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**Viscoelastic coupled systems of wave equations
with past history**

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..

Dedication

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I dedicate this humble work to my father Achoure, who has watched throughout my life to encourage me, helps and protects me. To the one who gave me life, the symbol of love who sacrificed herself for my happiness and my success, to my mother Ouarda, who has watched throughout my life to encourage me, gives me help and protects meter.

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To my family, To my friends.

To my colleagues.

To all those who are dear to me.

To everyone I love.

I dedicate this work.

Asma

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List of Symbols and Notations

Ω	Is a bounded domain of \mathbb{R}^n , $n \geq 1$ an integer.
$\Gamma = \partial\Omega$	Is the smooth boundary of Ω .
$\frac{\partial u}{\partial \nu}$	Denotes the outward normal derivative of u , i. e., $\frac{\partial u}{\partial \nu} = (\nabla u) \cdot \nu = \sum_{i,j=1}^n \frac{\partial u}{\partial x_j} \nu_i.$
u_{tt}	The second derivative of u with respect to time t .
\mathcal{H}	Hilbert space.
I	The identity operator.
$\langle \cdot, \cdot \rangle_{\mathcal{H}}$	The scalar product of \mathcal{H} .
$\mathcal{L}(\mathcal{H})$	The space of continuous linear operators from \mathcal{H} in \mathcal{H} .
$C(\mathbb{R}^+; \mathcal{H})$	The space of continuous functions from \mathbb{R}^+ in \mathcal{H} .
$C^1(\mathbb{R}^+; \mathcal{H})$	The space of continuously differentiable functions from \mathbb{R}^+ in \mathcal{H} .
$L^p(\Omega)$	The Lebesgue space, $1 \leq p \leq +\infty$.
$H_0^1(\Omega)$	Sobolev's space.
$H^{-1}(\Omega)$	Dual space of $H_0^1(\Omega)$.
$\langle \cdot, \cdot \rangle_2, \ \cdot\ _2$	Are the inner product and the norm on $L^2(\Omega)$, respectively.
$\langle \cdot, \cdot \rangle, \ \cdot\ $	Are the inner product and the norm on $H_0^1(\Omega)$, respectively.

Introduction

The changes are happening all around us. Some of them are natural (yesterday it was not raining, today it is), and some are man-made (last year there was no house standing in this place, now there is a house). The relationships between the quantities that participate in the "change" are called functions, and the differential equation is a mathematical equality that relates some functions to their derivatives. Solving them means finding these functions. After that, these functions are used to make our clocks tick the right time, and our planes don't fall, ...etc. In other words, partial differential equations are used in many fields.

In particular, many interesting physical phenomena such as viscoelasticity, genetic polarization in dielectrics, cluster dynamics or heat flow in real conductors, to name a few, are modeled by partial differential equations. Which are influenced by the past values of one or more variables in play so-called equations with memory. This type of equation is characterized by non local character, due to the presence of the memory term given by the time convolution of the unknown function against a suitable memory kernel.

The memory term may can produce an exponential stability for a viscoelastic problem of wave equation with past history [7] [8]. But it is not strong enough only on one equation to obtain this result for coupled systems, this result has been discussed by many authors [1] [9]. In [13] Cordeiro et all considered a coupled system of wave equations with past history given by

$$u_{tt} - \Delta u + \int_0^{+\infty} g(s)\Delta u(t-s) ds + \alpha v = 0, \quad \text{in } \Omega \times (0, +\infty), \quad (1)$$

$$v_{tt} - \Delta v + \alpha u = 0, \quad \text{in } \Omega \times (0, +\infty), \quad (2)$$

$$u = v = 0, \quad \text{on } \Gamma \times (0, +\infty), \quad (3)$$

$$(u(x, 0), v(x, 0)) = (u_0(x), v_0(x)), \quad \text{in } \Omega, \quad (4)$$

$$(u_t(x, 0), v_t(x, 0)) = (u_1(x), v_1(x)), \quad \text{in } \Omega. \quad (5)$$

Where Ω is an open bounded set of \mathbb{R}^n with smooth boundary Γ . The above model can be used to describe the evolution of a system consisting of two elastic membranes subject to an elastic force that attracts one membrane to the other with coefficient $\alpha > 0$, with the following memory term $\int_0^{+\infty} g(s)\Delta u(t-s) ds$. Cordeiro et all showed the lack of exponential decay. On the other hand, they proved that the solution of this system decays polynomially with optimal rate $t^{-\frac{1}{2}}$.

In this work, we will also consider the following viscoelastic coupled system with past history acting on both equations:

$$u_{tt} - \Delta u + \int_0^{+\infty} g(s) \Delta u(t-s) ds + \alpha v = 0, \quad \text{in } \Omega \times (0, +\infty), \quad (6)$$

$$v_{tt} - \Delta v + \int_0^{+\infty} g(s) \Delta v(t-s) ds + \alpha u = 0, \quad \text{in } \Omega \times (0, +\infty), \quad (7)$$

$$u = v = 0, \quad \text{on } \Gamma \times (0, +\infty), \quad (8)$$

$$(u(x, 0), v(x, 0)) = (u_0(x), v_0(x)), \quad \text{in } \Omega, \quad (9)$$

$$(u_t(x, 0), v_t(x, 0)) = (u_1(x), v_1(x)), \quad \text{in } \Omega. \quad (10)$$

Then, we will see the effect of the memory term in this system (6)-(10). For that, we divide our work in two chapters.

In the first chapter, we consider a coupled system (6)-(10) of wave with past history. Under certain hypotheses on the kernel g , we show the global existence and uniqueness of solutions by using the semigroup theory. Finally, we obtain the exponential decay of semigroup associated solution using Greahart-Prüss' theorem, see [12]. [12].

In the second chapter, we consider the system (1)-(5) and we will follow the result of [13], as we have explained some details in their proofs. We discuss the existence and uniqueness of the solution using semigroup theory with the same hypotheses on the kernel g . Then, we show the lack of exponential decay using Prüss's results [12]. Finally, we prove that the system is polynomially stable giving an optimal decay rate using the result of Borichev and Tomilov [2].

Chapter 1

Exponential stability for a viscoelastic coupled system with past history of wave equations

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In this chapter, we consider a viscoelastic coupled system for two wave equations with past history acting in two equations. Under certain conditions, we prove the global existence and uniqueness of the solution. Besides, we obtain the exponential stability of the solution

1.1 Problem statement

Let Ω be an open bounded set of \mathbb{R}^n with smooth boundary Γ . The object of this chapter is to study the existence and the exponential stability of the following system

$$u_{tt} - \Delta u + \int_0^{+\infty} g(s)\Delta u(t-s) ds + \alpha v = 0, \quad \text{in } \Omega \times (0, +\infty), \quad (1.1)$$

$$v_{tt} - \Delta v + \int_0^{+\infty} g(s)\Delta v(t-s) ds + \alpha u = 0, \quad \text{in } \Omega \times (0, +\infty), \quad (1.2)$$

$$u = v = 0, \quad \text{on } \Gamma \times (0, +\infty), \quad (1.3)$$

$$(u(x, 0), v(x, 0)) = (u_0(x), v_0(x)), \quad \text{in } \Omega, \quad (1.4)$$

$$(u_t(x, 0), v_t(x, 0)) = (u_1(x), v_1(x)), \quad \text{in } \Omega. \quad (1.5)$$

Where $s > 0$ is a fixed delay parameter, and α is a positive real number. The equations (1.4) and (1.5) are the initial conditions, with u_0, u_1, v_0 and v_1 are given.

In the study of the system (1.1)-(1.5), we will need to make some suitable hypotheses on the kernel g to achieve the desired results.

Hypotheses :

As in [10], we assume that the kernel g satisfies the following hypotheses

- $g(s) > 0$,
- $g \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$,
- $\exists q_0, q_1 > 0$ such that;

$$-q_0g(s) \leq g'(s) \leq -q_1g(s), \forall t \geq 0. \tag{1.6}$$

Remark 1.1.1. There are real functions that verify the above-mentioned hypotheses. For example we can take $g(t) = e^{-\mu t}$.

These hypotheses allow us to introduce the Hilbert space $L_g^2(\mathbb{R}^+; H_0^1(\Omega))$ (Hilbert space of $H_0^1(\Omega)$ -value functions on \mathbb{R}^+), endowed with the inner product

$$(f, h)_{L_g^2(\mathbb{R}^+; H_0^1(\Omega))} = \int_0^{+\infty} g(s) \int_{\Omega} \nabla f(x, s) \cdot \overline{\nabla h(x, s)} dx ds. \tag{1.7}$$

And from the hypotheses, we can write

$$b_0 = \int_0^{+\infty} g(s) ds < +\infty.$$

1.2 Well-posedness of the problem

In this section, we study the existence and uniqueness of solutions for the system (1.1)-(1.5). To do this, we reformulate it as a Cauchy linear problem by adding a new unknown. Then, we use the theory of the semigroup and we base ourselves on the theorem of Lumer-Phillips and Theorem (A.1).

Our first step will be to transform the system (1.1)-(1.5). For that we introduce as in [4] and [5], the new variables $\eta = \eta^t(s)$ and $\gamma = \gamma^t(s)$, the relative history of u and v respectively

$$\eta = \eta^t(s) = u(t) - u(t - s),$$

and

$$\gamma = \gamma^t(s) = v(t) - v(t - s),$$

where $s > 0$. Then, we have

$$\eta_t = u_t(t) - \frac{du}{d\tau}, \quad \eta_s = \frac{du}{d\tau},$$

and

$$\gamma_t = v_t(t) - \frac{dv}{d\tau}, \quad \gamma_s = \frac{dv}{d\tau}.$$

Hence, we get the equations

$$\eta_t + \eta_s - u_t = 0, \text{ in } \Omega \times (0, +\infty),$$

$$\gamma_t + \gamma_s - v_t = 0, \text{ in } \Omega \times (0, +\infty).$$

With the initial conditions

$$\begin{aligned} \eta_0(\cdot, s) &= u_0(\cdot) - u(\cdot, -s), \quad \text{in } \Omega \times (0, +\infty), \\ \gamma_0(\cdot, s) &= v_0(\cdot) - v(\cdot, -s), \quad \text{in } \Omega \times (0, +\infty). \end{aligned}$$

And, putting

$$\beta_0 = 1 - b_0 > 0.$$

Then from where the system (1.1)-(1.5) turns into the system

$$u_{tt} - \beta_0 \Delta u - \int_0^{+\infty} g(s) \Delta \eta(s) ds + \alpha v = 0, \quad \text{in } \Omega \times (0, +\infty), \quad (1.8)$$

$$v_{tt} - \beta_0 \Delta v - \int_0^{+\infty} g(s) \Delta \gamma(s) ds + \alpha u = 0, \quad \text{in } \Omega \times (0, +\infty), \quad (1.9)$$

$$\eta_t + \eta_s - u_t = 0, \quad \text{in } \Omega \times (0, +\infty), \quad (1.10)$$

$$\gamma_t + \gamma_s - v_t = 0, \quad \text{in } \Omega \times (0, +\infty), \quad (1.11)$$

$$u = v = \eta^t(s) = \gamma^s(s) = 0, \quad \text{on } \Gamma \times (0, +\infty), \quad \forall s > 0, \quad (1.12)$$

$$(u(x, 0), v(x, 0)) = (u_0(x), v_0(x)), \quad \text{in } \Omega, \quad (1.13)$$

$$(u_t(x, 0), v_t(x, 0)) = (u_1(x), v_1(x)), \quad \text{in } \Omega, \quad (1.14)$$

$$\eta_0(\cdot, s) = u_0(\cdot) - u(\cdot, -s), \quad \text{in } \Omega \times (0, +\infty), \quad (1.15)$$

$$\gamma_0(\cdot, s) = v_0(\cdot) - v(\cdot, -s), \quad \text{in } \Omega \times (0, +\infty). \quad (1.16)$$

Remark 1.2.1. The conditions (1.15), (1.16) and s, means that we took the initial conditions in past history in which the phenomenon occurred.

We study the existence and uniqueness of solutions for the system (1.8)-(1.16). And to give an accurate formulation of the evolution problem we introduce the product Hilbert spaces

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L_g^2(\mathbb{R}^+; H_0^1(\Omega)) \times L_g^2(\mathbb{R}^+; H_0^1(\Omega))$$

endowed with the following inner product

$$\begin{aligned} \langle U, V \rangle_{\mathcal{H}} &= \beta_0 \int_{\Omega} \nabla u_1 \cdot \nabla \overline{v_1} dx + \int_{\Omega} u_2 \overline{v_2} dx + \beta_0 \int_{\Omega} \nabla u_3 \cdot \nabla \overline{v_3} dx + \int_{\Omega} u_4 \overline{v_4} dx + \alpha \\ &\int_{\Omega} (u_1 \overline{v_3} + u_3 \overline{v_1}) dx + \int_0^{+\infty} g(s) \int_{\Omega} \nabla u_5(x, s) \cdot \nabla \overline{v_5(x, s)} dx ds + \int_0^{+\infty} g(s) \int_{\Omega} \nabla u_6(x, s) \cdot \nabla \overline{v_6(x, s)} dx ds. \end{aligned} \quad (1.17)$$

For all, $U = (u_1, u_2, u_3, u_4, u_5, u_6)^T$, $V = (v_1, v_2, v_3, v_4, v_5, v_6)^T \in \mathcal{H}$.

Now, we write the system (1.8)-(1.16) as a Cauchy linear problem in Hilbert space \mathcal{H} . We take $\mathcal{U} = (u, u_t, v, v_t, \eta, \gamma)^T \in \mathcal{H}$, and we define the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ given as follows

$$\mathcal{A} = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 \\ \beta_0 \Delta & 0 & -\alpha I & 0 & \mathcal{T}_{\eta} & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ -\alpha I & 0 & \beta_0 \Delta & 0 & 0 & \mathcal{T}_{\gamma} \\ 0 & I & 0 & 0 & -\partial_s & 0 \\ 0 & 0 & I & 0 & 0 & -\partial_s \end{bmatrix}$$

With domain

$$\mathcal{D}(\mathcal{A}) = \left\{ (u, \varphi, v, \psi, \eta, \gamma)^T \in \mathcal{H}; \beta_0 u + \int_0^{+\infty} g(s)\eta(s) ds \in H_0^1(\Omega) \cap H^2(\Omega) \right. \\ \left. \varphi \in H_0^1(\Omega), \beta_0 v + \int_0^{+\infty} g(s)\gamma(s) ds \in H_0^1(\Omega) \cap H^2(\Omega), \psi \in H_0^1(\Omega), \eta \text{ and } \gamma \in \mathcal{D}(\mathcal{T}) \right\}.$$

where

$$\mathcal{T}z = \int_0^{+\infty} g(s)\Delta z(s) ds, \quad \forall z \in \mathcal{D}(\mathcal{T}),$$

with

$$\mathcal{D}(\mathcal{T}) = \{z \in L_g^2(\mathbb{R}^+; H_0^1(\Omega)); z_s \in L_g^2(\mathbb{R}^+; H_0^1(\Omega)), z(0) = 0\}.$$

Remark 1.2.2. η_s and γ_s are the distributional derivative of η, γ with respect to the internal variable s , respectively.

Remark 1.2.3. We have $\mathcal{D}(\mathcal{A})$ is dense in \mathcal{H} .

Thus, the system (1.8)-(1.16) can be rewritten in the following

$$\begin{cases} \mathcal{U}_t(t) = \mathcal{A}\mathcal{U}(t), & \forall t > 0, \\ \mathcal{U}(0) = \mathcal{U}_0, \end{cases} \quad (1.18)$$

where $\mathcal{U}_0 = (u_0, u_1, v_0, v_1, \eta_0, \gamma_0)^T \in \mathcal{H}$.

Our goal now is to show that problem (1.18) is well positioned in terms of existence and uniqueness of solutions and this is what the following theorem ensured.

Theorem 1.2.4. *The operator \mathcal{A} is the infinitesimal generator of \mathcal{C}_0 - semigroup of contractions over the Hilbert space \mathcal{H} . Thus, for any initial data $\mathcal{U}_0 \in \mathcal{H}$, the problem (1.18) has a unique weak solution $\mathcal{U} \in C(\mathbb{R}^+; \mathcal{H})$.*

Moreover, if $\mathcal{U}_0 \in \mathcal{D}(\mathcal{A})$, then the solution $\mathcal{U} \in C(\mathbb{R}^+; \mathcal{D}(\mathcal{A})) \cap C^1(\mathbb{R}^+; \mathcal{H})$.

