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Structures on intuitionistic fuzzy relations

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Dedication

To my dear mother and father, :

To my dear mother and father, my pillars of strength, This work is a testament to your love, wisdom, and faith, For shaping me into the person I am today, I dedicate this work to you with gratitude and immense praise.

May your hearts be filled with pride and joy, Knowing that your love has nurtured your girl or boy, Forever grateful for the love you bestow, This dedication is a tribute to the love that continues to grow.

Thanks for All !

Gherbi & Hadjira ✍

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♡

Contents

Introduction

In the 1983, Atanassov created intuitionistic fuzzy sets (IFS) [3], a mathematical extension of classical fuzzy sets [25].

Several papers for example [2, 7, 10, 21, 23, 28] have researched intuitionistic fuzzy relations (IFR), and different structures have been studied. The characteristics that these relations satisfy are generally referred to in the structures of intuitionistic fuzzy relations. Ordered structures, equivalence relations, and tolerance relations are a few of the structures that have been researched. An intuitionistic fuzzy equivalence relation (IFER) is a reflexive, symmetric, and transitive intuitionistic fuzzy relation, according to the authors of [18]. They discuss IFERs' characteristics and offer several illustrations.

In [18], the authors propose a theoretical framework of hierarchical structures based on intuitionistic fuzzy tolerance relations. They use fuzzy quotient space theory to define these structures and provide some applications.

In [9], the authors propose a clustering algorithm based on intuitionistic fuzzy relations for tree structure evaluation. They use intuitionistic fuzzy relations to construct hierarchical structures for the evaluation of vague complicated humanistic systems.

Overall, the structures on intuitionistic fuzzy relations refer to the properties that these relations satisfy, and they have been studied in different contexts and applications, including decision support systems, medical diagnosis, and artificial intelligence. Some of the applications of intuitionistic fuzzy relations are:

- Decision support systems: Intuitionistic fuzzy relations have been used in decision support systems for management, marketing, medical diagnosis, gas pipeline networks, and transportation [1]. They have been used to model complex systems and provide decision-making tools that take into account the uncertainty and imprecision of the data.

- Medical diagnosis: Intuitionistic fuzzy sets have been used in medical diagnosis to model the uncertainty and vagueness of the medical data [17]. They have been used to develop decision-making tools that can assist medical professionals in diagnosing diseases and recommending treatments.

- Artificial intelligence: Intuitionistic fuzzy relations have been used in artificial intelligence to model complex systems and provide decision-making tools that take into account the uncertainty and imprecision of the data [15]. They have been used in areas such as knowledge representation and reasoning.

The main objective of this memory we study the structures of the intuitionistic fuzzy relations. We analyze the existent relations between the structures of a relation and the structures of its complementary one. We finish characterizing certain structures of intuitionistic relations according to the structures of two concrete fuzzy relations.
This memory is organized as follows:

- In the first chapter, we recall some concepts and well-known results on fuzzy subsets.
- In the second chapter, we introduce concepts of intuitionistic fuzzy subsets
- In the last chapter, We will start by studying the relation that exists between the properties of a relation and the properties of its complementary one. Next, we define the structures in the intuitionistic fuzzy relations, and we also study their main properties. Besides, we analyses the relation between the structures of $\boldsymbol{R}$ and $\boldsymbol{R_c}$. We conclude this chapter with two theorems that characterize the structures of the intuitionistic fuzzy relations according to the structures of the fuzzy relations $\boldsymbol{D_0(R)}$ and $\boldsymbol{D_1(R)}$. And study the composition of intuitionistic complementary fuzzy relations and its transitive closures .

Chapter 1

Generalities on fuzzy subsets

Contenu of chapter :

This chapter has been prepared based on the following references [8, 11, 12, 16, 19, 20, 22, 25, 26]

1-1 Fuzzy subset

A fuzzy subset is a set containing elements that have varying degrees of membership in the set. According to Zadeh [25], fuzzy subsets are formally defined as follows:

Definition 1.1.1: [25] **A fuzzy subset** A in X is characterized by an application:

$$\mu_A : X \longrightarrow [0,1],$$

Where $[0,1]$ means real numbers between 0 and 1. If x an element of X, $\mu_A(x)$ is the degree of membership of x. The fuzzy subset A in X may be represented as a subset of ordered pairs of generic element $x \in X$ and its grade of membership, i.e.,

$$A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}.$$

Example 1.1.1

Let $X = \{a,b,c,d\}$ be a universal set. $A_1 = \{\langle a,0.2\rangle, \langle b,0.5\rangle, \langle c,0.8\rangle, \langle d,0.1\rangle\}$, $A_2 = \{\langle a,0.4\rangle, \langle b,0.1\rangle, \langle c,0.0\rangle, \langle d,0.3\rangle\}$ and $A_3 = \{\langle a,0.7\rangle, \langle b,0.9\rangle, \langle c,0.6\rangle, \langle d,1.0\rangle\}$ be a fuzzy subsets in X.

Notation 1-1.1. Let $F(X)$ denoted the set of all fuzzy subsets of X.

1-2 Operations on fuzzy subsets

Since the theory of fuzzy subsets was developed as a generalization of classical set theory, it also supports operations like equality, inclusion, intersection, union, complement, addition, and multiplication of two fuzzy subsets. The following definitions are provided for the terms associated with operations on fuzzy subsets as follows (See [20, 25])

1-2-1 Equality and inclusion of fuzzy subsets

Definition 1.2.1: Let X be a non-empty set and let A and B two fuzzy subsets

1. We say A and B are equal. That is,

$$A = B \text{ if and only if } \mu_A(x) = \mu_B(x), \text{ for all } x \in X,$$

Or

$$A = B \Leftrightarrow (A \subseteq B \wedge B \subseteq A).$$

2. We say that A is included in B if and only if $\mu_A(x) \le \mu_B(x)$ for all $x \in X$. In symbols $A \subseteq B$ $\Leftrightarrow \mu_A(x) \le \mu_B(x)$, for all $x \in X$.
When a fuzzy subset A is included in B, then A is called a fuzzy subset of B.

Example 1.2.1

Equality:
Let $X = \{1,2,3\}$ and let A and B two fuzzy subsets defined by:

$$\begin{aligned} A &= \{(1,0.5),(2,0.7),(3,0.2)\}, \\ B &= \{(1,0.5),(2,0.7),(3,0.2)\} \end{aligned}$$

We have:

$$\begin{aligned} \mu_A(1) &= \mu_B(1) = 0.5 \\ \mu_A(2) &= \mu_B(2) = 0.7 \\ \mu_A(3) &= \mu_B(3) = 0.2 \end{aligned}$$

Then,

$$\mu_A(x) = \mu_B(x) \Leftrightarrow A = B, \text{ for all } x \in X.$$

Inclusion:
Let $X = \{1,2,3,4,5\}$ and let A and B two fuzzy subsets defined by:

$$\begin{aligned} A &= \{(1,0.5),(2,0.6),(3,0.2),(4,0.3),(5,0.4)\}, \\ B &= \{(1,0.5),(2,0.7),(3,0.2),(4,0.8),(5,0.6)\} \end{aligned}$$

We have:

$$\mu_A(1) \le \mu_B(1),\ \mu_A(2) \le \mu_B(2),\ \mu_A(3) \le \mu_B(3),\ \mu_A(4) \le \mu_B(4),\ \mu_A(5) \le \mu_B(5)$$

Then,

$$A \subseteq B, \text{ for all } x \in X.$$

1-2-2 Union and intersection of fuzzy subsets

Definition 2.2.1: Let X be a non-empty set and let A and B two fuzzy subsets. For all $x \in X$, the union and the intersection are defined as follows:

$$\begin{aligned}
\textit{Union:} \quad \mu_{A\cup B}(x) &= \max\{\mu_A(x), \mu_B(x)\} \\
&= \mu_A(x) \vee \mu_B(x); \\
\textit{Intersection:} \quad \mu_{A\cap B}(x) &= \min\{\mu_A(x), \mu_B(x)\} \\
&= \mu_A(x) \wedge \mu_B(x).
\end{aligned}$$

Example 2.2.1

Suppose we have two fuzzy subsets A and B defined over the universe of discourse $\{a, b, c, d\}$ as follows:

$$\begin{aligned}
A &= \{(a, 0.2), (b, 0.3), (c, 0.6), (d, 0.6)\}. \\
B &= \{(a, 0.9), (b, 0.9), (c, 0.4), (d, 0.5)\}.
\end{aligned}$$

The union of A and B is a fuzzy subset defined as:

$$A \cup B = \{(a, 0.9), (b, 0.9), (c, 0.6), (d, 0.6)\}.$$

The intersection of A and B is a fuzzy subset defined as:

$$A \cap B = \{(a, 0.2), (b, 0.3), (c, 0.4), (d, 0.5)\}.$$

1-2-3 Complement of a fuzzy subset

Definition 3.2.1: The complement of a fuzzy subset A is denoted by A^c and is defined by :

$$\mu_{A^c}(x) = 1 - \mu_A(x), \text{ for all } x \in X.$$

Example 3.2.1

Let $X = \{a, b, c, d\}$ and let fuzzy set $A = \{(a, 0.4), (b, 0.2), (c, 0.5), (d, 0.3)\}$. So,

$$\begin{aligned}
\mu_{A^c}(a) &= 1 - \mu_A(a) = 1 - 0.4 = 0.6 \\
\mu_{A^c}(b) &= 1 - \mu_A(b) = 1 - 0.2 = 0.8 \\
\mu_{A^c}(c) &= 1 - \mu_A(c) = 1 - 0.5 = 0.5 \\
\mu_{A^c}(d) &= 1 - \mu_A(d) = 1 - 0.3 = 0.7
\end{aligned}$$

The complement is:

$$A^c = \{(a, 0.6), (b, 0.8), (c, 0.5), (d, 0.7)\}.$$

Remark 1.2.1

Elements of X for wich $A(x) = A^c(x)$ are called equilibrium boints of A.

1-2-4 Addition

Definition 4.2.1: Let X be a non-empty set and let A and B two fuzzy subsets, the addition defined by for all $x \in X$:

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x).$$

Example 4.2.1

Let $X = \{a, b, c, d\}$ and let A, B be a fuzzy subset on a set X, defined by:

$$\begin{aligned} A &= \{\langle a, 0.2\rangle, \langle b, 0.5\rangle, \langle c, 0.4\rangle, \langle d, 0.3\rangle\}, \\ B &= \{\langle a, 0.6\rangle, \langle b, 0.2\rangle, \langle c, 0.3\rangle, \langle d, 0.7\rangle\}. \end{aligned}$$

We have:

$$\begin{aligned} \mu_{A+B}(a) &= 0.2 + 0.6 - 0.2 \cdot 0.6 = 0.68 \\ \mu_{A+B}(b) &= 0.5 + 0.2 - 0.5 \cdot 0.2 = 0.6 \\ \mu_{A+B}(c) &= 0.4 + 0.3 - 0.4 \cdot 0.3 = 0.58 \\ \mu_{A+B}(d) &= 0.3 + 0.7 - 0.3 \cdot 0.7 = 0.79 \end{aligned}$$

then, the addition for all $x \in X$:

$$\mu_{A+B}(x) = \{(a, 0.68), (b, 0.6), (c, 0.58), (d, 0.79)\}.$$

1-2-5 Multiplication

Definition 5.2.1: Let X be a non-empty set and let A and B two fuzzy subsets, the multiplication defined by for all $x \in X$:

$$\mu_{A \cdot B}(x) = \mu_A(x) \cdot \mu_B(x).$$

Example 5.2.1

Let $X = \{a, b, c, d\}$ and let A, B be a fuzzy subset on a set X, defined by:

$$\begin{aligned} A &= \{\langle a, 0.2\rangle, \langle b, 0.5\rangle, \langle c, 0.4\rangle, \langle d, 0.3\rangle\}, \\ B &= \{\langle a, 0.6\rangle, \langle b, 0.2\rangle, \langle c, 0.3\rangle, \langle d, 0.7\rangle\}. \end{aligned}$$

We have:

$$\begin{aligned}\mu_{A\cdot B}(a) &= 0.2\cdot 0.6 = 0.12\\ \mu_{A\cdot B}(b) &= 0.5\cdot 0.2 = 0.10\\ \mu_{A\cdot B}(c) &= 0.4\cdot 0.3 = 0.12\\ \mu_{A\cdot B}(d) &= 0.3\cdot 0.7 = 0.21\end{aligned}$$

Then, the multiplication for all $x \in X$:

$$\mu_{A\cdot B}(x) = \{(a, 0.12), (b, 0.10), (c, 0.12), (d, 0.21)\}.$$

1-3 Characteristics of fuzzy subset

We can characterize fuzzy subsets in more detail by referring to the features used in characterizing the membership functions.

1-3-1 The support

Definition 1.3.1: [22] Let A be a fuzzy subset on a set X. The support of A is the crisp subset on X given by:

$$\mathbf{Supp}(A) = \{x \in X |\ \mu_A(x) > 0\}.$$

1-3-2 The kernel

Definition 2.3.1: [22] Let A be a fuzzy subset on a set X. The kernel of A is the crisp subset on X given by :

$$Ker(A) = \{x \in X \mid \mu_A(x) = 1\}.$$

1-3-3 The height

Definition 3.3.1: [22] Let A be a fuzzy subset on a set X. The height of A is the highest value taken by its membership function given by:

$$H(A) = \sup_{x\in X} \mu_A(x).$$

Example 1.3.1

Let $X = \{a,b,c,d\}$ and let A be a fuzzy subset on a set X, defined by:

$$A = \{\langle a,0.3\rangle, \langle b,0.6\rangle, \langle c,0.8\rangle, \langle d,0.0\rangle\}$$

Then :
$\text{Supp}(A) = \{a,b,c\}$,
$\ker(A) = \emptyset$,
$H(A) = 0.8$.

Proposition 1.3.1. [11] The support and kernel of a fuzzy subset A satisfied the following :

(i) $\text{Supp}(A^c) = X - \ker(A)$;

(ii) $\ker(A^c) = X - supp(A)$.

