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Theme

Darboux problem for perturbed partial differential equation of fractional order

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Table des matières

Introduction

Fractional calculus is a mathematical branch which investigates the properties of derivatives and integrals of non-integer orders (also known as fractional derivatives and integrals, briefly differ-integrals). The interested readers in the subject should refer to the books (Samko et al. 1993 [18], Podlubny 1999 [9], Kilbas et al. 2006 [1], Diethelm 2010 [11]).

The differential equations of fractional order are generalizations of classical differential equations of integer order ; they are increasingly used in such fields as fluid flow, control theory of dynamical systems, diffusive transport akin to diffusion, probability and statistics... etc. The boundary value problem of fractional differential equations is recently approached by various researchers ([15], [12], [21], [16], [17]).

This thesis consists in studying the existence of solutions for partially disturbed hyperbolic functional differential equation withe fractional order, The results obtained are based on some fixed point theorems (Banach's contraction principal, the Leray-Schauder non linear alternative and Burton-Kird's theorem).

This memory consists of an introduction and three chapters .

The first chapter will be devoted to the basic notions and fundamental results of the theory of functional analysis, a reminder and some preliminary concepts will be introduced such as the Gamma function, the Beta function, which plays an important role in the theory of fractional calculus, and basic notions related to fractional calculus such as :
fractional integration, fractional derivation in the Rieman-Liouville , Caputo and katugampola sense , which are the most used .
In the second chapter, the question of the existence of the solution for the following perturbed problem will be addressed :

$$\begin{cases} ({}^c\mathcal{D}_0^r u)(x,y) = f(x,y,u_{(x,y)}) + g(x,y,u_{(x,y)}) \quad (x,y)\in J \\ u(x,y) = \phi(x,y) \quad (x,y)\in \tilde{J} \\ u(x,0)=\varphi(x), \quad u(0,y)=\psi(y), \quad (x,y)\in J. \end{cases}$$

where ${}^{c}\mathcal{D}_0^{r}$ is a fractional Caputo derivative of order $r = (r_1, r_2)$, $f, g : J \times C([-\alpha, 0] \times [-\beta, 0], \mathbb{R}) \longrightarrow \mathbb{R}$, $\phi \in C([-\alpha, 0] \times [-\beta, 0], \mathbb{R})$, $\varphi \in [0, a] \longrightarrow \mathbb{R}$,
$\psi \in [0, b] \longrightarrow \mathbb{R}$ are given function .
We presents a result based on a Burton-Kirk fixed point theorem for the sum of two operators (a contracting operator,a completely continuous operator) . And finally we present an illustrative example.

In the same way of second chapter we have studied the following perturbed problem :

$$\begin{cases} ({}^{c}\mathcal{D}_0^{r,\rho}u)(x,y) = f(x,y,u_{(x,y)}) + g(x,y,u_{(x,y)}) \quad (x,y) \in J \\ u(x,y) = \phi(x,y) \quad (x,y) \in \tilde{J} \\ u(x,0) = \varphi(x), \quad u(0,y) = \psi(y), \quad (x,y) \in J. \end{cases}$$

where ${}^{c}\mathcal{D}_0^{r,\rho}$ is a fractional caputo-Katugampola derivative of order $r = (r_1, r_2)$, $f, g : J \times C([-\alpha, 0] \times [-\beta, 0], \mathbb{R}) \longrightarrow \mathbb{R}$, $\phi \in C([-\alpha, 0] \times [-\beta, 0], \mathbb{R})$, $\varphi \in [0, a] \longrightarrow \mathbb{R}$, $\psi \in [0, b] \longrightarrow \mathbb{R}$ are given function
Also, we end this chapter with an illustrative example .

CHAPITRE 1

BASIC DEFINITIONS OF FRACTIONAL CALCULUS

In this chapter we introduce the fundamental concepts, fundamental definitions and results of fractional calculus (Gamma and Beta functions, Riemann-Liouville, Caputo, Katugampola, fractional integral and derivative).

1.1 Special Functions

1.1.1 Gamma Euler function

Definition 1.1.1 ([1]). The Gamma Euler function is a function which naturally extends the factorial to real numbers , and even complex numbers.
We define the Gamma function by

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt, \ \ (z \in \mathbb{C}, \ \ \mathrm{Re}(z) > 0).$$

With $t^{z-1} = e^{(z-1)\ln t}$.

Exemple 1.1.1.

1. $\Gamma(1) = \int_0^{+\infty} e^{-t} dt = 1.$
2. $\Gamma(\frac{1}{2}) = \int_0^{+\infty} t^{\frac{1}{2}-1} e^{-t} dt = \int_0^{+\infty} t^{-\frac{1}{2}} e^{-t} dt = 2\int_0^{+\infty} e^{-\tau^2} d\tau = \sqrt{\pi}.$

(setting the variable change $t = \tau^2$)

Lemma 1.1.1 ([1]). *The Gamma function is a class C^∞ function on $\mathbb{R}_+^*$, (resp. holomorphic on the half-plane $z \in \mathbb{C}, \ \ Re(z) > 0$) and*
$\forall k \in \mathbb{N}^*, \ \ \forall z \in \mathbb{R}_+^* (resp, \ \ z \in \mathbb{C}, \ \ Re(z) > 0)$

$$\Gamma^k(z) = \int_0^{+\infty} (lnt)^k t^{z-1} e^{-t} dt.$$

properties 1.1.1. For all $z \in \mathbb{C},\ \ Re(z) > 0,\ \ n \in \mathbb{N}$, we have

$$1.\,\Gamma(z+1) = z\Gamma(z).$$

$$2.\,\Gamma(n) = (n-1)!.$$

$$3.\,\Gamma(n+\frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{4^n\,n!}.$$

Remark 1.1.1. The determination of the Gamma function for non-integer negative values by the formula

$$\Gamma(z) = \frac{\Gamma(z+1)}{z},$$

and the transition from one interval to another (-1,0), (-2,-1), (-3,-1),...
The Gamma function does not exist for integer negative values.

Exemple 1.1.2.

1. $\Gamma(-\frac{1}{2}) = \frac{\Gamma(-\frac{1}{2}+1)}{-\frac{1}{2}} = \frac{\Gamma(\frac{1}{2})}{-\frac{1}{2}} = -2\sqrt{\pi}.$
2. $\Gamma(-\frac{3}{2}) = \frac{\Gamma(-\frac{3}{2}+1)}{-\frac{3}{2}} = \frac{\Gamma(-\frac{1}{2})}{-\frac{3}{2}} = \frac{-2\sqrt{\pi}}{-\frac{3}{2}} = \frac{4\sqrt{\pi}}{3}.$

1.1.2 Beta function

Definition 1.1.2 ([1]). The function of Beta is a type of integral of Euler defined by

$$B(p,q) = \int_0^1 t^{p-1}\,(1-t)^{q-1}dt\ \ (p,q \in \mathbb{C},\ \ Re(p) > 0,\ \ Re(q) > 0).$$

For all $p,\ \ q \in \mathbb{C}$, with $Re(p) > 0,\ \ Re(q) > 0$, we have

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Remark 1.1.2.

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \frac{\Gamma(q)\Gamma(p)}{\Gamma(q+p)} = B(q,p),\ \ (p,q \in \mathbb{C},\ \ Re(p) > 0,\ \ Re(q) > 0)$$

1.2 Fractional integral

This section will be devoted to elementary definitions and basic concepts relating to fractional calculation such as :
fractional Riemann-Liouville integration and fractional katugampola integration, the fractional derivation in the Riemann-Liouville sense , Caputo and katugampola, which are the most used.

Definition 1.2.1 ([7]). Let $\Omega = [a, b]$ $(0 < a < b < \infty)$ be a finite interval on the half-axis $\mathbb{R}^+$, and the parameter $0 \leq \gamma < 1$.

1. We denote by $C[a, b]$ the space of continuous functions g on Ω with the norm
$$\|g\|_C = max_{x\in\Omega} |g(x)| .$$
2. The weighted space $C_{\gamma,\rho}[a, b]$ of function g on $(a, b]$ is defined by
$$C_{\gamma,\rho}[a, b] = \left\{ g : (a, b] \longrightarrow \mathbb{R} : \left(\frac{x^\rho - a^\rho}{\rho}\right)^\gamma g(x) \in C[a, b] \right\}.$$
Where $0 \leq \gamma < 1$ and with the norm
$$\|g\|_{C_{\gamma,\rho}} = \left\| \left(\frac{x^\rho - a^\rho}{\rho}\right)^\gamma g(x) \right\|_C = max_{x\in\Omega} \left| \left(\frac{x^\rho - a^\rho}{\rho}\right)^\gamma g(x) \right|.$$
Where $C_{0,\rho}[a, b] = C[a, b]$.

1.2.1 Riemann-Liouville Fractional Integral

Definition 1.2.2 ([1]). Let $f : [a, b] \to \mathbb{R}$ a function defined on $[a, b]$ such that $f \in L^1([a, b])$. The fractional integral of Riemann-Liouville to the left of order $\alpha > 0$ of f is defined by :

$$(\mathcal{I}^\alpha_{a^+} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} f(s)ds, \quad \forall t \in [a, b].$$

In the same way we define the fractional integral of Riemann-Liouville to the right of order $\alpha > 0$ by :

$$(\mathcal{I}^\alpha_{b^-} f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (t - s)^{\alpha-1} f(s)ds, \quad \forall t \in [a, b] .$$

Function defined on $\mathbb{R}^+$ et $\mathbb{R}$.

$$\forall t \in \mathbb{R}^+; \;\; (\mathcal{I}^{\alpha}_{0^+} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

$$\forall t \in \mathbb{R}; \;\; (\mathcal{I}^{\alpha}_{+} f)(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t (t-s)^{\alpha-1} f(s) ds.$$

proposition 1.2.1. *For* $\alpha > 0$, $\beta > 0$, *we have*

1. $\left(\mathcal{I}^{\alpha}_{a^+} (t-a)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (t-a)^{\alpha+\beta-1}.$
2. $\left(\mathcal{I}^{\alpha}_{b^-} (b-t)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (b-t)^{\alpha+\beta-1}.$

Theorem 1.2.1 ([1])**.** *If* $f \in L^1([a,b])$, *then* $\mathcal{I}^{\alpha}_{a^+} f$ *exist for all* $\alpha > 0$ *and* $\mathcal{I}^{\alpha}_{a^+} \in L^1([a,b])$.

