

REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE
MINISTERE DE L'ENSEIGNEMENT SUPERIEUR ET DE LA RECHERCHE SCIENTIFIQUE
UNIVERSITE MOHAMED BOUDIAF - M'SILA

FACULTE DE TECHNOLOGIE
DEPARTEMENT ELECTRIQUE

FILIERE : AUTOMATIQUE
OPTION :ROBOTIQUE

N°:.....



Mémoire présenté pour l'obtention
Du diplôme de Master Académique

Par:

- BAKRI Alaeddine

Intitulé

**Improved Reaching Time for Higher-Order
Sliding Mode Controllers.**

Soutenu devant le jury composé de:

Dr.KHETTAB Khatir	Prof.	UniversitéM'sila	Président
Dr.BOUKHALFAAbdelouaheb	Prof.	UniversitéM'sila	Rapporteur
Dr. ROUBECHE Toufik	MCA	UniversitéM'sila	examineur

Année universitaire : 2024/2025

Remerciement

First and foremost, my thanks go to Allah Almighty, who helped me complete this work and gave me the courage, patience, and health during all these long years of study to enable me to reach this level.

We also thank the members of the jury for agreeing to judge our work.

We would like to thank Dr. Khettab Khatir and Dr. Boukhalifa Abdelouaheb for suggesting the topic of this dissertation and for their guidance throughout.

Finally, we express our sincere thanks to our dear parents and friends who, from near and far, supported and encouraged us throughout this dissertation.

DEDICACE

I dedicate this modest work:

To the dearest person in my life, my mother.

To the one who made me a man, my father.

To my dear brothers and sisters.

To Mr. Dr. Boukhalfa Abdelouaheb

To all my friends from my second-year Master's program...

Everyone who holds a place in my heart.

*To all the members of my family and everyone named Benzahia
and Belouadeh.*

I dedicate this work to all those who contributed to my success.

List of Figures

Figure1: Sliding Variable and Sliding Surface.....	6
Figure2: Simulation of Sliding Mode Control.....	7
Figure 3: System Response with Sliding Mode Control.....	9
Figure 4: Phase trajectory showing convergence to the sliding surface under external disturbance.....	12
Figure5: Kinematic car model.....	33
Figure6: Car trajectory.....	36
Figure7: Sliding variable.....	37
Figure8: Sliding derivatives versus time [s].....	37
Figure9: Comparison	37

Sommaire :

Dédicace

Remerciements

List of Figures

Introduction Générale.....	1
ChapitreI : Introduction An Overview of Classical Sliding Mode Control	
I.1Theoretical Background.....	3
I.2 Introduction:.....	4
1.3 Sliding Variable and Sliding Surface.....	5
I.4Dynamics of Sliding Mode Control.....	6
I.4.1 Linear Systems (First-Order Linear System)....	6
I.4.2 Figure: Simulation of Sliding Mode Control.....	7
I.5 Nonlinear Systems in Sliding Mode Control.....	8
I.6 First-Order Sliding Mode Control Input Design.....	9
I.7 Chattering Phenomenon.....	12
I.8 Summary.....	13

I.9 Conclusion.....	14
Chapitre II : Sliding Mode and Higher Order Sliding Mode Control	
II.1. Introduction.....	15
II.2. Sliding Mode Control.....	16
II.2.1 Design of Sliding Manifold and Controller.....	17
II.2.2 Controller Design.....	18
II.3 Higher Order Sliding Mode Control (HOSMC)	19
II.4 Adaptive Sliding Mode Controllers)	24
II.4.1 Adaptive First Order Sliding Mode Controller.....	24
II.4.2 Adaptive Supertwisting Controller.....	25
II.4.3 Adaptive Twisting Controller	25
II.5 Summary:	26
II.6 Conclusion:	26
Chapitre III: Higher Order Sliding Mode Controllers With Optimal Reaching	
III.1 Introduction:	27
III.2 RELIMINARIES P:	27
III.3 Second Order Sliding Mode Algorithms:.....	30
III.4 third-order sliding mode algorithm.....	31
III.5 Generic Order Sliding Mode Algorithms.....	31
III.6 System Model:.....	32
III.6.1 the Sliding Variable Definition:.....	34
III.7 Relative Degree Analysis.....	34
III.8. Simulation results:.....	36
III.8.1 Results discussion:.....	38
III.9 Conclusion.....	38
Conclusion générale.....	39
<i>Bibliographic References</i>	40

Introduction générale

Introduction générale

Sliding Mode Control (SMC) has long been recognized as a powerful methodology for the robust control of nonlinear and uncertain systems. It operates by driving the system states toward a predefined sliding manifold in the state space and maintaining them on that manifold thereafter. This is achieved through discontinuous control laws that exploit the system's structure to reject disturbances and model uncertainties. The resulting closed-loop system behavior known as the sliding mode ensures insensitivity to certain classes of perturbations and model inaccuracies, thus offering attractive robustness properties [1][2].

However, classical SMC is restricted by two fundamental limitations. Firstly, it is applicable only to systems with relative degree one with respect to the sliding variable. Secondly, it tends to induce chattering a phenomenon characterized by high-frequency oscillations in the control signal. Chattering can excite unmodeled dynamics, degrade performance, and lead to wear and damage in actuators, especially in mechanical and electromechanical systems [3].

