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Title

On Legendre Polynomials and Their Applications

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Dedications

- To my dear father:

To the man of my life, my examples of a respectful, honest father, eternal source of joy and happiness, the flame of my heart, no dedication can express the love and esteem. I have always had for you, I love you dad and I ask Almighty God to grant you health and wellness, and to make me always a source of your pride, and honor.

- To my dear Mother:

My mother who surrounded me with love, affection and who does everything to my success, May the Almighty give you health, happiness and long life so that I can fulfill you in turn.

- To my dear husband and my beloved son:

Thank you for your love, patience, and constant support throughout this journey. Your presence in my life has been my source of strength and happiness, encouraging me to continue and achieve this success. May Allah protect you and keep you as a belessing in my life.

- To my dear brothers:

Thank you for your love, encouragement, and continuous support.

- To my husband's family:

Thank you for your kindness, support, and for standing by my side throughout this journey.

- To all my family members:

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Contents

Notations

- Ω domain in $\mathbb{R}^n$.
- $\partial\Omega$ boundary of Ω.
- $L^2(\Omega)$ space of square integrable functions.
- $L^2_w(\Omega)$ weighted square integrable function space.
- l^2 The space of sequences such that: $\sum_{n=0}^{\infty} |x_n|^2 < \infty$.
- $\langle f, g\rangle_w$ weighted inner product of f and g.
- $w(x)$ weight function.
- $J_n^{\alpha,\beta}$ Jacobi polynomial of degree n.
- $G(x,t)$ generating function of Legendre polynomials.
- E_n sequence of subspaces of the Hilbert space H.
- $\bigoplus_n E_n$ Hilbert direct sum of the subspaces E_n.
- e_n orthonormal basis element.
- $\|u\|_2$ denotes the Euclidean norm.
- $\langle u, v\rangle$ inner product in Hilbert space.
- H Hilbert space.
- $dist(f, K)$ distance from f to the set K.
- P_M orthogonal projection onto M.
- H^* dual space of H.
- Δu Laplacian operator.
- $\frac{\partial u}{\partial x}$ partial derivative.
- $\frac{\partial u}{\partial n}$ normal derivative on $\partial\Omega$.
- $P_n(x)$ or $L_n(x)$ Legendre polynomial of degree n.
- $\psi(t)$ mother wavelet.

- $\psi_{a,b}(t)$ continuous wavelet.
- $\psi_{k,n}(t)$ discrete wavelet.
- $\Psi(x)$ vector of Legendre wavelets.
- M number of polynomials.
- k level of resolution.
- n translation parameter.
- δ_{ij} Kronecker delta.
- e^T transpose of the vector e.
- Ψ^T transpose of the matrix (or vector) Ψ.
- $\widetilde{E}$ transformed matrix.

Introduction

Orthogonal polynomials are a very vast branch of mathematics, of theoretical physics, and of mathematical physics. These polynomials have been a subject of study for a long time. At the beginning of the 19th century, more precisely in 1939, orthogonal polynomials received their first detailed study in [5]. Sequences of orthogonal polynomials appeared as solutions of equations in mathematical physics, particulary partial differential equations. Among the most important classical orthogonal polynomials, we mention Jacobi, Tchebyshev, Legendre, Hermite, etc.

Ordinary differential equations (ODEs) and partial differential equations (PDEs) are at the origin of the description of many natural phenomena in physics, chemistry, and biology...

We have several numerical methods to solve these equations, among which we cite the finite difference method, the finite element method, or iterative methods.

This cannot be done without efficiency in programming, since real systems are very complex, and require efficiency in the choice of the algorithm.

One of the solution methods, which is beginning to impose itself, is the Legendre Wavelet method. It was used in 2000 by M. Razzaghi and S. Yousefi [12] for solving variational problems in one dimension. The same authors applied it to solve an optimal control problem [12]. It has also been used for solving integro-differential equations such as the Fredholm integration equation.

In this work we will study Legendre polynomials and their classical properties (Rodriguez formula, generating function, orthogonality), and the use of these polynomials to solve partial differential equation by the Legendre wavelet method.

Our dissertation is organized into three chapters:

● In the first chapter, we will study some essential notions concerning differential equations, Hilbert spaces, orthogonal polynomial...etc, and many properties that will be used later in this work.

● In the second chapter, we introduces some essential notions on the Legendre polynomials. First, we present the Legendre differential equation. Then, we present the usual definitions for these polynomials and we cite some classical properties.

● In the last chapter, we study one-dimensional Legendre wavelets, as is known wavelets play a very important role in mathematical analysis, especially in the numerical solution of differential equations or patial differential equations.

● Finally, this work highlights the importance of Legendre polynomials and wavelets as powerful tools in the study and numerical solution of differential equations.

Chapter 1

Generalities On Orthogonal Polynomials

In this chapter, we will study some essential notions concerning differential equations, Hilbert spaces, orthogonal polynomial...etc, and many properties that will be used later in this work.

1.1 First-Order Differential Equations

Definition 1.1. An equation of the form

$$F(x, y, y') = 0$$

where y is a function of x and is the unknow, is called a **first-order differential equation**

Separable Equations
They are of the form

$$y' f(y) = g(x)$$

The general solution is given by

$$\int f(y)\,dy = \int g(x)\,dx \quad (\text{since} \quad y' = \frac{dy}{dx})$$

Homogeneous Equations
They are of the form

$$y' = F(\frac{y}{x})$$

The general solution is given by

$$x = Ce^{G(\frac{y}{x})}$$

Where $C \in \mathbb{R}$ and $G(t) = \int \frac{dt}{F(t)-t}$ (to prove it, we put $y = tx \Rightarrow dy = t\,dx + x\,dt$).

1.2 Linear Equations :

They are of the form

$$y' + yf(x) = g(x) \tag{1.1}$$

1.2.1 Integration of the Homogeneous Equation

(i.e., without a right-hand side)

$$y' + yf(x) = 0.$$

It is a separable variables equations. The general solution is given by

$$y = C\,\underbrace{e^{-\int f(x)\,dx}}_{y_1} \quad C \in \mathbb{R}$$

1.2.2 Integration of the Complete Equation (1.1)

$$\underbrace{y_G}_{\text{general solution of (1.1)}} = \underbrace{y}_{\text{general solution of (1.1) without the second member}} + \underbrace{y_0}_{\text{particular solution of (1.1)}}$$

How do we find y_0 ?

- By inspection (by simple observation).
  Otherwise
- By the method of variation of constants .

Method of variation of constants
We put $y_0(x) = C(x)y_1(x)$. After computation, we obtain

$$y_0(x) = e^{-\int f(x)\,dx} \cdot \int g(x)e^{\int f(x)\,dx}\,dx.$$

Example 1.1. : Integrate
$xy' - 2y = x$ $(-x$ is a particular solution)

1.2.3 Bernoulli Equations

They are of the form

$$y' + yf(x) = y^{\alpha}g(x)$$

- $y = 0$, is a particular solution .
- $\alpha = 0$, complete linear equation .
- $\alpha = 1$, linear equation without second member .
- $\alpha \neq 0, \quad \alpha \neq 1, \quad y \neq 0;$ we put $z = y^{1-\alpha}$.
  Then we obtain,
  $\frac{1}{1-\alpha} z' + zf(x) = g(x)$
  Thus, we are in the linear case.

Example 1.2. : Integrate.
$xy' - y = y^2 lnx.$

1.3 Second- Order Differential Equations

A second-order differential equation is any relation of the form $F(x, y, y', y'') = 0$ between the variable x, the function $g(x)$, and its first two derivatives .

Example 1.3. $y'' + wy = 0$ admits for solution on $\mathbb{R}$ the function $\phi_1(x) = \sin wx$ and $\phi_2(x) = \cos wx$

• One will admit that under certain conditions a second order differential equation admits an infinity of solutions depending on two arbitrary constants : $y = \phi(x, \lambda_1, \lambda_2)$.
The set of these solutions constitues the general integral and represents the equation of a family of curves depending on two parameters : which are called integral curves. A particular integral is obtained by imposing initial conditions. Most often they present the mselves in the form $y(x_0) = y_0$ and $y'(x_0) = y'_0$.
In this case, the curve is subjected to two conditions : to pass through a point (x_0, y_0) and to have a given coefficient of tangency y'_0.
Equations not Containing y
Let $F(x, y', y'') = 0$. One sets $y' = z(x)$, the equation becomes then $F(x, z, z') = 0$.

Example 1.4. $y'' + y'^2 = 0.$

we put $y' = z \Rightarrow z' + z^2 = 0$. we have

$$\begin{aligned}
&-\frac{dz}{z^2} = dx \\
&\Rightarrow \frac{1}{z} = x + x_0 \\
&\Rightarrow z = \frac{1}{x + x_0} \\
&\Rightarrow \frac{dy}{dx} = \frac{1}{x + x_0} \\
&\Rightarrow y = \ln(x + x_0) + y_0.
\end{aligned}$$

(x_0 and y_0 being constants).

Equations not Containing x

$F(y, y', y'') = 0$, and if one considers that y' is a function of y, we may set and the equation then becomes, with y as the new variable, a First-order equation in z. Let ... or ... and by integrating with respect to ...

1.4 Linear Second-Order Differential Equations

Definition 1.2. A linear Second- order differential equation is an equation of the form : $a(x)y'' + b(x)y' + c(x)y = f(x)$, where $a(x), b(x), c(x)$, and $f(x)$ are functions. Associated with this equation is the equation without right-hand side, called **the homogeneous equation :**

$$a(x)y'' + b(x)y' + c(x)y = 0. \tag{H}$$

We have the following fundamental theorem.

Theorem 1.1. *The general solution is obtained by adding to a particular solution of the complete equation, the general solution of the homogeneous equation* (H) *.*
$y_G = y_P + y_H$.