proposition 1.2.2 ([1])**.** *Let* $\alpha > 0$, $\beta > 0$, *and* $f \in L^1([a,b])$. *Then*

$$\mathcal{I}^{\alpha}_{a^+} \mathcal{I}^{\beta}_{a^+} f(t) = \mathcal{I}^{\beta}_{a^+} \mathcal{I}^{\alpha}_{a^+} f(t) = \mathcal{I}^{\alpha+\beta}_{a^+} f(t).$$

Proof we have

$$\begin{aligned}
\mathcal{I}^{\alpha}_{a^+} \mathcal{I}^{\beta}_{a^+} f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \left(\frac{1}{\Gamma(\beta)} \int_a^{\tau} (\tau-s)^{\beta-1} f(s) ds \right) d\tau, \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \int_a^{\tau} (t-\tau)^{\alpha-1} (\tau-s)^{\beta-1} f(s) ds d\tau, \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t f(s) \int_s^t (t-\tau)^{\alpha-1} (\tau-s)^{\beta-1} d\tau ds.
\end{aligned}$$

Setting the variable change $u = \frac{\tau-s}{t-s}$, so

$$\begin{aligned}
\mathcal{I}^{\alpha}_{a^+} \mathcal{I}^{\beta}_{a^+} f(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t f(s)(t-s)^{\alpha+\beta-1} \left(\int_s^t (1-u)^{\alpha-1} u^{\beta-1} du \right) ds, \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t f(s)(t-s)^{\alpha+\beta-1} B(\alpha,\beta) du ds, \left| B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \right|, \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \int_a^t f(s)(t-s)^{\alpha+\beta-1} ds, \\
&= \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-s)^{\alpha+\beta-1} f(s) ds, \\
&= \mathcal{I}^{\alpha+\beta}_{a^+} f(t).
\end{aligned}$$

1.2.2 Katugampola Fractional Integral

Definition 1.2.3 ([7])**.** Let $f : [a,b] \to \mathbb{R}$ a function defined on $[a,b]$ such that $f \in L^1([a,b])$. The fractional integral of Katugampola to the left of order $\alpha > 0$ of f is defined by :

$$({}^{\rho}\mathcal{I}^{\alpha}_{a^+} f)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1} (t^{\rho} - s^{\rho})^{\alpha-1} f(s) ds, \;\; \forall t \in [a,b], \;\; \rho > 0.$$

In the same way we define the fractional integral of Katugampola to the right of order $\alpha > 0$ by :

$$({}^{\rho}\mathcal{I}_{b^-}^{\alpha} f)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b s^{\rho-1}(s^{\rho} - t^{\rho})^{\alpha-1} f(s)ds, \ \ \forall t \in [a,b], \ \ \rho > 0.$$

proposition 1.2.3 **([7]).** *Let* $\alpha > 0, \ \ \beta > 0, \ \ \rho > 0 \ \ and \ \ f \in L^1([a,b])$. *Then*

$${}^{\rho}\mathcal{I}_{a^+}^{\alpha} {}^{\rho}\mathcal{I}_{a^+}^{\beta} f(t) = {}^{\rho}\mathcal{I}_{a^+}^{\beta} {}^{\rho}\mathcal{I}_{a^+}^{\alpha} f(t) = {}^{\rho}\mathcal{I}_{a^+}^{\alpha+\beta} f(t).$$

proposition 1.2.4 **([7]).** *For* $\alpha \geq 0 \ \ , \ \beta > 0,$ *we have*

1. $\left[{}^{\rho}\mathcal{I}_{a^+}^{\alpha} \left(\frac{t^{\rho}-a^{\rho}}{\rho} \right)^{\beta-1} \right](t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \left(\frac{t^{\rho}-a^{\rho}}{\rho} \right)^{\alpha+\beta-1}.$

2. $\left[{}^{\rho}\mathcal{I}_{b^-}^{\alpha} \left(\frac{b^{\rho}-t^{\rho}}{\rho} \right)^{\beta-1} \right](t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \left(\frac{b^{\rho}-t^{\rho}}{\rho} \right)^{\alpha+\beta-1}.$

Lemma 1.2.1 **([7]).** *For* $\alpha > 0 \ \ {}^{\rho}\mathcal{I}_{a^+}^{\alpha}$ *maps* $C[a,b]$ *into* $C[a,b]$.

Lemma 1.2.2 **([7]).** *Let* $\alpha > 0 \ \ and \ \ 0 \leq \gamma < 1$. *Then* ${}^{\rho}\mathcal{I}_{a^+}^{\alpha}$ *is bounded from* $C_{\gamma,\rho}[a,b]$ *into* $C_{\gamma,\rho}[a,b]$.

Lemma 1.2.3 **([7]).** *Let* $\alpha > 0 \ \ and \ \ 0 \leq \gamma < 1$. *If* $\gamma \leq \alpha$, *Then* ${}^{\rho}\mathcal{I}_{a^+}^{\alpha}$ *is bounded from* $C_{\gamma,\rho}[a,b]$ *into* $C[a,b]$.

1.3 Mixed Fractional integral

1.3.1 Mixed Riemann-Liouville Integral

Definition 1.3.1. [[10]] Let $r_1, \ \ r_2 > 0$ and $r = (r_1, r_2)$ for $f \in L^1(J, \mathbb{R})$
The left-sided mixed Riemann-Liouville integral of order r of f is defined by :

$$(\mathcal{I}_0^r f)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t)dsdt.$$

1.3.2 Mixed Katugampola Integral

Definition 1.3.2. let $r_1,\ \ r_2 > 0$, $\rho > 0$ and $r = (r_1, r_2)$ for $f \in L^1(J, \mathbb{R})$
The left-sided mixed Katugampola integral of order r of f is defined by :

$$({}^{\rho}\mathcal{I}_0^r f)(x,y) = \frac{\rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y s^{\rho-1} t^{\rho-1} (x^{\rho} - s^{\rho})^{r_1-1} (y^{\rho} - t^{\rho})^{r_2-1} f(s,t) ds dt.$$

1.4 Fractional derivatives

There are several definitions of fractional derivatives, we present in this section the definitions of the fractional derivatives of Rieman-Liouville , Caputo as well as Katugampola which ara the most used

1.4.1 Riemann-Liouville Fractional derivative

Definition 1.4.1 ([1]). Let $\alpha > 0$, and $n = [\alpha]+1$. The fractional derivative of Riemann-Liouville to the left of order α is defined by :

$$(\mathcal{D}_{a^+}^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s) ds, \ \ \forall t \in [a,b].$$

In the same way we define the fractional derivative of Riemann-Liouville to the right of order $\alpha > 0$ by :

$$(\mathcal{D}_{b^-}^{\alpha} f)(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (s-t)^{n-\alpha-1} f(s) ds, \ \ \forall t \in [a,b] \, .$$

proposition 1.4.1 ([1]). *For $\alpha \geq 0$, $\beta > 0$, we have*

1. $\left(\mathcal{D}_{a^+}^{\alpha} (t-a)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1}$.
2. $\left(\mathcal{D}_{b^-}^{\alpha} (b-t)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (b-t)^{\beta-\alpha-1}$.

1.4.2 Caputo Fractional Derivative

Definition 1.4.2 ([1]). Let $\alpha > 0$, and $n = [\alpha] + 1$. The fractional derivative of Caputo to the left of order α is defined by :

$$({}^{c}\mathcal{D}_{a^+}^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \ \ \forall t \in [a,b].$$

In the same way we define the fractional derivative of Caputo to the right of order $\alpha > 0$ by :

$$({}^{c}\mathcal{D}_{b^-}^{\alpha} f)(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} f^{(n)}(s) ds, \ \ \forall t \in [a,b] \, .$$

proposition 1.4.2. *For $\alpha \geq 0$, $\beta > 0$, we have*

1. $\left({}^c\mathcal{D}^{\alpha}_{a^+}(t-a)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}$, $\beta > n$
2. $\left({}^c\mathcal{D}^{\alpha}_{b^-}(b-t)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-t)^{\beta-\alpha-1}$, $\beta > n$

1.4.3 Katugampola Fractional Derivative

Definition 1.4.3 ([7]). Let $\alpha > 0$, $\rho > 0$, and $n = [\alpha] + 1$. The fractional derivative of katugampola to the left of order α is defined by :

$$({}^\rho\mathcal{D}^{\alpha}_{a^+}f)(t) = \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t s^{\rho-1}(t^\rho - s^\rho)^{n-1-\alpha}f(s)ds, \ \ \forall t \in [a,b].$$

In the same way we define the fractional derivative of katugampola to the right of order $\alpha > 0$ by :

$$({}^\rho\mathcal{D}^{\alpha}_{b^-}f)(t) = \frac{(-1)^n\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n \int_t^b s^{\rho-1}(s^\rho - t^\rho)^{n-\alpha-1}f(s)ds, \ \ \forall t \in [a,b]\,.$$

proposition 1.4.3 ([7]). *For $\alpha \geq 0$, $\beta > 0$, $\rho > 0$, we have,*

1. $\left[{}^\rho\mathcal{D}^{\alpha}_{a^+}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1}\right](t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-\beta-1}.$

2. $\left[{}^\rho\mathcal{D}^{\alpha}_{b^-}\left(\frac{b^\rho - t^\rho}{\rho}\right)^{\beta-1}\right](t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(\frac{b^\rho - t^\rho}{\rho}\right)^{\alpha-\beta-1}.$

Theorem 1.4.1. *(See [20]). Let $\alpha \in \mathbb{C}, Re\,(\alpha) \geq 0, n = [Re\,(\alpha)]$ and $\rho > 0$. Then, for $t > a$,*

$$1. \lim_{\rho\to 1}\left({}^\rho\mathcal{D}^{\alpha}_{a^+}f\right)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{f(s)\,ds}{(t-s)^{\alpha-n+1}} \tag{1.1}$$

$$2. \lim_{\rho\to 0^+}\left({}^\rho\mathcal{D}^{\alpha}_{a^+}f\right)(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_a^t \left(\ln\frac{t}{s}\right)^{n-\alpha-1} f(s)\,\frac{d}{ds} \tag{1.2}$$