Proof. (i)

$$\begin{aligned} Supp\,(A^c) &= \{x \in X \mid \mu_{A^c}(x) \neq 0\} \\ &= \{x \in X \mid 1-\mu_A(x) \neq 0\} \\ &= \{x \in X \mid \mu_A(x) \neq 1\} \\ &= \{x \in X \mid x \notin \ker(A)\} \\ &= X - \ker(A). \end{aligned}$$

(ii)

$$\begin{aligned} \ker\,(A^c) &= \{x \in X \mid \mu_{A^c}(x) = 1\} \\ &= \{x \in X \mid 1-\mu_A(x) = 1\} \\ &= \{x \in X \mid \mu_A(x) = 0\} \\ &= \{x \in X \mid x \notin Supp(A)\} \\ &= X - Supp(A). \end{aligned}$$

□

Example 2.3.1

Let $X = \{$The European world$\}$ and let the fuzzy subset,

$$A = \{\text{Les pays francophones}\}$$

Defined as follows:

$$A = \left\{ \begin{array}{l} (Allemagne, 0), (Belgique, 0.9), (Autriche, 0.7), (Bulgarie, 0.4), \\ (Chypre, 0.8), (Croatie, 0.7), (Danemark, 0.4), (Espagne, 0.2), \\ (Estonie, 0.3), (Finlande, 0.4), (France, 1), (Grce, 0.7), \\ (Hongrie, 0.1), (Irlande, 0.4), (Italie, 0), (Lettonie, 0.2), \\ (Lituanie, 0.1), (Luxembourg, 0.9), (Malte, 0.8), (Pays-Bas, 0), \\ (Pologne, 0), (Tchquie, 0.1), (Roumanie, 0.7), (Slovaquie, 0.8), \\ (Slovnie, 0.7), (Sude, 0.9) \end{array} \right\}$$

Then,

1. $h(A) = 1$ Donc, A est normalisé.

2. $$\begin{aligned} card(A) = & \; 0+0.9+0.7+0.4+0.8+0.7+0.4+0.2+0.3+0.4+1+0.7+0.1+ \\ & \; 0.4+0+0.2+0.1+0.9+0.8+0+0+0.1+0.7+0.8+0.7+0.9 \\ card(A) = & \; 12.2 \end{aligned}$$

3. $$Supp(A) = \left\{ \begin{array}{l} (Belgique, 0.9), (Autriche, 0.7), (Bulgarie, 0.4), \\ (Chypre, 0.8), (Croatie, 0.7), (Danemark, 0.4), \\ (Estonie, 0.3), (Finlande, 0.4), (France, 1), (Grce, 0.7), \\ (Hongrie, 0.1), (Irlande, 0.4), (Lettonie, 0.2), \\ (Lituanie, 0.1), (Luxembourg, 0.9), (Malte, 0.8), \\ (Pays-Bas, 0), (Tchquie, 0.1), (Roumanie, 0.7), \\ (Slovaquie, 0.8), (Slovnie, 0.7), (Sude, 0.9), (Espagne, 0.2) \end{array} \right\}$$

4. $$\ker A = \left\{ \; France \; \right\}$$

5. $$A_{0.8} \geq = \left\{ \begin{array}{l} (Belgique, 0.9), \\ (Chypre, 0.8), \\ (France, 1), \\ (Luxembourg, 0.9), \\ (Malte, 0.8) \\ (Slovaquie, 0.8), \\ (Sude, 0.9) \end{array} \right\}$$

$$A_{0.8} > = \left\{ \begin{array}{l} (Belgique, 0.9), \\ (France, 1), \\ (Luxembourg, 0.9), \\ (Sude, 0.9) \end{array} \right\}$$

1-4 Cartesian product on fuzzy subset

Definition 1.4.1: [20] The cartesian product applied to n fuzzy subsets can be defined as follows: Let $\mu_{A_1}, \mu_{A_2}, ..., \mu_{A_n}$ be membership functions of $A_1, A_2, ..., A_n$. Then, the membership degree of $(x_1, x_2, ..., x_n) \in X_1 \times X_2 \times ... \times X_n$ on the fuzzy subset $A_1, A_2, ..., A_n$ is,

$$\mu_{A_1 \times A_2 \times ... \times A_n}(x_1, x_2, ..., x_n) = \min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_n}(x_n)\}.$$

Example 1.4.1

Let $X_1 = \{\alpha, \beta\}$, $X_2 = \{a, b, c\}$ and let A, B be two fuzzy subsets defined on X_1, X_2 respectively given by:

$$\begin{aligned} A &= \{\langle \alpha, 0.2\rangle, \langle \beta, 0.7\rangle\}, \\ B &= \{\langle a, 0.5\rangle, \langle b, 0.9\rangle, \langle c, 1.0\rangle\}. \end{aligned}$$

Then we have:

$$A \times B = \left\{ \begin{array}{c} \langle(\alpha, a), 0.2\rangle, \langle(\alpha, b), 0.2\rangle, \langle(\alpha, c), 0.2\rangle, \\ \langle(\beta, a), 0.5\rangle, \langle(\beta, b), 0.7\rangle, \langle(\beta, c), 0.7\rangle \end{array} \right\}$$

1-5 Fuzzy relations and composition

1-5-1 Fuzzy relation

Definition 1.5.1: [26] Let X and Y be two non-empty sets. A fuzzy relation on $X \times Y$, denoted by R, is defined as the fuzzy subset

$$R = \{((x, y), \mu_R(x, y)), (x, y) \in X \times Y\},$$

Where the function $\mu_R : X \times Y \to [0, 1]$ is called a membership function. It gives the degree of membership of the ordered pair (x, y) in R associating with each pair (x, y) in $X \times Y$ a real number in interval $[0, 1]$.

Remark 1.5.1

A fuzzy relation R is a mapping from cartesian space $X \times Y$ to the interval $[0, 1]$. If $X = Y$ we say that R is a binary fuzzy relation in X.

Example 1.5.1

Let $X = \{x, y, z\}$ and R a binary fuzzy relation defined in X as:

μ_R	x	y	z
x	0.3	0.7	0.2
y	0.5	0.8	0.5
z	0.1	0.4	0.5

1-5-2 The α-cut sets and support of fuzzy relations

Definition 2.5.1: The α-cut sets and support of fuzzy relations is defined as in fuzzy subsets, i.e., **The α-cut** of a fuzzy relation R: $X \times Y \to [0,1]$ is defined as, for all $x \in X$ and $y \in Y$,

$$R_\alpha = \{(x,y) \in X \times Y \mid R(x,y) \geq \alpha\}.$$

In the same way, we define **the support** of a fuzzy relation $S(R)$ as,

$$S(R) = \{(x,y) \in X \times Y \mid R(x,y) > 0\}.$$

Example 2.5.1

Let $X = \{1,2,3\}$ and $Y = \{a,b,c\}$. We define the fuzzy relation R as follows:

$R(x,y)$	a	b	c
1	0.8	0.0	0.6
2	0.2	0.9	0.0
3	0.7	0.3	0.5

1. α-cut sets (R_α):
   For $\alpha = 0.5$, we want to find the elements in $X \times Y$ for which $R(x,y) \geq 0.5$.
   Then,
   $$R_{0.5} = \{(1,a),(1,c),(2,b),(3,a),(3,c)\}.$$
2. Support of a fuzzy relation $(S(R))$:
   The support of a fuzzy relation $S(R)$ consists of the elements (x,y) in $X \times Y$ for which $R(x,y)$ is greater than 0.
   Then,
   $$S(R) = \{(1,a),(1,c),(2,a),(2,b),(3,a),(3,b),(3,c)\}.$$

1-5-3 Composition of fuzzy relations

Definition 3.5.1: [20] Let R and S two fuzzy relations are defined on sets X, Y and Z. That is, $R \subseteq X \times Y$, $S \subseteq Y \times Z$. The composition $S \circ R$ of two relations R and S is expressed by the relation from X to Z and this composition is defined by the following.
For $(x, y) \in X \times Y$, $(y, z) \in Y \times Z$,

$$\begin{aligned} \mu_{S \circ R}(x, y) &= \max_y [\min\{\mu_R(x, y), \mu_S(x, y)\}] \\ &= \vee_y [\mu_R(x, y) \wedge \mu_S(x, y)]. \end{aligned}$$

Example 3.5.1

Let $X = \{1, 2, 3, 4\}$, $Y = \{a, b\}$ and $Z = \{\alpha, \beta\}$. Consider fuzzy relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, defined by its following tables:

R	a	b
1	0.1	0.3
2	1.0	0.4
3	0.6	0.9
4	0.0	0.7

S	α	β
a	0.2	0.4
b	0.9	0.5

The composition of two relations R and S is as follows:

$R \circ S$	α	β
1	0.3	0.3
2	0.4	0.4
3	0.9	0.5
4	0.7	0.5

1-6 Types of fuzzy relations

Definition 1.6.1: [20] Let R be a fuzzy relation and X, Y two non-empty sets we will say that R is:

(i) **Reflexive** if for all $x \in X$,

$$\mu_R(x,x) = 1.$$

(ii) **Symmetric** if for all $(x,y) \in X \times Y$,

$$\mu_R(x,y) = \mu_R(y,x).$$

(iii) **Antisymmetric** if for all $(x,y) \in X \times Y$,

$$\left\{\begin{array}{c} \mu_R(x,y) \\ \text{and} \\ \mu_R(y,x) \end{array}\right. \Longrightarrow x = y.$$

(iv) **Transitive** if for all $x, y \in X$,

$$R \circ R \subseteq R, i.e., \mu_{R \circ R}(x,y) \leq \mu_R(x,y).$$

The fuzzy reflexivity, symmetry, antisymmetry and transitivity notion were first defined by Zedeh [26].

Example 1.6.1

Let's consider a fuzzy relation R defined on sets X and Y.
Here's an example to illustrate the properties of fuzzy reflexivity, symmetry, antisymmetry, and transitivity:
Let $X = \{1,2,3\}$ and $Y = \{a,b,c\}$. We define the fuzzy relation R as follows:

$R(x,y)$	a	b	c
1	0.8	0.4	0.6
2	0.2	0.9	0.1
3	0.7	0.3	0.5

(i) **Reflexive:**
To check for reflexivity, we need to verify if $\mu_R(x,x) = 1$ for all x in X.

$$\mu_R(1,1) = 1,\ \mu_R(2,2) = 1,\ \mu_R(3,3) = 1$$

Since the membership values of $R(x,x)$ for all x in X are equal to 1, the fuzzy relation R is reflexive.

(ii) **Symmetric:**

To check for symmetry, we need to verify if $\mu_R(x, y) = \mu_R(y, x)$ for all (x, y) in $X \times Y$.

$$\begin{aligned} \mu_R(1, a) &= 0.8,\ \mu_R(a, 1) = 0.8 \\ \mu_R(2, b) &= 0.9,\ \mu_R(b, 2) = 0.9 \\ \mu_R(3, c) &= 0.5,\ \mu_R(c, 3) = 0.5 \end{aligned}$$

Since the membership values of R(x, y) are equal to the membership values of R(y, x) for all (x, y) in $X \times Y$, the fuzzy relation R is symmetric.

(iii) **Antisymmetric:**

To check for antisymmetry, we need to verify if whenever $\mu_R(x, y) > 0$ and $\mu_R(y, x) > 0$, it implies that $x = y$.

$$\begin{aligned} \mu_R(1, a) &= 0.8,\ \mu_R(a, 1) = 0.8 \\ \mu_R(2, b) &= 0.9,\ \mu_R(b, 2) = 0.9 \\ \mu_R(3, c) &= 0.5,\ \mu_R(c, 3) = 0.5 \end{aligned}$$

In all the cases where $\mu_R(x, y) > 0$ and $\mu_R(y, x) > 0$, x is equal to y. Hence, the fuzzy relation R is antisymmetric.

(iv) **Transitive:**

To check for transitivity, we need to verify if $\mu_{R\circ R}(x, y) \leq \mu_R(x, y)$ for all x and $y \in X$.

$R \circ R$ denotes the composition of R with itself.

$$\begin{aligned} \mu_{R\circ R}(1, c) &= \max[\min(R(1, a), R(a, c)),\ \min(R(1, b), R(b, c))] \\ &= \max[\min(0.8, 0.5),\ \min(0.4, 0.1)] \\ &= \max(0.5, 0.1) = 0.5 \end{aligned}$$

$\mu_R(1, c) = 0.6$

$$\begin{aligned} \mu_{R\circ R}(2, b) &= \max[\min(R(3, a), R(a, a)),\ \min(R(3, b), R(b, a))] \\ &= \max[\min(0.7, 0.8),\ \min(0.3, 0.2)] \\ &= \max(0.7, 0.2) = 0.9 \end{aligned}$$

$\mu_R(2, b) = 0.9$

$$\begin{aligned} \mu_{R\circ R}(3, a) &= \max[\min(R(1, a), R(a, c)),\ \min(R(1, b), R(b, c))] \\ &= \max[\min(0.8, 0.5),\ \min(0.4, 0.1)] \\ &= \max(0.5, 0.1) = 0.5 \end{aligned}$$

$\mu_R(3,a) = 0.7$

In all the cases where $\mu_{R\circ R}(x,y)$ is not greater than $\mu_R(x,y)$, the fuzzy relation R is transitive.

Then, the given fuzzy relation R, it is reflexive, symmetric, antisymmetric, and transitive.

1-6-1 Fuzzy equivalence relation

Definition 2.6.1: Let X be a non-empty set.
A binary fuzzy relation $R : X \times X \longrightarrow [0,1]$ is called **a fuzzy equivalence relation** in X if R is reflexive, transitive and symmetric.

Example 2.6.1

Let $X = \{1,2,3\}$. We define the fuzzy relation R as follows:

$R(x,y)$	1	2	3
1	1	0.1	0.3
2	0.1	1	0.9
3	0.3	0.9	1

1. **Reflexivity:**

   For a fuzzy relation to be reflexive, we need to verify if $\mu_R(x,x) = 1$ for all x in X.

   $$\mu_R(1,1) = 1,\ \mu_R(2,2) = 1,\ \mu_R(3,3) = 1$$

   Since the membership values of $R(x,x)$ for all x in X are equal to 1, the fuzzy relation R is reflexive.

2. **Symmetry:**

   For a fuzzy relation to be symmetric, we need to verify if $\mu_R(x,y) = \mu_R(y,x)$ for all (x,y) in $X \times X$.

   $$\begin{aligned}\mu_R(1,2) &= 0.8,\ \mu_R(2,1) = 0.8\\ \mu_R(1,3) &= 0.9,\ \mu_R(3,1) = 0.9\\ \mu_R(2,3) &= 0.5,\ \mu_R(3,2) = 0.5\end{aligned}$$

   Since the membership values of $R(x,y)$ are equal to the membership values of $R(y,x)$ for all (x,y) in $X \times X$, the fuzzy relation R is symmetric.

