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Entropy solotions of nonlinear parabolic systems

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Decication

I dedicate this modest work :

- To my parents,

- To my brothers and sister,

- To all my familly,

- To all friends and all my departement familly,

- To all my adorable ones that i have known during all my life,

Ali chikouche Imane

Contents

Notation

We introduce the necessary notations and definition which are used in the sequel..

H	Hilbert space.
E	Banach space.
X'	The dual topology X.
Ω	Open set of $\mathbb{R}$.
$L^p(\Omega)$	$= \Big\{ u : \Omega \to \mathbb{R}$ mesurable : $\int_\Omega \|u\|^p dx \leq \infty$ with $1 \leq p < \infty. \Big\}$
$\\|u\\|_{L^p}$	$= \Big(\int_\Omega \|u(x)\|^p dx \Big)^{\frac{1}{p}}$.
$L^\infty(\Omega)$	$= \{u : \Omega \to \mathbb{R}$ mesurable and $\exists C \ : \|u(x)\| \leq C \ a.e \ in \ \Omega\}$.
$\\|u\\|_{L^\infty}$	$= \inf\{C : \|u(x)\| \leq C$ a.e in $\Omega\}$.
$C(\Omega)$	Continuous function space in Ω.
$D(A)$	Domain of definition of a bounded operator A.
$< .,. >$	Define a scalar product.
$\mathcal{L}(X, Y)$	Set of continuous linear applications.
a.e.	Almost everywhere.
$\mathbb{R}^N$	Euclidean space of dimension N, where N is a nonzero natural number.
x	Vector in $\mathbb{R}^N, x = (x_1, x_2, ..., x_N), x_i \in \mathbb{R}, 1 \leq i \leq N$.
dx	Lebesgue measure in N-dimensional space.
$\partial\Omega = \Gamma$	boundary of Ω.
$\|E\|$	Or $mes(E)$ measure of the set E.
$\|.\|$	Hilbert norm .
χ_E	Characteristic function of set E.
$\frac{\partial u}{\partial t}$	Outward normal derivative.
$f_n \to f$	Denotes that the sequence $\{f_n\}$ converge to f .
$f_n \rightharpoonup f$	Denotes that the sequence $\{f_n\}$ converge weakly to f.
Supp u	Support of the function u.
$\int_\Omega f(x)dx$	Integral of f in Ω with respect to the Lebesgue measure .
$\|u\|_p$	$= [\int_\Omega \|u(x)\|^p dx]^{1/p} = \|u\|_{L^p}$.
$\mathcal{D}(\Omega)$	Space of infinitely differentiable functions on Ω with compact support in Ω.
$\nabla u(x) =$	$\Big(\frac{\partial u}{\partial x_1}(x), ..., \frac{\partial u}{\partial x_N}(x)\Big)$.
$\|\nabla u\| =$	$\Big[\Big\|\frac{\partial u}{\partial x_1}\Big\|^2 + ... + \Big\|\frac{\partial u}{\partial x_N}\Big\|^2\Big]^{1/2}$.

Introduction

In this memory we detail the paper [18]. Who speaks about the existence of entropy and renormalized solutions in a function space for nonlinear parabolic equations.

Considering the following equations system:

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} - \operatorname{div}(A_1(u_1, \nabla u_1)) \\
&- \operatorname{div}((k_1 u_1 + \delta_{1,2} u_2 + \delta_{1,3} u_3)\nabla u_1 + \delta'_{1,2} u_1 \nabla u_2 + \delta'_{1,3} u_1 \nabla u_3) \\
&= -\sigma(u_1, u_2, u_3) - \mu_1 u_1 + f \ \text{ in } \ Q_T, \\
&\frac{\partial u_2}{\partial t} - \operatorname{div}(A_2(u_2, \nabla u_2)) \\
&- \operatorname{div}((\delta_{2,1} u_1 + k_2 u_2 + \delta_{2,3} u_3)\nabla u_2 + \delta'_{2,1} u_2 \nabla u_1 + \delta'_{2,3} u_2 \nabla u_3) \\
&= \sigma(u_1, u_2, u_3) - \varrho u_2 - \mu_2 u_2 + g \ \text{ in } \ Q_T, \\
&\frac{\partial u_3}{\partial t} - \operatorname{div}(A_3(u_3, \nabla u_3)) \\
&- \operatorname{div}((\delta_{3,1} u_1 + \delta_{3,2} u_3 + k_3 u_3)\nabla u_3 + \delta'_{3,1} u_3 \nabla u_1 + \delta'_{3,2} u_3 \nabla u_2) \\
&= \varrho u_2 - \mu_3 u_3 + h \ \text{ in } \ Q_T,
\end{aligned}$$

With the initial and boundary conditions

$$u_i(x,0) = u_{i,0}(x) \ \text{ in } \Omega, \ i = 1,2,3,$$

$$u_i(x,t) = 0 \ \text{ on } \sum_T, i = 1,2,3,$$

Where $Q_T = \Omega \times (0,T)$, $\sum_T = \partial\Omega \times (0,T)$, and $T > 0$.

It is worth mentioning that to prove the existence of renormalized and entropy solutions to this system , we are in need of the following hypotheses.

The divergence form operator $A_i(s,\zeta) : \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Caratheodory function (that is,it is continuous with respect to s and ζ) such that

$(H1)$ $A_i(s,\zeta)\zeta \geq \alpha_i|\zeta|^2$, for every $\zeta \in \mathbb{R}^N$, where $\alpha_i > 0$ and $i = 1, 2, 3$,

$(H2)$ For any $k > 0$, there exists $\beta_k > 0$ and a function $C_k(x,t) \in L^2(Q_T)$ such that

$$|A_i(s,\zeta)| \leq C_k(x,t) + \beta_k|\zeta|, i = 1, 2, 3,$$

$(H3)$ $[A_i(s,\zeta) - A_i(s,\zeta')][\zeta - \zeta'] \geq 0, i = 1, 2, 3,$

$(H4)$ $u_{i,0}(x) \in L^1(\Omega), i = 1, 2, 3,$

$(H5)$ $f(x,t), g(x,t), h(x,t) \in L^1(Q_T).$

We are going to prove the existence of the weak and entropy solutions of the problem. To do this, we will approximate the problem by using the Faedo-Garalkin method, then we will prove some estimates uniform on the sequence of solutions of these problem, once this is done the linearity of the operator, as well as the boundedness and the continuity of the operator, we will make it possible to pass to the limit, thus finding the solution.

This memory is devided into three chapters as follows:

The first chapter is dedicated to giving some basic definitions and results with functional analysis tools essential to the achievement of the objectives for the study of the problem. For example we recall L^p and H spaces, and some theorems (Lebesgue convergence, weak convergence, Fatou,...). Then the Faedo-Garalkin method, and model example.

In the second chapter, we gives some inequalities of problems in the nonlinear limit with examples, that help us to prove the existence of the problem .

Finally, in the third chapter, we proof the existence of the renormalized and entropy solutions of nonlinear parabolic system in Ω.

Chapter 1

Preliminaries and basic tools

The purpose of this first chapter is to present a number of analytical tools that will be used throughout this memory. we will also take the opportunity to introduce the main notations.

1.1 Functional spaces

1.1.1 Lebesgue spaces

An L^p space is a space of functions for which the p-th power of their absolute value is Lebesgue integrable. L^p spaces are also called Lebesgue spaces.
Let Ω be an open subset of $\mathbb{R}^N$ and let p be a real number greater than or equal to 1. The Lebesgue space $L^p(\Omega)$ is defined as:

$$L^p(\Omega) = \{u : \Omega \longrightarrow \mathbb{R} \quad \text{is measurable and} \quad \int_\Omega |u(x)|^p < \infty\}$$

We equip it with the following norm

$$\|u\|_{L^p(\Omega)} = \|u\|_p = \left(\int_\Omega |u(x)|^p\right)^{\frac{1}{p}} \quad , \quad p \geq 1$$

By $L^\infty(\Omega)$ we denote the set

$$L^\infty(\Omega) = \{u : \Omega \longrightarrow \mathbb{R} \quad \text{measurable} \ , \exists M > 0 \quad |u(x)| \leq M \quad a.e\}$$

It is equipped with the following

$$\|u\|_{\infty,\Omega} = inf\{M > 0 : |u| \leq M \quad a.e \ \text{ in } \Omega\}$$

The space $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u,v)_{2,\Omega} = \int_\Omega uv dx$$

Remark 1.1. *Let* $1 \leq p \leq \infty$, *be denoted by* p' *the conjugate exponent of* p

$$i.e. \quad \frac{1}{p} + \frac{1}{p'} = 1$$

Let $1 \leq p \leq \infty$*.Then the dual space of* $L^p(\Omega)$ *is* $L^{p'}(\Omega)$*.*

1.1.2 Hilbert spaces

The Hilbert space $H^1(\Omega)$ is defined as:

$$H^1(\Omega) = \left\{ v \in L^2(\Omega) : \frac{\partial v}{\partial x_i} \in L^2(\Omega), i = 1, 2, ..., N \right\}$$

Where $\frac{\partial v}{\partial x_i}$ is the derivative of v.

The space $H^1(\Omega)$ is equipped with the scalar product

$$\langle u, v \rangle = \int_\Omega \left(uv + \sum_X \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right) dx$$

It is equipped with the following norm

$$\|v\|_{H^1(\Omega)} = \left[\int_\Omega \left(v^2 + \sum_X \left(\frac{\partial v}{\partial x_i} \right)^2 \right) dx \right]^{\frac{1}{2}}$$

Remark 1.2. *Let* $u \in H^1(\Omega)$*. The function* u *belongs to* $H_0^1(\Omega)$ *if and only if* $u = 0$ *in* $\partial\Omega$*.*

1.2 Main Convergence theorems

Theorem 1.1. ***(Fischer-Riesz):*** *The set* $L^p\left([a,b], \|.\|_p\right)$ *with* $1 \leq p < \infty$ *is a Banach space.*

Lemma 1.1. ***(Divergence and Green's formulas):*** *Let* Ω *be a domain in* $\mathbb{R}^N$*, and* $n(x)$ *its exterior normal. Let* u *and* v *be two regular functions, and* w *be a vector field defined on* Ω*. Then*

$$\int_\Omega divw \;\; dx = \int_{\partial\Omega} w.n \, d\sigma \quad \textit{(Divergence formula)}$$

$$\int_\Omega (\Delta u)_v \;\; dx = -\int_\Omega \nabla u.\nabla v \;\; dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \;\; d\sigma \quad \textit{(Green formula)}$$

Theorem 1.2. ***(Rellich-Kondrachov):****[13] Let* Ω *be a bounded open set of* $\mathbb{R}^N$ *and* $p \geq 1$*. If* $p < N$ *then for any* q *such that* $1 \leq q \leq p^*$ *(with* $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$ *), the injection of* $W_0^{1,p}(\Omega)$ *in* $L^q(\Omega)$ *is continuous and for all* q *such that* $1 \leq q \leq p^*$ *the injection is compact, that is the bounded of* $W_0^{1,p}(\Omega)$ *are relatively compact in* $L^q(\Omega)$*.*

Definition 1.1. *Let* (u_n) *be a sequence of measurable functions on* Ω *and* u *a measurable function on* Ω*.*

1. *The sequence* (u_n) *convergente almost everywhere on* Ω *to* u *if and only if*

$$meas\{x \in \Omega : u_n(x) \;\; \textit{does not converge to} \;\; u(x)\} = 0$$

2. *The sequence* (u_n) *is said to be Cauchy in measure if for every* $\epsilon > 0$ *and every* $\eta > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that for all* $m, n \geq n_0$*, so*

$$meas\{x \in \Omega : |u_n(x) - u(x)| > \eta\} \leq \epsilon$$

Lemma 1.2. *[23] Let* (u_n) *be a sequence of measurable functions of* Ω *in* $\mathbb{R}^N$*.If* (u_n) *of Cauchy in measure then there exists a sub-sequence of* (u_n) *converge almost everywhere.*

Lemma 1.3. *[22] Let f be a strictly positive measurable function. then for every $\epsilon > 0$, there exists $\delta > 0$ such that for every measurable $A \subset \Omega$, we have*

$$\int_A f\,dx \leq \delta \Rightarrow meas(A) \leq \epsilon \tag{1.1}$$

Lemma 1.4. *[11] Let (u_n) the sequence in $L^p(\Omega)$ with $1 < p < \infty$. Suppose that*

1. *(u_n) is bounded in $L^p(\Omega)$.*

2. *$u_n \to u$ a.e. on Ω*

Then, $u_n \to u$ in $L^q(\Omega)$, for all $1 \leq q < p$ and weakly in $L^p(\Omega)$, i.e.,

$$\lim_{n\to\infty} \int_\Omega u_n v\,dx = \int_\Omega uv\,dx \qquad \forall v \in L^{p'}(\Omega)$$

Theorem 1.3. ***(Fatou's Lemma [13]):*** *Let (u_n) be a sequence of non-negative measurable function on Ω. Then*

$$\int_\Omega \liminf_{n\to\infty} u_n(x)dx \leq \liminf_{n\to\infty} \int_\Omega u_n(x)dx$$

Theorem 1.4. ***(Giuseppe Vitali Convergence Theorem [12]):*** *Let $p \in [1, \infty[$ and (f_n) be a sequence of functions in $L^p(\Omega)$ such that*

1. *$f_n \to f$ a.e. in Ω*

2. *(f_n) equal-integration in Ω, i.e. $\forall \epsilon > 0, \exists \delta > 0, \forall E \subset \Omega$ with $|E| \leq \delta$, such that*

$$\int_E |f_n(x)|^p dx \leq \epsilon, \qquad \forall n \in \mathbb{N}$$

Or

$$\limsup_{|E|\to 0} \int_E |f_n(x)|^p dx = 0, \qquad \forall n \in \mathbb{N}$$

Then

$$f \in L^p(\Omega) \text{ and } f_n \to f \text{ in } L^p(\Omega) \text{ strongly}$$

Theorem 1.5. ***(Lebesgue's Dominated Convergence Theorem [13]):*** *Let $p \in [1, \infty[$, and (u_n) be a sequence of functions in $L^p(\Omega)$ such that*

1. $u_n \to u \quad$ *a.e. in* Ω

2. *There is a function $v \in L^p(\Omega)$ satisfying*

$$|u_n(x)| \leq v(x) \quad a.e. \ in\, \Omega \quad \forall n \in \mathbb{N}$$

Then $u_n \to u$ in $L^p(\Omega)$, that is

$$u \in L^p(\Omega), \quad and \quad \|u_n - u\|_{L^p(\Omega)} \to 0 \quad as \quad n \to \infty$$

1.2.1 Weak convergence

Definition 1.2. *Let E be a Banach space, E' its dual, and $\langle .,. \rangle$ the duality of EE'. We say that the sequence (x_n) in E weakly converges to $x \in E$ if and only if:*

$$\lim_{n\to\infty} \langle f, x_n \rangle \longrightarrow \langle f, x \rangle, \quad \forall f \in E'$$

We denote this as

$$x_n \rightharpoonup x \quad in\ E$$

Proposition 1.1.

1. *If $x_n \to x$ strongly then $x_n \rightharpoonup x$ in E.*

2. *If $x_n \rightharpoonup x$ in E then $\|x_n\|_E$ is bounded.*

3. *If $x_n \rightharpoonup x$, weakly in E, and $f_n \longrightarrow x$ strongly in E'. Then*

$$\langle f_n, x_n \rangle \longrightarrow \langle f, x \rangle \quad as\ n \longrightarrow +\infty$$

Theorem 1.6. *Let E be a reflexive Banach space. Then, for any bounded sequence $(x_n)_n \subset E$, there exists a sub-sequence on index that convergente weakly in E.*

1.3 Faedo-Galerkin method for a nonlinear boundary problem

Let Ω be an open bounded set of $\mathbb{R}^N$ and Γ his border, we have

$$(P) : \begin{cases} \frac{\partial^2 u}{\partial t^2} + Au + |u|^\rho u = f \quad \text{on } Q = \Omega \times]0, T[& \text{(P.1)} \\ u = 0 \quad \text{on } \sum = \Gamma \times]0, T[& \text{(P.2)} \\ u(x, 0) = u_0(x), \frac{\partial u}{\partial t}(x, 0) = u_1(x) \quad \forall x \in \Omega & \text{(P.3)} \end{cases}$$

Where u, f represent the displacement vector, the density of external forces respectively and u_0, u_1 and $\rho > 0$ are data. A is an operator defined by

$$\begin{cases} Au = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x, t) \frac{\partial u}{\partial x_j} \right) & \text{(A.1)} \\ a_{ij} \in C^1(\overline{\Omega} \times]0, T[), \quad a_{ij} = a_{ji} \ \ \forall i, j & \text{(A.2)} \\ \sum_{i,j=1}^{n} a_{ij}(x, t)\zeta_i \zeta_j \geq \alpha(\zeta_1^2 + ... + \zeta_n^2), \quad \alpha > 0, \zeta \in \mathbb{R} & \text{(A.3)} \end{cases}$$

The aim of this work is to search for a function $u = u(x, t);\ x \in \Omega,\ t \in]0, T[$ that has a real values solution of the problem (P) under certain hypotheses (H). In order to pose the problem, and to have the tools to resolve it, we have need to introduce some notions and some functional spaces that we will use later.

