \times n$ *invertible matrices forms a group called the **general linear group** we will denoted by* $GL_n(\mathbb{R})$ $=\{A \in M_n(\mathbb{R}), det(A) \neq 0\}$.

Definition 2.2.2 *(The Special linear group)*

*The **linear special group** of order n; denoted* $SL_n(\mathbb{R})$ *is :*

$$SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}), det(A) = 1\}$$

Example 2.2.3

$A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \in GL_2(\mathbb{R})$

2.2.2 The Orthogonal and Special Orthogonal groups

Definition 2.2.4 *(The Orthogonal group)*

- *A matrix A is a **orthogonal** if* $A^{-1} = A^t$.
- *The orthogonal group consists of the set of all orthogonal matrix, we will denoted by* $O(n)$ *for* $n \times n$ *orthogonal group.*

Example 2.2.5

- *The matrix* $A = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \in O_2(\mathbb{R})$
- *The matrix* $B = \frac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{pmatrix} \in O_3(\mathbb{R})$

Corollary 2.2.6

The determinant of an orthogonal matrix is ± 1.

Definition 2.2.7 *(Special Orthogonal Group)*

Special orthogonal group *of order* n *is the set of positive orthogonal matrices on* $M_n(\mathbb{R})$. *We call it* $SO_n(\mathbb{R})$.

2.3 Transformations

2.3.1 Linear Transformation

Definition 2.3.1 *(Linear Transformation)*

A vector function $T : \mathbb{R}^n \to \mathbb{R}^n$ *is called a* ***linear transformation****, or simply linear, if it satisfies the following conditions :*

1. *For all* $u, v \in \mathbb{R}^n$ *,we have* $T(u+v) = T(u) + T(v)$.
2. *For all* $v \in \mathbb{R}^n$ *and scalar* a *we have* $T(av) = aT(v)$.

Proposition 2.3.2

$T : \mathbb{R}^n \to \mathbb{R}^n$ *be a linear transformation if ,and only if there exists an* $n \times n$ *matrix* A *such that for all* $T(v) = Av$.

Proof.

- Suppose that T is linear transformation and consider the standard basis $\{e_1, e_2, \cdots, e_n\}$ of $\mathbb{R}^n$. For all i define $u_i = T(e_i)$ and let A be the matrix that has $u_1, u_2, \cdots, u_n$ as its columns . We claim that A is the desired matrix ie : $T(v) = Av$

holds for all $v \in \mathbb{R}^n$. Let $v = (x_1, x_2, \cdots, x_n)^t$ be some arbitrary element of $\mathbb{R}^n$. Then $v = x_1e_1 + x_2e_2 + \cdots + x_ne_n$, and we have

$$\begin{aligned} T(v) &= T(x_1e_1 + x_2e_2 + \cdots + x_ne_n) \\ &= x_1T(e_1) + x_2T(e_2) + \cdots + x_nT(e_n) \\ &= x_1u_1 + x_2u_2 + \cdots + x_nu_n \\ &= Av \end{aligned}$$

Then $A = \begin{pmatrix} | & | & \cdots & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & \cdots & | \end{pmatrix}$ ‘

- From the laws of matrix multiplication. Namely, by the distributive law , we have $A(v + w) = Av + Aw$, then $T(v + w) = T(v) + T(w)$. And by the compatibility of matrix multiplication and scalar multiplication, we have $A(av) = aAv$.

□

Example 2.3.3

If we let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the map given by

$$T(x_1, x_2) = (2x_1 + 5x_2, -4x_1 + 3x_2)$$

T is a linear transformation.

The column vectors $T(e_1) = (2, -4)^t$ and $T(e_2) = (5, 3)^t$, then T is given by the matrix $A = \begin{pmatrix} 2 & 5 \\ -4 & 3 \end{pmatrix}$

2.3.2 Orthogonal Transformation

Definition 2.3.4 *(Orthogonal Transformation)*

Let V be a vector space with an inner product. A linear transformation $T : V \to V$ is called an ***orthogonal transformation****, provided that T preserve length, that is, that $\|T(u)\| = \|u\|$ for all $u \in V$.*

Theorem 2.3.5

The following statements concerning a linear transformation

$T : V \to V$ *, where V is finite dimensional, are equivalent.*

1. *T is an orthogonal transformation.*

2. $< T(u), T(v) >=< u, v >$ *for all $u, v \in V$.*

3. *For some orthonormal basis $\{u_1, u_2, \cdots, u_n\}$ of V, the vectors $\{T(u_1), T(u_2), \cdots, T(u_n)\}$ are also form an orthonormal set .*

4. *The matrix A of T with respect to an orthonormal basis satisfies the condition $A^t.A = I$ where A^t is the matrix obtained from A by interchanging rows and columns (called the transpose of A) .*

Proof.

Statement (1) implies statement (2). We are given that $\|T(u)\| = \|u\|$ for all vectors $u \in V$. This implies that $< T(u), T(u) >=< u, u >$ for all vectors u.

Then $< T(u+v), T(u+v) >=< (u+v), (u+v) >$. We obtain $< T(u), T(u) > +2 < T(u), T(v) > + < T(v), T(v) >=< u, u > +2 < u, v > + < v, v >$. Since $< T(u), T(u) >=< u, u >$ and $< T(v), T(v) >=< v, v >$, the last equation implies that $< T(u), T(v) >=< u, v >$ for all u and v.

