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STUDY THE SOLUTIONS OF HIGHER-ORDER SINGULAR BOUNDARY VALUE PROBLEMS BY VARIATIONAL METHODS

Presented by:
Miss: Ines OUDINA
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In front of the jury composed of :

Rabah MECHETER	Ass. Pr (MCA),	University of M'sla	**President.**
Dahmane BOUAFIA	Ass. Pr (MCA),	University of M'sla	**Supervisor.**
Noureddine DECHOUCHA	Ass. L (MAA),	University of M'sla	**Examiner.**

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oudina inas nedjma 

DEDICATIONS



To my beloved parents, the source of my strength, patience, and inspiration.

To my dear brothers and sisters, who have always been my support and motivation.

To all those who stood by me with love, prayers, and encouragement.

I dedicate this modest work as a token of gratitude and appreciation.

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Notation

We introduce the necessary notations and definitions that are used later on.

$L^p(\Omega)$ $= \Big\{u : \Omega \to \mathbb{R} \ \text{ measurable} : |u|^p \in L^1(\Omega)\Big\}$ with $\quad 1 \leq p < \infty$.

$\|u\|_{L^p}$ $= \Big(\displaystyle\int_\Omega |u(t)|^p dt\Big)^{\frac{1}{p}}$.

$L^\infty(\Omega)$ $= \{u : \Omega \to \mathbb{R}$ measurable and $\exists C \ : |u(t)| \leq C$ *a.e on* $\Omega\}$.

$\|u\|_{L^\infty}$ $= \inf\{C : |u(t)| \leq C$ a.e on $\Omega\}$.

$C(\Omega)$: space of continuous functions on Ω.

$C_0^\infty(0,+\infty)$: the space of infinitely differentiable functions with compact support in $(0,+\infty)$.

H : Hilbert space.

$\langle .,. \rangle$: duality bracket.

$(.,.)$: inner product.

$H^m(0,1)$ $= \{u \in L^2(0,1) : |u^{(i)}| \in L^2(0,1) \quad \forall i \leq m\}$.

$H_0^m(0,1)$: the completion of $\mathcal{D}(0,1)$ in $H^m(0,1)$.

$C[0,T]$ $:= \Big\{u : [0,T] \to \mathbb{R}$is continuous$\Big\}$.

$\|u\|_\infty$ $= \displaystyle\sup_{t\in[0,T]} |u(t)|$.

X^* : dual topology .

$D(A)$: domain of definition of an unbounded operator A.

$R(A)$: image of A which is also noted by ImA

$\mathcal{L}(X,Y)$: set of continuous linear applications.

$X \hookrightarrow Y$: we write $X \hookrightarrow Y$ to mean that X is included in Y and the canonical injection of X into Y is continuous.

$X \hookrightarrow\hookrightarrow Y$: to imply that X is included in Y and that the canonical injection of X into Y is compact.

$u_n \rightharpoonup u_0$: u_n converge weakly to u_0.

i.e.	that is to say.
a.e.	almost everywhere.
dF	derivative in the sense of Frechet which is also noted by F'.
$d_G F$	derivative in the sense of Gateaux which sometimes is also noted by F'.
$\overline{B_\rho(0)}$	the closed ball with center 0 and radius ρ.
$(P.S)$	Palais-Smale condition.
$diamF$	Diameter of a set F.
l.s.c	lower Semi continuous .
u.s.c	upper Semi continuous .
w.l.s.c	Weakly lower semi continuous .
w.l.s.c	Weakly upper Semi continuous .

Introduction

Dfferential equations are among the most fundamental mathematical tools that emerged to describe complexe natural and physical phenomena.

They represent the relation between a function and its derivatives with respect to an independent variable, and they are widely used to model various processes in physics, engineering, economics, and biology. Their importance lies in providing a precise mathematical language for representing the dynamics of systems and continuous changes over time or space.

The study of differential equations dates back to the seventeenth century, during the scientific revolution. Isaac Newton and Gottfried Leibniz were among the pioneers who founded calculus, which provided the essential framework for defining and analysing such equations.

In the eighteenth and nineteenth centuries, mathematicians such as Euler, Lagrange, and Laplace contributed significantly to the development of mathematical analysis and the introduction of functional spaces by mathematicians.

Such as Hilbert and Sobolev, it became possible to deal with more general solutions, known as weak solutions, that is, solutions that remain strictly positive.

Within the domain educated, such mathematical modeling of natural and biological phenomena where the variables represent physical quantities that can not assume negative values, such as density, temperature, population size, or chemical concentration.

Consequently, investigating the existence, multiplicity and stability of positive solution for higher-orders differential equations has become a central topic that bridges both theoretical foundations and practical applications. This dissertation includes three chapters:

The first chapter is preliminaries which contains some basic tools which we frequently use later. We have mottoed the chapter by sections, the first section contains some notions about Banach operators such as Continuity of operators, Lower semi continuous functions , Differentiability in

Banach spaces and Lebesgue and Sobolev spaces . We have presented in the second section an va present the Critical point theory such as Mountain Pass Theorem.

Then, in the second chapter, we discuss the existence of positive solutions for following the singular linear boundary problem of higher order:

$$(LP)\begin{cases}(-1)^m u^{(2m)}(x) = \lambda p(x)u(x), & x \in (0,1)\\ u^{(i)}(0) = u^{(i)}(1) = 0, & i = 0,\dots,m-1\end{cases} \tag{1}$$

where $m > 1$, $p \in C(0,1)$, and it may be singular at $x = 0$ or $x = 1$.

Finally, in the last chapter we study some existence results of positive solutions for following a singular boundary problem of higher order

Finally, in the last chapter, we studied the existence of positive solutions to the following singular boundary value problem:

$$(NL)\begin{cases}(-1)^m u^{(2m)} = p(t)f(u(t)), & t \in (0,1),\\ u^{(i)}(0) = u^{(i)}(1) = 0, & i = 0,...,m-1,\end{cases} \tag{2}$$

where the function $p(t)$ can be singular at times $t = 0$ or $t = 1$. The analysis of this problem relies on variational methods. This is the subject of Yujun Cui's work in [4]. The author sought solutions in the Sobolev space and used an injection theorem that plays an important role in proving the results.

The first eigenvalue (LP) associated with the first eigenfunction ϕ plays an important role in proving the main results.

The first result assures us that the problem (NL) admits a positive solution under the following assumption:

(H_1) $f_\infty^+ < \lambda_1 < f_0^- \le f_0^+ < +\infty$

With,

$$f_0^- = \liminf_{t\to 0}\frac{f(t)}{t}, \quad f_0^+ = \limsup_{t\to 0}\frac{f(t)}{t},$$
$$f_\infty^- = \liminf_{t\to +\infty}\frac{f(t)}{t}, \quad f_\infty^+ = \limsup_{t\to +\infty}\frac{f(t)}{t}.$$

For the proof, we used the minimization theorem.

The second result assures us that the problem (NL) has a unique positive solution under the following hypotheses:

(H_1) $f_0^+ < \lambda_1 < f_\infty^- \le f_\infty^+ < +\infty$

(H_2) $p \geq 0$, $p \in C(0,1)$, $\exists\, t \in (0,1)$, $p(t) > 0$ and $\lim\limits_{s\to 0} s^{2m-1} \displaystyle\int_s^{1-s} p(\tau)d\tau = 0$,

(H_3) $f \geq 0$ and $f \in C[0,+\infty)$.

For the proof, we used Mountain pass theorem where the two conditions (H_2), (H_3) are used to prove the condition (P.S), while the condition (H_1) is used to prove the geometric conditions.

For the last result concerning problem (NP), we obtained a bounded positive solution, using an approximate problem and again, we used the principle of minimization under the following hypothesis:

(H_4) $f_0^+ < \lambda_1 < f_\infty^- \leq f_\infty^+ = +\infty$.

This work was supported by providing an illustrative example of the two studied problems

Chapter 1

Some preliminaries

1.1 Operators on Banach spaces

Let X and Y be two normed Banach spaces.

Definition 1.1. (Linear bounded operator)[6]

Let A be a linear operator such that $D(A) = X$ et $R(A) \subset Y$. We say that A is bounded if it is bounded on the unit ball $\overline{B}(0,1)$, i.e. If the set

$$\Big\{\|Ax\| : \quad x \in X,\ \|x\| \leq 1\Big\},$$

is bounded.

According to this definition, if A is bounded, there exists a constant $c > 0$ such that for all x where $\|x\| \leq 1$, we have the inequality

$$\|Ax\| \leq c.$$

Definition 1.2. (Dual space)[6]

The set of continuous linear functionals, defined on a normed vector space, constitutes a vector

space. It is called dual of the space X and we denote X^*.

Equipped X^* with the norm

$$\|f\|_{X^*} = \sup_{u\in X, \|u\|\leq 1} |\langle f, u\rangle|, \quad \forall f \in X^*$$

X provided with this norm is a Banach space and we have the inequality

$$|\langle f, u\rangle| \leq \|f\|_{X^*}\|u\|_X, \quad \forall f \in X^*, \quad \forall u \in X.$$

Definition 1.3. (Weak convergence) It is said that a sequence $(u_n) \in X$ converges weakly to u, if

$$\forall f \in X^*, \langle f, u_n\rangle \longrightarrow \langle f, u\rangle$$

and we write $u_n \rightharpoonup u$.

Proposition 1.1. *[7] Let (u_n) be a sequence of X. We have*

1. *If $u_n \longrightarrow u$, so $u_n \rightharpoonup u$.*

2. *If $u_n \rightharpoonup u$, so (u_n) is bounded and $\|u\| \leq \liminf_{n\longrightarrow\infty} \|u_n\|$.*

Proposition 1.2. *[7] When X is of finite dimension, a sequence (u_n) converges weakly if and only if it converges strongly.*

Theorem 1.1. *[7] A Banach space X is reflexive if and only if its closed ball is weakly compact.*

Corollary 1.1. *[7] If X is a reflexive Banach space then any bounded sequence $\{u_n\} \subset X$ with $\|u_n\| \leq M$, contains a subsequence which converges weakly to an element $u \in X$ satisfying $\|u\| \leq M$.*

1.1.1 Continuity of operators

We will consider operators T from X into Y and we will give a definition concerning the properties of the continuity of T.

The easy notion is the following

Definition 1.4. The operator $T : X \longrightarrow Y$ is said to be continuous at u, if for any sequence $\{u_n\} \subset X$ which converges to u, $(T(u_n))_n$ Converges to $T(u)$.

T is called continuous on $\Omega \subset X$ if T is continuous at any point $u \in \Omega$.

Definition 1.5. An operator $T : X \longrightarrow Y$ is said to be compact if it is continuous and has the property, for any (u_n) bounded in X, the sequence $(T(u_n))$ admits a convergent subsequence.

Definition 1.6. Let $(X, \|.\|_1)$ and $(Y, \|.\|_2)$ two Banach spaces. An operator $T : X \to Y$ is said to be completely continuous if it is continuous, and the image of any bounded set of X is relatively compact in Y.

1.1.2 Lower semicontinuous functions

Definition 1.7.

1. The functional $f : X \longrightarrow \mathbb{R}$ is said to be lower semi-continuous ($l.s.c$) at the point u_0, if whenever sequence $\{u_n\} \subset X$ such that $u_n \longrightarrow u_0$, it follows

$$f(u_0) \leq \liminf_{n \longrightarrow \infty} f(u_n).$$

2. The functional $f : X \longrightarrow \mathbb{R}$ is said to be weakly lower semi-continuous ($w.l.s.c.$) at the point u_0, if whenever sequence $\{u_n\} \subset X$ such that $u_n \rightharpoonup u_0$, it follows

$$f(u_0) \leq \liminf_{n \longrightarrow \infty} f(u_n).$$

3. The functional f is said to be upper semi-continuous ($u.s.c.$) at the point u_0, if whenever sequence $\{u_n\} \subset X$ such that $u_n \longrightarrow u_0$, it follows

$$f(u_0) \geq \limsup_{n \longrightarrow \infty} f(u_n).$$

4. The functional f is said to be weakly upper semi-continuous (*w.u.s.c.*) at the point u_0, if whenever sequence $\{u_n\} \subset X$ such that $u_n \rightharpoonup u_0$, it follows

$$f(u_0) \geq \limsup_{n \longrightarrow \infty} f(u_n).$$

Example 1.1. Let the functional φ be defined on a Hilbert space H as follows

$$\varphi : u \rightarrow \|u\|^2,$$

then φ is weakly lower semi-continuous (*w.l.s.c.*).

Indeed, let (u_n), such that $u_n \rightharpoonup u$, show that

$$\|u\|^2 \leq \liminf_{n \rightarrow +\infty} \|u_n\|^2.$$

We obtained

$$\begin{aligned} \|u_n - u\|^2 &= (u_n - u, u_n - u) \\ &= \|u_n\|^2 - 2(u_n, u) + \|u\|^2 \geq 0, \end{aligned}$$

We know that $H^* = H$, and $u_n \rightharpoonup u$, so

$$\forall u \in H : (u_n, u) \rightarrow (u, u) = \|u\|^2.$$

It follows that

$$\|u\|^2 \leq \liminf_{n \rightarrow +\infty} \|u_n\|^2.$$

1.1.3 Differentiability in Banach spaces

we introduce some definitions and fundamental properties **Fréchet differentiability** Let X and Y be two Banach spaces, U an open of X.

Definition 1.8. Let $u \in U$, and $\varphi : U \to Y$. We say that φ is differentiable at the point u if there exists a continuous linear mapping $A \in \mathcal{L}(X, Y)$ such that

$$R(h) = \varphi(u + h) - \varphi(u) - A.h, \quad \text{for } h \in X, \text{ is small,}$$

then

$$\frac{R(h)}{\|h\|} \to 0 \quad \text{when } \|h\| \to 0 \quad \text{i.e.}$$

$$\forall \varepsilon > 0, \ \exists \delta > 0 \quad \text{such that if } \|h\|_X \leq \delta, \text{ then } \|R(h)\|_Y \leq \varepsilon \|h\|_X.$$

If such an application $A \in \mathcal{L}(X, Y)$ exists, it is necessarily unique. We note by

$$A = d\varphi(u) \text{ or } A' = \varphi'.$$

It is called differential (in the sense of Fréchet) from φ in u, or a linear application tangent to φ in u. In the absence of differentiable additional precision will mean in the following differentiable in the sense of Fréchet.

Examples 1.1.

1. If $\varphi(u) = c$. Then φ Fréchet differentiable and $d\varphi(u) = 0, \ \forall u$.
2. If $A \in \mathcal{L}(X, Y)$ $A(u + h) - A(u) = A.h$ and so $dA(u) = A, \ \forall u \in X$.
3. If $X = H$ is a space of Hilbert and $\varphi(u) = \|u\|^2 = \langle u, u \rangle$ *then* $d\varphi(u)h = 2\langle u, h \rangle$.

   Therefore we have all the classical properties.

1.1.4 Gâteaux differentiability

Let us begin by defining the notion of directional derivative

Definition 1.9. Let $\varphi : U \to Y$, $u \in U$, $v \in X$ and $v \neq 0$.

We call the directional derivative in u of φ in the direction v, denoted $\partial_v \varphi(u)$, the limit where it exists

$$\partial_v \varphi(u) = \lim_{t \to 0} \frac{\varphi(u + tv) - \varphi(u)}{t}.$$

The notion of directional derivative is thus an extension of the notion of partial derivative. If φ is Fréchet differentiable, then for all $v \in X$ the directional derivative in the direction v is given by

$$\partial_v \varphi(u) = d\varphi(u)v.$$

Indeed $\varphi(u + tv) = \varphi(u) + d\varphi(u)(tv) + R(tv)$.