Proof. We will use the Lumer-Phillips theorem (A.2.10) and the theorem (A.2.9). For this purpose, we need the following two steps:

Step 1. \mathcal{A} is dissipative.

For any $\mathcal{U} = (u, u_t, v, v_t, \eta, \gamma)^T \in \mathcal{D}(\mathcal{A})$, using the inner product (1.17), we get

$$\langle \mathcal{A}\mathcal{U}, \mathcal{U} \rangle_{\mathcal{H}} = \langle \mathcal{U}_t, \mathcal{U} \rangle_{\mathcal{H}} = \beta_0 \int_{\Omega} \nabla u_t \cdot \nabla \bar{u} dx + \int_{\Omega} u_{tt} \bar{u}_t dx + \beta_0 \int_{\Omega} \nabla v_t \cdot \nabla \bar{v} dx + \int_{\Omega} v_{tt} \bar{v}_t dx \\ + \alpha \int_{\Omega} (u_t \bar{v} + v_t \bar{u}) dx + \int_0^{+\infty} g(s) \int_{\Omega} \nabla \eta_t(x, s) \cdot \nabla \overline{\eta(x, s)} dx ds + \int_0^{+\infty} g(s) \int_{\Omega} \nabla \gamma_t(x, s) \cdot \nabla \overline{\gamma(x, s)} dx ds.$$

Substituting u_{tt}, v_{tt}, η_t and γ_t from (1.8), (1.9), (1.10) and (1.11) respectively, getting

$$\langle \mathcal{A}\mathcal{U}, \mathcal{U} \rangle_{\mathcal{H}} = \beta_0 \int_{\Omega} \nabla u_t \cdot \nabla \bar{u} dx + \int_{\Omega} (\beta_0 \Delta u + \int_0^{+\infty} g(s)\Delta \eta(s) ds - \alpha v) \bar{u}_t dx + \beta_0 \int_{\Omega} \nabla v_t \cdot \nabla \bar{v} dx + \\ \int_{\Omega} (\beta_0 \Delta v + \int_0^{+\infty} g(s)\Delta \gamma(s) ds - \alpha u) \bar{v}_t dx + \alpha \int_{\Omega} (u_t \bar{v} + v_t \bar{u}) dx + \int_0^{+\infty} g(s) \int_{\Omega} \nabla (u_t - \eta_s(s)) \cdot \nabla \overline{\eta(s)} dx ds$$

$$+ \int_0^{+\infty} g(s) \int_{\Omega} \nabla(v_t - \gamma_s(s)) \cdot \nabla \overline{\gamma(s)} \, dx \, ds.$$

Using Green's formula, we obtain

$$\begin{aligned} \langle \mathcal{A}\mathcal{U}, \mathcal{U} \rangle_{\mathcal{H}} &= \beta_0 \int_{\Omega} \nabla u_t \cdot \nabla \bar{u} \, dx - \beta_0 \int_{\Omega} \nabla u \nabla \bar{u}_t \, dx - \alpha \int_{\Omega} v \bar{u}_t \, dx - \int_0^{+\infty} g(s) \int_{\Omega} \nabla \eta \nabla \bar{u}_t \, dx \, ds \\ &+ \beta_0 \int_{\Omega} \nabla v_t \cdot \nabla \bar{v} \, dx - \beta_0 \int_{\Omega} \nabla v \nabla \bar{v}_t \, dx - \int_0^{+\infty} g(s) \int_{\Omega} \nabla \gamma \nabla \bar{v}_t \, dx \, ds - \alpha \int_{\Omega} u \bar{v}_t \, dx + \alpha \int_{\Omega} (u_t \bar{v} + v_t \bar{u}) \, dx + \\ &\int_0^{+\infty} g(s) \int_{\Omega} \nabla u_t \nabla \bar{\eta(s)} \, dx \, ds - \int_0^{+\infty} g(s) \int_{\Omega} \nabla \eta_s(s) \cdot \nabla \bar{\eta(s)} \, dx \, ds + \int_0^{+\infty} g(s) \int_{\Omega} \nabla v_t \nabla \bar{\gamma(s)} \, dx \, ds \\ &- \int_0^{+\infty} g(s) \int_{\Omega} \nabla \gamma_s(s) \cdot \nabla \bar{\gamma(s)} \, dx \, ds, \end{aligned}$$

after simplifying, we get

$$\begin{aligned} \langle \mathcal{A}\mathcal{U}, \mathcal{U} \rangle_{\mathcal{H}} &= \beta_0 \int_{\Omega} \nabla u_t \cdot \nabla \bar{u} \, dx - \beta_0 \int_{\Omega} \nabla u_t \cdot \nabla \bar{u} \, dx + \beta_0 \int_{\Omega} \nabla v_t \cdot \nabla \bar{v} \, dx - \beta_0 \int_{\Omega} \nabla v_t \cdot \nabla \bar{v} \, dx + \alpha \int_{\Omega} (u_t \bar{v} + v_t \bar{u}) \, dx \\ &- \alpha \int_{\Omega} (u_t \bar{v} + v_t \bar{u}) \, dx + \int_0^{+\infty} g(s) \int_{\Omega} \nabla u_t \nabla \bar{\eta(s)} \, dx \, ds - \int_0^{+\infty} g(s) \int_{\Omega} \nabla u_t \nabla \bar{\eta(s)} \, dx \, ds \\ &+ \int_0^{+\infty} g(s) \int_{\Omega} \nabla v_t \nabla \bar{\gamma(s)} \, dx \, ds - \int_0^{+\infty} g(s) \int_{\Omega} \nabla v_t \nabla \bar{\gamma(s)} \, dx \, ds - \int_0^{+\infty} g(s) \int_{\Omega} \nabla \eta_s(s) \cdot \nabla \bar{\eta(s)} \, dx \, ds \\ &- \int_0^{+\infty} g(s) \int_{\Omega} \nabla \gamma_s(s) \cdot \nabla \bar{\gamma(s)} \, dx \, ds. \end{aligned}$$

Then

$$\begin{aligned} \langle \mathcal{A}\mathcal{U}, \mathcal{U} \rangle_{\mathcal{H}} &= - \int_0^{+\infty} g(s) \int_{\Omega} \nabla \eta_s(s) \cdot \nabla \bar{\eta(s)} \, dx \, ds - \int_0^{+\infty} g(s) \int_{\Omega} \nabla \gamma_s(s) \cdot \nabla \bar{\gamma(s)} \, dx \, ds + 2i \operatorname{Im} \left(\beta_0 \int_{\Omega} \nabla u_t \cdot \nabla \bar{u} \, dx \right. \\ &\left. + \int_{\Omega} \nabla v_t \cdot \nabla \bar{v} \, dx + \alpha \int_{\Omega} (u_t \bar{v} + v_t \bar{u}) \, dx + \int_0^{+\infty} g(s) \int_{\Omega} \nabla u_t \nabla \bar{\eta(s)} \, dx \, ds + \int_0^{+\infty} g(s) \int_{\Omega} \nabla v_t \nabla \bar{\gamma(s)} \, dx \, ds \right). \end{aligned}$$

Taking its real part, we arrive at

$$\begin{aligned} \operatorname{Re} \langle \mathcal{A}\mathcal{U}, \mathcal{U} \rangle_{\mathcal{H}} &= - \int_0^{+\infty} g(s) \int_{\Omega} \nabla \eta_s(s) \cdot \nabla \bar{\eta(s)} \, dx \, ds - \int_0^{+\infty} g(s) \int_{\Omega} \nabla \gamma_s(s) \cdot \nabla \bar{\gamma(s)} \, dx \, ds \\ &= - \frac{1}{2} \int_0^{+\infty} g(s) \frac{d}{ds} \int_{\Omega} (|\nabla \eta(s)|^2 + |\nabla \gamma(s)|^2) \, dx \, ds. \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}
 \operatorname{Re} \langle \mathcal{A}U, U \rangle_{\mathcal{H}} &= \left[-\frac{g(s)}{2} \int_{\Omega} (|\nabla \eta(s)|^2 + |\nabla \gamma(s)|^2) dx \right]_0^{+\infty} + \frac{1}{2} \int_0^{+\infty} g'(s) \int_{\Omega} (|\nabla \eta(s)|^2 + |\nabla \gamma(s)|^2) dx ds \\
 &= \frac{g(0)}{2} \int_{\Omega} (|\nabla \eta(0)|^2 + |\nabla \gamma(0)|^2) dx - \frac{1}{2} \lim_{s \rightarrow +\infty} g(s) \int_{\Omega} (|\nabla \eta(s)|^2 + |\nabla \gamma(s)|^2) dx \\
 &\quad + \frac{1}{2} \int_0^{+\infty} g'(s) \int_{\Omega} (|\nabla \eta(s)|^2 + |\nabla \gamma(s)|^2) dx ds.
 \end{aligned}$$

From the previous hypothesis on g we have, $g(+\infty) = 0$. And since $\eta, \gamma \in H_0^1(\Omega)$, and $\eta(0) = \gamma(0) = 0$. Then,

$$\frac{g(0)}{2} \int_{\Omega} (|\nabla \eta(0)|^2 + |\nabla \gamma(0)|^2) dx - \frac{1}{2} \lim_{s \rightarrow +\infty} g(s) \int_{\Omega} (|\nabla \eta(s)|^2 + |\nabla \gamma(s)|^2) dx = 0,$$

as a result

$$\operatorname{Re} \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = \frac{1}{2} \int_0^{+\infty} g'(s) \int_{\Omega} (|\nabla \eta(s)|^2 + |\nabla \gamma(s)|^2) dx ds.$$

Using (1.6), getting

$$\operatorname{Re} \langle \mathcal{A}U, U \rangle_{\mathcal{H}} \leq -\frac{q_1}{2} \int_0^{+\infty} g(s) \int_{\Omega} (|\nabla \eta(s)|^2 + |\nabla \gamma(s)|^2) dx ds \leq 0.$$

Therefore, \mathcal{A} is a dissipative operator.

Step 2. For all $\lambda > 0$, $(\lambda I - \mathcal{A})$ is maximal.

For this, let $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T$ be an element of \mathcal{H} . We show that there exists unique $U = (u, \varphi, v, \psi, \eta, \gamma)^T \in \mathcal{D}(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A})U = F,$$

then, in terms of its components, we can write

$$\lambda u - \varphi = f_1, \tag{1.19}$$

$$\lambda \varphi - \beta_0 \Delta u - \int_0^{+\infty} g(s) \Delta \eta(s) ds + \alpha v = f_2, \tag{1.20}$$

$$\lambda v - \psi = f_3, \tag{1.21}$$

$$\lambda \psi - \beta_0 \Delta v - \int_0^{+\infty} g(s) \Delta \gamma(s) ds + \alpha u = f_4, \tag{1.22}$$

$$\lambda \eta + \eta_s - \varphi = f_5, \tag{1.23}$$

$$\lambda \gamma + \gamma_s - \psi = f_6. \tag{1.24}$$

The equation (1.23) is ordinary differential equation of the first order, and the solution to its homogeneous equation is given by

$$\eta(s) = C e^{-\lambda s}, \text{ with } C \text{ is a constant.}$$

Then, we use the method of the variation of the constant, so the general solution of the differential equation is given by $\eta(s) = C(s)e^{-\lambda s}$, and since $\eta(0) = 0$, then $C(0) = 0$. After simplifications, we have

$$C'(s) = (f_5 + \varphi)e^{\lambda s},$$

therefore

$$C(., s) = C(0) + \int_0^s (f_5(., \tau) + \varphi(., \tau))e^{\lambda s} d\tau = \int_0^s e^{\lambda s} f_5(., \tau) d\tau + \varphi(., s) \int_0^s e^{\lambda s} d\tau.$$

Hence, the result will be as follows,

$$\eta(., s) = \frac{\varphi(., s)}{\lambda}(1 - e^{-\lambda s}) + \int_0^s e^{\lambda \tau} f_5(., \tau) d\tau. \quad (1.25)$$

Substituting φ from (1.19) and η from (1.25) into (1.20), getting

$$\lambda^2 u - \lambda f_1 - \beta_0 \Delta u + \alpha v - \int_0^{+\infty} g(s) \Delta \left[\frac{\lambda u - f_1}{\lambda} (1 - e^{-\lambda s}) + \int_0^s e^{\lambda(\tau-s)} f_5(\tau) d\tau \right] ds = f_2,$$

then

$$\lambda^2 u - \beta_g \Delta u + \alpha v = h_1, \quad (1.26)$$

where

$$h_1 = \lambda f_1 + f_2 + \int_0^{+\infty} g(s) \left[\frac{(e^{-\lambda s} - 1)}{\lambda} \Delta f_1 + \int_0^s e^{\lambda(\tau-s)} \Delta f_5(\tau) d\tau \right] ds.$$

With $\beta_g = \beta_0 + \int_0^{+\infty} g(s)(1 - e^{-\lambda s}) ds > 0$, is a positive constant in virtue of (1.6), and be $(1 - e^{-\lambda s})$ positive.

On the other hand, we also have the equation (1.24) is ordinary differential equation of the first order. Thus, in the same way we find

$$\gamma(., s) = \frac{\psi(., s)}{\lambda}(1 - e^{-\lambda s}) + \int_0^s e^{\lambda \tau} f_6(., \tau) d\tau. \quad (1.27)$$

Substituting ψ from (1.21) and γ from (1.27) into (1.22), we get

$$\lambda^2 v - \beta_g \Delta v + \alpha u = h_2, \quad (1.28)$$

where

$$h_2 = \lambda f_3 + f_4 + \int_0^{+\infty} g(s) \left[\frac{(e^{-\lambda s} - 1)}{\lambda} \Delta f_3 + \int_0^s e^{\lambda(\tau-s)} \Delta f_6(\tau) d\tau \right] ds.$$

First we prove that $u, v \in H_0^1(\Omega)$. To do this, multiplying (1.27), (1.28) by \tilde{u} , \tilde{v} respectively, and integrating over Ω . Then using Green's formula, we arrive at

$$\begin{cases} \lambda^2 \int_{\Omega} u \tilde{u} dx + \beta_g \int_{\Omega} \nabla u \cdot \nabla \tilde{u} dx + \alpha \int_{\Omega} v \tilde{u} dx = \langle h_1, \tilde{u} \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \\ \lambda^2 \int_{\Omega} v \tilde{v} dx + \beta_g \int_{\Omega} \nabla v \cdot \nabla \tilde{v} dx + \alpha \int_{\Omega} u \tilde{v} dx = \langle h_2, \tilde{v} \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \end{cases}, \quad \forall (\tilde{u}, \tilde{v}) \in V. \quad (1.29)$$

Next, the sum of the equations in (1.29) gives the following variational formulation:

$$a((u, v), (\tilde{u}, \tilde{v})) = L(\tilde{u}, \tilde{v}), \quad \forall (\tilde{u}, \tilde{v}) \in H_0^1(\Omega) \times H_0^1(\Omega),$$

where a is bilinear form and L is linear form, defined as

$$a((u, v), (\tilde{u}, \tilde{v})) = \lambda^2 \int_{\Omega} u \tilde{u} \, dx + \beta_g \int_{\Omega} \nabla u \cdot \nabla \tilde{u} \, dx + \lambda^2 \int_{\Omega} v \tilde{v} \, dx + \beta_g \int_{\Omega} \nabla v \cdot \nabla \tilde{v} \, dx + \alpha \int_{\Omega} (v \tilde{u} + u \tilde{v}) \, dx,$$

end

$$L(\tilde{u}, \tilde{v}) = \langle h_1, \tilde{u} \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} + \langle h_2, \tilde{v} \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}, \quad \forall (\tilde{u}, \tilde{v}) \in V.$$

With $V = H_0^1(\Omega) \times H_0^1(\Omega)$ is a Hilbert space equipped with the norm

$$\|(u, v)\|_V^2 = \|u\|^2 + \|v\|^2, \quad \forall (u, v) \in V.$$

Now we will use Lax-Milgram's theorem (A.3.7) to prove the existence and uniqueness of the solutions $(u, v) \in V$. For this we prove

1. The coercivity of a

We have, for all $(u, v) \in V$,

$$\begin{aligned} a((u, v), (u, v)) &= \lambda^2 \int_{\Omega} |u|^2 \, dx + \lambda^2 \int_{\Omega} |v|^2 \, dx + \beta_g \int_{\Omega} |\nabla u|^2 \, dx + \beta_g \int_{\Omega} |\nabla v|^2 \, dx + 2\alpha \int_{\Omega} uv \, dx \\ &= \lambda^2 \|u\|_2^2 + \lambda^2 \|v\|_2^2 + \beta_g \|u\|^2 + \beta_g \|v\|^2 + 2\alpha \int_{\Omega} uv \, dx. \end{aligned}$$

Using the Young's inequality, we find for all $\varepsilon > 0$,

$$\begin{aligned} a((u, v), (u, v)) &\geq \lambda^2 \|u\|_2^2 + \lambda^2 \|v\|_2^2 + \beta_g \|u\|^2 + \beta_g \|v\|^2 - \alpha\varepsilon \int_{\Omega} |u|^2 \, dx - \frac{\alpha}{\varepsilon} \int_{\Omega} |v|^2 \, dx \\ &= (\lambda^2 - \alpha\varepsilon) \|u\|_2^2 + (\lambda^2 - \frac{\alpha}{\varepsilon}) \|v\|_2^2 + \beta_g \|u\|^2 + \beta_g \|v\|^2. \end{aligned}$$

If $\alpha \leq \lambda^2$, we can choose $\varepsilon \in (\frac{\alpha}{\lambda^2}, \frac{\lambda^2}{\alpha})$ where $\lambda^2 - \alpha\varepsilon$ and $\lambda^2 - \frac{\alpha}{\varepsilon}$ are positive. And if $\alpha \geq \lambda^2$, we can choose $\varepsilon \in (\frac{\lambda^2}{\alpha}, \frac{\alpha}{\lambda^2})$ where $\lambda^2 - \alpha\varepsilon$ and $\lambda^2 - \frac{\alpha}{\varepsilon}$ are positive. Then we can find $\varepsilon > 0$, such that $\lambda^2 - \alpha\varepsilon$ and $\lambda^2 - \frac{\alpha}{\varepsilon}$ are positive.