Statement (2) implies (3). Let $\{u_1, u_2, \cdots, u_n\}$ be an orthonormal basis of V; then $< u_i, u_i >= 1$, $< u_i, u_j >= 0$, $i \neq j$

By statement (2) we have $< T(u_i), T(u_i) >= 1$, $< T(u_i), T(u_j) >= 0$, $i \neq j$, and $\{T(u_1), T(u_2), \cdots, T(u_n)\}$ is othonormal set .

Statement (3) implies (1). Suppose that for some orthonormal basis $\{u_1, u_2, \cdots, u_n\}$ of V the image vectors $\{(T(u_1), T(u_2), \cdots, T(u_n)\}$ form an orthonormal set. Let

$$v = \xi_1 u_1 + \xi_2 u_2 + \cdots + \xi_n u_n$$

be an arbitrary vectors in V . Then

$$\|v\|^2 =< v, v >=< (\xi_1 u_1 + \xi_2 u_2 + \cdots + \xi_n u_n), (\xi_1 u_1 + \xi_2 u_2 + \cdots + \xi_n u_n) >= \sum_1^n \xi_i^2$$

since $\{u_1, u_2, \cdots, u_n\}$ is an orthonormal set . Similarly, we have

$$\|T(v)\|^2 =< T(v), T(v) >=< \sum_1^n \xi_i T(u_i), \sum_1^n \xi_i T(u_i) >= \sum_1^n \xi_i^2$$

Thus statement (1)is proved and we have shown the equivalence of the first three statements.

Finally, we prove that statements (3) and (4) are equivalent. Suppose that statement (3) holds and let$\{u_1, u_2, \cdots, u_n\}$ be an orthonormal basis of V. Let

$$T(u_i) = \sum_{j=1}^n \alpha_{ji} u_j$$

Since $\{T(u_1), T(u_2), \cdots, T(u_n)\}$ is orthonormal set, we have

$$< T(u_i), T(u_i) >=< \sum_{j=1}^n \alpha_{ji} u_j, \sum_{j=1}^n \alpha_{ji} u_j >= \sum_{j=1}^n \alpha_{ji}^2 = 1$$

and if $i \neq j$, $< T(u_i), T(u_j) >=< \sum_{k=1}^n \alpha_{ki} u_k, \sum_{k=1}^n \alpha_{kj} u_k >= \sum_{k=1}^n \alpha_{ki}\alpha_{kj} = 0$. These equations imply that $A^t.A = I$, since the (i, k)th entry of A^t is α_{ki}.

Conversely, $A^t.A = I$ implies that the equations above are satisfied and hence that $\{T(u_{1)}, T(u_2), \cdots, T(u_n)\}$ is orthonormal set.

□

Lemma 2.3.6

If $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry such that $\alpha(0) = 0$ then α is linear.

Proof.

let $\{\alpha(e_1), \alpha(e_2), \cdots, \alpha(e_n)\}$ is an orthonormal basis of $\mathbb{R}^n$. If $a \in \mathbb{R}$ and $v \in \mathbb{R}^n$, then

$$< \alpha(av) - a\alpha(v), \alpha(e_i) >=< av, e_i > - < av, e_i >= 0$$

for each i.

Hence $\alpha(av) = a\alpha(v)$ and $\alpha(v + w) = \alpha(v) + \alpha(w)$ follows in the same way for all $v, w \in \mathbb{R}^n$. □

Example 2.3.7

Let us examine the orthogonal group on $\mathbb{R}^2$ a bit more closely. An element $A \in O(2)$ is *determined by its action on* $e_1 = (1,0)^t$ *and* $e_2 = (0,1)^t$.

If $Ae_1 = (a,b)^t$, *then* $a^2 + b^2 = 1$, *since the length of a vector must be preserved when it is multiplied by* A. *Since multiplication of an element of* $O(2)$ *preserves length and orthogonality,* $Ae_2 = \pm(-b,a)^t$. *If we choose* $Ae_2 = (-b,a)^t$, *then*

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

Where $0 \leq \theta < 360°$, *and* $detA = 1$. *The matrix* A *rotates a vector in* $\mathbb{R}^2$ *counterclockwise about the origin by an angle of* θ *(Figure 3.1)*

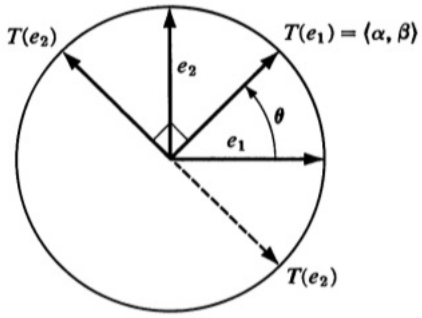

Figure 2.1: $O(2)$ acting on $\mathbb{R}^2$

If we chose $Ae_2 = (b,-a)^t$, *then we obtain the matrix*

$$A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \begin{pmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{pmatrix}$$

Hence $detA = -1$ *and* $A^2 = I_2$.

The first case gives a rotation about the origin through an angle θ *(or the identity when* $\theta = 0$*). The second gives reflection in the line* $\{(x,y) : y = (tan\frac{1}{2}\theta)x\}$ *at an angle* $\frac{1}{2}\theta$ *from the x-axis.*

Note that the determinant of the orthogonal matrix A must be either +1 or -1. If for rotations that preserve the orientation of the plane and -1 for the reflections that reverse the orientation.