Then

$$\frac{\varphi(u + tv) - \varphi(u)}{t} = d\varphi(u)(v) + \frac{R(tv)}{t}, \quad \frac{R(tv)}{t} \to 0, \text{ when } t \to 0.$$

Definition 1.10. We say that $\varphi : U \to Y$ is Gâteaux differentiable in u (G-differentiable in u), if there exists a continuous linear mapping A from X to Y, ($A \in \mathcal{L}(X, Y)$) such that for any $v \in X$, the directional derivative of φ in u in the direction v exists and is equal to $A(v)$, i.e.

$$\partial_v \varphi(u) = \lim_{t \to 0} \frac{\varphi(u + tv) - \varphi(u)}{t} = A(u), \quad t \to 0, \forall u \in X.$$

It is then verified that, if such an application A exists, it is unique. We notice

$$A = d_G \varphi(u).$$

Proposition 1.3. *[3] If φ is differentiable Fréchet in u, it is Gâteaux differentiable in u and*

$$d_G \varphi(u) = d\varphi(u).$$

The reciprocal is false in general, yet in finite dimension.

Example 1.2. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$

$$\begin{cases} \varphi(x, y) = \left[\frac{x^2 y}{x^4 + y^2}\right]^2, & if \quad y \neq 0 \\ \varphi(x, 0) = 0 \end{cases}$$

G-differentiability does not imply the continuity of φ ! (while of course differentiability in the sense of Fréchet implies).

On the other hand, if we know that φ is C^1 in the sense of Gâteaux, then φ is Fréchet differentiable on U and the two notions coincide and we have

Theorem 1.2. *[3],[19]*

If $\varphi : U \to Y$ is Gâteaux differentiable in an open neighborhood $U \subset X$, of u and $d_G\varphi : U \to X^*$ is continuous at u, then φ is Fréchet differentiable at u and the two derivatives at u coincide, i.e.,

$$\varphi'(v) = d_G\varphi(v), \quad \forall v \in U.$$

Definition 1.11. Let $u_0 \in X$. We say that u_0 is a local minimum for φ if there exists $\delta > 0$ such that

$$\varphi(u_0) \leq \varphi(v), \quad \forall v \in B(u_0, \delta), \quad v \neq u_0.$$

Definition 1.12. Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$. We say that φ is coercive if

$$\varphi(u) \to +\infty, \text{ when } \|u\| \to +\infty; \quad \text{i.e.}$$

$$\forall A > 0, \ \exists B > 0, \text{ such that } \|u\| \geq B \text{ implies } \varphi(u) \geq A.$$

1.1.5 Lebesgue and Sobolev spaces

Let Ω an open of $\mathbb{R}$, and $(\Omega, \mathcal{M}, dt)$ denote a measure space, where $\mathcal{M}$ is a $\sigma-$ algebra in Ω.

We assume that $f : \Omega \to \mathbb{R}$ is a measurable function and integrable function. We shall often write $\int f$ instead of $\int_\Omega f$, and we shall also use the notation

$$\|f\|_{L^1(\Omega)} = \int_\Omega |f(t)|dt, \text{ or } \|f\|_{L^1} = \int |f|.$$

Theorem 1.3. ***(Lebesgue's Dominated Convergence Theorem)****[7]*

Let (f_n) be a sequence of functions in L^1 that satisfy

(a) $(f_n) \to f$ *a.e. on* Ω.

(b) There is a function $g \in L^1(\Omega)$ such that for all n, $|f_n(t)| \leq g(t)$, a.e. on Ω. Then

$$f \in L^1(\Omega) \quad \text{and} \quad \|f_n - f\|_{L^1} \to 0, \ n \to \infty.$$

Definition 1.13. Let $p \in \mathbb{R}$ with $1 \leq p < \infty$, we put

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R}, \quad \text{measurable and } |f|^p \in L^1(\Omega)\}.$$

We notice $\|f\|_{L^p} = \Big(\int_\Omega |f(t)|^p dt \Big)^{\frac{1}{p}}$, and $\|.\|_{L^p}$ is a norm.

Definition 1.14. We put

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{R}, \quad \text{measurable} \quad \text{and} \quad \exists C : |f(t)| \leq C, \text{ a.e} \quad \text{on } \Omega\}.$$

We notice

$$\|f\|_{L^\infty} = \inf\{C : |f(t)| \leq C, \quad \text{a.e} \quad \text{on} \quad \Omega\}$$

Theorem 1.4. *[7]*

The space L^P is reflexive for $1 < p < \infty$, and its dual is L^q such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$

Theorem 1.5. ***(Cauchy-Schwarz inequality)****[7]*

Let f and $g \in L^2(\Omega)$. Then we have

$$\int_\Omega |fg| \leq \Big(\int_\Omega |f|^2 \Big)^{\frac{1}{2}} \Big(\int_\Omega |g|^2 \Big)^{\frac{1}{2}}.$$

Theorem 1.6. ***(Hölder's inequality)****[7]*

Let $1 \leq p \leq +\infty$, we denote by q the conjugate exponent, $\dfrac{1}{p} + \dfrac{1}{q} = 1$. *Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $1 \leq p \leq +\infty$. Then*

(i) $fg \in L^1(\Omega)$ *and*

(ii) $\displaystyle\int_\Omega |fg| \leq \Big(\int_\Omega |f|^p \Big)^{\frac{1}{p}} \Big(\int_\Omega |g|^q \Big)^{\frac{1}{q}} = \|f\|_p \|g\|_q.$

Theorem 1.7. ***(Hardy's Inequality)****[17]*

Suppose that $u \in \mathcal{H}_0^m(0, +\infty)$. So we have

$$\int_0^{+\infty} \left| \frac{u(x)}{x} \right|^p dx \leq \int_0^{+\infty} |u'(x)|^p dx.$$

Theorem 1.8. ***(Minkowski inequality)****[17]*

Let $f \in L^p(\mathbb{R}^N \times \mathbb{R}^N)$*, where* $0 < p < +\infty$*. So we have*

$$\Big[\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} f(x,y)dy\Big|^p dx\Big]^{\frac{1}{p}} \le \int_{\mathbb{R}^n}\Big[\int_{\mathbb{R}^n}|f(x,y)|^p dx\Big]^{\frac{1}{p}} dy.$$

Theorem 1.9. ***(Dirichlet formula)****[7]*

Let $\Omega_1 = [a,b], \Omega_2 = [c,d]$*,* $-\infty \le a < b \le \infty$*,* $-\infty \le c < d \le \infty$ *and let* $f(x,y)$ *be a measurable function defined on* $\Omega_1 \times \Omega_2$*. If at least one of the integrals*

$$\int_{\Omega_1} dx \int_{\Omega_2} f(x,y)dy, \int_{\Omega_2} dy \int_{\Omega_1} f(x,y)dx, \int\int_{\Omega_1\times\Omega_2} f(x,y)dxdy,$$

is absolutely convergent then they coincide.

The following particular case of Fubini's theorem holds, namely

$$\int_{\Omega_1} dx \int_{\Omega_2} f(x,y)dy = \int_{\Omega_2} dy \int_{\Omega_1} f(x,y)dx = \int\int_{\Omega_1\times\Omega_2} f(x,y)dxdy.$$

assuming that one of these integrals is absolutely convergent. The latter relation is called the Dirichlet formula.

Lemma 1.1. ***(Fatou's lemma)****[7]*

Let (f_n) *be a sequence of functions in* L^1 *that satisfy*

1. *for all* n*,* $f_n \ge 0$ *a.e. on* Ω
2. $\sup_n \int f_n < +\infty$.

For almost all $t \in \Omega$ *we set* $f(t) = \liminf_{n\to\infty} f_n(t) \le +\infty$*. Then* $f \in L^1$ *and*

$$\int f \le \liminf_{n\to\infty} \int f_n.$$

1.2 Critical point theory

Definition 1.15. Let φ be a differentiable functional from X to $\mathbb{R}$. A point $u \in X$ is said to be critical for φ if and only if

$$d\varphi(u) = 0.$$

Definition 1.16. Let the functional φ, of class C^1 be defined on X. A critical value of φ, is a number $c \in \mathbb{R}$, such that there exists $u \in X$, with

$$\varphi(u) = c, \quad d\varphi(u) = 0.$$

If we are able to determine a critical value, we have proved the existence of a critical point, in some situations it is useful to introduce the following definitions and theorems.

Definition 1.17. (Minimizing sequence) A minimizing sequence of a functional $\varphi : X \longrightarrow]-\infty, +\infty]$ is a sequence (u_n) such that

$$\lim_{n \longrightarrow +\infty} \varphi(u_n) = \inf_{u \in X} \varphi(u)$$

Theorem 1.10. *[6]*

Let φ be a functional defined on a reflexive Banach space X. Assume

(*i*) *φ is coercive, i.e., $\varphi(u) \to +\infty$ for $\|u\| \to +\infty$,*

(*ii*) *φ is weakly lower semi-continuous.*

Then φ is bounded from below on X, and achieved its infimum at some point u_0. If, moreover, φ is Gâteaux-differentiable at u_0, then $\varphi'(u_0) = 0$.

Theorem 1.11. *(**Minimization theorem**)[14]*

For the functional $\varphi : M \subseteq X \to [-\infty, +\infty]$ with $M \neq \emptyset$, $\min_{u \in M} \varphi(u) = \alpha$ has a solution in case the following conditions hold :

(i) X is a real reflexive Banach space,

(ii) M is bounded and weak sequentially closed,

(iii) φ is weak sequentially lower semi-continuous on M.

1.2.1 Palais-Smale's sequence and condition

Definition 1.18. [17]

Let X be a Banach space and $\varphi : X \to \mathbb{R}$ a C^1-functional. We say that φ satisfies the Palais-Smale condition, denoted $(P.S)$, if any sequence (u_n) in X such that

$$(\varphi(u_n)) \text{ is bounded, and } \varphi'(u_n) \to 0, \text{as } n \to +\infty \tag{1.1}$$

admits a convergent subsequence.

Any sequence satisfying (1.1) is called a Palais-Smale sequence.

Definition 1.19. [17]

Let X be a Banach space and $\varphi : X \to \mathbb{R}$ a C^1- functional, and $c \in \mathbb{R}$. The functional φ is said to satisfy the (local) Palais-Smale condition at the level c, denoted by $(P.S)_c$, if any sequence (u_n) in X such that

$$\varphi(u_n) \to c, \text{ and } \varphi'(u_n) \to 0, \text{as } n \to +\infty. \tag{1.2}$$

admits a convergent subsequence.

Theorem 1.12. *[3]Let X be a reflexive Banach space, and φ a functional defined on X such that*

1. *φ is $(w.l.s.c)$,*

2. *The minimizing sequence of φ is bounded on X,*

Then φ achieves its infimum on X.

Theorem 1.13. *[18]*

Let X be a Banach space and $\varphi : X \to \mathbb{R}$ a C^1- functional that is bounded from below, and let $c = \inf_X \varphi$. Assume that φ satisfies $(P.S)_c$. Then there exists $u \in X$ such that

$$\varphi(u) = \inf_{v \in X} \varphi(v), \quad \text{and} \quad \varphi'(u) = 0. \tag{1.3}$$

Theorem 1.14. ***(Mountain Pass Theorem)****[18]*

Let $\varphi \in C^1(X, \mathbb{R})$ satisfy the Palais-Smale condition. Assume that there exists $u_0, u_1 \in X$, and a bounded neighborhood Ω of u_0 satisfying $u_1 \notin \overline{\Omega}$ and

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf_{v \in \partial\Omega} \varphi(v).$$

Then there exists a critical value of φ, that is there exists $u \in X$ such that $\varphi'(u) = 0$ and $\varphi(u) > \max\{\varphi(u_0), \varphi(u_1)\}$.

1.2.2 Classical compactness theorem

Theorem 1.15. ***(Ascoli-Arzéla)****[5]*

Let (f_n) be a sequence of real functions defined on a compact metric space (X, d). We assume that the sequence (f_n) has the following two properties:

(1) *(f_n) is equicontinuous, i.e.,*

$$\forall \varepsilon > 0, \exists \eta > 0, \forall x, y \in X, \forall n \in \mathbb{N} : d(x, y) < \eta \Longrightarrow |f_n(x) - f_n(y)| < \varepsilon.$$

(2) *The sequence $(f_n(x))_n$ is bounded for all $x \in X$.*

Then there exists a subsequence of (f_n) that converges to a continuous function f in $C(X, \mathbb{R})$.

Theorem 1.16. ***(Rellich's theorem)****[7]*

If Ω is a regular bounded open of class C^1, then of any bounded sequence of $H^1(\Omega)$ we can extract a convergent subsequence in $L^2(\Omega)$ we say that the canonical injection of $H^1(\Omega)$ in $L^2(\Omega)$ is compact.

Chapter 2

Positive solutions of higher-order singular linear boundary value problems

In this chapter, we study some existence results of positive solutions for the singular linear boundary problem of higher order equation (2.1).To obtain our results, we use the variational method and the minimization theorem. We choose as an example the article [4] that was published by the author Y. Cui in **(2011)**.

2.1 Preliminaries

We consider the following problem:

$$\begin{cases} (-1)^m u^{(2m)}(x) = \lambda p(x) u(x), & x \in (0,1) \\ u^{(i)}(0) = u^{(i)}(1) = 0, & i = 0, \ldots, m-1 \end{cases} \tag{2.1}$$

where $m > 1$, $p \in C(0,1)$, and it may be singular at $x = 0$ or $x = 1$. We seek a positive classical solution for (2.1).

2.1.1 The basic hypothesis

The hypothesis on p in this memory is **(P)**, where:

1. $p \geq 0, p \in C(0,1), \exists x \in (0,1), p(x) > 0,$
2. and $\lim_{s \to 0} s^{2m-1} \int_s^{1-s} p(\tau) d\tau = 0.$

2.2 Variational construction of our problem

2.2.1 The appropriate spaces of our problem

In what follows, we will use the weighted space L^2_p, defined as:

$$L^2_p = \left\{ u \mid \int_0^1 p(s)u^2(s)ds < +\infty \right\}$$

with the norm:

$$\|u\|_{L^2_p} = \left(\int_0^1 p(s)u^2(s)ds \right)^{\frac{1}{2}}.$$

We also need the Sobolev space H^m_0, whose norm is denoted by $\|.\|_{H^m_0}$:

$$\|u\|_{H^m_0} = \left(\int_0^1 \left| u^{(m)}(s)ds \right|^2 \right)^{\frac{1}{2}}$$

H^m_0 is the completion of $C^\infty_0(0,1)$ with respect to this norm.

2.2.2 Somme embedding theorems in the Sobolev space $H^m_0(0,1)$

The study of Sobolev space embedding theorems are often interpreted as continuous or compact injections from one functional space into another. In fact, every embedding can be regarded as an injection that preserves additional structure such as continuity or compactness.
Continuous injections guarantee that boundedness in the input space, while compact injections ensure the existence of convergent subsequences. These results are fundamental in variational methods, since they allow the passage from weak to strong convergence and thus ensure the applicability of critical point theory to boundary value problems.