We define the space V by

$$V = H_0^1(\Omega) \cap L^p(\Omega) \tag{1.2}$$

Where $p = \rho+2$. The norm of space V is $\|v\|_{H_0^1(\Omega)} + |v|_{L^p(\Omega)}$ that is a Hilbert space. Let's remember that

$$H_0^1(\Omega) = \text{adhesion of } \mathcal{D}(\Omega) \text{ in } H^1(\Omega)$$
$$= \left\{ v \in H^1(\Omega); \quad v = 0 \text{ in } \Gamma \right\}$$

Where $H^1(\Omega)$ is a sobolev space of order 1 given as follows

$$\left(|v|^2_{L^2(\Omega)} + \sum_{i=1}^{n} |\frac{\partial v}{\partial x_i}|^2_{L^2(\Omega)} \right)^{1/2} = \|v\|_{H^1(\Omega)} \tag{1.3}$$

Then we have

$$\begin{cases} H_0^1(\Omega) \subset L^q(\Omega); \\ \frac{1}{q} = \frac{1}{2} - \frac{1}{n} \quad \text{if} \quad n \geq 3, \end{cases} \tag{1.4}$$

So that $V = H_0^1(\Omega)$ if $\rho \leq \frac{4}{n-2}$

Lemma 1.5. *The space V defined in* (1.2) *is separable (i.e admits a dense countable set)*

Now we introduce the functions spaces in x and t. Let X be a Banach space defined by

$$L^p(0,T;X) = \left\{ \begin{array}{l} f :]0,T[\to X \ \text{ measurable and such that} \\ \|f\|_{L^p(0,T;X)} = \begin{cases} (\int_0^T \|f(t)\|_X^p)^{1/p} < \infty \ \text{ if } \ 1 \leq p < \infty; \\ \text{sup ess } \|f(t)\|_X < \infty \ \text{ if } \ p = \infty, \end{cases} \end{array} \right\}$$

Which is a complete space. Naturally we have $L^p(0,T;L^p(\Omega)) = L^p(Q)$.

Lemma 1.6. *If* $f \in L^p(0,T;X)$ *and* $\frac{\partial f}{\partial t} \in L^p(0,T;X)(1 \leq p \leq \infty)$, *then* f *is after possible modification on a set of zero measure of* $(0,T)$, *coutinuous of* $[0,T] \to X$.

Lemma 1.7. ***(Gronwall lemma):*** *Let* $f \in L^\infty(0,T)$, $g \in L^1(0,T)$ *and* $f(t) \geq 0$, $g(t) \geq 0$.

$$\text{if } \ f(t) \leq c + \int_0^T f(s)g(s)ds; \ \text{ then } \ f(t) \leq c\exp\left(\int_0^T g(s)ds\right)$$

Notation 1.1. *To simplify the writing, we pose that*

$$u(x,t) = u; \quad f(x,t) = f; \quad \frac{\partial u}{\partial t} = u'; \quad \frac{\partial^2 u}{\partial t^2} = u''$$

$$\|u\|_{H_0^1(\Omega)} = \|u\|; \quad \|u\|_{L^p(\Omega)} = |u|_p;$$

We are now able to precisely formulate the problem (P), to study it, we will need the following hypotheses:

$$(H):\begin{cases} f\in L^2(Q) & (H.1)\\ u_0\in H_0^1(\Omega)\cap L^p(\Omega), \quad p=\rho+2 & (H.2)\\ u_1\in L^2(\Omega) & (H.3)\end{cases}$$

1.3.1 Existence and uniqueness

Variational formulation

Multiplying the equation $(P.1)$ by $v\in V$, and integrating on Ω, then using Green's formula we obtain the following variational formulation:

$$(u''(t),v)+a(u(t),v)+(|u(t)|^\rho u(t),v)=(f,v);\quad \forall v\in V, \tag{1.5}$$

Where

$$a(u,v)=(Au,v)=\sum_{i,j=1}^{n}\int_\Omega a_{ij}(x,t)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}dx,$$

The application $a:V\times V\to\mathbb{R}$ is a continuous and coercive bilinear form. Indeed, according to $(A.3)$ for $\zeta_i=\frac{\partial u}{\partial x_i}$, we find that

$$a(u,u)\geq \alpha_1\int_\Omega\sum_{i=1}^{n}|\frac{\partial u}{\partial x_i}|^2dx=\alpha_1|\nabla u|_2^2$$

But according to poincaré inequality[13] we have

$$a(u,u)\geq\alpha_1|\nabla u|_2^2\geq\alpha\|u\|^2 \tag{1.6}$$

1. **Existence**

Theorem 1.7. *Under the hypotheses (H), the problem (P) admits a solution u*

$$u\in L^\infty(0,T;H_0^1(\Omega)\cap L^p(\Omega)) \tag{1.7}$$

$$\frac{\partial u}{\partial t}\in L^\infty(0,T;L^2(\Omega)) \tag{1.8}$$

Remark 1.3. *The two expressions* $u(x,0) = u_0(x)$ *and* $u'(x,0) = u_1(x)$ *have a meaning.*

Indeed, for $(1.7), (1.8)$ *and Lemma* 1.6 *we conclude that* u *is continuous for* $[0,T] \to L^2(\Omega)$ *so that* $u(x,0) = u_0(x)$ *has a meaning.*

To verify that $u'(x,0) = u_1(x)$ *have a meaning, we must use the equation* $(P.1)$ *wich is written*

$$\frac{\partial^2 u}{\partial t^2} = f + Au - |u|^\rho u \tag{1.9}$$

As $A \in \mathcal{L}(H_0^1(\Omega)); H^{-1}(\Omega))$ *we have* $Au \in L^\infty(0,T; H^{-1}(\Omega))$ *and*

$$|u|^\rho u \in L^\infty(0,T; L^{p'}(\Omega)) \quad \text{with} \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

So that

$$\frac{\partial^2 u}{\partial t^2} \in L^2(0,T; L^2(\Omega)) + L^\infty(0,T; H^{-1}(\Omega) + L^{p'}(\Omega))$$

Hence

$$\frac{\partial^2 u}{\partial t^2} \in L^\infty(0,T; H^{-1}(\Omega) + L^{p'}(\Omega))$$

Which joins (1.8)*, by the lemma* (1.6)*, that* $\frac{\partial u}{\partial t}$ *is continuous for* $[0,T] \to H^{-1}(\Omega) + L^{p'}(\Omega)$*, so* $u'(x,0) = u_1(x)$ *has a meaning .*

proof of Theorem 1.7

The proof is based on the Faedo-Galerkin method, we detail (6) which consists of the following three steps:

- Search for approximate solutions.
- A priori estimates are established on these approximate solutions.
- We pass to the limit, because of compactness properties (in the non-linear terms).

Example

$$\frac{3d^2y}{dx} - \frac{dy}{dx} - 8 = 0 \qquad 0 \le x \le 1, \ \ y(0) = 1$$

Assume

$$y = c_0 + c_1x + c_2x^2 + c_3x^3, \quad y(0) = 1$$

$$\Rightarrow 1 = c_0 + c_1 \times 0 + c_2 \times 0 + c_3 \times 0 \quad \Rightarrow c_0 = 1$$

$$\Rightarrow 1 = c_0 + c_1 \times 0 + c_2 \times 0 + c_3 \times 0 \quad \Rightarrow c_0 = 1$$

$$y(1) = 2 = 2 = c_0 + c_1 + c_2 + c_3 \quad \Rightarrow c_1 = 1 - c_2 - c_3$$

$$\Rightarrow y = 1 + (1 - c_2 - c_3)x + c_2x^2 + c_3x^3$$

$$y = 1 + x + c_2(x^2 - x) + c_3(x^3 - x) \quad \text{............}(1)$$

$$\Rightarrow \frac{dy}{dx} = 1 + c_2(2x - 1) + c_3(3x^2 - 1)$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2c_2 + 6xc_3$$

Then

$$\frac{3d^2y}{dx} - \frac{dy}{dx} - 8 = 0$$

$$3(2c_2 + 6xc_3) - (1 + c_2(2x - 1) + c_3(3x^2 - 1)) + 8 = 0$$

$$\Rightarrow c_2(7 - 2x) + c_3(18x - 3x^2 + 1) + 7 = 0 \quad \text{............}(2)$$

$$\int_0^1 w_i \cdot (c_2(7 - 2x) + c_3(18x - 3x^2 + 1) + 7 = 0) = 0 \Rightarrow \quad w_1 = x^2 - x, \quad w_2 = x^3 - x$$

$$i = 1, w_1 = x^2 - x \Rightarrow \int_0^1 (x^2 - x)(c_2(7 - 2x) + c_3(18x - 3x^2 + 1) + 7 = 0) = 0$$

$$\Rightarrow \int_0^1 c_2(7 - 2x)(x^2 - x) + c_3(18x - 3x^2 + 1)(x^2 - x) + 7(x^2 - x) = 0$$

$$\Rightarrow c_2 + 10516c_2 = -10/67 \quad \text{............}(3)$$

$$i = 2, w_2 = x^3 - x \Rightarrow \int_0^1 (x^3 - x)(c_2(7 - 2x) + c_3(18x - 3x^2 + 1) + 7 = 0) = 0$$

$$\Rightarrow 204c_3 = -1075 \quad \text{............}(4)$$

For $(1), (2), (3)$ and (4), we have

$$c_2 = -0,905, \qquad c_3 = -0,146$$

Chapter 2

Some Inequalities used in Parabolic and Elliptic Equations

The Cauchy inequality is the familiar expression

$$2ab \leq a^2 + b^2$$

This can be proven very simply: noting that $(a-b)^2 \geq 0$, we have

$$0 \leq (a-b)^2 = a^2 - 2ab + b^2$$

which, after rearanging terms, is precisely the Cauchy inequality. In this note, we prove Young's inequality, which is a version of the Cauchy inequality that lets the power of 2 be replaced by the power of p for any $1 < p < \infty$ From Young's inequality follow the Minkowski inequality (the triangle inequality for the l^p-norms), and the Hölder inequalities.

2.1 Young's inequality

For all non-negative real numbers a, b and every $1 < p < \infty$, then we have

$$ab \leq \frac{p-1}{p} a^{\frac{p}{p-1}} + \frac{1}{p} b^p$$

The first thing to note is Young's inequality is a far-reaching generalization of Cauchy's inequality. In particular, if $p = 2$, then $\frac{1}{p} = \frac{p-1}{p} = \frac{1}{2}$, and we have Cauchy's inequality:

$$ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$$

Normally to use Young's inequality one chooses a specific p, and a and b are free-floating quantities. For instance, if $p = 5$, we get

$$ab \leq \frac{4}{5}a^{\frac{5}{4}} + \frac{1}{5}b^5$$

Before proving Young's inequality, we require a certain fact about the exponential function.

Lemma 2.1. ***(The interpolation inequality for*** e^x***):*** *If* $t \in [0, T]$*, then*

$$e^{ta+(1-t)b} \leq te^a + (1-t)e^b$$

Proof. The equation of the secant line through the points (a, e^a) and (b, e^b) on the graph of e^x is precisely

$$t \to (ta + (1-t)b, te^a + (1-t)e^b)$$

Obviously the graph of this line intersects the graph of e^x at precisely two points: (b, e^b) when $t = 0$, and (a, e^a) when $t = 1$. To parametize the graph of e^x so that the $x-$ value of this parametrization and that of the parametrization of the secant line are the same, we use

$$t \to (ta + (1-t)b, e^{ta+(1-t)b})$$

But because e^x is concave up, any secant line lies above the graph in between the points of intersection. This means precisely that the y-values of these two parametrized curves obey

$$e^{ta+(1-t)b} \leq te^a + (1-t)e^b$$

Which was to be proved.

Theorem 2.1. ***(Young's inequality):*** *Assume a and b are real numbers, and $p > 1$. Then*

$$ab \le \frac{p-1}{p} a^{\frac{p}{p-1}} + \frac{1}{p} b^p$$

Proof. There are a number of conceptually different ways to prove this inequality. Our method will use Lemma 2.1. Writing

$$\begin{aligned} ab &= e^{\log a + \log b} \\ &= Exp\Bigg(\frac{p-1}{p}\frac{p}{p-1}\log a + \left(1 - \frac{p-1}{p}\right)\left(\frac{1}{1-\frac{p-1}{p}}\right)\log b\Bigg), \end{aligned}$$

From Lemma 2.1. we get

$$\begin{aligned} ab &\le \frac{p-1}{p} Exp\left(\frac{p}{p-1}\log a\right) + \left(1 - \frac{p-1}{p}\right) Exp\left(\left(\frac{1}{1-\frac{p-1}{p}}\right)\log b\right) \\ &= \frac{p-1}{p} Exp\left(\frac{p}{p-1}\log a\right) + \frac{1}{p} Exp\left(p \log b\right) \\ &= \frac{p-1}{p} a^{\frac{p}{p-1}} + \frac{1}{p} b^p. \end{aligned}$$

Example 2.1. *For any $a, b > 0, p, q \in \mathbb{R} \setminus \{0\}, \frac{1}{p} + \frac{1}{q} = 1$, it yields that*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}, \quad \text{if } p > 1$$

and

$$ab \ge \frac{a^p}{p} + \frac{b^q}{q}, \quad \text{if } p < 1, p \ne 0$$

We have

$$ab = e^{\ln ab} = e^{\frac{1}{p}\ln a^p + \frac{1}{q}\ln b^q} \le \frac{1}{p} e^{\ln a^p} + \frac{1}{q} e^{\ln b^q} = \frac{1}{p} a^p + \frac{1}{q} b^q$$

2.2 Minkowski's Inequality

Theorem 2.2. *If $1 \leq p < \infty$, then whenever $f, g \in \nu_F$ we have*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Proof. To prove that $\|f+g\|_p \leq \|f\|_p + \|g\|_p$, we will replace g by tg, and use the observation that

$$\|f + g\|_p - \|f\|_p = \int_0^1 \frac{d}{dt}\|f + tg\|_p dt \tag{2.1}$$

$$\|f\| + \|g\|_p - \|f\|_p = \int_0^1 \frac{d}{dt}(\|f\|_p + t\|g\|_p) dt \tag{2.2}$$

And then all we need to prove is that

$$\frac{d}{dt}\|f + tg\|_p \leq \frac{d}{dt}(\|f\|_p + t\|g\|_p) \tag{2.3}$$

which is actually simpler. Note that the right side of (2.3) is just $\|Y\|_p$.