To address these issues, Higher Order Sliding Mode (HOSM) control was developed as a generalization of standard SMC. HOSM techniques extend the principle of sliding motion to the derivatives of the sliding variable, enabling the control of systems with higher relative degrees while significantly reducing or eliminating chattering [4], [5]. This is achieved by designing controllers that ensure finite-time convergence of both the sliding variable and a number of its time derivatives. Such controllers exhibit enhanced robustness and smoother control signals.

Among HOSM controllers, the second-order case has been extensively studied. Algorithms such as the super-twisting algorithm and the sub-optimal sliding mode controllers have proven effective in managing uncertain systems with second-order dynamics [6][7]. Furthermore, these approaches have been extended to multi-input-multi-output (MIMO) systems, enabling broader applicability [8].

Despite the successes of second-order HOSM controllers, the design of controllers for third and higher order systems remains a relatively underexplored area, primarily due to increased mathematical complexity and difficulties in ensuring finite-time stability. Recent studies have attempted to generalize HOSM algorithms by introducing new formulations based on quasi-continuous control [11], integral sliding modes [12], and optimal control theory [9][10]. A central theme in these developments is the concept of the auxiliary system, a mathematical construct representing a perturbed chain of integrators derived from the system output and its derivatives.

The HOSM problem is then reformulated as the stabilization of this auxiliary system in finite time [13].

In this context, the present work introduces the **Robust Fuller's Problem**, a novel generalization of the classical Fuller's time-optimal control problem. This problem considers a chain of integrators with bounded control and additive/multiplicative uncertainties, aiming to find feedback laws that guarantee optimal finite-time convergence to the sliding manifold. By solving this problem, the authors derive a family of HOSM controllers of arbitrary order, each ensuring optimal reaching times and robust performance. An important practical advantage of this approach is its capacity to assist in the systematic tuning of control parameters, a task often performed heuristically in traditional HOSM schemes.

Theoretical results presented in this study establish the convergence and stability properties of the proposed control laws, supported by homogeneity analysis and Lyapunov function construction. Furthermore, extensive simulation results demonstrate the superior performance of the derived controllers in benchmark nonlinear systems, including reduced chattering, improved convergence speed, and robustness against disturbances.

Chapitre I :

***Introduction An Overview of
Classical Sliding Mode Control***

I.1 Theoretical Background:

Sliding Mode Control (SMC) has long demonstrated its value in control engineering due to several notable advantages. These include its relatively simple design, the ability to decouple and independently control system motions (provided sliding conditions are maintained), invariance to both system dynamic characteristics and external disturbances, and its adaptability to various control tasks such as regulation, trajectory tracking [14], model following [15], and state observation [16].

Although extensively studied in numerous research papers [17, 18], survey articles [19], and books [20, 21, 22], SMC continues to be an active area of both theoretical investigation and practical application. The primary objective of this chapter is to introduce foundational concepts of sliding mode control, including attractivity, equivalent control, and sliding mode dynamics, supported by illustrative examples and applications [23].

At its core, sliding mode control is a direct consequence of employing discontinuous control strategies. During the early 1960s, discontinuous control particularly in the form of bang-bang control was a prominent topic among mechanical and control engineers. Notable early contributions include the work of Hamel [24] in France and that of Cypkin [25] and Emelyanov [26] in the former USSR, who rigorously addressed the oscillatory behavior inherent in bang-bang control systems. Initially, these phenomena were viewed as undesirable and efforts were focused on analyzing and mitigating them. However, this perspective soon evolved toward synthesis approaches, particularly in the areas of time-optimal control, system linearization, and robustness enhancement.

In time-optimal control, the emergence of discontinuities in the control signal at specific time instants was the outcome of solving variational problems. In contrast, the second approach of greater relevance to SMC relied on deliberately introducing discontinuities into the control law. The frequency of switching in these control systems varied depending on the specific control objective, such as system linearization. A notable early example includes the use of oscillating aerodynamic spoilers in the 1960s to control aircraft lift. These developments also led to the design of nonlinear corrective networks capable of overcoming classical limitations imposed by Bode's gain-phase trade-off, thereby enabling the realization of sliding modes.

Interestingly, although the motivations and objectives of these early approaches differed, they ultimately converged in their methodologies and theoretical underpinnings. The formal concept of sliding mode control emerged during the early 1960s in the context of designing robust control

laws well before the term robustness became widespread. In military aviation, control engineers sought control strategies that were inherently insensitive to system parameter variations. Classical linear networks of the time failed to provide sufficient compensation, particularly because high gain designs needed for insensitivity were constrained by Bode's gain-phase relationship, which couples and opposes amplitude and phase performance.

In 1962, building on a proposal by B. Hamel, research efforts began focusing on nonlinear compensators designed to surpass these constraints. These compensators typically acted on the error signal x of the feedback system and were defined by expressions such as:

$$u = |F1(x, \dot{x}, \dots)| \cdot \text{sgn}(F2(x, \dot{x}, \dots)) \quad (1.1)$$

where $| \cdot |$ denotes the absolute value, and $F1, F2$ are appropriately chosen linear filters. As a result, the control output was discontinuous, but modulated by a function of the system state and its derivatives. A simplified version of this can be

expressed as:

$$u = -|x| \cdot \text{sgn}(x + k\dot{x}) \quad (1.2)$$

I.2 Introduction:

Sliding Mode Control (SMC) is recognized as an effective approach for addressing complex control challenges in nonlinear and uncertain systems. The primary advantages of SMC include its robustness to parameter variations and external disturbances, finite-time convergence properties, and relatively straightforward implementation.