Theorem 1.4.2 ([7]). *Let $\alpha > 0$, $\beta > 0, \rho > 0$, such that : $0 < \alpha$, $\beta < 1$ and $f \in L^1([a,b])$. Then*

$${}^\rho\mathcal{D}^{\alpha}_{a^+}{}^\rho\mathcal{D}^{\beta}_{a^+}f(t) = {}^\rho\mathcal{D}^{\beta}_{a^+}{}^\rho\mathcal{D}^{\alpha}_{a^+}f(t) = {}^\rho\mathcal{D}^{\alpha+\beta}_{a^+}f(t).$$

1.4.4 Caputo-Katugampola Fractional Derivative

Definition 1.4.4 ([7]). Let $\alpha > 0,\ \rho > 0$ and $n = [\alpha] + 1$. The fractional derivative of Caputo-Katugampola to the left of order α is defined by :

$$({}^{c}\mathcal{D}^{\alpha,\rho}_{a^+} f)(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_a^t s^{(\rho-1)(1-n)} (t^\rho - s^\rho)^{n-\alpha-1} f^{(n)}(s) ds, \ \ \forall t \in [a, b].$$

In the same way we define the fractional derivative of Caputo-Katugampola to the right of order $\alpha > 0$ by :

$$({}^{c}\mathcal{D}^{\alpha,\rho}_{b^-} f)(t) = \frac{(-1)^n \rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_t^b s^{(\rho-1)(1-n)} (s^\rho - t^\rho)^{n-\alpha-1} f^{(n)}(s) ds, \ \ \forall t \in [a, b] \,.$$

proposition 1.4.4 ([7]). *For $\alpha \geq 0,\ \beta > 0,\ \rho > 0,$ we have,*

1. $\left[{}^{c}\mathcal{D}^{\alpha,\rho}_{a^+} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1}\right](t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-\beta-1}.$

2. $\left[{}^{c}\mathcal{D}^{\alpha,\rho}_{b^-} \left(\frac{b^\rho - t^\rho}{\rho}\right)^{\beta-1}\right](t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \left(\frac{b^\rho - t^\rho}{\rho}\right)^{\alpha-\beta-1}.$

Theorem 1.4.3 ([7]). *Let $x \in C^1([a, b])$. Then , the Caputo-Katugampola fractional derivatives ${}^{c}\mathcal{D}^{\alpha,\rho}_{a^+}$ and${}^{c}\mathcal{D}^{\alpha,\rho}_{b^-}$ exist and are continuous on the interval $[a, b]$. Moreover,*
${}^{c}\mathcal{D}^{\alpha,\rho}_{a^+} = 0$ at $t = a$, and ${}^{c}\mathcal{D}^{\alpha,\rho}_{b^-} = 0$ at $t = b$.

1.5 Mixed Fractional derivatives

1.5.1 Mixed Caputo fractional derivatives

Definition 1.5.1. [[10]] let $r_1,\ \ r_2 > 0$ and $r = (r_1, r_2)$ for $f \in L^1(J, \mathbb{R})$
The Caputo fractional order derivative of order r of f is defined by :

$$\begin{aligned}({}^{c}\mathcal{D}^{r}_{0} f)(x, y) &= \left(\mathcal{I}^{1-r}_{0} \frac{\partial^2}{\partial x \partial y} f\right)(x, y),\\ &= \frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)} \int_0^x \int_0^y (x-s)^{-r_1} (y-t)^{-r_2} \frac{\partial^2 f}{\partial s \partial t} ds dt.\end{aligned}$$

1.5.2 Mixed Caputo-Katugampola fractional derivatives

Definition 1.5.2. let $r_1,\ \ r_2 > 0$ and $r = (r_1, r_2)$ for $f \in L^1(J, \mathbb{R})$
The Caputo-Katugampola fractional order derivative of order r of f is defined by :

$$({}^{c}\mathcal{D}^{r,\rho}_{0} f)(x, y) = {}^{\rho}\mathcal{I}^{1-r}_{0} \left(t^{1-\rho} s^{1-\rho} \frac{\partial^2}{\partial x \partial y} f\right)(x, y).$$

1.6 Some fixed point theorem

In this section we have based on the different tools and results of the functional analysis theory used thereafter : Banach fixed point theorem , equicontinuity, Burton-Kirk theorem , Arzela-Ascoli theorem ,...

Definition 1.6.1. we call Banach space $(E, \|.\|_E)$ any normed and complete vector space for the distance deduced from the norm.
Let $J = [0, a] \times [0, b], \ \ a, \ \ b > 0 \ \ and \ \ (E, \|.\|_E)$ a Banach space .
We denote by $C(J, \mathbb{R})$ the Banach space of the continuous functions from J in E with the norm :

$$\|u\| = sup_{(x,y)\in J} \|u(x, y)\|_E \, .$$

Definition 1.6.2. A continuous function $f : J \times E \longrightarrow E$ defined on an open $J \times U$ of $\mathbb{R} \times E$ is called lipschitizienne if :
$\forall u_1, \ \ u_2 \in U, \ \ and \ \ x, \ \ y > 0 \ \ \exists \ \ c > 0,$ such that :

$$\|f(x, y, u_1) - f(x, y, u_2)\|_E \leq c \|u_1 - u_2\|_E \, .$$

Also if $0 < c \leq 1$, f is said to be contracting.

Exemple 1.6.1. $f(x, y, u) = xyu(x, y), \ \ x, \ \ y \in [0, a]; \ \ a > 0$ is of lipschitz type .
Indeed , $\forall \ \ u_1, u_2 \in U, \ \ and \ \ x, \ \ y > 0,$

$$|f(x, y, u_1) - f(x, y, u_2)| \leq a^2 |u_1 - u_2| \, .$$

Corollary 1.6.1. *Let f be a function of class C^1 on a bounded open $J \times U$ of $\mathbb{R} \times E$*
(The function f and its partial derivative $\frac{\partial f}{\partial u}$ with respect to u are continuous in a bounded$\Longrightarrow$bounded)
then, f is lipschitizienne .

Definition 1.6.3. Let be E and F two Banach spaces and $f : E \longrightarrow F$ a function.

— f is said to be compact if the image $f(E)$ is relatively compact in F.

— f is said to be completely continuous if it is continuous and the image of all bounded of E is relatively compact in F .

Theorem 1.6.1. ***(Banach fixed point theorem** .See [[14]])*
Let (F, d) a complete metric space and $f : F \longrightarrow F$ a strictly contractual map i.e., there is $0 < c < 1$ such that , for all $u, \ v \in F$

$$d(f(u), f(v)) < cd(u, v).$$

Then there is a unique fixed point u_0 i,e :

$$f(u_0) = u_0.$$

Theorem 1.6.2. ***(Schaefer's fixed point theorem.** See [13])*
Let E be a Banach space, and $F : E \to E$ a completely continuous operator. If the set

$$S = \{u \in E : u = \mu Fu, \text{for some } \mu \in (0, 1)\},$$

is bounded, the F has fixed points.

Theorem 1.6.3. ***(Nonlinear alternative Leray-Schouder type** [3])*
Let X a Banach space , U an open of a convex part D of X and $0 \in U$. Assume that,
$N : \bar{U} \longrightarrow D$ is a continuous and compact operator .
Then, only one of the following proposition is satisfied :
***(S1)** N has a fixed point in $\bar{U}$,*
***(S2)** there exists $v \in (0, 1)$ and $u \in \partial U$ with $u = vN(u)$.*

Theorem 1.6.4. ***(Burton-Kirk theorem**[2]) ()*
Let X be a Banach space and $F, G : X \longrightarrow X$ two operators satisfying :

1. *G is a contraction and,*
2. *F is completely continuous.*

Then either
***(S1)** the operator equation $u = F(u) + G(u)$ has a solution, or*
***(S2)** the set $\varepsilon = (u \in X : u = \lambda F(u) + \lambda G(\frac{u}{\lambda}))$ is unbounded for $\lambda \in (0, 1)$.*

Lemma 1.6.1. ***(Ascoli-ArzelÃ lemma** [8])*
Let $A \subset C(J, \mathbb{R})$ .A is relatively compact (i,e $\bar{A}$ is compact)if and only if :

1. *A is uniformly bounded .*
2. *A is equicontinuous .*

In the sequel we use the following version of Gronwall's lemma for two independent varaibles and singular kernel .

Lemma 1.6.2. *(**Gronwall's lemma** [17])*

Let $v : J \longrightarrow [0, \infty)$ *be a real function and w(.,.) be a non negative , locally integral funtion on J. If there are constants* $c > 0$ *and* $0 < r_1, \ r_2 < 1$ *such that*

$$v(x,y) \leq w(x,y) + c \int_0^x \int_0^y \frac{v(s,t)}{(x-s)^{r_1}(y-t)^{r_2}} dsdt,$$

Then there exists a constant $k = k(r_1, r_2)$ *such that ,*

$$v(x,y) \leq w(x,y) + ck \int_0^x \int_0^y \frac{w(s,t)}{(x-s)^{r_1}(y-t)^{r_2}} dsdt,$$

for every $(x, y) \in J$.

CHAPITRE 2

DARBOUX PROBLEM FOR PERTURBED OF CAPUTO FDE

In this chapter we study the following perturbed problem :

$$({}^c\mathcal{D}_0^r u)(x,y) = f(x,y,u_{(x,y)}) + g(x,y,u_{(x,y)}), \quad (x,y) \in J \tag{2.1}$$

$$u(x,y) = \phi(x,y), \quad (x,y) \in \tilde{J} \tag{2.2}$$

$$u(x,0) = \varphi(x), \quad u(0,y) = \psi(y). \quad (x,y) \in J \tag{2.3}$$

Where $J = [0,a] \times [0,b]$, $\tilde{J} = [-\alpha, a] \times [-\beta, b] \backslash [0,a] \times [0,b]$, $a, \ b, \ \alpha, \ \beta > 0$
$f, g : J \times C([-\alpha, 0] \times [-\beta, 0], \mathbb{R}) \longrightarrow \mathbb{R}$, $\phi \in C([-\alpha, 0] \times [-\beta, 0], \mathbb{R})$, $\varphi \in [0,a] \longrightarrow \mathbb{R}$,
$\psi \in [0,b] \longrightarrow \mathbb{R}$ are given function
We denote by $u_{(x,y)}$ the element of $C([-\alpha, 0] \times [-\beta, 0], \mathbb{R})$ defined by :
$\forall (s,t) \in [-\alpha, 0] \times [-\beta, 0]$

$$u_{(x,y)}(s,t) = u(x+s, y+t)$$

Next we consider the following nonlocal initial value problem :

$$({}^c\mathcal{D}_0^r u)(x,y) = f(x,y,u_{(x,y)}) + g(x,y,u_{(x,y)}), \quad (x,y) \in J \tag{2.4}$$

$$u(x,y) = \phi(x,y), \quad (x,y) \in \tilde{J} \tag{2.5}$$

$$u(x,0) + A(u) = \varphi(x), \quad u(0,y) + B(u) = \psi(y). \quad (x,y) \in J \tag{2.6}$$

where f , g , ϕ, ψ are as in the problem (2.1)-(2.3) and $A, \ B : C(J, \mathbb{R}) \longrightarrow \mathbb{R}$ are continuous functions.