Example 2.3.8

- *Rotations in* $\mathbb{R}^3$ *are orthogonal transformations, that rotations can occur about the* $x-$*axis,* $y-$*axis, or* $z-$*axis, represented by the following matrices respectively:*

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{pmatrix}, R_y(\theta) = \begin{pmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{pmatrix}$$

,

$$R_z(\theta) = \begin{pmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- *Reflections also are orthogonal transformations that flip vectors over a plane in* $\mathbb{R}^3$ *(reflection across the* $xy-$*plane,* $xz-$*plane,* $yz-$*plane) can be represented by the following matrices respeclively :*

$$M_{xy} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, M_{xz} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

,

$$M_{yz} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.4 Euclidean Group

2.4.1 Definition

Definition 2.4.1 *(Euclidean Group)*

Isometies (rigid motions) of euclidean n-space, that is, bijections

$T : \mathbb{R}^n \to \mathbb{R}^n$ *that preserve distance. The group of all isometries of* $\mathbb{R}^n$ *is called the* ***euclidean group*** *in n dimensions and is denoted* $E(n)$*.*

2.4.2 Translation

Definition 2.4.2 *(Translation)*

Translation by a vector $v \in \mathbb{R}^n$ *is the function* $t : \mathbb{R}^n \to \mathbb{R}^n$ *defined by* $t(u) = u + v$ *for all* $u \in \mathbb{R}^n$ *.*

Remark 2.4.3

The translations is a subgroup $T(n)$ *of* $E(n)$ *.*

Where :

If $t_1, t_2 \in T(n)$ defined by $t_1(u) = u + v$ and $t_2(u) = u + w$ for all $u \in \mathbb{R}^n$

$$\begin{aligned} t_1 \circ t_2^{-1}(u) &= t_1(u - w) \\ &= u - w + v \\ &= u + (v - w) \end{aligned}$$

so $t_1 \circ t_2^{-1}$ is a translation by the vector $w - v$ and therefore belongs to $T(n)$.

Theorem 2.4.4

For each $n \geq 1$, $T(n)$ *is an abelian normal group of* $E(n)$ *and* $E(n)/T(n) \cong O(n)$. *In fact, the map* $E(n) \to O(n)$ *given by*

$$\alpha \mapsto \textit{the standard matrix of } t_{\alpha(o)} \circ \alpha$$

is a surjective group morphism $E(n) \to O(n)$ *with kernel* $T(n)$

Proof of this theorem can be found in reference [7]

From this theorem if $\alpha \in E(n)$ and $\alpha(0) = w$ we have $t_w \circ \alpha \in G(n)$ (G(n) the group of all linear isometries of $\mathbb{R}^n$)

Hence every isometry α of $\mathbb{R}^n$ is the composition of a linear isometry $t_w^{-1} \circ \alpha$ followed by a translation t_w .

Chapter 3

Symmetry groups

3.1 The symmetry

Definition 3.1.1 *(Symmetry of Figure)*

By a figure X we mean a set of points in plane or space.

A symmetry of a figure X is a bijection T into it self such that :

1. *$T(X) = X$; that is, T send every point in X onto another point in X .*

2. *T preserves distance, that is, if $d(p, q)$ denoted the distance between the points p and q, then $d(T(p), T(q)) = d(p, q)$ for all $p, q \in X$.*

Theorem 3.1.2

The set G of all symmetries of a figure X form a group G with the operation of composition, called ***The symmetry group*** *of a figure .*

Proof.

- If $F, T \in G$. Then $F \circ T(p) = F(T(p))$, we have $FT(X) = F(T(X)) = F(X) = X$ and $d(F \circ T(p), F \circ T(q)) = d(F(T(p)), F(T(q))) = d(T(p), T(q)) = d(p, q)$ for all $p, q \in X$ and hence $F \circ T \in G$.

- We have already noted that the composition of symmetries is associative.

- The transformation 1 such that 1(p)=p for all $p \in X$ belongs, clearly to G and satisfies $T \circ 1 = 1 \circ T = T$; $T \in G$.

- If $T \in G$, and T is bijection, then there exist $T^{-1} \in G$ such that $T \circ T^{-1} = T^{-1} \circ T = 1$, and $d(p,q) = d(T^{-1} \circ T(p), T^{-1} \circ T(q)) = d(T^{-1}(p), T^{-1}(q))$.
  And $X = T^{-1} \circ T(X) = T^{-1}(X)$.

Then G is a group . $\square$

3.2 Some Types of symmetry

Symmetry may be viewed when you flip slide or turn an object. There are four types of symmetry then can be observed in various situations; they are :

Translation Symmetry

Rotational Symmetry.

Reflection symmetry.

Glide reflection symmetry .

1. **Translations** :

   A translation of X is a mapping that sends all points of X the same distance in the same direction .

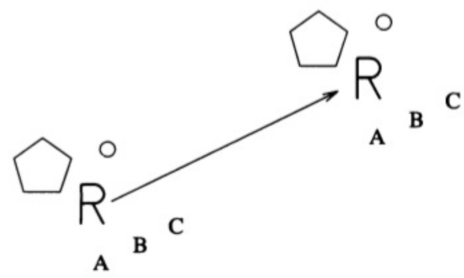

Figure 3.1: [6]The translation

2. **Rotations symmetry :**

   When an object is rotated in a particular direction around an point, then it is know as a rotational symmetry. Rotational symmetry existed when a shape turned, and the shape is identical to the origin .

The angle of rotational symmetry is the smallest angle at which the figure can be rotated to coincide with it self, when it is in rotation.