The core principle of sliding mode control involves directing the system trajectories toward a predefined region in the state space, known as the sliding surface, within a finite period. This surface is characterized by its attractivity; once the system state reaches the sliding surface, it becomes invariant to disturbances and uncertainties. Consequently, the behavior of the closed-loop system is governed by the properties of the sliding surface itself.

The design process of a sliding mode controller is typically carried out in two distinct stages. The first stage involves the formulation of the sliding surface based on the control objectives. This surface is generally expressed as a differential equation related to the system's output. The second stage entails the development of a discontinuous control law, which ensures that the system trajectories not only reach the sliding surface in finite time but also remain confined to it despite the presence of uncertainties and disturbances.

I.3 Sliding Variable and Sliding Surface

In sliding mode control (SMC), the sliding variable and sliding surface are key components that define the control strategy.

The sliding variable $\sigma(x, t)$ is a function of the system states and is used to define the sliding surface:

$$S = \{x \mid \sigma(x, t) = 0\} \tag{1.3}$$

When the system state reaches this surface, the behavior of the system is governed entirely by the surface's definition,

allowing for robustness against disturbances and model uncertainties.

I.3.1 Example:

Consider a first-order system defined by:

$$\dot{x} = u + d(t) \tag{1.3}$$

where x is the system state, u is the control input, and $d(t)$ is a bounded external disturbance.

Define the tracking error as:

$$e = x - x_{ref} \tag{1.4}$$

where x_{ref} is the desired reference signal. The sliding variable can be defined as:

$$\sigma = e = x - x_{ref} \tag{1.5}$$

The control objective is to drive σ to zero in finite time, ensuring that the state x tracks the reference x_{ref} . A suitable sliding mode control law is:

$$u = -K \cdot \text{sign}(\sigma) \tag{1.6}$$

where K is a positive gain chosen large enough to overcome the effect of disturbance $d(t)$.

Under this control law, the sliding variable σ will converge to zero in finite time, leading to: $x \rightarrow x_{ref}$

This example demonstrates the robustness and simplicity of sliding mode control in handling uncertain first-order systems.

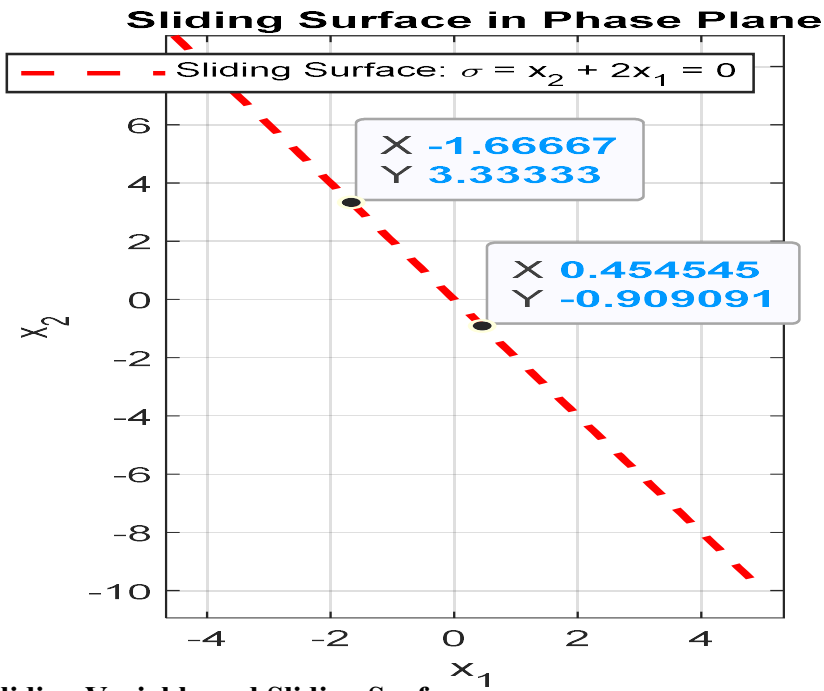


Figure1:Sliding Variable and Sliding Surface

I.4Dynamics of Sliding Mode Control:

1.1 I.4.1 Linear Systems (First-Order Linear System):

This section analyzes the behavior of linear systems under the application of Sliding Mode Control (SMC) techniques. This type of control is known for its robustness and effectiveness in maintaining system stability, especially in the presence of model uncertainties or external disturbances.

Consider a first-order linear system described by the differential equation:

$$\dot{x} = ax + bu \tag{1.7}$$

where:

- x is the state variable,
- u is the control input,
- a and b are system parameters.

The sliding surface is defined as:

$$s(x) = cx$$

To ensure sliding mode occurs, the control input u must be designed to satisfy the reaching condition

$$\dot{s} = c\dot{x} = c(ax + bu) < 0 \text{ when } s \neq 0 \tag{1.8}$$

A suitable control law is:

$$u = -k * \text{sign}(s) \tag{1.9}$$

where $k > 0$ is the control gain.