3. **Transitivity:**

   For a fuzzy relation to be transitive, we need to verify if $\mu_{R\circ R}(x,y) \leq \mu_R(x,y)$ for all x and y in X.

$$\begin{aligned}\mu_{R\circ R}(1,3) &= \max[\min(R(1,1),R(1,3)),\ \min(R(1,2),R(2,3))]\\ &= \max[\min(1,0.3),\ \min(0.1,0.9)]\\ &= \max(0.3,0.1)=0.3\end{aligned}$$

$\mu_R(1,3)=0.3$

$$\begin{aligned}\mu_{R\circ R}(2,3) &= \max[\min(R(2,1),R(1,3)),\ \min(R(2,2),R(2,3))]\\ &= \max[\min(0.1,0.3),\ \min(1,0.9)]\\ &= \max(0.1,0.9)=0.9\end{aligned}$$

$\mu_R(2,3)=0.9$

$$\begin{aligned}\mu_{R\circ R}(3,1) &= \max[\min(R(3,1),R(1,1)),\ \min(R(3,2),R(2,1))]\\ &= \max[\min(0.3,1),\ \min(0.9,0.1)]\\ &= \max(0.3,0.1)=0.3\end{aligned}$$

$\mu_R(3,1)=0.3$

   In all the cases where $\mu_{R\circ R}(x,y)$ is not greater than $\mu_R(x,y)$, the fuzzy relation R is transitive.

Then, the given fuzzy relation R, it is reflexive, symmetric, and transitive. Therefore, R is a fuzzy equivalence relation on set X.

Definition 3.6.1: [16] Let X be a non-empty set.
A binary fuzzy relation $R : X \times X \longrightarrow [0,1]$ is called :

(i) **Fuzzy pre-order relation** in X if it is a fuzzy reflexive and transitive relation.

(ii) **Fuzzy partial order relation** if R is fuzzy reflexive, antisymmetric and transitive.

(iii) **Fuzzy total order relation** if and only if either $R(x,y) > 0$ or $R(y,x) > 0$ for all $x, y \in X$.

If R is a fuzzy partial order relation on a set X, then (X,R) is called a **fuzzy partially ordered subset** or **fuzzy poset**. If R is a fuzzy total order relation in a set X , then (X,R) is called a **fuzzy totally ordered set** or a **fuzzy chain**.

1-7 Triangular norms and conorms

In this section, we will remind some basic definitions and notions concerning triangular norms and co-norms

1-7-1 Triangular norms

Definition 1.7.1: [19] **A triangular norm** (t-norm for short) is a binary operation T on the unit interval $[0,1]$, i.e., it is a function $T : [0,1]^2 \longrightarrow [0,1]$ such that for all $x, y, z \in [0,1]$ the following four axioms are satisfied :

(t_1) Commutativity i.e., $T(x,y) = T(y,x)$;

(t_2) Associativity i.e., $T(x,T(y,z)) = T(T(x,y),z)$;

(t_3) Monotonicity i.e., $T(x,y) \le T(x,z)$ whenever $y \le z$;

(t_4) Boundary condition i.e., $T(x,1) = x$.

Proposition 1.7.1. [8] Any t-norm T satisfies $T(0,x) = T(x,0) = 0$, for all $x \in [0,1]$.

Proof. From (t_4) we have $T(0,1) = 0$. Then from (t_3) it follows that:

$$T(0,x) \le T(0,1) = 0, \text{ for all } x \in [0,1],$$

i.e., $T(0,x) = 0$. Then from (t_1) we get $T(x,0) = 0$.

□

Example 1.7.1

The following are the four t-norms T_M,T_p,T_D and T_L given respectively by :

$T_M(x,y) = \min(x,y)$, (minimum)

$T_p(x,y) = x \cdot y$, (product)

$T_L(x,y) = \max(x+y-1,0)$, (Łukasiewicz t-norm)

$T_D(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1[^2, \\ \min(x,y) & \text{Otherwise.} \end{cases}$ (drastic product)

As a formal construction, triangular conorms are dual to triangular norms.

1-7-2 Triangular conorms

Definition 2.7.1: [19] A triangular conorm (t-conorm for short) is a binary operation S on the unit interval $[0,1]$, i.e., it is a function $S : [0,1]^2 \longrightarrow [0,1]$ such that for all $x,y,z \in [0,1]$ the following four axioms are satisfied :

(s_1) Commutativity : $S(x,y) = S(y,x)$;

(s_2) Associativity : $S(x,S(y,z)) = S(S(x,y),z)$;

(s_3) Monotonicity : $S(x,y) \leq S(x,z)$ whenever $y \leq z$;

(s_4) Boundary condition : $S(x,0) = x$.

Proposition 2.7.1. [8] Any t-conorm S satisfies $S(1,x) = S(x,1) = 1$, for all $x \in [0,1]$.

Proof. From (s_4) we have $S(1,0) = 1$ so, from (s_3) it follows that:

$$1 = S(1,0) \leq S(1,x),$$

Then,

$$S(1,x) = 1 = S(x,1), \text{ forall } x \in [0,1].$$

□

Example 2.7.1

The following are the four basic t-conorms S_M, S_p, S_L, and S_D given respectively by:

$S_M(x,y) = \max(x,y),$ (maximum)

$S_p(x,y) = x + y - x \cdot y,$ (probabilistic sum)

$S_L(x,y) = \min(x+y,1),$ (Łukasiewicz t-conorm, bounded sum)

$S_D(x,y) = \begin{cases} 1 & \text{if } (x,y) \in [0,1[^2, \\ \max(x,y) & \text{Otherwise.} \end{cases}$ (drastic sum)

Proposition 3.7.1. [8] We have $T(x,y) \leq x \wedge y$ and $S(x,y) \geq x \vee y$ for any t-norm T, t-conorm S, and any $x,y \in [0,1]$.

Proposition 4.7.1. [12] Let T be a triangular norm, the mapping defines as:

$$S : [0,1] \times [0,1] \longrightarrow [0,1]$$
$$S(x,y) \equiv 1 - T(1-x, 1-y)$$

Will be called dual t-conorm of T.

Remark 1.7.1

All t-norm is an intersection operator i.e., we can define $A \cap_T B$ by its membership function in the following way :

$$\mu_{A\cap_T B}(x) = T(\mu_A(x), \mu_B(x)), \text{ for all } x \text{ in } X.$$

All t-conorm is a union operator i.e., we can define $A \cup_S B$ by its membership function as follow:

$$\mu_{A\cup_S B}(x) = S(\mu_A(x), \mu_B(x)), \text{ for all } x \text{ in } X.$$

Example 3.7.1

Let $X = \{x, y, z\}$, let A and B be two fuzzy subsets on X given by:
$A = \{\langle x, 0.5\rangle, \langle y, 0.9\rangle, \langle z, 1.0\rangle\}$,
$B = \{\langle x, 0.2\rangle, \langle y, 0.4\rangle, \langle z, 0.1\rangle\}$.
We define the intersection and union respectively :

(i) $\mu_{A\cap_T B}(x) = T(\mu_A(x), \mu_B(x)) = \max(\mu_A(x) + \mu_B(x) - 1, 0),$ for all x in X,

(ii) $\mu_{A\cup_S B}(x) = S(\mu_A(x), \mu_B(x)) = \min(\mu_A(x) + \mu_B(x), 1),$ for all x in X.

So, we get

(i) $A \cap_T B = \{\langle x, 0.0\rangle, \langle y, 0.3\rangle, \langle z, 0.1\rangle\}$,

(ii) $A \cup_S B = \{\langle x, 0.7\rangle, \langle y, 1.0\rangle, \langle z, 1.0\rangle\}$.

1-7-3 Negation fonction

Definition 3.7.1: *A* function $N : [0,1] \to [0,1]$ is called **negation** if :

(i) N is non-increasing i.e., $x \leq y \Rightarrow N(y) \leq N(x)$, for all x, y in [0, 1],

(ii) $N(0) = 1$ and $N(1) = 0$.

The negation $N : [0,1] \to [0,1]$ is called **strict negation** if :

(i) N is a continuos function,

(ii) N is a strictly decreasing function i.e., $x < y \Longrightarrow N(x) > N(y)$.

The strict negation $N : [0,1] \to [0,1]$ is called **strong negation** if is a involution that is,

(i) $N(N(x)) = x$ for x in $[0,1]$.

Example 4.7.1

(i) **Negation function:**
Let's define the negation function $N(x) = 1 - x$ for x in the interval $[0,1]$.
Properties:

1. N is non-increasing: If $x \leq y$, then $1 - y \leq 1 - x$, which implies $N(y) \leq N(x)$.
2. $N(0) = 1 - 0 = 1$, and $N(1) = 1 - 1 = 0$.

Therefore, the function $N(x) = 1 - x$ satisfies the properties of a negation function.

(ii) **Strict negation function:**
Let's define the strict negation function $N(x) = \sqrt{(1 - x^2)}$ for x in the interval $[0,1]$.
Properties:

1. N is a continuous function.
2. N is strictly decreasing: If $x < y$, then $N(x) > N(y)$, as $N(x) = \sqrt{(1 - x^2)}$ and $N(y) = \sqrt{(1 - y^2)}$. Since $x < y$, it follows that $x^2 > y^2$, and hence $1 - x^2 < 1 - y^2$. Taking the square root, $\sqrt{(1 - x^2)} > \sqrt{(1 - y^2)}$, which implies $N(x) > N(y)$.

Therefore, the function $N(x) = \sqrt{(1 - x^2)}$ satisfies the properties of a strict negation function.

(iii) **Strong negation function:**
Let's define the strong negation function $N(x) = 1 - x$ for x in the interval $[0,1]$. Property:

1. $N(N(x)) = N(1 - x) = 1 - (1 - x) = x$.

Therefore, the function $N(x) = 1 - x$ satisfies the property of a strong negation function.

Definition 4.7.1: A t-norm T and a t-conorm S are said to be dual for strict negation if they satisfy the following formulas for all $x, y \in [0, 1]$:

$$\begin{aligned} S(x,y) &= N(T(N(x),N(y))). \\ T(x,y) &= N(S(N(x),N(y))). \end{aligned}$$

Example 5.7.1

Let's define the t-norm T and t-conorm S as follows:

$T(x,y) = \min(x,y)$ for x, y in the interval $[0,1]$.

$S(x,y) = \max(x,y)$ for x, y in the interval $[0,1]$.

$N(x) = 1 - x$ for x in the interval $[0,1]$.

We can verify that T and S are dual for strict negation using the given formulas:

$S(x,y) = N(T(N(x),N(y))) = N(T(1-x,1-y)) = N(\min(1-x,1-y)) = N(1-\max(x,y)) = 1-(1-\max(x,y)) = \max(x,y)$

$T(x,y) = N(S(N(x),N(y))) = N(S(1-x,1-y)) = N(\max(1-x,1-y)) = N(1-\min(x,y)) = 1-(1-\min(x,y)) = \min(x,y)$

Therefore, for these definitions of T, S, and N, we have $S(x,y) = N(T(N(x),N(y)))$ and $T(x,y) = N(S(N(x),N(y)))$, satisfying the formulas for strict negation duality.

Chapter 2

Generalities on intuitionistic fuzzy subsets

Contenu of chapter :

This chapter has been prepared based on the following references [5, 6, 8, 24]

2-1 Intuitionistic fuzzy subsets

Introduction

The concept of intuitionistic fuzzy subsets has found applications in various fields where uncertainty and imprecision are prevalent, such as decision-making, pattern recognition, expert systems, and artificial intelligence. The inclusion of a degree of hesitation in intuitionistic fuzzy subsets enables a more realistic representation of human cognitive processes, as humans often express ambiguity or doubt in their decision-making.

Definition 1.1.2: [5,6] Let X be a non-empty set. **An intuitionistic fuzzy subset** (IFS, for short) A on X is an object of the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$$

Where the functions:

$$\mu_A : X \longrightarrow [0,1]$$

And

$$\nu_A : X \longrightarrow [0,1]$$

Define the degree of membership and the degree of non-membership, of the element $x \in X$, respectively, and for every $x \in X$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

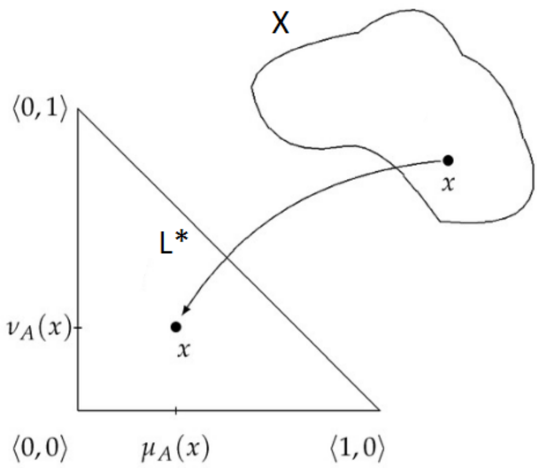

Figure2.1: Representation geometric of IFS

Definition 2.1.2: [5] The value of $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, is called the degree of non-determinacy (or uncertainty) of the element x to the intuitionistic fuzzy subset A. Clearly, in the case of ordinary fuzzy subsets, $\pi_A(x) = 0$, for every $x \in X$.

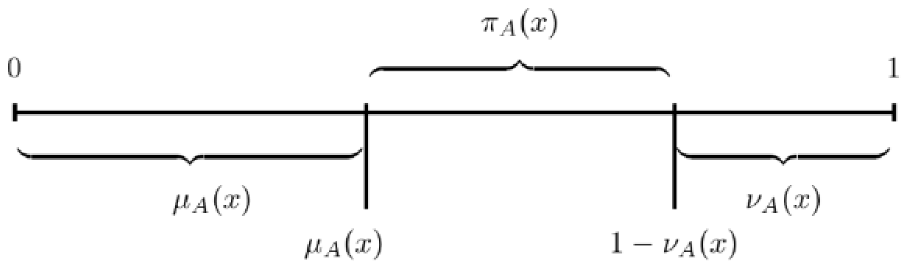

Figure2.2: The situation in the IFS case is different

Remark 1.1.2

Now, we can map this intuitionistic fuzzy subset to an L-fuzzy subset by defining the function $A : U \to L$ as follows:

$$A(x) = (\mu_A(x), \nu_A(x)),$$

For each x in the universe of discourse U.