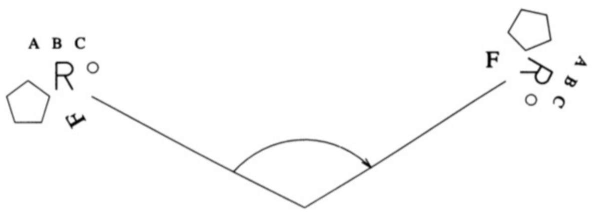

Figure 3.2: [6]The rotation

3. **Reflection Symmetry:**

   A figure is axially symmetrical if it can be bisected by one or more mirror axes, in this case the part on the left side of such an axis relate to the part on the right-hand side by begging its mirrors image. all points on the mirror axis remain fixed.

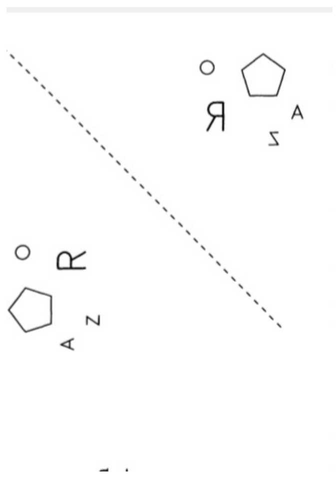

Figure 3.3: [6]The reflection

4. **[6]Glide-reflection:**

   A Glide-reflection is a translation in the direction of a line followed by reflection through the line.

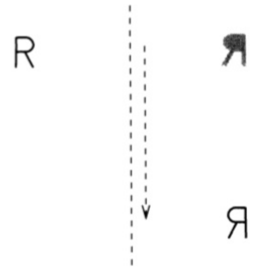

Figure 3.4: [6]The Glide-reflection

3.3 Symmetry in plane

3.3.1 Finite Symmetry groups in plane

Definition 3.3.1 *(Cyclic group)*

A group C_n is cyclic of order n, and consists of rotations through $k(\frac{360}{n})^\circ$, $0 \leq k < n$, around a fixed point p .

$$C_n = \{1, R, R^2, R^3, \dots, R^{n-1}\},\ R^n = 1$$

Example 3.3.2

The winged triod rotation group consists of 3 possible rotations.

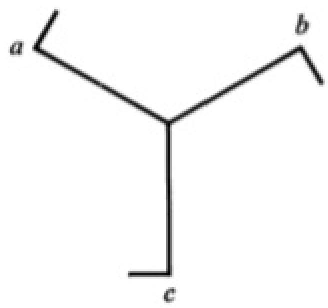

Figure 3.5: The cyclic group C_3

Definition 3.3.3 *(Dihedral Group)*

A group D_n has order $2n$ and contains the elements of C_n together with reflections through n axes that intersect at p and divide the plane into $2n$ equal angular regions. The group D_n are called ***Dihedral Group****.*

$$D_n = \{1, R, R^2, R^3, \ldots, R^{n-1}, S, SR, SR^2, \ldots, SR^{n-1}\} \ .$$

And thus symmetries are multiplied according to the rules $R^n = 1$, $S^2 = 1$ *and,* $SR = R^{-1}S$.

Example 3.3.4

There are 6 *possible symmetries for the triod.*

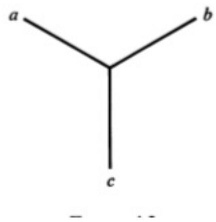

Figure 3.6: The Dihedral group D_3

Example 3.3.5

The symmetry of an equilateral triangle admits:

- *Three axes of symmetry: The equilateral triangle has three lines of reflection symmetry. Each line of symmetry passes through a vertex and the midpoint of the opposite side.*
- *Three rotational Symmetry around M the point of intersection the medians of its sides, by angle of:* $120°$, $240°$ *or* $0°$ *(* $0°$ *which is equivalent to no rotation, and it will still appear identical).*

The group of symmetries of the equilateral triangle is a simple of symmetry group in $O(2)$. *Assume that the origin is at the center of the triangle and one vertex is* $(0, 1)$.
The six orthogonal transformation in two dimensions; which leave the triangle fixed are represented by the follow matrices :

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, B = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$$

$$C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, D = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, E = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$$

The motions can also be represented by the action on the vertices $(O, 1)$, $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$, $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$ *, labeled 1, 2 and 3 respectively. For example* $A : (123)$

Using similar cyclic notation, the action of the other symmetries on the others vertices are shown in the following figure.

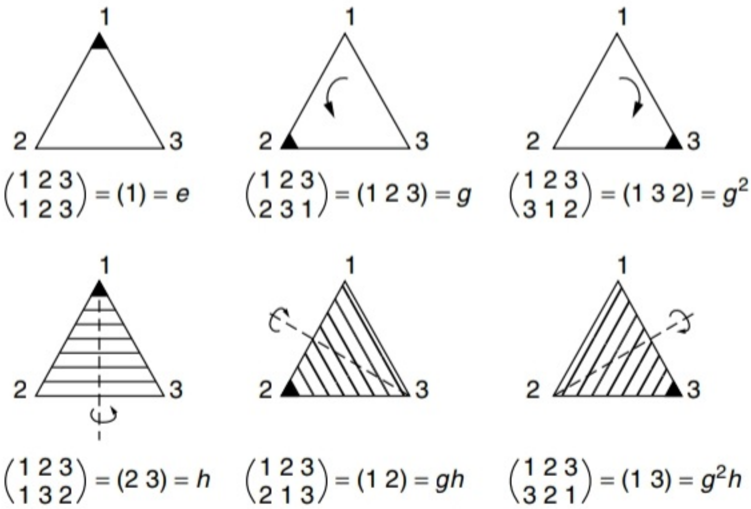

Figure 3.7: Symmetries of an equilateral triangle

Theorem 3.3.6

A finite symmetry group of a plane figure is either a cyclic group C_n *or, dihedral group* D_n.