Substituting into the expression for \dot{s} :

$$\dot{s} = ca * x - cbk * \text{sign}(s) \tag{1.10}$$

To fulfill the condition $s\dot{s} < 0$, the product cbk must be large enough to dominate the term $ca * x$.

Once the system reaches the sliding surface (i.e., when $s(x) = 0$), the dynamics during the sliding mode can be derived using the concept of equivalent control, which represents the average of the discontinuous control input. Assuming $\dot{s} = 0$, we obtain:

$$c(ax + bu_{eq}) = 0 \rightarrow u_{eq} = -ca/cb * x$$

Substituting u_{eq} back into the system equation yields:

$$\dot{x} = a * x + b * u_{eq} = x(a - bca/cb) = x(a - a) = 0 \tag{1.11}$$

I.4.2 Figure: Simulation of Sliding Mode Control:

The figure below illustrates the behavior of a first-order linear system under sliding mode control. The system state $x(t)$, sliding function $s(t)$, and control signal $u(t)$ are plotted over time. It is observed that the state converges to the sliding surface and remains there, demonstrating ideal sliding behavior.

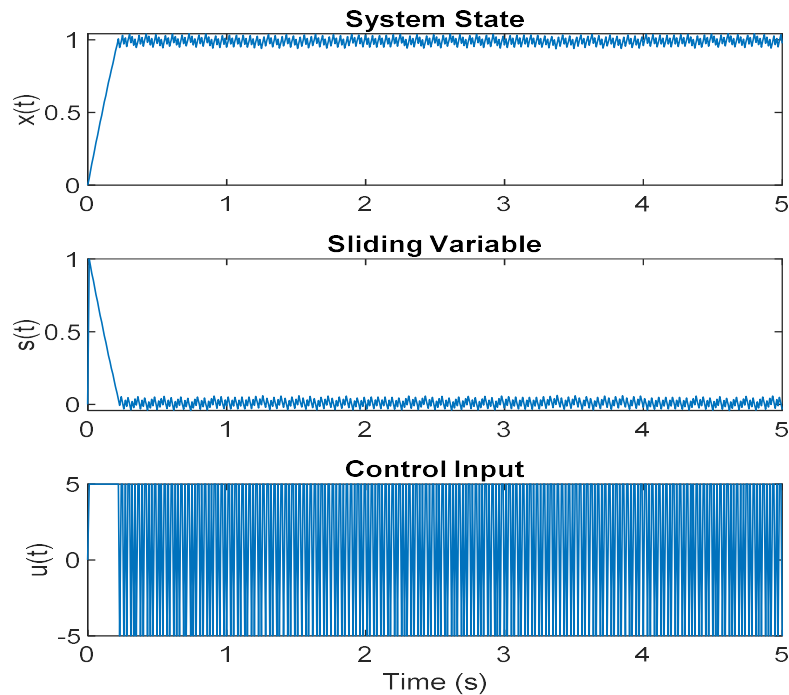


Figure2: Simulation of Sliding Mode Control

Hence, the system remains stationary on the sliding surface, achieving ideal stability.

This demonstrates the effectiveness of sliding mode control in mitigating the effects of model uncertainties and maintaining desired system performance even under varying conditions[27].

I.5 Nonlinear Systems in Sliding Mode Control:

Sliding Mode Control (SMC) is widely recognized for its robustness and effectiveness in handling nonlinear systems. In the case of nonlinear systems, especially those affine in control, the concept of sliding motion can still be applied, albeit with greater complexity compared to linear systems.

Consider a general nonlinear system of the form:

$$dx/dt = f(x) + g(x)u(t) \tag{1.12}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input, and $f(x)$, $g(x)$ are smooth nonlinear functions.

A sliding surface is defined as:

$$S = \{x \in \mathbb{R}^n : s(x) = 0\}, \tag{1.13}$$

where $s(x) \in \mathbb{R}^m$ is a differentiable function. The goal is to design a control law $u(t)$ such that the system's trajectories reach and maintain the condition $s(x) = 0$. When in sliding mode, the equivalent control ensures:

$$ds/dt = (\partial s/\partial x)(f(x) + g(x)u_{eq}) = 0. \tag{1.14}$$

I.5.1 Example:

Consider the nonlinear system:

$$dx/dt = -x^3 + u \tag{1.15}$$

Define the sliding surface as:

$$s(x) = x \tag{1.16}$$

Then, to maintain $s(x) = 0$, the sliding condition requires:

$$\frac{ds}{dt} = -x^3 + u = 0 \tag{1.17}$$

Thus, the equivalent control is:

$$u_{eq} = x^3 \tag{1.18}$$

To ensure reachability and robustness, we use a discontinuous control law:

$$u = x^3 - k \cdot \text{sign}(s(x)), \tag{1.19}$$

where $k > 0$.

This example illustrates how sliding mode control can be used to stabilize a first-order nonlinear system. By ensuring that the control law forces the system state to the sliding surface and keeps it there, robust performance can be achieved even in the presence of nonlinearities and disturbances.