Proposition 2.1. *1. The space H^m_0 continuously embeds into C^1, that is: $H^m_0 \hookrightarrow C^1$.*

2. The space H^m_0 compactly embeds into C^1, that is: $H^m_0 \hookrightarrow\hookrightarrow C^1$.

Proof. 1. Let $u \in H_0^m$. By the Cauchy-Schwarz inequality:

$$\begin{aligned} |u(x)| &= \left| \int_0^x u'(s)ds \right| \leq \int_0^x |u'(s)|\, ds \\ &\leq \left(\int_0^x 1^2 ds \right)^{\frac{1}{2}} \left(\int_0^x |u'(s)|^2\, ds \right)^{\frac{1}{2}} \\ &\leq C_1 \|u\|_{H_0^1} \leq C_1 \|u\|_{H_0^2}. \end{aligned}$$

Because $\|u\|_{\mathrm{H}_0^1} \leq \|u\|_{\mathrm{H}_0^2}$, Thus, we have:

$$\|u\|_C = \sup_{x \in (0,1)} |u(x)| \leq C_1 \|u\|_{H_0^1} \leq C_1 \|u\|_{H_0^2} \qquad (*)$$

which means $H_0^1 \hookrightarrow C$. Similarly, by the Cauchy-Schwarz inequality, we have:

$$\begin{aligned} |u'(x)| &= \left| \int_0^x u''(s)ds \right| \leq \int_0^x |u''(s)|\, ds \\ &\leq \left(\int_0^x 1^2 ds \right)^{\frac{1}{2}} \left(\int_0^s |u''(s)|^2\, ds \right)^{\frac{1}{2}} \leq C_1 \|u\|_{H_0^2}. \end{aligned}$$

Thus,

$$\|u'\|_C = \sup_{x \in (0,1)} |u(x)| \leq C_1 \|u\|_{H_0^2} \qquad (**)$$

From (*) and (**), we obtain:

$$\|u\|_{C^1} = \max_{x \in (0,1)} (\|u\|_C + \|u'\|_C) \leq C_3 \|u\|_{H_0^2}$$

This implies that $H_0^2 \hookrightarrow C^1$.

By induction, we can show that $H_0^m \hookrightarrow C^{m-1}$, and since $C^i \hookrightarrow C^{i-1}, \quad \forall i \in \{1, \ldots, m-1\}$, we conclude that $H_0^m \hookrightarrow C^1$.

2. Let $\mathcal{B}$ be the unit ball in H_0^m. We show that $\mathcal{B}$ is relatively compact. We verify:

   (a) $\mathcal{B}$ is uniformly bounded, which follows from (1).

(b) $\mathcal{B}$ is equicontinuous. Indeed, for any $u \in \mathcal{B}$, we have:

$$\begin{aligned}
|u(x_2) - u(x_1)| &\leq \left| \int_{x_1}^{x_2} u'(s)ds \right| \leq \int_{x_1}^{x_2} |u'(s)|\, ds \\
&\leq \left(\int_{x_1}^{x_2} 1^2 ds \right)^{\frac{1}{2}} \left(\int_{x_1}^{x_2} |u'(s)|^2\, ds \right)^{\frac{1}{2}} \\
&= |x_2 - x_1|^{\frac{1}{2}} \left(\int_{x_1}^{x_2} |u'(s)|^2\, ds \right)^{\frac{1}{2}} \\
&\leq |x_2 - x_1|^{\frac{1}{2}} \left(\int_{0}^{1} |u'(s)|^2\, ds \right)^{\frac{1}{2}} \\
&= |x_2 - x_1|^{\frac{1}{2}} \|u\|_{H_0^1} \leq |x_2 - x_1|^{\frac{1}{2}} \|u\|_{H_0^m} \\
&\leq |x_2 - x_1|^{\frac{1}{2}} \to 0, \text{when} x_1 \to x_2
\end{aligned}$$

□

Theorem 2.1. *1. H_0^m is continuously injected into L_p^2 if $s^{2m-1} \int_s^{1-s} p(\tau)d\tau$ is bounded when $s \to 0$,*

2. H_0^m is compactly injected into L_p^2 if

$$\lim_{s \to 0} s^{2m-1} \int_s^{1-s} p(\tau)d\tau = 0 \tag{2.2}$$

Proof. In what follows, we will prove the injection theorem 2.1. To simplify the presentation, we assume that the problem (2.1) has only one singular point at $x = 0$, that is, $p \in C(0, 1]$.

Step 1: Continuous injection

We will show that this injection is continuous. It is easy to see that

$$\left| u^{(m-1)}(x) \right|^2 = \left(\int_0^x u^{(m)}(s)ds \right)^2 \leq x \int_0^x \left| u^{(m)}(s) \right|^2 ds \leq x \|u\|_{H_0^m}^2$$

By recurrence, we have

$$\left| u^{(m-i)}(x) \right|^2 \leq \frac{x^{2i-1}}{2^{i-1}(i-1)!} \|u\|_{H_0^m}^2, \quad i = 1, 2, \ldots, m.$$

By a change of variable, we also have

$$\begin{aligned} u(x) &= x\int_0^1 u'(\theta_1 x)\,d\theta_1 = x\int_0^1\left[\theta_1 x\int_0^1 u''(\theta_1\theta_2 x)\,d\theta_2\right]d\theta_1 \\ &= \dots \\ &= x^{m-1}\int_0^1\dots\int_0^1 u^{(m-1)}(\theta_1\dots\theta_{m-1}x)\times\theta_1^{m-2}\theta_2^{m-3}\dots\theta_{m-2}d\theta_{m-1}\dots d\theta_1 \end{aligned} \tag{2.3}$$

From Hardy's inequality and Minkowski's inequality (See Theorems (1.7) and (1.8) in chapter I) and for $\delta\in(0,1]$, we obtain

$$\begin{aligned} \left[\int_0^\delta\left|\frac{u(x)}{x^m}\right|^2 dx\right]^{\frac12} &= \left[\int_0^\delta\left|\int_0^1\dots\int_0^1\frac{u^{(m-1)}(\theta_1\dots\theta_{m-1}x)}{x}\theta_1^{m-2}\theta_2^{m-3}\dots\theta_{m-2}d\theta_{m-1}\dots d\theta_1\right|^2 dx\right]^{\frac12} \\ &\le \int_0^1\left[\int_0^\delta\left|\int_0^1\dots\int_0^1\frac{u^{(m-1)}(\theta_1\dots\theta_{m-1}x)}{x}\theta_1^{m-2}\theta_2^{m-3}\dots\theta_{m-2}d\theta_{m-1}\dots d\theta_1\right|^2 dx\right]^{\frac12} d\theta_{m-1} \\ &\le \dots \\ &\le \int_0^1\dots\int_0^1\left[\int_0^\delta\left|\frac{u^{(m-1)}(\theta_1\dots\theta_{m-1}x)}{x}\right|^2 dx\right]^{\frac12} d\theta_1\dots d\theta_{m-1} \\ &\le \int_0^1\dots\int_0^1\left[\int_0^1\left|u^{(m)}(x)\right|^2 dx\right]^{\frac12} d\theta_1\dots d\theta_{m-1} \le \|u\|_{H_0^m} \end{aligned} \tag{2.4}$$

and

$$\left[\int_0^\delta\left|\frac{u'(x)}{x^{m-1}}\right|^2 dx\right]^{\frac12} \le \|u\|_{H_0^m} \tag{2.5}$$

By integration by parts, we have for $\epsilon>0$,

$$\int_\varepsilon^1 p(x)u^2(x)dx = \underbrace{u^2(\varepsilon)\int_\varepsilon^1 p(\tau)d\tau}_{I_1} + \underbrace{2\int_\varepsilon^1 u'(x)u(x)\left(\int_x^1 p(\tau)d\tau\right)dx}_{I_2} \tag{2.6}$$

From (2.3), (2.4), (2.5) and $x^{2m-1}\displaystyle\int_x^1 p(s)ds<+\infty$, for the first term I_1 of (2.6), we get

$$I_1 \le c_1\|u\|_{H_0^m}^2 \tag{2.7}$$

For the second term I_2 of equation (2.6), we have

$$\begin{aligned} I_2 &\leq c_2 \int_0^1 \left| \frac{u(x)}{x^m} \| \frac{u'(x)}{x^{m-1}} \right| dx \\ &\leq c_2 \left(\int_0^1 \left| \frac{u(x)}{x^m} \right|^2 \right)^{\frac{1}{2}} \left(\int_0^1 \left| \frac{u'(x)}{x^{m-1}} \right|^2 \right)^{\frac{1}{2}} \\ &\leq c_2 \|u\|^2_{H_0^{m.}} \end{aligned} \tag{2.8}$$

From (2.7) and (2.8), and by taking the limit as $\epsilon \to 0$, we get

$$\|u\|_{L_p^2} \leq c \|u\|_{H_0^m}. \tag{2.9}$$

where $c = \sqrt{c_1 + c_2}$, so we have a continuous injection $H_0^m \hookrightarrow L_p^2$, provided that the quantity $x^{2m-1} \int_x^1 p(s)ds < +\infty$, remains bounded.

Step 2: Compact injection

We will now show that this injection is compact.

Suppose now that $x^{2m-1} \int_x^1 p(s)ds \to 0$ as $x^{2m-1} \int_x^1 p(s)ds \to 0$. Let $(u_n) \subset H_0^m$ be a bounded sequence. We can assume that (u_n) converges uniformly to a function u in $C^1(0,1)$.

We obtain from (2.6):

$$\int_0^1 p(x) \left| u_n(x) - u(x) \right|^2 dx = \underbrace{2 \int_0^1 \left(u_n'(x) - u'(x) \right) \left(u_n(x) - u(x) \right) \left(\int_x^1 p(\tau) d\tau \right) dx}_{I_3} \tag{2.10}$$

Since $\lim_{x \to 0} x^{2m-1} \int_x^1 p(s)ds = 0$, then, for all $\varepsilon > 0$, there exists $\delta > 0$ such that:

$0 < x^{2m-1} \int_x^1 p(s)ds < \varepsilon$ for $0 < x < \delta$. From (2.4), (2.5), we have

$$\begin{aligned}
I_3 \leq \leq & 2\int_0^1 \left|u_n'(x) - u'(x)\right|\left|u_n(x) - u(x)\right|\left(\int_x^1 p(\tau)d\tau\right)dx \\
= & 2\int_0^\delta \left|\frac{u_n'(x) - u'(x)}{x^{m-1}}\right| \left|\frac{u_n(x) - u(x)}{x^m}\right| \left(x^{2m-1}\int_x^1 p(\tau)d\tau\right)dx \\
& + 2\int_\delta^1 \left|u_n'(x) - u'(x)\right|\left|u_n(x) - u(x)\right|\left(\int_x^1 p(\tau)d\tau\right)dx \\
\leq & 2\varepsilon \left(\int_0^\delta \left|\frac{u_n'(x) - u'(x)}{x^{m-1}}\right|^2 dx\right)^{\frac{1}{2}} \left(\int_0^\delta \left|\frac{u_n(x) - u(x)}{x^m}\right|^2 dx\right)^{\frac{1}{2}} \\
& + C_\varepsilon \int_0^1 \left|u_n'(x) - u'(x)\right|\left|u_n(x) - u(x)\right| dx \\
\leq & 2\varepsilon \left\|u_n - u\right\|_{H_0^m}^2 + C_\varepsilon \left\|u_n' - u'\right\|_{C(0,1)} \left\|u_n - u\right\|_{C(0,1)}
\end{aligned}$$

Thus, $\int_0^1 p(x)\left|u_n(x) - u(x)\right|^2 dx \to 0$ as $n \to +\infty$. □

2.2.3 The Euler-Lagrange functional

In what follows, the variational method is used (see [7]).

For everything $u \in H_0^1(0,1)$ we define the functional J of Euler-Lagrange associated our problem (2.1) by:

$$J(u) = \frac{1}{2}\int_0^1 u^2(t) - \frac{\lambda}{2}\int_0^1 p(t)u^2(t)dt.$$

Here, in the linear problem, we put $f(u) = u$, so, $F(u) \overset{\text{def}}{=} \int_0^u f(s)ds = \frac{1}{2}u^2$.

Since p, G, and f they are positive, Thus,

$$u(x) = \frac{1}{2}\int_0^1 G(x,s)p(s)u^2(s)ds > 0, \quad \forall x \in (0,1).$$

2.3 Eigenvalue problem

2.3.1 Existence of positive solution via minimization theorem

Theorem 2.2. *Let*

$$\lambda_1 = \inf_{u \in H_0^m} \frac{\int_0^1 \left|u^{(m)}(x)\right|^2 dx}{\int_0^1 p(x)u^2(x)dx}$$

Then, $\lambda_1 > 0$ *is attained if* (2.2) *is satisfied.*

Proof. We now show that if (2.2) is verified, then $\lambda_1 > 0$ is attained (see [5]).

Step 1: The minimizing sequence is bounded. For a fixed $u \in H_0^m$, we define $I_1(u) = \|u\|^2_{H_0^m}$, $I_2(u) = \|u\|^2_{L^2_p}$ And we define the functional quotient $J : H_0^m \to \mathbb{R}$ by

$$J(u) = \frac{I_1(u)}{I_2(u)}$$

We have

$$\lambda_1 = \inf_{u \in H_0^m} J(u)$$

Poincaré's inequality immediately gives $\lambda_1 > 0$. Let (u_n) be a minimizing sequence. Clearly, $(|u_n|)$ is also a minimizing sequence for J, so we can assume that $u_n(x) \geq 0$ almost everywhere in $(0, 1)$. Since the functional J is homogeneous of degree zero, i.e.,

$$J(su) = J(u), \quad \text{for each } s \in \mathbb{R},$$

we can normalize u_n by setting $\|u_n\|_{L^2_p} = 1$ for all n.

Then, we see that $I_1(u_n)$, the norm of u_n in H_0^m, must be bounded independently of n, because $(u_n)_n$ is a minimizing sequence, i.e., $\lim_{n\to+\infty} J(u_n) = \inf_{u \in H_0^m} J(u) = \lambda_1$, so the sequence $(J(u_n))_n$ is bounded in $\mathbb{R}$, and since $J(u_n) = \|u_n\|^2_{H_0^m}$, this implies that (u_n) is bounded in H_0^m.

Step 2: J is (l.s.c).

By Sobolev's embedding (Theorem 2.1), we deduce, by passing to subsequences, that:

- $u_n \to u$ in $H_0^m(0,1)$,
- $u_n \to u$ in $L^2(0,1)$,
- $L^2(0,1)$ almost everywhere in $(0,1)$.

In particular, $\|u\|_{L^2} = 1$ and $u_n(x) \geq 0$ almost everywhere in $(0,1)$.

Then, by the lower semi-continuity of the norm, we have

$$\lambda_1 \leq J(u) = I_1(u) \leq \liminf_n I_1(u_n) = \liminf_n J(u_n) = \lambda_1,$$

which means J is (l.s.c). Therefore, by Theorem (1.12), J attains its infimum, so there exists $\varphi_1 \in H_0^m$ such that $\varphi_1 \neq 0$ and

$$J(\varphi_1) = \lambda_1 = \frac{\|\varphi_1\|^2_{H_0^m}}{\|\varphi_1\|^2_{L^2_p}}$$

The proof of the theorem is complete. □

2.4 Green's Function

We now give the formula for the Green function of the boundary problem (**??**).

Theorem 2.3. *[6]*

The Green function of the boundary problem 2.1is given by

$$G(x,s) = \begin{cases} P(x,s) & 0 \leq x \leq s \leq 1 \\ P(x,s) + \dfrac{(-1)^m}{(2m-1)!}(x-s)^{2m-1}, & 0 \leq s \leq x \leq 1 \end{cases} \tag{2.11}$$

where

$$P(x,s) = \sum_{j=0}^{m-1} \frac{s^{m-j-1}x^{m+j}}{(m-1)!\,j!\,(m-j-1)!} \times \int_s^1 (1-\tau)^{m-1}(-\tau)^j d\tau \tag{2.12}$$

Furthermore, the Green's function can be written in the following integral form:

$$G(x,s)=\begin{cases}\dfrac{1}{[(m-1)!]^2}\displaystyle\int_0^{x(1-s)}\tau^{m-1}(\tau+s-x)^{m-1}d\tau, & 0\le x\le s\le 1;\\ \dfrac{1}{[(m-1)!]^2}\displaystyle\int_0^{s(1-x)}\tau^{m-1}(\tau+x-s)^{m-1}d\tau, & 0\le s\le x\le 1\end{cases}\tag{2.13}$$

Proof. Step 1: We show that equation (2.11) is verified.