Computing the left side is slightly tougher:

$$\begin{aligned}
\frac{d}{dt}\|f + tg\|_p &= \frac{d}{dt}\left(\sum_{i=1}^{\infty} |f_i - tg_i|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^{\infty} |f_i - tg_i|^p\right)^{\frac{1-p}{p}} \sum_{i=1}^{\infty} |f_i - tg_i|^{p-1}.\, sgn(f_i - tg_i).\, g_i \\
&= \|f - tg\|_p^{1-p}. \sum_{i=1}^{\infty} |f_i - tg_i|^{p-1}.\, sgn(f_i - tg_i).\, y_i
\end{aligned}$$

But of course $sgn(f_i - tg_i).\, g_i \leq |g_i|$, so we have

$$\frac{d}{dt}\|f + tg\|_p \leq \sum_{i=1}^{\infty} \left(\frac{|f_i - tg_i|}{\|f - tg\|_p}\right)^{p-1} |g_i|$$

To proceed from here, we manipulate this expression so that eventually we can use Young's inequality to our advantage. We have

$$\begin{aligned}\frac{d}{dt}\|f+tg\|_p &\leq \sum_{i=1}^{\infty}\left(\frac{|f_i-tg_i|}{\|f-tg\|_p}\right)^{p-1}\frac{|g_i|}{\|g\|_p^{\frac{p-1}{p}}}\|g\|_p^{\frac{p-1}{p}} \\ &= \sum_{i=1}^{\infty}\left(\frac{|f_i-tg_i|}{\|f-tg\|_p}\|g\|_p^{\frac{1}{p}}\right)^{p-1}\cdot\frac{|g_i|}{\|g\|_p^{\frac{p-1}{p}}}\end{aligned}$$

When $p=1$ we get directly that

$$\begin{aligned}\frac{d}{dt}\|f+tg\|_1 &\leq \sum_{i=1}^{\infty}|g_i| \\ &= \|g\|_1 \\ &= \frac{d}{dt}(\|f\|_1+t\|g\|_1)\end{aligned}$$

As desired. When $1<p<\infty$ we apply Young's inequality to get

$$\begin{aligned}\frac{d}{dt}\|f+tg\|_p &\leq \sum_{i=1}^{\infty}\left(\frac{p-1}{p}\left(\frac{|f_i-tg_i|}{\|f-tg\|_p}\|f\|_p^{\frac{1}{p}}\right)^{(p-1)\frac{p}{p-1}}+\frac{1}{p}\left(\frac{|g_i|}{\|g\|_p^{\frac{p-1}{p}}}\right)^p\right) \\ &= \frac{p-1}{p}\sum_{i=1}^{\infty}\frac{|f_i-tg_i|^p}{\|f-tg\|_p^p}\|f\|_p+\frac{1}{p}\sum_{i=1}^{\infty}\frac{|g_i|^p}{\|g\|_p^{p-1}} \\ &= \frac{p-1}{p}\left(\frac{\|g\|_p}{\|f-tg\|_p^p}\cdot\sum_{i=1}^{\infty}|f_i-tg_i|^p\right)+\frac{1}{p}\left(\frac{1}{\|g\|_p^{p-1}}\cdot\sum_{i=1}^{\infty}|g_i|^p\right)\end{aligned}$$

Finally note that $\sum_{i=1}^{\infty}|f_i-tg_i|^p$ equals precisely $\|g\|_p^p$. Therfore

$$\begin{aligned}\frac{d}{dt}\|f+tg\|_p &\leq \frac{p-1}{p}\left(\frac{\|g\|_p}{\|f-tg\|_p^p}\cdot\|f-tg\|_p^p\right)+\frac{1}{p}\left(\frac{1}{\|g\|_p^{p-1}}\cdot\|g\|_p^p\right) \\ &= \frac{p-1}{p}\|g\|_p+\frac{1}{p}\|g\|_p \\ &= \|g\|_p\end{aligned}$$

Therefore, as desired, we have proved that

$$\frac{d}{dt}\|f+tg\|_p \leq \frac{d}{dt}(\|f\|_p+t\|g\|_p)$$

As the theorem follows from (2.1) and (2.2).

Example 2.2. *If $p > 1$, then*

$$\left(\int_\Omega |f+g|^p dx\right)^{\frac{1}{p}} \leq \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}} + \left(\int_\Omega |g|^p dx\right)^{\frac{1}{p}}$$

It yields that

$$\begin{aligned}\left(\int_\Omega |f+g|^p dx\right)^{\frac{1}{p}} &= \sup_{\|\varphi\|_q=1} \int_\Omega |f+g|\varphi dx \\ &\leq \sup_{\|\varphi\|_q=1} \int_\Omega \left(|f|_\varphi + |g|_\varphi\right) dx \\ &\leq \sup_{\|\varphi\|_q=1} \int_\Omega |f|_\varphi dx + \sup_{\|\varphi\|_q=1} \int_\Omega |g|_\varphi dx \\ &= \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}} + \left(\int_\Omega |g|^p dx\right)^{\frac{1}{p}}\end{aligned}$$

2.3 Hölder's inequality

Theorem 2.3. *If $f, g \in \nu_F$, then*

$$\sum_{i=1}^{\infty} f_i g_i \leq \|f\|_{\frac{p}{p-1}} \|g\|_p$$

Proof. By Young's inequality we have

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{f_i}{\|f\|_{\frac{p}{p-1}}} \frac{g_i}{\|g\|_p} &\leq \sum_{i=1}^{\infty} \frac{|f_i|}{\|f\|_{\frac{p}{p-1}}} \frac{|g_i|}{\|g\|_p} \\ &\leq \sum_{i=1}^{\infty} \left(\frac{p-1}{p} \frac{|f_i|^{\frac{p}{p-1}}}{\|f\|_{\frac{p}{p-1}}^{\frac{p}{p-1}}} + \frac{1}{p} \frac{|g_i|^p}{\|g\|_p^p}\right) \\ &= \frac{p-1}{p} \frac{1}{\|f\|_{\frac{p}{p-1}}^{\frac{p}{p-1}}} \sum_{i=1}^{\infty} |f_i|^{\frac{p}{p-1}} + \frac{1}{p} \frac{1}{\|g\|_p^p} \sum_{i=1}^{\infty} |g_i|^p \\ &= \frac{p-1}{p} + \frac{1}{p}\end{aligned}$$

Thus we have shown that

$$\frac{1}{\|f\|_{\frac{p}{p-1}}\|g\|_p}\sum_{i=1}^{\infty} f_i g_i = \sum_{i=1}^{\infty} \frac{f_i}{\|f\|_{\frac{p}{p-1}}}\frac{g_i}{\|g\|_p}$$
$$\leq 1$$

So after multiplying both sides by $\|f\|_{\frac{p}{p-1}}\|g\|_p$, we get

$$\sum_{i=1}^{\infty} f_i g_i \leq \|f\|_{\frac{p}{p-1}}\|g\|_p$$

Which was to be proved.

Example 2.3. *The following is a problem from Royden and Fitzpatrick's Real Analysis book. We find the values of the parameter λ for wich*

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon^\lambda}\int_0^\epsilon f(x)dx = 0, \quad \forall f \in L^p[0,1]$$

By Hölder's inequality

$$\frac{1}{\epsilon^\lambda}\int_0^\epsilon f(x)dx \leq \frac{1}{\epsilon^\lambda}\|f\|_{p^{\epsilon^{1-1/p}}}$$
$$= \|f\|_{p^{\epsilon^{1-1/p-\lambda}}}$$

So if $\lambda < 1 - 1/p$, the limit holds. For the case, $\lambda > 1 - 1/p$, we can consider the following counterexample, $f(x) = \lambda x^{\lambda-1}$. It is simple to see that $f \in L^p[0,1]$, but

$$\frac{1}{\epsilon^\lambda}\int_0^\epsilon \lambda x^{\lambda-1}dx = 1$$

Finally in the case $\lambda = 1 - 1/p$ the limit is zero.

2.4 Poicaré inequality

Theorem 2.4. *Let Ω a bounded subset of $\mathbb{R}^N$, when $1 < p < \infty$, $\exists C > 0$, $\forall u \in H_0^1$, we have*

$$\|u\|_{L^p(\Omega)} \leq \|\nabla u\|_{L^p(\Omega)}$$

Proof. Since Ω is bounded we have for any $j \in \{1, 2, ..., n\}$ and $u \in H_0^1$

$$\begin{aligned}\int_\Omega |u|^p dx &= \int_\Omega \frac{\partial}{\partial x_j}(x_j)|u|^p dx \\ &= -p \int_\Omega x_j |u|^{p-1} sgn(u) u_{x_j} dx \\ &\leq C \int_\Omega |u|^{p-1} |\nabla u| dx\end{aligned}$$

For some constant $C > 0$.Now noticing that

$$\frac{1}{p} + \frac{1}{\left(\frac{p}{p-1}\right)} = 1$$

It follows by Hölder's inequality that

$$\begin{aligned}\|u\|_{L^p(\Omega)}^p &\leq C \left(\int_\Omega (|u|^{p-1})^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}} \\ &= \|u\|_{L^p(\Omega)}^{p-1} \|\nabla u\|_{L^p(\Omega)}\end{aligned}$$

Chapter 3

Renormalized and Entropy Solutions of Nonlinear Parabolic Systems

In this chapter, we study the existence of renormalized and entropy solutions of nonlinear parabolic systems in Ω.

3.1 Positions of the problem

Let Ω a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$, we have the following parabolic system

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} - \operatorname{div}(A_1(u_1, \nabla u_1))\\
&- \operatorname{div}((k_1u_1 + \delta_{1,2}u_2 + \delta_{1,3}u_3)\nabla u_1 + \delta'_{1,2}u_1\nabla u_2 + \delta'_{1,3}u_1\nabla u_3)\\
&= -\sigma(u_1, u_2, u_3) - \mu_1u_1 + f \ \text{ in } \ Q_T,\\
&\frac{\partial u_2}{\partial t} - \operatorname{div}(A_2(u_2, \nabla u_2))\\
&- \operatorname{div}((\delta_{2,1}u_1 + k_2u_2 + \delta_{2,3}u_3)\nabla u_2 + \delta'_{2,1}u_2\nabla u_1 + \delta'_{2,3}u_2\nabla u_3)\\
&= \sigma(u_1, u_2, u_3) - \varrho u_2 - \mu_2u_2 + g \ \text{ in } \ Q_T,\\
&\frac{\partial u_3}{\partial t} - \operatorname{div}(A_3(u_3, \nabla u_3))\\
&- \operatorname{div}((\delta_{3,1}u_1 + \delta_{3,2}u_3 + k_3u_3)\nabla u_3 + \delta'_{3,1}u_3\nabla u_1 + \delta'_{3,2}u_3\nabla u_2)\\
&= \varrho u_2 - \mu_3u_3 + h \ \text{ in } \ Q_T,
\end{aligned} \tag{3.1}$$

With the initial and boundary conditions

$$u_i(x, 0) = u_{i,0}(x) \ \text{ in } \Omega, \ i = 1, 2, 3,$$

$$u_i(x,t) = 0 \ \text{ on } \sum_T, i = 1,2,3,$$

Where $Q_T = \Omega \times (0,T)$, $\sum_T = \partial\Omega \times (0,T)$, and $T > 0$.

It is worth mentioning that to prove the existence of renormalized and entropy solutions to system (3.1), we are in need of the following hypotheses.

The divergence form operator $A_i(s,\zeta) : \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Caratheodory function (that is,it is continuous with respect to s and ζ) such that

$(H1)$ $A_i(s,\zeta)\zeta \geq \alpha_i|\zeta|^2$, for every $\zeta \in \mathbb{R}^N$, where $\alpha_i > 0$ and $i = 1,2,3$,

$(H2)$ For any $k > 0$, there exists $\beta_k > 0$ and a function $C_k(x,t) \in L^2(Q_T)$ such that

$$|A_i(s,\zeta)| \leq C_k(x,t) + \beta_k|\zeta|, i = 1,2,3,$$

$(H3)$ $[A_i(s,\zeta) - A_i(s,\zeta')][\zeta - \zeta'] \geq 0, i = 1,2,3,$

$(H4)$ $u_{i,0}(x) \in L^1(\Omega), i = 1,2,3,$

$(H5)$ $f(x,t), g(x,t), h(x,t) \in L^1(Q_T)$.

We define the truncation function that we have used after that we give the definition of the renormalized solutions and the entropy solutions of the reaction-diffusion system with cross-diffusion terms (3.1).

$$T_k(z) = \begin{cases} k & \text{if } z \geq k \\ z & \text{if } |z| \leq k \\ -k & \text{if } z \leq -k \end{cases}$$

$$\widetilde{T}_k(z) = \int_0^z T_k(s)ds = \begin{cases} \frac{z^2}{2} & \text{if } |z| \leq k \\ k|z| - \frac{k^2}{2} & \text{if } |z| \geq k \end{cases}$$

3.2 Renormalized and Entropy solutions

Definition 3.1. *A renormalized solution of* (3.1) *is a triple function* (u_1, u_2, u_3) *satisfying the following conditions* $u_1, u_2, u_3 \geq 0$ *for a.e. in* $(x,t) \in Q_T$. *For* $i = 1,2,3$,

$$u_i \in L^\infty(0,T;L^1(\Omega)) \cap C([0,T], L^1(\Omega)),$$

$$T_k(u_i) \in L^2(0,T;H_0^1(\Omega)), \quad \textit{for any} \;\; k \geq 0,$$

$$\int_{\{n\leq|u_i|\leq n+1\}} A_i(u_i,\nabla u_i)\,dxdt \to 0 \quad \textit{as } n \to \infty.$$

For all $S(u_i) \in C^\infty(\mathbb{R})$ *with supp* S' *compact,*

$$\begin{aligned}
&\frac{\partial S(u_1)}{\partial t} - div(S'(u_1)A_1(u_1,\nabla u_1)) + s''(u_1)A_1(u_1,\nabla u_1)\nabla u_1 \\
&- div(S'(u_1)((k_1u_1+u_2+u_3)\nabla u_1 + u_1\nabla u_2 + u_1\nabla u_3)) \\
&+ S''(u_1)((k_1u_1+u_2+u_3)\nabla u_1 + u_1\nabla u_2 + u_1\nabla u_3)\nabla u_1 \\
&= (-\sigma(u_1,u_2,u_3) - \mu_1 u_1 + f)S'(u_1) \quad \textit{ in } \; D'(Q_T), \\
&\frac{\partial S(u_2)}{\partial t} - div(S'(u_2)A_2(u_2,\nabla u_2)) + s''(u_2)A_2(u_2,\nabla u_2)\nabla u_2 \\
&- div(S'(u_2)((u_1+k_2u_2+u_3)\nabla u_2 + u_2\nabla u_1 + u_2\nabla u_3)) \\
&+ S''(u_2)((u_1+k_2u_2+u_3)\nabla u_2 + u_2\nabla u_1 + u_2\nabla u_3)\nabla u_2 \\
&= (\sigma(u_1,u_2,u_3) - \varrho u_2 - \mu_2 u_2 + g)S'(u_2) \quad \textit{ in } \; D'(Q_T), \\
&\frac{\partial S(u_3)}{\partial t} - div(S'(u_3)A_3(u_3,\nabla u_3)) + s''(u_3)A_3(u_3,\nabla u_3)\nabla u_3 \\
&- div(S'(u_3)((u_1+u_2+k_3u_3)\nabla u_3 + u_3\nabla u_1 + u_3\nabla u_2)) \\
&+ S''(u_3)((u_1+u_2+k_3u_3)\nabla u_3 + u_3\nabla u_1 + u_3\nabla u_2)\nabla u_3 \\
&= (\varrho u_2 - \mu 3 u_3 + h)S'(u_3) \quad \textit{ in } \; D'(Q_T),
\end{aligned} \tag{3.2}$$

And the initial conditions $S(u_i(x,0)) = S(u_{i,0}(x)), i = 1,2,3,$ *in* Ω *hold.*

Definition 3.2. *An entropy solution of* (3.1) *is a triple function* (u_1,u_2,u_3) *satisfying the following conditions, that is, for* $i = 1,2,3,$

$$u_i \in L^\infty(0,T;L^1(\Omega)) \cap C([0,T],L^1(\Omega))$$

For any $k > 0$ and for all $\phi_i \in C^1(Q_T)$ with $\phi_i = 0$ in $\sum_T$,

$$
\begin{aligned}
&\int_\Omega \widetilde{T}_k(u_1 - \phi_1)(T)dx - \int_\Omega \widetilde{T}_k(u_1 - \phi_1)(0)dx + \int_0^T \langle \phi_{1t}, T_k(u_1 - \phi_1) \rangle dt \\
&+ \int_{Q_T} A_1(u_1, \nabla u_1)\nabla T_k(u_1 - \phi_1)dxdt + \int_{Q_T} ((k_1 u_1 + u_2 + u_3)\nabla u_1 + u_1 \nabla u_2 \\
&+ u_1 \nabla u_3)\nabla T_k(u_1 - \phi_1)dxdt \\
&= \int_{Q_T} (-\sigma(u_1, u_2, u_3) - \mu_1 u_1 + f)T_k(u_1 - \phi_1)dxdt, \\
&\int_\Omega \widetilde{T}_k(u_2 - \phi_2)(T)dx - \int_\Omega \widetilde{T}_k(u_2 - \phi_2)(0)dx + \int_0^T \langle \phi_{2t}, T_k(u_2 - \phi_2) \rangle dt \\
&+ \int_{Q_T} A_2(u_2, \nabla u_2)\nabla T_k(u_2 - \phi_2)dxdt + \int_{Q_T} ((u_1 + k_2 u_2 + u_3)\nabla u_2 + u_2 \nabla u_1 \\
&+ u_2 \nabla u_3)\nabla T_k(u_2 - \phi_2)dxdt \\
&= \int_{Q_T} (\sigma(u_1, u_2, u_3) - \varrho u_2 - \mu_2 u_2 + g)T_k(u_2 - \phi_2)dxdt, \\
&\int_\Omega \widetilde{T}_k(u_3 - \phi_3)(T)dx - \int_\Omega \widetilde{T}_k(u_3 - \phi_3)(0)dx + \int_0^T \langle \phi_{3t}, T_k(u_3 - \phi_3) \rangle dt \\
&+ \int_{Q_T} A_3(u_3, \nabla u_3)\nabla T_k(u_3 - \phi_3)dxdt + \int_{Q_T} ((u_1 + u_2 + k_3 u_3)\nabla u_3 + u_3 \nabla u_1 \\
&+ u_3 \nabla u_2)\nabla T_k(u_3 - \phi_3)dxdt \\
&= \int_{Q_T} (\varrho u_2 - \mu_3 u_3 + h)T_k(u_1 - \phi_1)dxdt,
\end{aligned}
$$

It should be remarked that as far as the equivalence between the renormalized and entropy solutions of reaction-diffusion system with cross-diffusion terms is concerned.