1.4.1 Linear Equations with Constant Coefficients

They are of the form

$$ay'' + by' + cy = f(x) \quad \text{where} \quad a, b, c \in \mathbb{R} \tag{1.2}$$

Homogeneous Equations (i.e., $f(x) = 0$).
We look for the general solution in the form, $y = e^{rx}$. We replace y by e^{rx} in (1.2) without the

right-hand side, we obtain

$$e^{rx}(ar^2 + br + c) = 0 \leftrightarrow \underbrace{ar^2 + br + c}_{\text{characteristic equation (*)}} = 0.$$

- **(*) Has two distinct real roots r_1 and r_2**
  Let $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$. The general solution of equation (1.2) without the right-hand side is.
  $$y = C_1 y_1 + C_2 y_2; \quad C_1, C_2 \in \mathbb{R}$$

- **r_1 and r_2 are complex numbers**
  $r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta$. The general soluion of the homogeneous equation is .
  $$\begin{aligned} & e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x) \\ & = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x \\ & = C_1 y_1 + C_2 y_2; \quad C_1, C_2 \in \mathbb{R} \end{aligned}$$

- **(*) Has a double root**
  The solution is of the form, $y = z(x)e^{rx}$. We replace y by its value in (1.2) without the right-hand side, we find $z = C_1 x + C_2, C_1$ and $C_2 \in \mathbb{R}$. The general solution of the homogeneous equation is
  $$\begin{aligned} & C_1 x e^{rx} + C_2 e^{rx} \\ & = C_1 y_1 + C_2 y_2; \quad C_1, C_2 \in \mathbb{R} \end{aligned}$$

Complete Equation (i.e., $f(x) \neq 0$)

$$\underbrace{y_G}_{\text{general solution of (1.2)}} = \underbrace{y}_{\text{Gneral solution of (1.2) without right-hand side}} + \underbrace{y_0}_{\text{Particular solution of (1.2)}}$$

Case where $f(x) \in R_n[X]$

- $c \neq 0$. We replace y by a polynomial of degree n, then we identify.
- $c = 0, b \neq 0$. We replace y by a polynomial of degree $n+1$, then we identify.
- $c = 0, b = 0, a \neq 0$. We integrate $f(x)$ twice in succession .

Case where $f(x) = e^{\alpha x} P_n(x)$.
We look for the particular solution by setting $y = e^{\alpha x} P(x)$ (P polynomial) . (1.2) is written
$aP'' + P'(2\alpha a + b) + P(a\alpha^2 + b\alpha + c) = P_n(x)$.
We are in the first case.
Case where $f(x) = A \cos \alpha x + B \sin \alpha x; \quad \alpha, A \quad \text{and} \quad B \quad \text{in} \quad \mathbb{R}$.

- $i\alpha$ is not a root of the characteristic equation. We look for y_0 in the form

$$y_0 = A' \cos \alpha x + B' \sin \alpha x.$$

- $i\alpha$ is root of the characteristic equation (necessarily simple).

We look for y_0 in the form

$$y_0 = x(A' \cos \alpha x + B' \sin \alpha x).$$

In the general case, to integrate equation (1.2), we apply the method of variation of constants. Let y be the general solution of (1.2) without the second member ($\Rightarrow y = C_1 y_1 + C_2 y_2$) .
We look for a particular solution by the method of variation of constants. We se

$$y = C_1(x)y_1(x) + C_2(x)y_2(x) \quad \text{with} \quad C_1'(x)y_1 + C_2'(x)y_2 = 0.$$

After calculation, we obtain

$$\begin{aligned} & ay_0'' + by_0' + cy_0 \\ & = C_1 \underbrace{(ay_1'' + by_1' + cy_1)}_{=0} + C_2 \underbrace{(ay_2'' + by_2' + cy_2)}_{=0} + C_1'(x)y_1'(x) + C_2'(x)y_2'(x) \\ & = f(x). \end{aligned}$$

We will therefore have two equations with two unknowns

$$\begin{cases} C_1'(x)y_1(x) + C_2'(x)y_2(x) = 0 \\ C_1'(x)y_1'(x) + C_2'(x)y_2'(x) = \frac{f(x)}{a} \end{cases}$$

This system has one and only one solution because y_1 and y_2 are linearly independent .

Example 1.5. Integrate $y'' + y = x \sin x$.

1. $y'' + y = 0 \Rightarrow y = C_1 \underbrace{\sin x}_{y_1} + C_2 \underbrace{\cos x}_{y_2}$
   $(r^2 + 1 = 0 \quad \leftrightarrow \quad r_1 = i, r_2 = -i)$.

2. $\begin{cases} C_1'(x) \sin x + C_2'(x) \cos x = 0 \\ C_1'(x) \cos x - C_2'(x) \sin x = x \sin x \end{cases}$
   $\leftrightarrow$
   $\begin{cases} C_1'(x) = \frac{1}{2} x \sin x \\ C_2'(x) = x \sin^2 x \end{cases}$

$\Rightarrow$ (integration by parts)

$$\begin{cases} C_1(x) = -\frac{1}{4}x\cos 2x + \frac{1}{8}\sin^2 x + C_1 \\ C_2(x) = -\frac{1}{4}(x\sin 2x + \frac{1}{2}\cos 2x - x^2) + C_2 \end{cases}$$

$$\begin{aligned} &y_G \\ &= C_1\sin x + C_2\cos x + C_1(x)\sin x + C_2(x)\cos x \\ &= \frac{\sin x}{4}(-x\cos 2x + \frac{1}{2}\sin^2 x + C_1) - \frac{\cos x}{4}(x\sin 2x + \frac{1}{2}\cos 2x - x^2 + C_2), \quad C_1 \quad \text{and} \quad C_2 \quad \text{in} \quad \mathbb{R}. \end{aligned}$$

1.5 Hilbert Spaces

1.5.1 Definitions and Elementary Properties. Projection onto a Closed Convex Set

Definition 1.3. Let H be a vector space. A scalar product $\langle u, v\rangle$ is a bilinear form on $H \times H$ with values in $\mathbb{R}$ (i.e., a map from $H \times H$ to $\mathbb{R}$ that is linear in both variables) such that

$$\begin{aligned} \langle u, v\rangle = \langle v, u\rangle \quad \forall u, v \in H \quad &\text{(symmetry)}, \\ \langle u, u\rangle \geq 0 \quad \forall u \in H \quad &\text{(positive)}, \\ \langle u, u\rangle = 0 \quad \Longleftrightarrow u = 0 \quad &\text{(definite)}. \end{aligned}$$

Let us recall that a scalar product satisfies the Cauchy-Schwarz inequality

$$|u, v| \leq \langle u, u\rangle^{\frac{1}{2}} \langle v, v\rangle^{\frac{1}{2}}, \quad \forall u, v \in H.$$

It follows from the Cauchy-Schwarz inequality that the quantity

$$\boxed{\|u\|_H = \langle u, u\rangle^{\frac{1}{2}},}$$

is a norm.

Definition 1.4. A Hilbert space is a vector space H equipped with a scalar product such that H is complete for the norm $\| \quad \|$.

Basic example $L^2(\Omega)$ equipped with the scalar product

$$\langle u, v\rangle = \int_\Omega u(x)v(x)d\mu,$$

is a Hilbert space. In particular, l^2 is a Hilbert space.

Theorem 1.2. ***(projection onto a closed convex set)*** *Let $K \subset H$ be a nonempty closed convex set. Then for every $f \in H$ there exists a unique element $u \in K$ such that*

$$\|f-u\|_H = \min_{v \in K} \|f-v\| = dist(f, K). \tag{1.3}$$

Moreover, u is characterized by the property

$$u \in K \quad and \quad \langle f-u, v-u \rangle \leq 0, \quad \forall v \in k. \tag{1.4}$$

Corollary 1.1. *Assume that $M \subset H$ is a closed convex subspace. Let $f \in H$. Then $u = P_M(f)$ is characterized by*

$$\boxed{u \in M \quad and \quad \langle f-u, v \rangle = 0, \quad \forall v \in M.}$$

Moreover, P_M is a linear operator, called the orthogonal projection.

Proof. See [4] □

1.6 The Dual Space of a Hilbert Space

It is very easy, in a Hilbert space, to write down continuous linear functionals. Pick any $f \in H$, then the map $u \mapsto \langle f, u \rangle$ is a continuous linear functional on H. It is a remarkable fact that all continuous linear functionals on H are obtained in this fashion:

Theorem 1.3. ***(Riesz-Fréchet representation theorem)*** *Given any $L \in H^*$ there exists a unique $f \in H$ such that*

$$L(u) = \langle f, u \rangle, \quad \forall u \in H.$$

Moreover;

$$\|f\| = \|L\|_{H^*}.$$

Proof. See [4] □

Definition 1.5. A bilinear form a : $H \times H \to \mathbb{R}$ is said to be

(i) continuous if there is a constant C such that

$$|a(u,v)| \leq C \|u\| \|v\|, \quad \forall u, v \in H,$$

(ii) coercive if there is a constant $\alpha > 0$ such that

$$a(v,v) \geq \alpha \|v\|^2, \quad \forall v \in H.$$

Theorem 1.4. ***(Stampacchia)*** *Assume that $a(u,v)$ is a continuous coercive bilinear form on H. Let $K \subset H$ be a nonempty closed and convex subset. Then, given any $\phi \in H^*$. There exists a unique element $u \in K$ such that*

$$a(u, v-u) \geq \langle \phi, v-u \rangle \quad \forall v \in K. \tag{1.5}$$

Moreover, if a is symmetric, then u is characterized by the property

$$\boxed{u \in K \quad \text{and} \quad \frac{1}{2}a(u,u) - \langle \phi, u \rangle = \min_{v \in K} \left\{ \frac{1}{2}a(v,v) - \langle \phi, v \rangle \right\}.} \tag{1.6}$$

Proof. See [4] □

Corollary 1.2. ***(Lax-Milgram)*** *Assume that $a(u,v)$ is a continuous coercive bilinear form on H. Then, given any $\phi \in H^*$. There exists a unique element $u \in H$ such that*

$$a(u,v) = \langle \phi, v \rangle \quad \forall v \in H. \tag{1.7}$$

Moreover, if a is symmetric, then u is characterized by the property

$$\boxed{u \in H \quad \text{and} \quad \frac{1}{2}a(u,u) - \langle \phi, u \rangle = \min_{v \in K} \left\{ \frac{1}{2}a(v,v) - \langle \phi, v \rangle \right\}.} \tag{1.8}$$

Proof. See [4] □

1.7 Hilbert Sums. Orthonormal Bases

Definition 1.6. Let $(E_n)_{n\geq 1}$ be a sequence of closed subspaces of H. One says that H is the Hilbert sum of the E_n's and one writes $H = \bigoplus_n E_n$ if

(a) the spaces E_n are mutually orthogonal, i.e.,

$$(u,v) = 0 \quad \forall u \in E_n, \quad \forall v \in E_m, \quad m \neq n,$$

(b) the linear space spanned by $\bigcup_{n=1}^{\infty} E_n$ is dense in H.