Proof.

Assume that G is a finite symmetry group of a plane figure, and assume first that G has order n and contains only rotations. Each rotation exept the identity can be assumed to be clockwise through a positive angle of less then $360°$. Let α be the one with the smallest angle. Then G contains $e = \alpha^0, \alpha, \alpha^2, \cdots$; in fact, every element of G will appear in this list: For suppose that $\beta \in G$ and $\beta \neq \alpha^k$ for every k. If the angles for α and β

are θ_α and θ_β, respectively, then $t\theta_\alpha < \theta_\beta < (t+1)\theta_\alpha$, for some positive integer t. But then $\beta\alpha^{-t} \in G$ and $\beta\alpha^{-t}$ corresponds to a positive clockwise rotation through an angle less then θ_α, a contradiction. Therefore, $G = \{e = \alpha^0, \alpha, \alpha^2, \cdots, \alpha^{n-1}\}$ with $\alpha^n = e$ and $\theta_\alpha = (\frac{360}{n})^\circ$.

Now assume that G contains a reflection ρ, and let H denote the set of rotations in G. Then H is subgroup of G we can assume that $H = \{e = \alpha^0, \alpha, \alpha^2, \cdots, \alpha^{n-1}\}$ for some rotation α. Neccessarily, G contains $e = \alpha^0, \alpha, \alpha^2, \cdots, \alpha^{n-1}, \rho, \rho\alpha, \rho\alpha^2, \cdots, \rho\alpha^{n-1}$. Each element $\rho\alpha^k$ $(0 \le k < n)$ is reflection, and these elements are all distinct by the left concellation law in G.

Let μ be a reflection in G. Then $\rho\mu$, is a rotation. Therefore $\rho\mu = \alpha^k$ for some k so that $\mu = \rho^{-1}\alpha^k = \rho\alpha^k$. It follows from this that

$$G = \{e = \alpha^0, \alpha, \alpha^2, \cdots, \alpha^{n-1}, \rho, \rho\alpha, \rho\alpha^2, \cdots, \rho\alpha^{n-1}\}$$

. □

Example 3.3.7 *Objects whose symmetry groups are C_n and D_n .*

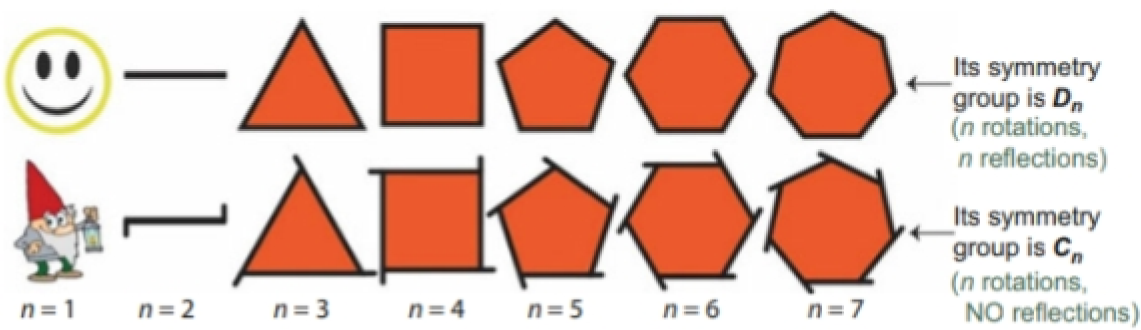

Figure 3.8: Objects whose symmetry groups are C_n and D_n

3.3.2 Infinite symmetry in plane

There are 7 types of infinite symmetry in the plane

type 1 The group in Figure 3.9 consists of translations only. If x denotes translation through the smallest possible distance, then the group is infinite cyclic with generator x. We may write it as

$$F_1 = \{x^n : n \in \mathbb{Z}\}$$

Figure 3.9: [11]Type 1

type 2. The group in Figure 3.10 is also infinite cyclic. It is generated by a glide-reflection; if this glide-reflection is denoted by x, then the even powers of x are translations. We may write the symmetry group of this type as

$$F_2 = \{x^n : n \in \mathbb{Z}\}$$

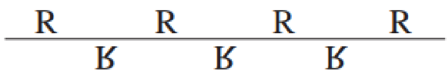

Figure 3.10: [11]type 2

Type 3. The group in Figure 3.11 is generated by a translation say x and a reflation say y through a vertical line (such as the dotted line in the figure). This group is an infinite dihedral group. Verify that $xy = yx^{-1}$. We may write the symmetry group of this type as

$$F_3 = \{x^n y^m : n \in \mathbb{Z}, m = 0, 1\}$$

Figure 3.11: [11]Type 3

Type 4. The group in Figure 3.12 is generated by a translation(say x) and a rotation (say y) through $180°$ around a point such as p in the figure. This is also an infinite dihedral group. We may write the symmetry group of this type as

$$F_4 = \{x^n y^m : n \in \mathbb{Z}, m = 0, 1\}$$

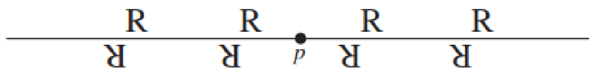

Figure 3.12: [11] Type 4

Type 5. The group in Figure 3.13 is generated by a glide-reflection(say x) and a rotation(say y) through $180°$ around a point such as p in the figure. This is also an infinite dihedral group. We may write the symmetry group of this type as

$$F_5 = \{x^n y^m : y \in \mathbb{Z}, m = 0, 1\}$$

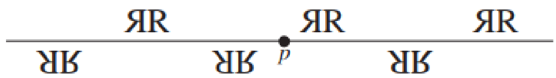

Figure 3.13:

[11]Type 6. The group in Figure 3.14 is generated by translation (say x) and reflection (say y) through an axis of symmetry (which is horizontal in the figure). This group is abelian (Verify that $xy = yx$). We may write the symmetry group of this type as

$$F_6 = \{x^n y^m : n \in \mathbb{Z}, m = 0, 1\}$$

Figure 3.14: [11]Type 6

Type 7. The group in Figure 3.15 is generated by translation (say x) and hotizontal reflections say y; and z, as an vertical reflection. We may write the symmetry group of this type as

$$F_7 = \{x^n y^m z^k : n \in \mathbb{Z}, m = 0, 1, k = 0, 1\}$$

Figure 3.15: [11]Type 7

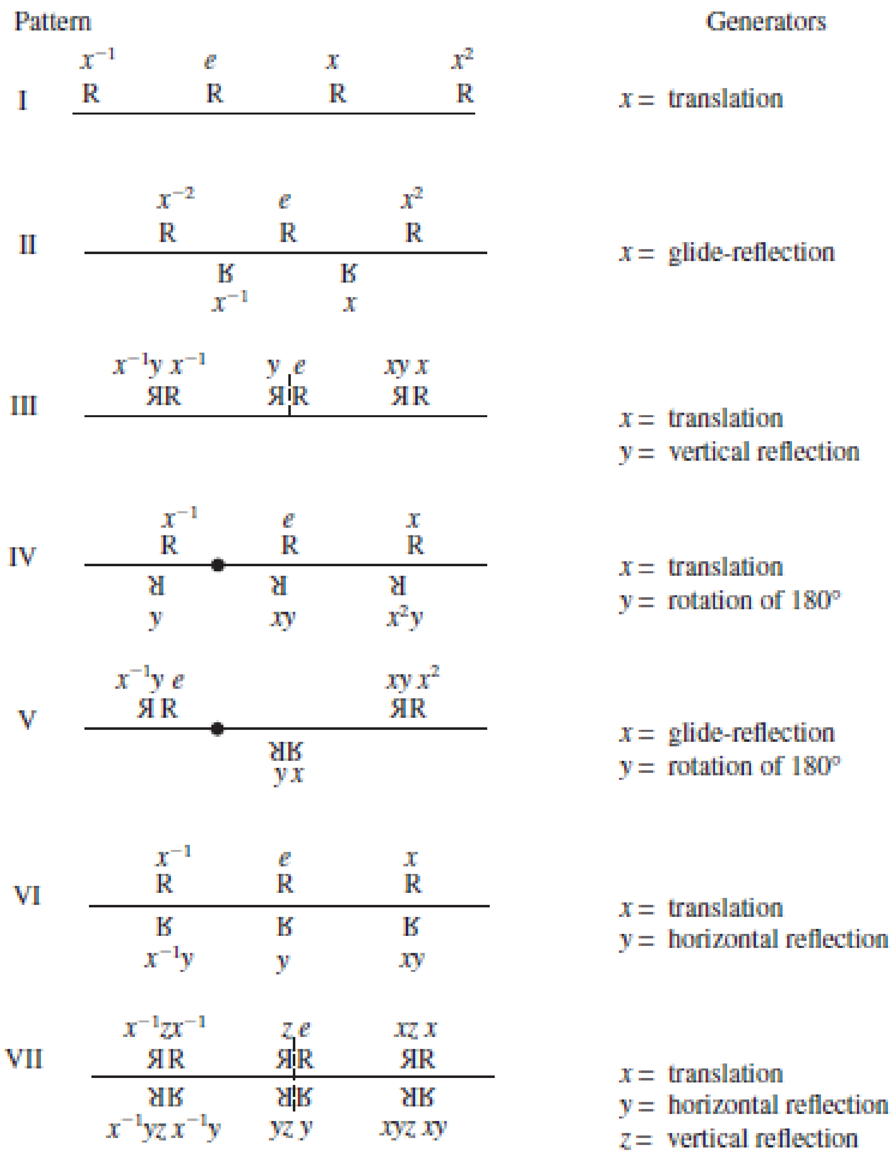

Figure 3.16: The infinite plane group and their groups of symmetries

Example 3.3.8 *Some examples of the infinite plane groups.*

Figure 3.17: Some examples of the infinite plane groups

3.4 Finite Groups of rotations in space

3.4.1 Regular Solids

Definition 3.4.1 *(Regular Solid)*

A ***regular solid*** *is polyhedron in which all faces are congruent regular polygons and all vertices are incident with the same number of faces. there are five such solids.*