Theorem 3.1. *Under the hypotheses $(H1) - (H5)$, there exists a renormalized solution of* (3.1) *in the sense of Definition* 3.1

Theorem 3.2. *Under the hypotheses $(H1) - (H5)$, the renormalized solution of* (3.1) *is also an entropy solution of the same system in the sense of Definition* 3.2.

Remark 3.1. *The renormalized solution of* (3.1) *is equivalent to the entropy solution of given cross-diffusion epidemic system* (3.1).

3.3 Approximation problem

In this section, first we introduce an approximation problem for the given reaction-diffusion system (3.1) and then we prove the existence of solutions of the approximation problem.

For $\epsilon > 0$, let us introduce the following approximations on the data:

$(H6)$ $f^\epsilon, g^\epsilon, h^\epsilon \in L^2(Q_T)$ and $f^\epsilon \to f, g^\epsilon \to g$ and $h^\epsilon \to h$ a.e. in Q_T and strongly in $L^1(Q_T)$ as ϵ tends to zero.

$(H7)$ $u_{i,0}^\epsilon \in L^2(\Omega), i = 1,2,3,$ and $u_{i,0}^\epsilon \to u_{i,0}, i = 1,2,3,$ a.e. in Ω and strongly in $L^1(\Omega)$ as ϵ tends to zero.

$$
\begin{aligned}
&\frac{\partial {u_1}^\epsilon}{\partial t} - \operatorname{div}(A_1(u_1^\epsilon, \nabla u_1^\epsilon)) - \operatorname{div}((k_1 F_\epsilon^+(u_1^\epsilon) + F_\epsilon^+(u_2^\epsilon) + F_\epsilon^+(u_3^\epsilon))\nabla u_1^\epsilon \\
&+ F_\epsilon^+(u_1^\epsilon)\nabla u_2^\epsilon + F_\epsilon^+(u_1^\epsilon)\nabla u_3^\epsilon) \\
&= -\sigma(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon) - \mu_1 u_1^\epsilon + f^\epsilon \quad \text{in } Q_T, \\
&\frac{\partial {u_2}^\epsilon}{\partial t} - \operatorname{div}(A_2(u_2^\epsilon, \nabla u_2^\epsilon)) - \operatorname{div}((F_\epsilon^+(u_1^\epsilon) + k_2 F_\epsilon^+(u_2^\epsilon) + F_\epsilon^+(u_3^\epsilon))\nabla u_2^\epsilon \\
&+ F_\epsilon^+(u_2^\epsilon)\nabla u_1^\epsilon + F_\epsilon^+(u_2^\epsilon)\nabla u_3^\epsilon) \\
&= \sigma(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon) - \varrho u_2^\epsilon - \mu_2 u_2^\epsilon + g^\epsilon \quad \text{in } Q_T, \\
&\frac{\partial {u_3}^\epsilon}{\partial t} - \operatorname{div}(A_3(u_3^\epsilon, \nabla u_3^\epsilon)) - \operatorname{div}((F_\epsilon^+(u_1^\epsilon) + F_\epsilon^+(u_2^\epsilon) + k_3 F_\epsilon^+(u_3^\epsilon))\nabla u_3^\epsilon \\
&+ F_\epsilon^+(u_3^\epsilon)\nabla u_1^\epsilon + F_\epsilon^+(u_3^\epsilon)\nabla u_2^\epsilon) \\
&= \varrho u_2^\epsilon - \mu_3 u_3^\epsilon + h^\epsilon \quad \text{in } Q_T,
\end{aligned}
\tag{3.3}
$$

With the initial and boundary conditions

$$u_i^\epsilon(x,0) = u_{i,0}^\epsilon(x) \ \text{ in } \Omega, \ i = 1,2,3,$$

$$u_i^\epsilon(x,t) = 0 \ \text{ on } \sum_T, \ i = 1,2,3,$$

Where $F_\epsilon^+(a) = \max(0, \frac{a}{1+\epsilon|a|})$.

Lemma 3.1. *Assume that* $u_{i,0}^\epsilon \in L^2(\Omega), i = 1,2,3,$ *and* $f^\epsilon, g^\epsilon, h^\epsilon \in L^2(Q_T)$. *Then the approximation problem* (3.3) *admits a unique weak solution*

$$u_i^\epsilon \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap C([0,T],L^2(\Omega)),\ \ i = 1,2,3,$$

With $u_{it}^\epsilon \in L^2(0,T;H^{-1}(\Omega))$ *such that, for any* $\phi_i \in L^2(0,T;H_0^1(\Omega)),\ \ i = 1,2,3,$

$$\begin{aligned}
&\int_0^T \langle \partial_t u_1^\epsilon, \phi_1\rangle dt + \int_{Q_T} A_1(u_1^\epsilon, \nabla u_1^\epsilon)\nabla\phi_1\, dxdt\\
&+ \int_{Q_T} (\alpha_{1,1}\nabla u_1^\epsilon + \alpha_{1,2}\nabla u_2^\epsilon + \alpha_{1,3}\nabla u_3^\epsilon)\nabla\phi_1\, dxdt\\
&= \int_{Q_T} (-\sigma(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon) - \mu_1 u_1^\epsilon + f^\epsilon)\phi_1\, dxdt,\\
&\int_0^T \langle \partial_t u_2^\epsilon, \phi_2\rangle dt + \int_{Q_T} A_2(u_2^\epsilon, \nabla u_2^\epsilon)\nabla\phi_2\, dxdt\\
&+ \int_{Q_T} (\alpha_{2,1}\nabla u_1^\epsilon + \alpha_{2,2}\nabla u_2^\epsilon + \alpha_{2,3}\nabla u_3^\epsilon)\nabla\phi_2\, dxdt\\
&= \int_{Q_T} (\sigma(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon) - \varrho u_2^\epsilon - \mu_2 u_2^\epsilon + g^\epsilon)\phi_2\, dxdt,\\
&\int_0^T \langle \partial_t u_3^\epsilon, \phi_3\rangle dt + \int_{Q_T} A_3(u_3^\epsilon, \nabla u_3^\epsilon)\nabla\phi_3\, dxdt\\
&+ \int_{Q_T} (\alpha_{3,1}\nabla u_1^\epsilon + \alpha_{3,2}\nabla u_2^\epsilon + \alpha_{3,3}\nabla u_3^\epsilon)\nabla\phi_3\, dxdt\\
&= \int_{Q_T} (\varrho u_2^\epsilon - \mu_3 u_3^\epsilon + h^\epsilon)\phi_3\, dxdt,
\end{aligned}$$

Proof. The proof of this lemma is not new, we present here a self-contained sketch of the proof for the sake of simplicity and readability. For more details regarding this proof, one can refer the article [4] and some details related to the following method see [8]. Here the proof is relies on using the Faedo-Galerkin approximation method.

Now we look for the finite-dimensional approximate solutions to problem (3.3) in the form of sequences $\{u_{i,n}^\epsilon\},\ i = 1,2,3,$ defined for $t \geq 0$ and $x \in \overline{\Omega}$ by

$$u_{i,n}^\epsilon = \sum_{l=1}^{n} C_{i,n,l}(t) w_l(x),\ \ i = 1,2,3,$$

our aim is to determine the set of coefficients $\{C_{i,n,l}\}_{l=1}^{n},\ i=1,2,3$, such that for $m=1,2,...,n$,

$$\begin{aligned}
&\int_\Omega \partial_t u^\epsilon_{1,n} w_m dx + \int_\Omega A_1(u^\epsilon_{1,n}, \nabla u^\epsilon_{1,n})\nabla w_m dx \\
&+ \int_\Omega (\alpha_{1,1,n}\nabla u^\epsilon_{1,n} + \alpha_{1,2,n}\nabla u^\epsilon_{2,n} + \alpha_{1,3,n}\nabla u^\epsilon_{3,n})\nabla w_m dx \\
&= \int_\Omega (-\sigma(u^\epsilon_{1,n}, u^\epsilon_{2,n}, u^\epsilon_{3,n}) - \mu_1 u^\epsilon_{1,n} + f^\epsilon) w_m dx, \\
&\int_\Omega \partial_t u^\epsilon_{2,n} w_m dx + \int_\Omega A_2(u^\epsilon_{2,n}, \nabla u^\epsilon_{2,n})\nabla w_m dx \\
&+ \int_\Omega (\alpha_{2,1,n}\nabla u^\epsilon_{1,n} + \alpha_{2,2,n}\nabla u^\epsilon_{2,n} + \alpha_{2,3,n}\nabla u^\epsilon_{3,n})\nabla w_m dx \\
&= \int_\Omega (\sigma(u^\epsilon_{1,n}, u^\epsilon_{2,n}, u^\epsilon_{3,n}) - \varrho u^\epsilon_{2,n} - \mu_2 u^\epsilon_{2,n} + g^\epsilon) w_m dx, \\
&\int_\Omega \partial_t u^\epsilon_{3,n} w_m dx + \int_\Omega A_3(u^\epsilon_{3,n}, \nabla u^\epsilon_{3,n})\nabla w_m dx \\
&+ \int_\Omega (\alpha_{3,1,n}\nabla u^\epsilon_{1,n} + \alpha_{3,2,n}\nabla u^\epsilon_{2,n} + \alpha_{3,3,n}\nabla u^\epsilon_{3,n})\nabla w_m dx \\
&= \int_\Omega (\varrho u^\epsilon_{2,n} - \mu_3 u^\epsilon_{3,n} + h^\epsilon) w_m dx,
\end{aligned} \tag{3.4}$$

Where $\alpha_{i,i,n} = k_i F^+_\epsilon(u^\epsilon_{i,n}) + \sum_{j=1,j\neq i}^{3} F^+_\epsilon(u^\epsilon_{j,n})$, for $i=1,2,3$, and with initial conditions $u^\epsilon_{i,n}(x,0) = u^\epsilon_{i,0,n}(x) := \sum_{l=1}^{n} C_{i,n,l}(0) w_l(x),\ i=1,2,3$. Further it should be remarked that the above form of the basis satisfies the required boundary conditions of the approximation problem (3.3). Now, (3.4) can be rewritten in the form

$$\begin{aligned}
C'_{1,n,m}(t) &= -\int_\Omega A_1(u^\epsilon_{1,n}, \nabla u^\epsilon_{1,n})\nabla w_m dx \\
&- \int_\Omega (\alpha_{1,1,n}\nabla u^\epsilon_{1,n} + \alpha_{1,2,n}\nabla u^\epsilon_{2,n} + \alpha_{1,3,n}\nabla u^\epsilon_{3,n})\nabla w_m dx \\
&- \int_\Omega (-\sigma(u^\epsilon_{1,n}, u^\epsilon_{2,n}, u^\epsilon_{3,n}) - \mu_1 u^\epsilon_{1,n} + f^\epsilon) w_m dx \\
&=: G_1^m(t, \{C_{1,n,l}\}_{l=1}^n, \{C_{2,n,l}\}_{l=1}^n, \{C_{3,n,l}\}_{l=1}^n), \\
C'_{2,n,m}(t) &= -\int_\Omega A_2(u^\epsilon_{2,n}, \nabla u^\epsilon_{2,n})\nabla w_m dx \\
&- \int_\Omega (\alpha_{2,1,n}\nabla u^\epsilon_{1,n} + \alpha_{2,2,n}\nabla u^\epsilon_{2,n} + \alpha_{2,3,n}\nabla u^\epsilon_{3,n})\nabla w_m dx \\
&- \int_\Omega (\sigma(u^\epsilon_{1,n}, u^\epsilon_{2,n}, u^\epsilon_{3,n}) - \varrho u^\epsilon_{2,n} - \mu_2 u^\epsilon_{2,n} + g^\epsilon) w_m dx \\
&=: G_2^m(t, \{C_{1,n,l}\}_{l=1}^n, \{C_{2,n,l}\}_{l=1}^n, \{C_{3,n,l}\}_{l=1}^n),
\end{aligned}$$

$$
\begin{aligned}
C'_{3,n,m}(t) = &- \int_\Omega A_3(u^\epsilon_{3,n}, \nabla u^\epsilon_{3,n}) \nabla w_m dx \\
&- \int_\Omega (\alpha_{3,1,n} \nabla u^\epsilon_{1,n} + \alpha_{3,2,n} \nabla u^\epsilon_{2,n} + \alpha_{3,3,n} \nabla u^\epsilon_{3,n}) \nabla w_m dx \\
&- \int_\Omega (\varrho u^\epsilon_{2,n} - \mu_3 u^\epsilon_{3,n} + h^\epsilon) w_m dx \\
&=: G^m_3(t, \{C_{1,n,l}\}^n_{l=1}, \{C_{2,n,l}\}^n_{l=1}, \{C_{3,n,l}\}^n_{l=1}),
\end{aligned}
$$

Let $\rho \in (0, T)$ and set $U = [0, \rho]$. Choose $r > 0$ large enough so that the ball $B_r \subset \mathbb{R}^n$ contains $\{C_{i,n,l}(0)\}, i = 1, 2, 3$; then set $V = \overline{B}_r$ The components of G_1 can be bounded on $U \times V$, from $(H1 - H7)$, we obtain

$$
\begin{aligned}
&|G^m_1(t, \{C_{1,n,l}\}^n_{l=1}, \{C_{2,n,l}\}^n_{l=1}, \{C_{3,n,l}\}^n_{l=1})| \\
&\leq (\int_\Omega |A_1(\sum_{l=1}^n C_{1,n,l} w_l, \sum_{l=1}^n C_{1,n,l} \nabla w_l)|^2 dx)^{1/2} (\int_\Omega |\nabla w_m|^2 dx)^{1/2} \\
&+ \frac{(k_1 + 2)}{\epsilon} (\int_\Omega |\sum_{l=1}^n C_{1,n,l} \nabla w_l|^2 dx)^{1/2} (\int_\Omega |\nabla w_m|^2 dx)^{1/2} \\
&+ \frac{1}{\epsilon} ((\int_\Omega |\sum_{l=1}^n C_{2,n,l} \nabla w_l|^2 dx)^{1/2} \\
&+ (\int_\Omega |\sum_{l=1}^n C_{3,n,l} \nabla w_l|^2 dx)^{1/2} (\int_\Omega |\nabla w_m|^2 dx)^{1/2} \\
&+ (\sigma_1 + \mu_1)(\int_\Omega |\sum_{l=1}^n C_{1,n,l} w_l|^2 dx)^{1/2}) (\int_\Omega |w_m|^2 dx)^{1/2} \\
&+ (\int_\Omega |f^\epsilon|^2 dx)^{1/2} (\int_\Omega |w_m|^2 dx)^{1/2} \\
&\leq c(r, n)
\end{aligned}
$$

Where the constant depends only on r and n. Similarly one can easly obtain

$$
|G^m_2(t, \{C_{1,n,l}\}^n_{l=1}, \{C_{2,n,l}\}^n_{l=1}, \{C_{3,n,l}\}^n_{l=1})| \leq c(r, n),
$$

$$
|G^m_3(t, \{C_{1,n,l}\}^n_{l=1}, \{C_{2,n,l}\}^n_{l=1}, \{C_{3,n,l}\}^n_{l=1})| \leq c(r, n),
$$

Then the standard ODE theory shows that $\{C_{i,n,l}\}^n_{l=1}$, $i = 1, 2, 3$, respectively satisfies (3.4) for a.e $t \in [0, \rho')$. Moreover we have

$$
\begin{aligned}
& C_{1,n,l}(t) \\
& = C_{1,n,l}(0) + \int_0^t G_1^l(\tau, \{C_{1,n,m}(\tau)\}_{m=1}^n, \{C_{2,n,m}(\tau)\}_{m=1}^n, \{C_{3,n,m}(\tau)\}_{m=1}^n) d\tau, \\
& C_{2,n,l}(t) \\
& = C_{2,n,l}(0) + \int_0^t G_2^l(\tau, \{C_{1,n,m}(\tau)\}_{m=1}^n, \{C_{2,n,m}(\tau)\}_{m=1}^n, \{C_{3,n,m}(\tau)\}_{m=1}^n) d\tau, \\
& C_{3,n,l}(t) \\
& = C_{3,n,l}(0) + \int_0^t G_3^l(\tau, \{C_{1,n,m}(\tau)\}_{m=1}^n, \{C_{2,n,m}(\tau)\}_{m=1}^n, \{C_{3,n,m}(\tau)\}_{m=1}^n) d\tau,
\end{aligned}
$$