Let $h(x)\in(0,1)$. Then the following linear boundary value problem:

$$\begin{cases}(-1)^m u^{(2m)}=h(x), & x\in(0,1)\\ u^{(i)}(0)=u^{(i)}(1)=0, & i=0,\dots,m-1\end{cases}\tag{2.14}$$

admits a unique solution $u(x)$ which is defined by:

$$u(x)=\int_0^1 G(x,s)h(s)ds,\quad 0\le x\le 1\tag{2.15}$$

Since $u\in C^{2m}(0,1)$, we have the following expression:

$$u(x)=\sum_{i=0}^{2m-1}\frac{a_i x^i}{i!}+\frac{(-1)^m}{(2m-1)!}\int_0^x(x-s)^{2m-1}h(s)ds,\quad 0\le x\le 1\tag{2.16}$$

By virtue of the boundary conditions in (2.14), i.e.,

$$u^{(i)}(0)=0,\quad i=0,\dots,m-1$$

we obtain:

$$a_0=a_1=\ldots=a_{m-1}=0$$

By virtue of the other boundary conditions, i.e.,

$$u^{(i)}(1)=0,\quad i=0,\dots,m-1$$

we also have the system:

$$\begin{aligned}&\frac{a_m}{(m-j)!}+\frac{a_{m+1}}{(m+1-j)!}+\ldots+\frac{a_{2m-1}}{(2m-1-j)!}\\ &+\frac{(-1)^m}{(2m-1-j)!}\int_0^1(1-s)^{2m-1-j}h(s)ds=0,\quad 0\le j\le m-1\end{aligned}\tag{2.17}$$

Thus, using Cramer's rule, we obtain:

$$a_{m+j} = \frac{D_{j+1}}{D}, \quad j = 0, 1 \ldots, m-1$$

where D is the determinant of the coefficient matrix of system (2.17), and D_k, for $1 \leq k \leq m$, is the determinant of D when the k^{th} column of D is replaced by $\left(\bar{h}_1, \bar{h}_1, \ldots, \bar{h}_m\right)^T$, where:

$$\bar{h}_i = \frac{(-1)^{m-1}}{(2m-i)!} \int_0^1 (1-s)^{2m-i} h(s) ds, \quad i = 1, \ldots, m \tag{2.18}$$

Now, we will compute D and $\dfrac{D_{j+1}}{D}$ for $j = 0, 1, \ldots, m-1$.

$$D = \begin{vmatrix} \frac{1}{m!} & \frac{1}{(m+1)!} & \cdots & \frac{1}{(2m-2)!} & \frac{1}{(2m-1)!} \\ \frac{1}{(m-1)!} & \frac{1}{m!} & \cdots & \frac{1}{(2m-3)!} & \frac{1}{(2m-2)!} \\ \cdot & \cdot & \cdots & \cdot & \\ \cdot & \cdot & \cdots & \cdot & \\ \cdot & \cdot & \cdots & \cdot & \\ \frac{1}{2!} & \frac{1}{3!} & \cdots & \frac{1}{m!} & \frac{1}{(m+1)!} \\ \frac{1}{1!} & \frac{1}{2!} & \cdots & \frac{1}{(m-1)!} & \frac{1}{m!} \end{vmatrix}$$

Let L_i be the iii-th row of D multiplied by $(2m-1-i)$. Then, replacing L_i by $L_i - L_{i+1}$, we

obtain:

$$D = \begin{vmatrix} \frac{(2m-2)!(m-1)}{m!} & \frac{(2m-2)!(m-2)}{(m+1)!} & . & . & . & 1 & 0 \\ \frac{(2m-3)!(m-1)}{(m-1)!} & \frac{(2m-3)!(m-2)}{m!} & . & . & . & 1 & 0 \\ . & . & . & . & . & & \\ . & . & . & . & . & & \\ . & . & . & . & . & & \\ \frac{m!(m-1)}{2!} & \frac{m!(m-2)}{3!} & . & . & . & 1 & 0 \\ \frac{m!}{1!} & \frac{m!}{2!} & . & . & \frac{m!}{(m-1)!} & 1 & \end{vmatrix}$$

$$= (m-1)!(m-2)!\ldots 2! \begin{vmatrix} \frac{m!}{(m-1)!} & 1 \end{vmatrix}$$

$$= \left(\prod_{i=1}^{m-1} i!\right).$$

We can prove this result by induction. Similarly, we can obtain for all $j = 0, 1, 2, ..., m?1$,

$$D_{j+1} = \int_0^1 \frac{(m+j)!}{(m-1)!j!} s^{m-j-1} \times \sum_{i=0}^{j} C_j^i \frac{(-1)^{j-1}}{m+i} (1-s)^{m+i} h(s)ds \times \left(\prod_{i \neq m-j-1}^{m-1} i!\right)$$

and

$$\frac{D_{j+1}}{D} = \int_0^1 \frac{(m+j)!}{(m-1)!j!} s^{m-j-1} \times \sum_{i=0}^{j} C_j^i \frac{(-1)^{j-1}}{m+i} (1-s)^{m+i} h(s)ds \times \left(\prod_{i \neq m-j-1}^{m-1} i!\right) / \left[\left(\prod_{i=1}^{m-1} i!\right)\right]$$

which simplifies to

$$\frac{D_{j+1}}{D} = \int_0^1 \frac{(m+j)!}{(m-1)!j!(m-j-1)!} s^{m-j-1} \times \sum_{i=0}^{j} C_j^i \frac{(-1)^{j-1}}{m+i} (1-s)^{m+i} h(s)ds$$

where $C_j^i = \dfrac{j!}{i!(j-i)!}$ are the binomial coefficients. Thus, we obtain for $j = 0, 1, \ldots, m-1$,

$$a_{m+j} = \int_0^1 \frac{(m+j)!}{(m-1)!j!(m-j-1)!} s^{m-j-1} \times (1-s)^m \sum_{i=0}^{j} \frac{C_j^i (-1)^{j-i}}{(m+i)} (1-s)^i h(s)ds \tag{2.19}$$

so we can write $u(x)$ as follows:

$$\begin{aligned}
u(x) &= \sum_{j=0}^{m-1} \frac{x^{m+j}}{(m+j)!} \left(\int_0^1 \frac{(m+j)! s^{m-1-j}(1-s)^m}{(m-1)!\, j!} \sum_{i=0}^{j} C_j^i \frac{(-1)^{j-i}}{m+i} (1-s)^i h(s) ds \right) \\
&\quad + \frac{(-1)^m}{(2m-1)!} \int_0^x (x-s)^{2m-1} h(s) ds \\
&= \int_0^1 \sum_{j=0}^{m-1} \left(\frac{x^{m+j} s^{m-1-j}(1-s)^m}{(m-1)!\, j!} \sum_{i=0}^{j} C_j^i \frac{(-1)^{j-i}}{m+i} (1-s)^i \right) h(s) ds \\
&\quad + \frac{(-1)^m}{(2m-1)!} \int_0^x (x-s)^{2m-1} h(s) ds \\
&= \int_0^x \left(\sum_{j=0}^{m-1} \frac{x^{m+j} s^{m-1-j}(1-s)^m}{(m-1)!\, j!} \sum_{i=0}^{j} C_j^i \frac{(-1)^{j-i}}{m+i} (1-s)^i + \frac{(-1)^m}{(2m-1)!} (x-s)^{2m-1} \right) h(s) ds \\
&\quad + \int_x^1 \left(\sum_{j=0}^{m-1} \frac{x^{m+j} s^{m-1-j}(1-s)^m}{(m-1)!\, j!} \sum_{i=0}^{j} C_j^i \frac{(-1)^{j-i}}{m+i} (1-s)^i \right) h(s) ds.
\end{aligned}$$

We define:

$$P(x,s) = \sum_{j=0}^{m-1} \frac{1}{(m-1)!\, j!\, (m-j-1)!} s^{m-j-1} x^{m+j} \times \sum_{i=0}^{j} \frac{C_j^i (-1)^{j-i}}{m+i} (1-s)^{m+i}$$

Using the binomial formula $\displaystyle\sum_{i=0}^{j} \frac{C_j^i (-1)^{j-i}}{(m+i)} (1-s)^{m+i} = \int_s^1 (1-\tau)^{m-1} (-\tau)^j d\tau$, we obtain:

$$P(x,s) = \sum_{j=0}^{m-1} \frac{1}{(m-1)!\, j!\, (m-j-1)!} s^{m-j-1} x^{m+j} \times \int_s^1 (1-\tau)^{m-1} (-\tau)^j d\tau \tag{2.20}$$

Then, from (2.16) - (2.20) and for an arbitrary $h(x)$, we conclude that:

$$G(x,s) = \begin{cases} P(x,s), & 0 \le x \le s \le 1 \\ P(x,s) + \dfrac{(-1)^m}{(2m-1)!} (x-s)^{2m-1}, & 0 \le s \le x \le 1 \end{cases} \tag{2.21}$$

Step 2: We show that (2.13) is verified.

Since:

$$\sum_{i=0}^{j} \frac{C_j^i (-1)^{j-i}}{(m+i)} (1-s)^{m+i} = \int_s^1 (1-\tau)^{m-1} (-\tau)^j d\tau$$

we have:

$$\begin{aligned} P(x,s) &= x^m \frac{1}{(m-1)!}\int_s^1 (1-\tau)^{m-1}\sum_{j=0}^{m-1}\frac{1}{j!(m-j-i)!}\times(-x\tau)^j s^{m-j-1}d\tau \\ &= x^m \frac{1}{(m-1)!(m-1)!}\int_s^1 (1-\tau)^{m-1}\sum_{j=0}^{m-1} C_{m-1}^j \times(-x\tau)^j s^{m-j-1}d\tau \\ &= x^m \frac{1}{(m-1)!(m-1)!}\int_s^1 (1-\tau)^{m-1}\times[s-(\tau x)]^{m-1}d\tau \\ &= \frac{1}{(m-1)!(m-1)!}\int_s^1 [x(1-\tau)]^{m-1}\times[(s-x)+x(1-\tau)]^{m-1}d\tau \end{aligned}$$

For a fixed x, let $u = x(1-\tau)$, then the above expression is rewritten as:

$$P(x,s) = \frac{1}{[(m-1)!]^2}\int_0^{x(1-s)} u^{m-1}(u+s-x)^{m-1}du. \tag{2.22}$$

Let $v = u+s-x$, and s, x are fixed. Then, (2.22) will be of the form

$$\frac{1}{[(m-1)!]^2}\int_{s-x}^{(1-x)s} v^{m-1}(x-s+v)^{m-1}dv = \frac{1}{[(m-1)!]^2}\left[\int_0^{(1-x)s} v^{m-1}(x-s+v)^{m-1}dv - \int_0^{s-x} v^{m-1}(x-s+v)^{m-1}dv\right]$$

Let $v = (s-x)\lambda$. Then,

$$\begin{aligned} \frac{1}{[(m-1)!]^2}\int_0^{s-x} v^{m-1}(x-s+v)^{m-1}dv &= \frac{(-1)^{m-1}(s-x)^{2m-1}}{[(m-1)!]^2}\int_0^1 \lambda^{m-1}(1-\lambda)^{m-1}d\lambda \\ &= \frac{(-1)^m(x-s)^{2m-1}}{[(m-1)!]^2}\int_0^1 \lambda^{m-1}(1-\lambda)^{m-1}d\lambda \end{aligned} \tag{2.23}$$

By using integration by parts, it is easy to obtain

$$\int_0^1 \lambda^{m-1}(1-\lambda)^{m-1}d\lambda = \frac{[(m-1)!]^2}{(2m-1)!}$$

Thus, (2.23) will be of the form

$$\frac{(-1)^m(x-s)^{2m-1}}{[(m-1)!]^2}\int_0^1 \lambda^{m-1}(1-\lambda)^{m-1}d\lambda = \frac{(-1)^m(x-s)^{2m-1}}{(2m-1)!}, \quad (x>s) \tag{2.24}$$

By combining the results from (2.21) to (2.24), we obtain

$$G(x,s) = \begin{cases} \dfrac{1}{[(m-1)!]^2}\displaystyle\int_0^{x(1-s)} \tau^{m-1}(\tau+s-x)^{m-1}d\tau, & 0\le x\le s\le 1 \\ \dfrac{1}{[(m-1)!]^2}\displaystyle\int_0^{(1-x)s} \tau^{m-1}(\tau+x-s)^{m-1}d\tau, & 0\le s\le x\le 1 \end{cases} \tag{2.25}$$

□

Lemma 2.1. *Let*

$$\phi(x) = \frac{1}{2m-1}x^m(1-x)^m$$
$$\eta(s) = \frac{1}{[(m-1)!]^2}s^m(1-s)^m$$

Then for all $x, s \in (0,1)$, we have

$$\phi(x)\eta(s) \leq G(x,s) \leq \eta(s)$$

$\phi(x)$ *We can prove this result by induction. Similarly, we can obtain for all*

$$D_{j+1} = \int_0^1 \frac{(m+j)!}{(m-1)!j!}s^{m-j-1}\sum_{i=0}^{j} C_j^i \frac{(-1)^{j-i}}{m+i}(1-s)^{m+i}h(s)ds \times \left(\frac{\prod_{i\neq m-j-1}^{m-1} i!}{\prod_i^{m-1} i!}\right)$$

and

$$\frac{D_{j+1}}{D} = \int_0^1 \frac{(m+j)!}{(m-1)!j!}s^{m-j-1} \times \sum_{i=0}^{j} C_j^i \frac{(-1)^{j-i}}{m+i}(1-s)^{m+i}h(s)ds \times \left(\frac{\prod_{i\neq m-j-1}^{m-1} i!}{\prod_i^{m-1} i!}\right)$$

which simplifies to

$$\frac{D_{j+1}}{D} = \int_0^1 \frac{(m+j)!}{(m-1)!j!(m-j-1)!}s^{m-j-1} \times \sum_{i=0}^{j} C_j^i \frac{(-1)^{j-i}}{m+i}(1-s)^{m+i}h(s)ds$$

where $C_j^i = \dfrac{j!}{i!(j-i)!}$ are the binomial coefficients. Thus, we obtain for $j = 0, 1, ..., m-1$,

$$a_{m+j} = \int_0^1 \frac{(m+j)!}{(m-1)!j!(m-j-1)!}s^{m-j-1}(1-s)^m \sum_{i=0}^{j} C_j^i \frac{(-1)^{j-i}}{m+i}(1-s)^i h(s)ds,$$

so we can write $u(x)$ *as follows:*

$$\begin{aligned}
u(x) &= \sum_{j=0}^{m-1} \frac{x^{m+j}}{(m+j)!} \left(\int_0^1 \frac{(m+j)!}{(m-1)!j!} s^{m-j-1}(1-s)^m \sum_{i=0}^{j} C_j^i \frac{(-1)^{j-i}}{m+i} (1-s)^i h(s) ds \right) \\
&\qquad + \frac{(-1)^m}{(2m-1)!} \int_0^x (x-s)^{2m-1} h(s) ds \\
&= \int_0^1 \sum_{j=0}^{m-1} \left(\frac{x^{m+j} s^{m-j-1}(1-s)^m}{(m-1)!j!} \sum_{i=0}^{j} C_j^i \frac{(-1)^{j-i}}{m+i} (1-s)^i h(s) \right) ds \\
&\qquad + \frac{(-1)^m}{(2m-1)!} \int_0^x (x-s)^{2m-1} h(s) ds \\
&= \int_0^x \sum_{j=0}^{m-1} \frac{x^{m+j} s^{m-j-1}(1-s)^m}{(m-1)!j!} \sum_{i=0}^{j} C_j^i \frac{(-1)^{j-i}}{m+i} (1-s)^i h(s) ds \\
&+ \int_x^1 \sum_{j=0}^{m-1} \frac{x^{m+j} s^{m-j-1}(1-s)^m}{(m-1)!j!} \sum_{i=0}^{j} C_j^i \frac{(-1)^{j-i}}{m+i} (1-s)^i h(s) ds
\end{aligned}$$