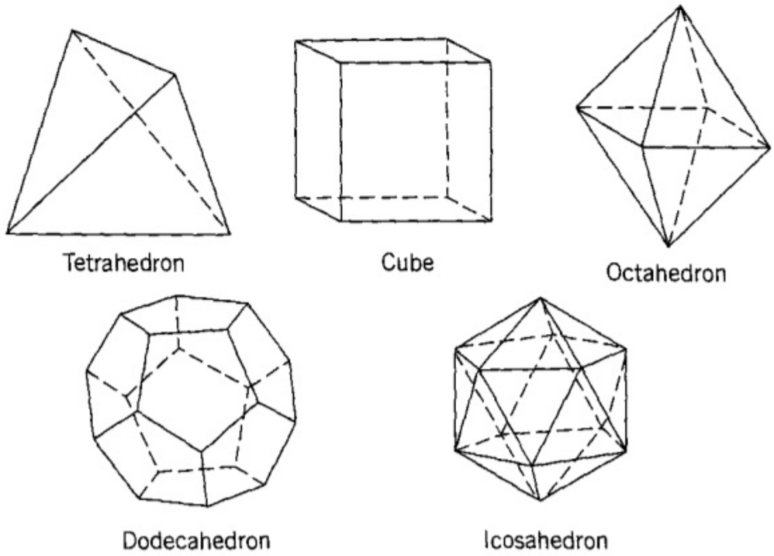

Figure 3.18: [11] regular solids

Polyhedron	Number of Vertices	Number of Edges	Number of Faces	Faces	Number of Faces at Each Vertex
Tetrahedron	4	6	4	Triangles	3
Cube	8	12	6	Squares	3
Octahedron	6	12	8	Triangles	4
Dodecahedron	20	30	12	Pentagons	3
Icosahedron	12	30	20	Triangles	5

Figure 3.19: [11]regular solids

Definition 3.4.2 *(Dual Polyhedron)*

Given any polyhedron, we can construct its **dual** *polyhedron in the following way. The vertices of the dual are the centers of the faces of the original polyhedron. Two centers are joined by an edge if the corresponding faces meet in an edge.*

Example 3.4.3

The dual of a regular tetrahedron is another regular tetrahedron. The dual of a cube is an octahedron, and the dual of an octahedron is a cube. The dodecahedron and icosahedron are also duals of each others ...

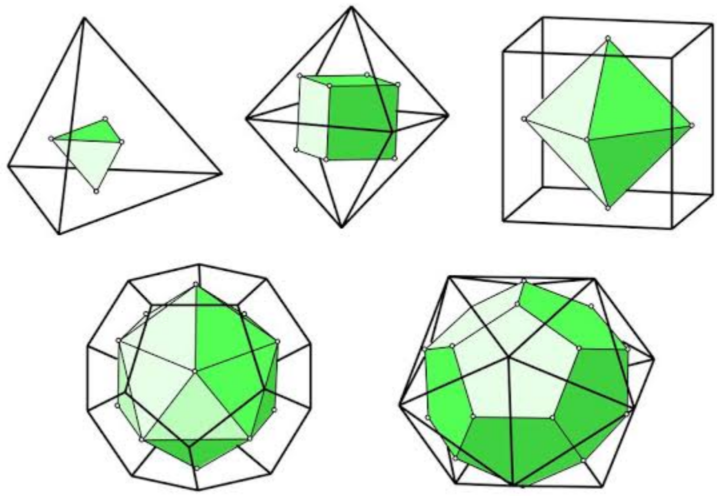

Figure 3.20: [9] Dual of solids

Definition 3.4.4 *A line is* ***n-fold axis*** *of symmetry for a geometric figure if the rotation by* $\frac{2\pi}{n}$ *about this line symmetry.*

3.4.2 Proper Rotations Of Regular Solids

Rotation Group of the tetrahedron

The rotation group of the **tetrahedron** has order 12 and is called the tetrahedron group. The rotational axis / Axis of symmetry of of tetrahedron are in the following table :

n-fold rotation axis $n =$	**Axis passes through**	**number of n-fold axis**	**The order of rotation**	**Number of the group elements**
2	The centers of a pair of opposite edges	3	2	3 non identity
3	A vertex and the centroid of the opposite faces	4	3	8 non identity

With the identity we have 12 rotations.

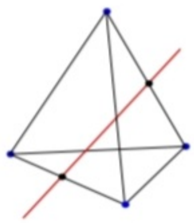

Figure 3.21: [9]two-fold axes of the tetrahedron

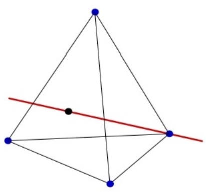

Figure 3.22: [9]Three-fold axes of the tetrahedron

Theorem 3.4.5 *The group of proper rotations of a regular tetrahedron is isomorphic to* A_4.

Proof of this theorem can be found in reference [7]

The rotation group of the cube

The rotation group of the **cube** has order 24. It is the same as the rotation group of the octahedron, and it called the octahedral group.

The rotational axis / Axis of symmetry of cube are in the following table :

n-fold rotation axis $n =$	**Axis passes through**	**number of n-fold axis**	**The order of rotation**	**Number of the group elements**
2	The centers of opposite edges	6	2	6 non identity
3	pairs of opposite vertices	4	3	8 non identity
4	the centroids of opposite faces	3	element has order 2 and others has order 4	9 non identity

With the identity we have 24 rotations.

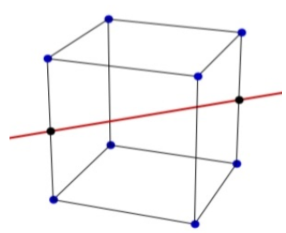

Figure 3.23: [9]Two-fold axes of the cube

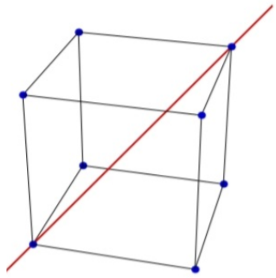

Figure 3.24: [9]Three-fold axes of the cube

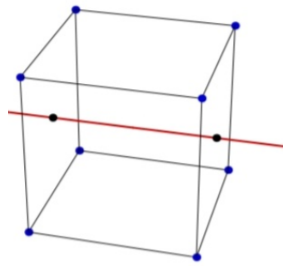

Figure 3.25: [9]Four-fold axes of the cube

Theorem 3.4.6

The rotation group of the cube is isomorphic to the permutation group S_4.