2.1 Fundamental lemma

Lemma 2.1.1. *[17] Let $f \in L^1(J,\mathbb{R})$. A function $u \in C(J,\mathbb{R})$ is a solution of problem :*

$$({}^c\mathcal{D}_0^r u)(x,y) = f(x,y), \quad (x,y) \in J \tag{2.7}$$

$$u(x,0) = \varphi(x), \quad u(0,y) = \psi(y). \quad (x,y) \in J \tag{2.8}$$

With

$$\varphi(0) = \psi(0),$$

if and only if $u(x,y)$ satisfies

$$u(x,y) = \mu(x,y) + I_0^r(f)(x,y). \quad (x,y) \in J \tag{2.9}$$

Where

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0).$$

Proof : (see[17])

Let $u(x,y)$ be a solution of problem (2.7)-(2.8). Then, we have

$$I_0^{1-r}(D_{xy}^2 u)(x,y) = f(x,y),$$

hence, we obtain

$$I_0^r(I_0^{1-r}(D_{xy}^2 u)(x,y)) = (I_0^r f)(x,y),$$

Then

$$I_0^1 D_{xy}^2 u(x,y) = (I_0^r f)(x,y),$$

since

$$I_0^1 D_{xy}^2 u(x,y) = u(x,y) - u(x,0) - u(0,y) + u(0,0),$$

we have

$$u(x,y) = \mu(x,y) + (I_0^r f)(x,y).$$

Now let u(x,y) satisfy (2.9). It is clear that u(x,y) satisfies (2.7)-(2.8).

Corollary 2.1.1. *Let* $f : J \times C\left([-\alpha, 0] \times [-\beta, 0], \mathbb{R}\right) \to \mathbb{R}, \left(x, y, u_{(x,y)}\right) \mapsto f\left(x, y, u_{(x,y)}\right)$.
The function $u \in C\left([-\alpha, a] \times [-\beta, b], R\right)$ *is a solution of problem (2.1)-(2.3) if and only if* u *satisfies the equation*

$$\begin{aligned} u\left(x, y\right) = &\frac{1}{\Gamma\left(r_1\right)\Gamma\left(r_2\right)} \int_0^x \int_0^y \left(x-s\right)^{r_1-1} (y-t)^{r_2-1} f\left(s, t, u_{(s,t)}\right) dsdt \\ &+ \varphi\left(x\right) + \psi\left(y\right) - \varphi\left(0\right), \end{aligned}$$

for almost all $(x, y) \in J$ *and the condition* (2) *on* $\tilde{J}$

2.2 Existence of solutions

Definition 2.2.1. A function $u \in C([-\alpha, a] \times [-\beta, b], \mathbb{R})$ is said to be a solution of (2.1)-(2.3) if u satisfies Eqs.(2.1) and (2.3) on J and the condition (2.2) on $\tilde{J}$.

Now, we present the conditions for the existence of solutions to the problem (2.1)-(2.3).

Theorem 2.2.1. *[17] Assume that the following hypotheses are verified :*
(H$_1$) $f, \ \ g : J \times C([-\alpha, 0] \times [-\beta, 0], \mathbb{R})$ *are jointly continuous.*
(H$_2$) *There existe* $k > 0$ *such that for* $(x, y) \in J$

$$|g(x, y, u) - g(x, y, v)| \leq k \left\|u - v\right\|_C,$$

for any $u, \ \ v \in C([-\alpha, 0] \times [-\beta, 0], \mathbb{R})$.
(H$_3$) *There existe* $p, \ \ q \in C(J, \mathbb{R}_+)$ *such that :*

$$|f(x, y, u)| \leq p(x, y) + q(x, y) \left\|u\right\|_C,$$

for $(x, y) \in J$ *and* $u \in C([-\alpha, a] \times [-\beta, b], \mathbb{R})$.

if

$$\frac{k a^{r_1} b^{r_2}}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} < 1. \tag{2.10}$$

Then the IVP (2.1)-(2.3) has at least one solution on$[-\alpha, a] \times [-\beta, b]$

Proof : consider the operators $F, \quad G : C([-\alpha, a] \times [-\beta, b], \mathbb{R}) \longrightarrow C([-\alpha, a] \times [-\beta, b], \mathbb{R})$ defined by :

$$F(u)(x,y) = \begin{cases} \phi(x,y) & (x,y) \in \tilde{J} \\ \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u_{(s,t)}) dsdt & (x,y) \in J, \end{cases}$$

and

$$G(u)(x,y) = \begin{cases} 0 & (x,y) \in \tilde{J} \\ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} g(s,t,u_{(s,t)}) dsdt & (x,y) \in J. \end{cases}$$

Then the problem of finding the solution of the IVP (2.1)-(2.3) is reduced to finding the solution of the operator equation

$$F(u)(s,t) + G(u)(s,t) = u(s,t), \quad (s,t) \in J$$

we will shall show that the operators F and G satisfy all the condition of theorem(3.5).
The proof will be given in several steps.
Step 1 : F is continuous
Let (u_n) be a sequence such that $u_n \longrightarrow u$ in $C([-\alpha, a] \times [-\beta, b], \mathbb{R})$.Let $\eta > 0$ such that $\|u_n\|_\infty \leq \eta$.
Then

$$\begin{aligned} |F(u_n)(x,y) - F(u)(x,y)| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \left| f(s,t,u_{n(s,t)}) - f(s,t,u_{(s,t)}) \right| dsdt \\ &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} sup_{(s,t)\in J} \left| f(s,t,u_{n(s,t)}) \right. \\ &\left. - f(s,t,u_{(s,t)}) \right| dsdt \\ &\leq \frac{\left\| f(.,.,u_{n(.,.)} - f(.,.,u_{(.,.)}) \right\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dsdt \\ &\leq \frac{a^{r_1} b^{r_2}}{r_1 r_2 \Gamma(r_1)\Gamma(r_2)} \left\| f(.,.,u_{n(.,.)} - f(.,.,u_{(.,.)}) \right\|_\infty . \end{aligned}$$

Since f is a continuous function, we have

$$\|F(u_n) - F(u)\|_\infty \leq \frac{a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \left\| f(.,.,u_{n(.,.)} - f(.,.,u_{(.,.)}) \right\|_\infty \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Step 2 : F maps bounded sets into bounded sets in C.
Indeed , for any $\eta_* > 0$,there existe $l > 0$ such that, for each $u \in B_{\eta_*} = [u \in C([-\alpha, a] \times [-\beta, b], \mathbb{R}) : \|u\|_\infty \leq \eta_*]$, we have $\|F(u)\|_\infty \leq l$.
By (H_3) we have for each $(x, y) \in J$:

$$\begin{aligned}
|F(u)(x,y)| &\leq |\mu(x,y)| \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \left| f(s,t,u_{(s,t)}) \right| dsdt \\
&\leq |\mu(x,y)| \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dsdt.
\end{aligned}$$

Thus

$$\|F(u)\|_\infty \leq \|\mu\|_\infty + \frac{\|p\|_\infty + \|q\|_\infty \eta^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} = l.$$

Step 3 : F maps bounded sets into equicontinuous sets in C
Let $(x_1, y_1), \ (x_2, y_2) \in (0, a] \times (0, b], \ x_1 < x_2, \ y_1 < y_2$, B_{η^*} be a bounded set of $C([-\alpha, a] \times [-\beta, b], \mathbb{R})$as in Step 2 , and let $u \in B_{\eta^*}$. Then

$$\begin{aligned}
|F(u)(x_2,y_2) - F(u)(x_1,y_1)| &\leq |\mu(x_2,y_2) - \mu(x_1,y_1)| \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1} \int_0^{y_1} \big[(x_2-s)^{r_1-1}(y_2-t)^{r_2-1} \\
&-(x_1-s)^{r_1-1}(y_1-t)^{r_2-1} \big] \left| f(s,t,u_{(s,t)}) \right| dsdt \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} [(x_2-s)^{r_1-1}(y_2-t)^{r_2-1} \left| f(s,t,u_{(s,t)}) \right| dsdt \\
&\leq |\mu(x_2,y_2) - \mu(x_1,y_1)| \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1} \int_0^{y_1} \big[(x_1-s)^{r_1-1}(y_1-t)^{r_2-1} \\
&-(x_2-s)^{r_1-1}(y_2-t)^{r_2-1} \big] dsdt \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1} dsdt \\
&\leq |\mu(x_2,y_2) - \mu(x_1,y_1)| + \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \big[(x_2-x_1)^{r_1}(y_2-y_1)^{r_2} \\
&+x_1^{r_1}y_1^{r_2} - x_2^{r_1}y_2^{r_2}] + \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} (x_2-x_1)^{r_1}(y_2-y_1)^{r_2} \\
&\leq |\mu(x_2,y_2) - \mu(x_1,y_1)| \\
&+ \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} [2(x_2-x_1)^{r_1}(y_2-y_1)^{r_2} + x_1^{r_1}y_1^{r_2} - x_2^{r_1}y_2^{r_2}].
\end{aligned}$$

As $x_1 \longrightarrow x_2, y_1 \longrightarrow y_2$ the right-hand side of the above inequality tends to zero.As a consequence of steps 1 to 3 together with the Arzela-Ascoli theorem .

We can conclude that $F : C([-\alpha, a] \times [-\beta, b], \mathbb{R}) \longrightarrow C([-\alpha, a] \times [-\beta, b], \mathbb{R})$ is continuous and completely continuous.