Therefore and for the chosen ε , we have

$$\begin{aligned} a((u, v), (u, v)) &\geq \beta_g (\|u\|^2 + \|v\|^2) \\ &\geq \beta_g \|(u, v)\|_V^2. \end{aligned}$$

So we have the coercivity.

2. The continuity of a

Using the Cauchy-Schwarz's inequality, we get

$$\begin{aligned} |a((u, v), (\tilde{u}, \tilde{v}))| &\leq \lambda^2 \|u\|_2 \|\tilde{u}\|_2 + \lambda^2 \|v\|_2 \|\tilde{v}\|_2 + \beta_g \|u\| \|\tilde{u}\| \\ &\quad + \beta_g \|v\| \|\tilde{v}\| + \alpha \|v\|_2 \|\tilde{u}\|_2 + \alpha \|u\|_2 \|\tilde{v}\|_2. \end{aligned}$$

Then, we find that

$$|a((u, v), (\tilde{u}, \tilde{v}))| \leq C(\|u\| \|\tilde{u}\| + \|v\| \|\tilde{v}\| + \|v\| \|\tilde{u}\| + \|u\| \|\tilde{v}\|)$$

$$\leq C(\|u\| + \|v\|)(\|\tilde{u}\| + \|\tilde{v}\|)$$

$$\leq C \|(u, v)\|_V \|(\tilde{u}, \tilde{v})\|_V,$$

where C is a positive constant. So we have the continuity.

3. The continuity of \mathbf{L} .

Using the Cauchy-Schwarz's inequality, we get

$$|L(\tilde{u}, \tilde{v})| \leq C(\|\tilde{u}\| + \|\tilde{v}\|)$$

$$\leq C \|(\tilde{u}, \tilde{v})\|_V,$$

where C is a positive constant. So we have the continuity of \mathbf{L} .

Thus, through Lax-Milgram's theorem (A.3.7), we obtain the existence and uniqueness of $u, v \in H_0^1(\Omega)$. Then from (1.19) and (1.21), we have $\varphi, \psi \in H_0^1(\Omega)$. Hence, from (1.25) and (1.27) we guarantee the existence and uniqueness of η and γ .

Now we will prove that $\eta, \gamma \in \mathcal{D}(\mathcal{T})$. First, we prove that $\eta \in \mathcal{D}(\mathcal{T})$.

From (1.23), we have

$$\lambda \nabla \eta(t, s) + \nabla \eta_s(t, s) = \nabla \varphi(t) + \nabla f_5(t, s),$$

by multiplication to $\nabla \eta(t, s)$ and integrating on Ω , getting

$$\lambda \int_{\Omega} |\nabla \eta(t, s)|^2 dx + \int_{\Omega} \nabla \eta(t, s) \nabla \eta_s(t, s) dx = \int_{\Omega} \nabla \eta(t, s) \nabla \varphi(t) dx + \int_{\Omega} \nabla \eta(t, s) \nabla f_5(t, s) dx.$$

Using Cauchy-Schwarz's inequality for the second term to the left of the above equality, we get

$$\lambda \int_{\Omega} |\nabla \eta(t, s)|^2 dx + \int_{\Omega} \nabla \eta(t, s) \nabla \eta_s(t, s) dx \leq \|\eta(s)\| \|\varphi\| + \|\eta(s)\| \|f_5(s)\|.$$

After using Young's inequality, we find for all $\varepsilon > 0$,

$$\lambda \int_{\Omega} |\nabla \eta(t, s)|^2 dx + \int_{\Omega} \nabla \eta(t, s) \nabla \eta_s(t, s) dx \leq \varepsilon \|\eta(s)\|^2 + K_{1\varepsilon} \|\varphi\|^2 + \varepsilon \|\eta(s)\|^2 + K_{2\varepsilon} \|f_5(s)\|^2,$$

then, we have for all $\varepsilon > 0$,

$$(1 - 2\varepsilon)\lambda \|\eta(s)\|^2 + \int_{\Omega} \nabla \eta(t, s) \nabla \eta_s(t, s) dx \leq K_{1\varepsilon} \|\varphi\|^2 + K_{2\varepsilon} \|f_5(s)\|^2.$$

Now, by multiplication to g and integrating on $(0, +\infty)$, getting for all $\varepsilon > 0$,

$$(1-2\varepsilon)\lambda \int_0^{+\infty} g(s) \|\eta\|^2 ds + \underbrace{\int_0^{+\infty} g(s) \int_{\Omega} \nabla \eta(t, s) \nabla \eta_s(t, s) dx ds}_{I_1} \leq K_{1\varepsilon} \int_0^{+\infty} g(s) \|\varphi\|^2 ds + K_{2\varepsilon} \int_0^{+\infty} g(s) \|f_5\|^2 ds$$

$$\leq K(\|\varphi\|^2 + \|f_5\|_{2,g}^2),$$

where $K = \max(K_{1\varepsilon}b_0, K_{2\varepsilon})$. On the other hand, we have

$$I_1 = \frac{1}{2} \int_0^{+\infty} g(s) \frac{d}{ds} \int_{\Omega} |\nabla \eta(s)|^2 dx ds.$$

Integrating by parts and using (1.6), we obtain

$$I_1 \geq \frac{q_1}{2} \int_0^{+\infty} g(s) \|\eta\|^2 ds.$$

Hence, getting for all $\varepsilon > 0$,

$$(1 - 2\varepsilon)\lambda \int_0^{+\infty} g(s) \|\eta\|^2 ds + \frac{q_1}{2} \int_0^{+\infty} g(s) \|\eta\|^2 ds \leq K(\|\varphi\|^2 + \|f_5\|_{2,g}^2),$$

then, we have for all $\varepsilon > 0$,

$$(\lambda - 2\lambda\varepsilon - \frac{q_1}{2}) \|\eta\|_{2,g}^2 \leq K(\|\varphi\|^2 + \|f_5\|_{2,g}^2).$$

We choose ε where $(\lambda - 2\varepsilon\lambda - \frac{q_1}{2}) > 0$, then we obtain

$$\|\eta\|_{2,g}^2 \leq C(\|\varphi\|^2 + \|f_5\|_{2,g}^2),$$

where C is a positive constant. From where it follows that $\eta \in L_g^2(\mathbb{R}^+; H_0^1(\Omega))$.

Now from (1.23), we obtain

$$\nabla \eta_s = \nabla f_5 + \nabla \varphi - \lambda \nabla \eta,$$

then,

$$|\nabla \eta_s| \leq |\nabla \varphi| + |\nabla f_5| + |\nabla \eta|.$$

Using Young's inequality and integrating on Ω , getting

$$\|\eta_s\|^2 \leq 2\|f_5\|^2 + 4\|\varphi\|^2 + 4\lambda\|\eta\|^2,$$

by multiplication to g and integrating on $(0, +\infty)$, we find

$$\begin{aligned} \|\eta_s\|_{2,g}^2 &\leq 2\|f_5\|_{2,g}^2 + 4b_0\|\varphi\|^2 + 4\lambda\|\eta\|_{2,g}^2 \\ &\leq C(\|f_5\|_{2,g}^2 + \|\varphi\|^2 + \|\eta\|_{2,g}^2). \end{aligned}$$

Where C is a positive constant.

Hence,

$$\eta_s \in L_g^2(\mathbb{R}^+; H_0^1(\Omega)).$$

From (1.25), we have $\eta(0) = 0$, then $\eta \in \mathcal{D}(\mathcal{T})$. Next we will prove that $\gamma \in \mathcal{D}(\mathcal{T})$.

From (1.24), we have

$$\lambda \nabla \gamma(t, s) + \nabla \gamma_s(t, s) = \nabla \psi(t) + \nabla f_6(t, s).$$

By following the same steps of the previous proof, we arrive at

$$\|\gamma\|_{2,g}^2 \leq C_1(\|\psi\|^2 + \|f_6\|_{2,g}^2),$$

and

$$\|\gamma_s\|_{2,g}^2 \leq C_2(\|f_6\|_{2,g}^2 + \|\psi\|^2 + \|\gamma\|_{2,g}^2),$$

with C_1 and C_2 are positive constants.

Hence,

$$\gamma, \gamma_s \in L_g^2(\mathbb{R}^+; H_0^1(\Omega)).$$

Now from (1.27), we have $\gamma(0) = 0$.

In the end, we have proved that $\gamma \in \mathcal{D}(\mathcal{T})$. And in the last step we still have to proof that

$$\beta_0 u + \int_0^{+\infty} g(s)\eta(s) ds \text{ and } \beta_0 v + \int_0^{+\infty} g(s)\gamma(s) ds \in H^2(\Omega).$$

From (1.20) and (1.22), we obtain

$$\Delta(\beta_0 u + \int_0^{+\infty} g(s)\eta(s) ds) = \lambda\varphi + \alpha v - f_2,$$

and

$$\Delta(\beta_0 v + \int_0^{+\infty} g(s)\gamma(s) ds) = \lambda\psi + \alpha u - f_4.$$

Hence and from where it follows that

$$\beta_0 u + \int_0^{+\infty} g(s)\eta(s) ds, \beta_0 v + \int_0^{+\infty} g(s)\gamma(s) ds \in L^2(\Omega).$$

Then, getting

$$\beta_0 u + \int_0^{+\infty} g(s)\eta(s) ds, \beta_0 v + \int_0^{+\infty} g(s)\gamma(s) ds \in H^2(\Omega).$$

Thus, $(\lambda I - \mathcal{A})$ is maximal.

Then, thanks to the Lumer-Phillips theorem (A.2.10) the operator \mathcal{A} generates a C_0 -semigroup of contractions on \mathcal{H} . Therefore, theorem (A.2.9) guarantees that the Cauchy problem (1.18) has a unique solution. The proof is complete. \blacksquare

1.3 Exponential stability

In this section, we will show the exponential stability of (1.18). The method that we will use in the following theorem is based on the Gearhart-Prüss theorem (A.2.11).

We found in the previous section that the operator \mathcal{A} generates a C_0 -semigroup of contractions $S(t) = e^{t\mathcal{A}}$ on \mathcal{H} . Here we will show that $S(t)$ is exponentially stable, which is assured by the following theorem.

Theorem 1.3.1. *The semigroup associated to the problem (1.18) $S(t)$ is exponentially stable, i.e., there exist positive constants M, m such that*

$$\|S(t)\|_{\mathcal{L}(\mathcal{H})} \leq M e^{-mt}, \text{ with } M \leq 1.$$

Proof. We will use the theorem (A.2.11). First, we prove that $i\mathbb{R} = \{i\beta : \beta \in \mathbb{R}\} \subset \rho(\mathcal{A})$. It is equivalent to prove that $i\mathbb{R} \cap \sigma(\mathcal{A}) = \emptyset$. Using contradiction arguments. Suppose that $i\lambda$ is imaginary eigenvalue of \mathcal{A} . Then, we have

$$\mathcal{A}\mathcal{U} = i\lambda\mathcal{U},$$

where $\mathcal{U} = (u, \varphi, v, \psi, \eta, \gamma)^T \in \mathcal{H}$. So, we have

$$(i\lambda I - \mathcal{A})\mathcal{U} = 0,$$

that is,

$$i\lambda u - \varphi = 0, \quad (1.30)$$

$$i\lambda\varphi - \beta_0\Delta u - \int_0^{+\infty} g(s)\Delta\eta(s) ds + \alpha v = 0, \quad (1.31)$$

$$i\lambda v - \psi = 0, \quad (1.32)$$

$$i\lambda\psi - \beta_0\Delta v - \int_0^{+\infty} g(s)\Delta\gamma(s) ds + \alpha u = 0, \quad (1.33)$$

$$i\lambda\eta + \eta_s - \varphi = 0, \quad (1.34)$$

$$i\lambda\gamma + \gamma_s - \psi = 0. \quad (1.35)$$

In the previous section we take $F = 0$. Then, we arrive at

$$a((u, v), (\tilde{u}, \tilde{v})) = 0, \quad \forall (\tilde{u}, \tilde{v}) \in H_0^1(\Omega) \times H_0^1(\Omega).$$

Since we have the bilinear form a is continuous and coercive on $H_0^1(\Omega)$. Then, Lax-Milgram's theorem give us $u = v = 0$. From (1.30)-(1.32), we get $\varphi = \psi = 0$. And then getting $\eta = \gamma = 0$.

Hence, $\mathcal{U} = 0$, here we have a contradiction. Thus, we have $i\mathbb{R} \subset \rho(\mathcal{A})$. The next step consists to proof that

$$\limsup_{|\lambda| \rightarrow +\infty} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < +\infty. \quad (1.36)$$

We use the argument contradiction again to prove that, suppose that (1.36) is not true. Then there exists a sequence $\lambda_n \in \mathbb{R}$ with $i\lambda_n \in \rho(\mathcal{A})$ and $|\lambda_n| \rightarrow +\infty$, such that

$$\|(i\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \rightarrow +\infty, \quad \text{as } n \rightarrow +\infty.$$

This means that,

$$\|(i\lambda_n I - \mathcal{A})\|_{\mathcal{L}(\mathcal{H})} \rightarrow 0, \quad \text{as } n \rightarrow +\infty,$$

that is, there exists also a sequence of complex function $\mathcal{U}_n = (u_n, \varphi_n, v_n, \psi_n, \eta_n, \gamma_n)^T \in \mathcal{D}(\mathcal{A})$ with $\|\mathcal{U}_n\|_{\mathcal{H}} = 1$ such that

$$(i\lambda_n I - \mathcal{A})\mathcal{U}_n \rightarrow 0 \quad \text{in } \mathcal{H}, \quad \text{as } n \rightarrow +\infty, \quad (1.37)$$

i.e.,

$$i\lambda_n u_n - \varphi_n \rightarrow 0 \quad \text{in } H_0^1(\Omega), \quad (1.38)$$

$$i\lambda_n\varphi_n - \beta_0\Delta u_n - \int_0^{+\infty} g(s)\Delta\eta_n(s) ds + \alpha v_n \rightarrow 0 \quad \text{in } L^2(\Omega), \quad (1.39)$$

$$i\lambda_n v_n - \psi_n \rightarrow 0 \quad \text{in } H_0^1(\Omega), \quad (1.40)$$

$$i\lambda_n \psi_n - \beta_0 \Delta v_n - \int_0^{+\infty} g(s) \Delta \gamma_n(s) ds + \alpha u_n \rightarrow 0 \text{ in } L^2(\Omega), \quad (1.41)$$

$$i\lambda_n \eta_n + \partial_s \eta_n - \varphi_n \rightarrow 0 \text{ in } L_g^2(\mathbb{R}^+; H_0^1(\Omega)), \quad (1.42)$$

$$i\lambda_n \gamma_n + \partial_s \gamma_n - \psi_n \rightarrow 0 \text{ in } L_g^2(\mathbb{R}^+; H_0^1(\Omega)). \quad (1.43)$$