Theorem 1.5. *Assume that H is the Hilbert sum of the E_n's. Given $u \in H$, set*

$$u_n = P_{E_n} u$$

and

$$S_n = \sum_{k=1}^{n} u_k.$$

Then we have

$$\lim_{n\to\infty} S_n = u \tag{1.9}$$

and

$$\sum_{k=1}^{\infty} |u_k|^2 = |u|^2 \quad \text{(Bassel-Parseval's identity).} \tag{1.10}$$

It is convenient to use the following lemma.

Lemma 1.1. *Assume that (v_n) is any sequence in H such that*

$$(v_m, v_n) = 0 \quad \forall m \neq n, \tag{1.11}$$

$$\sum_{k=1}^{\infty} |v_k|^2 < \infty. \tag{1.12}$$

set

$$S_n = \sum_{k=1}^{n} v_k.$$

Then

$$S = \lim_{n\to\infty} S_n \quad \text{exists}$$

and, moreover,

$$|S|^2 = \sum_{k=1}^{\infty} |v_k|^2 . \tag{1.13}$$

Proof. See [4] □

Definition 1.7. A sequence $(e_n)_{n\geq 1}$ in H is said to be an orthonormal basis of H if it satisfies the following properties:

(i) $|e_n| = 1 \quad \forall n$ and $(e_m, e_n) = 0 \quad \forall m \neq n$,

(ii) the linear space spanned by e_n 's is dense in H .

Corollary 1.3. *Let (e_n) be an orthonormal basis. Then for every $u \in H$, we have*

$$u = \sum_{k=1}^{\infty} (u, e_k) e_k, \quad \text{i.e.,} \quad u = \lim_{n\to\infty} \sum_{k=1}^{n} (u, e_k) e_k$$

and

$$|u|^2 = \sum_{k=1}^{\infty} |(u, e_k)|^2 .$$

Conversely, given any sequence $(\alpha_n) \in l^2$, the series $\sum_{k=1}^{\infty} \alpha_k e_k$ converges to some element $u \in H$ such that $(u, e_k) = \alpha_k \quad \forall k$ and $|u|^2 = \sum_{k=1}^{\infty} \alpha_k^2$.

Proof. See [4] □

Theorem 1.6. *Every separable Hilbert space has an orthonormal basis.*

Proof. See [4] □

1.8 Frobenius Method

In this section, we will study the Frobenius method to solve certain differential equations

Theorem 1.7. *[15] Consider a differential equation of Fuchs types*

$$y'' + \frac{a(x)}{x}y' + \frac{b(x)}{x^2}y = 0, \tag{1.14}$$

Where the functions

$$a(x) = a_0 + a_1x + ...,$$

$$b(x) = b_0 + b_1x + ...,$$

Are analytic on $-R < x < R$. If r_1 and $r_2 \quad (r_1 \geq r_2)$ are the roots of the characteristic equation (also called the indicial equation)

$$r^2 + (a_0 - 1)r + b_0 = 0 \tag{1.15}$$

Then equation (1.14) *always admit a solution of the form*

$$y_1(x) = x^{r_1}\sum_{m=0}^{+\infty} c_m x^m = x^{r_1}(c_0 + c_1x + c_2x^2 + ...), \quad c_0 \neq 0, \tag{1.16}$$

The radius of convergence of equation (**??**) *is at least R. The form of the second solution depends on the difference $r_1 - r_2$. We distinguish three cases*
Case 1 : *if $0 < r_1 - r_2$ is not an integer, then*

$$y_2(x) = x^{r_2}\sum_{m=0}^{+\infty} c^*_m x^m, \quad c^*_0 \neq 0. \tag{1.17}$$

Case 2 : *if $r_1 = r_2$, then*

$$y_2(x) = y_1(x)\ln(x) + x^{r_2}\sum_{m=1}^{+\infty} A_m x^m, \quad x > 0, \quad A_1 \neq 0$$

.

Case 3 : *if* $0 < r_1 - r_2 = p$*, a positive integer, then*

$$y_2(x) = ky_1(x)\ln(x) + x^{r_2}\sum_{m=0}^{+\infty} c_m x^m, \quad x > 0, \quad c_0 \neq 0$$

. The solution in case 1 and in case 3, when the logarithmic terms is admet $(k = 0)$*, is obtained by the Frobenius method itself. The solution in case 2 and 3, where the logarithmic term appears, is obtained by the method of generalized series.*

Definition 1.8. (Regular Singular Point)
The singular point 0 is said to be regular if the functions

$$x(\frac{a(x)}{x}) = a(x), \quad x^2(\frac{b(x)}{x^2}) = b(x)$$

. Are analytic in a neighborhood of 0 .
Hence, equation (1.14) admits a regular singular point at $x = 0$.

Determination of the coefficients of the solution by recurrence
Let us set

$$y(x) = c_0x^r + c_1x^{r+1} + c_2x^{r+2} + ...$$

In

$$x^2y'' + xa(x)y' + b(x)y = 0.$$

Then

$$\begin{aligned} x^2y'' &= r(r-1)c_0x^r + (r+1)rc_1x^{r+1} + (r+2)(r+1)c_2x^{r+2} + ..., \\ xa(x)y' &= \left[a_0 + a_1x + a_2x^2 + ...\right]\left[rc_0x^r + (r+1)c_1x^{r+1} + ...\right], \\ b(x)y &= \left[b_0 + b_1x + b_2x^2 + ...\right]\left[c_0x^r + c_1x^{r+1} + ...\right], \end{aligned}$$

Hence

$$0 = 0, \quad \text{For all } x$$

The coefficient of x^r

$$r(r-1)c_0 + a_0rc_0 + b_0c_0 = 0.$$

Thus ,

$$\left[r(r-1) + a_0r + b_0\right]c_0 = 0.$$

If $c_0 \neq 0$, then r is a solution of the indicial equation:

$$r(r-1) + a_0r + b_0 = 0.$$

Then c_0 remains arbitrary.
The coefficient of x^{r+1} We have the equation

$$(r+1)rc_1 + a_1rc_0 + a_0(r+1)c_1 + b_1c_0 + b_0c_1 = 0.$$

Which can be solve for c_1 in terms of c_0 :

$$\left[(r+1)r + a_0(r+1) + b_0\right] c_1 = -(a_1r + b_1)c_0.$$

The coefficient of x^{r+2} gives the equation

$$(r+2)(r+1)c_2 + a_0(r+2)c_2 + a_1(r+1)c_1 + a_2rc_0 + b_0c_2 + b_1c_1 + b_2c_0 = 0.$$

Which can be solve for c_2 in terms of c_0 and c_1 :

$$\left[(r+2)(r+1) + a_0(r+2) + b_0\right] c_2 = -\left[a_1(r+1) + b_1\right] c_1 - \left[a_2r + b_2\right] c_0.$$

Thus, by recurrence, we find the coefficients c_s of x^{r+s} in terms of $c_0, ..., c_{s-1}$.

Remark 1.1. If r_1 and r_2 are the solutions of the indicial equation (1.15), then the coefficients c_0 and c_0^*, A_1 and c_0 of the nonzero solutions $y_1(x)$ and $y_2(x)$ are nonzero.

1.9 Review Of Orthogonal Polynomials

Polynomial Space
The study of orthogonal polynomials in one variable requires working in the vector space of polynomials in one real variable $\mathbb{R}[x]$. A polynomial $P_n(x)$ is of degree n if it can be written in the form

$$P_n(x) = \sum_{i=0}^{n} a_i x^i,$$

with $a_n \neq 0$.
Orthogonal Polynomials
Let $I =]a, b[$ be an interval of $\mathbb{R}$ and $W : I \to \mathbb{R}^+$ a weight function which is strictly positive and continuous such that

$$\forall n \in \mathbb{N}, \int_I |x|^n w(x)\, dx < +\infty.$$

We define the Hilbert space $\mathbb{E} = L^2_w(I)$ by

$$\mathbb{E} = \left\{ f | \int_I f^2(x)w(x)dx < +\infty \right\}.$$

The space $\mathbb{E}$ is endowed with the inner product

$$\langle f, g \rangle_w = \int_I f(x)g(x)w(x)dx.$$

The orthogonal polynomials with respect to the weight function $w(x)$ are the polynomials P_n of degree n satisfying the orthogonality relation in $\mathbb{E}$

$$\langle P_n, P_m \rangle = \int_I P_n(x)P_m(x)w(x)dx = 0, \quad n \neq m, \quad n, m = 0, 1, 2, ...$$

Jacobi Polynomials

Consider the Hilbert space $L^2_{w^{(\alpha,\beta)}}([-1, 1])$ with the weight function

$$w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta.$$

For the weight function $w^{(\alpha,\beta)}$ to be integrable on th interval $[-1, 1]$, we impose the following condition on the parameters α and β [5]

$$\alpha, \beta > -1$$

Definition 1.9. [18] Jacobi polynomials are the most general classical orthogonal polynomials, denoted by $J_n^{(\alpha,\beta)}$.

They are solutions of the second-order homogeneous differential equation

$$(1-x^2)y'' + (\beta - \alpha - (\alpha + \beta + 2)x)y' + n(n + \alpha + \beta + 1)y = 0,$$

Where n is the degree of the polynomial.