For $A = \dfrac{1}{[(m-1)!]^2}$, *we also have*

$$\frac{\Lambda}{2m-1} x^m (1-x)^m s^m (1-s)^m \le G(x,s) \le \Lambda s^m (1-s)^m$$

Proof. By applying the binomial formula and integrating term by term in (2.11), we obtain

$$G(x,s) = \begin{cases} \dfrac{1}{[(m-1)!^2} \displaystyle\sum_{i=0}^{m-1} C_{m-1}^i (s-x)^i \frac{[x(1-s)]^{2m-i-1}}{(2m-i-1)} & 0 \le x \le s \le 1 \\ \dfrac{1}{[(m-1)]^2} \displaystyle\sum_{i=0}^{m-1} C_{m-1}^i (x-s)^i \frac{[s(1-x)]^{2m-1}}{(2m-i-1)}, & 0 \le s \le x \le 1 \end{cases} \tag{2.26}$$

Lower bound for $G(x,s)$ by replacing $2m-1-i$ with $2m-1$ in (2.26) and using the binomial

formula, we obtain

$$G(x,s) \geq \begin{cases} \dfrac{1}{[(m-1)!]^2(2m-1)}(x(1-s))^m(s(1-x))^{m-1}, & 0 \leq x \leq s \leq 1, \\ \dfrac{1}{[(m-1)!]^2(2m-1)}(x(1-s))^{m-1}(s(1-x))^m, & 0 \leq s \leq x \leq 1. \end{cases}$$

$$G(x,s) \geq \begin{cases} \dfrac{1}{[(m-1)!]^2(2m-1)}x^m(1-x)^m(1-s)^m s^{m-1}, & 0 \leq x \leq s \leq 1, \\ \dfrac{1}{[(m-1)!]^2(2m-1)}x^m(1-x)^m(1-s)^m s^m, & 0 \leq s \leq x \leq 1. \end{cases}$$

$$= \phi(x)\eta(s)$$

Upper bound for $G(x,s)$ On the other hand, by replacing $2m-1-j$ with 1 in (2.26) and applying the binomial formula again, we obtain

$$G(x,s) \leq \begin{cases} \dfrac{1}{[(m-1)!]^2}x^m(1-x)^{m-1}(1-s)^m s^{m-1}, & 0 \leq x \leq s \leq 1 \\ \dfrac{1}{[(m-1)!]^2}x^{m-1}(1-x)^m(1-s)^{m-1}s^m, & 0 \leq s \leq x \leq 1 \end{cases}$$

$$\leq \begin{cases} \dfrac{x}{s}\eta(s), & 0 \leq x \leq s \leq 1 \\ \dfrac{1-x}{1-s}\eta(s), & 0 \leq s \leq x \leq 1 \end{cases}$$

$$\leq \eta(s)$$

□

2.5 Positive classical solution and weak solution for linear problem (LP)

In what follows, the variational method is used for obtained our results (see [7]).

2.5.1 The Euler-Lagrange functional

We define $f(u) = 0$, if $u \leq 0$. Let F be the primitive function of f, i.e., $F(u) \overset{\text{def}}{=} \int_0^u f(s)ds$. There fore, for everything $u \in H_0^1(0,1)$ we define the functional J by:

$$J(u) = \frac{1}{2}\int_0^1 |u^{(m)}(x)|^2 - \lambda \int_0^1 p(x)F(u(x))dx.$$

Here, in the linear problem (LP), we put $f(u) = u$, so, $F(u) \overset{\text{def}}{=} \int_0^u f(s)ds = \frac{1}{2}u^2$. Thus,

$$J(u) = \frac{1}{2}\int_0^1 |u^{(m)}(x)|^2 - \frac{\lambda}{2}\int_0^1 p(x)u^2(x)dx.$$

Since p, G they are positives, then:

$$u(x) = \int_0^1 G(x,s)p(s)u(s)ds > 0, \quad \forall x \in (0,1).$$

Lemma 2.2. *Let*

$$\lambda_1 = \inf_{u \in H_0^m} \frac{\int_0^1 \left|u^{(m)}(x)\right|^2 dx}{\int_0^1 p(x)u^2(x)dx}$$

Then there exists a function $\varphi \in H_0^m, \varphi > 0$, such that

$$\lambda_1 = \frac{\int_0^1 \left|\varphi^{(m)}(x)\right|^2 dx}{\int_0^1 p(x)\varphi^2(x)dx}, \quad \forall x \in (0,1)$$

and φ satisfies

$$\begin{cases} (-1)^m \varphi^{(2m)}(x) = \lambda_1 p(x)\varphi(x), & x \in (0,1) \\ \varphi^{(i)}(0) = \varphi^{(i)}(1) = 0, & i = 0, \ldots, m-1 \end{cases} \tag{2.27}$$

Proof. **-1 Existence:** The existence of a minimizer φ is a consequence of Theorem 2.2, and φ satisfies the weak form of (2.27), i.e.,

$$\int_0^1 \varphi^{(m)}(x)v^{(m)}(x)dx = \lambda_1 \int_0^1 p(x)\varphi(x)v(x)dx, \quad \forall v \in H_0^m \tag{2.28}$$

By regularity theory, from (2.28), we obtain the problem (2.27). Obviously, φ is also the solution of the integral equation:

$$\varphi(x) = \lambda_1 \int_0^1 G(x,s)p(s)\varphi(s)ds$$

-1 Uniqueness: Let ω be the solution of the following equation:

$$\begin{cases} (-1)^m \omega^{(2m)}(x) = \lambda_1 p(x)|\varphi(x)|, & x \in (0,1) \\ \omega^{(i)}(0) = \omega^{(i)}(1) = 0, & i = 0, \dots, m-1 \end{cases} \tag{2.29}$$

Then,

$$\omega(x) = \lambda_1 \int_0^1 G(x,s)p(s)|\varphi(s)|ds \geq \left| \lambda_1 \int_0^1 G(x,s)p(s)\varphi(s)ds \right| = |\varphi(x)|.$$

On the other hand, let

$$\mu = \inf_{u \in I_0^m} \frac{\int_0^1 \left|u^{(m)}(x)\right|^2 dx}{\int_0^1 p(x)\left(u^+\right)^2(x)dx}$$

Then μ is attained; indeed, consider the space $L_p^{2,+}$ defined by

$$L_p^{2,+} = \left\{ u \mid \int_0^1 p(x)\left(u^+(x)\right)^2 dx < +\infty \right\}, \quad \text{where } u^+ = \max\{u, 0\}$$

Now, consider the functional J defined as follows:

$$J : H_0^m \to \mathbb{R}$$
$$u \mapsto \frac{\|u\|^2_{I_0^m}}{\|u\|^2_{L_p^{2,+}}}$$

From the proof of Theorem 3.1 (see chapter 3), let $u \in H_0^m$, be fixed. Define $I_1(u) = \|u\|^2_{I_0^m}$ and $I_2(u) = \|u\|^2_{L_p^{2+}}$, then

$$J(u) = \frac{I_1(u)}{I_2(u)}$$

Thus, we have

$$\mu = \inf_{u \in II_a^m} J(u)$$

Note that $H_0^m(0,1) \hookrightarrow L_p^{2,+}(0,1)$, since

$$u^+(x) \leq |u(x)|, \quad \forall x \in (0,1)$$

and since

$$\left(u^+(x)\right)^2 \leq |u(x)|^2 = (u(x))^2, \quad \forall x \in (0,1)$$

it follows that

$$\int_0^1 p(x)\left(u^+(x)\right)^2 dx \leq \int_0^1 p(x)(u(x))^2 dx \tag{2.30}$$

Thus, from (2.30), we obtain $\|u\|^2_{L^{2,+}_p} \leq \|u\|^2_{L^2_p}$, meaning $L^2_p \hookrightarrow L^{2,+}_p$. Since $H^m_0 \hookrightarrow\hookrightarrow L^2_p \hookrightarrow L^{2,+}_p$, we conclude that

$$H^m_0 \hookrightarrow\hookrightarrow L^{2,+}_p \tag{2.31}$$

Poincaré's inequality immediately gives $\mu > 0$.

Let (u_n) be a minimizing sequence. Clearly, $(|u_n|)$ is also a minimizing sequence for J, so we may assume that $u_n(x) \geq 0$ almost everywhere in $(0,1)$.

Since the functional J is homogeneous of degree zero, we can normalize u_n by setting for all $n \in \mathbb{N}, \|u_n\|^2_{L^{2,+}_p} = 1$. Next, we observe that $I_1(u_n)$, the norm of u_n in H^m_0, must be bounded independently of n, since (u_n) is a minimizing sequence, i.e., $\lim\limits_{n\to\infty} J(u_n) = \inf\limits_{u\in H^m_0} J(u) = \mu$ Thus, the sequence $(J(u_n))_n$ is bounded in $\mathbb{R}$, and since $J(u_n) = \dfrac{\|u_n\|^2_{H^m_0}}{\|u_n\|^2_{L^{2,+}_p}}$, this implies that (u_n) is bounded in H^m_0.

By Sobolev's embedding theorem (2.31), we deduce that, passing to subsequences,

- $u_n \to u$ in $H^m_0(0,1)$,
- $u_n \to u$ in $L^{2,+}_p(0,1)$,
- $u_n(x) \to u(x)$ almost everywhere in $(0,1)$.

In particular, $I_2(u) = 1$ and $u_n(x) \geq 0$ almost everywhere in $(0,1)$.

By the weak lower semi-continuity of the norm, we have

$$\mu \leq J(u) = I_1(u) \leq \liminf_n I_1(u_n) = \liminf_n J(u_n) = \mu$$

This means that J is weakly lower semi-continuous (w.l.s.c). Thus, by Theorem (1.12), J attains its infimum, i.e.,

$$\exists \psi \in H_0^m \quad \text{such that} \quad \mu = \frac{\|\psi\|^2_{H_0^m}}{\|\psi\|^2_{L_p^{2,+}}}$$

Consequently, $\psi \in H_0^m$ satisfies the weak form

$$\int_0^1 \psi^{(m)}(x)v^{(m)}(x)dx = \mu \int_0^1 p(x)\left(\psi^+\right)(x)v(x)dx, \quad \forall v \in H_0^m \tag{2.32}$$

It is evident that $\int_0^1 p(x)\left(u^+\right)^2(x)dx > 0$, and we obtain $\mu \geq \lambda_1$ because (2.30) gives

$$\begin{aligned} 0 < \int_0^1 p(x)\left(u^+(x)\right)\Big)^2 dx \leq \int_0^1 p(x)(u(x))^2 dx & \\ \Rightarrow \frac{\int_0^1 \left|u^{(m)}(x)\right|^2 dx}{\int_0^1 p(x)\left(u^+(x)\right)^2 dx} & \geq \frac{\int_0^1 \left|u^{(m)}(x)\right|^2 dx}{\int_0^1 p(x)(u(x))^2 dx}. \end{aligned}$$

Thus, it follows that

$$\mu = \inf_{u \in H_0^m} \frac{\int_0^1 \left|u^{(m)}(x)\right|^2 dx}{\int_0^1 p(x)\left(u^+(x)\right)^2 dx} \geq \inf_{u \in H_0^m} \frac{\int_0^1 \left|u^{(m)}(x)\right|^2 dx}{\int_0^1 p(x)(u(x))^2 dx} = \lambda_1$$

Similarly, we have

$$\begin{aligned} \psi(x) &= \mu \int_0^1 G(x,s)p(s)\left(\psi^+\right)(s)ds \\ &\geq \frac{x^m(1-x)^m\mu}{(2m-1)[(m-1)!]^2} \int_0^1 s^m(1-s)^m p(s)\left(\psi^+\right)(s)ds > 0 \end{aligned} \tag{2.33}$$

Then, we obtain $\mu = \lambda_1$. Indeed,

$$\begin{aligned} \mu \int_0^1 p(s)\psi(s)\omega(s)ds &\leq \mu \int_0^1 p(s)\left(\psi^+\right)(s)\omega(s)ds, (\psi \text{ minimizer }) \\ &= \int_0^1 \psi^{(m)}(s)\omega^{(m)}(s)ds \\ &= \lambda_1 \int_0^1 p(s)\psi(s)|\varphi(s)|ds \\ &\leq \lambda_1 \int_0^1 p(s)\psi(s)\omega(s)ds, (\text{ since } \omega \geq |\varphi|) \end{aligned}$$

Thus, $\mu \leq \lambda_1$ and $\mu = \lambda_1$. From (2.33), it follows that $\varphi(x) > 0, \forall x \in (0,1)$. □

Chapter 3

Positive solutions of higher-order singular boundary value problems

For works related to singular boundary problems, see references ([9], [10], [11], [12]) and those cited therein

3.1 Preliminaries

We consider the following problem:

$$\begin{cases} (-1)^m u^{(2m)}(x) = \lambda p(x) f(u(x)), & x \in (0,1) \\ u^{(i)}(0) = u^{(i)}(1) = 0, & i = 0, \dots, m-1 \end{cases} \tag{3.1}$$

where $m > 1$, $p \in C(0,1)$, and it may be singular at $x = 0$ or $x = 1$. We seek a positive classical solution for ((3.1)).

3.1.1 The basic hypothesis

We assume same hypothesis on p as in chapter 2, and donet by **(P)**, where:

1. $p \geq 0, p \in C(0,1), \exists x \in (0,1), p(x) > 0,$

2. and $\lim_{s\to 0} s^{2m-1}\int_s^{1-s} p(\tau)d\tau = 0.$

For the nonlinear function f, we assume:

(F) : $f \geq 0$ and $f \in C[0,+\infty)$, and we define:

$$f_0^- = \liminf_{u\to 0}\frac{f(u)}{u}, \quad f_0^+ = \limsup_{x\to 0}\frac{f(u)}{u}$$
$$f_\infty^- = \liminf_{u\to +\infty}\frac{f(u)}{u}, \quad f_\infty^+ = \limsup_{u\to +\infty}\frac{f(u)}{u}$$

3.2 Variational construction of our problem

3.2.1 The appropriate spaces of our problem

Use the same Hilbert spaces as in chapter two, we will use the weighted space L_p^2, defined as:

$$L_p^2 = \left\{u \mid \int_0^1 p(s)u^2(s)ds < +\infty\right\}$$

with the norm:

$$\|u\|_{L_p^2} = \left(\int_0^1 p(s)u^2(s)ds\right)^{\frac{1}{2}}.$$

We also need the Sobolev space H_0^m, whose norm is denoted by $\|.\|_{H_0^m}$:

$$\|u\|_{H_0^m} = \left(\int_0^1 \left|u^{(m)}(s)ds\right|^2\right)^{\frac{1}{2}}$$

H_0^m is the completion of $C_0^\infty(0,1)$ with respect to this norm.

3.2.2 The Euler-Lagrange functional

For everything $u \in H_0^n(0,1)$ we define the functional J by:

$$J(u) = \frac{1}{2}\int_0^1 |u^{(m)}(x)|^2 - \lambda\int_0^1 p(x)F(u(x))dx.$$

We define $f(u) = 0$, if $u \leq 0$, and let F be the primitive function of f, i.e., $F(u) \overset{\text{def}}{=} \int_0^u f(s)ds$. Since they are positive, then according operational writing the problem (3.1) equivalent to:

$$u(x) = \int_0^1 G(x,s)p(s)F(u(s))ds > 0, \quad \forall x \in (0,1).$$

Lemma 3.1. *Suppose that* (P) *and* (F) *are satisfied.* J *is well defined on* H_0^m *and is of class* C^1.