Now taking the inner product (1.17) of $(i\lambda_n I - \mathcal{A})\mathcal{U}_n$ with \mathcal{U}_n and then taking its real part, we get

$$\begin{aligned} \operatorname{Re} \langle (i\lambda_n I - \mathcal{A})\mathcal{U}_n, \mathcal{U}_n \rangle_{\mathcal{H}} &= \operatorname{Re} \langle i\lambda_n I - \mathcal{U}_n, \mathcal{U}_n \rangle_{\mathcal{H}} - \operatorname{Re} \langle \mathcal{A}\mathcal{U}_n, \mathcal{U}_n \rangle_{\mathcal{H}} \\ &= -\operatorname{Re} \langle \mathcal{A}\mathcal{U}, \mathcal{U} \rangle_{\mathcal{H}}. \end{aligned}$$

Then,

$$\operatorname{Re} \langle (i\lambda_n I - \mathcal{A})\mathcal{U}_n, \mathcal{U}_n \rangle_{\mathcal{H}} = -\frac{1}{2} \int_0^{+\infty} g'(s) \int_{\Omega} (|\nabla \eta_n|^2 + |\nabla \gamma_n|^2) dx ds.$$

From (1.37), and as \mathcal{U}_n is bounded in \mathcal{H} , we have

$$\operatorname{Re} \langle (i\lambda_n I - \mathcal{A})\mathcal{U}_n, \mathcal{U}_n \rangle_{\mathcal{H}} \rightarrow 0 \text{ in } \mathbb{R}.$$

Using (1.6), we get

$$\frac{q_1}{2} \int_0^{+\infty} g(s) \int_{\Omega} (|\nabla \eta_n|^2 + |\nabla \gamma_n|^2) dx ds \rightarrow 0,$$

then, we find that

$$\eta_n \rightarrow 0 \text{ in } L_g^2(\mathbb{R}^+; H_0^1(\Omega)). \quad (1.44)$$

$$\gamma_n \rightarrow 0 \text{ in } L_g^2(\mathbb{R}^+; H_0^1(\Omega)). \quad (1.45)$$

Thus, it follows that

$$\eta_n \rightarrow 0 \text{ in } H_0^1(\Omega). \quad (1.46)$$

$$\gamma_n \rightarrow 0 \text{ in } H_0^1(\Omega). \quad (1.47)$$

On the other hand, taking the inner product of (1.38) with $\beta_0 u_n$ in $H_0^1(\Omega)$, we obtain

$$\langle (i\lambda_n u_n - \varphi_n), \beta_0 u_n \rangle = i\lambda_n \beta_0 \int_{\Omega} |\nabla u_n|^2 dx - \beta_0 \langle \varphi_n, u_n \rangle.$$

From (1.38), and as u_n is bounded in $H_0^1(\Omega)$, we have

$$\langle (i\lambda_n u_n - \varphi_n), \beta_0 u_n \rangle \rightarrow 0. \quad (1.48)$$

Now, taking the inner product of (1.39) with φ_n in $L^2(\Omega)$ and using Green's formula, we get

$$\begin{aligned} \left\langle (i\lambda_n \varphi_n - \beta_0 \Delta u_n - \int_0^{+\infty} g(s) \Delta \eta_n(s) ds + \alpha v_n), \varphi \right\rangle_2 &= i\lambda_n \int_{\Omega} |\varphi_n|^2 dx + \beta_0 \langle u_n, \varphi_n \rangle + \\ &\int_0^{+\infty} g(s) \langle \eta_n, \varphi_n \rangle ds + \alpha \langle v_n, \varphi_n \rangle_2. \end{aligned}$$

From (1.40), and as φ_n is bounded in $H_0^1(\Omega)$, we have

$$\left\langle (i\lambda_n\varphi_n - \beta_0\Delta u_n - \int_0^{+\infty} g(s)\Delta\eta_n(s) ds + \alpha v_n), \varphi \right\rangle_2 \rightarrow 0. \quad (1.49)$$

From (1.48) and (1.49), we find

$$\begin{aligned} i(\beta_0 \int_{\Omega} |\nabla u_n|^2 dx + \int_{\Omega} |\varphi_n|^2 dx) + \frac{2\beta_0}{\lambda_n} \mathcal{I}m(\langle u_n, \varphi_n \rangle) + \frac{\alpha}{\lambda_n} \langle v_n, \varphi_n \rangle_2 \\ + \frac{1}{\lambda_n} \int_0^{+\infty} g(s) \langle \eta_n, \varphi_n \rangle ds \rightarrow 0. \end{aligned} \quad (1.50)$$

We can deduce from (1.38) that $iu_n - \frac{1}{\lambda_n}\varphi_n \rightarrow 0$ in $H_0^1(\Omega)$. On the other hand, we have

$$\left\| \|u_n\| - \frac{1}{\lambda_n} \|\varphi_n\| \right\| \leq \left\| u_n - \frac{\varphi_n}{\lambda_n} \right\| \rightarrow 0,$$

it follows that $\frac{1}{\lambda_n} \|\varphi_n\|$ is bounded. By using (1.46), the last term in (1.50) converges to zero. Then,

$$i(\beta_0 \int_{\Omega} |\nabla u_n|^2 dx + \int_{\Omega} |\varphi_n|^2 dx) + \frac{2\beta_0}{\lambda_n} \mathcal{I}m(\langle u_n, \varphi_n \rangle) + \frac{\alpha}{\lambda_n} \langle v_n, \varphi_n \rangle_2 \rightarrow 0. \quad (1.51)$$

And with the same steps, taking the inner product of (1.40) with $\beta_0 v_n$ in $H_0^1(\Omega)$, then adding it to the inner product of (1.41) with ψ_n in $L^2(\Omega)$ and using Green's formula, we get

$$\begin{aligned} i(\beta_0 \int_{\Omega} |\nabla v_n|^2 dx + \int_{\Omega} |\psi_n|^2 dx) + \frac{2\beta_0}{\lambda_n} \mathcal{I}m(\langle v_n, \psi_n \rangle) + \frac{\alpha}{\lambda_n} \langle u_n, \psi_n \rangle_2 \\ + \frac{1}{\lambda_n} \int_0^{+\infty} g(s) \langle \gamma_n, \psi_n \rangle ds \rightarrow 0. \end{aligned} \quad (1.52)$$

From (1.40), we deduce that $\frac{1}{\lambda_n} \|\psi_n\|$ is bounded. By using (1.47), the last term in (1.52) converges to zero. Hence,

$$i(\beta_0 \int_{\Omega} |\nabla v_n|^2 dx + \int_{\Omega} |\psi_n|^2 dx) + \frac{2\beta_0}{\lambda_n} \mathcal{I}m(\langle v_n, \psi_n \rangle) + \frac{\alpha}{\lambda_n} \langle u_n, \psi_n \rangle_2 \rightarrow 0. \quad (1.53)$$

Now we want to arrive at

$$\frac{\varphi_n}{\lambda_n}, \frac{\psi_n}{\lambda_n} \rightarrow 0 \text{ in } H_0^1(\Omega).$$

For this, we will show that $\frac{\varphi_n}{\lambda_n}, \frac{\psi_n}{\lambda_n}$, in $L_g^2(\mathbb{R}^+; H_0^1(\Omega))$, and this is clear because we have

$$\|\varphi_n\|_{2,g} = b_0 \left\| \frac{\varphi_n}{\lambda_n} \right\|,$$

and

$$\|\psi_n\|_{2,g} = b_0 \left\| \frac{\psi_n}{\lambda_n} \right\|.$$

Then, taking the inner product (1.7) of (1.42) with $\frac{\varphi_n}{\lambda_n}$ and using that $\frac{\varphi_n}{\lambda_n}$ is bounded in $L^2_g(\mathbb{R}^+; H_0^1(\Omega))$, we get

$$\begin{aligned} \left\langle (i\lambda_n\eta_n + \partial_s\eta_n - \varphi_n), \frac{\varphi_n}{\lambda_n} \right\rangle_{2,g} &= i \int_0^{+\infty} g(s) \langle \eta_n, \varphi_n \rangle ds + \frac{1}{\lambda_n} \int_0^{+\infty} g(s) \langle \partial_s\eta_n, \varphi_n \rangle ds - \\ &\int_0^{+\infty} \frac{g(s)}{\lambda_n} \|\varphi_n\| ds \rightarrow 0. \end{aligned}$$

Therefore, we have

$$\frac{i}{\lambda_n} \int_0^{+\infty} g(s) \langle \eta_n, \varphi_n \rangle ds + \frac{1}{\lambda_n^2} \int_0^{+\infty} g(s) \langle \partial_s\eta_n, \varphi_n \rangle ds - \frac{b_0}{\lambda_n^2} \|\varphi_n\|^2 \rightarrow 0. \quad (1.54)$$

By using (1.46) and the fact that $\frac{1}{\lambda_n} \|\varphi_n\|$ is bounded, we have that the first term of (1.54) converges to zero. This yields

$$b_0 \left\| \frac{\varphi_n}{\lambda_n} \right\|^2 - \underbrace{\frac{1}{\lambda_n^2} \int_0^{+\infty} g(s) \langle \partial_s\eta_n, \varphi_n \rangle ds}_{I_1} \rightarrow 0. \quad (1.55)$$

The second term (I_1) in (1.55) converges to zero. Indeed, we have

$$I_1 = \frac{1}{\lambda_n^2} \int_0^{+\infty} g(s) \partial_s \langle \eta_n, \varphi_n \rangle ds.$$

Integrating by parts, getting

$$I_1 = -\frac{1}{\lambda_n^2} \int_0^{+\infty} g'(s) \langle \eta_n, \varphi_n \rangle ds.$$

From (1.6), there exist a positive real number $C > 0$ such that $|g'(s)| < Cg(s)$, (where $C \geq q_1$ and $-C \leq q_0$). Then, we obtain

$$|I_1| \leq \frac{C}{|\lambda_n|} \int_0^{+\infty} g(s) \left| \left\langle \eta_n, \frac{\varphi_n}{\lambda_n} \right\rangle \right| ds.$$

Using Cauchy Schwarz's formula, we obtain

$$\begin{aligned} |I_1| &\leq \frac{C}{|\lambda_n|} \int_0^{+\infty} g(s) \|\eta_n\| \left\| \frac{\varphi_n}{\lambda_n} \right\| ds \\ &= \frac{C}{|\lambda_n|} \int_0^{+\infty} \left(g^{\frac{1}{2}}(s) \|\eta_n\| \right) \left(g^{\frac{1}{2}}(s) \left\| \frac{\varphi_n}{\lambda_n} \right\| \right) ds \\ &\leq \frac{C}{|\lambda_n|} \left(\int_0^{+\infty} g(s) \|\eta_n\|^2 \right)^{\frac{1}{2}} \left(\int_0^{+\infty} g(s) \left\| \frac{\varphi_n}{\lambda_n} \right\|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

From where, we have

$$|I_1| \leq \frac{C\sqrt{b_0}}{|\lambda_n|} \left\| \frac{\varphi_n}{\lambda_n} \right\| \|\eta_n\|_{2,g}.$$

Using again that $\frac{1}{\lambda_n} \|\varphi_n\|$ is bounded and (1.44), then $|I_1| \rightarrow 0$. This with (1.55), leads to

$$\frac{\varphi_n}{\lambda_n} \rightarrow 0 \text{ in } H_0^1(\Omega). \quad (1.56)$$

Now, taking the inner product (1.7) of (1.43) with $\frac{\psi_n}{\lambda_n}$ and using that $\frac{\psi_n}{\lambda_n}$ is bounded in $L_g^2(\mathbb{R}^+; H_0^1(\Omega))$, we get

$$\begin{aligned} \left\langle (i\lambda_n\gamma_n + \partial_s\gamma_n - \psi_n), \frac{\psi_n}{\lambda_n} \right\rangle_{2,g} &= \frac{i}{\lambda_n} \int_0^{+\infty} g(s) \langle \gamma_n, \psi_n \rangle ds + \frac{1}{\lambda_n^2} \int_0^{+\infty} g(s) \langle \partial_s\gamma_n, \psi_n \rangle ds \\ &\quad - \int_0^{+\infty} \frac{g(s)}{\lambda_n^2} \|\psi_n\|^2 ds \rightarrow 0. \end{aligned} \quad (1.57)$$

By using (1.47) and $\frac{1}{\lambda_n} \|\psi_n\|$ is bounded, we have that the first term of (1.57) converges to zero. Then, getting

$$b_0 \left\| \frac{\psi_n}{\lambda_n} \right\|^2 - \underbrace{\frac{1}{\lambda_n^2} \int_0^{+\infty} g(s) \langle \partial_s\gamma_n, \psi_n \rangle ds}_{I_2} \rightarrow 0. \quad (1.58)$$

In the same way as before, we can show that

$$|I_2| \leq \frac{C\sqrt{b_0}}{|\lambda_n|} \left\| \frac{\gamma_n}{\lambda_n} \right\| \|\gamma_n\|_{2,g}.$$

From (1.45) and as $\frac{1}{\lambda_n} \|\psi_n\|$ is bounded, then $|I_2| \rightarrow 0$. Hence,

$$\frac{\psi_n}{\lambda_n} \rightarrow 0 \text{ in } H_0^1(\Omega). \quad (1.59)$$

Now, since u_n and v_n are bounded in $H_0^1(\Omega)$ and by using (1.56) and (1.59), we obtain that the last two terms in (1.51) and (1.53) converges to zero. Thus,

$$\beta_0 \int_{\Omega} |\nabla u_n|^2 dx + \int_{\Omega} |\varphi_n|^2 dx \rightarrow 0,$$

and

$$\beta_0 \int_{\Omega} |\nabla v_n|^2 dx + \int_{\Omega} |\psi_n|^2 dx \rightarrow 0.$$

Then

$$u_n, v_n \rightarrow 0 \text{ in } H_0^1(\Omega).$$

Using Cauchy–Schwarz’s and Poincaré’s inequalities, we get

$$|\langle u_n, v_n \rangle| \leq K \|u_n\| \|v_n\| \rightarrow 0,$$

with K is a positive constant. Now from what we found, getting

$$\| \mathcal{U}_m \|_{\mathcal{H}}^2 = \beta_0 \int_{\Omega} |\nabla u_n|^2 dx + \int_{\Omega} |\varphi_n|^2 dx + \beta_0 \int_{\Omega} |\nabla v_n|^2 dx + \int_{\Omega} |\psi_n|^2 dx + 2\alpha \mathcal{R}e \langle u_n, v_n \rangle +$$

$$\|\eta_n\|_{2,g} + \|\gamma_n\|_{2,g} \rightarrow 0.$$

And this contradicts because we have $\|\mathcal{U}_m\|_{\mathcal{H}}^2 = 1$. Therefore, the proof is achieved. ■

Chapter 2

Optimal polynomial decay for a viscoelastic coupled system of wave equations with past history

Contents

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In this chapter, we consider a viscoelastic coupled systems of two wave equations with past history acting only in one equation. Under certain conditions, we study the global existence and uniqueness of the solution. After that, we prove the lack of exponential decay. Finally, we show that the system is polynomially stable with rate $t^{-\frac{1}{2}}$. Moreover we show that the rate is optimal.