Jacobi polynomials are explicitly defined by the following three formulas:

1. **Rodrigues' Formula:**

$$(1-x)^\alpha(1+x)^\beta J_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}((1-x)^{(n+\alpha)}(1+x)^{(n+\beta)}).$$

2. **Analytic Formula :**

$$J_n^{(\alpha,\beta)}(x) = \frac{1}{2^n}\sum_{i=0}^{n}\binom{n+\alpha}{i}\binom{n+\beta}{n-i}(x-1)^{n-i}(1+x)^i.$$

3. **Recurrence Relation :**

$$J_0^{(\alpha,\beta)} = 1,$$

$$J_1^{(\alpha,\beta)} = \frac{\alpha+\beta+2}{2}x + \frac{\alpha-\beta}{2},$$

$$J_{n+1}^{(\alpha,\beta)}(x) = (a_n x + b_n)J_n^{(\alpha,\beta)}(x) - c_n J_{n-1}^{(\alpha,\beta)}(x), \quad n \geq 1.$$

With

$$a_n = \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)},$$

$$b_n = \frac{(2n+\alpha+\beta+1)(\alpha^2-\beta^2)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)},$$

$$c_n = \frac{(\alpha+n)(\beta+n)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}.$$

The first of these polynomials :
$J_0^{(\alpha,\beta)}(x) = 1,$
$J_1^{(\alpha,\beta)}(x) = \frac{1}{2}\left[2(\alpha+1) + (\alpha+\beta+2)(x-1)\right],$
$J_2^{(\alpha,\beta)}(x) \;=\; \frac{1}{8}\left[4(\alpha+1)(\alpha+2) + 4(\alpha+\beta+3)(\alpha+2)(x-1) + (\alpha+\beta+3)(\alpha+\beta+4)(x-1)^2\right].$
Standard orthogonal polynomials are particular cases of Jacobi polynomials. The most well-known cases are the Gegenbauer (or ultraspherical) polynomials, obtained by setting $\alpha = \beta$. Legendre polynomials P_n are obtained by setting $\alpha = \beta = 0$, and all types of Chebyshev polynomials T_n^i with $i = 1, 2, 3$, and 4

$$\begin{cases} T_n^1(x) = J_n^{(\frac{-1}{2},\frac{-1}{2})}, \\ T_n^2(x) = J_n^{(\frac{1}{2},\frac{1}{2})}, \\ T_n^3(x) = J_n^{(\frac{-1}{2},\frac{1}{2})}, \\ T_n^4(x) = J_n^{(\frac{1}{2},\frac{-1}{2})}. \end{cases}$$

CHAPTER 2

LEGENDRE POLYNOMIALS

In the second chapter, we introduces some essential notions on the Legendre polynomials. First, we present the Legendre differential equation. Then, we present the usual definitions for these polynomials and we cite some classical properties.

2.1 Legendre Equations

Legendre polynomials are polynomial solutions on the inteval $[-1, 1]$, of the homogeneous linear differential equation of order 2 with variable coefficients called the Legendre differential equation. This equation si given by

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0, \quad n \in \mathbb{R} \tag{2.1}$$

Before searching for the solution of equation (2.1), let us first look for the domain of convergence. For this, we use the following theorem : if we have a differential equation of the form

$$x^2\frac{d^2y}{dx^2} - xq(x)\frac{dy}{dx} + p(x)y = 0, \tag{2.2}$$

and if $q(x)$ and $p(x)$ can be developed in the following forms

$$q(x) = \sum_{i=0}^{\infty} q_i x^i, \quad p(x) = \sum_{i=0}^{\infty} p_i x^i.$$

Therefore, the domain of convergence of the solution of equation (2.2) is the same as the domain of convergence of $q(x)$ and $p(x)$.
To determine the two functions $q(x)$ and $p(x)$, we write equation (2.1) in the form

$$x^2\frac{d^2y}{dx^2} - x\frac{2x^2}{(1-x^2)}\frac{dy}{dx} + \frac{n(n+1)x^2}{(1-x^2)}y = 0. \tag{2.3}$$

By comparing equation (2.3) with equation (2.2), we deduce that.

$$q(x)=\frac{2x^2}{(1-x^2)},\quad p(x)=\frac{n(n+1)x^2}{(1-x^2)}.$$

The expansion into a power series of the functions $q(x)$ and $p(x)$ is

$$q(x)=2x^2\frac{1}{(1-x^2)}=2x^2\sum_{k=0}^{\infty}x^{2k},$$
$$p(x)=n(n+1)x^2\frac{1}{(1-x^2)}=n(n+1)x^2\sum_{k=0}^{\infty}x^{2k}.$$

Thus, the two functions $q(x)$ and $p(x)$ are convergent only in the interval $]-1,1[$.
Equation (2.1) is solved by the Frobenius method which consists in searching for solutions in the form of a power series

$$y(x)=\sum_{i=0}^{+\infty}a_ix^i=a_0+a_1x+a_2x^2+.... \tag{2.4}$$

The only singular points of this equation are 1 and -1. Consequently.

$$y'(x)=\left(\sum_{i=0}^{+\infty}a_ix^i\right)'=\sum_{i=1}^{+\infty}ia_ix^{i-1}, \tag{2.5}$$

and

$$y''(x)=\left(\sum_{i=0}^{+\infty}a_ix^i\right)''=\sum_{i=2}^{+\infty}i(i-1)a_ix^{i-2}. \tag{2.6}$$

By substituting the expressions (2.4), (2.5) and (2.6) into equation (2.1), we obtain

$$(1-x^2)\sum_{i=2}^{+\infty}i(i-1)a_ix^{i-2}-2x\sum_{i=1}^{+\infty}ia_ix^{i-1}+n(n+1)\sum_{i=0}^{+\infty}a_ix^i=0. \tag{2.7}$$

By index changes, we are led to $\sum_{i=0}^{+\infty}b_ix^i$, where the sequence (b_i) is written as a function of the sequence (a_i). We have

$$x\sum_{i=1}^{+\infty}ia_ix^{i-1}=\sum_{i=0}^{+\infty}ia_ix^i, \tag{2.8}$$

and

$$\begin{aligned}(1-x^2)\sum_{i=2}^{+\infty} i(i-1)a_i x^{i-2} &= \sum_{i=2}^{+\infty} i(i-1)a_i x^{i-2} - x^2\sum_{i=2}^{+\infty} i(i-1)a_i x^{i-2} \\ &= \sum_{i=0}^{+\infty}(i+2)(i+1)a_{i+2}x^i - \sum_{i=0}^{+\infty} i(i-1)a_i x^i.\end{aligned} \tag{2.9}$$

By substituting the expressions (2.8) and (2.9) into equation (2.7), we obtain

$$\sum_{i=0}^{+\infty}(i+2)(i+1)a_{i+2}x^i - \sum_{i=0}^{+\infty} i(i-1)a_i x^i - 2\sum_{i=0}^{+\infty} i a_i x^i + n(n+1)\sum_{i=0}^{+\infty} a_i x^i = 0,$$

$$\sum_{i=0}^{+\infty}\left[(i+2)(i+1)a_{i+2} + \left[n(n+1) - i(i-1) - 2i\right]a_i\right]x^i = 0,$$

$$\sum_{i=0}^{+\infty}\left[(i+2)(i+1)a_{i+2} + \left[n(n+1) - i(i+1)\right]a_i\right]x^i = 0.$$

$$\sum_{i=0}^{+\infty} b_i x^i = 0 \Rightarrow b_i = 0,$$

Such that

$$b_i = (i+2)(i+1)a_{i+2} + \left[n(n+1) - i(i+1)\right]a_i = 0.$$

We therefore have the recurrence relation.

$$\begin{aligned}a_{i+2} &= a_i\frac{i(i+1) - n(n+1)}{(i+1)(i+2)} \\ &= a_i\frac{(i-n)(n+i+1)}{(i+1)(i+2)}\end{aligned}$$

On sees that there are two series of solution, one series contains only even powers of x, the other is a series of odd power. For deducing the form of the general term a_i as a function of a_0

(or a_1) and i, we consider some particular cases.

$$\begin{aligned}
a_2 &= -a_0\frac{n(n+1)}{1.2}\\
a_3 &= a_1\frac{(1-n)(n+2)}{2.3}\\
a_4 &= -a_2\frac{(n-2)(n+3)}{3.4}\\
&= a_0\frac{n(n-2)(n+1)(n+3)}{1.2.3.4}\\
a_5 &= -a_3\frac{(n-3)(n+4)}{4.5}\\
&= a_1\frac{(n-1)(n-3)(n+2)(n+4)}{1.2.3.4.5}
\end{aligned}$$

Form the pervious equations, one can deduce the general term which is given by

$$a_{2i} = \frac{(-1)^i a_0}{(2i)!}n(n-2)(n-4)...(n-2i+2)(n+1)(n+3)...(n+2i-1),$$

$$a_{2i+1} = \frac{(-1)^i a_1}{(2i+1)!}(n-1)(n-3)...(n-2i+1)(n+2)(n+4)...(n+2i).$$

Thus, the general solution can be written in the form.

$$\begin{aligned}
y(x) &= \sum_{i=0}^{+\infty} a_i x^i\\
&= \sum_{i=0}^{+\infty} a_{2i}x^{2i} + \sum_{i=0}^{+\infty} a_{2i+1}x^{2i+1}\\
&= a_0\sum_{i=1}^{+\infty} a_{2i}x^{2i} + a_1 x + \sum_{i=1}^{+\infty} a_{2i+1}x^{2i+1}\\
&= a_0\left\{1 + (-1)^i\frac{n(n-2)(n-4)...(n-2i+2)(n+1)(n+3)...(n+2i-1)}{(2i)!}x^i\right\}\\
&+ a_1\left\{x + \sum_{i=1}^{+\infty}(-1)^i\frac{(n-1)(n-3)...(n-2i+1)(n+2)(n+4)...(n+2i)}{(2i+1)!}x^{2i+1}\right\}\\
&= a_0 y_1(x) + a_1 y_2(x).
\end{aligned}$$