Step 4 : G is a contraction.

Let $u, \ \ v \in C([-\alpha, a] \times [-\beta, b], \mathbb{R})$. Then

for each $(x, y) \in J$ we have :

$$\begin{aligned}
|G(u)(x,y) - G(v)(x,y)| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \left| g(s,t,u_{(s,t)}) - g(s,t,v_{(s,t)}) \right| dsdt \\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} k \left\| u - v \right\|_\infty dsdt \\
&\leq \frac{k \left\| u - v \right\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dsdt \\
&\leq \frac{k a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \left\| u - v \right\|_\infty .
\end{aligned}$$

Which by (2.10) , implies that G is contraction.

Step 5 :A priori bounds.

Now it remains to show that the set

$$\varepsilon = \left\{ u \in C(J, \mathbb{R}) : u = \lambda F(u) + \lambda G(\frac{u}{\lambda}), \forall \ 0 < \lambda < 1 \right\}.$$

Is bounded.

Let $u \in \varepsilon$, then

$$u = \lambda F(u) + \lambda G(\frac{u}{\lambda}).$$

$\forall \ 0 < \lambda < 1$. Thus for each $(x, y) \in J$ we have :

$u(x,y) = \lambda \mu(x,y) + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u_{(s,t)}) dsdt$

$$+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} g(s,t,\frac{u_{(s,t)}}{\lambda}) dsdt.$$

This implies by **(H_2)** and **(H_3)** that , for each $(x,y) \in J$

$$\begin{aligned}
|u(x,y)| &\leq |\mu(x,y)| + \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \left\|u_{(s,t)}\right\|_C dsdt \\
&+ \frac{\|q\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dsdt \\
&+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \left|g(s,t,\frac{u_{(s,t)}}{\lambda}) - g(s,t,0)\right| dsdt \\
&+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} |g(s,t,0)| dsdt \\
&\leq |\mu(x,y)| + \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \left\|u_{(s,t)}\right\|_C dsdt \\
&+ \frac{\|q\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dsdt \\
&+ \frac{k}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \left\|u_{(s,t)}\right\|_C dsdt + \frac{g^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}.
\end{aligned}$$

Where $g^* = sup_{(s,t)\in J} |g(s,t,0)|$.

We consider the function τ defined by

$$\tau(x,y) = sup \{|u(s,t)| : 0 \leq s \leq x, 0 \leq t \leq y\}, 0 \leq x \leq a, 0 \leq y \leq b$$

Let $(x^*, y^*) \in [-\alpha, x] \times [-\beta, y]$ such that

$$\tau(x,y) = |u(x^*, y^*)|, (x^*, y^*) \in J$$

then by the previous inequality, we have for $(x,y) \in J$

$$\tau(x,y) \leq |\mu(x,y)| + \frac{a^{r_1} b^{r_2}(\|q\|_\infty + g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)} + \frac{(\|p\|_\infty + k)}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \tau(s,t) dsdt.$$

If $(x^*, y^*) \in [-\alpha, a] \times [-\beta, b]$, then

$$\tau(x,y) = \|\phi\|_C .$$

And the previous inequality holds.

Then from lemma (1.6.2) there exists $k^* = k^*(r_1, r_2)$ such that we have

$$\begin{aligned}
\tau(x,y) &\leq |\mu(x,y)| + \frac{a^{r_1} b^{r_2}(\|q\|_\infty + g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)} + \frac{(\|p\|_\infty + k)}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dsdt. \\
&\leq |\mu(x,y)| + \frac{a^{r_1} b^{r_2}(\|q\|_\infty + g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)} + \frac{k^*(\|p\|_\infty + k) a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\
&= R^*.
\end{aligned}$$

Since for every $(x, y) \in J$

$$\left\|u_{(x,y)}\right\|_C \leq \tau(x, y).$$

We have

$$\begin{aligned}\|u\|_\infty &\leq max(\|\phi\|_C, R^*) \\ &= A.\end{aligned}$$

This shows that the set ε is bounded . as a consequence of theorem (1.6.4) , we deduce that $F + G$ has a fixed point u which is a solution to problem (2.1)-(2.3) .
Now, we present (without proof)two existence results for the nonlocal problem (2.4)-(2.6) .

Definition 2.2.2. [17] A function $u \in C([-\alpha, a] \times [-\beta, b], \mathbb{R})$ is said to be a solution of (2.4)-(2.6) if u satisfies Eqs (2.4)and (2.6) on J and the condition (2.5) on $[-\alpha, 0] \times [-\beta, 0]$.

Theorem 2.2.2. *[17] Assume **(H_1)-(H_2)**and following conditions :*
***(H'_1)** there existe $\tilde{k} > 0$ such that*

$$|A(u) - A(v)| \leq \tilde{k} \|u - v\|_\infty \forall \; u, v \in C(J, \mathbb{R}).$$

***(H''_1)** there existe $k^* > 0$ such that*

$$|B(u) - B(v)| \leq k^* \|u - v\|_\infty \forall \; u, \; v \in C(J, \mathbb{R}).$$

Hold if

$$\tilde{k} + k^* + \frac{ka^{r_1}b^{r_2}}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} < 1.$$

Then there exists at least one solution for IVP(2.4)-(2.6) on $[-\alpha, a] \times [-\beta, b]$.

Theorem 2.2.3. *[17] Assume **(H_1)-(H_3)** and following conditions :*
***(H'_3)** there existe $\tilde{d} > 0$ such that*

$$|A(u)| \leq \tilde{d}(1 + \|u\|_\infty)\forall \;\; u \in C(J, \mathbb{R}).$$

***(H''_3)** there existe $d^* > 0$ such that*

$$|B(u)| \leq d^*(1 + \|u\|_\infty)\forall \; u \in C(J, \mathbb{R}).$$

Hold if (2.10) is satisfied, then there exists at least one solution for IVP(2.4)-(2.6) on $[-\alpha, a] \times [-\beta, b]$.

2.3 Example

We consider the following perturbed problem such that : $J = [0,1] \times [0,1]$, $\tilde{J} = [-1,1] \times [-2,1] \backslash [0,1] \times [0,1]$

$$({}^{c}\mathcal{D}_{0}^{r})(x,y) = \frac{1 + 3e^{x+y+2}(|u(x-1,y-2)| + 2)}{3e^{x+y+2}(|u(x-1,y-2)| + 2)} \quad if(x,y) \in J \tag{2.11}$$

$$u(x,y) = x + y \quad (x,y) \in \tilde{J} \tag{2.12}$$

$$u(x,0) = x, \quad u(0,y) = y^2, \quad (x,y) \in J \tag{2.13}$$

Set

$$f(x,y,u_{(x,y)}) = \frac{|u(x-1,y-2)| + 2}{1 + |u(x-1,y-2)|} \quad (x,y) \in J,$$

and

$$g(x,y,u_{(x,y)}) = \frac{1}{3e^{x+y+2}(1 + |u(x-1,y-2)|)} \quad (x,y) \in J.$$

For each $u, \ \bar{u} \in \mathbb{R}$ and $(x,y) \in J$ we have

$$\left| g(x,y,u_{(x,y)}) - g(x,y,\bar{u}_{(x,y)}) \right| \leq \frac{\|u - \bar{u}\|_C}{3e^2}.$$

Hence condition **(H_2)** is satisfied with $k = \frac{1}{3e^2}$.
we shall that condition (2.10) holds with $a = b = 1$. Indeed

$$\frac{1}{3e^2\Gamma(r_1+1)\Gamma(r_2+1)} < 1.$$

Which is satisfied for each $(r_1, r_2) \in (0,1] \times (0,1]$. Also the function f is continuous on $J \times [0,\infty)$ and
$|f(x,y,w)| \leq |w| + 2 \quad \forall (x,y) \in J \times [0,\infty)$
Thus conditions **(H_1)** and **(H_3)** hold.
The theorem (2.2.1) implies that the problem (2.11)-(2.13) has a solution on $[-1,1] \times [-2,1]$ (see [17]).

DARBOUX PROBLEM FOR PERTURBED OF CAPUTO-KATUGAMPOLA FDE

In this chapter we study the following perturbed problem :

$$({}^{c}\mathcal{D}_{0}^{r,\rho}u)(x,y) = f(x,y,u_{(x,y)}) + g(x,y,u_{(x,y)}), \quad (x,y) \in J \tag{3.1}$$

$$u(x,y) = \phi(x,y), \quad (x,y) \in \tilde{J} \tag{3.2}$$

$$u(x,0) = \varphi(x), \quad u(0,y) = \psi(y). \quad (x,y) \in J \tag{3.3}$$

Where $J = [0,a] \times [0,b]$, $\tilde{J} = [-\alpha,a] \times [-\beta,b] \backslash [0,a] \times [0,b]$, $a,\ \ b,\ \ \alpha,\ \ \beta,\ \ \rho > 0$
$({}^{c}\mathcal{D}_{0}^{r,\rho}u)$ is the fractional Caputo-Katugampola derivative of order $r = (r1, r2) \in (0,1] \times (0,1]$
$f, g : J \times C([-\alpha,0] \times [-\beta,0],\mathbb{R}) \longrightarrow \mathbb{R}$, $\phi \in C([-\alpha,0] \times [-\beta,0],\mathbb{R})$, $\varphi \in [0,a] \longrightarrow \mathbb{R}$,
$\psi \in [0,b] \longrightarrow \mathbb{R}$ are given function.
Next we consider the following nonlocal initial value problem :

$$(c^{\rho}\mathcal{D}_{0}^{r}u)(x,y) = f(x,y,u_{(x,y)}) + g(x,y,u_{(x,y)}), \quad (x,y) \in J \tag{3.4}$$

$$u(x,y) = \phi(x,y), \quad (x,y) \in \tilde{J} \tag{3.5}$$

$$u(x,0) + A(u) = \varphi(x), \quad u(0,y) + B(u) = \psi(y). \quad (x,y) \in J \tag{3.6}$$

Where f , g , ϕ , ψ are as in the problem (3.1)-(3.3) and $A,\ \ B : C(J,\mathbb{R}) \longrightarrow \mathbb{R}$ are continuous functions.