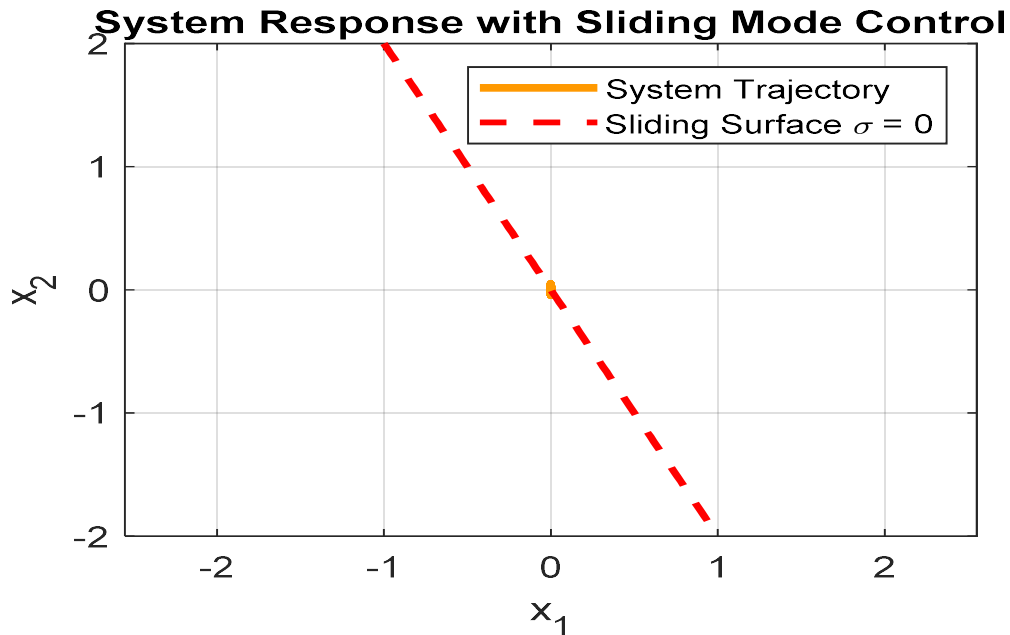


Figure3: System Response with Sliding Mode Control

Figure: The plot shows the trajectory of the state variable $x(t)$ under sliding mode control. The system converges to the sliding surface $s(x) = 0$, demonstrating stable behavior despite the system nonlinearity

I.6 First-Order Sliding Mode Control Input Design:

To ensure that the system trajectories reach and evolve on the predefined sliding surface, a control law must be designed that is robust against system uncertainties and disturbances. This can be accomplished by employing Lyapunov-based methods.

Definition 1.1.3: A function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is termed a candidate Lyapunov function if:

$$- \quad V(0) = 0, -V(x) > 0, \text{ for all } x \in \mathbb{X} \setminus \{0\} \quad (1.20).$$

The sign of the time derivative $\dot{V}(x)$ provides insights into the system's stability. For the sliding variable σ , a typical Lyapunov candidate is:

$$V(\sigma) = (1/2) \sigma^2, \tag{1.21}$$

which is clearly positive definite. For asymptotic stability, the derivative of V must be negative definite:

$$V(\sigma) = \sigma \sigma' < 0 \tag{1.22}$$

To ensure finite – time convergence, a stricter condition known as the η

– attractiveness condition is imposed, as proposed by Utkin [1992]:

$$\sigma \sigma' \leq -\eta |\sigma|, \eta > 0 \tag{1.23}$$

This implies:

$$V \leq -\eta \sqrt{2V} \tag{1.24}$$

and thus, integrating yields:

$$\sqrt{2V(t)} - \sqrt{2V(0)} \leq -\eta t \tag{1.25}$$

from which we derive the finite reaching time:

$$t_F \leq \frac{|\sigma(0)|}{\eta} \tag{1.26}$$

A control law satisfying this condition will drive σ to zero in finite time. Such a control input takes the discontinuous form:

$$u = -K \text{sign}(\sigma), \tag{1.27}$$

where K is a gain chosen to ensure the η attractive condition is met. From the system

dynamics, the required gain must satisfy:

$$K \geq (|a(x,t)| + \eta) / b(x,t) \tag{1.28}$$

According to Assumption 1.1.2, if $|a(x,t)| \leq a_M$ and $b(x,t) \geq b_m$, then a sufficient condition is:

$$K \geq (a_M + \eta) / b_m \tag{1.29}$$

Thus, with a control input of the above form and an appropriately chosen gain, the system state is guaranteed to reach the sliding surface in finite time and remain on it, ensuring robust convergence of the tracking error.

I.6.1 Example:

Consider a nonlinear system described by the following dynamics:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + u + d(t) \\ y &= x_1 \end{aligned} \tag{1.30}$$

where $d(t)$ is a bounded external disturbance, for example, $d(t) = 0.5 \sin(3t)$. The objective is to design a control input u such that the system output $y = x_1$ converges to zero in finite time.

Define the sliding variable:

$$\sigma = x^2 + c^1 x^1 \tag{1.31}$$

where $c_1 > 0$. Then, the sliding surface is defined as $\sigma = 0$, and once on this surface, the system dynamics reduce to:

$$\dot{x}^1 = -c^1 x^1 \Rightarrow x^1(t) = x^1(0)e^{-c^1 t} \tag{1.32}$$

which ensures exponential convergence to the origin.