This proves that the functions $(u_{1,n}^\epsilon, u_{2,n}^\epsilon, u_{3,n}^\epsilon)$ are well-defined and approximate solutions to the problem (3.3) on $[0, \rho')$. Set $\phi_{i,n}(x,t) = \sum_{l=1}^n b_{i,n,l}(t) w_l(x), i = 1, 2, 3$, where the coefficients $b_{i,n,l}$, $i = 1, 2, 3$, are absolutely continuous functions. Then, from (3.4), the approximate solutions satisfy the weak formulation

$$
\begin{aligned}
& \int_\Omega \partial_t u_{1,n}^\epsilon \phi_{1,n} dx + \int_\Omega A_1(u_{1,n}^\epsilon, \nabla u_{1,n}^\epsilon) \nabla \phi_{1,n} dx \\
& + \int_\Omega (\alpha_{1,1,n} \nabla u_{1,n}^\epsilon + \alpha_{1,2,n} \nabla u_{2,n}^\epsilon + \alpha_{1,3,n} \nabla u_{3,n}^\epsilon) \nabla \phi_{1,n} dx \\
& = \int_\Omega (-\sigma(u_{1,n}^\epsilon, u_{2,n}^\epsilon, u_{3,n}^\epsilon) - \mu_1 u_{1,n}^\epsilon + f^\epsilon) \nabla \phi_{1,n} dx, \\
& \int_\Omega \partial_t u_{2,n}^\epsilon \phi_{2,n} dx + \int_\Omega A_2(u_{2,n}^\epsilon, \nabla u_{2,n}^\epsilon) \nabla \phi_{2,n} dx \\
& + \int_\Omega (\alpha_{2,1,n} \nabla u_{1,n}^\epsilon + \alpha_{2,2,n} \nabla u_{2,n}^\epsilon + \alpha_{2,3,n} \nabla u_{3,n}^\epsilon) \nabla \phi_{2,n} dx \\
& = \int_\Omega (\sigma(u_{1,n}^\epsilon, u_{2,n}^\epsilon, u_{3,n}^\epsilon) - \varrho u_{2,n}^\epsilon - \mu_2 u_{2,n}^\epsilon + g^\epsilon) \nabla \phi_{2,n} dx, \\
& \int_\Omega \partial_t u_{3,n}^\epsilon \phi_{3,n} dx + \int_\Omega A_3(u_{3,n}^\epsilon, \nabla u_{3,n}^\epsilon) \nabla \phi_{3,n} dx \\
& + \int_\Omega (\alpha_{3,1,n} \nabla u_{1,n}^\epsilon + \alpha_{3,2,n} \nabla u_{2,n}^\epsilon + \alpha_{3,3,n} \nabla u_{3,n}^\epsilon) \nabla \phi_{3,n} dx \\
& = \int_\Omega (\varrho u_{2,n}^\epsilon - \mu_3 u_{3,n}^\epsilon + h^\epsilon) \nabla \phi_{3,n} dx,
\end{aligned} \tag{3.5}
$$

From now on, $\widetilde{T}$ is an arbitrary time in the existence interval $[0, \rho')$ of Faedo-Galerkin solutions. Take $\phi_{i,n} = u_{i,n}^\epsilon, i = 1, 2, 3$, respectively in (3.5) and using, Gronwall's lemma, And Young's inequality, we obtain

$$\|u_{i,n}^{\epsilon}\|_{L^{\infty}(0,\widetilde{T};L^2(\Omega))} + \|u_{i,n}^{\epsilon}\|_{L^2(0,\widetilde{T};H_0^1(\Omega))} \leq c,$$

$$\|\partial_t u_{i,n}^{\epsilon}\|_{L^2(0,\widetilde{T};H^{-1}(\Omega))} + \|\sigma(u_{1,n}^{\epsilon}, u_{2,n}^{\epsilon}, u_{3,n}^{\epsilon})\|_{L^2(Q_T)} \leq c,$$

For some constant $c > 0$ and $i = 1, 2, 3,$. Moreover one can easily show, using similar approach of [4] with the above estimates, the global existence of approximate solutions of the problem (3.3). Hence, from the above arguments as $n \to \infty$, for $i = 1, 2, 3$, we obtain

$$u_{i,n}^{\epsilon} \rightharpoonup u_i^{\epsilon} \quad \text{weakly-* in } L^{\infty}(Q_T),$$

$$u_{i,n}^{\epsilon} \to u_i^{\epsilon} \quad \text{a.e in } Q_T \text{ and strongly in } L^2(Q_T),$$

$$u_{i,n}^{\epsilon} \rightharpoonup u_i^{\epsilon} \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)),$$

$$A_i(u_{i,n}^{\epsilon}, \nabla u_{i,n}^{\epsilon}) \rightharpoonup \eta_i \quad \text{weakly in } L^2(Q_T),$$

$$\partial_t u_{i,n}^{\epsilon} \rightharpoonup \partial_t u_i^{\epsilon} \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)),$$

Using similar type of arguments as in [5], with the monotonicity assumption on A_i, we can show that $A_i(u_i^{\epsilon}, \nabla u_i^{\epsilon}) = \eta_i, i = 1, 2, 3$. Since the solutions $u_i^{\epsilon} \in L^2(0,T;H_0^1(\Omega)) \cap L^{\infty}(0,T;L^2(\Omega))$, $i = 1, 2, 3$, using the approximation problem, we conclude that $u_i^{\epsilon} \in C([0,T], L^2(\Omega)), i = 1, 2, 3$. This established the existence of weak solutions $(u_1^{\epsilon}, u_2^{\epsilon}, u_3^{\epsilon})$ of the regularized problem (3.3).

Lemma 3.2. *Under hypotheses* $(H6)$ *and* $(H7)$*, the functions* $T_k(u_i^{\epsilon})$ *and* $\frac{\partial S(u_i^{\epsilon})}{\partial t}, i = 1, 2, 3,$ *are bounded in* $L^2(0,T;H_0^1(\Omega))$ *and* $L^1(Q_T) \cap L^2(0,T;H^{-1}(\Omega))$ *respectively.*

Proof. Taking $T_k(u_1^{\epsilon})$ as a test function in the first equation of (3.3) and integrating over $Q_t = \Omega \times (0,t)$, we obtain

$$
\begin{aligned}
&\int_{Q_t} u^\epsilon_{1s} T_k(u^\epsilon_1)dxds + \int_{Q_t} A_1(u^\epsilon_1, \nabla u^\epsilon_1)\nabla T_k(u^\epsilon_1)dx\,ds \\
&+ \int_{Q_t} (\alpha_{1,1}\nabla u^\epsilon_1 + \alpha_{1,2}\nabla u^\epsilon_2 + \alpha_{1,3}\nabla u^\epsilon_3)\nabla T_k(u^\epsilon_1)dx\,ds \\
&= \int_{Q_t} (-\sigma(u^\epsilon_1, u^\epsilon_2, u^\epsilon_3) - \mu_1 u^\epsilon_1 + f^\epsilon)T_k(u^\epsilon_1)dx\,ds, \\
&\int_\Omega \widetilde{T}_k(u^\epsilon_1)(t)dx + \alpha_1 \int_{Q_t} |\nabla T_k(u^\epsilon_1)|^2 dx\,ds \\
&+ \int_{Q_t} (\alpha_{1,1}\nabla u^\epsilon_1 + \alpha_{1,2}\nabla u^\epsilon_2 + \alpha_{1,3}\nabla u^\epsilon_3)\nabla T_k(u^\epsilon_1)dx\,ds \\
&\leq \int_\Omega \widetilde{T}_k(u^\epsilon_{1,0})(x)dx + \int_{Q_t} (-\sigma(u^\epsilon_1, u^\epsilon_2, u^\epsilon_3) - \mu_1 u^\epsilon_1 + f^\epsilon)T_k(u^\epsilon_1)dx\,ds
\end{aligned}
\tag{3.6}
$$

Similarly by considering the second and third equations of (3.3), we obtain

$$
\begin{aligned}
&\int_\Omega \widetilde{T}_k(u^\epsilon_2)(t)dx + \alpha_2 \int_{Q_t} |\nabla T_k(u^\epsilon_2)|^2 dx\,ds \\
&+ \int_{Q_t} (\alpha_{2,1}\nabla u^\epsilon_1 + \alpha_{2,2}\nabla u^\epsilon_2 + \alpha_{2,3}\nabla u^\epsilon_3)\nabla T_k(u^\epsilon_2)dx\,ds \\
&\leq \int_\Omega \widetilde{T}_k(u^\epsilon_{2,0})(x)dx + \int_{Q_t} (\sigma(u^\epsilon_1, u^\epsilon_2, u^\epsilon_3) - \varrho u^\epsilon_2 - \mu_2 u^\epsilon_2 + g^\epsilon)T_k(u^\epsilon_2)dx\,ds \\
&\int_\Omega \widetilde{T}_k(u^\epsilon_3)(t)dx + \alpha_3 \int_{Q_t} |\nabla T_k(u^\epsilon_3)|^2 dx\,ds \\
&+ \int_{Q_t} (\alpha_{3,1}\nabla u^\epsilon_1 + \alpha_{3,2}\nabla u^\epsilon_2 + \alpha_{3,3}\nabla u^\epsilon_3)\nabla T_k(u^\epsilon_3)dx\,ds \\
&\leq \int_\Omega \widetilde{T}_k(u^\epsilon_{3,0})(x)dx + \int_{Q_t} (\varrho u^\epsilon_2 - \mu_3 u^\epsilon_3 + h^\epsilon)T_k(u^\epsilon_3)dx\,ds
\end{aligned}
\tag{3.7}
$$

Summing the above three inequalities and using the Young's inequality, the properties of the functions $f^\epsilon, g^\epsilon, h^\epsilon, u^\epsilon_{i,0}(x), i = 1, 2, 3, \sigma(u^\epsilon_1, u^\epsilon_2, u^\epsilon_3)$, (3.4), the boundedness of approximate solutions with the definition of the functions $\widetilde{T}_k(u^\epsilon_i), i = 1, 2, 3$, we have

$$
\int_{Q_t} |\nabla T_k(u^\epsilon_i)|^2 dx\,ds \leq c, \tag{3.8}
$$

For $i = 1, 2, 3$, and any constant $c > 0$. This proves that $T_k(u^\epsilon_i), i = 1, 2, 3$, are bounded in $L^2(0, T; H^1_0(\Omega))$. Now, multiplying the first equation of (3.3) by $S'(u^\epsilon_1)$, we obtain

$$\begin{aligned}\frac{\partial S(u_1^\epsilon)}{\partial t} &= \operatorname{div}(S'(u_1^\epsilon)A_1(u_1^\epsilon,\nabla u_1^\epsilon)) - S''(u_1^\epsilon)A_1(u_1^\epsilon,\nabla u_1^\epsilon)\nabla u_1^\epsilon \\ &+ \operatorname{div}(S'(u_1^\epsilon)(\alpha_{1,1}\nabla u_1^\epsilon + \alpha_{1,2}\nabla u_2^\epsilon + \alpha_{1,3}\nabla u_3^\epsilon)) \\ &- S''(u_1^\epsilon)(\alpha_{1,1}\nabla u_1^\epsilon + \alpha_{1,2}\nabla u_2^\epsilon + \alpha_{1,3}\nabla u_3^\epsilon)\nabla u_1^\epsilon \\ &+ S'(u_1^\epsilon)(-\sigma(u_1^\epsilon,u_2^\epsilon,u_3^\epsilon) - \mu_1 u_1^\epsilon + f^\epsilon).\end{aligned} \tag{3.9}$$

This may be rewritten in thr following way by using the definition of $T_k(u_i^\epsilon), i = 1, 2, 3,$

$$\begin{aligned}&\frac{\partial S(u_1^\epsilon)}{\partial t} \\ &= \operatorname{div}(S'(u_1^\epsilon)A_1(T_k(u_1^\epsilon),\nabla T_k(u_1^\epsilon))) - S''(u_1^\epsilon)A_1(T_k(u_1^\epsilon),\nabla T_k(u_1^\epsilon))\nabla T_k(u_1^\epsilon) \\ &+ \operatorname{div}(S'(u_1^\epsilon)((k_1F_\epsilon^+(T_k(u_1^\epsilon)) + F_\epsilon^+(T_k(u_2^\epsilon)) + F_\epsilon^+(T_k(u_3^\epsilon)))\nabla T_k(u_1^\epsilon) \\ &+ F_\epsilon^+(T_k(u_1^\epsilon))\nabla T_k(u_2^\epsilon) + F_\epsilon^+(T_k(u_1^\epsilon))\nabla T_k(u_3^\epsilon))) - S''(u_1^\epsilon)((k_1F_\epsilon^+(T_k(u_1^\epsilon)) \\ &+ F_\epsilon^+(T_k(u_2^\epsilon)) + F_\epsilon^+(T_k(u_3^\epsilon)))\nabla T_k(u_1^\epsilon) + F_\epsilon^+(T_k(u_1^\epsilon))\nabla T_k(u_2^\epsilon) \\ &+ F_\epsilon^+(T_k(u_1^\epsilon))\nabla T_k(u_3^\epsilon))\nabla T_k(u_1^\epsilon) \\ &+ (-\sigma(T_k(u_1^\epsilon),T_k(u_2^\epsilon),T_k(u_3^\epsilon)) - \mu_1 T_k(u_1^\epsilon) + f^\epsilon)S'(u_1^\epsilon).\end{aligned} \tag{3.10}$$

For any $S \in C^\infty(\mathbb{R})$ with supp S' compact,(3.10) shows that $\frac{\partial S(u_1^\epsilon)}{\partial t}$ is bounded in $L^11(Q_T) \cap L^2(0,T;H^{-1}(\Omega))$, from the result (3.8). Similar arguments on $\frac{\partial S(u_1^\epsilon)}{\partial t}, i = 1, 2, 3,$ proves the desired result. This completes the proof.

Lemma 3.3. *The solutions $(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon)$ of the regularized problem* (3.3) *are non negative.*

Proof. To prove the non-negativity of the solutions, we consider $u_i^{-\epsilon} = \sup(-u_i^\epsilon, 0), i = 1, 2, 3.$ Now, multipliying the first equation of (3.3) by $-T_k(u_1^{-\epsilon})$ and integrating over Ω, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \widetilde{T}_k(u_1^{-\epsilon})(t)dx + \int_\Omega A_1(u_1^{-\epsilon},\nabla u_1^{-\epsilon})\nabla T_k(u_1^{-\epsilon})dx + \int_\Omega((k_1F_\epsilon^+(u_1^{-\epsilon}) + F_\epsilon^+(u_2^{-\epsilon}) \\ &+ F_\epsilon^+(u_3^{-\epsilon}))\nabla u_1^{-\epsilon} + F_\epsilon^+(u_1^{-\epsilon})\nabla u_2^{-\epsilon} + F_\epsilon^+(u_1^{-\epsilon})\nabla u_3^{-\epsilon})\nabla T_k(u_1^{-\epsilon})dx \\ &= \int_\Omega(-\sigma(u_1^{-\epsilon},u_2^{-\epsilon},u_3^{-\epsilon}) - \mu_1 u_1^\epsilon + f^\epsilon) - \mu_1 u_1^\epsilon + f^\epsilon) \\ &\frac{d}{dt}\int_\Omega \widetilde{T}_k(u_1^{-\epsilon})(t)dx + \alpha_1\int_\Omega |\nabla T_k(u_1^{-\epsilon})|^2dx + \int_\Omega((k_1F_\epsilon^+(u_1^{-\epsilon}) + F_\epsilon^+(u_2^{-\epsilon}) \\ &+ F_\epsilon^+(u_3^{-\epsilon}))\nabla u_1^{-\epsilon} + F_\epsilon^+(u_1^{-\epsilon})\nabla u_2^{-\epsilon} + F_\epsilon^+(u_1^{-\epsilon})\nabla u_3^{-\epsilon})\nabla T_k(u_1^{-\epsilon})dx\end{aligned}$$

$$\leq \int_\Omega (\sigma(u_1^{-\epsilon}, u_2^{-\epsilon}, u_3^{-\epsilon}) + \mu_1 u_1^\epsilon + f^\epsilon) T_k(u_1^{-\epsilon}) dx.$$

By considering $-T_k(u_2^{-\epsilon}), -T_k(u_3^{-\epsilon})$ respectively as the test functions of the other two equations of (3.3), the definition of the functions $\sigma(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon), f^\epsilon, g^\epsilon, h^\epsilon$, and finally from the non-negativity of the terms of the LHS in resulting inequalities, we obtain

$$\frac{d}{dt}\int_\Omega \widetilde{T}_k(u_i^{-\epsilon})(t) dx \leq 0, \quad \text{for } i = 1,2,3.$$

The non-negativity of the initial conditions $u_{i,0}^{-\epsilon}, i = 1,2,3$, with the above inequalities prove the non-negativity of the solutions $(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon)$.