3.1 Fundamental lemma

Lemma 3.1.1. *Let* $f \in L^1(J,\mathbb{R})$ *. A function* $u \in C(J,\mathbb{R})$ *is a solution of problem :*

$$(c^{\rho}\mathcal{D}_0^r u)(x,y) = f(x,y), \quad (x,y) \in J \tag{3.7}$$

$$u(x,0) = \varphi(x), \quad u(0,y) = \psi(y). \quad (x,y) \in J \tag{3.8}$$

With

$$\varphi(0) = \psi(0),$$

if and only if u(x,y) satisfies

$$u(x,y) = \mu(x,y) + {}^{\rho}\mathcal{I}_0^r(f)(x,y). \quad (x,y) \in J \tag{3.9}$$

Where

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0).$$

Proof : Let $u(x,y)$ be a solution of problem (1.7)-(1.8).Then, we have :

$${}^{\rho}\mathcal{I}_0^{1-r}(t^{1-\rho}s^{1-\rho}D^2_{xy}u)(x,y) = f(x,y),$$

hence, we obtain

$${}^{\rho}\mathcal{I}_0^{r}({}^{\rho}\mathcal{I}_0^{1-r}(t^{1-\rho}s^{1-\rho}D^2_{xy}u)(x,y)) = ({}^{\rho}\mathcal{I}_0^r f)(x,y),$$

then

$${}^{\rho}\mathcal{I}_0^{1}(t^{1-\rho}s^{1-\rho}D^2_{xy}u)(x,y) = ({}^{\rho}\mathcal{I}_0^r f)(x,y),$$

since

$${}^{\rho}\mathcal{I}_0^{1}(t^{1-\rho}s^{1-\rho}D^2_{xy}u)(x,y) = u(x,y) - u(x,0) - u(0,y) + u(0,0),$$

we have

$$u(x,y) = \mu(x,y) + ({}^{\rho}\mathcal{I}_0^r f)(x,y).$$

Now let $u(x,y)$ satisfy (3.9). It is clear that $u(x,y)$ satisfies (3.7)-(3.8).

Corollary 3.1.1. *Let* $f : J \times C\left([-\alpha, 0] \times [-\beta, 0], \quad \mathbb{R}\right) \to \mathbb{R}, \left(x, y, u_{(x,y)}\right) \mapsto f\left(x, y, u_{(x,y)}\right)$.
The function $u \in C\left([-\alpha, a] \times [-\beta, b], R\right)$ *is a solution of problem (3.1)-(3.3) if and only if* u *satisfies the equation*

$$\begin{aligned} u(x,y) = & \frac{\rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y s^{\rho-1} t^{\rho-1} (x^\rho - s^\rho)^{r_1-1} (y^\rho - t^\rho)^{r_2-1} f(s,t)\, dsdt \\ & + \varphi(x) + \psi(y) - \varphi(0). \end{aligned}$$

For almost all $(x, y) \in J$ *and the condition* (2) *on* $\tilde{J}$.

3.2 Existence of solutions

Definition 3.2.1. A function $u \in C([-\alpha, a] \times [-\beta, b], \mathbb{R})$ is said to be a solution of (3.1)-(3.3) if u satisfies Eqs.(3.1) and (3.3) on J and the condition (3.2) on $\tilde{J}$.

Now, we present the conditions for the existence of solutions to the problem (3.1)-(3.3).

Theorem 3.2.1. *Assume that the following hypotheses are verified :*
(H₁) $f, g : J \times C([-\alpha, 0] \times [-\beta, 0], \mathbb{R})$ *are jointly continuous.*
(H₂) *There existe* $k > 0$ *such that for* $(x, y) \in J$

$$|g(x, y, u) - g(x, y, v)| \leq k \left\| u - v \right\|_C .$$

For any $u, \quad v \in C([-\alpha, 0] \times [-\beta, 0], \mathbb{R})$.
(H₃) *There existe* $p, q \in C(J, \mathbb{R}_+)$ *such that :*

$$|f(x, y, u)| \leq p(x, y) + q(x, y) \left\| u \right\|_C .$$

For $(x, y) \in J$ *and* $u \in C([-\alpha, a] \times [-\beta, b], \mathbb{R})$.

If

$$\frac{k a^{\rho r_1} b^{\rho r_2} \rho^{(1-r_1)+(1-r_2)}}{\rho^2 \Gamma(r_1 + 1) \Gamma(r_2 + 1)} < 1. \tag{3.10}$$

Then the IVP (3.1)-(3.3) has at least one solution on$[-\alpha, a] \times [-\beta, b]$.

Proof : consider the operators $F, \quad G : C([-\alpha, a] \times [-\beta, b], \mathbb{R}) \longrightarrow C([-\alpha, a] \times [-\beta, b], \mathbb{R})$ defined by :

$$F(u)(x,y) = \begin{cases} \phi(x,y) & (x,y) \in \tilde{J} \\ \mu(x,y) + \frac{\rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y s^{\rho-1} t^{\rho-1} (x^\rho - s^\rho)^{r_1-1} (y^\rho - t^\rho)^{r_2-1} f(s,t,u_{(s,t)}) dsdt & (x,y) \in J. \end{cases}$$

and

$$G(u)(x,y) = \begin{cases} 0 & (x,y) \in \tilde{J} \\ \frac{\rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y s^{\rho-1} t^{\rho-1} (x^\rho - s^\rho)^{r_1-1} (y^\rho - t^\rho)^{r_2-1} g(s,t,u_{(s,t)}) dsdt & (x,y) \in J. \end{cases}$$

Then the problem of finding the solution of the IVP (3.1)-(3.3) is reduced to finding the solution of the operator equation

$$F(u)(s,t) + G(u)(s,t) = u(s,t), \quad (s,t) \in J$$

we will shall show that the operators F and G satisfy all the condition of Burton-Kirk theorem . The proof will be given in several steps.

Step 1 : F is continuous

Let (u_n) be a sequence such that $u_n \longrightarrow u$ in $C([-\alpha, a] \times [-\beta, b], \mathbb{R})$.Let $\eta > 0$ such that $\|u_n\|_\infty \leq \eta$. Then

$$\begin{aligned} |F(u_n)(x,y) - F(u)(x,y)| &\leq \frac{\rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y s^{\rho-1} t^{\rho-1} (x^\rho - s^\rho)^{r_1-1} (y^\rho - t^\rho)^{r_2-1} \\ &\times \left| f(s,t,u_{n(s,t)}) - f(s,t,u_{(s,t)}) \right| dsdt \\ &\leq \frac{\rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)} \int_0^a \int_0^b s^{\rho-1} t^{\rho-1} (x^\rho - s^\rho)^{r_1-1} (y^\rho - t^\rho)^{r_2-1} \\ &\times sup_{(s,t)\in J} \left| f(s,t,u_{n(s,t)}) - f(s,t,u_{(s,t)}) \right| dsdt \\ &\leq \frac{\rho^{(1-r_1)+(1-r_2)} \left\| f(.,.,u_{n(.,.)} - f(.,.,u_{(.,.)} \right\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \\ &\times \int_0^a \int_0^b s^{\rho-1} t^{\rho-1} (x^\rho - s^\rho)^{r_1-1} (y^\rho - t^\rho)^{r_2-1} dsdt \\ &\leq \frac{a^{\rho r_1} b^{\rho r_2} \rho^{(1-r_1)+(1-r_2)}}{\rho^2 r_1 r_2 \Gamma(r_1)\Gamma(r_2)} \left\| f(.,.,u_{n(.,.)} - f(.,.,u_{(.,.)} \right\|_\infty . \end{aligned}$$

Since f is a continuous function, we have

$$\|F(u_n) - F(u)\|_\infty \leq \frac{a^{\rho r_1} b^{\rho r_2} \rho^{(1-r_1)+(1-r_2)}}{\rho^2 \Gamma(r_1+1)\Gamma(r_2+1)} \left\| f(.,.,u_{n(.,.)} - f(.,.,u_{(.,.)} \right\|_\infty \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Step 2 : F maps bounded sets into bounded sets in C.
Indeed , for any $\eta_* > 0$, there existe $l > 0$ such that, for each $u \in B_{\eta_*} = [u \in C([-\alpha, a] \times [-\beta, b], \mathbb{R}) : \|u\|_\infty \leq \eta_*]$, we have $\|F(u)\|_\infty \leq l$.
By **(H_3)** we have for each $(x, y) \in J$,

$$\begin{aligned}|F(u)(x,y)| &\leq |\mu(x,y)| + \frac{\rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y s^{\rho-1}t^{\rho-1}(x^\rho - s^\rho)^{r_1-1}(y^\rho - t^\rho)^{r_2-1} \left|f(s,t,u_{(s,t)})\right| dsdt \\ &\leq |\mu(x,y)| + \frac{\|p\|_\infty + \|q\|_\infty \eta^* \rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)} \int_0^a \int_0^b s^{\rho-1}t^{\rho-1}(x^\rho - s^\rho)^{r_1-1}(y^\rho - t^\rho)^{r_2-1} dsdt,\end{aligned}$$

thus

$$\|F(u)\|_\infty \leq \|\mu\|_\infty + \frac{\|p\|_\infty + \|q\|_\infty \eta^* a^{\rho r_1} b^{\rho r_2} \rho^{(1-r_1)+(1-r_2)}}{\rho^2 \Gamma(r_1+1)\Gamma(r_2+1)} = l.$$