We emphasize that this solution is valid in the interval $-1 < x < 1$. In order to extend the solution to the point $x = -1$ and $x = 1$, we use the following observation :

- If n is even, i.e., $n = 2i$, we have

$$a_2 \neq 0, a_4 \neq 0, ..., a_{2i} \neq 0,$$

$$a_{2i+2} = a_{2i+4} = \text{..........} = 0.$$

- If n is odd, i.e., $n = 2i + 1$

$$a_3 \neq 0, a_5 \neq 0, \text{...........}, a_{2i+1} \neq 0,$$

$$a_{2i+3} = a_{2i+5} = \text{.................} = 0.$$

Therefore for $n = 2i$, the polynomial $y_2(x)$ is zero and $y_1(x)$ becomes finite for every x inside the interval $[-1, 1]$. Conversely, for $n = 2i + 1$, the polynomial $y_1(x)$ is zero while the polynomial $y_2(x)$ becomes finite for every x inside the interval $[-1, 1]$. The solution of the Legendre equation in the interval $[-1, 1]$ is therefore given either by $y_1(x)$ (in the case where n is even) or by $y_2(x)$ (if n is odd). $y(x)$ can be written in the general form

$$\begin{aligned} y(x) &= a_n x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} + \cdots + \begin{cases} a_0, & \text{if} \quad n \quad \text{is even}, \\ a_1 x, & \text{if} \quad n \quad \text{is odd}, \end{cases} \\ &= \sum_{k=0}^{\left[\frac{n}{2}\right]} a_{n-2k} x^{n-2k}. \end{aligned} \tag{2.10}$$

Where

$$\left[\frac{n}{2}\right] = \begin{cases} \dfrac{n}{2}, & \text{if} \quad n \quad \text{is even}, \\ \dfrac{n-1}{2}, & \text{if} \quad n \quad \text{is odd} \end{cases}$$

a_{n-2k} can be deduced from the general form of a_i. We have

$$a_i = -a_{i+2} \frac{(i+2)(i+1)}{(n-i)(n+i+1)} \tag{2.11}$$

By replacing i by $n-2$ in equation (2.11), we find

$$a_{n-2} = -a_n \frac{n(n-1)}{2(2n-1)}.$$

For $i = n-4$, we obtain

$$\begin{aligned} a_{n-4} &= -a_{n-2} \frac{(n-2)(n-3)}{4(2n-3)} \\ &= a_n \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}. \end{aligned}$$

Form the equations of a_{n-2} and a_{n-4}, we can deduce the general form.

$$a_{n-2k} = (-1)^k \frac{n(n-1)(n-2)...(n-2k+1)}{2.4...(2k)(2n-1)(2n-3)...(2n-2k+1)}.$$

Replacing a_{n-2k} in equation (2.10), we obtain

$$y(x) = a_n \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{n(n-1)(n-2)...(n-2k+1)}{2.4...(2k)(2n-1)(2n-3)...(2n-2k+1)} x^{n-2k}$$

We can simplify the expression of $y(x)$ by writing the numerator and the denominator differently.

$$n(n-1)...(n-2k+1) = n(n-1)...(n-2k+1)\frac{(n-2k)!}{(n-2k)!} = \frac{n!}{(n-2k)!}$$

$$2.4.6.........2k = (2.1)(2.2)(2.3)...(2.k) = 2^k k!.$$

And

$$\begin{aligned}(2n-1)(2n-3)...(2n-2k+1) &= \frac{2n(2n-1)(2n-2)(2n-3)...(2n-2k+1)(2n-2k)!}{2n(2n-2)...(2n-2k+2)(2n-2k)!} \\ &= \frac{(2n)!}{2^k n(n-1)...(n-k+1)(2n-2k)!} \\ &= \frac{(2n)!(n-k)!}{2^k n!(2n-2k)!}\end{aligned}$$

Therefore

$$\begin{aligned} y(x) &= a_1 \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{n!}{(n-2k)!} \frac{1}{2^k k!} \frac{2^k n!(2n-2k)!}{(2n)!(n-k)!} x^{n-2k} \\ &= a_1 \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{(n!)^2(2n-2k)!}{k!(n-2k)!(2n)!(n-k)!} x^{n-2k}. \end{aligned}$$

Choosing a_1 to be

$$a_1 = \frac{(2n)!}{2^n (n!)^2}.$$

We obtain the final result for the solution of the Legendre equation (2.1)

$$P_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}, \quad -1 \leq x \leq 1.$$

2.2 Properties Of Legendre Polynomials

2.2.1 Rodrigue Formula

Proposition 2.1. *[2] The Rodrigue formula is given by*

$$p_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n. \tag{2.12}$$

Proof. If $k \leq n$, we have

$$\frac{d^k}{dx^k}x^n = \frac{n!}{(n-k)!}x^{n-k}.$$

According to the Newton binomial formula, we have

$$(x^2-1)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!}(-1)^k x^{2n-2k}.$$

Therefore

$$\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n = \frac{1}{2^n n!}\sum_{k=0}^{n} \frac{n!}{(n-k)!k!}(-1)^k \frac{d^n}{dx^n}x^{2n-2k}.$$

But

$$\frac{d^n}{dx^n}x^{2(n-k)} = 0 \quad \text{if} \quad 2(n-k) < n, \quad \text{i.e., if} \quad k > \frac{n}{2}.$$

We can therefore replace $\sum_{k=0}^{n}$ by $\sum_{k=0}^{\left[\frac{n}{2}\right]}$ if $k \leq \frac{n}{2}$, we have

$$\begin{aligned}\frac{d^n}{dx^n}x^{2(n-k)} &= [2(n-k)]\,[2(n-k)-1]\,[2(n-k)-2] \ldots\ldots [2(n-k)-n+1]\,x^{2(n-k)-n}.\\ &= \frac{(2n-2k)!}{(n-2k)!}x^{n-2k}\end{aligned}$$

Then

$$\begin{aligned}\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n &= \sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k \frac{(2n-2k)!}{2^n n!(n-k)!(n-2k)!}x^{n-2k}\\ &= p_n(x)\end{aligned}$$

■ The first Legendre polynomials are

$$\begin{aligned}P_0(x) &= 1,\\ P_1(x) &= x,\\ P_2(x) &= \frac{1}{2}(3x^2-1),\\ P_3(x) &= \frac{1}{2}(5x^3-3x),\\ P_4(x) &= \frac{1}{8}(35x^4-30x^2+3),\\ P_5(x) &= \frac{1}{8}(63x^5-70x^3+15x).\end{aligned}$$

□

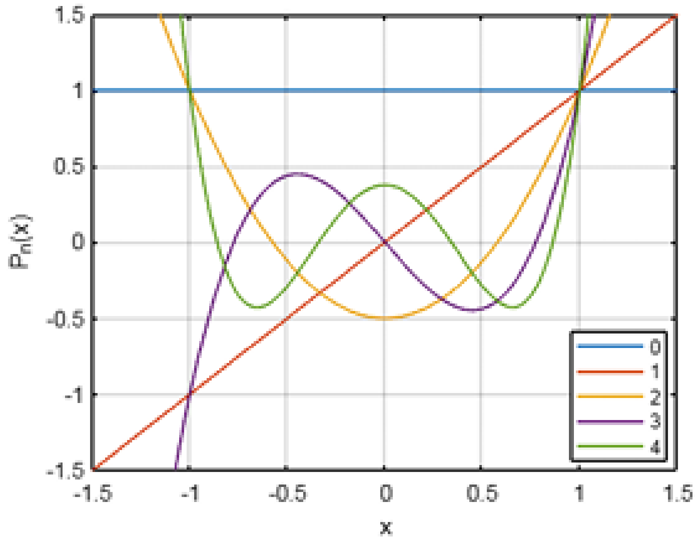

Figure 2.1: Legendre polynomials of degree 0 to 4

Example 2.1. We determine the polynomial $p_2(x)$ using Rodrigued' formula

$$\begin{aligned} p_2(x) &= \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d^2}{dx^2}(x^4-2x^2+1) \\ &= \frac{1}{8}\frac{d}{dx}(4x^3-4x) = \frac{1}{8}(12x^2-4) \\ &= \frac{1}{2}(3x^2-1). \end{aligned}$$

2.2.2 Generating Function

Proposition 2.2. *[2] The generating function of Legendre polynomials is given by*

$$G(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{+\infty} p_n(x)t^n, \quad |x| \leq 1, \quad |t| < 1. \tag{2.13}$$

Proof. We have

$$\begin{aligned} \frac{1}{\sqrt{1-2xt+t^2}} &= (1-2xt+t^2)^{\frac{-1}{2}} \\ &= (1-(2xt-t^2))^{\frac{-1}{2}} \\ &= (1-u)^{\frac{-1}{2}} \end{aligned}$$

with $u = 2xt - t^2 = t(2x-t)$.