The Freshet derivative of J *has the following form:*

$$\langle J'(u), \varphi \rangle = \int_0^1 u^{(m)}(x)\varphi^{(m)}(x)dx - \int_0^1 p(x)f(u(x))\varphi(x)dx, \forall \varphi \in H_0^m \tag{3.2}$$

Proof. Step 1: We show that the functional J is Gâteaux differentiable. Let $\varphi \in H_0^m$ and $t \in \mathbb{R}$, then we have

$$\begin{aligned} J(u+t\varphi) - J(u) &= \frac{1}{2}\int_0^1 \left|(u+t\varphi)^{(m)}\right|^2 dx - \int_0^1 p(x)F(u+t\varphi)dx - \frac{1}{2}\int_0^1 \left|u^{(m)}\right|^2 dx + \int_0^1 p(x)F(u)dx \\ &= \frac{1}{2}\left(t^2\int_0^1 \left|\varphi^{(m)}\right|^2 + 2t\int_0^1 u^{(m)}\varphi^{(m)}\right)dx - \int_0^1 p(x)[F(u+t\varphi) - F(u) \mid dx \\ &= \frac{1}{2}\left(t^2\int_0^1 \left|\varphi^{(m)}\right|^2 + 2t\int_0^1 u^{(m)}\varphi^{(m)}\right)dx - t\int_0^1 p(x)f(u+\theta t\varphi)\varphi dx \end{aligned}$$

where $0 < \theta(x) < 1, \forall x \in (0,1)$. Furthermore :

$$\begin{aligned} \langle d_G J(u), \varphi \rangle &= \lim_{t\to 0} \frac{J(u+t\varphi) - J(u)}{t} \\ &= \int_0^1 u^{(m)}(x)\varphi^{(m)}(x)dx - \int_0^1 p(x)f(u(x))\varphi(x)dx, \forall \varphi \in H_0^m \end{aligned}$$

Step 2: We show that the functional $d_G J$ is continuous.

Let $(u_n)_n \subset H_0^m$ such that $u_n \to u$ as $n \to +\infty$, then we have

$$\begin{aligned} \langle d_G J(u_n) - d_G J(u), \varphi \rangle &= \int_0^1 u_n^{(m)}(x)\varphi^{(m)}(x)dx - \int_0^1 p(x)f(u_n(x))\varphi(x)dx - \int_0^1 u^{(m)}(x)\varphi^{(m)}(x)dx \\ &\quad + \int_0^1 p(x)f(u(x))\varphi(x)dx \\ &= \int_0^1 \left(u_n^{(m)}(x) - u^{(m)}(x)\right)\varphi^{(m)}(x)dx - \int_0^1 p(x)\left(f(u_n(x)) - f(u(x))\right)\varphi(x)dx \end{aligned}$$

Since the function f is continuous, we have

$$\langle d_G J(u_n) - d_G J(u), \varphi \rangle \to 0, \text{ as } \quad n \to +\infty, \quad \forall \varphi \in H_0^m$$

That is, $d_G J \in C^1\left(H_0^m, \mathbb{R}\right)$, hence we conclude that J is Fréchet differentiable.

That is, $\langle d_G J(u), \varphi \rangle = \langle J'(u), \varphi \rangle = \int_0^1 u^{(m)}(x)\varphi^{(m)}(x)dx - \int_0^1 p(x)f(u(x))\varphi(x)dx, \forall \varphi \in H_0^m$.

□

3.2.3 Classical Solution and Weak Solution

In what follows, the variational method is used (see [7]).

We define $f(u) = 0$, if $u \leq 0$. Let F be the primitive function of f, i.e., $F(u) \overset{\text{def}}{=} \int_0^u f(s)ds$

Since they are positive, then:

$$u(x) = \int_0^1 G(x,s)p(s)f(u(s))ds > 0, \quad \forall x \in (0,1).$$

Definition 3.1. A function u is called a classical solution of the boundary problem (3.1) if $u \in C^{2m-1}[0,1] \cap C^{2m}(0,1)$ and satisfies both the equation and the boundary conditions.

3.3 Existence results for nonlinear problem (NP) under the conditions $f_\infty^\pm \leq +\infty$

Theorem 3.1. *Suppose that **(P)** and **(F)** are satisfied. Suppose further that:*

$$f_0^+ < \lambda_1 < f_\infty^- \leq f_\infty^+ \leq +\infty \tag{3.3}$$

or

$$f_\infty^+ < \lambda_1 < f_0^- \leq f_0^+ \leq +\infty \tag{3.4}$$

Then, the boundary problem (3.1) *admits a classical positive solution.*

First, we verify condition $(P.S)$.

Lemma 3.2. *Suppose that* $(P),(F)$ *are verified and let* $f_0^+ < \lambda_1 < f_\infty^- \leq f_\infty^+ < +\infty$ *or* $f_\infty^+ < \lambda_1 < f_0^- \leq f_0^+ < +\infty$, *then* J *satisfies the Palais-Smale* $(P.S)$ *condition.*

Proof. Let (u_n) be a Palais-Smale $(P.S)$ sequence, then we have

$$\langle J'(u_n), \varphi \rangle = \int_0^1 u_n^{(m)}(x)\varphi^{(m)}(x)dx - \int_0^1 p(x)f(u_n(x))\varphi(x)dx \to 0, \quad n \to +\infty, \quad \forall \varphi \in H_0^m \tag{3.5}$$

Step 1: f satisfies the sublinear condition.

Since $f_\infty^+ < +\infty$ and $f_0^+ < +\infty$, then there exists a constant c such that

$$0 \le f(u) \le c|u|, \quad 0 \le F(u) \overset{\text{def}}{=} \int_0^u f(s)ds \le cu^2.$$

Indeed, from the hypothesis $f_\infty^+ < +\infty$, for $\varepsilon > 0$, there exists $M > 0$, such that

$$f(u) \le (f_\infty^+ + \varepsilon)\, u, \quad u \ge M$$

And from the hypothesis $f_0^+ < +\infty$, for $\varepsilon > 0$, there exists $\delta > 0$, such that

$$f(u) \le \left(f_0^+ + \varepsilon\right) u, \quad u \le \delta$$

Let $c_1 = \max\left\{f_\infty^+ + \varepsilon, f_0^+ + \varepsilon\right\}$, then we have

$$f(u) \le c_1 u, \quad \forall u \in [0, \delta] \cup [M, +\infty)$$

Since the function $f(u) - c_1 u$ is continuous on $[\delta, M]$, there exists a constant c_2 such that

$$f(u) - c_1 u \le c_2, \quad u \in [\delta, M]$$

that is,

$$f(u) \le c_1 u + c_2, \quad u \in [\delta, M]$$

Thus, for all $c \ge \dfrac{\delta c_1 + c_2}{\delta}$, we obtain

$$f(u) \le c_1 u + c_2 \le cu, \quad \forall u \in [0, +\infty)$$

Step 2: The sequence $\left(\left\|u_n^+\right\|_{L_p^2}\right)$ is bounded. Indeed, if $\left\|u_n^+\right\|_{L_p^2}$ is not bounded, then there exists a sub-sequence of (u_n) also denoted by (u_n), such that $\lim\limits_{n\to+\infty} \left\|u_n^+\right\|_{L_p^2} = +\infty$.

We define $v_n = \dfrac{u_n}{\|u_n\|_{L^2}}$, so that $\|v_n\|_{L^2} = 1$.

Taking $\varphi = v_n$ in (3.5), we get that $\|v_n\|_{H_0^m}$ is bounded.

Since

$$\int_0^1 v_n^{(m)} \left\|u_n^+\right\|_{L_p^2} v_n^{(m)} dx = \int_0^1 p(x) f(u_n) v_n dx$$

and also

$$\begin{aligned}
\|v_n\|_{H_0^m}^2 &= \int_0^1 p(x) \frac{f(u_n)}{\|u_n^+\|_{L_p^2}} v_n dx \\
&\leq \int_0^1 p(x) \frac{c(u_n)}{\|u_n^+\|_{L_p^2}} v_n dx \\
&\leq c \times \underbrace{\frac{1}{\|u_n^+\|_{L_p^2}} \left(\int_0^1 p(x) (u_n^+)^2 dx \right)^{\frac{1}{2}}}_{=1} \underbrace{\left(\int_0^1 p(x) (v_n)^2 dx \right)^{\frac{1}{2}}}_{=1}, \text{ Cauchy-Shwartz} \\
&\leq c.
\end{aligned}$$

Suppose that $v_n \rightharpoonup v$ in H_0^m, then $v_n \to v$ in $C(0,1)$ and in L_p^2, $\left\|v^+\right\|_{L_p^2} = 1$. From (3.5), we get

$$\int_0^1 v_n^{(m)}(x) \varphi^{(m)}(x) dx = \int_0^1 p(x) \frac{f(u_n(x))}{\|u_n^+\|_{L_p^2}} \varphi dx \tag{3.6}$$

We first consider the case $f_0^+ < \lambda_1 < f_\infty^- < f_\infty^+ < +\infty$. Let $\varepsilon > 0$ and $f_\infty^- - \varepsilon > \lambda_1$, and we choose a constant $M > 0$ such that $f(u) \geq (f_\infty^- - \varepsilon) u, \forall u > M$.

Let $\varphi \in H_0^m$ and $\varphi > 0$. From (3.5), we get

$$\begin{aligned}
\int_0^1 v_n^{(m)}(x) \varphi^{(m)} dx &= \int_0^1 p(x) \frac{f\left(\left\|u_n^+\right\|_{L_p^2} v_n\right)}{\|u_n^+\|_{L_p^2}} \varphi dx \\
&= \int_{\|u_n^+\|_{L_p^2} v_n < M} p(x) \frac{f\left(\left\|u_n^+\right\|_{L_p^2} v_n\right)}{\|u_n^+\|_{L_p^2}} \varphi dx + \int_{\|u_n^+\|_{L_p^2} v_n \geq M} p(x) \frac{f\left(\left\|u_n^+\right\|_p v_n\right)}{\|u_n^+\|_{L_p^2}} \varphi dx \\
&\geq \int_{\|u_n^+\|_{L_p^2} v_n < M} p(x) \frac{f\left(\left\|u_n^+\right\|_p v_n\right)}{\|u_n^+\|_{L_p^2}} \varphi dx + (f_\infty^- - \varepsilon) \int_{\|u_n^1\|_{L_p^2} v_n \geq M} p(x) v_n^+ \varphi dx \\
&\geq (f_\infty^- - \varepsilon) \int_0^1 p(x) v_n^+ \varphi dx - c \int_{\|u_n^+\|_{L_p^2} v_n < M} p(x) v_n^+ \varphi dx
\end{aligned} \tag{3.7}$$

Let $n \to +\infty$, we get:

$$\int_0^1 v^{(m)}(x) \varphi^{(m)} dx \geq (f_\infty^- - \varepsilon) \int_0^1 p(x) v^+ \varphi dx, \quad \forall \varphi \in H_0^m, \varphi \geq 0$$

In particular, let φ be the minimizer in Lemma 2.1, then

$$\lambda_1 \int_0^1 p(x)v^+\varphi dx \geq \lambda_1 \int_0^1 p(x)v\varphi dx = \int_0^1 v^{(m)}(x)\varphi^{(m)}dx \geq (f_\infty^- - \varepsilon)\int_0^1 p(x)v^+\varphi dx$$

This implies that $v^+ \equiv 0$. Consequently, $\left\|v^+\right\|_{L_p^2} = 0$, which is a contradiction.

Now, suppose that $f_\infty^+ < \lambda_1 < f_0^- < f_0^+ < +\infty$. Let $\varepsilon > 0$ and $f_\infty^+ + \varepsilon < \lambda_1$.. Similar to (3.7), we obtain:

$$\int_0^1 v_n^{(m)}(x)\varphi^{(m)}dx \leq (f_\infty^+ + \varepsilon)\int_0^1 p(x)v_n^+\varphi dx + c\int_{\|u_n^+\|_{L_p^2}v_n \leq M} p(x)v_n^+\varphi dx$$

And as $n \to +\infty$, we have:

$$\int_0^1 v^{(m)}(x)\varphi^{(m)}dx \leq (f_\infty^+ + \varepsilon)\int_0^1 p(x)v_n^+(x)\varphi(x)dx, \quad \forall \varphi \in H_0^m, \varphi \geq 0 \tag{3.8}$$

Taking $\varphi = v^+$ in (3.8), and by the definition of λ_1, we obtain:

$$\lambda_1 \int_0^1 p(x)\left(v^+\right)^2 dx \leq \int_0^1 \left|\left(v^+\right)^{(m)}\right|^2 dx \leq (f_\infty^+ + \varepsilon)\int_0^1 p(x)\left(v^+\right)^2 dx$$

Since $\int_0^1 p(x)\left(v^+\right)^2 dx = 1$, we obtain a contradiction, i.e. $\left(\left\|u_n^+\right\|_{L_p^2}\right)$ is bounded. **Step 3:** We now show that the sequence (u_n) satisfies the $(P.S)$ condition. From Step 2, we know that $\left\|u_n^+\right\|_{L_p^2}$ is bounded. Taking $\varphi = u_n$ in (3.5) and noting that $\left|f(u_n)\right| \leq cu_n^+$, we obtain a bound on $\|u_n\|_{H_0^m}$.

Let $u_n \rightharpoonup u$ in H_0^m. By the embedding theorem 2.1, we can assume that $u_n \to u$ in L_p^2, and since $f(u_n) \to f(u)$ in L_p^2, then from (3.5) applied to $u_n \to u$ in H_0^m. we get

$$\begin{aligned}
\left|\left\langle J'(u_n - u), u_n - u\right\rangle\right| &= \left|\int_0^1 \left|(u_n - u)^{(m)}\right|^2 dx - \int_0^1 p(x)f(u_n - u)(u_n - u)\,dx\right| \\
&\leq \left|\int_0^1 \left|(u_n - u)^{(m)}\right|^2 dx\right| + \left|\int_0^1 p(x)f(u_n - u)(u_n - u)\,dx\right|
\end{aligned}$$

By applying the Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned}
\left|\int_0^1 p(x)f(u_n - u)(u_n - u)\,dx\right| &= \left|\int_0^1 \sqrt{p(x)}f(u_n - u)\,\sqrt{p(x)}(u_n - u)\,dx\right| \\
&\leq \left(\int_0^1 p(x)\left|f(u_n - u)\right|^2 dx\right)^{\frac{1}{2}}\left(\int_0^1 p(x)(u_n - u)^2 dx\right)^{\frac{1}{2}} \\
&= \left(\int_0^1 p(x)\left|f(u_n - u)\right|^2 dx\right)^{\frac{1}{2}}\|u_n - u\|_{L_p^2}
\end{aligned}$$

Since

$$u_n - u \xrightarrow{L^2_p} 0 \Rightarrow f(u_n - u) \xrightarrow{L^2_p} f(0) = 0$$

we obtain

$$\begin{aligned} \int_0^1 \left|(u_n - u)^{(m)}\right|^2 dx &\to 0, \text{ in } n \to +\infty \\ \Rightarrow u_n &\xrightarrow{H^m_0} u, \text{ in } n \to +\infty \end{aligned}$$

This means that J satisfies the Palais-Smale $(P.S)$ condition. □

3.3.1 Existence of positive solution by Mountain Pass Theorem under the conditions $f^+_\infty < \lambda_1 < f^-_\infty \leq f^+_\infty < +\infty$

In what follows, we will prove theorem 3.1 in the case where $f^\pm_\infty < +\infty$ using a variational method, particularly by the Mountain Pass Theorem (1.14). We note that the condition $(P.S)$ has already been proven in the lemma 3.2 above. It remains to verify the following geometrical conditions:

1. There exist constants $\alpha, \rho > 0$ such that $J(u) \geq \alpha$ for all u with $\|u\|_{H^m_0} = \rho$.
2. There exists an element e_1 such that $J(e_1) \leq 0$ and $\|e_1\|_{H^m_0} > \rho$.