2.1 Global existence

Here we will proof the existence and uniqueness of the following system

$$u_{tt} - \Delta u + \int_0^{+\infty} g(s)\Delta u(t-s) ds + \alpha v = 0, \quad \text{in } \Omega \times (0, +\infty), \quad (2.1)$$

$$v_{tt} - \Delta v + \alpha u = 0, \quad \text{in } \Omega \times (0, +\infty), \quad (2.2)$$

$$u = v = 0, \quad \text{on } \Gamma \times (0, +\infty), \quad (2.3)$$

$$(u(x, 0), v(x, 0)) = (u_0(x), v_0(x)), \quad \text{in } \Omega, \quad (2.4)$$

$$(u_t(x, 0), v_t(x, 0)) = (u_1(x), v_1(x)), \quad \text{in } \Omega. \quad (2.5)$$

For the study of the system (2.1)-(2.5), and in the same way with the system (1.1)-(1.5) we introduce the new variable $\eta = \eta^t(s)$, the relative history of u

$$\eta = \eta^t(s) = u(t) - u(t-s), \quad \text{where } s > 0.$$

We will take the same assumptions to the kernel g that we considered in the first chapter. And we also put

$$\beta_0 = 1 - \int_0^{+\infty} g(s) ds > 0,$$

then the system (2.1)-(2.5) can be rewritten as

$$u_{tt} - \beta_0 \Delta u - \int_0^{+\infty} g(s) \Delta \eta(s) ds + \alpha v = 0, \quad \text{in } \Omega \times (0, +\infty), \quad (2.6)$$

$$v_{tt} - \Delta v + \alpha u = 0, \quad \text{in } \Omega \times (0, +\infty), \quad (2.7)$$

$$\eta_t + \eta_s - u_t = 0, \quad \text{in } \Omega \times (0, +\infty), \quad (2.8)$$

$$u = v = \eta^t(s) = 0, \quad \text{on } \Gamma \times (0, +\infty), \quad \forall s > 0, \quad (2.9)$$

$$(u(x, 0), v(x, 0)) = (u_0(x), v_0(x)), \quad \text{in } \Omega, \quad (2.10)$$

$$(u_t(x, 0), v_t(x, 0)) = (u_1(x), v_1(x)), \quad \text{in } \Omega, \quad (2.11)$$

$$\eta_0(\cdot, s) = u_0(\cdot) - u(\cdot, -s), \quad \text{in } \Omega \times (0, +\infty). \quad (2.12)$$

Now, to give an accurate formulation of the evolution problem we introduce the product Hilbert spaces

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L_g^2(\mathbb{R}^+; H_0^1(\Omega))$$

endowed with the following inner product

$$\begin{aligned} \langle U, V \rangle = & \beta_0 \int_{\Omega} \nabla u_1 \cdot \nabla \bar{v}_1 dx + \int_{\Omega} u_2 \bar{v}_2 dx + \int_{\Omega} \nabla u_3 \cdot \nabla \bar{v}_3 dx + \int_{\Omega} u_4 \bar{v}_4 dx + \alpha \int_{\Omega} (u_1 \bar{v}_3 + u_3 \bar{v}_1) dx + \\ & \int_0^{+\infty} g(s) \int_{\Omega} \nabla u_5(x, s) \cdot \nabla \overline{v_5(x, s)} dx ds. \end{aligned} \quad (2.13)$$

Where $U = (u_1, u_2, u_3, u_4, u_5)^T$, $V = (v_1, v_2, v_3, v_4, v_5)^T \in \mathcal{H}$

We take $\mathcal{U} = (u, u_t, v, v_t, \eta)^T \in \mathcal{H}$, then the system (2.6)-(2.12) becomes

$$\begin{cases} \mathcal{U}_t(t) = \mathcal{B}\mathcal{U}(t), \quad \forall t > 0, \\ \mathcal{U}(0) = \mathcal{U}_0, \end{cases} \quad (2.14)$$

where $\mathcal{U}_0 = (u_0, u_1, v_0, v_1, \eta_0)^T \in \mathcal{H}$, and the operator \mathcal{B} is defined by $\mathcal{B} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$

$$\mathcal{B} = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ \beta_0 \Delta & 0 & -\alpha I & 0 & \mathcal{T} \\ 0 & 0 & 0 & I & 0 \\ -\alpha I & 0 & \Delta & 0 & 0 \\ 0 & I & 0 & 0 & \partial_s \end{bmatrix}$$

With domain

$$\mathcal{D}(\mathcal{B}) = \left\{ (u, \varphi, v, \psi, \eta)^T \in \mathcal{H}; \beta_0 u + \int_0^{+\infty} g(s) \eta(s) ds \in H_0^1(\Omega) \cap H^2(\Omega) \right\}$$

$$\varphi \in H_0^1(\Omega), \psi \in H_0^1(\Omega), \eta \in \mathcal{D}(\mathcal{T})\}.$$

And

$$\mathcal{T}\eta = \int_0^{+\infty} g(s)\Delta\eta(s) ds, \quad \forall \eta \in \mathcal{D}(\mathcal{T}),$$

where

$$\mathcal{D}(\mathcal{T}) = \{\eta \in L_g^2(\mathbb{R}^+; H_0^1(\Omega)); \eta_s \in L_g^2(\mathbb{R}^+; H_0^1(\Omega)), \eta(0) = 0\}.$$

Remark 2.1.1. We have $\mathcal{D}(\mathcal{A})$ is dense in \mathcal{H} .

The existence of the solution of the problem (2.14) is ensured in the following theorem.

Theorem 2.1.2. *The operator \mathcal{B} is the infinitesimal generator of C_0 - semigroup of contractions over the Hilbert space \mathcal{H} . Thus, for any initial data $\mathcal{U}_0 \in \mathcal{H}$, the problem (2.14) has a unique weak solution $\mathcal{U} \in C(\mathbb{R}^+; \mathcal{H})$.*

Moreover, if $\mathcal{U}_0 \in \mathcal{D}(\mathcal{B})$, then the solution $\mathcal{U} \in C(\mathbb{R}^+; \mathcal{D}(\mathcal{B})) \cap C^1(\mathbb{R}^+; \mathcal{H})$.

Proof. We use the Lumer-Phillips theorem to show that, \mathcal{B} is generates a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$.

Step 1. \mathcal{B} is dissipative. For $\mathcal{U} = (u, u_t, v, v_t, \eta)^T \in \mathcal{D}(\mathcal{B})$, using the inner product (2.13), we get

$$\begin{aligned} \langle \mathcal{B}\mathcal{U}, \mathcal{U} \rangle &= \langle \mathcal{U}_t, \mathcal{U} \rangle = \beta_0 \int_{\Omega} \nabla u_t \cdot \nabla \bar{u} dx + \int_{\Omega} u_{tt} \bar{u}_t dx + \int_{\Omega} \nabla v_t \cdot \nabla \bar{v} dx + \int_{\Omega} v_{tt} \bar{v}_t dx \\ &\quad + \alpha \int_{\Omega} (u_t \bar{v} + v_t \bar{u}) dx + \int_0^{+\infty} g(s) \int_{\Omega} \nabla \eta_t(x, s) \cdot \nabla \overline{\eta(x, s)} dx ds. \end{aligned}$$

Substituting u_{tt} , v_{tt} and η from (2.6), (2.7), and (2.8) respectively, getting

$$\begin{aligned} \langle \mathcal{B}\mathcal{U}, \mathcal{U} \rangle &= \beta_0 \int_{\Omega} \nabla u_t \cdot \nabla \bar{u} dx + \int_{\Omega} (\beta_0 \Delta u + \int_0^{+\infty} g(s)\Delta\eta(s) ds - \alpha v) \bar{u}_t dx + \int_{\Omega} \nabla v_t \cdot \nabla \bar{v} dx + \int_{\Omega} (\Delta v - \alpha u) \bar{v}_t dx \\ &\quad + \alpha \int_{\Omega} (u_t \bar{v} + v_t \bar{u}) dx + \int_0^{+\infty} g(s) \int_{\Omega} \nabla (u_t - \eta_s(s)) \cdot \nabla \overline{\eta(s)} dx ds. \end{aligned}$$

Using Green's formula, we obtain

$$\begin{aligned} \langle \mathcal{B}\mathcal{U}, \mathcal{U} \rangle &= \beta_0 \int_{\Omega} \nabla u_t \cdot \nabla \bar{u} dx - \beta_0 \int_{\Omega} \nabla u \nabla \bar{u}_t dx - \alpha \int_{\Omega} v \bar{u}_t dx - \int_0^{+\infty} g(s) \int_{\Omega} \nabla \bar{u}_t \nabla \eta dx ds \\ &\quad + \int_{\Omega} \nabla v_t \cdot \nabla \bar{v} dx - \int_{\Omega} \nabla v \nabla \bar{v}_t dx - \alpha \int_{\Omega} u \bar{v}_t dx + \alpha \int_{\Omega} (u_t \bar{v} + v_t \bar{u}) dx + \int_0^{+\infty} g(s) \int_{\Omega} \nabla u_t \nabla \overline{\eta(s)} dx ds \\ &\quad - \int_0^{+\infty} g(s) \int_{\Omega} \nabla \eta_s(s) \cdot \nabla \overline{\eta(s)} dx ds. \end{aligned}$$

After simplifying, we get

$$\begin{aligned} \langle \mathcal{B}\mathcal{U}, \mathcal{U} \rangle = & - \int_0^{+\infty} g(s) \int_{\Omega} \nabla \eta_s(s) \cdot \nabla \overline{\eta(s)} \, dx \, ds + 2i \mathcal{I}m(\beta_0) \int_{\Omega} \nabla u_t \cdot \nabla \bar{u} \, dx + \int_{\Omega} \nabla v_t \cdot \nabla \bar{v} \, dx + \alpha \int_{\Omega} (u_t \bar{v} + v_t \bar{u}) \, dx \\ & + \int_0^{+\infty} g(s) \int_{\Omega} \nabla u_t \nabla \overline{\eta(s)} \, dx \, ds. \end{aligned}$$

Taking its real part, we arrive at

$$\mathcal{R}e \langle \mathcal{B}\mathcal{U}, \mathcal{U} \rangle = - \int_0^{+\infty} g(s) \int_{\Omega} \nabla \eta_s(s) \cdot \nabla \overline{\eta(s)} \, dx \, ds = - \frac{1}{2} \int_0^{+\infty} g(s) \frac{d}{ds} \int_{\Omega} |\nabla \eta(s)|^2 \, dx \, ds.$$

Integrating by parts, we have

$$\begin{aligned} \mathcal{R}e \langle \mathcal{B}\mathcal{U}, \mathcal{U} \rangle &= \left[-\frac{1}{2} g(s) \int_{\Omega} |\nabla \eta(s)|^2 \, dx \right]_0^{+\infty} + \frac{1}{2} \int_0^{+\infty} g'(s) \int_{\Omega} |\nabla \eta(s)|^2 \, dx \, ds \\ &= \frac{1}{2} g(0) \int_{\Omega} |\nabla \eta(0)|^2 \, dx - \frac{1}{2} \lim_{s \rightarrow +\infty} g(s) \int_{\Omega} |\nabla \eta(s)|^2 \, dx + \frac{1}{2} \int_0^{+\infty} g'(s) \int_{\Omega} |\nabla \eta(s)|^2 \, dx \, ds. \end{aligned}$$

From the previous hypotheses on g we have,

$$\frac{1}{2} g(0) \int_{\Omega} |\nabla \eta(0)|^2 \, dx - \frac{1}{2} \lim_{s \rightarrow +\infty} g(s) \int_{\Omega} |\nabla \eta(s)|^2 \, dx = 0,$$

as a result

$$\mathcal{R}e \langle \mathcal{B}\mathcal{U}, \mathcal{U} \rangle = \frac{1}{2} \int_0^{+\infty} g'(s) \int_{\Omega} |\nabla \eta(s)|^2 \, dx \, ds.$$

Using (1.6), we have

$$\mathcal{R}e \langle \mathcal{B}\mathcal{U}, \mathcal{U} \rangle \leq -\frac{q_1}{2} \int_0^{+\infty} g(s) \int_{\Omega} |\nabla \eta(s)|^2 \, dx \, ds \leq 0.$$

Then, \mathcal{B} is a dissipative operator.

Step 2. for all $\lambda > 0$, $(\lambda I - \mathcal{B})$ is maximal. For this, let $F = (f_1, f_2, f_3, f_4, f_5)^T \in \mathcal{H}$. We show that there exists unique $\mathcal{U} = (u, \varphi, v, \psi, \eta)^T \in \mathcal{D}(\mathcal{B})$ such that

$$(\lambda I - \mathcal{B})\mathcal{U} = F,$$

then, we can write

$$\lambda u - \varphi = f_1, \tag{2.15}$$

$$\lambda \varphi - \beta_0 \Delta u - \int_0^{+\infty} g(s) \Delta \eta(s) \, ds + \alpha v = f_2, \tag{2.16}$$

$$\lambda v - \psi = f_3, \tag{2.17}$$

$$\lambda \psi - \Delta v + \alpha u = f_4, \tag{2.18}$$

$$\lambda \eta + \eta_S - \varphi = f_5. \tag{2.19}$$

We note that the equation (2.19) is ordinary differential equation of the first order, and the solution to its homogeneous equation is given by

$$\eta(s) = Ce^{-\lambda s}, \text{ with } C \text{ is a constant.}$$

Then, we use the method of the variation of the constant, so the general solution of the differential equation is given by $\eta(s) = C(s)e^{-\lambda s}$. After simplifications, we get

$$C(., s) = C(0) + \int_0^s (f_5(., \tau) + \varphi(., \tau))e^{\lambda \tau} d\tau = \int_0^s e^{\lambda \tau} f_5(., \tau) d\tau + \varphi(., \tau) \int_0^s e^{\lambda \tau} d\tau.$$

Hence, the general solution of the differential equation is given by

$$\eta(., s) = \frac{\varphi(., \tau)}{\lambda}(1 - e^{-\lambda s}) + \int_0^s e^{\lambda \tau} f_5(., \tau) d\tau. \quad (2.20)$$

Now, from (2.15) getting $\varphi = \lambda u - f_1$. Substituting φ and η into (2.17), we have

$$\lambda^2 u - \lambda f_1 - \beta_0 \Delta u + \alpha v - \int_0^{+\infty} g(s) \Delta \left[\frac{\lambda u - f_1}{\lambda} (1 - e^{-\lambda s}) + \int_0^s e^{\lambda(\tau-s)} f_5(\tau) d\tau \right] ds = f_2,$$

then

$$\lambda^2 u - \beta_g \Delta u + \alpha v = k_1. \quad (2.21)$$

Where

$$k_1 = \lambda f_1 + f_2 + \int_0^{+\infty} g(s) \left[\frac{(e^{-\lambda s} - 1)}{\lambda} \Delta f_1 + \int_0^s e^{\lambda(\tau-s)} \Delta f_5(\tau) d\tau \right] ds,$$

and $\beta_g = \beta_0 + \int_0^{+\infty} g(s)(1 - e^{-\lambda s}) ds > 0$. Substituting ψ from (2.18) into (2.19), gives us

$$\lambda^2 v - \Delta v + \alpha u = k_2. \quad (2.22)$$

Where

$$k_2 = f_4 + \lambda f_5.$$

First we prove that $u, v \in H_0^1(\Omega)$. To do this, multiplying (2.21) and (2.22) by \tilde{u}, \tilde{v} respectively, and integrating over Ω . Then using Green's formula, we arrive at

$$\begin{cases} \lambda^2 \int_{\Omega} u \tilde{u} dx + \beta_g \int_{\Omega} \nabla u \cdot \nabla \tilde{u} dx + \alpha \int_{\Omega} v \tilde{v} dx = \langle k_1, \tilde{u} \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \\ \lambda^2 \int_{\Omega} v \tilde{v} dx + \int_{\Omega} \nabla v \cdot \nabla \tilde{v} dx + \alpha \int_{\Omega} u \tilde{u} dx = \langle k_2, \tilde{v} \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \end{cases}, \forall (\tilde{u}, \tilde{v}) \in V. \quad (2.23)$$

The sum of the equations in (2.23) gives the following variational formulation:

$$a((u, v), (\tilde{u}, \tilde{v})) = L(\tilde{u}, \tilde{v}), \forall (\tilde{u}, \tilde{v}) \in H_0^1(\Omega) \times H_0^1(\Omega),$$

where a is bilinear form and L is linear form, defined as

$$a((u, v), (\tilde{u}, \tilde{v})) = \lambda^2 \int_{\Omega} u \tilde{u} dx + \beta_g \int_{\Omega} \nabla u \cdot \nabla \tilde{u} dx + \lambda^2 \int_{\Omega} v \tilde{v} dx + \int_{\Omega} \nabla v \cdot \nabla \tilde{v} dx + \alpha \int_{\Omega} (v \tilde{u} + u \tilde{v}) dx,$$

end

$$L(\tilde{u}, \tilde{v}) = \langle k_1, \tilde{u} \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} + \langle k_2, \tilde{v} \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}, \forall (\tilde{u}, \tilde{v}) \in V.$$

With $V = H_0^1(\Omega) \times H_0^1(\Omega)$

Now we will use Lax- Milgram's theorem to prove the existence and uniqueness of the solutions $(u, v) \in V$. For this we prove

1. The coercivity of \mathbf{a}

We have

$$\begin{aligned} a((u, v), (u, v)) &= \lambda^2 \int_{\Omega} |u|^2 dx + \lambda^2 \int_{\Omega} |v|^2 dx + \beta_g \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |\nabla v|^2 dx + 2\alpha \int_{\Omega} uv dx \\ &= \lambda^2 \|u\|_2^2 + \lambda^2 \|v\|_2^2 + \beta_g \|u\|^2 + \|v\|^2 + 2\alpha \int_{\Omega} uv dx. \end{aligned}$$

Using the Young's inequality, we find

$$\begin{aligned} \forall \varepsilon > 0, \quad a((u, v), (u, v)) &\geq \lambda^2 \|u\|_2^2 + \lambda^2 \|v\|_2^2 + \beta_g \|u\|^2 + \|v\|^2 - \alpha\varepsilon \int_{\Omega} |u|^2 dx - \frac{\alpha}{\varepsilon} \int_{\Omega} |v|^2 dx \\ &\geq (\lambda^2 - \alpha\varepsilon) \|u\|_2^2 + (\lambda^2 - \frac{\alpha}{\varepsilon}) \|v\|_2^2 + \beta_g \|u\|^2 + \|v\|^2, \end{aligned}$$

and as in the first chapter we can find $\varepsilon > 0$, such that $(\lambda^2 - \alpha\varepsilon)$ and $(\lambda^2 - \frac{\alpha}{\varepsilon})$ are positive.