Step 3 : F maps bounded sets into equicontinuous sets in C
Let (x_1, y_1), $(x_2, y_2) \in (0, a] \times (0, b]$, $x_1 < x_2$, $y_1 < y_2$, B_{η^*} be a bounded set of $C([-\alpha, a] \times [-\beta, b], \mathbb{R})$ as in Step 2 , and let $u \in B_{\eta^*}$ .Then

$$\begin{aligned}|F(u)(x_2,y_2) - F(u)(x_1,y_1)| &\leq |\mu(x_2,y_2) - \mu(x_1,y_1)| + \frac{\rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)} \\ &\times \int_0^{x_1} \int_0^{y_1} [(x_2^\rho - s^\rho)^{r_1-1}(y_2^\rho - t^\rho)^{r_2-1} \\ &- (x_1^\rho - s^\rho)^{r_1-1}(y_1^\rho - t^\rho)^{r_2-1}] s^{\rho-1}t^{\rho-1} \left|f(s,t,u_{(s,t)})\right| dsdt \\ &+ \frac{\rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} [(x_2^\rho - s^\rho)^{r_1-1}(y_2^\rho - t^\rho)^{r_2-1}] s^{\rho-1}t^{\rho-1} \\ &\times \left|f(s,t,u_{(s,t)})\right| dsdt \\ &\leq |\mu(x_2,y_2) - \mu(x_1,y_1)| + \frac{\|p\|_\infty + \|q\|_\infty \eta^* \rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)} \\ &\times \int_0^{x_1} \int_0^{y_1} s^{\rho-1}t^{\rho-1}[(x_1^\rho - s^\rho)^{r_1-1}(y_1^\rho - t^\rho)^{r_2-1} \\ &- (x_2^\rho - s^\rho)^{r_1-1}(y_2^\rho - t^\rho)^{r_2-1}] dsdt + \frac{\|p\|_\infty + \|q\|_\infty \eta^* \rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)} \\ &\times \int_{x_1}^{x_2} \int_{y_1}^{y_2} s^{\rho-1}t^{\rho-1}(x_2^\rho - s^\rho)^{r_1-1}(y_2^\rho - t^\rho)^{r_2-1} dsdt \\ &\leq |\mu(x_2,y_2) - \mu(x_1,y_1)| + \frac{\|p\|_\infty + \|q\|_\infty \eta^* \rho^{(1-r_1)+(1-r_2)}}{\rho^2 \Gamma(r_1+1)\Gamma(r_2+1)} \\ &\times [(x_2^\rho - x_1^\rho)^{r_1-1}(y_2^\rho - y_1^\rho)^{r_2-1} + x_1^{\rho r_1} y_1^{\rho r_2} - x_2^{\rho r_1} y_2^{\rho r_2}] \\ &+ \frac{\|p\|_\infty + \|q\|_\infty \eta^* \rho^{(1-r_1)+(1-r_2)}}{\rho^2 \Gamma(r_1+1)\Gamma(r_2+1)} (x_2^\rho - x_1^\rho)^{r_1-1}(y_2^\rho - y_1^\rho)^{r_2-1} \\ &\leq |\mu(x_2,y_2) - \mu(x_1,y_1)| + \frac{\|p\|_\infty + \|q\|_\infty \eta^* \rho^{(1-r_1)+(1-r_2)}}{\rho^2 \Gamma(r_1+1)\Gamma(r_2+1)} \\ &\times [2(x_2^\rho - x_1^\rho)^{r_1-1}(y_2^\rho - y_1^\rho)^{r_2-1} + x_1^{\rho r_1} y_1^{\rho r_2} - x_2^{\rho r_1} y_2^{\rho r_2}].\end{aligned}$$

As $x_1 \longrightarrow x_2, \quad y_1 \longrightarrow y_2$ the right-hand side of the above inequality tends to zero. As a consequence of steps 1 to 3 together with the Arzela-Ascoli theorem . we can conclude that$F : C([-\alpha, a] \times [-\beta, b], \mathbb{R}) \longrightarrow C([-\alpha, a] \times [-\beta, b], \mathbb{R})$ is continuous and completely continuous.

Step 4 : G is a contraction.

Let $u, \quad v \in C([-\alpha, a] \times [-\beta, b], \mathbb{R})$. Then , for each $(x, y) \in J$ we have :

$$\begin{aligned}
|G(u)(x,y) - G(v)(x,y)| &\leq \frac{\rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y s^{\rho-1}t^{\rho-1}(x^\rho - s^\rho)^{r_1-1}(y^\rho - t^\rho)^{r_2-1} \\
&\times \left| g(s,t,u_{(s,t)}) - g(s,t,v_{(s,t)}) \right| dsdt \\
&\leq \frac{\rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)} \int_0^a \int_0^b s^{\rho-1}t^{\rho-1}(x^\rho - s^\rho)^{r_1-1}(y^\rho - t^\rho)^{r_2-1} k \left\| u - v \right\|_\infty dsdt \\
&\leq \frac{k \left\| u - v \right\|_\infty \rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)} \int_0^a \int_0^b s^{\rho-1}t^{\rho-1}(x^\rho - s^\rho)^{r_1-1}(y - t)^{r_2-1} dsdt \\
&\leq \frac{k a^{\rho r_1} b^{\rho r_2} \rho^{(1-r_1)+(1-r_2)}}{\rho^2 \Gamma(r_1+1)\Gamma(r_2+1)} \left\| u - v \right\|_\infty .
\end{aligned}$$

Which by (3.10) , implies that G is contraction

Step 5 : A priori bounds.

Now it remains to show that the set

$$\varepsilon = \left\{ u \in C(J, \mathbb{R}) : u = \lambda F(u) + \lambda G(\frac{u}{\lambda}), \forall \ 0 < \lambda < 1 \right\}.$$

Is bounded.

Let $u \in \varepsilon$, then

$$u = \lambda F(u) + \lambda G(\frac{u}{\lambda}).$$

$\forall \ 0 < \lambda < 1$. Thus for each $(x, y) \in J$ we have :

$$\begin{aligned}
u(x,y) = \lambda \mu(x,y) &+ \frac{\lambda \rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y s^{\rho-1}t^{\rho-1}(x^\rho - s^\rho)^{r_1-1}(y^\rho - t^\rho)^{r_2-1} f(s,t,u_{(s,t)}) dsdt \\
&+ \frac{\lambda \rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y s^{\rho-1}t^{\rho-1}(x^\rho - s^\rho)^{r_1-1}(y^\rho - t^\rho)^{r_2-1} g(s,t,\frac{u_{(s,t)}}{\lambda}) dsdt.
\end{aligned}$$

This implies by **(H_2)** and **(H_3)** that , for each $(x,y)\in J$

$$
\begin{aligned}
|u(x,y)| &\leq |\mu(x,y)| + \frac{\|p\|_\infty \rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y s^{\rho-1}t^{\rho-1}(x^\rho-s^\rho)^{r_1-1}(y^\rho-t^\rho)^{r_2-1}\left\|u_{(s,t)}\right\|_C dsdt\\
&+\frac{\|q\|_\infty \rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y s^{\rho-1}t^{\rho-1}(x^\rho-s^\rho)^{r_1-1}(y^\rho-t^\rho)^{r_2-1}dsdt\\
&+\frac{\lambda\rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y s^{\rho-1}t^{\rho-1}(x^\rho-s^\rho)^{r_1-1}(y^\rho-^\rho t)^{r_2-1}\left|g(s,t,\frac{u_{(s,t)}}{\lambda})-g(s,t,0)\right|dsdt\\
&+\frac{\lambda\rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y s^{\rho-1}t^{\rho-1}(x^\rho-s^\rho)^{r_1-1}(y^\rho-t^\rho)^{r_2-1}|g(s,t,0)|\,dsdt\\
&\leq |\mu(x,y)| + \frac{\|p\|_\infty \rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y s^{\rho-1}t^{\rho-1}(x^\rho-s^\rho)^{r_1-1}(y^\rho-t^\rho)^{r_2-1}\left\|u_{(s,t)}\right\|_C dsdt\\
&+\frac{\|q\|_\infty \rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y s^{\rho-1}t^{\rho-1}(x^\rho-s^\rho)^{r_1-1}(y^\rho-t^\rho)^{r_2-1}dsdt\\
&+\frac{k\rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y s^{\rho-1}t^{\rho-1}(x^\rho-s^\rho)^{r_1-1}(y^\rho-t^\rho)^{r_2-1}\left\|u_{(s,t)}\right\|_C dsdt\\
&+\frac{g^* a^{\rho r_1}b^{\rho r_2}\rho^{(1-r_1)+(1-r_2)}}{\rho^2\Gamma(r_1+1)\Gamma(r_2+1)}.
\end{aligned}
$$

Where $g^*=sup_{(s,t)\in J}\,|g(s,t,0)|$.
We consider the function τ defined by :

$$\tau(x,y)=sup\,\{|u(s,t)|:0\leq s\leq x,0\leq t\leq y\}\,.0\leq x\leq a,0\leq y\leq b$$

Let $(x^*,y^*)\in[-\alpha,x]\times[-\beta,y]$ such that

$$\tau(x,y)=|u(x^*,y^*)|\,,(x^*,y^*)\in J$$

then by the previous inequality, we have for $(x,y)\in J$

$$
\begin{aligned}
\tau(x,y)&\leq |\mu(x,y)|+\frac{a^{\rho r_1}b^{\rho r_2\rho^{(1-r_1)+(1-r_2)}}(\|q\|_\infty+g^*)}{\rho^2\Gamma(r_1+1)\Gamma(r_2+1)}\\
&+\frac{(\|p\|_\infty+k)\rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y s^{\rho-1}t^{\rho-1}(x^\rho-s^\rho)^{r_1-1}(y^\rho-t^\rho)^{r_2-1}\tau(s,t)dsdt.
\end{aligned}
$$

If $(x^*,y^*)\in[-\alpha,a]\times[-\beta,b]$,then

$$\tau(x,y)=\|\phi\|_C\,.$$

And the previous inequality holds.

Then from lemma (1.6.2) there exists $k^* = k^*(r_1, r_2)$ such that we have

$$\begin{aligned}
\tau(x,y) &\leq |\mu(x,y)| + \frac{a^{\rho r_1} b^{\rho r_2}(\|q\|_\infty + g^*)\rho^{(1-r_1)+(1-r_2)}}{\rho^2\Gamma(r_1+1)\Gamma(r_2+1)} \\
&+ \frac{(\|p\|_\infty + k)\rho^{(1-r_1)+(1-r_2)}}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y s^{\rho-1}t^{\rho-1}(x^\rho - s^\rho)^{r_1-1}(y^\rho - t^\rho)^{r_2-1} ds dt \\
&\leq |\mu(x,y)| + \frac{a^{\rho r_1} b^{\rho r_2}(\|q\|_\infty + g^*)\rho^{(1-r_1)+(1-r_2)}}{\rho^2\Gamma(r_1+1)\Gamma(r_2+1)} + \frac{k^*(\|p\|_\infty + k)a^{\rho r_1} b^{\rho r_2}\rho^{(1-r_1)+(1-r_2)}}{\rho^2\Gamma(r_1+1)\Gamma(r_2+1)} \\
&= R^*.
\end{aligned}$$

Since for every $(x,y) \in J$

$$\left\|u_{(x,y)}\right\|_C \leq \tau(x,y).$$

We have

$$\begin{aligned}
\|u\|_\infty &\leq max(\|\phi\|_C, R^*) \\
&= A.
\end{aligned}$$

This shows that the set ε is bounded . As a consequence of theorem (1.6.4), we deduce that $F+G$ has a fixed point u which is a solution to problem (3.1)-(3.3) .
Now, we present (without proof) two existence results for the nonlocal problem (3.4)-(3.6)

Definition 3.2.2. A function $u \in C([-\alpha, a] \times [-\beta, b], \mathbb{R})$ is said to be a solution of (3.4)-(3.6) if u satisfies Eqs (3.4)and (3.6) on J and the condition (3.5) on $[-\alpha, 0] \times [-\beta, 0]$.