The negative Newton's binomial formula is given by

$$(1-u)^{-n} = 1 + nu + \frac{n(n+1)}{2!}u^2 + \frac{n(n+1)(n+2)}{3!}u^3 + \ldots + \frac{n(n+1)(n+2)\ldots(2n-1)}{n!}u^n + \ldots.$$

Then

$$\begin{aligned}(1-u)^{\frac{-1}{2}} &= 1+\frac{1}{2}u+\frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}u^2+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}u^3+...+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{(2n-1)}{2}}{n!}u^n+....\\ &= 1+\frac{1}{2}u+\frac{1.3}{2.4}u^2+\frac{1.3.5}{2.4.6}u^3+...+\frac{1.3.5...(2n-1)}{2.4.6...2n}u^n+....\\ &= \sum_{n=0}^{+\infty}\frac{1.3.5...(2n-1)}{2^n n!}u^n.\end{aligned} \tag{2.14}$$

By multiplying the numerator and the denominator of equation (2.14) by $(2^n n!)$, we obtain

$$\begin{aligned}(1-u)^{\frac{-1}{2}} &= \sum_{n=0}^{+\infty}\frac{1.3.5...(2n-1)}{2^n n!}\times\frac{2^n n!}{2^n n!}\times u^n\\ &= \sum_{n=0}^{+\infty}\frac{1.2.3.4.5...(2n-1).(2n)}{2^n n!\times 2^n n!}u^n\\ &= \sum_{n=0}^{+\infty}\frac{(2n)!}{2^{2n}(n!)^2}u^n.\end{aligned}$$

Therefore

$$\begin{aligned}(1-(2xt-t^2))^{\frac{-1}{2}} &= \sum_{n=0}^{+\infty}\frac{(2n)!}{2^{2n}(n!)^2}(2xt-t^2)^n\\ &= \sum_{n=0}^{+\infty}\frac{(2n)!}{2^{2n}(n!)^2}t^n(2x-t)^n.\end{aligned} \tag{2.15}$$

Using Newton's binomial formula, we obtain

$$(2x-t)^n = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(2x)^{n-k}(-t)^k. \tag{2.16}$$

By substituting (2.16) into equation (2.15), we find

$$\begin{aligned}\frac{1}{\sqrt{1-2xt+t^2}} &= \sum_{n=0}^{+\infty}\frac{(2n)!}{2^{2n}(n!)^2}\sum_{k=0}^{n}(-1)^k\frac{n!}{k!(n-k)!}(2x)^{n-k}(t)^{n+k}\\ &= \sum_{n=0}^{+\infty}\sum_{k=0}^{n}\frac{(-1)^k(2n)!}{2^{n+k}k!(n-k)!n!}x^{n-k}t^{n+k}.\end{aligned} \tag{2.17}$$

We put $m=n+k \Longleftrightarrow n=m-k$ if $k=0$ then $m=n$. The series in the expression (2.17) is given starting from $k=0$ by

$$\sum_{m=0}^{+\infty}\sum_{k=0}^{m}\frac{(-1)^k(2m-2k)}{2^m k!(m-k)!(m-2k)!}x^{m-2k}t^m.$$

- If $m-2k<0$ then $(m-2k)!=\infty$,
- If $m-2k\geq 0$ then $(m-2k)!\neq\infty$.

$$\begin{aligned} m-2k\geq 0 \quad &\Longleftrightarrow \quad k\leq \frac{m}{2} \\ &\Longleftrightarrow \begin{cases} k=0,1,2,...,\frac{m}{2}, & \text{if} \quad m \quad \text{is even,} \\ k=0,1,2,..,\frac{m-1}{2}, & \text{if} \quad m \quad \text{is odd} \end{cases} \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{\sqrt{1-2xt+t^2}} &= \sum_{m=0}^{+\infty}\sum_{k=0}^{\left[\frac{m}{2}\right]} \frac{(-1)^k(2m-2k)!}{2^m k!(m-k)!(m-2k)!}x^{m-2k}t^m \\ &= \sum_{m=0}^{+\infty} p_m(x)t^m \end{aligned}$$

□

Example 2.2. We determine $p_n(0)$ using the generating function. We put $x=0$ in equation (2.13), we obtain

$$\begin{aligned} G(0,t) &= \frac{1}{\sqrt{1-t^2}} \\ &= \sum_{n=0}^{+\infty} p_n(0)t^n \\ &= p_0(0)+p_1(0)t+p_2(0)t^2+p_3(0)t^3+.... \end{aligned}$$

And since we have

$$\begin{aligned} \frac{1}{\sqrt{1-t^2}} &= \sum_{n=0}^{+\infty}(-1)^n\frac{(2n)!}{2^{2n}(n!)^2}t^n \\ &= 1-\frac{1}{2}t^2+\frac{3}{8}t^4+.... \end{aligned}$$

Then

$$p_n(0)=\begin{cases} 0, & \text{if} \quad n \quad \text{is odd.} \\ (-1)^n\frac{(2n)!}{2^{2n}(n!)^2}, & \text{if} \quad n \quad \text{is even.} \end{cases}$$

Example 2.3. We determine $p_n(-1)$ using the generating function. We put $x=-1$ in equation (2.13), we obtain

$$\begin{aligned} G(-1,t) &= \frac{1}{\sqrt{1+2t+t^2}} = \frac{1}{1+t} \\ &= 1-t+t^2-t^3+t^4+.... \end{aligned}$$

Thus, $p_n(-1)=(-1)^n$.

2.2.3 Recurrence Relation

Proposition 2.3. *[3] Legendre polynomials satisfy the following recurrence relation*

$$(n+1)p_{n+1}(x) = (2n+1)xp_n(x) - np_{n-1}(x). \tag{2.18}$$

Proof. We have

$$(1-2xt+t^2)^{\frac{-1}{2}} = \sum_{n=0}^{+\infty} p_n(x)t^n. \tag{2.19}$$

By differentiating both sides of equation (2.19) with respect to t, we obtain

$$\frac{-1}{2}(1-2xt+t^2)^{\frac{-3}{2}}(-2x+2t) = \sum_{n=0}^{+\infty} p_n(x)nt^{n-1}. \tag{2.20}$$

By multiplying both sides of equation (2.20) by $(1-2xt+t^2)$, we obtain

$$\begin{aligned} \frac{-1}{2}(-2x+2t)(1-2xt+t^2)^{\frac{-3}{2}+1} &= (1-2xt+t^2)\sum_{n=0}^{+\infty} p_n(x)nt^{n-1}, \\ (x-t)(1-2xt+t^2)^{\frac{-1}{2}} &= (1-2xt+t^2)\sum_{n=0}^{+\infty} p_n(x)nt^{n-1}. \end{aligned} \tag{2.21}$$

By substituting equation (2.19) into the pervious equation (2.21), we obtain

$$\begin{aligned} (x-t)\sum_{n=0}^{+\infty} p_n(x)t^n &= (1-2xt+t^2)\sum_{n=0}^{+\infty} np_n(x)t^{n-1}, \\ x\sum_{n=0}^{+\infty} p_n(x)t^n - \sum_{n=0}^{+\infty} p_n(x)t^{n+1} &= \sum_{n=0}^{+\infty} np_n(x)t^{n-1} - 2x\sum_{n=0}^{+\infty} np_n(x)t^n + \sum_{n=0}^{+\infty} np_n(x)t^{n+1}. \end{aligned}$$

By the following changes of index

$$\begin{aligned} p_n(x)t^{n+1} &= p_{n-1}(x)t^n, \\ np_n(x)t^{n-1} &= (n+1)p_{n+1}(x)t^n, \\ np_n(x)t^{n+1} &= (n-1)p_{n-1}(x)t^n, \end{aligned}$$

Then, we find

$$\sum_{n=0}^{+\infty} \left[xp_n(x) - p_{n-1}(x)\right]t^n = \sum_{n=0}^{+\infty} \left[(n+1)p_{n+1}(x) - 2xnp_n(x) + (n-1)p_{n-1}(x)\right]t^n.$$

Which implies

$$xp_n(x) - p_{n-1}(x) = (n+1)p_{n+1}(x) - 2xnp_n(x) + (n-1)p_{n-1}(x).$$

Therefore

$$\begin{aligned}(n+1)p_{n+1}(x) - 2xnp_n(x) + (n-1)p_{n-1}(x) - xp_n(x) + p_{n-1}(x) = 0,\\ (n+1)p_{n+1}(x) - (2n+1)xp_n(x) + np_{n-1}(x) = 0.\end{aligned}$$

Hence

$$(n+1)p_{n+1}(x) = (2n+1)xp_n(x) - np_{n-1}(x).$$

□

Example 2.4. Using the recurrence relation to determine $p_2(x)$ and $p_3(x)$ such that $p_0(x) = 1$ and $p_1(x) = x$.
We take $n = 1$ in formula (2.18), we obtain

$$2p_2(x) = 3xp_1(x) - p_0(x) = 3x^2 - 1.$$

Thus, $p_2(x) = \frac{1}{2}(3x^2 - 1)$.
We take $n = 2$, we find

$$3p_3(x) = 5xp_2(x) - 2p_1(x) = \frac{5}{2}x(3x^2 - 1) - 2x = \frac{1}{2}(15x^3 - 9x).$$

Thus, $p_3(x) = \frac{1}{2}(5x^2 - 3x)$.

2.2.4 Orthogonality of Legendre Polynomials

Theorem 2.1. *[2] Legendre polynomials are orthogonal on the interval $[-1, 1]$ with a uniform weight. If $m, n \in \mathbb{N}^*$, then*

$$\langle p_m(x), p_n(x)\rangle = \int_{-1}^{1} p_m(x)p_n(x)\,dx = \begin{cases} 0, & \textit{if} \quad m \neq n, \\ \frac{2}{2n+1}, & \textit{if} \quad m = n. \end{cases}$$

Proof.

1. If $m \neq n$

   Let us consider the Legendre equations

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + m(m+1)y = 0, \quad m \in \mathbb{R}, \tag{2.22}$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] n(n+1)y = 0, \quad n \in \mathbb{R}. \tag{2.23}$$

These are respectively satisfied by $p_m(x)$ and $p_n(x)$

$$(1-x^2)p_m''(x) - 2xp_m'(x) + m(m+1)p_m(x) = 0, \tag{2.24}$$

$$(1-x^2)p_n''(x) - 2xp_n'(x) + n(n+1)p_n(x) = 0. \tag{2.25}$$

By multiplying equation (2.24) by $p_n(x)$ and equation (2.25) by $p_m(x)$ and subtracting the two equation, we obtain

$$\begin{gathered}
(1-x^2)\left[p_n(x)p_m''(x) - p_m(x)p_n''(x)\right] - 2x\left[p_n(x)p_m'(x) - p_m(x)p_n'(x)\right] \\
+ \left[m(m+1) - n(n+1)\right] p_m(x)p_n(x) = 0, \\
(1-x^2)\frac{d}{dx}\left[p_n(x)p_m'(x) - p_n'(x)p_m(x)\right] - 2x\left[p_n(x)p_m'(x) - p_n'(x)p_m(x)\right] \\
+ \left[(m^2-n^2) - (m-n)\right] p_m(x)p_n(x) = 0, \\
\frac{d}{dx}(1-x^2)\left[p_n(x)p_m'(x) - p_n'(x)p_m(x)\right] = (n-m)\left[n+m+1\right] p_m(x)p_n(x).
\end{gathered}$$

By integrating both sides with respect to x between -1 and 1, we obtain

$$\begin{aligned}
\left[(1-x^2)(p_n(x)p_m'(x) - p_n'(x)p_m(x))\right]_{-1}^{1} &= \int_{-1}^{1}(n-m)(m+n+1)p_m(x)p_n(x)\,dx, \\
0 &= (n-m)(m+n+1)\int_{-1}^{1} p_m(x)p_n(x)\,dx.
\end{aligned}$$

Since $m \neq n$, then $\int_{-1}^{1} p_m(x)p_n(x)\,dx = 0$.