Example 1.1.2

[6] Let X be the set of all countries with elective governments. Assume that we know for every country $x \in X$ the percentage of the electorate that has voted for the corresponding government. Denote it by $M(x)$ and let $\mu(x) = \frac{M(x)}{100}$ (degree of membership, validity, etc.).
Let $\nu(x) = 1 - \mu(x)$.
This number corresponds to the part of the electorate who have not voted for the government. Using only the fuzzy subset theory, we cannot consider this value in more detail. However, if we define $\nu(x)$ (degree of non-membership, non-validity, etc.) as the number of votes given to parties or persons outside the government, then we can show the part of electorate who have not voted at all or who have given bad voting-paper and the corresponding number will be $\pi(x) = 1 - \mu(x) - \nu(x)$ (degree of indeterminacy, uncertainty, etc.). Thus, we can construct the set $\{\langle x, \mu(x), \nu(x)\rangle \mid x \in X\}$ and obviously,

$$0 \le \mu(x) + \nu(x) \le 1.$$

Example 2.1.2

Let $X = \{a, b\}$, $I = \{0, \frac{1}{2}, 1\} \subset [0, 1]$ we will find the intuitionistic fuzzy subsets of X.
For all $x, y \in I$ we define the set:

$$\tilde{\wp} = \{\wp : X \longrightarrow I^2 \mid x + y \le 1\}$$

The set of all intuitionistic fuzzy subsets of X.
So, $\mid \tilde{\wp} \mid = (\mid I_2 \mid)^{|X|} - \mid \{(x, y) \mid x + y > 1\} \mid$. Consequently we need to find **36** intuitionistic

fuzzy subsets.

$A_1 = \{\langle a,0,0\rangle, \langle b,0,0\rangle\}$
$A_2 = \{\langle a,0,0\rangle, \langle b,0,\frac{1}{2}\rangle\}$
$A_3 = \{\langle a,0,0\rangle, \langle b,0,1\rangle\}$
$A_4 = \{\langle a,0,0\rangle, \langle b,\frac{1}{2},0\rangle\}$
$A_5 = \{\langle a,0,0\rangle, \langle b,\frac{1}{2},\frac{1}{2}\rangle\}$
$A_6 = \{\langle a,0,0\rangle, \langle b,1,0\rangle\}$

$A_7 = \{\langle a,0,\frac{1}{2}\rangle, \langle b,0,0\rangle\}$
$A_8 = \{\langle a,0,\frac{1}{2}\rangle, \langle b,0,\frac{1}{2}\rangle\}$
$A_9 = \{\langle a,0,\frac{1}{2}\rangle, \langle b,0,1\rangle\}$
$A_{10} = \{\langle a,0,\frac{1}{2}\rangle, \langle b,\frac{1}{2},0\rangle\}$
$A_{11} = \{\langle a,0,\frac{1}{2}\rangle, \langle b,\frac{1}{2},\frac{1}{2}\rangle\}$
$A_{12} = \{\langle a,0,\frac{1}{2}\rangle, \langle b,1,0\rangle\}$

$A_{13} = \{\langle a,\frac{1}{2},\frac{1}{2}\rangle, \langle b,0,0\rangle\}$
$A_{14} = \{\langle a,\frac{1}{2},\frac{1}{2}\rangle, \langle b,0,\frac{1}{2}\rangle\}$
$A_{15} = \{\langle a,\frac{1}{2},\frac{1}{2}\rangle, \langle b,0,1\rangle\}$
$A_{16} = \{\langle a,\frac{1}{2},\frac{1}{2}\rangle, \langle b,\frac{1}{2},0\rangle\}$
$A_{17} = \{\langle a,\frac{1}{2},\frac{1}{2}\rangle, \langle b,\frac{1}{2},\frac{1}{2}\rangle\}$
$A_{18} = \{\langle a,\frac{1}{2},\frac{1}{2}\rangle, \langle b,1,0\rangle\}$

$A_{19} = \{\langle a,0,1\rangle, \langle b,0,0\rangle\}$
$A_{20} = \{\langle a,0,1\rangle, \langle b,0,\frac{1}{2}\rangle\}$
$A_{21} = \{\langle a,0,1\rangle, \langle b,0,1\rangle\}$
$A_{22} = \{\langle a,0,1\rangle, \langle b,\frac{1}{2},0\rangle\}$
$A_{23} = \{\langle a,0,1\rangle, \langle b,\frac{1}{2},\frac{1}{2}\rangle\}$
$A_{24} = \{\langle a,0,1\rangle, \langle b,1,0\rangle\}$

$A_{25} = \{\langle a,\frac{1}{2},0\rangle, \langle b,0,0\rangle\}$
$A_{26} = \{\langle a,\frac{1}{2},0\rangle, \langle b,0,\frac{1}{2}\rangle\}$
$A_{27} = \{\langle a,\frac{1}{2},0\rangle, \langle b,0,1\rangle\}$
$A_{28} = \{\langle a,\frac{1}{2},0\rangle, \langle b,\frac{1}{2},0\rangle\}$
$A_{29} = \{\langle a,\frac{1}{2},0\rangle, \langle b,\frac{1}{2},\frac{1}{2}\rangle\}$
$A_{30} = \{\langle a,\frac{1}{2},0\rangle, \langle b,1,0\rangle\}$

$A_{31} = \{\langle a,1,0\rangle, \langle b,0,0\rangle\}$
$A_{32} = \{\langle a,1,0\rangle, \langle b,0,\frac{1}{2}\rangle\}$
$A_{33} = \{\langle a,1,0\rangle, \langle b,0,1\rangle\}$
$A_{34} = \{\langle a,1,0\rangle, \langle b,\frac{1}{2},0\rangle\}$
$A_{35} = \{\langle a,1,0\rangle, \langle b,\frac{1}{2},\frac{1}{2}\rangle\}$
$A_{36} = \{\langle a,1,0\rangle, \langle b,1,0\rangle\}$

Definition 3.1.2: [8] Intuitionistic fuzzy subsets can be seen as L-fuzzy subsets by considering the lattice $L \subseteq [0,1]^2$,

$$L = \{(x,y) \in [0,1]^2 \mid x + y \leq 1\},$$

where the inequality relation generating the lattice structure is defined by:

$$(x,y) \leq_L (z,t) \Leftrightarrow [x \leq z \text{ and } y \geq t].$$

Moreover, L in this case is a complete, completely distributive lattice.

Example 3.1.2

Let's say that the intuitionistic fuzzy subset (μ_A, ν_A) we defined earlier has the following membership and non-membership degrees:

$$\begin{aligned} \mu_A(1.60) &= 0.1, \mu_A(1.85) = 0.8, \mu_A(2.10) = 1.0, \\ \nu_A(1.60) &= 0.9, \nu_A(1.85) = 0.1, \nu_A(2.10) = 0.0. \end{aligned}$$

Then,

$$\begin{aligned} A(1.60) &= (0.1, 0.9) \\ A(1.85) &= (0.8, 0.1) \\ A(2.10) &= (1.0, 0.0). \end{aligned}$$

In this example, the first component of each pair represents the degree to which the height x can be considered tall, and the second component represents the degree to which it cannot be considered non-tall.

2-2 Operations on intuitionistic fuzzy subsets

For two intuitionistic fuzzy subsets A and B on a set X, several operations are defined in the following way(see [5, 6]) :

• **Inclusion**

$$A \subset B \text{ if } \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x), \text{ for any } x \in X.$$

• **Complement**

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}.$$

• **Equality**

$$A = B \text{ if } \mu_A(x) = \mu_B(x) \text{ and } \nu_A(x) = \nu_B(x), \text{ for any } x \in X.$$

• **Complement**

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}.$$

• **Empty set**

$$A = \emptyset \text{ if } \mu_A(x) = 0 \text{ and } \nu_A(x) = 1.$$

- **Intersection**

$$A \cap B = \{\langle x,\ \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\}.$$

- **Union**

$$A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}.$$

- **Addition**

$$A \oplus B = \{x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x), \nu_A(x) \cdot \nu_B(x) \mid x \in X\}.$$

- **Multiplication**

$$A \otimes B = \{x, \mu_A(x) \cdot \mu_B(x), \mu_A(x) + \mu_B(x) \nu_A(x) \cdot \nu_B(x) \mid x \in X\}.$$

Example 1.2.2

Let $X = \{a, b, c\}$ and Let A, B be two intuitionistic fuzzy subsets given by:

$$A = \{\langle a, 0.3, 0.5\rangle, \langle b, 0.6, 0.4\rangle, \langle c, 0.9, 0.0\rangle\},$$
$$B = \{\langle a, 0.7, 0.1\rangle, \langle b, 0.8, 0.1\rangle, \langle c, 1.0, 0.0\rangle\}.$$

Then, we have:

$$\begin{aligned}
A \cap B &= \{\langle a, 0.3, 0.5\rangle, \langle b, 0.6, 0.4\rangle, \langle c, 0.9, 0.0\rangle\},\\
A \cup B &= \{\langle a, 0.7, 0.1\rangle, \langle b, 0.8, 0.1\rangle, \langle c, 1.0, 0.0\rangle\},\\
A^c &= \{\langle a, 0.5, 0.3\rangle, \langle b, 0.4, 0.6\rangle, \langle c, 0.0, 0.9\rangle\},\\
B^c &= \{\langle a, 0.1, 0.7\rangle, \langle b, 0.1, 0.8\rangle, \langle c, 0.0, 1.0\rangle\},\\
A \oplus B &= \{\langle a, 0.79, 0.05\rangle, \langle b, 0.92, 0.04\rangle, \langle c, 1.0, 0.0\rangle\},\\
A \otimes B &= \{\langle a, 0.21, 0.55\rangle, \langle b, 0.48, 0.46\rangle, \langle c, 0.9, 0.0\rangle\}.
\end{aligned}$$

2-2-1 The support

Definition 1.2.2: [24] Let A be an intuitionistic fuzzy subset on universe X. **The support** of A is the crisp subset of X given by:

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0 \text{ or } (\mu_A(x) = 0 \text{ and } \nu_A(x) < 1)\}.$$

2-2-2 The kernel

Definition 2.2.2: [24] Let A be an intuitionistic fuzzy subset on universe X. **The kernel** of A is the crisp subset of X given by:

$$\ker(A) = \{x \in X \mid \mu_A(x) = 1 \text{ and } \nu_A(x) = 0\}.$$

Example 2.2.2

From **Example** 2-2 we have :

$\mathrm{Ker}(A) = \emptyset$,

$\mathrm{Supp}(A) = \{a, b, c\}$.

2-3 Cartesian product on intuitionistic fuzzy subset

The cartesian product applied to n intuitonistic fuzzy subsets can be defined as follows:

Definition 1.3.2: Let $\mu_{A1}, \mu_{A2}, ..., \mu_{An}$ be membership functions of $A_1, A_2, ..., A_n$. Then the membership degree of $(x_1, x_2, ..., x_n) \in X_1 \times X_2 \times ... \times X_n$ on the intuitonistic fuzzy subset $A_1, A_2, ..., A_n$ is,

$$\begin{aligned} \mu_{A_1 \times A_2 \times ... \times A_n}(x_1, x_2, ..., x_n) &= \min\left\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_n}(x_n)\right\} \\ &= \mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2) \wedge ... \wedge \mu_{A_n}(x_n), \end{aligned}$$

And the non-membership degree is,

$$\begin{aligned} \nu_{A_1 \times A_2 \times ... \times A_n}(x_1, x_2, ..., x_n) &= \max\left\{\nu_{A_1}(x_1), \nu_{A_2}(x_2), ..., \nu_{A_n}(x_n)\right\} \\ &= \nu_{A_1}(x_1) \vee \nu_{A_2}(x_2) \vee ... \vee \nu_{A_n}(x_n). \end{aligned}$$

Example 1.3.2

Let's consider two intuitionistic fuzzy subsets A and B defined on the universe $X = \{1, 2, 3\}$, with membership and non-membership functions given by:

$$\begin{aligned} \mu_A(1) &= 0.8,\ \mu_A(2) = 0.5,\ \mu_A(3) = 0.2 \\ \nu_A(1) &= 0.1,\ \nu_A(2) = 0.3,\ \nu_A(3) = 0.6 \end{aligned}$$

$$\begin{aligned} \mu_B(1) &= 0.6,\ \mu_B(2) = 0.4,\ \mu_B(3) = 0.3 \\ \nu_B(1) &= 0.2,\ \nu_B(2) = 0.5,\ \nu_B(3) = 0.7 \end{aligned}$$

The Cartesian product of A and B, denoted by $A \times B$, is a new intuitionistic fuzzy subset defined on the universe

$$X \times X = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

The membership and non-membership functions of $A \times B$ are given by:

$$\begin{aligned}
\mu_{A\times B}(1,1) &= \min\{0.7, 0.6\} = 0.6, \ \nu_{A\times B}(1,1) = \max\{0.1, 0.2\} = 0.2\\
\mu_{A\times B}(1,2) &= \min\{0.7, 0.4\} = 0.4, \ \nu_{A\times B}(1,2) = \max\{0.1, 0.5\} = 0.5\\
\mu_{A\times B}(1,3) &= \min\{0.7, 0.3\} = 0.3, \ \nu_{A\times B}(1,3) = \max\{0.1, 0.7\} = 0.7\\
\mu_{A\times B}(2,1) &= \min\{0.5, 0.6\} = 0.5, \ \nu_{A\times B}(2,1) = \max\{0.3, 0.2\} = 0.3\\
\mu_{A\times B}(2,2) &= \min\{0.5, 0.4\} = 0.4, \ \nu_{A\times B}(2,2) = \max\{0.3, 0.5\} = 0.5\\
\mu_{A\times B}(2,3) &= \min\{0.5, 0.3\} = 0.3, \ \nu_{A\times B}(2,3) = \max\{0.3, 0.7\} = 0.7\\
\mu_{A\times B}(3,1) &= \min\{0.2, 0.6\} = 0.2, \ \nu_{A\times B}(3,1) = \max\{0.6, 0.2\} = 0.6\\
\mu_{A\times B}(3,2) &= \min\{0.2, 0.4\} = 0.2, \ \nu_{A\times B}(3,2) = \max\{0.6, 0.5\} = 0.6\\
\mu_{A\times B}(3,3) &= \min\{0.2, 0.3\} = 0.2, \ \nu_{A\times B}(3,3) = \max\{0.6, 0.7\} = 0.7
\end{aligned}$$