Proof. Step 1: Verifying Condition (i).

Let $\varepsilon > 0$, and assume $f^+_0 + \varepsilon < \lambda_1$. Then, there exists $\sigma > 0$ such that:

$$F(u) \overset{def}{=} \int_0^u f(s)ds \geq \frac{1}{2}\left(f^+_0 + \varepsilon\right)u^2, \quad \forall 0 \leq u \leq \sigma.$$

For $\rho \ll 1, \|u\|_{H^m_0} = \rho$, we have:

$$\begin{aligned} J(u) &= \frac{1}{2}\int_0^1 \left(u^{(m)}\right)^2 dx - \int_0^1 p(x)F(u)dx \\ &\geq \frac{1}{2}\int_0^1 \left(u^{(m)}\right)^2 dx - \frac{1}{2}\left(f^+_0 + \varepsilon\right)\int_0^1 p(x)u^2(x)dx \\ &\geq \frac{1}{2}\left(1 - \frac{\int_0^+ + \varepsilon}{\lambda_1}\right)\int_0^1 \left(u^{(m)}\right)^2 dx \\ &= \frac{1}{2}\left(1 - \frac{f^+_0 + \varepsilon}{\lambda_1}\right)\rho^2 > 0. \end{aligned}$$

Thus, setting $\alpha = \frac{1}{2}\left(1 - \frac{\rho_0^+ + e}{\lambda_1}\right)\rho^2$, we obtain:

$$J(u) \geq \alpha \quad \forall u \in H_0^m, \quad \|u\|_{H_0^m} = \rho$$

This confirms that J satisfies condition (i).

Step 2: Verifying Condition (ii)

Let φ be the eigenfunction from Lemma 2.2. Given the assumption $f_\infty^- > \lambda_1$, we have:

$$F(u) \overset{\text{def}}{=} \int_0^u f(s)ds \geq \frac{1}{2}\left(f_\infty^- - \varepsilon\right)u^2 - C, \quad \forall u > 0.$$

Indeed, if $f_\infty^- > \lambda_1$, then

$$\forall \varepsilon > 0, \exists M > 0, \text{ such that } : f(u) \geq \left(f_\infty^- - \varepsilon\right)u, \quad \forall u \geq M$$

Since

$$F(u) \overset{def}{=} \int_0^u f(s)ds \geq \frac{1}{2}\left(f_\infty^- - \varepsilon\right)u^2, \quad \forall u > M$$

and given that the function $F(u) - \frac{1}{2}\left(f_\infty^- - \varepsilon\right)u^2$ is continuous on $(0, M]$, there exists a constant $C > 0$, such that:

$$F(u) - \frac{1}{2}\left(f_\infty^- - \varepsilon\right)u^2 \geq -C, \quad \forall u \in (0, M]$$

Thus, it follows that:

$$F(u) \geq \frac{1}{2}\left(f_\infty^- - \varepsilon\right)u^2 - C, \quad \forall u \in (0, +\infty)$$

Moreover, we have:

$$\begin{aligned}\int_0^1 p(x)F(\varphi)dx &\geq \frac{1}{2}\left(f_\infty^- - \varepsilon\right)\int_0^1 p(x)(\varphi)^2 dx - C\int_0^1 p(x)dx \\ &= \frac{1}{2}\left(f_\infty^- - \varepsilon\right)\frac{\|\varphi\|_{H_0^m}^2}{\lambda_1} - C\int_0^1 p(x)dx\end{aligned}$$

Thus, for $T > 0$, we have:

$$\begin{aligned} J(T\varphi) &= \frac{1}{2}T^2 \int_0^1 \left(\varphi^{(m)}\right)^2 dx - \int_0^1 p(x)F(T\varphi)dx \\ &\leq \frac{1}{2}T^2 \int_0^1 \left(\varphi^{(m)}\right)^2 dx - \frac{1}{2}T^2 \left(f_\infty^- - \varepsilon\right) \int_0^1 p(x)(\varphi)^2 dx + C\int_0^1 p(x)dx \\ &= \frac{1}{2}T^2\|\varphi\|_{H_0^m}^2 - \frac{1}{2}T^2 \left(f_\infty^- - \varepsilon\right) \frac{\|\varphi\|_{H_0^m}^2}{\lambda_1} + C\int_0^1 p(x)dx \\ &= \frac{1}{2}T^2\|\varphi\|_{H_0^m}^2 \left(1 - \frac{f_\infty^- - \varepsilon}{\lambda_1}\right) + C\int_0^1 p(x)dx \end{aligned}$$

Since

$$\frac{1}{2}\|\varphi\|_{H_0^m}^2 \left(1 - \frac{f_\infty^- - \varepsilon}{\lambda_1}\right) < 0, \quad \forall \varphi \in H_0^m$$

Then, we can choose $e_1 = T_1\varphi$ and T_1 sufficiently large such that:

$$\frac{1}{2}T_1^2\|\varphi\|_{H_0^m}^2 \left(1 - \frac{f_\infty^- - \varepsilon}{\lambda_1}\right) + C\int_0^1 p(x)dx < 0$$

That is,

$$J(e_1) < 0, \quad \text{et } \|e_1\|_{H_0^m} = T_1 > \rho$$

Now, by the Mountain Pass Lemma, we define:

$$c = \inf_{\gamma\in\Gamma} \sup_{x\in(0,1)} J(\gamma(x))$$

where

$$\Gamma = \left\{\gamma \mid \gamma \in C\left((0,1), H_0^m\right), \gamma(0) = 0, \gamma(1) = T_1\varphi\right\}$$

$c \geq \alpha$ is a critical value of J, and J admits a critical point u with $J(u) = c$, u is a classical positive solution to our boundary problem (3.1). □

3.3.2 Existence of positive solution for nonlinear problem via minimization theorem under the conditions $f_\infty^+ < \lambda_1 < f_0^- \leq f_0^+ < +\infty$

Proof. In this case, we will use the minimization theorem.

Step 1: We show that the functional J is coercive, meaning $J(u) \to +\infty$ when $\|u\|_{H_0^m} \to +\infty$. We

use a contradiction argument.

Suppose there exists a sequence $(u_n) \subset H_0^m$, such that $\|u_n\|_{H_0^m} \to +\infty$ but $(u_n) \leq c_1 < +\infty$.

Let $v_n = \dfrac{u_n}{\|u_n^{+n}\|_{L_p^2}}$, so that $\left\|v_n^+\right\|_{L_p^2} = 1$ and

$$\begin{aligned} c_1 \geq J(u_n) &= \frac{1}{2}\int_0^1 \left|u_n^{(m)}\right|^2 dx - \int_0^1 p(x)F(u_n(x))\,dx \\ &\geq \frac{1}{2}\int_0^1 \left|u_n^{(m)}\right|^2 dx - \int_0^1 p(x)\left(u_n(x)\right)^2 dx \end{aligned}$$

Dividing by $\left\|u_n^+\right\|_{L_p^2}^2$ we get

$$\int_0^1 p(x)\frac{u_n^2(x)}{\|u_n^+\|_{L_p^2}^2}dx \geq o(1) + \frac{1}{2}\lambda_1$$

Let $\varepsilon > 0$ and $f_\infty^+ + \varepsilon < \lambda_1$, and choose a constant $M > 0$ such that $F(u) \leq \dfrac{1}{2}\left(f_\infty^+ + \varepsilon\right)u^2, \forall u >$ M. Thus, we have

$$\int_0^1 p(x)\frac{u_n^2(x)}{\|u_n^+\|_{L_p^2}^2}dx \leq \frac{1}{2}\left(f_\infty^+ + \varepsilon\right)\int_0^1 p(x)\left(v_n^+\right)^2 dx + c\int_{\|u_n^+\|_{L_p^2} v_n \leq M} p(x)\left(v_n^+\right)^2 dx$$

If we assume $v_n \to v$ in L_p^2, then $\displaystyle\int_0^1 p(x)\left(v^+\right)^2 dx - 1$ and $\lambda_1 \leq f_\infty^+ + \varepsilon$, which is a contradiction.

Step 2 : We show that J is weakly lower semi-continuous.

Let (u_n) be a sequence weakly converging to u in $H_0^m(0,1)$. meaning $u_n \rightharpoonup u$, quand

Let (u_n) be a sequence weakly converging to u in $H_0^m(0,1)$, meaning $u_n \rightharpoonup u$

As $n \to +\infty$, we have:

$$\|u\|_{I_0^m} \leq \liminf_{n \longrightarrow +\infty} \|u_n\|_{I_0^m} \qquad \text{(see example (I.1) in Chapter I)}$$

According to the embedding theorem 2.1, the sequence (u_n) converges uniformly to u on $[0,1]$. Therefore, we have:

$$\lim_{n\to+\infty}\int_0^1 p(x)F(u_n(x))\,dx = \int_0^1 p(x)F(u(x))dx$$

We conclude that:

$$J(u) \leq \liminf_{n \longrightarrow +\infty} J(u_n)$$

Thus J is weakly lower semi-continuous (w.l.s.c), and by Theorem (1.10), J has a minimizer u_0.

Step 3 : It remains to show that the trivial solution $u \equiv 0$ is not a local minimizer, which implies that u_0 is a nontrivial classical positive solution.

Let φ be the eigenfunction from Lemma 2.2, then we have:

$$\int_0^1 \left|\varphi^{(m)}\right|^2 dx = \lambda_1 \int_0^1 p(x)\varphi^2 dx$$

For sufficiently small $\sigma > 0$, we define:

$$J(\sigma\varphi) = \frac{1}{2}\sigma^2 \int_0^1 \left|\varphi^{(m)}\right|^2 dx - \int_0^1 p(x)F(\sigma\varphi)dx$$

Let $\varepsilon > 0$ and suppose $f_0^- - \varepsilon > \lambda_1$. We choose a constant $\sigma > 0$ such that:

$$F(u) \geq \frac{1}{2}\left(f_0^- - \varepsilon\right)u^2, \quad \forall 0 < u < \sigma$$

As a result, we obtain:

$$\begin{aligned} J(\sigma\varphi) &\leq \frac{1}{2}\sigma^2 \int_0^1 \left|\varphi^{(m)}\right|^2 dx - \frac{1}{2}\left(f_0^- - \varepsilon\right)\sigma^2 \int_0^1 p(x)\varphi^2(x)dx \\ &= \frac{1}{2}\sigma^2 \|\varphi\|^2_{H_0^m} - \frac{1}{2\lambda_1}\left(f_0^- - \varepsilon\right)\sigma^2 \|\varphi\|^2_{H_0^m} \\ &= \frac{1}{2}\sigma^2 \|\varphi\|^2_{H_0^m}\left(1 - \frac{f_0^- - \varepsilon}{\lambda_1}\right) < .0 \end{aligned}$$

□

3.3.3 Existence of a positive solution via minimization theorem under the condition $f_\infty^+ = +\infty$

Theorem 3.2. *Suppose that **(P)** is satisfied. Suppose further that:*

$$f_\infty^+ = +\infty \tag{3.9}$$

Then, the boundary problem 2.1 admits a classical positive solution.

Proof. We consider the case $f_0^+ < \lambda_1 < f_\infty^- \leq f_\infty^+ = +\infty$. For $\delta > 0$, we define:

$$\lambda_\delta = \inf_{u \in H_0^m} \frac{\int_0^1 \left|u^{(m)}\right|^2 dx}{\int_\delta^{1-\delta} p(x)u^2(x)dx}$$

where $\lambda_\delta \to \lambda_1$ as $\delta \to 0$. There exists a function $\varphi_\delta \in H_0^m$, that satisfies $\varphi_\delta(x) > 0, x \in (0,1)$,

$$\int_0^1 \varphi_\delta^{(m)}(x)v^{(m)}dx = \lambda_\delta \int_\delta^{1-\delta} p(x)\varphi_\delta(x)v(x)dx, \quad v \in H_0^m$$

To prove the existence of such λ_δ and φ_δ, we define the functional:

$$J : H_0^m \to \mathbb{R}$$
$$u \mapsto \frac{\|u\|^2_{H_0^m(0,1)}}{\|u\|^2_{L_p^2(\delta,1-\delta)}}$$

According to the proof of Theorem 2.2, for a fixed $u \in H_0^m$, we define $I_1(u) = \|u\|^2_{H_0^m}$, $I_2(u) = \|u\|^2_{L_p^2(\delta,1-\delta)}$, so that:

$$J(u) = \frac{I_1(u)}{I_2(u)}$$

Thus,

$$\lambda_\delta = \inf_{u \in H_o^m} J(u)$$

We note that $H_0^m(0,1) \hookrightarrow\hookrightarrow L_p^2(\delta, 1-\delta)$, since:

$$\int_\delta^{1-\delta} p(x)u^2(x)dx \leq \int_0^1 p(x)u^2(x)dx$$

Hence $\|u\|_{L_p^2(\delta,1-\delta)} \leq \|u\|_{L_p^2(0,1)}$, i.e., $L_p^2(0,1) \hookrightarrow L_p^2(\delta, 1-\delta)$, Since $H_0^m(0,1) \hookrightarrow\hookrightarrow L_p^2(0,1)$, it follows that:

$$H_0^m(0,1) \hookrightarrow\hookrightarrow L_p^2(0,1) \hookrightarrow L_p^2(\delta, 1-\delta) \tag{3.10}$$

By Poincaré's inequality, we immediately obtain $\lambda_\delta > 0$. Let (u_n) be a minimizing sequence. Clearly, $(|u_n|)$ is also a minimizing sequence for J, so we can assume $u_n(x) \geq 0$ almost everywhere in (0,1).

Since the functional J is homogeneous of degree zero, we can normalize u_n by setting $\|u\|^2_{L^2_p(\delta,1-\delta)} = 1$, for all n.

Next, we observe that $I_1(u_n)$, the norm of u_n in H^m_0, must be bounded independently of nnn, since $(u_n)_n$ is a minimizing sequence, i.e., , $\lim_{n\to+\infty} J(u_n) = \inf_{u\in H^m_0} J(u) = \lambda_\delta$ so the sequence $(J(u_n))_n$ is bounded in $\mathbb{R}$. Since $(u_n) = \|u_n\|^2_{H^m_0}$, this implies that (u_n) is bounded in H^m_0.

By the Sobolev embedding (3.10), we deduce, by passing to subsequences, that:

- $u_n \to u$ in $H^m_0(0,1)$,
- $u_n \to u$ in $L^2_p(\delta, 1-\delta)$,
- $u_n(x) \to u(x)$ almost everywhere in $(0,1)$

In particular, $I_2(u) = 1$ and $u_n(x) \geq 0$ almost everywhere in $(0,1)$.