The derivative of the sliding variable is:

$$\dot{\sigma} = \dot{x}^2 + c^1 \dot{x}^1 = -x^1 + u + d(t) + c^1 x^2 \tag{1.33}$$

Substituting $\sigma = x_2 + c_1 x_1$, we can rewrite:

$$\dot{\sigma} = -x_1 + u + d(t) + c_1(\sigma - c_1 x_1) = u + a(x, t) \tag{1.34}$$

$$\text{where } (x, t) = -x^1 + d(t) + c^1(\sigma - c^1 x^1) \tag{1.35}.$$

The control input is designed as:

$$u = -K \text{sign}(\sigma) \tag{1.36}$$

To ensure finite – time convergence, choose K such that:

$$K \geq |a(x, t)| + \eta \tag{1.37}$$

Assuming $|d(t)| \leq d_{\max}$ and $|x_1|, |\sigma|$ bounded, then:

$$K \geq |-x^1 + d_{\max}^1 (\sigma - c^1 x^1)| + \eta \tag{1.38}$$

Choosing a sufficiently large K ensures finite – time convergence of $\sigma \rightarrow 0$, and hence

$$x^1 \rightarrow 0$$

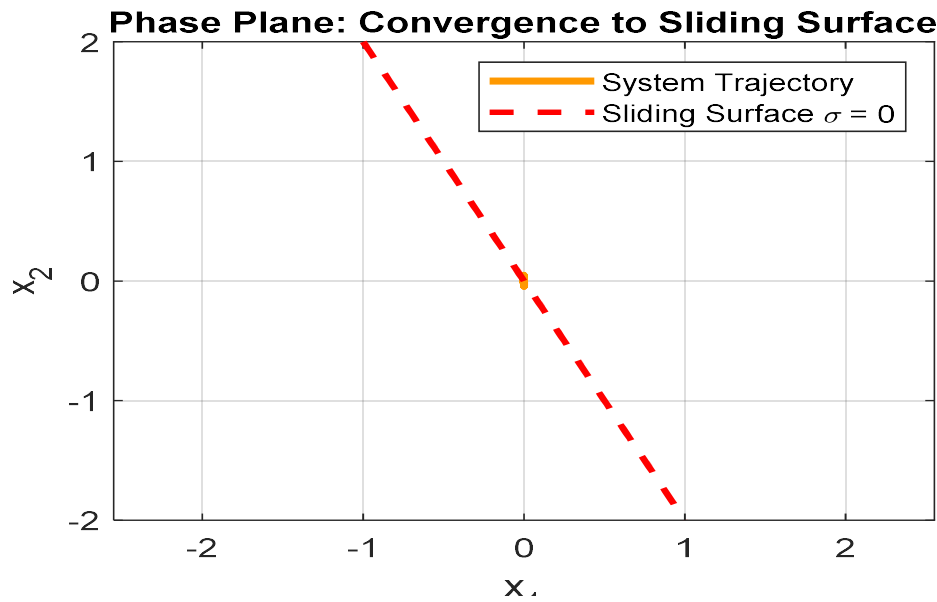


Figure 4: Phase trajectory showing convergence to the sliding surface under external disturbance.

I.7 Chattering Phenomenon:

The chattering phenomenon is a well-known issue in classical sliding mode control. Ideally, sliding mode control relies on an infinitely fast switching control law to maintain the system state on the sliding surface. However, in real-world applications, infinite switching frequency is unattainable due to hardware and physical limitations. As a result, the control signal alternates rapidly but within a finite frequency, causing a high-frequency oscillation around the sliding surface known as chattering.

Chattering is undesirable for several reasons. It can cause wear and tear in mechanical systems, generate heat losses in power electronics, and reduce overall control precision. The phenomenon is primarily attributed to two factors: unmodeled fast system dynamics and the use of digital control implementations with finite sampling intervals[28].

To mitigate chattering, several techniques have been developed. A common approach is to approximate the discontinuous sign function near the sliding surface with a continuous function. For instance, the sign function can be replaced by a saturation function within a boundary layer defined by $\pm\delta$. In this case, the control law provides a smooth transition through the sliding surface, producing what is known as a "pseudo-sliding mode" [Burton and Zinober, 1986; Yu and Potts, 1992].

Three commonly used smooth approximations are:

1. Saturation Function:

$$sat(\sigma, \delta) = sign(\sigma), \text{ if } |\sigma| > \delta; \text{ otherwise, } \sigma - \delta \quad (1.39)$$

2. Arctangent Function:

$$v(\sigma, \delta) = \left(\frac{2}{\pi}\right) * arctan\left(\sigma - \delta\right) \quad (1.40)$$

3. Hyperbolic Tangent Function:

$$v(\sigma, \delta) = tanh(\sigma / \delta) \quad (1.41)$$

Although these continuous functions effectively reduce chattering, they may also degrade the robustness of the controller by creating a trade-off between precision and smoothness. Alternatively, high-order sliding mode controllers have been proposed to attenuate chattering by acting on higher-order derivatives of the sliding variable, thus preserving the robustness of sliding mode control while offering a smoother control action.

I.8 Summary:

Sliding Mode Control (SMC) has proven to be a highly effective strategy for dealing with complex control problems, especially in the presence of system nonlinearities, modeling uncertainties, and external disturbances. By enforcing system trajectories to converge to and remain on a predefined sliding surface, SMC ensures robust performance and finite time convergence two features that are particularly valuable in real-world control applications.