Then, we have:

$$A \times B = \left\{ \begin{array}{c} \langle (1,1), 0.6, 0.2\rangle, \langle (1,2), 0.4, 0.5\rangle, \langle (1,3), 0.3, 0.7\rangle, \\ \langle (2,1), 0.5, 0.3\rangle, \langle (2,2), 0.4, 0.5\rangle, \langle (2,3), 0.3, 0.7\rangle, \\ \langle (3,1), 0.2, 0.6\rangle, \langle (3,2), 0.2, 0.6\rangle, \langle (3,3), 0.2, 0.7\rangle \end{array} \right\}.$$

Example 2.3.2

Let $X_1 = \{\alpha, \beta\}$, $X_2 = \{a, b, c\}$, and let A_1, A_2 be two intuitionistic fuzzy subsets defined on X_1, X_2 respectively given by:

$$\begin{aligned}
A_1 &= \{\langle \alpha, 0.2, 0.6\rangle, \langle \beta, 0.7, 0.2\rangle\};\\
A_2 &= \{\langle a, 0.5, 0.4\rangle, \langle b, 0.9, 0.1\rangle, \langle c, 1.0, 0.0\rangle\}.
\end{aligned}$$

Then, we have:

$$A_1 \times A_2 = \left\{ \begin{array}{c} \langle (\alpha, a), 0.2, 0.6\rangle, \langle (\alpha, b), 0.2, 0.6\rangle, \langle (\alpha, c), 0.2, 0.6\rangle, \\ \langle (\beta, a), 0.5, 0.4\rangle, \langle (\beta, b), 0.7, 0.2\rangle, \langle (\beta, c), 0.7, 0.2\rangle \end{array} \right\}.$$

Chapter 3

Structures on intuitionistic fuzzy relations

Contenu of chapter :

This chapter has been prepared based on the following references [3, 4, 12–15, 27]

3-1 Intuitionistic fuzzy relation

Introduction

Intuitionistic fuzzy relations are a generalization of fuzzy relations that allow for the representation of uncertainty in both membership and non-membership degrees.
Intuitionistic fuzzy relations were first introduced in a paper by Atanassov in 1986 [3] .
Since then, they have been studied extensively and have found applications in various fields, including decision-making, pattern recognition, and image processing.

This section contains the basic definitions and properties of intuitionistic fuzzy relations which introduced by Burillo and Bustince [12, 13] as a natural generalization of fuzzy relation.

Definition 1.1.3: [12, 15] Let X and Y be two non-empty sets. **An intuitionistic fuzzy relation** from X to Y (IFR, for short) is an intuitionistic fuzzy subset of $X \times Y$, i.e., is an expression R given by:

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle \mid (x, y) \in X \times Y\}$$

Where,

$$\mu_R : X \times Y \longrightarrow [0, 1]$$

And

$$\nu_R : X \times Y \longrightarrow [0, 1]$$

Satisfy the condition $0 \leq \mu_R(x, y) + \nu_R(x, y) \leq 1$, for every $(x, y) \in X \times Y$. The value $\mu_R(x, y)$ is called the degree of membership of (x, y) in R and $\nu_R(x, y)$ is called the degree of non-membership of (x, y) in R.

Notation 3-1.1. We will denote by IFR$(X \times Y)$ the set of all the intuitionistic fuzzy subsets in $X \times Y$.

Example 1.1.3

Let $X = \{x, y, z\}$ and R be intuitionistic fuzzy relation on X given by:

μ_R	x	y	z
x	0.2	0.5	0.1
y	0	0.4	0.7
z	1	0.6	0.6

ν_R	x	y	z
x	0.6	0.4	0.3
y	0.7	0.2	0.1
z	0	0.3	0.2

Remark 1.1.3

This definition of intuitionistic fuzzy relation includes, as a particular case, binary relation and fuzzy relation in a way similar intuitionistic fuzzy relation.

3-1-1 Inverse of intuitionistic fuzzy relation

Definition 2.1.3: [12] Given a binary intuitionistic fuzzy relation between X and Y, we can define R^{-1} between Y and X by means of

$$\mu_{R^{-1}}(y,x) = \mu_R(x,y)$$
$$\nu_{R^{-1}}(y,x) = \nu_R(x,y), \forall (x,y) \in X \times Y$$

To which we will call **inverse relation** of R.

Example 2.1.3

Suppose we have an intuitionistic fuzzy relation R between sets $X = \{1,2,3\}$ and $Y = \{a,b,c\}$ defined as follows:

$$R = \{\langle(1,a), 0.7, 0.3\rangle, \langle(1,b), 0.5, 0.2\rangle, \langle(2,b), 0.6, 0.4\rangle, \langle(3,c), 0.9, 0.1\rangle\},$$

The inverse relation R^{-1}:

$$\begin{aligned} \mu_{R^{-1}}(a,1) &= \mu_R(1,a) = 0.7, \ \nu_{R^{-1}}(a,1) = \nu_R(1,a) = 0.3 \\ \mu_{R^{-1}}(b,1) &= \mu_R(1,b) = 0.5, \ \nu_{R^{-1}}(b,1) = \nu_R(1,b) = 0.2 \\ \mu_{R^{-1}}(b,2) &= \mu_R(2,b) = 0.6, \ \nu_{R^{-1}}(b,2) = \nu_R(2,b) = 0.4 \\ \mu_{R^{-1}}(c,3) &= \mu_R(3,c) = 0.9, \ \nu_{R^{-1}}(c,3) = \nu_R(3,c) = 0.1 \end{aligned}$$

Then,

$$R^{-1} = \{\langle(a,1), 0.7, 0.3\rangle, \langle(b,1), 0.5, 0.2\rangle, \langle(b,2), 0.6, 0.4\rangle, \langle(c,3), 0.9, 0.1\rangle\}.$$

3-2 Operations on intuitionistic fuzzy relations

Let R and P be two intuitionistic fuzzy relations from a universe X to a universe Y, for every $(x,y) \in X \times Y$, we can define (see [12]).

1. $R \subseteq P \Leftrightarrow \mu_R(x,y) \leq \mu_P(x,y)$ and $\nu_R(x,y) \geq \nu_P(x,y)$,
2. $R \cup P = \{\langle(x,y), \mu_R(x,y) \vee \mu_P(x,y), \nu_R(x,y) \wedge \nu_P(x,y)\rangle | \ x \in X, y \in Y\}$,
3. $R \cap P = \{\langle(x,y), \mu_R(x,y) \wedge \mu_P(x,y), \nu_R(x,y) \vee \nu_P(x,y)\rangle | \ x \in X, y \in Y\}$,
4. $R_c = \{\langle(x,y), \nu_R(x,y), \mu_R(x,y)\rangle | \ x \in X, y \in Y\}$.

Proporties 3-2.1. [12] Let R, P and Q be three intuitionistic fuzzy relations from a universe X to a universe Y.

(i) If $R \subseteq P$, then $R^{-1} \subseteq P^{-1}$,

(ii) $(R \cup P)^{-1} = R^{-1} \cup P^{-1}$,

(iii) $(R \cap P)^{-1} = R^{-1} \cap P^{-1}$,

(iv) $(R^{-1})^{-1} = R$,

(v) $R \cap (P \cup Q) = (R \cap P) \cup (R \cap Q)$ and $R \cup (P \cap Q) = (R \cup P) \cap (R \cup Q)$,

(vi) $R \subseteq (R \cup P), P \subseteq (R \cup P), R \cap P \subseteq R$ and $R \cap P \subseteq P$,

(vii) If $P \subseteq R$ and $Q \subseteq R$, then $P \cup Q \subseteq R$,

(viii) If $R \subseteq P$ and $R \subseteq Q$, then $R \subseteq p \cap Q$.

Proof. (i) If $R \subseteq P$, then $\mu_{R^{-1}}(y,x) = \mu_R(x,y) \leq \mu_P(x,y) = \mu_{P^{-1}}(y,x)$ for every (x,y) of $X \times Y$, analogously

$$\nu_{R^{-1}}(y,x) = \nu_R(x,y) \geq \nu_P(x,y) = \nu_{P^{-1}}(y,x)$$

For every (x,y) of $X \times Y$.

(ii)

$$\begin{aligned} \mu_{(R\cup P)^{-1}}(y,x) &= \mu_{R\cup P}(x,y) \\ &= \mu_R(x,y) \vee \mu_P(x,y) \\ &= \mu_{R^{-1}}(y,x) \vee \mu_{P^{-1}}(y,x) \\ &= \mu_{R^{-1}\cup P^{-1}}(y,x). \end{aligned}$$

The proof for $\nu_{(R\cup P)^{-1}}(y,x) = \nu_{R^{-1}\cup P^{-1}}(y,x)$ is done in a similar way.

(v) We will use the fact that $([0,1], \leq, \wedge, \vee)$ is a distrubitive lattice

$$\begin{aligned} \mu_{(R\cap(P\cup Q))}(x,y) &= \mu_R(x,y) \wedge \{\mu_P(x,y) \vee \mu_Q(x,y)\} \\ &= \{\mu_R(x,y) \wedge \mu_P(x,y)\} \vee \mu_R(x,y) \wedge \mu_Q(x,y) \\ &= \mu_{(R\cap P)}(x,y) \vee \mu_{(R\cap Q)}(x,y) \\ &= \mu_{(R\cap P)\cup(R\cap Q)}(x,y). \end{aligned}$$

The proof is analogous to the previous one, in the case of:

$$\nu_{R\cap(P\cup Q)}(x,y) = \nu_{(R\cap P)\cup(R\cap Q)}(x,y).$$

The rest of the items are proved in a way similar to the previous ones.

□

In 1986, Atanassov established different ways of changing an intuitionistic fuzzy set into a fuzzy set and he defined the following operator:
If $E \in$ IFSs(X), then

$$D_p(E) = \{\langle x, \mu_E(x) + p \cdot \pi_E(x),\ 1 - \mu_E(x) - p \cdot \pi_E(x)\rangle \mid x \in X\} \tag{1.3}$$

With $p \in [0,1]$. Obviously, $D_p(E) \in$ FSs(X).
A study of the properties of this operator (we will call it Atanassov's operator) is made in [4, 14].
Let E be an intuitionistic fuzzy set and D_p the operator given in the previous definition, then the family of all fuzzy sets associated to E through the operator D_p, will be denoted by $\{D_p(E)\}_{p\in[0,1]}$. It is clear that $\{D_p(E)\}_{p\in[0,1]}$ is a totally ordered family of fuzzy sets.
It is worth pointing out that for any intuitionistic fuzzy set E:

$$\mu_{(D_o(E)}(x) = \mu_E(x), \qquad \mu_{(D_1(E)_c}(x) = \nu_E(x) \tag{2.3}$$

Are verified for every x in X, which will be used very much in the last part of this paper.

3-3 Composition of intuitionistic fuzzy relations

Definition 1.3.3: [12] Let α, β, λ and ρ be t-norms or t-conorms not necessarily dual two-two, $R \in$ IFR $(X \times Y)$ and $P \in IFR(Y \times Z)$. We will call **composed relation** $P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R \in$ IFR $(X \times Z)$ to the one defined by:

$$P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R = \left\{ \left\langle (x,z), \mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z), \nu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) \right\rangle \mid x \in X, z \in Z \right\}$$

Where

$$\begin{aligned} \mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) &= \underset{y}{\alpha}\{\beta\ [\mu_R(x,y), \mu_P(y,z)]\} \\ \nu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) &= \underset{y}{\lambda}\{\rho\ [\nu_R(x,y), \nu_P(y,z)]\} \end{aligned}$$

Whenever

$$0 \le \mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) + \nu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) \le 1, \text{forall } (x,z) \in X \times Z.$$

Proposition 1.3.3. [12] In the conditions of the **Definition** 3-3, if λ^* and ρ^* are respectively, the dual forms of λ and ρ and $\alpha \le \lambda^*$, $\beta \le \rho^*$, then

$$0 \le \mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) + \nu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) \le 1, \text{ for all } (x,z) \in X \times Z.$$

Proof. We know that:

$$\mu_R(x,y) \leq 1 - \nu_R(x,y), \text{ for all } (x,y) \in X \times Y$$

And

$$\mu_P(y,z) \leq 1 - \nu_P(y,z), \text{ for all } (y,z) \in Y \times Z$$

Taking as hypothesis $\alpha \leq \lambda^*$ and $\beta \leq \rho^*$, we have:

$$\beta\ [\mu_R(x,y), \mu_P(y,z)] \leq \rho^*[1 - \nu_R(x,y), 1 - \nu_P(y,z)],$$

$$\begin{aligned} \underset{y}{\alpha}\{\beta[\mu_R(x,y), \mu_P(y,z)]\} &\leq \underset{y}{\lambda^*}\{\rho^*[1-\nu_R(x,y), 1-\nu_P(y,z)]\} \\ &= 1 - \underset{y}{\lambda}\{1 - \rho^*[1-\nu_R(x,y), 1-\nu_P(y,z)]\} \\ &= 1 - \underset{y}{\lambda}\{\rho\ [\nu_R(x,y), \nu_P(y,z)]\} \end{aligned}$$

Therefore

$$\underset{y}{\alpha}\{\beta[\mu_R(x,y), \mu_P(y,z)]\} + \underset{y}{\lambda}\{\rho[\nu_R(x,y), \nu_P(x,y)]\},$$

i.e.,

$$0 \leq \mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) + \nu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) \leq 1, \text{ for all } (x,z) \in X \times Z.$$

□

Example 1.3.3

Suppose we have the following intuitionistic fuzzy relations:

$$R = \{\langle(x_1,y_1), 0.8, 0.2\rangle, \langle(x_1,y_2), 0.5, 0.4\rangle, \langle(x_2,y_2), 0.6, 0.3\rangle, \langle(x_2,y_3), 0.9, 0.1\rangle\}.$$

$$P = \{\langle(y_1,z_1), 0.7, 0.2\rangle, \langle(y_1,z_2), 0.6, 0.3\rangle, \langle(y_2,z_2), 0.4, 0.5\rangle, \langle(y_2,z_3), 0.8, 0.1\rangle\}.$$

Let's assume that α, β, λ, and ρ are the min t-norm and max t-conorm, respectively, to compute $P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R(x,z)$ for a specific (x,z) pair, let's take (x_1,z_3) as an example:

$$P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R(x_1,z_3) = \left\{\left\langle (x_1,z_3), \mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x_1,z_3), \nu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x_1,z_3) \right\rangle \mid x_1 \in X, z_3 \in Z\right\}$$

Where

$$\begin{aligned} \mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x_1,z_3) &= \underset{y}{\alpha}\{\beta\ [\mu_R(x_1,y), \mu_P(y,z_3)]\} \\ \nu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x_1,z_3) &= \underset{y}{\lambda}\{\rho\ [\nu_R(x_1,y), \nu_P(y,z_3)]\} \end{aligned}$$

$$\begin{aligned} \mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x_1,z_3) &= \underset{y}{\alpha}\{\beta[\mu_R(x_1,y), \mu_P(y,z_3)]\} \\ &= \underset{y}{\alpha}\{[\beta(0.8, 0.8)]\} \\ &= \underset{y}{\alpha}\{[0.8]\} \\ &= 0.8 \end{aligned}$$

$$\begin{aligned}
\nu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x_1,z_3) &= \underset{y}{\lambda}\{\rho[\nu_R(x_1,y),\nu_P(y,z_3)]\} \\
&= \underset{y}{\lambda}\{[\rho(0.2,0.1)]\} \\
&= \underset{y}{\lambda}\{[0.2]\} \\
&= 0.2
\end{aligned}$$

$$0 \leq 0.8 + 0.2 \leq 1.$$

Therefore,

$$P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R(x_1,z_3) = \{(x_1,z_3), 0.8, 0.2\}.$$

3-4 Properties of intuitionistic fuzzy relations

We will study now the properties of intuitionistic fuzzy relations in a set X.