2. If $m = n$.
   Using the generating function, we have

$$(1-2xt+t^2)^{\frac{-1}{2}} = \sum_{n=0}^{+\infty} p_n(x)t^n. \tag{2.26}$$

$$(1-2xt+t^2)^{\frac{-1}{2}} = \sum_{m=0}^{+\infty} p_m(x)t^m. \tag{2.27}$$

By multiplying equation (2.26) by equation (2.27), we obtain

$$\begin{aligned} \frac{1}{1-2xt+t^2} &= \sum_{n=0}^{+\infty} p_n(x)t^n \sum_{m=0}^{+\infty} p_m(x)t^m \\ &= \sum_{n,m=0}^{+\infty} p_n(x)p_m(x)t^{n+m}. \end{aligned} \tag{2.28}$$

By integrating both sides of equation (2.28) with respect to x from -1 to 1, we obtain

$$\sum_{n,m=0}^{+\infty} \int_{-1}^{1} p_n(x)p_m(x)t^{n+m}\,dx = \int_{-1}^{1} \frac{1}{1-2xt+t^2}\,dx$$

We have $n = m$, then

$$\begin{aligned} \sum_{n=0}^{+\infty} \int_{-1}^{1} t^{2n}\left[p_n(x)\right]^2\,dx &= \int_{-1}^{1} \frac{1}{1-2xt+t^2}\,dx \\ &= \left[\frac{1}{-2t}\ln(1+t^2-2xt)\right]_{-1}^{1} \\ &= \frac{1}{-2t}\ln(1+t^2-2t) - \frac{1}{-2t}\ln(1+t^2+2t) \\ &= -\frac{1}{2t}\left[\ln(1-t)^2 - \ln(1+t)^2\right] \\ &= -\frac{1}{2t}2\left[\ln(1-t) - \ln(1+t)\right] \\ &= \frac{1}{t}\left[\ln(1+t) - \ln(1-t)\right]. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{n=0}^{+\infty} \int_{-1}^{1} t^{2n}\left[p_n(x)\right]^2\,dx &= \frac{1}{t}\left[\left(t - \frac{t^2}{2} + \frac{t^3}{3} + \ldots\right) - \left(-t - \frac{t^2}{2} - \frac{t^3}{3} + \ldots\right)\right] \\ &= \frac{2}{t}\left[t + \frac{t^3}{3} + \frac{t^5}{5} + \ldots\right] \\ &= 2\left[1 + \frac{t^2}{3} + \frac{t^4}{5} + \ldots + \frac{t^{2n}}{2n+1} + \ldots\right] \\ &= 2\sum_{n=0}^{+\infty} \frac{t^{2n}}{2n+1}. \end{aligned}$$

Therefore, $\|p_n(x)\|_2^2 = \frac{2}{2n+1}$.

□

■

CHAPTER 3

LEGENDRE WAVELETS

In this chapter, we study the Legendre wavelets, as it has become known.

In recent times, Legendre wavelets have been applied to different problems such as vibrational problems, diffusion computation, mathematical physics, and integrals. The main characteristic of Legendre wavelets in variational problems is that they reduce these to a system of algebraic equations.

3.1 Wavelets

Definition 3.1. Wavelets are a family of functions formed from the dilation and translation of the same function $\psi(t) \in L^2(\mathbb{R})$, called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we obtain the following family of continuous wavelets (see [10] and [11]).

$$\psi_{a,b}(t) = |a|^{\frac{-1}{2}}\,\psi\left(\frac{t-b}{a}\right); \quad a, b \in \mathbb{R}; \quad a \neq 0.$$

For a restriction of the parameters a and b to discrete values such as $a = a_1^{-k}, b = nb_1a_1^{-k}$, where $a_1 > 1$ and $b_1 > 0$, we obtain the following family of discrete wavelets :

$$\psi_{k,n}(t) = |a_1|^{\frac{k}{2}}\,\psi(a_1^k t - nb_1).$$

These basis functions are called wavelets. This family $\psi_{k,n}(t) \triangleq \psi_{kn}(t)$ constitutes a wevelet basis for $L^2(\mathbb{R})$.

Remark 3.1. For $a_1 = 2, b_1 = 1, n, k$ positive integers, the family of wavelets becomes orthonormal(i.e. $\langle\psi_{k,n}(t), \psi_{l,m}(t)\rangle = \delta_{kl}\delta_{nm}$).

3.2 Shifted Legendre Polynomials

For a partical use of Legendre polynomials on the interval of interest $[0, 1]$, it is necessary to make a change of variable of the domain of definition: $x = 2t - 1, \quad 0 \leq t \leq 1$. Thus, we obtain new Legendre polynomials.

Definition 3.2. The so-called shifted (or modified) Legendre polynomials, denoted $L_n(t)$ are polynomials defined on the interval $[0, 1]$ as follows:

$$L_n(t) = P_n(2t - 1), \quad n \in \mathbb{N}^*$$

Such that

$$\begin{cases} L_0(t) = 1 \\ L_1(t) = 2t - 1 \\ L_{n+1}(t) = \frac{(2n+1)}{(n+1)}(2t - 1)L_n(t) - \frac{n}{(n+1)}L_{n-1}(t), n \in \mathbb{N}^* \end{cases} \tag{3.1}$$

Properties 3.1.

1. The shifted Legendre polynomials are orthogonal polynomials and their orthogonality condition is:

$$\int_0^1 L_m(t)L_n(t)\, dt = \frac{1}{2m + 1}\delta_{mn} \tag{3.2}$$

Where $m, n \in \mathbb{N}^*$ and δ_{mn} denotes the Kronecker symbol.

2. The analytical form of the polynomial $L_n(t)$ of degree n is given by:

$$L_n(t) = \sum_{i=0}^{m} b_{ni}t^i \tag{3.3}$$

With

$$b_{ni} = (-1)^{n+i}\frac{(n + i)!}{(n - i)!(i!)^2}; \quad i = 1, 2, 3, ..., n$$

Examples 3.1. The first six shifted Legendre polynomials defined on the interval $[0, 1]$

$L_0(t) = 1$

$L_1(t) = 2t - 1$

$L_2(t) = 6t^2 - 6t + 1$

$L_3(t) = 20t^3 - 30t^2 + 12t - 1$

$L_4(t) = 70t^4 - 140t^3 + 90t^2 - 20t - 1$

$L_5(t) = 252t^5 - 360t^4 + 560t^3 - 210t^2 + 30t - 1$

3.3 One- Dimensional Legendre Wavelets

Definition 3.3. The Legendre wavelets $\psi(k,n,m,x) \triangleq \psi_{nm}(x)$ are functions constructed from Legendre polynomials and form a basis for $L^2([0,1])$, and they have four arguments: n argument, $k \in \mathbb{N}^*, m$ is the order of the Legendre polynomials and $x \in [0,1]$. They are defined on the interval $[0,1]$:

$$\psi_{nm}(x) = \begin{cases} \sqrt{m+\frac{1}{2}}2^{\frac{k}{2}}P_m(2^kx-2n+1), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}} \\ 0, & \text{Otherwise.} \end{cases} \tag{3.4}$$

Or [11]

$$\psi_{nm}(x) = \begin{cases} \sqrt{m+\frac{1}{2}}2^{\frac{k}{2}}L_m(2^kx-n+1), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}} \\ 0, & \text{Otherwise.} \end{cases}$$

Where $m = 0,1,...,M-1, M \in \mathbb{N}^*, n = 1,2,...,2^{k-1}$.

Remark 3.2.

1. In the writing of $\psi_{nm}(x)$, the term (x) will be omitted for convenience. Thus we set $\psi_{nm}(x) = \psi_{nm}$.
2. The coefficient $\sqrt{m+\frac{1}{2}}$ in (3.4) is to ensure orthonormality, $a = 2^{\frac{k}{2}}$ is the dilatation parameter and $b = n2^{\frac{k}{2}}$ is the translation parameter.

Definition 3.4. The vector of Legendre wavelets is called the column vector $\Psi(x)$ of dimension $2^{k-1}M$ given by:

$$\Psi(x) = \left[\psi_{10},...,\psi_{1(M-1)},\psi_{20},...,\psi_{2(M-1)},...,\psi_{2^{K-1}0},...,\psi_{2^{K-1}(M-1)}\right]^T \tag{3.5}$$

Remark 3.3. The vector of Legendre Wavelets has the most important role in the approximation of functions of $L^2([0,1])$.

Properties 3.2. For $m = 0,1,...,M-1, M \in \mathbb{N}^*$, $n = 1,2,...,2^{k-1}$and $k \in \mathbb{N}^*$ we have the following properties:

1. $\psi_{nm}(x)\psi_{n'm'} = 0 \quad \text{if} \quad n \neq n'$.
2. $\psi_{n0}(x)\psi_{nm}(x) = \sqrt{2}\psi_{nm}(x)$.
3. $\psi_{n1}(x)\psi_{n1}(x) = \frac{4}{\sqrt{10}}\psi_{n2}(x) + \sqrt{2}\psi_{n0}(x)$.