Proof of this theorem can be found in reference [7]

Proposition 3.4.7

The octahedron is dual to the cube, so its group of rotations is also isomorphic to S_4.

The rotation group of the icosahedron

The rotation group of the **icosahedron** has order 60. It is the same as the rotation group of the dodecahedron, and is called icosahedral group.

The rotational axis / Axis of symmetry of cube are in the following table :

n-fold rotation axis $n =$	**Axis passes through**	**number of n-fold axis**	**The order of rotation**	**Number of the group elements**
2	The centers of opposite edges	15	2	15 non identity
3	pairs of opposite vertices	10	3	20 non identity
5	the centroids of opposite faces	6	5	24 non identity

With the identity we have 60 rotations.

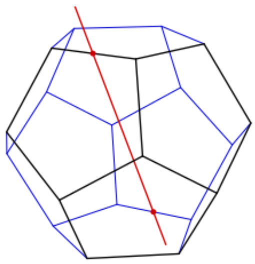

Figure 3.26: [9]Two-fold axes of the dodecahedron

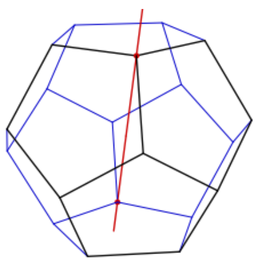

Figure 3.27: [9]Three-fold axes of the dodecahedron

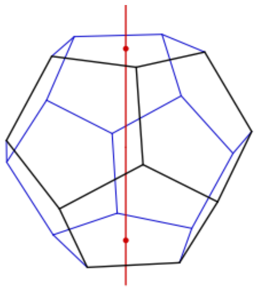

Figure 3.28: [9]Five-fold axes of the dodecahedron

Theorem 3.4.8

The rotation groups if the dodecahedron and the icosahedron are isomorphic to the group A_5 .

Proof of this theorem can be found in reference [7]

Theorem 3.4.9

A finite rotation group of a three-dimensional figure is either a cyclic group, a dihedral group, the tetrahedron group, the octahedral group, or the icosahedral group.

Bibliography

[1] Alen Slomson, An Introduction to combinatiorics, Chapman and hall,(1994).

[2] Amstrong,M.A, Groups and symmetry, World publishing corporation, (2011).

[3] Andrew Barker, Groups and symmetry .University of Glasgow,(2000).

[4] Bijan Davvaz, Groups and Symmetry Theory and Application, Springer,(2021).

[5] Charles w. Curis, Linear Algebra, Springer -verlag,(1984) .

[6] David W. Farmer, Group and symmetry, American Mathematical society ,(1995).

[7] Gilbert, W. J., Nicholson, W. K, Modern Algebra with Applications,John Wiley and Sons, (2004).

[8] John R.Durbin, Modern Algebra An Introduction - 6th ed, john wiley and sons,(2005).

[9] Joseph,A.Gallian, Contemporary abstract algebra 7th-ed, Books/cole,(2010).

[10] Kristopher Tapp, Symmetry A mathematical Exploration, springer,(2012).

[11] M,Frederick, Algebra abstract and concrete, iniversity of Iowa, (2006).

[12] Peter.Selinger, Matrix theory and linear algebra,(2006) .

[13] Thomas,W.Judson Stephen, Abstract algebra theory and application, University of Austin state , (2010).

[14] willard Millar, Symmetry groups and thier applications , Academic pres Elsivier,(1972).

Abstract:

Symmetry is a fundamental concept that permeates many areas of mathematics, art, science, and nature. Where, Symmetries, on the other hand, are transformations that leave an object or a system unchanged in some way. These transformations can be represented as elements of a group, and the study of symmetries is closely related to the study of groups.
The work of this master's memo will consist in studying the interplay between groups and symmetries.
Keywords: Symmetry group, group action, distance, euclidean group.

ملخص

التماثل مفهوم أساسي يتخلل العديد من مجالات الرياضيات والفن والعلوم والطبيعة. حيث، التماثلات، من ناحية أخرى، هي التحويلات التي تترك جسمًا أو نظامًا دون تغيير بطريقة ما. يمكن تمثيل هذه التحويلات كعناصر مجموعة، وترتبط دراسة التماثلات ارتباطًا وثيقًا بدراسة الزمر.
ويتمثل عمل مذكرة الماجستير هذه في دراسة التفاعل بين الزمر والتناظرات.
الكلمات المفتاحية: زمرة التماثل، عمل الزمر، المسافة، الزمرة الإقليدية.

Résumé:

La symétrie est un concept fondamental qui imprègne de nombreux domaines des mathématiques, de l'art, de la science et de la nature. Les symétries sont des transformations qui laissent un objet ou un système inchangé d'une certaine manière. Ces transformations peuvent être représentées comme des éléments d'un groupe, et l'étude des symétries est étroitement liée à l'étude des groupes.
L'objectif de ce mémoire de maîtrise est d'étudier l'interaction entre les groupes et les symétries.
Mots clés : Groupe de symétrie, action de groupe, distance, groupe euclidien.