3-4-1 Identity relation and the complementary relation

Definition 1.4.3:

(i) The relation $\Delta \in$ IFR$(X \times X)$ is called **the relation of identity** if,

$$\mu_\Delta(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y, \end{cases}$$

$$\nu_\Delta(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y, \end{cases}$$

For all $x, y \in X \times X$.

(ii) The complementary relation Δ_c defined by

$$\mu_{\Delta_c}(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y, \end{cases}$$

$$\nu_{\Delta_c}(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y, \end{cases}$$

For all $x, y \in X \times X$.

Remark 1.4.3

Is evident $\Delta = \Delta^{-1}$.

Example 1.4.3

Let's consider the set $X = \{1, 2, 3\}$.

(i) The relation of identity $\Delta \in$ IFR $(X \times X)$ can be defined as follows:

$$\Delta = \left\{ \begin{array}{c} \langle(1,1),1,0\rangle, \langle(1,2),0,1\rangle, \langle(1,3),0,1\rangle, \\ \langle(2,1),0,1\rangle, \langle(2,2),1,0\rangle, \langle(2,3),0,1\rangle, \\ \langle(3,1),0,1\rangle, \langle(3,2),0,1\rangle, \langle(3,3),1,0\rangle \end{array} \right\}$$

The membership degree $\mu_\Delta(x, y) = 1$ if $x = y$ and 0 otherwise, and the non-membership degree $\nu_\Delta(x, y) = 0$ if $x = y$ and 1 otherwise for all $x, y \in X \times X$.

(ii) The complementary relation $\Delta_c \in$ IFR$(X \times X)$ can be defined as follows:

$$\Delta_c = \left\{ \begin{array}{c} \langle(1,1),0,1\rangle, \langle(1,2),1,0\rangle, \langle(1,3),1,0\rangle, \\ \langle(2,1),1,0\rangle, \langle(2,2),0,1\rangle, \langle(2,3),1,0\rangle, \\ \langle(3,1),1,0\rangle, \langle(3,2),1,0\rangle, \langle(3,3),0,1\rangle \end{array} \right\}$$

The membership degree $\mu_{\Delta_c}(x, y) = 0$ if $x = y$ and 1 otherwise, and the non-membership degree $\nu_{\Delta_c}(x, y) = 1$ if $x = y$ and 0 otherwise for all $x, y \in X \times X$.

3-4-2 Reflexivity and Antireflexivity

Definition 2.4.3: [12, 15] Let R be an intuitionistic fuzzy relation on X, we will say that R is :

1. **Reflexive**, if for every $x \in X$, $\mu_R(x, x) = 1$. Just notice that for every $x \in X$, $\nu_R(x, x) = 0$,
2. **Antireflexive**, if for every $x \in X$, then $\left\{ \begin{array}{l} \mu_R(x,x) = 0 \\ \nu_R(x,x) = 1, \end{array} \right.$ that is to say, if its complementary R_c is reflexive.

Example 2.4.3

Suppose we have a set $X = \{a, b, c\}$, and we define an intuitionistic fuzzy relation R on X as follows:

1. **Reflexive:**

$$\mu_R(a, a) = 1,\ \mu_R(b, b) = 0.8,\ \mu_R(c, c) = 0.9$$
$$\nu_R(a, a) = 0,\ \nu_R(b, b) = 0.2,\ {}_R(c, c) = 0.1$$

we can see that $\mu_R(a, a) = 1$, $\mu_R(b, b) = 0.8$, and $\mu_R(c, c) = 0.9$. Since all the diagonal elements have a membership grade of 1, the relation R is reflexive.

2. **Antireflexive:**

$$\mu_{R_c}(a,a)=0,\ \mu_{R_c}(b,b)=0.2,\ \mu_{R_c}(c,c)=0.1$$
$$\nu_{R_c}(a,a)=1,\ \nu_{R_c}(b,b)=0.8,\ \nu_{R_c}(c,c)=0.9$$

We have $\mu_{R_c}(a,a)=0$, $\mu_{R_c}(b,b)=0.2$, and $\mu_{R_c}(c,c)=0.1$,Since all the diagonal elements have a membership grade of 0, in addition, $\nu_{R_c}(a,a)=1$, $\nu_{R_c}(b,b)=0.8$, and $\nu_{R_c}(c,c)=0.9$ and the diagonal elements have a membership grade of 1. Therefore, the complementary relation R is antireflecxive, that is to say, if its complementary R_c is reflexive.

3-4-3 Symmetry and Antisymmetry

Definition 3.4.3: [12, 15] Let R be an intuitionistic fuzzy relation on X, we will say that R is :

1. **Symmetric**, if $R=R^{-1}$, that is, for every (x,y) in $X\times X$,

$$\begin{cases}\mu_R(x,y)=\mu_R(y,x)\\ \nu_R(x,y)=\nu_R(y,x)\end{cases}$$

In a contrary manner we will say that it ia asymmetric.

2. **Antisymmetrical intuitionistic** if, for every (x,y) in $X\times X$, $x\neq y$ then

$$\begin{cases}\mu_R(x,y)\neq\mu_R(y,x)\\ \nu_R(x,y)\neq\nu_R(y,x)\\ \pi_R(x,y)=\pi_R(y,x),\end{cases}$$

Where $\pi_R(x,y)=1-\mu_R(x,y)-\nu_R(x,y)$.

Example 3.4.3

Suppose we have a set $X=\{a,b,c\}$, and we define an intuitionistic fuzzy relation R on X as follows:

1. **Symmetric:**

μ_R	a	b	c
a	1	0.5	0.7
b	0.5	0.8	0.6
c	0.7	0.6	0.9

ν_R	a	b	c
a	0	0.1	0.4
b	0.1	0.2	0.3
c	0.4	0.3	0.1

From the above comparisons, we can see that for every pair (x,y) in $X\times X$, the membership grades and non-membership grades of R are equal to those of R^{-1}.

Therefore, the relation R is symmetric.

2. **Antisymmetric:**

μ_R	a	b	c
a	1	0.9	0.6
b	0.6	0.8	0.8
c	0.5	0.7	0.9

ν_R	a	b	c
a	0.0	0.1	0.2
b	0.4	0.2	0.0
c	0.3	0.1	0.1

$$\begin{aligned}
\pi_R(a,b) &= 1-0.9-0.1,\ \pi_R(b,a)=1-0.6-0.4\,,\\
\pi_R(a,c) &= 1-0.6-0.2,\ \pi_R(c,a)=1-0.5-0.3\,,\\
\pi_R(b,c) &= 1-0.8-0.0,\ \pi_R(c,b)=1-0.7-0.1\,.
\end{aligned}$$

From the above comparisons, we can see that for every pair (x,y) in $X \times X$ where $x \neq y$, the membership grades and non-membership grades of R are not equal to those of R^{-1}. However, the hesitancy grades (π_R) of R are equal to those of R^{-1}.
Therefore, the relation R is antisymmetric.

Theorem 1.4.3: [12] Let R be an element of IFR$(X \times X)$. R is antisymmetrical fuzzy intuitionistic if and only if,

$$\text{For every } (x,y) \in X \times X,\ x \neq y \text{ then, } \mu_R(x,y) \neq \mu_R(y,x).$$

Proof. As $\nu_R(x,y) = 1-\mu_R(x,y)$ and $\pi_R(x,y)=0$ for every $(x,y) \in X \times X$, then

$$\mu_R(x,y) \neq \mu_R(y,x) \text{ if and only if } \begin{cases} \mu_R(x,y) \neq \mu_R(y,x) \\ \nu_R(x,y) \neq \nu_R(y,x) \\ \pi_R(x,y) = \pi_R(y,x). \end{cases}$$

□

3-4-4 Perfect antisymmetrical fuzzy intuitionistic relation

Definition 4.4.3: [12] Let R be an intuitionistic fuzzy relation on X, we will say that R is **Perfect antisymmetrical fuzzy intuitionistic relation** if,
For every $x, y \in X$ with $x \neq y$ and

$$\begin{cases} \mu_R(x,y) > 0 \\ or \\ \mu_R(x,y) = 0 \text{ and } \nu_R(x,y) < 1, \end{cases}$$

Then

$$\begin{cases} \mu_R(y,x) = 0 \\ \text{and} \\ \nu_R(y,x) = 1. \end{cases}$$

Example 4.4.3

Let $X = \{x, y, z\}$, let R be the intuitionistic fuzzy relation given by:

μ_R	x	y	z
x	0.4	0.3	0.1
y	0.0	0.5	0.0
z	0.0	0.0	0.1

ν_R	x	y	z
x	0.5	0.7	0.4
y	1	0.3	0.6
z	1	1	0.7

R is a perfect antisymmetrical intuitionistic fuzzy relation.

3-4-5 Transitivity and c-transitivity

Definition 5.4.3: [12, 15] Lets take α t-conorme, β t-norme, λ t-norme and ρ t-conorme.

- We will say that $R \in IFR(X \times X)$ is **transitive** if $R \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} R \subseteq R$.
- We will say that $R \in IFR(X \times X)$ is **c-transitive** if $R \subseteq R \underset{\alpha,\beta}{\overset{\lambda,\rho}{\circ}} R$.

3-4-6 Intuitionistic fuzzy ordering

Definition 6.4.3: [27] Let X be a non-empty crisp set and

$$R = \{\langle (x,y), \mu_R(x,y), \nu_R(x,y)\rangle \mid x,y \in X\}$$

Be an intuitionistic fuzzy relation on X. Then R is called **an intuitionistic fuzzy ordering** or a partial intuitionistic fuzzy order if it is reflexive, transitive and perfect antisymmetrical fuzzy intuitionistic.

Example 5.4.3

Let $X = \{a,b,c,d\}$. Then the intuitionistic fuzzy relation R defined on X by

$$R = \{\langle (x,y), \mu_R(x,y), \nu_R(x,y)\rangle \mid x,y \in X\}$$

Where μ_R and ν_R given by the following tables :

μ_R	a	b	c	d
a	1.0	0.0	0.0	0.45
b	0.0	1.0	0.0	0.2
c	0.0	0.0	1.0	0.0
d	0.0	0.0	0.0	1.0

ν_R	a	b	c	d
a	0.0	1.0	0.25	0.3
b	0.2	0.0	0.5	0.2
c	1.0	1.0	0.0	0.6
d	1.0	1.0	1.0	0.0

Is intuitionistic fuzzy ordering on X .

Example 6.4.3

Let $m,n \in X = \{1,2,3,4\}$ and let μ_R, ν_R defined as follow :

$$\mu_R(m,n) = \begin{cases} 1, & \text{If } m = n \\ 1 - \frac{m}{n}, & \text{If } m < n \\ 0, & \text{If } m > n, \end{cases} \quad \text{and} \quad \nu_R(m,n) = \begin{cases} 0, & \text{If } m = n \\ \frac{m}{2n}, & \text{If } m < n \\ 1, & \text{If } m > n. \end{cases}$$

The intuitionistic fuzzy relation R on X is an intuitionistic fuzzy ordering.

Now, we recall the notion of linear or total intuitionistic fuzzy ordering as follows:

3-4-7 Intuitionistic fuzzy ordering linear or total

Now, we recall the notion of linear or total intuitionistic fuzzy ordering as follows:

Definition 7.4.3: [27] An intuitionistic fuzzy ordering R is **linear** (or total) on X if for every $x, y \in X$, we have:

$$\mu_R(x,y) > 0 \text{ and } \nu_R(x,y) = 0$$

Or

$$\mu_R(y,x) > 0 \text{ and } \nu_R(y,x) = 0.$$

Example 7.4.3

Let's consider the set $X = \{a, b, c\}$ and an intuitionistic fuzzy ordering R defined as:

$$R = \left\{ \begin{array}{c} \langle (a,a), 1, 0 \rangle, \langle (a,b), 0.8, 0 \rangle, \langle (a,c), 0.6, 0.6 \rangle, \\ \langle (b,a), 0.4, 0.2 \rangle, \langle (b,b), 1, 0 \rangle, \langle (b,c), 0, 0.7 \rangle, \\ \langle (c,a), 0.5, 0 \rangle, \langle (c,b), 0, 0.3 \rangle, \langle (c,c), 1, 0 \rangle \end{array} \right\}$$

Let's check the conditions for linearity or totality:

1. For the pair (a, b):

$$\begin{aligned} \mu_R(a,b) &= 0.8 > 0, \ \nu_R(a,b) = 0 \\ \mu_R(b,a) &= 0.4 > 0, \ \nu_R(b,a) = 0.2 \neq 0 \end{aligned}$$

The binary (a, b) checks the condition so that: $\mu_R(a,b) > 0$ and $\nu_R(a,b) = 0$.

2. For the pair (a, c):

$$\begin{aligned} \mu_R(a,c) &= 0.6 > 0, \ \nu_R(a,c) = 0.6 \neq 0 \\ \mu_R(c,a) &= 0.5 > 0, \ \nu_R(c,a) = 0 \end{aligned}$$

The binary (a, c) checks the condition so that: $\mu_R(c,a) > 0$ and $\nu_R(c,a) = 0$.