Definition 3.3. *Let us define the Lipschitz continuous function*

$$\Theta_n(z) = T_{n+1}(z) - T_n(z) = \begin{cases} 0 \quad \text{if } |z| \leq n, \\ (|z| - n) sgn(z) \quad \text{if } n \leq |z| \leq n+1, \\ sgn(z) \quad \text{if } |z| \geq n+1. \end{cases}$$

Remark 3.2. *From the above definition, one can easily understand that the function $\Theta(z)$ satisfies $\|\Theta(z)\|_{L^\infty(\mathbb{R})} \leq 1$, for any $n \geq 1$ and $\Theta(z) \to 0$, for any z when $n \to \infty$.*

3.4 Passage to the limit

Lemma 3.4. *The Lipschitz continuous function $\Theta_n(u_i), i = 1,2,3,$ for some $n > 0$ and $\epsilon > 0$ satisfies*

$$\lim_{n\to\infty} \limsup_{\epsilon\to 0} \int_0^t \int_{(n \leq |u_i^\epsilon| \leq n+1} A_i(u_i^\epsilon, \nabla u_i^\epsilon) \nabla u_i^\epsilon \ dxds = 0$$

And $\Theta_n(u_i) \to 0$, for $i = 1,2,3$, strongly in $L^2(0,T; H_0^1(\Omega))$ as $n \to \infty$.

Proof. Using $\Theta_n(u_i^\epsilon)$ as a test function in the first equation of (3.3) and integrating over Ω

and then over $(0,t)$, we have

$$\begin{aligned} &\int_\Omega \widetilde{\Theta}_n(u_1^\epsilon)(t)dx + \int_{Q_T} A_1(u_1^\epsilon, \nabla u_1^\epsilon)\nabla\theta_n(u_1^\epsilon)\ dxds \\ &+ \int_{Q_T} (\alpha_{1,1}\nabla u_1^\epsilon + \alpha_{1,2}\nabla u_2^\epsilon + \alpha_{1,3}\nabla u_3^\epsilon)\nabla\Theta_n(u_1^\epsilon)\ dxds \\ &= \int_{Q_T} (-\sigma(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon) - \mu_1 u_1^\epsilon + f^\epsilon)\Theta_n(u_1^\epsilon)\ dxds + \int_\Omega \widetilde{\Theta}_n(u_{1,0}^\epsilon(x))dx, \end{aligned} \tag{3.11}$$

For almost all t in $(0,T)$ and $\epsilon < \frac{1}{n+1}$. Since $\widetilde{\Theta}_n(u_1^\epsilon) \geq 0, i = 1,2,3$, for all $x \in \Omega$, by taking in account of (3.4), we obtain the inequality

$$\begin{aligned} &\int_{Q_T} A_1(u_1^\epsilon, \nabla u_1^\epsilon)\nabla\Theta_n(u_1^\epsilon)\ dxds \\ &\leq \int_{Q_T} (\sigma(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon) + \mu_1 u_1^\epsilon + f^\epsilon)\Theta_n(u_1^\epsilon)\ dxds + \int_\Omega \widetilde{\Theta}_n(u_{1,0}^\epsilon(x))dx, \end{aligned} \tag{3.12}$$

For all t in $(0,T)$ and $\epsilon < \frac{1}{n+1}$. For any subsequences u_i^ϵ(still denoted by $u_i^\epsilon, i = 1,2,3$), Lemma 3.2 shows that

$$\begin{aligned} &(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon) \to (u_1, u_2, u_3) \quad \text{a.e in } Q_T, \\ &T_k(u_i^\epsilon) \rightharpoonup T_k(u_i) \ \text{ weakly in } L^2(0,T; H_0^1(\Omega)), i = 1,2,3, \\ &\Theta_n(u_i^\epsilon) \rightharpoonup \Theta_n(u_i) \ \text{ weakly in } L^2(0,T; H_0^1(\Omega)), i = 1,2,3, \end{aligned} \tag{3.13}$$

as $\epsilon \to 0$ for any $k > 0$ and $n \geq 1$. From $(H2)$,

$$|A_i(T_k(u_i^\epsilon), \nabla T_k(u_i^\epsilon))| \leq C_k(x,t) + \beta_k|\nabla T_k(u_i^\epsilon)|, \quad i = 1,2,3,$$

Where $C_k \in L^2(Q_T)$. It shows that $A_i(T_k(u_i^\epsilon), \nabla T_k(u_i^\epsilon)), i = 1,2,3$, is bounded in$L^2(Q_T)$. Therfore,

$$A_i(T_k(u_i^\epsilon), \nabla T_k(u_i^\epsilon)) \rightharpoonup \eta_{i,k} \ \text{ weakly in } L^2(Q_T), i = 1,2,3, \tag{3.14}$$

as $\epsilon \to 0$ where $\eta_{i,k} \in L^2(Q_T), i = 1,2,3$. From (3.6), (3.7) with Lemma 3.2 we obtain

$$\sum_{i=1}^{3} \int_\Omega \widetilde{T}_k(u_i^\epsilon)(t)dx \leq \int_{Q_T} (\sigma(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon) + \mu_1 u_1^\epsilon + f^\epsilon) T_k(u_1^\epsilon)\, dxds$$
$$+ \int_{Q_T} (\sigma(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon) + \mu_1 u_1^\epsilon + f^\epsilon) T_k(u_1^\epsilon)\, dxds$$
$$+ \int_{Q_T} (\varrho u_2^\epsilon + \mu_3 u_3^\epsilon + h^\epsilon) T_k(u_3^\epsilon)\, dxds + \sum_{i=1}^{3} \int_\Omega \widetilde{T}_k(u_{i,0}^\epsilon(x))dx.$$

Using the boundedness of the solutions $(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon)$, Lemma 3.2, the properties of $\sigma(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon), f^\epsilon, g^\epsilon, h^\epsilon$, and Young's inequality, we obtain

$$\sum_{i=1}^{3} \int_\Omega \widetilde{T}_k(u_i^\epsilon)(t)dx$$
$$\leq C_k + k(\|f^\epsilon\|_{L^2(Q_T)} + \|g^\epsilon\|_{L^2(Q_T)} + \|h^\epsilon\|_{L^2(Q_T)} + \sum_{i=1}^{3} \|u_{i,0}(x)\|_{L^2(\Omega)}),$$

where C_k is a constant independent of ϵ. Taking $\liminf$ as ϵ tends to zero in the above estimate and using (3.13) and Lemma 3.2, we obtain

$$\sum_{i=1}^{3} \int_\Omega \widetilde{T}_k(u_i)(t)dx$$
$$\leq C_k + k(\|f\|_{L^1(Q_T)} + \|g\|_{L^1(Q_T)} + \|h\|_{L^1(Q_T)} + \sum_{i=1}^{3} \|u_{i,0}(x)\|_{L^1(\Omega)}),$$

By using the definition of $\widetilde{T}_k(u_i)$, we deduce that

$$\sum_{i=1}^{3} k \int_\Omega |u_i(x,t)|dx \leq C_k + \frac{k^2}{2}\ \text{meas}(\Omega) + k(\|f\|_{L^1(Q_T)} + \|g\|_{L^1(Q_T)}$$
$$+ \|h\|_{L^1(Q_T)} + \sum_{i=1}^{3} \|u_{i,0}(x)\|_{L^1(\Omega)}), \tag{3.15}$$

For almost all t in $(0,T)$. This proves that $u_i \in L^\infty(0,T; L^1(\Omega)), i = 1,2,3$. Now equation (3.12) with (3.13) proves that

$$
\begin{aligned}
&\int_{Q_t} A_1(u_1^\epsilon, \nabla u_1^\epsilon)\nabla\Theta_n(u_1^\epsilon)\,dxds \\
&\le \int_{Q_t} (\sigma(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon) + \mu_1 u_1 + f)\Theta_n(u_1)\,dxds + \int_\Omega \widetilde{\Theta}_n(u_{1,0}(x))dx.
\end{aligned} \tag{3.16}
$$

Using $(H1), \nabla\Theta_n(u_1^\epsilon) = \chi_{\{n\le|u_1^\epsilon|\le n+1\}}\nabla u_1^\epsilon$ and the weak convergence in (3.13), we obtain

$$
\begin{aligned}
&\alpha_1 \int_{Q_t} |\nabla\Theta_n(u_1)|^2 dxds \\
&\le \int_{Q_t} (\sigma(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon) + \mu_1 u_1 + f)\Theta_n(u_1)\,dxds + \int_\Omega \widetilde{\Theta}_n(u_{1,0}(x))dx.
\end{aligned} \tag{3.17}
$$

Since $\Theta_n(u_1) \to 0$, as $n \to \infty$ shows that $\Theta_n(u_1) \to 0$, weakly in $L^2(0,T; H_0^1(\Omega))$. This leads to the right-hand side of each term of (3.17), that is,

$$
\int_{Q_t} (\sigma(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon)\Theta_n(u_1)\,dxds \to 0, \quad \int_{Q_t} \mu_1 u_1\Theta_n(u_1)\,dxds \to 0,
$$

$$
\int_{Q_t} f\Theta_n(u_1)\,dxds \to 0,
$$

as $n \to \infty$ and $\|\Theta_n(u_{1,0})\|_{L^1(\Omega)} \le \|u_{1,0}\|_{L^1(\Omega)}$ impliying that $\int_\Omega \widetilde{\Theta}_n(u_{1,0})dx \to 0$ as $n \to \infty$ which follows from the Lebesgue convergence theorem. Hence, passing to the lim inf in (3.16) and (3.17), we obtain

$$
\lim_{n\to\infty} \limsup_{\epsilon\to 0} \int_0^t \int_{\{n\le|u_1^\epsilon|\le n+1\}} A_1(u_1^\epsilon, \nabla u_1^\epsilon)\nabla\Theta_n(u_1^\epsilon)\,dxds \to 0,
$$

$\Theta_n(u_1) \to 0$, strongly in $L^2(0,T; H_0^1(\Omega))$ as $n \to \infty$. Similarly, by considering $\Theta_n(u_2^\epsilon), \Theta_n(u_3^\epsilon)$ respectively test functions in the second and third equations of (3.3) and by adopting the above type of arguments to prove that for $i = 1, 2, 3$,

$$
\begin{aligned}
&\lim_{n\to\infty} \limsup_{\epsilon\to 0} \int_0^t \int_{\{n\le|u_i^\epsilon|\le n+1\}} A_i(u_i^\epsilon, \nabla u_i^\epsilon)\nabla\Theta_n(u_i^\epsilon)\,dxds \to 0, \\
&\Theta_n(u_i) \to 0, \quad \text{strongly in } L^2(0,T; H_0^1(\Omega)) \text{ as } n \to \infty.
\end{aligned} \tag{3.18}
$$

This proves the desired result of the Lemma.

Definition 3.4. *We define the time regularization of the function, for $i = 1, 2, 3$, by*

$$(T_k(u_i))_\gamma = \gamma \int_{-\infty}^{t} e^{\gamma(s-t)} T_k \overline{(u_i(x,s))} ds,$$

Where $\overline{u_i(x,s)} = \begin{cases} u_i(x,s) & \text{if } s > 0, \\ u_{i,0}(x) & \text{if } s < 0. \end{cases}$
Let us consider the unique solution $(T_k(u_i))_\gamma \in L^\infty(Q_T) \cap L^2(0,T; H_0^1(\Omega))$ of the monotone problem

$$\begin{aligned} &\frac{\partial}{\partial t}(T_k(u_i))_\gamma + \gamma((T_k(u_i))_\gamma - T_k(u_i)) = 0 \quad \text{in } Q_T, \\ &(T_k(u_i(x,0)))_\gamma = (T_k(u_{i,0}(x)) \quad \text{in } \Omega, \end{aligned} \tag{3.19}$$

for $\gamma > 0$ and $k > 0$. From (3.19) *and Lemma* 3.2*, we have $\frac{\partial}{\partial t}(T_k(u_i))_\gamma \in L^2(0,T; H_0^1(\Omega))$.*

Remark 3.3. *For $i = 1, 2, 3$, we have $(T_k(u_i))_\gamma \to T_k(u_i))$ a.e in Q_T, weak-* in $L^\infty(Q_T)$ and strongly in $L^2(0,T; H_0^1(\Omega))$ as $\gamma \to \infty$ and also*

$$\|(T_k(u_i))_\gamma\|_{L^\infty(Q_T)} \le \max(\|T_k(u_i))\|_{L^\infty(Q_T)}, \|T_k(u_{i,0}))\|_{L^\infty(\Omega)}) \le k$$

for any $\gamma > 0$ and $k \ge 0$.

Lemma 3.5. *Let $k \ge 0$ be fixed and S be an increasing $C^\infty(\mathbb{R})$ function such that $S(z) = z$ for $|z| \le k$ with suppS' compact. Then, for $i = 1, 2, 3$,*

$$\lim_{n\to\infty} \limsup_{\epsilon\to 0} \int_0^T \int_{Q_t} \frac{\partial S(u_i^\epsilon)}{\partial t}(T_k(u_i^\epsilon) - (T_k(u_i))_\gamma)\, dx\, ds\, dt \ge 0.$$

The proof of the above Lemma is similar to that of the Lemma in [7], and is omitted here.

Lemma 3.6. *For $i = 1, 2, 3$, $\eta_{i,k}$, as defined in* (3.14)*, the subsequences u_i^ϵ (still denoted by u_i^ϵ) satisfy*

$$\begin{aligned} &\limsup_{\epsilon\to 0} \int_0^T \int_{Q_t} A_i(T_k(u_i^\epsilon), \nabla T_k(u_i^\epsilon)) \nabla T_k(u_i^\epsilon)\, dx\, ds\, dt \\ &\le \int_0^T \int_{Q_t} \eta_{i,k} \nabla T_k(u_i)\, dx\, ds\, dt. \end{aligned} \tag{3.20}$$

Proof. Let S_n be a sequence of increasing $C^\infty(\mathbb{R})$ functions such that

$$\begin{aligned} &S_n(z) = z, \quad \text{for } |z| \le n, \\ &suppS'_n \subset [-(n+1),(n+1)], \quad \|S''_n\|_L^\infty(\mathbb{R}) \le 1. \end{aligned} \tag{3.21}$$

Multiply the first equation of (3.3) by $S'_n(u_1^\epsilon)$ to obtain

$$\begin{aligned} \frac{\partial S_n(u_1^\epsilon)}{\partial t} &= \mathrm{div}(S'_n(u_1^\epsilon)A_1(u_1^\epsilon,\nabla u_1^\epsilon)) - S''_n(u_1^\epsilon)A_1(u_1^\epsilon,\nabla u_1^\epsilon)\nabla u_1^\epsilon \\ &+ \mathrm{div}(S'_n(u_1^\epsilon)(\alpha_{1,1}\nabla u_1^\epsilon + \alpha_{1,2}\nabla u_2^\epsilon + \alpha_{1,3}\nabla u_3^\epsilon)) \\ &- S''_n(u_1^\epsilon)(\alpha_{1,1}\nabla u_1^\epsilon + \alpha_{1,2}\nabla u_2^\epsilon + \alpha_{1,3}\nabla u_3^\epsilon)\nabla u_1^\epsilon \\ &+ S'_n(u_1^\epsilon)(-\sigma(u_1^\epsilon,u_2^\epsilon,u_3^\epsilon) - \mu_1 u_1^\epsilon + f^\epsilon). \end{aligned} \tag{3.22}$$

From (3.22), we understand that $\frac{\partial S_n(u_1^\epsilon)}{\partial t} \in L^1(Q_T) + L^2(0,T;H^{-1}(\Omega))$. For fixed $k>0, \gamma>0$, and $\epsilon>0$, we set

$$W_{i,\gamma}^\epsilon = T_k(u_i^\epsilon) - (T_k(u_i))_\gamma, \quad i=1,2,3. \tag{3.23}$$

Multipliying,(3.22) by $W_{i,\gamma}^\epsilon$ and integrating over $Q_t \times (0,T)$, we obtain

$$\int_Q \frac{\partial S_n(u_1^\epsilon)}{\partial t} W_{1,\gamma}^\epsilon \, dx\, ds\, dt = I_1 + I_2 + I_3 + I_4 + I_5, \tag{3.24}$$

Where

$$\begin{aligned} I_1 &= -\int_Q S'_n(u_1^\epsilon)A_1(u_1^\epsilon,\nabla u_1^\epsilon)\nabla W_{1,\gamma}^\epsilon \, dx\, ds\, dt, \\ I_2 &= -\int_Q S''_n(u_1^\epsilon)A_1(u_1^\epsilon,\nabla u_1^\epsilon)\nabla u_1^\epsilon W_{1,\gamma}^\epsilon \, dx\, ds\, dt, \\ I_3 &= -\int_Q S'_n(u_1^\epsilon)(\alpha_{1,1}\nabla u_1^\epsilon + \alpha_{1,2}\nabla u_2^\epsilon + \alpha_{1,3}\nabla u_3^\epsilon)\nabla W_{1,\gamma}^\epsilon \, dx\, ds\, dt, \\ I_4 &= -\int_Q S''_n(u_1^\epsilon)(\alpha_{1,1}\nabla u_1^\epsilon + \alpha_{1,2}\nabla u_2^\epsilon + \alpha_{1,3}\nabla u_3^\epsilon)\nabla u_1^\epsilon W_{1,\gamma}^\epsilon \, dx\, ds\, dt, \\ I_5 &= \int_Q S'_n(u_1^\epsilon)(-\sigma(u_1^\epsilon,u_2^\epsilon,u_3^\epsilon) - \mu_1 u_1^\epsilon + f^\epsilon)W_{1,\gamma}^\epsilon \, dx\, ds\, dt, \end{aligned}$$

Where $Q = Q_t \times (0,T)$. By (3.23) for fixed $\gamma > 0, W_{i,\gamma}^\epsilon \rightharpoonup T_k(u_i) - (T_k(u_i))_\gamma, i=1,2,3$, weakly in $L^2(0,T;H_0^1(\Omega))$ as $\epsilon > 0$. By Remark 3.3, we conclude that $\|W_{i,\gamma}^\epsilon\|_{L^\infty(Q_T)} \le 2k$, for any $\epsilon > 0$

and $\gamma > 0$. This boundedness of $W^\epsilon_{1,\gamma}$ shows that for fixed $\gamma > 0, W^\epsilon_{i,\gamma} \rightharpoonup T_k(u_i) - (T_k(u_i))_\gamma$, $i = 1, 2, 3$, a.e in Q_T and $L^\infty(Q_T)$ weak* as $\epsilon > 0$.