Examples 3.2. Expression of the Legendre wavelets of $L^2([0,1])$. According to (3.4) and taking $k=2$ and $M=3$ we obtain

$$\left.\begin{array}{l} \psi_{10}(x) = \sqrt{2} \\ \psi_{11}(x) = \sqrt{6}(4x-1) \\ \psi_{12}(x) = \sqrt{10}\left[\frac{3}{2}(4x-1)^2 - \frac{1}{2}\right] \end{array}\right\} \quad 0 \leq x < \frac{1}{2},$$

$$\left.\begin{array}{l}\psi_{20}(x)=\sqrt{2}\\ \psi_{21}(x)=\sqrt{6}(4x-3)\\ \psi_{22}(x)=\sqrt{10}\left[\frac{3}{2}(4x-3)^2-\frac{1}{2}\right]\end{array}\right\}\quad \frac{1}{2}\leq x<1,$$

Where $\{\psi_{10}\,\psi_{11}\,\psi_{12}\,\psi_{20}\,\psi_{21}\,\psi_{22}\}$ is a wavelet basis for $L^2([0,1])$, therefore the vector of Legendre wavelets is written in the following form

$$\begin{aligned}\Psi(x)&=[\psi_{10},\psi_{11},\psi_{12},\psi_{20},\psi_{21},\psi_{22}]^T\\ &=\left[\sqrt{2},\sqrt{6}(4x-1),\sqrt{10}\left(\frac{3}{2}(4x-1)^2-\frac{1}{2}\right),\sqrt{2},\sqrt{6}(4x-3),\sqrt{10}\left(\frac{3}{2}(4x-3)^2-\frac{1}{2}\right)\right]^T\end{aligned} \tag{3.6}$$

3.4 The Product of Two vectors of Legendre Wavelets

The Product Matrix

According to [12], the product of two vectors of Legendre Wavelets $\Psi(x),\Psi^T(x)$ can be developed into a simple expression as follows:
$\Psi(x)\Psi^T(x)=$

$$\begin{pmatrix}\psi_{10}\psi_{10} & \cdots & \psi_{1(M-1)}\psi_{10} & \cdots & \psi_{10}\psi_{2^{k-1}0} & \cdots & \psi_{10}\psi_{2^{k-1}(M-1)}\\ \vdots & \ddots & \vdots & & \vdots & & \vdots\\ \psi_{1(M-1)}\psi_{10} & \cdots & \psi_{1(M-1)}\psi_{1(M-1)} & \cdots & \psi_{1(M-1)}\psi_{2^{k-1}0} & \cdots & \psi_{1(M-1)}\psi_{2^{k-1}(M-1)}\\ \vdots & & \vdots & \ddots & \vdots & & \vdots\\ \psi_{2^{k-1}0}\psi_{10} & \cdots & \psi_{2^{k-1}0}\psi_{1(M-1)} & \cdots & \psi_{2^{k-1}0}\psi_{2^{k-1}0} & \cdots & \psi_{2^{k-1}0}\psi_{2^{k-1}(M-1)}\\ \vdots & & \vdots & & \vdots & \ddots & \vdots\\ \psi_{2^{k-1}(M-1)}\psi_{10} & \cdots & \psi_{2^{k-1}(M-1)}\psi_{1(M-1)} & \cdots & \psi_{2^{k-1}(M-1)}\psi_{2^{k-1}0} & \cdots & \psi_{2^{k-1}(M-1)}\psi_{2^{k-1}(M-1)}\end{pmatrix}$$

By using the following orthonormality property

$$\psi_{nm}(x)\psi_{n'm'}=0 \quad \text{if} \quad n\neq n'$$

We obtain:

$$\Psi(x)\Psi^T(x)=A \tag{3.7}$$

Where A is a matrix of dimension $2^{k-1}M$ called the product matrix, defined by [12]

$$A = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{2^{k-1}} \end{pmatrix}$$

Where $A_i, i = 1, 2, ..., 2^{k-1}$ are matrices of dimension M given by:

$$A_i = \begin{pmatrix} \psi_{i0}\psi_{i0} & \cdots & \psi_{i(M-1)}\psi_{i0} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \psi_{i0}\psi_{i(M-1)} & \cdots & \psi_{i(M-1)}\psi_{i(M-1)} \end{pmatrix}$$

And O is the zero matrix of dimension M .

Example 3.1. For $M = 3, k = 2$ and by using the following properties

$$\begin{cases} \psi_{n0}(x)\psi_{nm}(x) = \sqrt{2}\psi_{nm}(x) \\ \psi_{n1}(x)\psi_{n1}(x) = \frac{4}{\sqrt{10}}\psi_{n2}(x) + \sqrt{2}\psi_{n0}(x). \end{cases}$$

We obtain

$$\Psi(x)\Psi^T(x) = A = \begin{pmatrix} A_1 & O_{3\times 3} \\ O_{3\times 3} & A_2 \end{pmatrix} \tag{3.8}$$

Where A_1, A_1 and $O_{3\times 3}$ are 3×3 matrices given by:

$$A_1 = \begin{pmatrix} \sqrt{2}\psi_{10} & \sqrt{2}\psi_{11} & \sqrt{2}\psi_{12} \\ \sqrt{2}\psi_{11} & \frac{4}{\sqrt{10}}\psi_{12} + \sqrt{2}\psi_{10} & \frac{4}{\sqrt{10}}\psi_{11} \\ \sqrt{2}\psi_{12} & \frac{4}{\sqrt{10}}\psi_{11} & \frac{20}{7\sqrt{10}}\psi_{12} + \sqrt{2}\psi_{10} \end{pmatrix}$$

$$A_2 = \begin{pmatrix} \sqrt{2}\psi_{20} & \sqrt{2}\psi_{21} & \sqrt{2}\psi_{22} \\ \sqrt{2}\psi_{21} & \frac{4}{\sqrt{10}}\psi_{22} + \sqrt{2}\psi_{20} & \frac{4}{\sqrt{10}}\psi_{21} \\ \sqrt{2}\psi_{22} & \frac{4}{\sqrt{10}}\psi_{21} & \frac{20}{7\sqrt{10}}\psi_{22} + \sqrt{2}\psi_{20} \end{pmatrix}$$

And

$$O_{3\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Operational Matrix Of The Product

For any given vector e of dimension $2^{k-1}M$, the product of two vector functions of Legendre Wavelets has the following property:

$$e^T\Psi\Psi^T = \Psi^T\widetilde{E}, \tag{3.9}$$

Where $\widetilde{E}$ is a matrix $(2^{k-1}M)\times(2^{k-1}M)$ depending on the vector e, which is called the operational matrix of the product of vector functions of Legendre Wavelets. (see [14]).

Example 3.2. For $M=3, k=2$, we can write for a vector e of dimension 6,

$$e^T = (e_1, e_2, e_3, e_4, e_5, e_6)$$

We have

$$e^T\Psi\Psi^T = \Psi^T\widetilde{E},$$

Where $\widetilde{E}$ the operational matrix of the product of Legendre Wavelet vector functions is of dimension 6 which is written in the following form:

$$\widetilde{E} = \begin{pmatrix} \sqrt{2}e_1 & \sqrt{2}e_2 & \sqrt{2}e_3 & 0 & 0 & 0 \\ \sqrt{2}e_2 & \frac{4}{\sqrt{10}}e_3+\sqrt{2}e_1 & \frac{4}{\sqrt{10}}e_2 & 0 & 0 & 0 \\ \sqrt{2}e_3 & \frac{4}{\sqrt{10}}e_2 & \frac{20}{7\sqrt{10}}e_3+\sqrt{2}e_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}e_4 & \sqrt{2}e_5 & \sqrt{2}e_6 \\ 0 & 0 & 0 & \sqrt{2}e_5 & \frac{4}{\sqrt{10}}e_6+\sqrt{2}e_4 & \frac{4}{\sqrt{10}}e_5 \\ 0 & 0 & 0 & \sqrt{2}e_6 & \frac{4}{\sqrt{10}}e_5 & \frac{20}{7\sqrt{10}}e_6+\sqrt{2}e_4 \end{pmatrix} \tag{3.10}$$

General Conclusions and perspectives

In this dissertation, we studied the simplest example of orthogonal polynomials, which are the Legendre polynomials. These polynomials are the solutions of second-order linear differential equations.

In the last chapter, we studied the Legendre Wavelets method for solving an ordinary differential equation and a partial differential equation.

The results obtained show that the Legendre wavelet method is an effective and powerful tool for approximating solutions of differential equations. This approach provides good accuracy and simplifies the computational process.

As prespectives, this work can be extended by applying Legendre wavelets to more complex problems, such as nonlinear partial differential equations or higher-dimensional systems.

Futher research may also focus on improving numerical efficiency and comparing this method with other numerical techniques.

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ملخص:

تتناول هذه المذكرة دراسة حول كثيرات الحدود من نوع ليجندر و أهم خصائصها الأساسية (صيغة رودريغز، دالة التوليد، العلاقة التراجعية، كثيرات الحدود ليجندر المعدلة) و قد قمنا بتطبيق كثيرات الحدود ليجندر من أجل حل المعادلات التفاضلية باستخدام طريقة التموجات.

الكلمات المفتاحية: كثيرات الحدود المتعامدة – كثيرات الحدود ليجندر – المعادلات التفاضلية – طريقة التموجات

Dans ce mémoire, on va étude des polynômes de Legendre et de leurs principales propriétés (formule de Rodrigues, fonction génératrice, relation de récurrence, équation des polynômes de Legendre). Nous avons également appliqué les polynômes de Legendre afin de résoudre des équations différentielles en utilisant la méthode des ondelettes.

Mots-Clés: Polynômes Orthogonaux, Polynômes de Legendre , Equations Différentielles , Méthode Des Ondelettes.

In this dissertation, we studied the Legendre polynomials and their main properties (Rodrigues'formula, generating function, recurrence relation, Legendre polynomials equation). We have also applied Legendre polynomials in order to solve differential equations using the wavelets method.

Keywords: Orthogonal Polynomials , Legendre Polynomials, Differential Equations, Wavelet Method.