3. For the pair (b, c):

$$\begin{aligned} \mu_R(b,c) &= 0 \not> 0, \ \nu_R(b,c) = 0.7 \neq 0 \\ \mu_R(c,b) &= 0 \not> 0, \ \nu_R(c,b) = 0.3 \neq 0 \end{aligned}$$

Binary (b, c) does not meet any of the conditions.

Definition 8.4.3: [27] An intuitionistic fuzzy ordered set (X, μ_R, ν_R) in which R is linear is called a **linearly intuitionistic fuzzy ordered set or an intuitionistic fuzzy chain**.

Conversly, we obtain the following definition of incomparable elements.

Definition 9.4.3: [27] Let (X, μ_R, ν_R) be a non-empty intuitionistic fuzzy ordered set and let a, b be two elements of X. We say that a and b are incomparable in (X, μ_R, ν_R) if,

$$\mu_R(a,b) = 0 \text{ or } \nu_R(a,b) > 0$$

And

$$\mu_R(b,a) = 0 \text{ or } \nu_R(b,a) > 0.$$

Definition 10.4.3: Let R be an element of IFR$(X \times X)$. β t-conorm and ρ t-norm.

(1) We will call transitive closure of R, to the minimum intuitionistic fuzzy relation $\hat{R}$ on $X \times X$ which contains R and it is transitive. Besides

$$\hat{R} = R \vee R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \vee R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \vee \cdots \vee R^n.$$

(2) We will call c-transitive closure of R to the biggest c-transitive relation $\overset{\vee}{R} \in$ IFR$(X \times X)$ contained in R. Besides

$$\overset{\vee}{R}_c = R_c \wedge R_c \underset{\vee,\wedge}{\overset{\wedge,\vee}{\circ}} R_c \wedge R_c \underset{\vee,\wedge}{\overset{\wedge,\vee}{\circ}} R_c \underset{\vee,\wedge}{\overset{\wedge,\vee}{\circ}} R \wedge \cdots \wedge R^n.$$

3-5 Relation between an intuitionistic fuzzy relation and its complementary relation

We will start this section by studying the relation that exists between the properties of a relation and the properties of its complementary one. Next, we define the structures in the intuitionistic fuzzy relations, and we also study their main properties. Besides, we analyse the relation between the structures of R and R_c.

We conclude this section with two theorems that characterize the structures of the intuitionistic fuzzy relations according to the structures of the fuzzy relations $D_0(R)$ and $D_1(R)$.

Now we are going to study the composition of intuitionistic complementary fuzzy relations and its transitive closures .

Theorem 1.5.3: Let R,P be two elements of IFR$(X \times X)$, β a t-norm and ρ a t-conorm, it is verified that

$$(R \underset{\wedge,\rho}{\overset{\vee,\beta}{\circ}} P)_c = R_c \underset{\vee,\beta}{\overset{\wedge,\rho}{\circ}} P_c.$$

Proof. As $\mu_{R_c}(x,z) = \nu_R(x,z)$ and $\nu_{R_c}(x,z) = \mu_R(x,z)$ for every (x,z) in $X \times X$, we have

$$R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} P = \left\{\left\langle (x,z), \bigvee_y \{\beta[\mu_P(x,y), \mu_R(y,z)]\}, \bigwedge_y \{\rho[\nu_p(x,y), \nu_R(y,z)]\} \right\rangle | (x,z) \in X \times X \right\}.$$

$$\left(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} P\right)_c = \left\{\left\langle (x,z), \bigwedge_y \{\rho[\nu_P(x,z), \nu_R(x,z)]\}, \bigvee_y \{\beta[\mu_p(x,z), \mu_R(x,z)]\} \right\rangle\right\} = R_c \overset{\wedge,\rho}{\underset{\vee,\beta}{\circ}} P_c.$$

□

Theorem 2.5.3: Let R be an intuitionistic fuzzy relation in $(X \times X)$ and R_c its complementary relation, $\beta = \wedge$ and $\rho = \vee$

$$\overset{\vee}{R}_c = (\hat{R})_c$$

holds.

Proof. We know that

$$R = \{\langle (x,y), \mu_R(x,y), \nu_R(x,y)\rangle \mid (x,y) \in X \times X\},$$
$$R_c = \{\langle (x,y), \nu_R(x,y), \mu_R(x,y)\rangle \mid (x,y) \in X \times X\},$$

Calculating $\hat{R}$ and $\overset{\vee}{R}_c$, we have $\hat{R} = R \vee R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R \vee R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R \vee \cdots \vee R^n$, therefore

$$\mu_{\hat{R}}(x,y) = \mu_R(x,y) \vee \mu_{R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R}(x,y) \vee \cdots \vee \mu_{R^n}(x,y),$$

$$\nu_{\hat{R}}(x,y) = \nu_R(x,y) \vee \nu_{R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R}(x,y) \vee \cdots \vee \nu_{R^n}(x,y),$$

With

$$\mu_{R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R}(x,y) = \bigvee_z \{\wedge[\mu_R(x,z), \mu_R(z,y)]\},$$

$$\nu_{R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R}(x,y) = \bigwedge_z \{\vee[\nu_R(x,z), \nu_R(z,y)]\},$$

$\overset{\vee}{R}_c = R_c \wedge R_c \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R_c \wedge R_c \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R_c \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R \wedge \cdots \wedge R^n$, therefore

$$\mu_{\overset{\vee}{R}_c}(x,y) = \mu_{R_c}(x,y) \wedge \mu_{R_c \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R_c}(x,y) \wedge \cdots \wedge \mu_{R_c^n}(x,y),$$

$$\nu_{\overset{\vee}{R}_c}(x,y) = \nu_{R_c}(x,y) \vee \nu_{R_c \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R_c}(x,y) \vee \cdots \vee \nu_{R_c^n}(x,y),$$

With

$$\mu_{R_c \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R_c}(x,y) = \bigwedge_z \{\vee[\mu_{R_c}(x,z), \mu_{R_c}(z,y)]\},$$

$$\nu_{R_c \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R_c}(x,y) = \bigvee_z \{\wedge[\nu_{R_c}(x,z), \nu_{R_c}(z,y)]\},$$

As $\mu_{R_c}(x,z) = \nu_R(x,z)$ and $\nu_{R_c}(x,z) = \mu_R(x,z)$, we have

$$\mu_{R_c \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R_c}(x,y) = \bigwedge_z \{\vee[\mu_{R_c}(x,z), \mu_{R_c}(z,y)]\} = \bigwedge_z \{\vee[\nu_R(x,z), \nu_R(z,y)]\} = \nu_{R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R}(x,y),$$

$$\nu_{R_c \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R_c}(x,y) = \bigvee_z \{\wedge[\nu_{R_c}(x,z), \nu_{R_c}(z,y)]\} = \bigvee_z \{\wedge[\mu_R(x,z), \mu_R(z,y)]\} = \mu_{R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R}(x,y),$$

Therefore

$$\mu_{\overset{\vee}{R_c}}(x,y) = \mu_{R_c}(x,y) \wedge \mu_{R_c \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R_c}(x,y) \wedge \cdots \wedge \mu_{R_c^n}(x,y) = \nu_{\hat{R}}(x,y)$$

$$\nu_{\overset{\vee}{R_c}}(x,y) = \nu_{R_c}(x,y) \vee \nu_{R_c \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R_c}(x,y) \vee \cdots \vee \nu_{R_c^n}(x,y) = \mu_{\hat{R}}(x,y)$$

That is,

$$\begin{aligned}\overset{\vee}{R_c} &= \{\left\langle (x,y), \mu_{\overset{\vee}{R_c}}(x,y), \nu_{\overset{\vee}{R_c}}(x,y) \right\rangle \mid (x,y) \in X \times Y\} \\ &= \{\left\langle (x,y), \nu_{\hat{R}}(x,y), \mu_{\hat{R}}(x,y) \right\rangle \mid (x,y) \in X \times Y\} = (\hat{R})_c.\end{aligned}$$

□

The next theorem establishes the relation that exists between the properties of an intuitionistic fuzzy relation and its complementary one.

Theorem 3.5.3: $R \in \mathrm{IFR}(X \times X)$

(i) R is reflexive if and only if R_c is antireflexive,

(ii) R is symmetrical if and only if R_c is symmetrical,

(iii) R is antisymmetrical intuitionistic if and only if R_c is antisymmetrieal intuitionistic, and

(iv) R is transitive if and only if R_c is c-transitive.

Proof. It is enough to remember the definitions. □

3-6 Structures in intuitionistic fuzzy relation

In this section we will have $\beta = \wedge$ and $\rho = \vee$.

Definition 1.6.3: A intuitionistic fuzzy relation R on the cartesian set $(X \times X)$, is called:

(1) an intuitionistic tolerance relation on $X \times X$ if R is reflexive and symmetric,

(2) an intuitionistic atolerance relation on $X \times X$ if R is symmetric and antireflexive,

(3) an intuitionistic preorder if it is reflexive and transitive,

(4) an intuitionistic order if it is reflexive, transitive and antisymmetricai intuitionistic,

(5) an intuitionistic perfect ordering if it is reflexive, transitive and perfect antisymmetrical intuitionistic,

(6) an intuitionistic strict order if R is antireflexive, transitive and antisymmetrical intuitionistic,

(7) an intuitionistic similarity relation on $X \times X$ if R is reflexive, symmetric and transitive, and

(8) an intuitionistic dissimilarity relation on $X \times X$ if R is symmetric, antireflexive and transitive.

We can build a definition similar to the previous one using c-transitive property, where the structures of similarity, preorder, intuitionistic order, intuitionistic strict order, dissimilarity and intuitionistic perfect order will have to willful the same properties that the ones given in the definition, but substituting the transitive property for the c-transitive one. Whenever we use a structure respecting the c-transitive property, we will specify it.

We know that [14], If $R \in \mathrm{IFR}(X \times X\ )$ is reflexive, then

$$R^n = \overbrace{R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \cdots \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R}^{n-\text{times}}$$

With $n = 1, 2, \ldots$, it is reflexive. Besides

Theorem 1.6.3: Let R_1 be a reflexive intuitionistic fuzzy relation in $X \times X$, then

(i) $(R_1)^{-1}$ is reflexive,

(ii) $R_1 \vee R_2$ is reflexive for every $R_2 \in \mathrm{IFR}(X \times X)$, and

(iii) $R_1 \wedge R_2$ is reflexive if and only if $R_2 \in \mathrm{IFR}(X \times X)$ is reflexive.

Proof. Just notice that

$$\begin{aligned}
\mu_{R_1 \vee R_2}(x,x) &= \mu_{R_1}(x,x) \vee \mu_{R_2}(x,x) = 1 \vee \mu_{R_2}(x,x) = 1 \\
\nu_{R_1 \vee R_2}(x,x) &= \nu_{R_1}(x,x) \wedge \nu_{R_2}(x,x) = 0 \wedge \nu_{R_2}(x,x) = 0 \\
\mu_{R_1 \wedge R_2}(x,x) &= \mu_{R_1}(x,x) \wedge \mu_{R_2}(x,x) = 1 \wedge \mu_{R_2}(x,x) = \mu_{R_2}(x,x) \\
\nu_{R_1 \wedge R_2}(x,x) &= \nu_{R_1}(x,x) \vee \nu_{R_2}(x,x) = 0 \vee \nu_{R_2}(x,x) = \nu_{R_2}(x,x).
\end{aligned}$$

□

Corollary 3-6.1. If R_1,R_2 $\in$ RIF($X \times X$) are intuitionistic tolerance relations, then so are $R_1 \vee R_2$, $R_1 \wedge R_2$ and $\hat{R}_1$.

Proof. It is enough to remember **Theorem** 3-6 □

Theorem 2.6.3: Let R_1,R_2 be two elements of lFR($X \times X$), R_1 intuitionistic tolerance relation, R_2 intuitionistic A similarity relation and $R_1 \leqslant R_2$, then $\hat{R}_1 \leqslant R_2$.

Proof. $R_1 \leqslant R_2 \in$ IFR($X \times X$), by means of the monotony of the composition of intuitionistic fuzzy relations [14], $R_1 \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R_1 \leqslant R_2$, $R_1 \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R_1 \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R_1 \leqslant R_2, \ldots$, as R_1 is reflexive and symmetrical intuitionistic, by **Theorem** 3-6 and [14], it is fulfilled that

$$R_1 \vee R_1 \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R_1 \vee \cdots \vee R^n$$

Is reflexive and symmetrical intuitionistic, therefore

$$\hat{R}_1 = R_1 \vee R_1 \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R_1 \vee \cdots \vee R_1^n \leqslant R_2 \vee R_2 \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R_2 \vee \cdots \vee R_2^n = R_2.$$

□

In the section [14] is proof that if $R, P \in$ IFR($X \times X$) and $R \leqslant P$, then $\hat{R} \leqslant \hat{P}$ and $\overset{\vee}{R} \leqslant \overset{\vee}{P}$. Besides, we know that [14] if R is reflexive and transitive, then $R \underset{\wedge,\rho}{\overset{\vee,\beta}{\circ}} R$.

Next theorem is proved with all this.

Theorem 3.6.3: In every intuitionistic preorder:

(i) $R = R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R$

(ii) $R = R^k$ for every $k = 1, 2, \ldots$, are verified.

The next theorem establishes the relation that exists between the structures of $R \in$ IFR($X \times X$) and the ones of its complementary relation.

Theorem 4.6.3: Let R be an intuitionistic fuzzy relation on ($X \times X$):

(i) R is an intuitionistic order if and only if R_c is an intuitionistic strict order respecting the c-transitive property,

(ii) R is an intuitionistic tolerance relation if and only if R_c is an intuitionistic atolerance relation, and

(iii) R is an intuitionistic tolerance relation if and only if $\overset{\vee}{R}_c$ is an intuitionistic dissimilarity relation respecting the c-transitive property.

Proof. It is a direct consequence of **Theorem** 3-5. □

Theorem 5.6.3: $R \in \text{IFR}(X \times X)$ is an intuitionistic similarity relation if and only if R_c is an intuitionistic dissimilarity relation respecting the c-transitive property.