By the definition of S_n, we have supp $S''_n \subset [-(n+1), (n+1)] \cup [n, (n+1)]$, for any $n \geq 1$.

As a consequence

$$|I_2| \leq T\|S''\|_{L^\infty(\mathbb{R})}\|W^\epsilon_{i,\gamma}\|_{L^\infty(Q_T)} \int_{\{(x,t);n\leq|u^\epsilon_1|\leq n+1\}} A_1(u^\epsilon_1, \nabla u^\epsilon_1)\nabla u^\epsilon_1 \, dx \, ds,$$

for any $n \geq 1, \epsilon \leq \frac{1}{n+1}$ and $\gamma > 0$. From $\|W^\epsilon_{1,\gamma}\|_{L^\infty(Q_T)} \leq 2k$ and (3.21), we easily obtain

$$\limsup_{\gamma\to\infty} \limsup_{\epsilon\to 0} |I_2| \leq c \limsup_{\epsilon\to 0} \int_{\{(x,t);n\leq|u^\epsilon_1|\leq n+1\}} A_1(u^\epsilon_1, \nabla u^\epsilon_1)\nabla u^\epsilon_1 \, dx \, ds,$$

for any $n \geq 1$, where the constant c depends on T and k. Hence, by Lemma 3.4, we achieve that

$$\lim_{n\to\infty} \limsup_{\gamma\to\infty} \limsup_{\epsilon\to 0} \int_Q S''_n(u^\epsilon_1)A_1(u^\epsilon_1, \nabla u^\epsilon_1)\nabla u^\epsilon_1 W^\epsilon_{1,\gamma} \, dx \, ds \, dt = 0. \tag{3.25}$$

For some $n \geq 1, I_3$ can be rewritten as

$$I_3 = \int_Q S'_n(u^\epsilon_1)((k_1F^+_\epsilon(T_{n+1}(u^\epsilon_1)) + F^+_\epsilon(T_{n+1}(u^\epsilon_2)) + F^+_\epsilon(T_{n+1}(u^\epsilon_3))\nabla T_{n+1}(u^\epsilon_1)$$
$$+ F^+_\epsilon(T_{n+1}(u^\epsilon_1))\nabla T_{n+1}(u^\epsilon_2) + F^+_\epsilon(T_{n+1}(u^\epsilon_1))\nabla T_{n+1}(u^\epsilon_3))\nabla W^\epsilon_{1,\gamma} \, dx \, ds \, dt,$$

a.e in Q_T. Since supp $S'_n \subset [-(n+1), (n+1)]$, for $i = 1, 2, 3$, the definition of $F^+_\epsilon(u^\epsilon_i)$ andthe results (3.13) lead to $S'_n(u^\epsilon_1)F^+_\epsilon(T_{n+1}(u^\epsilon_i)) \to S'_n(u_1)T_{n+1}(u_i)$ a.e in Q_T and in $L^\infty(Q_T)$ weak* as $\epsilon \to 0$. This proves

$$\lim_{\epsilon\to 0} I_3$$
$$= \int_Q S'_n(u_1)((k_1T_{n+1}(u_1) + T_{n+1}(u_2) + T_{n+1}(u_3))\nabla T_{n+1}(u_1)$$
$$+ T_{n+1}(u_1)\nabla T_{n+1}(u_2) + T_{n+1}(u_1)\nabla T_{n+1}(u_3))(\nabla T_k(u_1) - \nabla(T_k(u_1))_\gamma) \, dx \, ds \, dt,$$

for any $\gamma > 0$. By using the remark 3.3, this leads to

$$\lim_{\gamma\to\infty}\lim_{\epsilon\to 0}\int_Q S_n'(u_1^\epsilon)(\alpha_{1,1}\nabla u_1^\epsilon + \alpha_{1,2}\nabla u_2^\epsilon + \alpha_{1,3}\nabla u_3^\epsilon)\nabla W_{1,\gamma}^\epsilon \, dx\, ds\, dt = 0. \tag{3.26}$$

Similarly, we can show that

$$\begin{aligned}\lim_{\gamma\to\infty}\lim_{\epsilon\to 0}\int_Q S_n''(u_1^\epsilon)(\alpha_{1,1}\nabla u_1^\epsilon + \alpha_{1,2}\nabla u_2^\epsilon + \alpha_{1,3}\nabla u_3^\epsilon)\nabla u_1^\epsilon W_{1,\gamma}^\epsilon \, dx\, ds\, dt = 0,\\ \lim_{\gamma\to\infty}\lim_{\epsilon\to 0}\int_Q S_n'(u_1^\epsilon)(\sigma(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon) + \mu_1 u_1^\epsilon)W_{1,\gamma}^\epsilon \, dx\, ds\, dt = 0.\end{aligned} \tag{3.27}$$

Since $fS_n'(u_1^\epsilon) \in L^1(Q_T)$, (3.13) and Remark 3.3 lead to

$$\lim_{\gamma\to\infty}\lim_{\epsilon\to 0}\int_Q f^\epsilon S_n'(u_1^\epsilon)W_{1,\gamma}^\epsilon \, dx\, ds\, dt = 0. \tag{3.28}$$

Consequently, from Lemma 3.5 and the definition of $W_{1,\gamma}^\epsilon$, we have

$$\lim_{\gamma\to\infty}\lim_{\epsilon\to 0}\int_Q \frac{\partial S_n(u_1^\epsilon)}{\partial t} W_{1,\gamma}^\epsilon \, dx\, ds\, dt \geq 0 \quad \text{for any } n \geq k. \tag{3.29}$$

It is easy to understand that, from (3.25) – (3.28) along with (3.24) and (3.29), we obtain

$$\lim_{\gamma\to\infty}\lim_{\epsilon\to 0} I_1 = \lim_{\gamma\to\infty}\lim_{\epsilon\to 0}\int_Q S_n'(u_1^\epsilon)A_1(u_1^\epsilon, \nabla u_1^\epsilon)\nabla W_{1,\gamma}^\epsilon \, dx\, ds\, dt \leq 0. \tag{3.30}$$

Since

$$S_n'(u_1^\epsilon)A_1(u_1^\epsilon, \nabla u_1^\epsilon)\nabla T_k(u_1^\epsilon) = A_1(u_1^\epsilon, \nabla u_1^\epsilon)\nabla T_k(u_1^\epsilon)$$

for $k \leq \frac{1}{\epsilon}$ and $k \leq n$ because of the definition of S_n and from (3.30), we obtain

$$\begin{aligned}&\limsup_{\epsilon\to 0}\int_Q A_1(u_1^\epsilon, \nabla u_1^\epsilon)\nabla T_k(u_1^\epsilon)\, dx\, ds\, dt\\ &\leq \lim_{\gamma\to\infty}\lim_{\epsilon\to 0}\int_Q S_n'(u_1^\epsilon)A_1(u_1^\epsilon, \nabla u_1^\epsilon)\nabla (T_k(u_1^\epsilon))_\gamma \, dx\, ds\, dt \quad \text{for } k \leq n.\end{aligned}$$

For $\epsilon \le 1/(n+1)$, we know that

$$S_n'(u_1^\epsilon)A_1(u_1^\epsilon, \nabla u_1^\epsilon) = S_n'(u_1^\epsilon)A_1(T_{n+1}(u_1^\epsilon), \nabla T_{n+1}(u_1^\epsilon))$$

a.e in Q. Due to (3.14), we have $S_n'(u_1^\epsilon)A_1(T_{n+1}(u_1^\epsilon), \nabla T_{n+1}(u_1^\epsilon)) \rightharpoonup S_n'(u_1)\eta_{1,n+1}$ weakly in $L^2(Q_T)$ as $\epsilon \to 0$. This help us to prove that, for any $n \le 1$,

$$\begin{aligned} &\lim_{\gamma\to\infty}\lim_{\epsilon\to 0}\int_Q S_n'(u_1^\epsilon)A_1(u_1^\epsilon, \nabla u_1^\epsilon)\nabla(T_k(u_1^\epsilon))_\gamma \, dx\, ds\, dt \\ &= \int_Q S_n'(u_1)\eta_{1,n+1}\nabla T_k(u_1)\, dx\, ds\, dt \\ &= \int_Q \eta_{1,n+1}\nabla T_k(u_1)\, dx\, ds\, dt. \end{aligned} \tag{3.31}$$

For any $k \le n$, we have

$$A_1(T_{n+1}(u_1^\epsilon), \nabla T_{n+1}(u_1^\epsilon))_{\chi(|u_1^\epsilon|\le k)} = A_1(T_k(u_1^\epsilon), \nabla T_k(u_1^\epsilon))_{\chi(|u_1^\epsilon|\le k)} \text{ a.e in } Q_T.$$

The above equation with (3.13) and (3.14) implies that ${\eta_{1,n+1}}_{\chi(|u_1^\epsilon|\le k)} = {\eta_{1,k}}_{\chi(|u_1^\epsilon|\le k)}$ a.e in $Q_T - \{|u_1^\epsilon| = k\}$ for $k \le n$ as $\epsilon \to 0$. Therfore, (3.31) becomes

$$\limsup_{\epsilon\to 0}\int_Q A_1(T_k(u_1^\epsilon), \nabla T_k(u_1^\epsilon))\nabla T_k(u_1^\epsilon)\, dx\, ds\, dt = \int_Q \eta_{1,k}\nabla T_k(u_1)\, dx\, ds\, dt.$$

Similar arguments as we used to obtain the previous equation lead to, for $i = 1, 2, 3$,

$$\limsup_{\epsilon\to 0}\int_Q A_i(T_k(u_i^\epsilon), \nabla T_k(u_i^\epsilon))\nabla T_k(u_i^\epsilon)\, dx\, ds\, dt = \int_Q \eta_{i,k}\nabla T_k(u_i)\, dx\, ds\, dt.$$

This completes the proof of the Lemma.

Lemma 3.7. *For any $k > 0$ and $i = 1, 2, 3$, we have*

$$\begin{aligned} &\limsup_{\epsilon\to 0}\int_Q [A_i(T_k(u_i^\epsilon), \nabla T_k(u_i^\epsilon)) - A_i(T_k(u_i^\epsilon), \nabla T_k(u_i))] \\ &\times [\nabla T_k(u_i^\epsilon) - \nabla T_k(u_i)]\, dx\, ds\, dt = 0. \end{aligned} \tag{3.32}$$

Proof. The monotone assumption $(H3)$ shows that, for $i = 1, 2, 3$,

$$\limsup_{\epsilon\to 0} \int_Q [A_i(T_k(u_i^\epsilon), \nabla T_k(u_i^\epsilon)) - A_i(T_k(u_i^\epsilon), \nabla T_k(u_i))] \times [\nabla T_k(u_i^\epsilon) - \nabla T_k(u_i)]\, dx\, ds\, dt \geq 0 \tag{3.33}$$

for any $k \geq 0$. We remark that $(H2)$ and first result of (3.13) implies that

$$A_i(T_k(u_i^\epsilon), \nabla T_k(u_i)) \to A_i(T_k(u_i), \nabla T_k(u_i)),$$

a.e in Q_T as $\epsilon \to 0$ and that $A_i(T_k(u_i^\epsilon), \nabla T_k(u_i)) \leq C_k(x,t) + \beta_k |\nabla T_k(u_i)|$ a.e. in Q_T, uniformly with respect to ϵ. Then, by Lemma 3.6, (3.13) and (3.14), we have, for $i = 1, 2, 3$,

$$\limsup_{\epsilon\to 0} \int_Q [A_i(T_k(u_i^\epsilon), \nabla T_k(u_i^\epsilon)) - A_i(T_k(u_i^\epsilon), \nabla T_k(u_i))] \times [\nabla T_k(u_i^\epsilon) - \nabla T_k(u_i)]\, dx\, ds\, dt = 0$$

This completes the proof.

Lemma 3.8. *For fixed $k \geq 0$ and $i = 1, 2, 3,$ we have*

$$\eta_{i,k} = A_i(T_k(u_i), \nabla T_k(u_i)) \quad a.e \ in \ Q_T,$$

$$A_i(T_k(u_i^\epsilon), \nabla T_k(u_i^\epsilon))\nabla T_k(u_i^\epsilon) \rightharpoonup A_i(T_k(u_i), \nabla T_k(u_i))\nabla T_k(u_i) \quad weakly \ in \ L^1(Q_T).$$

Proof. For any $k > 0$ and $0 < \epsilon < \frac{1}{k}$, from Lemma 3.6, convergence (3.13) implies that, for $i = 1, 2, 3$,

$$\limsup_{\epsilon\to 0} \int_Q A_i(T_k(u_i^\epsilon), \nabla T_k(u_i^\epsilon))\nabla T_k(u_i^\epsilon)\, dx\, ds\, dt = \int_Q \eta_{i,k} \nabla T_k(u_i)\, dx\, ds\, dt$$

Using Minty's type arguments and (3.13), (3.14), from the above equation we obtain $A_k(T_k(u_i), \nabla T_k(u_i)) = \eta_{i,k}$, for $i = 1, 2, 3$, and any $k \geq 0$. This proves the first result of the present Lemma.

For any $k \geq 0, T' < T$, Lemma 3.7 shows that, for $i = 1, 2, 3$,

$$[A_i(T_k(u_i^\epsilon), \nabla T_k(u_i^\epsilon)) - A_i(T_k(u_i^\epsilon), \nabla T_k(u_i))]\,[\nabla T_k(u_i^\epsilon) - \nabla T_k(u_i)] \to 0$$

Strongly in $L^1(\Omega \times (0, T'))$, as $\epsilon \to 0$. By (3.13) and with the first result of this Lemma, for $i = 1, 2, 3$, we obtain

$$\begin{aligned}
&A_i(T_k(u_i^\epsilon), \nabla T_k(u_i^\epsilon))\nabla T_k(u_i) \rightharpoonup A_i(T_k(u_i), \nabla T_k(u_i))\nabla T_k(u_i) \quad \text{weakly in } L^1(Q_T),\\
&A_i(T_k(u_i^\epsilon), \nabla T_k(u_i))\nabla T_k(u_i^\epsilon) \rightharpoonup A_i(T_k(u_i), \nabla T_k(u_i))\nabla T_k(u_i) \quad \text{weakly in } L^1(Q_T),\\
&A_i(T_k(u_i^\epsilon), \nabla T_k(u_i))\nabla T_k(u_i) \to A_i(T_k(u_i), \nabla T_k(u_i))\nabla T_k(u_i) \quad \text{strongly in } L^1(Q_T),
\end{aligned}$$

as $\epsilon \to 0$. Hence

$$A_i(T_k(u_i^\epsilon), \nabla T_k(u_i^\epsilon))\nabla T_k(u_i^\epsilon) \rightharpoonup A_i(T_k(u_i), \nabla T_k(u_i))\nabla T_k(u_i)$$

Weakly in $L^1(\Omega \times (0, T'))$, for any $T' < T$ as $\epsilon \to 0$. According to the definition of the function $A_i(s, \varsigma)$ and f, g, h, the assummptions hold true for all time T. Hence

$$A_i(T_k(u_i^\epsilon), \nabla T_k(u_i^\epsilon))\nabla T_k(u_i^\epsilon) \rightharpoonup A_i(T_k(u_i), \nabla T_k(u_i))\nabla T_k(u_i)$$

weakly in $L^1(Q_T)$ holds.