Therefore and for the chosen ε , we have

$$\begin{aligned} a((u, v), (u, v)) &\geq \beta_g \|u\|^2 + \|v\|^2 \\ &\geq C \|(u, v)\|_V^2, \end{aligned}$$

with $C = \min(\beta_g, 1)$. So we have the coercivity.

2. The continuity of \mathbf{a}

Using the Cauchy-Schwarz's inequality, we get

$$\begin{aligned} |a((u, v), (\tilde{u}, \tilde{v}))| &\leq \lambda^2 \|u\|_2 \|\tilde{u}\|_2 + \lambda^2 \|v\|_2 \|\tilde{v}\|_2 + \beta_g \|u\| \|\tilde{u}\| \\ &\quad + \|v\|^2 \|\tilde{v}\|^2 + \alpha \|v\|_2 \|\tilde{u}\|_2 + \|u\|_2 \|\tilde{v}\|_2. \end{aligned}$$

Then, we get

$$\begin{aligned} |a((u, v), (\tilde{u}, \tilde{v}))| &\leq C(\|u\| \|\tilde{u}\| + \|v\| \|\tilde{v}\| + \|v\| \|\tilde{u}\| + \|u\| \|\tilde{v}\|) \\ &\leq C(\|u\| + \|v\|)(\|\tilde{u}\| + \|\tilde{v}\|) \\ &\leq C \|(u, v)\|_V \|(\tilde{u}, \tilde{v})\|_V, \end{aligned}$$

with C is a positive constant. So we have the continuity.

3. The continuity of \mathbf{L} .

Using the Cauchy-Schwarz's inequality, we get

$$\begin{aligned} |L(\tilde{u}, \tilde{v})| &\leq C(\|\tilde{u}\| + \|\tilde{v}\|) \\ &\leq C \|(\tilde{u}, \tilde{v})\|_V, \end{aligned}$$

where C is a positive constant. Then we have the continuity of \mathbf{L} .

Thus, through Lax- Milgram's theorem, we obtain the existence and uniqueness of $u, v \in H_0^1(\Omega)$. Consequently from (2.15) and (2.18), we get $\varphi, \psi \in H_0^1(\Omega)$. Hence, the existence and uniqueness of η . And now we will prove that $\eta \in \mathcal{D}(\mathcal{T})$.

From (2.19), we have

$$\lambda \nabla \eta(t, s) + \nabla \eta_s(t, s) = \nabla \varphi(t) + \nabla f_5(t, s).$$

In the same way as in the first chapter, we find that

$$\|\eta\|_{2,g}^2 \leq C(\|\varphi\|^2 + \|f_5\|_{2,g}^2),$$

and

$$\|\eta_s\|_{2,g}^2 \leq C(\|f_5\|_{2,g}^2 + \|\varphi\|^2 + \|\eta\|_{2,g}^2),$$

where C_1 and C_2 are positive constants. Then

$$\eta, \eta_s \in L_g^2(\mathbb{R}^+; H_0^1(\Omega)).$$

From (2.20), we have $\eta(0) = 0$. Hence, we have proved that $\eta \in \mathcal{D}(\mathcal{T})$. And in the end, we will prove that

$$v, \beta_0 u + \int_0^{+\infty} g(s)\eta(s) ds \in H^2(\Omega).$$

From (2.17) and (2.19), we obtain

$$\Delta(\beta_0 u + \int_0^{+\infty} g(s)\eta(s) ds) = \lambda \varphi + \alpha v - f_2,$$

and

$$\Delta v = \lambda \psi + \alpha u - f_4.$$

Hence and from where it follows that

$$\Delta(\beta_0 u + \int_0^{+\infty} g(s)\eta(s) ds) \text{ and } \Delta v \in L^2(\Omega),$$

then

$$v \text{ and } \beta_0 u + \int_0^{+\infty} g(s)\eta(s) ds \in H^2(\Omega).$$

Thus, $(\lambda I - \mathcal{B})$ is maximal.

Therefore, by Lumer-Phillips theorem (A.2.10) we get that \mathcal{B} generates a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ of contraction in the space \mathcal{H} and according to theorem (A.2.9) the Cauchy problem (2.14) admits a unique solution. The proof is complete. \blacksquare

2.2 Lack of exponential decay

Our starting point is to show that the semigroup associated to the system (2.6) -(2.12) is not exponential stable. To show this we will use the Prüss's theorem (A.2.10) to prove the

lack of exponential stability.

To do this, let us consider the spectral problem:

$$\begin{cases} -\Delta w_m = \lambda_m w_m & \text{in } \Omega \\ w_m = 0 & \text{on } \Gamma \end{cases}, \forall m \geq 0 \quad (2.24)$$

where $\{\lambda_m\}_{m \geq 0}$ is real sequence, and check that $\lim_{m \rightarrow +\infty} \lambda_m = +\infty$.

The following theorem explains the results to be obtained in this section.

Theorem 2.2.1. *Let $T(t)$ be C_0 -semigroup of contractions generated by \mathcal{A} . Then $T(t)$ is not exponentially stable.*

Proof. For proof, we will use the theorem (A.2.11). Suppose that

$$i\mathbb{R} = \{i\beta : \beta \in \mathbb{R}\} \subset \rho(\mathcal{B}).$$

And we will prove that there exists a sequence of imaginary number $\lambda_m \in \rho(\mathcal{B})$, such that

$$\|(\lambda_m I - \mathcal{B})^{-1}\|_{\mathcal{L}(\mathcal{H})} \rightarrow +\infty, \text{ as } m \rightarrow +\infty.$$

This means that, prove that there exists a sequence of complex numbers $\lambda_m \in \rho(\mathcal{B})$ and a sequence of data $F_m = (f_m^1, f_m^2, f_m^3, f_m^4, f_m^5)^T \in \mathcal{H}$, with $\|F_m\|_{\mathcal{H}} \leq 1$, such that

$$\|(\lambda_m I - \mathcal{B})^{-1} F_m\|_{\mathcal{H}} = \|\mathcal{U}_m\|_{\mathcal{H}} \rightarrow +\infty,$$

where $\mathcal{U}_m = (u_m, \varphi_m, v_m, \psi_m, \eta_m)^T \in \mathcal{D}(\mathcal{B})$ not bounded, and

$$(\lambda_m I - \mathcal{B})\mathcal{U}_m = F_m. \quad (2.25)$$

To simplify the notation we will omit m . Then, the equation (2.25) is equivalent to

$$\begin{cases} i\lambda u - \varphi = f^1, \\ i\lambda\varphi - \beta_0 \Delta u - \int_0^{+\infty} g(s) \Delta \eta(s) ds + \alpha v = f^2, \\ i\lambda v - \psi = f^3, \\ i\lambda\psi - \Delta v + \alpha u = f^4, \\ i\lambda\eta + \eta_S - \varphi = f^5. \end{cases} \quad (2.26)$$

Taking $f^1 = f^3 = f^5 = 0$ and $f^2 = f^4 = w_m$, then we obtain $\varphi = i\lambda u$ and $\psi = i\lambda v$, hence the above system becomes

$$\begin{cases} -\lambda^2 u - \beta_0 \Delta u - \int_0^{+\infty} g(s) \Delta \eta(s) ds + \alpha v = w_m, \\ -\lambda^2 v - \Delta v + \alpha u = w_m, \\ i\lambda\eta + \eta_S - \varphi = 0. \end{cases} \quad (2.27)$$

We look for solutions of the form

$$u = aw_m, \quad v = bw_m, \quad \varphi = cw_m, \quad \psi = dw_m, \quad \eta(x, s) = \gamma(s)w_m,$$

where $a, b, c, d \in \mathbb{R}$ and $\gamma(s)$ depend on λ . Now from (2.24) and (2.27), we get $a, b, \gamma(s)$ satisfy

$$\begin{cases} -\lambda^2 a + \beta_0 a \lambda_m + \lambda_m \int_0^{+\infty} g(s) \gamma(s) ds + \alpha b = 1, \\ -\lambda^2 b + \lambda_m b + \alpha a = 1, \\ \gamma_S + i\lambda\gamma - i\lambda a = 0. \end{cases} \quad (2.28)$$

Solving the third equation which of (2.28), we get $\gamma(s) = Ce^{-i\lambda s} + a$, with C is a constant and since $\eta(0) = 0$, then we get

$$\gamma(s) = a - ae^{-i\lambda s}.$$

Therefore, we have

$$\int_0^{+\infty} g(s)\gamma(s) ds = \int_0^{+\infty} g(s)(a - ae^{-i\lambda s}) ds = ab_0 - a \int_0^{+\infty} g(s)e^{-i\lambda s} ds.$$

Now, choosing $\lambda = \sqrt{\lambda_m}$ then from the second equation of (2.28) we obtain $a = \frac{1}{\alpha}$.

Substituting a into the first equation of (2.28), getting

$$b = \frac{\lambda_m(1 - \beta_0)}{\alpha^2} - \frac{\lambda_m}{\alpha} \int_0^{+\infty} g(s)e^{i\lambda s} ds + \frac{1}{\alpha}.$$

Hence, we find

$$c = \frac{i\sqrt{\lambda_m}}{\alpha},$$

and

$$d = i\sqrt{\lambda_m} \left(\frac{\lambda_m(1 - \beta_0)}{\alpha^2} - \frac{\lambda_m}{\alpha} \int_0^{+\infty} g(s)e^{i\lambda s} ds + \frac{1}{\alpha} \right).$$

Recalling that

$$\varphi = cw_m = \frac{i\sqrt{\lambda_m}}{\alpha} w_m,$$

we get

$$\|\varphi\|_2^2 = \frac{\lambda_m}{\alpha^2}, \text{ because we have } \|w_m\|_2 = 1.$$

Therefore, we get

$$\lim_{m \rightarrow +\infty} \|U_m\|_{\mathcal{H}}^2 \geq \lim_{m \rightarrow +\infty} \|\varphi\|_2^2 = \lim_{m \rightarrow +\infty} \frac{\lambda_m}{\alpha^2} = +\infty.$$

Than

$$\|U_m\|_{\mathcal{H}}^2 \rightarrow +\infty, \text{ as } m \rightarrow +\infty.$$

Thus, there is no exponential stability. This completes the proof. ■

2.3 Polynomial decay and optimally result

In this section we study the polynomial decay associated to the system (2.6) -(2.12) and subsequently we find the optimal rate of decay.

We have as the operator \mathcal{B} generate a C_0 - semigroup $T(t)$ of contraction on \mathcal{H} . Thus according to the Lumer-Phillips theorem, the solution of (2.6)-(2.12) \mathcal{U} that verifies the resolvent equation

$$(i\lambda I - \mathcal{B})\mathcal{U} = F, \text{ with } \lambda \in \mathbb{R} \text{ and } F \in \mathcal{U},$$

that is,

$$i\lambda u - \varphi = f_1, \quad (2.29)$$

$$i\lambda\varphi - \beta_0\Delta u - \mathcal{T}\eta + \alpha v = f_2, \quad (2.30)$$

$$i\lambda v - \psi = f_3, \quad (2.31)$$

$$i\lambda\psi - \Delta v + \alpha u = f_4, \quad (2.32)$$

$$i\lambda\eta + \eta_S - \varphi = f_5. \quad (2.33)$$

In the next step we shall show three important lemmas to proof the main result.

Lemma 2.3.1. *The solutions of the system (2.6)-(2.12), satisfy*

$$\int_{\Omega} \int_0^{+\infty} g(s) |\nabla\eta|^2 ds dx \leq K|\lambda|^2 \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}}.$$

Where K is a positive constant and $|\lambda| > 1$.

Proof. Multiplying the equality (2.30) by $\bar{\varphi}$ and integrating by parts on Ω , we get

$$i\lambda \int_{\Omega} |\varphi|^2 dx + \beta_0 \underbrace{\int_{\Omega} \nabla u \cdot \nabla \bar{\varphi} dx}_{=I_1} + \alpha \underbrace{\int_{\Omega} v \cdot \bar{\varphi} dx}_{=I_2} + \underbrace{\int_{\Omega} \int_0^{+\infty} g(s) \nabla \eta \cdot \nabla \bar{\varphi} ds dx}_{I_3} = \int_{\Omega} f_2 \bar{\varphi} dx. \quad (2.34)$$

Substituting φ given in (2.29) into I_1 and I_2 , we have

$$I_1 = -i\lambda\beta_0 \int_{\Omega} |u|^2 dx - \beta_0 \int_{\Omega} \nabla u \cdot \nabla \bar{f}_1 dx. \quad (2.35)$$

And

$$I_2 = -i\lambda\alpha \int_{\Omega} v \bar{u} dx - \alpha \int_{\Omega} v \cdot \bar{f}_1 dx. \quad (2.36)$$

Now, substituting φ given in (2.33) into I_3 , getting

$$I_3 = -i\lambda \int_{\Omega} \int_0^{+\infty} g(s) |\nabla\eta|^2 ds dx + \int_{\Omega} \int_0^{+\infty} g(s) \nabla \eta \cdot \nabla \bar{\eta}_s ds dx - \int_{\Omega} \int_0^{+\infty} g(s) \nabla \eta \cdot \nabla \bar{f}_5 ds dx.$$

Integrating by parts, we obtain

$$I_3 = -i\lambda \int_{\Omega} \int_0^{+\infty} g(s) |\nabla\eta|^2 ds dx - \frac{1}{2} \int_{\Omega} \int_0^{+\infty} g'(s) |\nabla\eta|^2 ds dx - \int_{\Omega} \int_0^{+\infty} g(s) \nabla \eta \cdot \nabla \bar{f}_5 ds dx. \quad (2.37)$$

After Substituting (2.35) and (2.37) into (2.34), we arrive at

$$\begin{aligned} & i\lambda \int_{\Omega} |\varphi|^2 dx - i\lambda\beta_0 \int_{\Omega} |u|^2 dx - i\lambda\alpha \int_{\Omega} v \bar{u} dx - i\lambda \int_{\Omega} \int_0^{+\infty} g(s) |\nabla\eta|^2 ds dx - \frac{1}{2} \int_{\Omega} \int_0^{+\infty} g'(s) |\nabla\eta|^2 ds dx \\ & = \beta_0 \int_{\Omega} \nabla u \cdot \nabla \bar{f}_1 dx + \alpha \int_{\Omega} v \cdot \bar{f}_1 dx + \int_{\Omega} \int_0^{+\infty} g(s) \nabla \eta \cdot \nabla \bar{f}_5 ds dx + \int_{\Omega} f_2 \bar{\varphi} dx. \end{aligned} \quad (2.38)$$

Taking the real part on the left side of the above equality, we find

$$-\frac{1}{2} \int_{\Omega} \int_0^{+\infty} g'(s) |\nabla\eta|^2 ds dx = \beta_0 \int_{\Omega} \nabla u \cdot \nabla \bar{f}_1 dx + \alpha \int_{\Omega} v \cdot \bar{f}_1 dx + \int_{\Omega} \int_0^{+\infty} g(s) \nabla \eta \cdot \nabla \bar{f}_5 ds dx + \int_{\Omega} f_2 \bar{\varphi} dx.$$

Using (1.6) on the right side, we find

$$\frac{q_1}{2} \int_{\Omega} \int_0^{+\infty} g(s) |\nabla\eta|^2 ds dx \leq \beta_0 \int_{\Omega} \nabla u \cdot \nabla \bar{f}_1 dx + \alpha \int_{\Omega} v \cdot \bar{f}_1 dx + \int_{\Omega} \int_0^{+\infty} g(s) \nabla \eta \cdot \nabla \bar{f}_5 ds dx + \int_{\Omega} f_2 \bar{\varphi} dx.$$