Proof. It is enough to remember the definition of similarity and dissimilarity relations. □

3-7 Characterization of intuitionistic fuzzy structures

Next theorem gives necessary and sufficient conditions between the possible structures of the intuitionistic fuzzy relation R and the fuzzy relations associated to this by Atanassov's operators for $p = 0$ and $p = 1$, (2.3). The composition that we consider for the following Theorems is $R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R$, being β any t-norm and β^* its dual t-conorm.
With that composition, let us remember that transitivity is defined as $R \geqslant R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R$ and c-transitivity as $R \leqslant R \overset{\wedge,\beta^*}{\underset{\vee,\beta}{\circ}} R$.

Theorem 1.7.3: Let R be an intuitionistic fuzzy, relation in $X \times X$

(i) R is an intuitionistic preorder if and only if

  (a) $D_0(R)$ is a fuzzy preorder and

  (b) $(D_1(R))_c$ is antireflexive and c-transitive.

(ii) R is an intuitionistic similarity relation if and only if

  (a) $D_0(R)$ is a fuzzy similarity relation and

  (b) $(D_1(R))_c$ is a fuzzy dissimilarity relation respecting the c-transitive.

Proof. In the proof we are going to use (1.3) and (2.3).

($\Rightarrow$)

- If R is reflexive, then $\mu_R(x,x) = 1$ and $\nu_R(x,x) = 0$ for all $x \in X$ therefore $\mu_{D_0(R)}(x,x) = \mu_R(x,x) = 1$ and $\mu_{(D_0R))_c}(x,x) = \nu_R(x,x) = 0$ $\forall x \in X$, then $D_o(R)$ is reflexive and $(D_1(R))_c$ is antireflexive.
- If R is transitive, then $R \geqslant R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R$, therefore

$$\begin{aligned}\mu_{D_0(R)}(x,y) &= \mu_R(x,y) \geqslant \bigvee_z \{\beta[\mu_R(x,z),\mu_R(z,y)]\}\\ &= \bigvee_z \{\beta[\mu_{D_0(R)}(x,z),\mu_{D_0(R)}(z,y)]\} = \mu_{D_0(R)\overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} D_0(R)}(x,y),\end{aligned}$$

And, therefore, $D_0(R)$ is transitive. As we have

$$\begin{aligned}\mu_{(D_1(R))_c}(x,y) &= \nu_R(x,y) \leqslant \bigwedge_z \{\beta^*[\nu_R(x.z),\nu_R(z,y)]\}\\ &= \bigwedge_z \{\beta^*[\mu_{(D_1(R))_c}(x,z),\mu_{(D_1(R))_c}(z,y)]\} = \mu_{(D_1(R))_c \overset{\wedge,\beta^*}{\underset{\vee,\beta}{\circ}} (D_1(R))_c}(x,y),\end{aligned}$$

Then $(D_1(R))_c$ is c-transitive.

- If R is a symmetrical intuitionistic relation, then $\mu_R(x,y) = \mu_R(y,x)$ and $\nu_R(x,y) = \nu_R(y,x)\ \forall(x,y)$, therefore $\mu_{D_0(R)}(x,y) = \mu_{D_0(R)}(y,x)$ and $\mu_{(D_1(R))_c}(x,y) = \mu_{(D_1(R))_c}(y,x)$, then $D_0(R)$ and $(D_1(R))_c$ are symmetrical relations.

($\Leftarrow$)

If $D_0(R)$ is a fuzzy preorder and $(D_1(R))_c$ is an antireflexive and c-transitive relation, we get

- $D_0(R)$ is a reflexive relation and $(D_1(R))_c$ is an antireflexive relation, then $\mu_{D_0(R)}(x,x) = 1$ and $\nu_{D_0(R)}(x,x) = 0\ \forall x \in X$, taking into account that $\mu_R(x,x) = \mu_{D_0(R)}(x,x) = 1$ and $\nu_R(x,x) = \mu_{(D_1(R))_c}(x,x) = 0\ \ \forall x \in X$, then R is a reflexive relation.
- As $D_0(R)$ is transitive fuzzy, then

$$D_0(R) \geqslant D_0(R) \underset{\wedge,\beta^*}{\overset{\vee,\beta}{\circ}} D_0(R)$$

Therefore

$$\mu_{D_0(R)}(x,y) \geqslant \bigvee_z \{\beta[\mu_{D_0(R)}(x,z), \mu_{D_0(R)}(z,y)]\},$$

As $\mu_R(x,y) = \mu_{D_0(R)}(x,y)\ \forall(x,y) \in X \times Y$, we get

$$\mu_R(x,y) \geqslant \bigvee_z \{\beta[\mu_R(x,z), \mu_R(z,y)]\},$$

Then

$$\mu_R(x,y) \geqslant \mu_{R \underset{\wedge,\beta^*}{\overset{\vee,\beta}{\circ}} R}(x,y)]\}, \quad \forall(x,y) \in X \times X$$

$(D_1(R))_c$ is c-transitive, Then

$$\mu_{(D1(R))_c}(x,y) \leqslant \bigwedge_z \{\beta^*[\mu_{(D1(R))_c}(x,z), \mu_{(D1(R))_c}(z,y)]\},$$

As $\nu_R(x,y) = \mu_{(D1(R))_c}(x,y)\ \forall(x,y) \in X \times X$, then

$$\nu_R(x,y) \leqslant \bigwedge_z \{\beta^*[\nu_R(x,z), \nu_R(z,y)]\},$$

Therefore, R is a transitive relation.

- If $D_0(R)$ is a symmetrical relation, then

$$\forall(x,y) \in X \times X, \mu_{D_0(R)}(x,y) = \mu_{D_0(R)}(y,x).$$

If $(D_1(R))_c$ is a symmetrical relation, then

$$\mu_{(D_1(R))_c}(x,y) = \mu_{(D_1(R))_c}(y,x), \text{for every } (x,y) \text{ in } X \times X.$$

Taking into account that for every $(x,y) \in X \times X$

$$\mu_R(x,y) = \mu_{D_0(R)}(x,y) \quad \text{and} \quad \nu_R(x,y) = \nu_{(D_1(R))_c}(x,y),$$

The result is that R is a symmetrical relation.

□

Theorem 2.7.3: If R is an intuitionistic order, then

(a) $D_0(R)$ is an order relathion and

(b) $(D_1(R))_c$ is a strict order relation respecting the c-transitive property.

Proof. Reflexivity and transitivity are proved in the same way as in the previous theorem. If R is antisymmetrical intuitionistic, then for every $(x,y) \in X \times X$ with $x \neq y$, we get

$$\mu_R(x,y) \neq \mu_R(y,x), \qquad \nu_R(x,y) \neq \nu_R(y,x), \qquad \pi_R(x,y) = \pi_R(y,x),$$

Therefore,

$$\mu_{D_0(R)}(x,y) \neq \mu_{D_0(R)}(y,x), \qquad \mu_{(D_1(R))_c}(x,y) \neq \mu_{(D_1(R))_c}(y,x)$$

So that $D_0(R)$ and $(D_1(R))_c$ are intuitionistic antisymmetrical relations.

The reciprocal version of this theorem is not true because of the condition $\pi_R(x,y) = \pi_R(y,x)$ of the intuitionistic antisymmetry. □

Corollary 3-7.1. Let R be an element of IFR($X \times X$)

(i) R is an intuitionistic preorder if and only if

  (a) $D_0(R)$ is a fuzzy preorder and

  (b) $D_1(R)$ is a fuzzy preorder.

(ii) R is an intuitionistic similarity relation if and only if

  (a) $D_0(R)$ is a fuzzy similarity relation and

  (b) $D_1(R)$ is a fuzzy similarity relation.

Proof. It is a consequence of **Theorems** 3-5 and 3-7. □

Corollary 3-7.2. If R is an intuitionistic order, then

(a) $D_o(R)$ is a fuzzy order and

(b) $D_1(R)$ is a fuzzy order.

Proof. It is a consequence of **Theorems** 3-5 and 3-7. □

Conclusion

The properties of the intuitionistic fuzzy relations presented in this section show that the definition of these properties does not always coincide with the definition of the properties of fuzzy relations. It happens that there are properties like perfect intuitionistic antisymmetry that recover Zadeh's perfect antisymmetry for the case fuzzy, while intuitionistic antisymmetry does not recover Kaufmann's antisymmetry for the fuzzy case.
On the other hand, Atanassov's operators (1.3) have allowed us to characterize the structure of intuitionistic fuzzy relations in function with only two values of the index $\boldsymbol{p}$, ($\boldsymbol{p} = \mathbf{0}$ and $\boldsymbol{p} = \mathbf{1}$), (2.3). The study of the characterization of the structures of intuitionistic fuzzy relations from fuzzy relations and from Atanassov's operators (1.3) with any value of $\boldsymbol{p}$ can be found in [14]

Bibliography

[1] Muhammad Akram, Ather Ashraf, and Mansoor Sarwar. Novel applications of intuitionistic fuzzy digraphs in decision support systems. *The Scientific World Journal*, 2014, 2014.

[2] Abdelaziz Amroune and Brahim Ziane. More on intuitionistic fuzzy sublattices and their ideals. *Facta Universitatis, Series: Mathematics and informatics*, pages 871–888, 2019.

[3] KrassimirT Atanassov. Intuitionistic fuzzy sets, fuzzy sets and systems, vol. 20. 1986.

[4] KrassimirT Atanassov. 2 operators on intuitionistic fuzzy-sets. *DOKLADI NA BOLGARSKATA AKADEMIYA NA NAUKITE*, 41(5):35–38, 1988.

[5] KrassimirT Atanassov. *On intuitionistic fuzzy sets theory*, volume 283. Springer, 2012.

[6] KrassimirT Atanassov and KrassimirT Atanassov. *Intuitionistic fuzzy sets*. Springer, 1999.

[7] KrassimirT Atanassov and ValentinaS Atanassova. Intuitionistic fuzzy relations and their applications. *Information Sciences*, 178(4):985–987, 2008.

[8] Barnabas Bede and Barnabas Bede. *Fuzzy Analysis*. Springer, 2013.

[9] Ismat Beg and Tabasam Rashid. A clustering algorithm based on intuitionistic fuzzy relations for tree structure evaluation. *International Journal of Applied and Computational Mathematics*, 3:3131–3145, 2017.

[10] Debmalya Bhattacharya and SamarendraPrasad Biswas. Intuitionistic fuzzy graph in decision-making. *Information Sciences*, 198:69–95, 2012.

[11] Bernadette Bouchon-Meunier. *La logique floue et ses applications*. Editions Addison-Wesley France, 1995.

[12] PedroJ Burillo and Humberto Bustince. Intuitionistic fuzzy relations (part i). *Mathware and soft computing*, 2(1):5–38, 1995.

[13] Pedro BurilloLópez and HumbertoNicanor BustinceSola. Intuitionistic fuzzy relations (part ii) effect of atanassov's operators on the properties of the intuitionistic fuzzy relations. *Mathware & soft computing. 1995 Vol. 2 Núm. 2 p. 117-148*, 1995.

[14] HBustince. *Conjuntos Intuicionistas e Intervalo valorados difusos: propiedades y construcci on. Relaciones Intuicionistas Fuzzy*. PhD thesis, Universidad Pública de Navarra Pamplona, Spain, 1994.

[15] Humberto Bustince and PBurillo. Structures on intuitionistic fuzzy relations. *Fuzzy sets and systems*, 78(3):293–303, 1996.

[16] Inheung Chon. Fuzzy partial order relations and fuzzy lattices. *Korean Journal of Mathematics*, 17(4):361–374, 2009.

[17] SupriyaKumar De, Ranjit Biswas, and AkhilRanjan Roy. An application of intuitionistic fuzzy sets in medical diagnosis. *Fuzzy sets and Systems*, 117(2):209–213, 2001.

[18] Xiufang Jia, Rujing Wang, Liangtu Song, and Wei Li. Hierarchical structures based on intuitionistic fuzzy tolerance relations. In *2012 IEEE International Conference on Granular Computing*, pages 211–215. IEEE, 2012.

[19] ErichPeter Klement, Radko Mesiar, and Endre Pap. *Triangular norms*, volume8. Springer Science & Business Media, 2013.

[20] KwangHyung Lee. *First course on fuzzy theory and applications*, volume27. Springer Science & Business Media, 2004.

[21] PrasantaK Panigrahi and AmitaRani Sahoo. A novel approach for solving intuitionistic fuzzy relation equations. In *2012 Ninth International Conference on Fuzzy Systems and Knowledge Discovery*, pages 307–311. IEEE, 2012.

[22] Witold Pedrycz and Fernando Gomide. *An introduction to fuzzy sets: analysis and design*. MIT press, 1998.

[23] SankarKumar Roy and Susanta Dutta. Intuitionistic fuzzy clustering: recent advances and perspectives. *Applied Soft Computing*, 40:428–445, 2016.

[24] Sankar Sahoo and Madhumangal Pal. Different types of products on intuitionistic fuzzy graphs. *Pacific Science Review A: Natural Science and Engineering*, 17(3):87–96, 2015.

[25] LotfiA Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[26] LotfiA Zadeh. Similarity relations and fuzzy orderings. *Information sciences*, 3(2):177–200, 1971.

[27] Lemnouer Zedam, Adelaziz Amroune, and BDavvaz. Szpilrajn theorem for intuitionistic fuzzy orderings. *Annals of Fuzzy Mathematics and Informatics*, 9(5):703–718, 2015.

[28] Brahim Ziane and Abdelaziz Amroune. Representation and construction of intuitionistic fuzzy tpreorders and fuzzy weak t-orders. *Discussiones Mathematicae: General Algebra & Applications*, 41(1), 2021.

ملخص :

في هذا المذكرة ، ندرس العناصر الهيكلية للعلاقات الحدسية الغامضة. ندرس الروابط الموجودة بين المكونات الهيكلية للعلاقة وتلك الخاصة بمكونها التكميلي. وفقاً لهيكليتين من العلاقات الغامضة الملموسة ، نكمل وصف بعض الهياكل العلائقية الحدسية.

Abstract :

In this Memory, we examine the fuzzy intuitionistic relations' structural elements. We examine the existing connections between a relation's structural components and those of its complimentary component. In accordance with the structures of two concrete fuzzy relations, we complete describing certain intuitionistic relational structures.

Résumé :

Dans ce mémoire, nous examinons les éléments structuraux des relations intuitionnistes floues. Nous examinons les connexions existantes entre les composantes structurelles d'une relation et celles de sa composante complémentaire. Conformément aux structures de deux relations floues concrètes, nous complétons la description de certaines structures relationnelles intuitionnistes.