Lemma 3.9. *For any $n \geq 0$, and $i = 1, 2, 3$,*

$$\int_{\{(x,t)\in Q_T; n\leq |u_i|\leq n+1\}} A_i(u_i, \nabla u_i)\nabla u_i \, dx\, dt \to 0 \quad \textit{as } n \to \infty.$$

Proof. For $i = 1, 2, 3$,

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\{(x,t)\in Q_T; n\le |u_i|\le n+1\}} A_i(u_i^\epsilon, \nabla u_i^\epsilon)\nabla u_i^\epsilon \,dx\,dt \\
&= \lim_{\epsilon\to 0}\int_Q A_i(u_i^\epsilon, \nabla u_i^\epsilon)\nabla (T_{n+1}(u_i^\epsilon) - T_n(u_i^\epsilon))\,dx\,dt \\
&= \int_{Q_T} A_i(u_i, \nabla u_i)\nabla T_{n+1}(u_i)\,dx\,dt - \int_{Q_T} A_i(u_i, \nabla u_i)\nabla T_n(u_i)\,dx\,dt \\
&= \int_{\{(x,t)\in Q_T; n\le |u_i|\le n+1\}} A_i(u_i, \nabla u_i)\nabla u_i\,dx\,dt \ \text{ for any } \ n \ge 0.
\end{aligned}
$$

Using Lemma 3.4 and from the above equality, we have

$$
\int_{\{(x,t)\in Q_T; n\le |u_i|\le n+1\}} A_i(u_i, \nabla u_i)\nabla u_i\,dx\,dt \to 0 \ \text{ as } n \to \infty.
$$

This completes the proof.

3.5 Existence of Renormalized Solutions

Proof of the Theorem 3.1 From system (3.3), we have

$$
\begin{aligned}
&\frac{\partial S(u_1^\epsilon)}{\partial t} - \operatorname{div}(S'(u_1^\epsilon)A_1(u_1^\epsilon, \nabla u_1^\epsilon)) + S''(u_1^\epsilon)A_1(u_1^\epsilon, \nabla u_1^\epsilon)\nabla u_1^\epsilon \\
&- \operatorname{div}(S'(u_1^\epsilon)(\alpha_{1,1}\nabla u_1^\epsilon + \alpha_{1,2}\nabla u_2^\epsilon + \alpha_{1,3}\nabla u_3^\epsilon)) \\
&+ S''(u_1^\epsilon)(\alpha_{1,1}\nabla u_1^\epsilon + \alpha_{1,2}\nabla u_2^\epsilon + \alpha_{1,3}\nabla u_3^\epsilon)\nabla u_1^\epsilon \\
&= (-\sigma(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon) - \mu_1 u_1^\epsilon + f^\epsilon)S'(u_1^\epsilon), \\
&\frac{\partial S(u_2^\epsilon)}{\partial t} - \operatorname{div}(S'(u_2^\epsilon)A_2(u_2^\epsilon, \nabla u_2^\epsilon)) + S''(u_2^\epsilon)A_2(u_2^\epsilon, \nabla u_2^\epsilon)\nabla u_2^\epsilon \\
&- \operatorname{div}(S'(u_2^\epsilon)(\alpha_{2,1}\nabla u_1^\epsilon + \alpha_{2,2}\nabla u_2^\epsilon + \alpha_{2,3}\nabla u_3^\epsilon)) \\
&+ S''(u_2^\epsilon)(\alpha_{2,1}\nabla u_1^\epsilon + \alpha_{2,2}\nabla u_2^\epsilon + \alpha_{2,3}\nabla u_3^\epsilon)\nabla u_2^\epsilon \\
&= (\sigma(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon) - \varrho u_2^\epsilon - \mu_2 u_2^\epsilon + g^\epsilon)S'(u_2^\epsilon),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial S(u_3^\epsilon)}{\partial t} - \operatorname{div}(S'(u_3^\epsilon)A_3(u_3^\epsilon, \nabla u_3^\epsilon)) + S''(u_3^\epsilon)A_3(u_3^\epsilon, \nabla u_3^\epsilon)\nabla u_3^\epsilon \\
&- \operatorname{div}(S'(u_3^\epsilon)(\alpha_{3,1}\nabla u_1^\epsilon + \alpha_{3,2}\nabla u_2^\epsilon + \alpha_{3,3}\nabla u_3^\epsilon)) \\
&+ S''(u_3^\epsilon)(\alpha_{3,1}\nabla u_1^\epsilon + \alpha_{3,2}\nabla u_2^\epsilon + \alpha_{3,3}\nabla u_3^\epsilon)\nabla u_3^\epsilon \\
&= (\varrho u_2^\epsilon - \mu_3 u_3^\epsilon + h^\epsilon)S'(u_3^\epsilon),
\end{aligned}
$$

Since S is a bounded and continuous function with supp $S' \subset [-k, k]$, using the boundedness of $S''(u_i), i = 1, 2, 3$, along with the results (3.13), (3.14), using the hypotheses $(H5), (H6), (H7)$ and the Lemma 3.8, 3.9, we conclude that the equations (3.2) of Definition 3.1 hold. By Lemma 3.2 and Aubin type Lemma, we obtain that $S(u_i^\epsilon(x, 0)) = S(u_{i,0}^\epsilon(x)), i = 1, 2, 3$, converges to $S(u_{i,0}^\epsilon(x))$ strongly in $H^{-1,s}(\Omega)$, where $s < \inf(2, \frac{N}{N-1})$. Then $(H4), (H7)$ and the smoothness of S prove the strong convergence in $L^1(\Omega)$. Hence we conclude that $S(u_i(x, 0)) = S(u_{i,0}(x)), i = 1, 2, 3$. This establishes the existence of renormalized solutions of the reaction-diffusion system (3.1).

3.6 Existence of entropy Solutions

Proof of the theorem 3.2 Take $T_k(u_i^\epsilon - \phi_i), i = 1, 2, 3$, as the test functions, respectively in the equations (3.2) and for $k > 0, \phi_i \in C'(\overline{Q}_T)$ with $\phi_i = 0, i = 1, 2, 3$, in $\sum_T$. Now, integrating the equation (3.2) over Q_T, we obtain

$$
\begin{aligned}
&\int_0^T \langle u_{1t}^\epsilon, T_k(u_1^\epsilon - \phi_1)\rangle dt + \int_{Q_T} A_1(u_1^\epsilon, \nabla u_1^\epsilon)\nabla T_k(u_1^\epsilon - \phi_1)dx\,dt \\
&+ \int_{Q_T} (\alpha_{1,1}\nabla u_1^\epsilon + \alpha_{1,2}\nabla u_2^\epsilon + \alpha_{1,3}\nabla u_3^\epsilon)\nabla T_k(u_1^\epsilon - \phi_1)dx\,dt \\
&= \int_{Q_T} (-\sigma(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon) - \mu_1 u_1^\epsilon + f^\epsilon)T_k(u_1^\epsilon - \phi_1)dx\,dt, \\
&\int_0^T \langle u_{2t}^\epsilon, T_k(u_2^\epsilon - \phi_2)\rangle dt + \int_{Q_T} A_2(u_2^\epsilon, \nabla u_2^\epsilon)\nabla T_k(u_2^\epsilon - \phi_2)dx\,dt \\
&+ \int_{Q_T} (\alpha_{2,1}\nabla u_1^\epsilon + \alpha_{2,2}\nabla u_2^\epsilon + \alpha_{2,3}\nabla u_3^\epsilon)\nabla T_k(u_2^\epsilon - \phi_2)dx\,dt \\
&= \int_{Q_T} (\sigma(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon) - \varrho u_2^\epsilon - \mu_2 u_2^\epsilon + g^\epsilon)T_k(u_2^\epsilon - \phi_2)dx\,dt,
\end{aligned}
\tag{3.34}
$$

$$
\begin{aligned}
&\int_0^T \langle u_{3t}^\epsilon, T_k(u_3^\epsilon - \phi_3)\rangle dt + \int_{Q_T} A_3(u_3^\epsilon, \nabla u_3^\epsilon)\nabla T_k(u_3^\epsilon - \phi_3) dx\, dt \\
&+ \int_{Q_T} (\alpha_{3,1}\nabla u_1^\epsilon + \alpha_{3,2}\nabla u_2^\epsilon + \alpha_{3,3}\nabla u_3^\epsilon)\nabla T_k(u_3^\epsilon - \phi_3) dx\, dt \\
&= \int_{Q_T} (\varrho u_2^\epsilon - \mu_3 u_3^\epsilon + h^\epsilon) T_k(u_3^\epsilon - \phi_3) dx\, dt,
\end{aligned}
$$

Note that, if $L = k + \|\phi_1\|_{L^\infty(Q_T)}$ and $u_{1t}^\epsilon = (u_1^\epsilon - \phi_1)_t + \phi_{1t}$; we have

$$
\begin{aligned}
&\int_{Q_T} A_1(u_1^\epsilon, \nabla u_1^\epsilon)\nabla T_k(u_1^\epsilon - \phi_1) dx\, dt \\
&= \int_{Q_T} A_1(T_L(u_1^\epsilon), \nabla T_L(u_1^\epsilon))\nabla T_k(u_1^\epsilon - \phi_1) dx\, dt. \\
&\int_0^T \langle u_{1t}^\epsilon, T_k(u_1^\epsilon - \phi_1)\rangle dt \\
&= \int_\Omega \widetilde{T}_k(u_1^\epsilon - \phi_1)(T) dx - \int_\Omega \widetilde{T}_k(u_1^\epsilon - \phi_1)(0) dx \\
&+ \int_0^T \langle \phi_{1t}, T_k(T_L(u_1^\epsilon) - \phi_1)\rangle dt.
\end{aligned} \tag{3.35}
$$

From (3.35), the first equation of (3.34) can be re-written as

$$
\begin{aligned}
&\int_\Omega \widetilde{T}_k(u_1^\epsilon - \phi_1)(T) dx - \int_\Omega \widetilde{T}_k(u_1^\epsilon - \phi_1)(0) dx + \int_0^T \langle \phi_{1t}, T_k(T_L(u_1^\epsilon) - \phi_1)\rangle dt \\
&+ \int_{Q_T} A_1(T_L(u_1^\epsilon), \nabla T_L(u_1^\epsilon))\nabla T_k(T_L(u_1^\epsilon) - \phi_1) dx\, dt \\
&+ \int_{Q_T} (\alpha_{1,1}\nabla T_L(u_1^\epsilon) + \alpha_{1,2}\nabla T_L(u_2^\epsilon) + \alpha_{1,3}\nabla T_L(u_3^\epsilon))\nabla T_k(T_L(u_1^\epsilon) - \phi_1) dx\, dt \\
&= \int_{Q_T} (-\sigma(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon) - \mu_1 u_1^\epsilon + f^\epsilon) T_k(u_1^\epsilon - \phi_1) dx\, dt.
\end{aligned}
$$

Using the fect that $\widetilde{T}_k$ is Lipschitz continuous, $(H7)$ and tha results (3.13), we obtain

$$
\int_\Omega \widetilde{T}_k(u_1^\epsilon - \phi_1)(T) dx \to \int_\Omega \widetilde{T}_k(u_1 - \phi_1)(T) dx,
$$

$$
\int_\Omega \widetilde{T}_k(u_1^\epsilon - \phi_1)(0) dx \to \int_\Omega \widetilde{T}_k(u_1 - \phi_1)(0) dx, \quad \text{as } \epsilon \to 0.
$$

Finally, by considering the strong convergence of f^ϵ, (3.13), (3.14), and the definition of $\sigma(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon)$ and the Lemma 3.8, we obtain the following result by extracting the limit as $\epsilon \to 0$

$$\begin{aligned}
&\int_\Omega \widetilde{T}_k(u_1-\phi_1)(T)dx - \int_\Omega \widetilde{T}_k(u_1-\phi_1)(0)dx + \int_0^T \langle \phi_{1t}, T_k(u_1-\phi_1)\rangle dt \\
&+ \int_{Q_T} A_1(u_1, \nabla u_1)\nabla T_k(u_1-\phi_1)dx\,dt \\
&+ \int_{Q_T} ((k_1u_1+u_2+u_3)\nabla u_1 + u_1\nabla u_2 + u_1\nabla u_3)\nabla T_k(u_1-\phi_1)dx\,dt \\
&= \int_{Q_T} (-\sigma(u_1,u_2,u_3) - \mu_1 u_1 + f)T_k(u_1-\phi_1)dx\,dt.
\end{aligned} \tag{3.36}$$

Similar arguments on the other two equations of the (3.34) lead to

$$\begin{aligned}
&\int_\Omega \widetilde{T}_k(u_2-\phi_2)(T)dx - \int_\Omega \widetilde{T}_k(u_2-\phi_2)(0)dx + \int_0^T \langle \phi_{2t}, T_k(u_2-\phi_2)\rangle dt \\
&+ \int_{Q_T} A_2(u_2, \nabla u_2)\nabla T_k(u_2-\phi_2)dx\,dt + \int_{Q_T} ((u_1+k_2u_2+u_3)\nabla u_2 + u_2\nabla u_1 \\
&+ u_2\nabla u_3)\nabla T_k(u_2-\phi_2)dx\,dt \\
&= \int_{Q_T} (\sigma(u_1,u_2,u_3) - \varrho u_2 - \mu_2 u_2 + g)T_k(u_2-\phi_2)dx\,dt, \\
&\int_\Omega \widetilde{T}_k(u_3-\phi_3)(T)dx - \int_\Omega \widetilde{T}_k(u_3-\phi_3)(0)dx + \int_0^T \langle \phi_{3t}, T_k(u_3-\phi_3)\rangle dt \\
&+ \int_{Q_T} A_3(u_3, \nabla u_3)\nabla T_k(u_3-\phi_3)dx\,dt + \int_{Q_T} ((u_1+u_2+k_3u_3)\nabla u_3 + u_3\nabla u_1 \\
&+ u_3\nabla u_2)\nabla T_k(u_3-\phi_3)dx\,dt \\
&= \int_{Q_T} (\varrho u_2 - \mu_3 u_3 + h)T_k(u_3-\phi_3)dx\,dt,
\end{aligned}$$

for all $k > 0$ and for $i = 1, 2, 3, \phi_i \in C'(\overline{Q}_T)$ with $\phi_i = 0$ in $\sum_T$. This completes the existence of entropy solutions of the system (3.1)

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Abstract

In this work, we study the existence of renormalized and entropy (weak) solutions of nonlinear parabolic system defined by

$$\frac{\partial u_1}{\partial t} - div(A_1(u_1, \nabla u_1) - div(k_1 u_1 + f(u)) = -\sigma(u_1, u_2, u_3) - \mu_1 u_1 + f \qquad \text{in } \ Q_T$$

The same equations with u_2 and u_3. The method of solving our problem consist of obtaining local estimates for suitable approximate problems and then passing to the limit.
This problem is taken from the article [18], that is published in 2013. Which will be used in detailed proofs.

Résumé

Dans ce travail, nous étudions l'éxistence de la solution entropique et renormalisation (faible) d'un system parabolique non linéaire défini par

$$\frac{\partial u_1}{\partial t} - div(A_1(u_1, \nabla u_1) - div(k_1 u_1 + f(u)) = -\sigma(u_1, u_2, u_3) - \mu_1 u_1 + f \qquad \text{dans } \ Q_T$$

Les mêmes équations avec u_2 et u_3. La méthode de résolution de notre problème consiste à obtenir des estimations locales pour des problèmes approximatifs appropriés puis à passer à la limite.
Ce problème est pris à partir de l'article [18], qui est publié dans 2013. Qui sera utilisé dans les preuves détaillées.

ملخص

في هذه المذكرة، قمنا بدراسة وجود الحلول الضعيفة و الأنتروبية، في فضاء تابعي متعلق بمعادلات زائدية غير خطية و غير متجانسة المعرفة كالتالي:

$$\frac{\partial u_1}{\partial t} - div(A_1(u_1, \nabla u_1) - div(k_1 u_1 + f(u)) = -\sigma(u_1, u_2, u_3) - \mu_1 u_1 + f \qquad \text{في } Q_T$$

و نفس المعادلات مع u_2 وu_3 . طريقة حل هذه المعادلات تعتمد على تقريب الجملة الى متتاليات جزئية ثم نبرهن أن الجملة المقربة تتمتع بحل ضعيف ، و من ثم الحصول على التقديرات لمتتالية الحلول التقريبية، و في الأخير نمر الى النهاية .
هذه المعادلات مأخوذة من المقال [16]، المنشور سنة 2013، و هذا حتى يصبح العمل يراعى في الطرح المنهجي و المفصل للبراهين.