This chapter has provided a comprehensive overview of classical SMC, beginning with its theoretical foundation and historical development. The analysis of both linear and nonlinear systems under SMC has highlighted the technique's versatility and efficacy. Moreover, the discussion on the chattering phenomenon and its mitigation has illustrated the importance of practical implementation considerations.

I.9 Conclusion:

SMC is a powerful, practical control method especially suited for nonlinear and uncertain environments. Its core strength lies in its ability to maintain system performance despite modeling imperfections, making it a preferred technique in many modern control applications. Moreover, the inherent robustness of SMC allows it to effectively reject external disturbances and parameter variations without the need for exact system models. This attribute is particularly beneficial in systems where conventional linear control strategies fail to guarantee stability or performance. As research continues to advance, variations such as higher-order and adaptive sliding mode control are expanding the applicability of SMC to increasingly complex and high-performance systems.

Chapitre II :

***Sliding Mode and Higher
Order Sliding Mode Control***

II.1.Introduction:

Nonlinear dynamic systems are inherently subject to parametric uncertainties, making their precise modeling and control a complex task. These uncertainties often result from fluctuating operational conditions and external disturbances, which impact the physical characteristics of such systems. While the bounds of these uncertainties might be known in some cases, they may remain undefined in others. Consequently, control system design must account for these variations to ensure robust performance across diverse operational scenarios.

Sliding Mode Control (SMC) is a robust control methodology particularly effective for nonlinear systems. It is distinguished by its robustness against parameter variations and external disturbances. The fundamental principle of SMC involves applying a discontinuous control signal that drives the system states toward a predefined sliding manifold in the state space. Once on this manifold, the system dynamics are governed by a reduced-order system that is insensitive to certain types of uncertainties [29,30,31].

Despite its advantages, SMC is often criticized for a phenomenon known as chattering—high-frequency oscillations of finite amplitude around the sliding manifold. These oscillations are problematic as they may excite unmodeled dynamics, leading to system instability or degradation in performance [32]

To mitigate the chattering issue, Higher Order Sliding Mode Control (HOSMC) has been introduced. Unlike conventional SMC, HOSMC applies the discontinuous control to higher derivatives of the sliding variable. This ensures not only the convergence of the sliding variable to the origin but also the convergence of its higher-order derivatives. Since the control input is no longer directly discontinuous, the chattering effect is substantially reduced [33].

This chapter provides an introductory overview of SMC and HOSMC concepts. Initially, the basic theories and performance aspects of Sliding Mode Control are discussed. Subsequently, the principles of HOSMC are introduced, accompanied by examples illustrating existing HOSMC controllers, along with their respective benefits and limitations. The final section of the chapter presents adaptive first-order and second-order SMC strategies, highlighting the advantages of incorporating adaptability in sliding mode control schemes[34],[52].

II.2. Sliding Mode Control:

Sliding Mode Control (SMC) is a robust control strategy designed to compel the state variables of a dynamic system to reach a predetermined manifold (or surface) within the state-space in finite time and to remain on that surface thereafter. This manifold encapsulates a set of relationships among the system's state variables and governs the desired behavior of the system. Once the system's states reside on this manifold, its behavior becomes entirely determined by the surface itself, rendering the system dynamics invariant to perturbations and uncertainties[35].

The design of an SMC controller typically involves two primary steps:

1. **Manifold Design:** A suitable manifold is defined based on the control objectives and the desired dynamic and static properties of the closed-loop system.
2. **Control Law Synthesis:** A discontinuous control law is developed to ensure that the state trajectories reach the manifold in finite time and persist on it, despite the presence of external disturbances and modeling uncertainties.

Consider a nonlinear control-affine system represented in the Brunovsky canonical form as follows:

$$\begin{cases} \dot{x}_1 = x_2, \\ \vdots \\ \dot{x}_n = \psi(x, t) + \varphi(x, t) + (x, t)u, \\ y = x_1, \end{cases} \quad (2.1)$$

where $x \in X \subset \mathbb{R}^n$ denotes the state vector within an open set X of \mathbb{R}^n , and $u \in U \subset \mathbb{R}$ is the control input defined in an open set U . The output y represents a measurable and smooth function of the system states.

The nominal dynamics of the system are captured by the known function $\psi(x, t)$, while $\varphi(x, t)$ and $\gamma(x, t)$ are sufficiently smooth yet uncertain functions, defined for all $x \in X$.

In terms of input-output representation, the system can be rewritten as:

$$y^{(n)} = \psi(\tilde{y}, t) + \varphi(\tilde{y}, t) + \gamma(\tilde{y}, t)u, \quad (2.2)$$

where $\tilde{y} = [y, \dot{y}, \dots, y^{(n-1)}]$. It is assumed that $\gamma(x, t)$ and $\varphi(x, t)$ are bounded by positive constants γ_m, γ_M , and φ^- , respectively, such that:

$$0 < \gamma_m \leq \gamma(x, t) \leq \gamma_M, \quad |\varphi(x, t)| \leq \varphi^- \quad (2.3)$$

Therefore, the system can be described by the following differential inclusion:

$$y^{(n)} \in \psi(\tilde{y}, t) + [-\bar{\varphi}, \bar{\varphi}] + [\gamma_m, \gamma_M]u \quad (2.4)$$

Here, $\bar{\varphi}$ represents the bound on parameter uncertainties due to simplifications, unmodeled dynamics, or external disturbances, while γ_m and γ_M denote the lower and upper bounds of the control gain uncertainty, respectively.