On the other hand we have

$$\int_{\Omega} \int_0^{+\infty} g(s) \nabla \eta \cdot \nabla \overline{f_5} \, ds \, dx = \int_{\Omega} \int_0^{+\infty} (g(s)^{\frac{1}{2}} \nabla \eta) \cdot (g(s)^{\frac{1}{2}} \nabla \overline{f_5}) \, ds \, dx,$$

then after using the Cauchy-Schwartz's and Poincaré's inequalities, we get

$$\frac{q_1}{2} \int_{\Omega} \int_0^{+\infty} g(s) |\nabla \eta|^2 \, ds \, dx \leq \beta_0 \|u\| \|f_1\| + \alpha \|u\| \|f_1\|_2 + \|\eta\|_{2,g} \|f_5\|_{2,g} + \|\varphi\|_2 \|f_2\|_2.$$

Hence, we find

$$\frac{q_1}{2} \int_{\Omega} \int_0^{+\infty} g(s) |\nabla \eta|^2 \, ds \, dx \leq \beta_0 \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + C \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + 2 \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}},$$

where C is a positive constant. As a result, we have

$$\int_{\Omega} \int_0^{+\infty} g(s) |\nabla \eta|^2 \, ds \, dx \leq K \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}},$$

with K is a positive constant. Now, for all $|\lambda| > 1$, we can write that

$$\int_{\Omega} \int_0^{+\infty} g(s) |\nabla \eta|^2 \, ds \, dx \leq K |\lambda|^2 \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}}.$$

The proof is complete. ■

Lemma 2.3.2. *For any $\varepsilon > 0$, there exists a positive constant K_{ε} such that*

$$\begin{aligned} & \beta_0 \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} |\nabla v|^2 \, dx + \alpha \int_{\Omega} (u\bar{v} + v\bar{u}) \, dx \\ & \leq \int_{\Omega} (|\varphi|^2 + |\psi|^2) \, dx + \varepsilon b_0 \int_{\Omega} |\nabla u|^2 \, dx + K_{\varepsilon} |\lambda|^2 \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + K \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}}. \end{aligned}$$

Proof. Multiplying the equalities (2.30) and (2.32) by \bar{u} and \bar{v} respectively, then integrating by parts on Ω and summing up the results, we get

$$\begin{aligned} & i\lambda \int_{\Omega} \varphi \bar{u} \, dx + \beta_0 \int_{\Omega} |\nabla u|^2 \, dx + \alpha \int_{\Omega} v \bar{u} \, dx + \int_{\Omega} \int_0^{+\infty} g(s) \nabla \eta \nabla \bar{u} \, ds \, dx \\ & + i\lambda \int_{\Omega} \psi \bar{v} \, dx + \int_{\Omega} |\nabla v|^2 \, dx + \alpha \int_{\Omega} u \bar{v} \, dx = \int_{\Omega} f^2 \bar{u} \, dx + \int_{\Omega} f^4 \bar{v} \, dx. \end{aligned} \quad (2.39)$$

Substituting $i\lambda \bar{u}$ given in (2.29) and $i\lambda \bar{v}$ given in (2.31) into (2.3), we find

$$\begin{aligned} & - \int_{\Omega} \varphi (\bar{\varphi} + \bar{f}^1) \, dx + \beta_0 \int_{\Omega} |\nabla u|^2 \, dx + \alpha \int_{\Omega} v \bar{u} \, dx + \int_{\Omega} \int_0^{+\infty} g(s) \nabla \eta(s) \nabla \bar{u} \, ds \, dx \\ & - \int_{\Omega} \psi (\bar{\psi} + \bar{f}^3) \, dx + \int_{\Omega} |\nabla v|^2 \, dx + \alpha \int_{\Omega} u \bar{v} \, dx = \int_{\Omega} f^2 \bar{u} \, dx + \int_{\Omega} f^4 \bar{v} \, dx. \end{aligned}$$

Then, we have

$$\beta_0 \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} |\nabla v|^2 \, dx + \alpha \int_{\Omega} (v\bar{u} + u\bar{v}) \, dx = \int_{\Omega} (|\varphi|^2 + |\psi|^2) \, dx - \int_{\Omega} \int_0^{+\infty} g(s) \nabla \eta(s) \nabla \bar{u} \, ds \, dx$$

$$+ \int_{\Omega} \varphi \overline{f^1} dx + \int_{\Omega} f^2 \overline{u} dx + \int_{\Omega} \psi \overline{f^3} dx + \int_{\Omega} f^4 \overline{v} dx.$$

Hence, getting

$$\begin{aligned} \beta_0 \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |\nabla v|^2 dx + \alpha \int_{\Omega} (v\overline{u} + u\overline{v}) dx &\leq \int_{\Omega} (|\varphi|^2 + |\psi|^2) dx + \int_{\Omega} \int_0^{+\infty} g(s) |\nabla \eta| |\nabla \overline{u}| ds dx \\ &+ \int_{\Omega} |\varphi| |\overline{f^1}| dx + \int_{\Omega} |f^2| |\overline{u}| dx + \int_{\Omega} |\psi| |\overline{f^3}| dx + \int_{\Omega} |f^4| |\overline{v}| dx. \end{aligned}$$

Now, using Cauchy–Schwarz’s inequality, we find

$$\begin{aligned} \beta_0 \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |\nabla v|^2 dx + \alpha \int_{\Omega} (v\overline{u} + u\overline{v}) dx &\leq \int_{\Omega} (|\varphi|^2 + |\psi|^2) dx + \int_0^{+\infty} g(s) \|\eta\| \|u\| ds \\ &\|\varphi\| \|f^1\| + \|f^2\|_2 \|u\| + \|\psi\|_2 \|f^3\| + \|f^4\|_2 \|v\|, \end{aligned}$$

then, we arrive at

$$\begin{aligned} \beta_0 \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |\nabla v|^2 dx + \alpha \int_{\Omega} (v\overline{u} + u\overline{v}) dx &\leq \int_{\Omega} (|\varphi|^2 + |\psi|^2) dx + \underbrace{\int_0^{+\infty} g(s) \|\eta\| \|u\| ds}_{I_4} \\ &+ K \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}}. \end{aligned}$$

And now using Young’s inequality on I_4 , we get

$$\forall \varepsilon > 0, I_4 = \int_0^{+\infty} g(s) (C_\varepsilon \|\eta\|^2 + \varepsilon \|u\|^2) ds.$$

According to lemma (2.3.1), we have

$$\forall \varepsilon > 0, I_4 \leq K_\varepsilon |\lambda|^2 \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + \varepsilon b_0 \|u\|^2,$$

where K_ε is a positive constant. Then our conclusion follows. \blacksquare

Lemma 2.3.3. *Under the conditions of the previous lemma, we have*

$$\frac{b_0}{2} \int_{\Omega} |\varphi|^2 dx \leq \varepsilon \int_{\Omega} (|u|^2 + |v|^2) dx + K_\varepsilon |\lambda|^2 \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + K \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}},$$

and

$$\left(\frac{1}{2} - \frac{K}{|\lambda|}\right) \int_{\Omega} |\psi|^2 dx \leq K_\varepsilon |\lambda|^2 \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + K \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}},$$

with $|\lambda| > 1$ large enough.

Proof. Multiplying the equation (2.33) by $\int_0^{+\infty} g(s) ds \overline{\varphi}$ and integrating on Ω , we find

$$i\lambda \underbrace{\int_0^{+\infty} g(s) \int_{\Omega} \eta(s) \overline{\varphi} dx ds}_{I_6} - b_0 \int_{\Omega} |\varphi|^2 dx + \int_0^{+\infty} g(s) \int_{\Omega} \eta_s(s) \overline{\varphi} dx ds = \int_0^{+\infty} g(s) \int_{\Omega} f_6(s) \overline{\varphi} dx ds.$$

On the other hand, noting that

$$\int_0^{+\infty} g(s) \int_{\Omega} \eta_s(s) \overline{\varphi} dx ds = - \int_0^{+\infty} g'(s) \int_{\Omega} \eta(s) \overline{\varphi} dx ds.$$

Substituting $i\lambda\bar{\varphi}$ from (2.30) into I_6 and using Green's formula, we obtain

$$\begin{aligned} \frac{b_0}{2} \int_{\Omega} |\varphi|^2 dx &= \beta_0 \int_0^{+\infty} g(s) \int_{\Omega} \nabla \eta(s) \nabla \bar{u} dx ds + \alpha \int_0^{+\infty} g(s) \int_{\Omega} \eta(s) \bar{v} dx ds - \int_0^{+\infty} g'(s) \int_{\Omega} \eta(s) \bar{\varphi} dx ds \\ &\quad + \int_{\Omega} \left(\int_0^{+\infty} g(s) \nabla \eta(s) ds \right)^2 dx ds - \int_0^{+\infty} g(s) \int_{\Omega} \eta(s) \overline{f^5(s)} dx. \end{aligned}$$

Using (1.6) and by following the same steps of the previous proofs, we arrive at that, for all $\varepsilon > 0$,

$$\frac{b_0}{2} \int_{\Omega} |\varphi|^2 dx \leq \varepsilon \int_{\Omega} (|u|^2 + |v|^2) dx + K_{\varepsilon} \|\eta\|_{2,g}^2 + K \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}},$$

where K is a positive constant and $|\lambda| > 1$. Using the lemma (2.3.2), follows the first inequality. To show the second inequality, we substitute the equation (2.29) into (2.33). This gives,

$$i\lambda\eta - i\lambda u + \eta_s = f_5 - f_1. \quad (2.40)$$

Substitute u given in (2.28) into (2.40). Then, we get,

$$i\lambda\alpha\eta - \lambda^2\varphi - i\lambda\Delta v + \alpha\eta_s = \alpha(f_5 - f_1) + i\lambda f_4. \quad (2.41)$$

Multiplying the equation (2.41) by $\int_0^{+\infty} g(s) ds \bar{\psi}$ and integrating on Ω and proceeding as to obtain the first estimate, we have

$$\frac{1}{2} \int_{\Omega} |\psi|^2 dx \leq \frac{K}{|\lambda|} \int_{\Omega} |\psi|^2 dx + K_{\varepsilon} |\lambda|^2 \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + K \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}},$$

where K is a positive constant and $|\lambda| > 1$. From where follows the second inequality. The proof is now complete. ■

The next theorem is one of the main results in this chapter.

Theorem 2.3.4. *The semigroup associated to the system (2.6)-(2.12) is polynomially stable and*

$$\|T(t)\mathcal{U}_0\|_{\mathcal{H}} \leq \frac{K}{\sqrt{t}} \|\mathcal{U}_0\|_{\mathcal{D}(\mathcal{B})}.$$

Moreover, this result is optimal.

Proof. Choosing $\varepsilon > 0$ small enough and for $|\lambda| > 1$ large enough, and from lemmas (2.3.1), (2.3.2) and (2.3.3), we get

$$\begin{aligned} &\bullet \int_{\Omega} \int_0^{+\infty} g(s) |\nabla \eta|^2 ds dx \leq K |\lambda|^2 \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}}, \\ &\bullet \beta_0 \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |\nabla v|^2 dx + \alpha \int_{\Omega} (u\bar{v} + v\bar{u}) dx \leq \int_{\Omega} (|\varphi|^2 + |\psi|^2) dx + K_{\varepsilon} |\lambda|^2 \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + K \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}}, \end{aligned}$$

$$\bullet \frac{b_0}{2} \int_{\Omega} |\varphi|^2 dx \leq K_{\varepsilon} |\lambda|^2 \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + K \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}},$$

and

$$\bullet \frac{1}{2} \int_{\Omega} |\psi|^2 dx \leq K_{\varepsilon} |\lambda|^2 \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + K \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}},$$

as $\varepsilon \rightarrow 0$, and $|\lambda| \rightarrow +\infty$.

On the other hand, we have

$$\begin{aligned} \|T(t)\mathcal{U}_0\|_{\mathcal{H}}^2 &= \|\mathcal{U}\|_{\mathcal{H}}^2 = \beta_0 \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} (|\varphi|^2 + |\psi|^2) dx + \int_{\Omega} |\nabla v|^2 dx + \alpha \int_{\Omega} (u\bar{v} + v\bar{u}) dx \\ &\quad + \int_{\Omega} \int_0^{+\infty} g(s) |\nabla \eta|^2 ds dx. \end{aligned}$$

From where it follows that

$$\|\mathcal{U}\|_{\mathcal{H}}^2 \leq K|\lambda|^2 \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + K \|\mathcal{U}\|_{\mathcal{H}} \|F\|_{\mathcal{H}},$$

where K is a positive constant. Using Young's inequality, we get

$$\forall \varepsilon > 0, \quad \|\mathcal{U}\|_{\mathcal{H}}^2 \leq \varepsilon \|\mathcal{U}\|_{\mathcal{H}}^2 + K_{1\varepsilon} |\lambda|^4 \|F\|_{\mathcal{H}}^2 + \varepsilon K \|\mathcal{U}\|_{\mathcal{H}}^2 + K_{2\varepsilon} \|F\|_{\mathcal{H}}^2,$$

after choosing ε where $1 - \varepsilon(1 + K) > 0$, we find

$$\|\mathcal{U}\|_{\mathcal{H}}^2 \leq K(1 + |\lambda|^4) \|F\|_{\mathcal{H}}^2,$$

with $K = \max(K_{1\varepsilon}, K_{2\varepsilon})$. Since $|\lambda| > 1$, then we have $1 + |\lambda|^4 \leq 2|\lambda|^4$. Hence,

$$\|\mathcal{U}\|_{\mathcal{H}} \leq K|\lambda|^2 \|F\|_{\mathcal{H}}, \quad \text{with } K \text{ is a positive constant.}$$

That can be written as

$$\|(\lambda I - \mathcal{B})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq K|\lambda|^2,$$

that is

$$\|(\lambda I - \mathcal{B})^{-1}\|_{\mathcal{L}(\mathcal{H})} = K\mathcal{O}(|\lambda|^2), \quad \text{as } \lambda \rightarrow +\infty. \quad (2.42)$$

Now, using the theorem (A.2.12), then the condition (2.42) is equivalent to

$$\begin{aligned} \|S(t)\mathcal{B}^{-1}\|_{\mathcal{L}(\mathcal{H})} &= K\mathcal{O}\left(t^{-\frac{1}{2}}\right), \quad \text{as } t \rightarrow +\infty \\ \Rightarrow \|S(t)\mathcal{B}^{-1}\|_{\mathcal{L}(\mathcal{H})} &\leq \frac{K}{\sqrt{t}}, \quad \text{as } t \rightarrow +\infty, \end{aligned}$$

then, we have

$$\begin{aligned} \frac{\|S(t)\mathcal{B}^{-1}F\|_{\mathcal{H}}}{\|F\|_{\mathcal{H}}} &\leq \frac{K}{\sqrt{t}}, \quad \forall F \in \mathcal{H}, (\text{with } \|F\|_{\mathcal{H}} \neq 0), \quad \text{as } t \rightarrow +\infty \\ \Rightarrow \|S(t)\mathcal{B}^{-1}F\|_{\mathcal{H}} &\leq \frac{K}{\sqrt{t}} \|F\|_{\mathcal{H}}, \quad \text{as } t \rightarrow +\infty. \end{aligned}$$

Taking $F = \mathcal{B}U_0$, we find

$$\|S(t)\mathcal{U}_0\|_{\mathcal{H}} \leq \frac{K}{\sqrt{t}} \|\mathcal{B}U_0\|_{\mathcal{H}}.$$

And as $\mathcal{B} : \mathcal{D}(\mathcal{B}) \rightarrow \mathcal{H}$, is a continuous operator, then

$$\|\mathcal{B}U_0\|_{\mathcal{H}} \leq C \|\mathcal{U}_0\|_{\mathcal{D}(\mathcal{B})}, \quad \text{with } C \text{ is a positive constant.}$$

We finally get

$$\|T(t)\mathcal{U}_0\|_{\mathcal{H}} \leq \frac{K}{\sqrt{t}} \|\mathcal{U}_0\|_{\mathcal{D}(\mathcal{B})}, \quad \text{where } K \text{ is a positive constant.}$$

Therefore the solution decays polynomially.