Then, by the weak lower semicontinuity of the norm, we have

$$\lambda_\delta \leq J(u) = I_1(u) \leq \liminf_n I_1(u_n) = \liminf_n J(u_n) = \lambda_\delta,$$

That is, J is lower semicontinuous. By Theorem (1.12), J attains its infimum, so there exists $\varphi_\delta \in H^m_0$ such that $\varphi_\delta \neq 0$ and

$$J(\varphi_\delta) = \lambda_\delta = \frac{\left\|\varphi_\delta\right\|^2_{H^{m(0,1)}_0}}{\left\|\varphi_\delta\right\|^2_{L^2_p(\delta,1-\delta)}}$$

To show that $\lambda_\delta \to \lambda_1$, let φ be the eigenfunction from Lemma 2.2. Then,

$$\lambda_1 \leq \lambda_\delta \leq \frac{\int_0^1 \left|\varphi^{(m)}\right|^2 dx}{\int_\delta^{1-\delta} p(x)\varphi^2(x)dx} \to \frac{\int_0^1 \left|\varphi^{(m)}\right|^2 dx}{\int_0^1 p(x)\varphi^2(x)dx} \to \lambda_1, \text{ as } \delta \to 0.$$

We choose δ small enough so that $\lambda_\delta < \int_\infty^-$, and there exists $x \in [\delta, 1-\delta]$ such that $p(x) \neq 0$.

We choose M, Λ such that:

- $\lambda_\delta < \Lambda < f^-_\infty$,

- $f(u) \geq \Lambda u$ when $u \geq \dfrac{\delta^m(1-\delta)^m}{2m-1}M$.

We define a function

$$g(u) = \begin{cases} \Lambda(u-M) + f(M), & u \geq M \\ f(u), & u \leq M \end{cases}$$

It is easy to see that $g(u) \geq \Lambda u$ when $u \geq \dfrac{\delta^{mm}(1-\delta)^m}{2m-1}M$. By the previous result, there exists a solution u that satisfies

$$\begin{cases} (-1)^m u^{(2m)}(x) = p(x)g(u(x)), & x \in (0,1) \\ u^{(i)}(0) = u^{(i)}(1) = 0, & i = 0, \ldots, m-1 \end{cases} \tag{3.11}$$

or the weak form

$$\int_0^1 u^{(m)}(x)\varphi^{(m)}(x)dx = \int_0^1 p(x)g(u(x))\varphi(x)dx, \quad \forall \varphi \in H_0^m$$

According to (3.11), we have

$$\begin{aligned} u(x) &= \int_0^1 G(x,s)p(s)g(u(s))ds \\ &\geq \frac{x^m(1-x)^m}{2m-1}\int_0^1 G(\tau,s)p(s)g(u(s))ds \\ &= \frac{x^m(1-x)^m}{2m-1}u(\tau), \forall x, \tau \in (0,1) \end{aligned} \tag{3.12}$$

since $G(x,s) \geq \dfrac{x^m(1-x)^m}{2m-1}G(\tau,s), \quad \forall x, \tau, s \in (0,1)$

We show that the solution u of (3.11) satisfies $0 \leq u \leq M$ and is thus a solution of (3.1). Suppose this is not true, meaning $\|u\|_{C(0,1)} > M$. By (3.12), we have $u(x) \geq \dfrac{x^m(1-x)^m}{2m-1}M$ for $x \in [\delta, 1-\delta]$ and $g(u(x)) \geq \Lambda u(x)$ for $x \in [\delta, 1-\delta]$.

Now, for all $v \in H_0^m, v \geq 0$, we have

$$\begin{aligned}\int_0^1 u^{(m)}(x)v^{(m)}(x)dx &= \int_0^1 p(x)g(u(x))v(x)dx \\ &\geq \int_\delta^{1-\delta} p(x)g(u(x))v(x)dx \\ &\geq \Lambda \int_\delta^{1-\delta} p(x)u(x)v(x)dx\end{aligned}$$

In particular, choosing $v = \varphi_\delta$, the minimizer which is the eigenfunction associated with λ_δ, we get

$$\lambda_\delta \int_\delta^{1-\delta} p(x)u(x)\varphi_\delta(x)dx = \int_0^1 u^{(m)}(x)\varphi_\delta^{(m)}(x)dx \geq \Lambda \int_\delta^{1-\delta} p(x)u(x)\varphi_\delta(x)dx$$

This implies that$\lambda_\delta \int_\delta^{1-\delta} p(x)u(x)\varphi_\delta(x)dx = 0$ and since $p(x)u(x)\varphi_\delta(x) \equiv 0$ in $[\delta, 1-\delta]$, this is a contradiction.

We have shown that $\max_{x\in(0,1)} u \leq M$ and u is the desired solution for our original problem. □

3.4 Application

In numerous articles that have addressed the problem with singular boundary conditions (3.1) (see [8]), the following hypothesis is made:

$$\int_0^1 p(\tau)\tau^m(1-\tau)^m d\tau < +\infty \tag{3.13}$$

We note that (3.13) implies **(P)**. Another sufficient condition for (P) to be true is the following:

$$\lim_{s\to 0} s^{2m}p(s) = \lim_{s\to 1}(1-s)^{2m}p(s) = 0 \tag{3.14}$$

Next, we provide an estimation of λ_1. For convenience, we always assume that (3.14) is satisfied. Without loss of generality, let us assume that φ, the normalized positive eigenfunction corresponding to λ_1, satisfies $\|\varphi\|_{H_0^m} = \varphi(x_0) = 1$, then:

$$\begin{aligned}1 = \varphi(x_0) - \lambda_1 \int_0^1 G(x_0, s)\, p(s)\varphi(s)ds \leq \lambda_1 \int_0^1 G(x_0, s)\, p(s)ds \\ \leq \lambda_1 \max_{x\in(0,1)} \int_0^1 G(x,s)p(s)ds \leq \lambda_1 A \int_0^1 s^m(1-s)^m p(s)ds.\end{aligned}$$

Thus

$$\lambda_1 \geq \left(A \int_0^1 s^m (1-s)^m p(s) ds \right)^{-1}$$

On the other hand, it is easy to see that there exists $x_1 \in \left[\frac{1}{4}, \frac{3}{4}\right]$ such that $\varphi(x_1) = \min_{x \in \left|\frac{1}{4}, \frac{3}{4}\right|} \varphi(x) > 0$ et $G(x_1, s) \geq \frac{3^m \Lambda}{(2m-1)16^m} s^m \times (1-s)^m$ for all $s \in (0,1)$, hence

$$\begin{aligned} \varphi(x_1) &= \lambda_1 \int_0^1 G(x_1, s)\, p(s) \varphi(s) ds \geq \lambda_1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(x_1, s)\, p(s) \varphi(s) ds \\ &\geq \lambda_1 \varphi(x_1) \int_{\frac{1}{4}}^{\frac{3}{4}} G(x_1, s)\, p(s) ds \geq \lambda_1 \frac{3^m A}{(2m-1)16^m} \varphi(x_1) \int_{\frac{1}{4}}^{\frac{3}{4}} s^m (1-s)^m p(s) ds. \end{aligned}$$

Consequently,

$$\lambda_1 \leq \left(\frac{3^m A}{(2m-1)16^m} \int_{\frac{1}{4}}^{\frac{3}{4}} s^m (1-s)^m p(s) ds \right)^{-1}$$

Thus, we have

$$\begin{aligned} m_1 &\overset{d!}{=} \left(A \int_0^1 s^m (1-s)^m p(s) ds \right)^{-1} \leq \lambda_1 \\ &\leq \left(\frac{3^m A}{(2m-1)16^m} \int_{\frac{1}{4}}^{\frac{3}{4}} s^m (1-s)^m p(s) ds \right)^{-1} \overset{def}{=} M_1 \end{aligned}$$

Example 3.1. We consider

$$\begin{cases} u^{(4)}(x) - \dfrac{1}{2\pi} \left(x - x^2 \right)^{-2-\frac{1}{2}} u\left(1 + \dfrac{c}{1+u^2} \right), \quad x \in (0,1) \\ u(0) - u(1) - u'(0) - u'(1) = 0 \end{cases} \tag{3.15}$$

where $c > 511$, We define

$$p(x) = \frac{1}{2\pi} \left(x - x^2 \right)^{-2-\frac{1}{2}}, \quad f(u) = u\left(1 + \frac{c}{1+u^2} \right).$$

Then, we have

$$\begin{aligned} A &= 1 \\ m_1 &= \left(A \int_0^1 s^2 (1-s)^2 p(s) ds \right)^{-1} = 2 \\ M_1 &= \left(\frac{3^2 A}{(4-1)16^2} \int_{\frac{1}{4}}^{\frac{3}{4}} s^2 (1-s)^2 p(s) ds \right)^{-1} \simeq 512 \end{aligned}$$

Also $f_0^- = f_0^+ = 1 + c \in (512, +\infty), f_\infty^+ = 1 \in (0, 2), \quad \lambda_1 \in [2, 512]$, since
$m_1^{-1} = \left(\frac{1}{2\pi}\int_0^1 s^{-\frac{1}{2}}(1-s)^{-\frac{1}{2}}ds\right)^{-1} = \left(\frac{1}{2\pi}\pi\right)^{-1} = 2$, and

$$M_1^{-1} = \left(\frac{3}{16^2}\int_{\frac{1}{4}}^{\frac{3}{4}} s^{-\frac{1}{2}}(1-s)^{-\frac{1}{2}}ds\right)^{-1}$$
$$= \left(\frac{3}{16^2}\int_0^1 \left[\frac{1}{2}\left(x+\frac{1}{2}\right)\right]^{-\frac{1}{2}}\left[1-\frac{1}{2}\left(x+\frac{1}{2}\right)\right]^{-\frac{1}{2}}\frac{1}{2}dx\right)^{-1}, \text{ où } x = 2s - \frac{1}{2}$$

where $x = 2s - \frac{1}{2}$,

$$M_1^{-1} = \left(\frac{3}{16^2} \times \frac{1}{2\pi} \times \frac{\pi}{2}\right) \simeq 512$$

This implies that condition (3.4) is satisfied. Consequently, according to theorem 3.1, the boundary value problem (3.15) has a classical positive solution.

Conclusion

In this dissertation, we studied the existence of positive solutions for higher-order singular boundary value problems. The work was based on variational methods and on the results established by Yujun Cui.

First, we presented the mathematical tools required for our study, including Banach and Sobolev spaces, lower semi-continuity, critical point theory, the minimization theorem, and the Mountain Pass Theorem.

Next, we investigated a higher-order singular linear boundary value problem. By constructing an appropriate variational framework and studying the associated eigenvalue problem, we proved the existence of a positive solution. The Green function associated with the problem was also established and used to derive qualitative properties of the solutions.

Finally, we considered a nonlinear higher-order singular boundary value problem. Using variational techniques, minimization methods, and the Mountain Pass Theorem, we obtained several existence results for positive solutions under different assumptions on the nonlinear term. An illustrative example was presented to demonstrate the applicability of the theoretical results.

The results obtained in this work contribute to the study of singular higher-order boundary value problems and highlight the effectiveness of variational methods in proving the existence of positive solutions.

As a perspective, it would be interesting to investigate multiplicity results, uniqueness conditions, and the existence of positive solutions for more general classes of higher-order singular differential equations.

Bibliography

[1] J. Zhang Y. Yang. *Existence of solutions for some fourth-order boundary value problems with parameters. Nonlinear Anal.* **69(2008),** 1364-1375.

[2] Z. Liang F. Li, Y. Li. *Existence of solutions to nonlinear hammerstein integral equations and applications. J. Math. Anal. Appl.* 323**(2006),** 209-227.

[3] P.H. Rabinowitz. *Minimax methods in critical point theory with applications to differential equations. CBMS regional conferece series in mathematics. American mathematical society, providence, RI.* **65(1986)**.

[4] Y. Cui. *Positive solutions of higher-order singular boundary value problems. J. Appl. Math. Comput.* **37(2011),** 193-205.

[5] V.Trenoguine. *Analyse fonctionnelle. Traduction fran aise, Edition. Mir. Moscou.* **(1985)**.

[6] Kolmogorov. A, Fomine. S. *A. l ment de la th orie des fonctions et de l'analyse fonctionnelle. Traduction fran ais, edition Mir. Moscou.* **(1977).**

[7] H. Brezis. *Analyse fonctionnelle, th orie et applications. Masson, Paris.* **(1983).**

[8] X. Yang. *Green's function and positive solutions for higher-order ode. Appl. Math. Comput.* **136(2003),** 379-393.

[9] Agarwal. R. P, Perera. K, O'Regan. D. *Positive solutions of higher-order singular problems. Differ. Equ.* **41(2005),** 739-743.

[10] Agarwal. R. P, O'Regan. D. *Nonlinear superlinear singular and nonsingular second order boundary value problem. J. Differ. Equ.* **143(1998),** 60-95.

[11] Jiaquan .L, Pingan. Z, Ming. X. *Positive solutions for singular boundary value problem of second order.* Chin. Ann. Math. **3(2004),** 383-39.

[12] Pei. H. R, Sun. J. P. *Positive solution for fourth-order m-point nonhomogeneous boundary value problems.* J. Appl. Math. Comput. **32(2010),** 443-452.

[13] Marino. Badiale, Enrico. Serra. *Semilinear Elliptic Equations for Beginners Existence Results via the Variational Approach. Springer-Verlag London Limited.* **(2011).**

[14] E. Zeidler. *Nonlinear functional analysis and its applications, III : Variational methods and optimization. Springer, New York.* **(1985).**

[15] Jaroslav. Milota, Pavel. Dr bek. *Methods of Nonlinear Analysis Applications to Differential Equations. Birkh user advanced texts, Basler.* **(2007).**

[16] D. C. Lay, A. E. Taylor. *Introduction to functional analysis. Wiley.* **(1980).**

[17] G. Pyolya G. H. Hardy, J. E. Littlewood. *Inequalities. Cambridge university press, Cambridge.* **(1952).**

[18] Y. Jabri. *The Mountan Pass Theorem Variants, Generalizations and Some Applications. Cambridge University Press, New York.* **(2003).**

[19] Radu Precup. Methods in nonlinear integral equations. Kluwer Academic, Dorecha **(2002).**

Abstract

This dissertation is devoted to the study of positive solutions for higher-order singular boundary value problems using variational methods.

The studied problems are:

$$(-1)^m u^{(2m)}(x) = \lambda p(x)u(x), \quad x \in (0,1)$$

and

$$(-1)^m u^{(2m)}(x) = p(x)f(u(x)), \quad x \in (0,1)$$

with suitable boundary conditions.

In the first chapter, we presented some preliminaries from functional analysis and variational methods. In the second chapter, we studied the linear problem and proved the existence of positive solutions using minimization techniques and Green's function. Finally, in the third chapter, we investigated the nonlinear problem and established existence results by using variational methods and the Mountain Pass Theorem.
Keywords: Variation method. Critical point. Existence of solutions. Higher order. Palais-Smale condition. Minimization theory. Mountain Pass theorem.

ملخص

تتناول هذه المذكرة دراسة الحلول الموجبة لمسائل القيم الحدية الشاذة من الرتبة العليا باستعمال الطرق التباينية.

والمسائل المدروسة هي:

$$(-1)^m u^{(2m)}(x) = \lambda p(x)u(x), \quad x \in (0,1)$$

و

$$(-1)^m u^{(2m)}(x) = p(x)f(u(x)), \quad x \in (0,1)$$

مع شروط حدودية مناسبة.

في الفصل الأول، قدمنا بعض المفاهيم الأساسية المتعلقة بالتحليل الدالي والطرق التباينية.

أما في الفصل الثاني، فقد درسنا المشكلة الخطية وأثبتنا وجود حلول موجبة باستعمال تقنيات التصغير ودالة غرين.

وفي الفصل الثالث، تناولنا المشكلة غير الخطية وأثبتنا نتائج الوجود باستعمال الطرق التباينية ونظرية الممر الجبلي (Mountain Pass Theorem).
كلمات مفتاحية: طريقة التغاير .نقطة حرجة . وجود الحلول. درجة العليا . شرط بالي-سمال. نظرية التصغير. نظرية ممر الجبل