Theorem 3.2.2. *Assume **(H_1)-(H_3)** and following conditions :*
(H'_1) *there existe $\tilde{k} > 0$ such that*

$$|A(u) - A(v)| \leq \tilde{k} \|u - v\|_\infty, \forall \ u, v \in C(J, \mathbb{R})$$

(H''_1) *there existe $k^* > 0$ such that*

$$|B(u) - B(v)| \leq k^* \|u - v\|_\infty, \forall \ u, v \in C(J, \mathbb{R})$$

hold if

$$\tilde{k} + k^* + \frac{k a^{\rho r_1} b^{\rho r_2}\rho^{(1-r_1)+(1-r_2)}}{\rho^2\Gamma(r_1+1)\Gamma(r_2+1)} < 1.$$

Then there exists at least one solution for IVP(3.4)-(3.6) on $[-\alpha, a] \times [-\beta, b]$.

Theorem 3.2.3. *Assume (**H_1**)-(**H_3**) and following conditions :*
(H'_3) *there existe $\tilde{d} > 0$ such that*

$$|A(u)| \leq \tilde{d}(1 + \|u\|_{\infty}).\forall \ u \in C(J, \mathbb{R})$$

(H''_3) *there existe $d^* > 0$ such that*

$$|B(u)| \leq d^*(1 + \|u\|_{\infty}).\forall \ u \in C(J, \mathbb{R})$$

Hold if (3.10) is satisfied, then there exists at least one solution for IVP(3.4)-(3.6) on $[-\alpha, a] \times [-\beta, b]$.

3.3 Example

We consider the following perturbed problem such that : $J = [0,1] \times [0,1], \tilde{J} = [-1,1] \times [-2,1] \backslash [0,1] \times [0,1]$

$$({}^c\mathcal{D}_0^r)(x,y) = \frac{1 + 4e^{x+y+3}(|u(x-1, y-2)| + 2)}{4e^{x+y+3}(|u(x-1, y-2)| + 2)}, \quad if (x,y) \in J \tag{3.11}$$

$$u(x,y) = x + y, \quad (x,y) \in \tilde{J} \tag{3.12}$$

$$u(x,0) = x, \quad u(0,y) = y^2. \ \ (x,y) \in J \tag{3.13}$$

Set

$$f(x, y, u_{(x,y)}) = \frac{|u(x-1, y-2)| + 2}{1 + |u(x-1, y-2)|}, \quad (x,y) \in J$$

and

$$g(x, y, u_{(x,y)}) = \frac{1}{4e^{x+y+3}(1 + |u(x-1, y-2)|)}. \quad (x,y) \in J$$

For each $u, \ \ \bar{u} \in \mathbb{R}$ and $(x,y) \in J$ we have

$$\left| g(x, y, u_{(x,y)}) - g(x, y, \bar{u}_{(x,y)}) \right| \leq \frac{\|u - \bar{u}\|_C}{4e^3}.$$

Hence condition **(H_2)** is satisfied with $k = \frac{1}{4e^3}$. we shall that condition (3.10) holds with $a = b = \rho = 1$. Indeed

$$\frac{1}{4e^3\Gamma(r_1+1)\Gamma(r_2+1)} < 1.$$

Which is satisfied for each $(r_1, r_2) \in (0,1] \times (0,1]$. Also the function f is continuous on $J \times [0, \infty)$ and

$$|f(x,y,w)| \leq |w| + 2 \quad \forall (x,y) \in J \times [0, \infty).$$

Thus conditions **(H_1)** and **(H_3)** hold.The theorem(3.2.1) implies that the problem (3.11)-(3.13) has a solution on $[-1,1] \times [-2,1]$.

General conclusion

In this memory we considered the existence of solutions of some Darboux problems of partial hyperbolic differential equations disturbed fractional. Sufficient conditions for the existence of the solutions of our problems have been given.

Bibliographie

[1] A. A. Kilbas, H. M. Srivastava and J. J. Trujllo. Theory and Application of Fractional Differential Equations. *Elsevier*, 2006.

[2] .A.Burton, C.Kirk, A fixed point theorem of Krasnoselskii-Schaefer type, *Math .Nachr.*189 : 23-31, 1998.

[3] A.Granas , J.Dugundji, Fixed point theory, Springer -Verlag, New York, 2003.

[4] A.N.Vityuk, A.V.Golushkov, Existence of solution of systems of partial differential equations of fractional order, *Nonlinear Oscil.*7(3) :318-325,2004

[5] B. Zhanbing, L. Haishen. Positive Solutions for Boundary Value Problem of Nonlinear Fractional Differential Equation. *J. Math. Anal. Appll*, 3(11) :495-505, 2005.

[6] C. Yu , Gao, Existence of fractional differential equations, *J.Math .Anal.appl.*310 : 26-29, 2005 .

[7] D. S. Oliveira and E. Capelas de Oliveiral. On a Caputo-Type Fractional Derivative. *Department of Applied Mathematics, Imecc-Unicamp, 13083-859, Campinas, SP, Brazil*, :1-13, 2015.

[8] J.K.Hale ,S.Verduyn Lunel ,Introduction to Functional Differential Equations,*Applied Mathematical Sciences* ,Springer-Verlag,New York :99,1993.

[9] I. Podlubny. *Fractional differential equations.* Academic Press, 1999.

[10] I. Podlubny, I. Petras, B.M. Vinagre ,P. O'Leary, L.Dorcak, Analogue realizations of fractional-order controllers.Fractional order calculus and its applications,*Nonlinear Dynam* .29 : 281-296 , 2002 .

[11] K. Diethelm. *The Analysis of Fractional Differential Equations.* Springer, 2010.

[12] M. Benchohra, J. E. Lazreg. Existence Results for Nonlinear Implicit Fractional Differential Equations. *Surveys in Mathematics and its Applications*, 9 :79-92, 2014.

[13] M. Janaki, K. Kanagarajan and D. Vivek. Analytic Study on Fractional Implicit Differential Equations with Impulses via Katugampola Fractional Derivative. *Int. J. Math. And App*, 36 :53-63, 2018.

[14] N. Benhamidouche, Y. Arioua and B. Bast. Initial Value Problem for Nonlinear Implicit Fractional Differential Equations with Katugampola Derivative. *Applied Mathematics E-Notes*, article acceptais.

[15] R. Almeida, A. B. Malinowska and Odzijewicz. Fractional Differential Equations with Dependence on the Caputo-Katugampola Derivative. *J. Comput Nonlinear Dynam*, 11(6), 2016.

[16] S. A. Murad and S. Hadid. Existence and Uniqueness Theorem for Fractional Differential Equation with Integral Boundary Condition. *J. Frac. Calc. Appl*, 3(6) :1-9, 2012.

[17] S.Abbas, M.Benchohra. Darboux problem for perturbed partial differential equations of fractional order with finite delay.*Mathematics laboratory ,University of Saida ,Sidi Bel-AbbÃ¨s*, 3 : 597-604, 2009.

[18] S. G. Samko, A. A. Kilbas and O. I. Marichev. *Fractional Integrals and Derivatives : Theory and Applications.* Gordon and Breach Science Publishers, 1993.

[19] S. Zhang, Positive solutions for boundary-value problems of nonlinear fractional differential equations, Electron *.J.Differential Equations* (36) 1-12 , 2006 .

[20] U. N. Katugampola. A new Approach to Generalized Fractional Derivatives. *Mathematical Analysis and Applications*, 6(4) :1-15, 2014.

[21] U. N. Katugampola. Existence and Uniqueness Results for a Class of Generalized Fractional Differential Equations. *Department of Mathematics, University of Delaware, Newark DE 19716, USA*.

[22] U. N. Katugampola. New Approach to a Aenaralized Fractional Integra. *Applied Mathematics and Computation*, 3(218) :860-865, 2011.

[23] Y. Arioua. Initial Value Problem for a Coupled System of Katugampola type Fractional Differential Equations. *Laboratory for Pure and Applied Mathematics*, :1-19, article acceptais.

ملخص:

الهدف من هذه المذكرة، هو دراسة مشكلة داربو للمعادلات التفاضلية الجزئية المضطربة مع مشتق كسري ، وتستند هذه النتائج إلى نظرية ذات نقطة صامدة لمجموع الانكماش.

وفي الأخير ترفق النتائج المحصل عليها بأمثلة توضيحية.

كلمات مفتاحية: مشكلة داربو، المعادلات التفاضلية الوظيفية، الترتيب الكسري، حلول، التكامل المختلط لريمان ليوفيل، التكامل المختلط لكتوغامبولا، مشتق من ترتيب كسري لكابوتو، مشتق من ترتيب كسري لكتو غامبولا، النقطة الصامدة.

Résumé :

L’objectif de cette mémoire est étudier l’existence des solutions de problème de Darboux pour les équations aux dérivées partielles perturbées avec order fractionnaire. Ces résultats sont basés sur un théorème à point fixe pour la sommes des opérateurs de contraction et compact.
Nous concluons les résultats obtenus par des exemples illustratifs.

Mots-Clés : Problème de Darboux, équations différentielles fonctionnelles, order fractionnel, solution, mixte Riman-Liouville intégrale, mixte Katugampola intégral, dérivées fractionnaires de Caputo, dérivées fractionnaires de Katugampola, pint fixe.

Abstract :

The objective of this memory is to study the existence of solution for Darboux problem for perturbed partial differential equations of fractional order. These results are based on a fixed-point theorem for the sum of contraction and compact operators.
We conclude the results obtained by illustrative examples.

Keywords : Darboux problem, functional differential equations, solution, left-sided mixed Riman-Liouville integral, Caputo fractional order derivative, Katugampola fractional order derivative, fixed point.