We prove now that the rate of decay is optimal, we will argue by contradiction. Suppose that the rate $t^{\frac{-1}{2}}$ can be improved; for example that the rate is $t^{\frac{-1}{2-\varepsilon}}$ for some $0 < \varepsilon < 2$, it's mean

$$\|T(t)\mathcal{U}_0\|_{\mathcal{H}} \leq \frac{K}{t^{\frac{1}{2-\varepsilon}}} \|\mathcal{U}_0\|_{\mathcal{D}(\mathcal{B})}.$$

From the theorem (A.2.13), we get

$$\overline{\lim}_{|\lambda| \rightarrow +\infty} |\lambda|^{-2+\frac{\varepsilon}{2}} \|(\lambda I - \mathcal{B})^{-1}\|_{\mathcal{L}(\mathcal{H})} < +\infty, \quad \text{with } i\mathbb{R} \subset \rho(\mathcal{B}).$$

But this does not happen, and for proof that let us consider the spectral problem:

$$\begin{cases} -\Delta w_n = \lambda_n^4 w_n & \text{in } \Omega \\ w_n = 0 & \text{on } \Gamma \end{cases}, \quad \forall n \geq 0, \quad (2.43)$$

where $\{\lambda_n\}_{n \geq 0}$ is real sequence, and check that $\lim_{n \rightarrow +\infty} \lambda_n = +\infty$. We will show that there exists a sequence of values λ_n such that

$$|\lambda_n|^{-2+\frac{\varepsilon}{2}} \|(\lambda_n I - \mathcal{B})^{-1}\|_{\mathcal{L}(\mathcal{H})} \rightarrow +\infty, \quad \text{as } n \rightarrow +\infty.$$

It is equivalent to prove that there exists a sequence $(\lambda_n) \subset \mathbb{R}$ with $\lim_{n \rightarrow +\infty} |\lambda_n| = +\infty$ and a sequence of data $F_n \in \mathcal{H}$, with $\|F_n\|_{\mathcal{H}} \leq 1$, such that

$$\lim_{n \rightarrow +\infty} |\lambda_n|^{-2+\frac{\varepsilon}{2}} \|(\lambda_n I - \mathcal{B})^{-1} F_n\|_{\mathcal{H}} = \lim_{n \rightarrow +\infty} |\lambda_n|^{-2+\frac{\varepsilon}{2}} \|\mathcal{U}_n\|_{\mathcal{H}} \rightarrow +\infty,$$

where $\mathcal{U}_n \in \mathcal{D}(\mathcal{B})$ not bounded, and

$$(\lambda_n I - \mathcal{B})\mathcal{U}_n = F_n.$$

Following the same steps as in the proof of theorem (2.2.1) with choosing $\lambda = \lambda_n^2$, we get

$$\|\varphi_n\|_2 = \frac{\lambda_n^2}{\alpha},$$

then, we have

$$\|\mathcal{U}_n\|_{\mathcal{H}} \geq \|\varphi_n\|_2 = \frac{\lambda_n^2}{\alpha}.$$

Hence, getting

$$|\lambda_n|^{-2+\frac{\varepsilon}{2}} \|\mathcal{U}_n\|_{\mathcal{H}} \geq \frac{|\lambda_n|^{\frac{\varepsilon}{2}}}{\alpha} \rightarrow +\infty, \quad \text{as } n \rightarrow +\infty.$$

Therefore the rate cannot be improved. The proof is now complete. ■

Conclusion

In this memory, two coupled systems of wave equations with past history terms are considered. In the first one, we assumed the past history terms in both equations. Moreover, in the second system we took the past history term, only in one equation. Under suitable hypotheses on the kernel, we showed the global existence and uniqueness of solutions by using semigroup theory. Taking into account the Gearhart-Prüss theorem, the exponential stability of the first system is concluded. However, we proved the lack of exponential decay to the second system by using the recent results due to Borichev-Tomilov, which allows us to deduce the optimal polynomial decay.

Appendix A

Basic concepts in Semigroup of linear operators

Contents

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A.1 Unbounded operators

Let \mathbb{H} be a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and the norm $\|\cdot\|_{\mathbb{H}}$, and \mathcal{A} is a linear mapping from $\mathcal{D}(\mathcal{A}) \subset \mathbb{H}$ into \mathbb{H} , where $\mathcal{D}(\mathcal{A})$ is the domain of \mathcal{A} .

Definition A.1.1. The resolvent set $\rho(\mathcal{A})$ of an operator \mathcal{A} is the set

$$\rho(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \lambda I - \mathcal{A} \text{ is invertible}\}.$$

The spectrume of operator \mathcal{A} , denoted $\sigma(\mathcal{A})$, is the complement of the resolvent set $\rho(\mathcal{A})$

$$\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \lambda \text{ is an eigenvalue}\},$$

this means that for all $\lambda \in \sigma(\mathcal{A})$ there exists a nonzero $\varphi \in \mathbb{H}$, such that $\mathcal{A}\varphi = \lambda\varphi$.

Corollary A.1.2. *The resolvent set $\rho(\mathcal{A})$ is open, and the spectrume is closed.*

Definition A.1.3. [11] Let \mathcal{A} be a linear operator on \mathbb{H} . \mathcal{A} is dissipative if and only if

$$\forall \varphi \in \mathcal{D}(\mathcal{A}), \operatorname{Re} \langle \mathcal{A}\varphi, \varphi \rangle_{\mathbb{H}} < 0.$$

A.2 Semigroup on a Hilbert space

Definition A.2.1. [11] A family $\{T(t)\}_{t>0}$ of bounded linear operators on \mathbb{H} is a semigroup of bounded linear operator when the following conditions hol:

(i) $T(0) = I$, (I is the identity operator on \mathbb{H}).

(ii) $T(t + s) = T(t)T(s)$ for every $t, s \geq 0$ (the semigroup property).

If $\lim_{t \rightarrow 0^0} \|T(t) - I\|_{\mathcal{L}(\mathbb{H})} = 0$, then we say that $T(t)$ is uniformly continuous.

Example A.2.2. We consider the Banach space ℓ^p , $p \geq 1$ defined by

$$\ell^p = \left\{ (x_n)_{n \in \mathbb{N}^*}; \sum_{n \geq 1} |x_n|^p < +\infty \right\}.$$

Equipped with the norm

$$\|x_n\|_{\ell^p} = \left(\sum_{n \geq 1} |x_n|^p \right)^{\frac{1}{p}}.$$

We define a family of linear operators $\{T(t)\}_{t \geq 0}$ by

$$T(t)x_n = e^{-a_n t} x_n, \quad \forall x_n \in \ell^p, \quad \forall t \geq 0, \quad \text{with } \{a_n\}_{n \geq 1} \text{ is a positive number sequence.}$$

If $t = 0$, we have

$$T(0)x_n = e^0 x_n = x_n,$$

then $T(0) = Id_{\ell^p}$. And,

$$\forall t, s \geq 0; T(t+s)x_n = e^{-a_n(t+s)} x_n = e^{-a_n t} e^{-a_n s} x_n = T(t)T(s)x_n,$$

then

$$T(t+s) = T(t)T(s), \quad \forall x_n \in \ell^p, \quad \forall t, s \geq 0.$$

So $\{T(t)\}_{t \geq 0}$ is a semigroup of bounded linear operators on the space ℓ^p .

Definition A.2.3. [11] Let \mathcal{A} be a linear operator, defined by

$$\mathcal{A}x = \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} = \left. \frac{d^+ T(t)x}{dt} \right|_{t=0}, \quad \text{for } x \in \mathcal{D}(\mathcal{A}),$$

with

$$\mathcal{D}(\mathcal{A}) = \left\{ x \in X : \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}.$$

We say that \mathcal{A} is the infinitesimal generator of the semigroup $T(t)$, and $\mathcal{D}(\mathcal{A})$ is the domain of \mathcal{A} .

Theorem A.2.4. Let $T(t)$ and $S(t)$ two uniformly continuous semigroups bounded linear operators, if

$$\lim_{t \rightarrow 0^+} \frac{T(t) - I}{t} = \mathcal{A} = \lim_{t \rightarrow 0^+} \frac{S(t) - I}{t},$$

then $T(t) = S(t)$, $\forall t \geq 0$.

Definition A.2.5. [11] A semigroup $\{T(t)\}_{t \geq 0}$ of bounded linear operators on \mathbb{H} is a strongly continuous semigroup of bounded linear operators if

$$\lim_{t \rightarrow 0^+} \|T(t)\varphi - \varphi\|_{\mathbb{H}} = 0.$$

A strongly continuous semigroup of bounded linear operators on \mathbb{H} will be called a semigroup of class C_0 or simply a C_0 -semigroup.

Theorem A.2.6. [11] Let $T(t)$ be a C_0 -semigroup. There exist constants $\omega \geq 0$ and $M \geq 1$ such that

$$\|T(t)\|_{\mathbb{H}} \leq Me^{\omega t}, \quad \forall t \geq 0.$$

Definition A.2.7. [11] We say that a C_0 -semigroup $\{T(t)\}_{t \geq 0}$ is of contraction if

$$\|T(t)\|_{\mathbb{H}} \leq 1, \quad \forall t \geq 0.$$

Definition A.2.8. (Cauchy abstract homogeneous problem) Let \mathcal{A} be a linear operator from $\mathcal{D}(\mathcal{A}) \subset \mathbb{H}$ into \mathbb{H} , the initial value problem

$$\begin{cases} U'(t) = \mathcal{A}U(t), \quad \forall t > 0, \\ U(0) = U_0, \end{cases} \quad (\text{A.1})$$

says an abstract homogeneous Cauchy problem associated with \mathcal{A} , with $U_0 \in \mathbb{H}$.

Theorem A.2.9. [11] Let \mathcal{A} be the infinitesimal generator of $T(t)$, a C_0 -semigroup on \mathbb{H} . For any initial data $U_0 \in \mathbb{H}$, the problem (A.1) has a unique weak solution $U \in C(\mathbb{R}^+; \mathbb{H})$. Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$, then the solution

$$U \in C(\mathbb{R}^+; \mathcal{D}(\mathcal{A})) \cap C^1(\mathbb{R}^+; \mathbb{H}).$$

Theorem A.2.10. (Lumer-Phillips)[11]. Let \mathcal{A} be a linear operator with dense domain $\mathcal{D}(\mathcal{A})$ in \mathbb{H} .

(a) If \mathcal{A} is dissipative and there is $\lambda_0 > 0$ such that $(\lambda_0 I - \mathcal{A})$ is maximal, then \mathcal{A} is the infinitesimal generator of a C_0 -semigroup of contractions on \mathbb{H} .

(b) If \mathcal{A} is the infinitesimal generator of a C_0 -semigroup of contractions on \mathbb{H} , then $(\lambda_0 I - \mathcal{A})$ is maximal for all $\lambda > 0$, and \mathcal{A} is dissipative. Moreover, for every $x \in \mathcal{D}(\mathcal{A})$, $\operatorname{Re} \langle \mathcal{A}x, x \rangle < 0$.

Theorem A.2.11. (Gearhart-Prüss)[12] Let $\{T(t)\}_{t \geq 0}$ be a C_0 -semigroup of contractions on Hilbert space \mathbb{H} . Then $T(t)$ is exponentially stable if and only if

(i) The resolvent set $\rho(\mathcal{A})$ of \mathcal{A} contains the imaginary axis $i\mathbb{R} \subset \rho(\mathcal{A})$.

(ii) $\limsup_{|\lambda| \rightarrow +\infty} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathbb{H})} < +\infty$.

Theorem A.2.12. [2] Let $\{T(t)\}_{t \geq 0}$ be a bounded C_0 -semigroup on a Hilbert space \mathbb{H} with generator \mathcal{A} such that $i\mathbb{R} \subset \rho(\mathcal{A})$. Then for a fixed $\alpha > 0$ the following conditions are equivalent:

(i) $\|(isI - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{A})} = \mathcal{O}(|s|^{-\alpha})$, as $s \rightarrow +\infty$.

(ii) $\|T(t)\mathcal{A}^{-1}\|_{\mathcal{L}(\mathcal{H})} = \mathcal{O}\left(t^{-\frac{1}{\alpha}}\right)$, as $t \rightarrow +\infty$.

Theorem A.2.13. [6] Let $T(t)$ be a C_0 -semigroup of contractions of linear operators on Hilbert space \mathbb{H} , with infinitesimal generator \mathcal{A} such that $i\mathbb{R} \subset \rho(\mathcal{A})$. If

$$\|T(t)U_0\|_{\mathbb{H}} \leq \frac{K}{t^\beta} \|U_0\|_{\mathcal{D}(\mathcal{A})}, \quad \text{where } K \text{ and } \beta \text{ are positive constants.}$$

Then we have that for any $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that

$$\frac{1}{|\lambda|^{\varepsilon + \frac{1}{\beta}}} \|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < C_\varepsilon.$$

A.3 Principal theorems and inequalities

Let Ω be an open bounded set of \mathbb{R}^n with smooth boundary Γ .

Theorem A.3.1. (Hölder's inequality) Let $1 \leq p \leq +\infty$, and q its conjugate exponent. Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$ and

$$\int_{\Omega} |f(x)g(x)| \, dx \leq \|f\|_p \|g\|_q.$$

Corollary A.3.2. (Cauchy-Schwarz's inequality) By taking $p = q = 2$, we obtain

$$\int_{\Omega} |f(x)g(x)| \, dx \leq \|f\|_2 \|g\|_2.$$

Theorem A.3.3. (Young's inequality) Let a, b two real numbers. we have, $\forall \varepsilon > 0$

$$|ab| \leq \varepsilon a^2 + K_{\varepsilon} b^2.$$

Theorem A.3.4. (Poincaré's inequality) we assume that Ω is a bounded open set of \mathbb{R}^n . Then there exists a positive constant C (dependent on Ω) such that

$$\|\varphi\| \leq C \|\nabla\varphi\|, \quad \forall \varphi \in H_0^1(\Omega).$$

Lemma A.3.5. (Green's formula) Let Ω a bounded open set of \mathbb{R}^n with a smooth boundary Γ . for $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, we have

$$\int_{\Omega} \Delta u v \, dx = \int_{\Omega} \frac{\partial u}{\partial \nu} v \, d\Gamma - \int_{\Omega} \nabla u \nabla v \, dx.$$

Remark A.3.6. If $v \in H_0^1(\Omega)$, then Green's formula is reduced to

$$\int_{\Omega} \Delta u v \, dx = - \int_{\Omega} \nabla u \nabla v \, dx.$$

Theorem A.3.7. (Lax Milgram's theorem)[3] Let a be a bilinear, continuous and coercive form. Then for all $v \in \mathbb{H}'$ (Dual space of \mathbb{H}), there exists a unique $u \in \mathbb{H}$ such that

$$a(u, v) = \langle v, u \rangle, \quad \forall v \in \mathbb{H}.$$

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المخلص:

في هذا العمل، نعتبر جملتي معادلات مشكلة من معادلاتي الأمواج، في الجملة الأولى نضع حد التأخر في كلتا المعادلتين، بينما في الجملة الثانية نعتبر حد التأخر في المعادلة الأولى فقط. و بفرض بعض الشروط على المعطيات، نبرهن وجود و وحدانية الحل الكلي للجملتين المعبرتين وذلك باستخدام نظرية نصف الزمر. بعد ذلك، نبرهن التناقص الأسي للجملة الأولى، بينما في الجملة الثانية نفقد التناقص الأسي و نحصل فقط على التناقص الحدودي الأمثل.

الكلمات المفتاحية: جمل معادلاتي الأمواج، التناقص الأسي، التناقص الحدودي المثالي، نظرية نصف الزمر، حد التأخر.

Abstract:

In this work, two coupled systems for two wave equations have been considered; In the first system we have made the past history in two equations, however in the second system we have considered the past history only in the first equation. Under some hypotheses in given data we have showed the global existence and uniqueness of solutions of two systems by using the semigroup theory. After, we have proved the exponential decay for the first system and for the second system we have obtained the lack exponential decay of solutions, consequently we have got the optimal polynomial decay.

Keywords: Coupled system of wave equations, Exponential decay, Optimal polynomial decay, Semigroup theory, Past history.

Résumé:

Dans ce travail, deux systèmes couplés pour les équations d'onde ont été considérés, dans le premier système nous avons pris des termes de retard dans les deux équations, cependant dans le second système nous avons considéré un terme de retard uniquement dans la première équation. Sous certaines hypothèses sur les données, nous avons montré l'existence globale et l'unicité des solutions de deux systèmes en utilisant la théorie de semi-groupes. En suite, nous avons prouvé la décroissance exponentielle pour le premier système et pour le second système nous avons perdu la décroissance exponentielle des solutions, par contre nous avons obtenu la décroissance polynomiale optimale.

Mots-clés: Système couplé pour les équations d'ondes, Décroissance exponentielle, Décroissance polynomiale optimale, Théorie de semigroupes, Terme de retard.