II.2.1 Design of Sliding Manifold and Controller:

Sliding Mode Control (SMC) involves designing a sliding manifold that guides the system states to follow desired behavior despite uncertainties and external perturbations. Let $s(x, t): X \times \mathbb{R}^+ \rightarrow \mathbb{R}$ be a smooth, measurable output-feedback function, known as the sliding variable. The control objective is to force this sliding variable to zero. The set

$$S = \{x \in X \mid s(x, t) = 0\} \quad (2.5)$$

defines an $(n-1)$ -dimensional submanifold of the state-space X , referred to as the sliding surface.

Definition II.2.1: There exists an ideal sliding regime on S if there exists a finite time T_s such that all trajectories of System (1.1) satisfy $s(x, t) = 0$ for all $t \geq T_s$. Once the system reaches the surface S , its dynamics reduce to a lower-dimensional system, known as the reduced system, which is fully determined by the selected sliding surface.

For the system given in Equation (1.1), the control objective is to force the output y to track a sufficiently smooth reference signal y_{ref} . This implies driving the tracking error $e = y - y_{ref}$ to zero. A common choice for the sliding surface is a linear combination of the tracking error and its derivatives:

$$s(x, t) = e^{(n-1)} + l_{n-2}e^{(n-2)} + \dots + l^1\dot{e} + l^0e, \quad (2.6)$$

where $l_i > 0$ for $i = 0, \dots, n-2$ are constants such that the polynomial

$$P(\theta) = \theta^{n-1} + l_{n-2}\theta^{n-2} + \dots + l^1\theta + l^0, \quad (2.7)$$

is Hurwitz. After achieving the sliding regime, the reduced system is governed by the stable differential equation:

$$e^{(n-1)} + l_{n-2}e^{(n-2)} + \dots + l_1\dot{e} + l_0e = 0, \quad (2.8)$$

ensuring exponential convergence of the tracking error to zero.

II.2.2 Controller Design:

The control input u should be formulated such that the state trajectories are driven to the sliding surface S and remain on it, despite disturbances and uncertainties. The dynamics of the sliding variable $s(x, t)$ are expressed as:

$$\dot{s}(x, t) \in \psi(x, t) + [-\bar{\varphi}, \bar{\varphi}] + [\gamma_m, \gamma_M]u + \sum_{i=0}^{n-2} l_i e^{(i+1)} - y_{ref}^{(n)} \quad (2.9)$$

To ensure that the system's trajectories are directed toward S , a condition known as the attractivity condition must be fulfilled. This involves defining a Lyapunov function $V(s)$, a positive, continuously differentiable, radially-unbounded function that satisfies $V(0) = 0$ and $V(\infty) = \infty$. A typical choice is the quadratic function:

$$V(s) = (1/2)s^2 \quad (2.10)$$

Its time derivative must satisfy:

$$\dot{V} = s\dot{s} \leq 0 \quad (2.11)$$

For finite – time convergence, a stronger condition is applied, termed η – attractivity:

$$\dot{V} = s\dot{s} < -\eta|s|, \eta > 0 \quad (2.12)$$

This condition is met if the control law takes the form:

$$u = -U\text{sign}(s) \quad (2.13),$$

where U is a sufficiently large constant designed to compensate for the uncertainties, perturbations, and mismatches in the model. A suitable choice for U is:

$$U \geq \max_{x \in X} \left(\frac{1}{\gamma_m} \left(\left| \psi(x, t) + \sum_{i=0}^{n-2} l_i e^{(i+1)} - y_{ref}^{(n)} \right| + \bar{\varphi} + \eta \right) \right). \quad (2.14)$$

System Behavior Phases

1. Reaching Phase ($t \in [0, T_s]$): During this period, the system state converges to the sliding surface S .
2. Sliding Phase ($t \in [T_s, \infty]$): Once on S , the system evolves along the surface, with dynamics governed by the reduced system.

In an ideal SMC regime, the control law switches at an infinite frequency, which is not practical due to inherent system delays. This leads to a phenomenon known as chattering, where the state

oscillates around the sliding surface. To mitigate chattering, higher-order sliding mode control techniques can be employed, which will be discussed in the subsequent section.

II.3 Higher Order Sliding Mode Control (HOSMC)

Higher Order Sliding Mode Control (HOSMC) is a powerful technique used to mitigate the adverse effects of chattering inherent in classical sliding mode control systems. Unlike conventional Sliding Mode Control (SMC), HOSMC applies discontinuous control actions not directly to the system input but to a higher-order derivative of the sliding variable. As a result, both the sliding variable and its derivatives converge to zero, and the phenomenon of chattering is substantially reduced [33].

II.3.1 Problem Formulation

Consider an uncertain nonlinear system described by the following equations:

$$\begin{cases} \dot{x}(t) = f(x, t) + g(x, t)u \\ y(t) = s(x, t) \end{cases} \quad (2.15)$$

where $x \in \mathbb{R}^n$ represents the state vector, $u \in \mathbb{R}$ is the control input, and $s(x, t)$ is a smooth output-feedback function. The functions $f(x, t)$ and $g(x, t)$ are uncertain yet smooth. The control objective is framed in terms of the system's relative degree, which is defined as follows.

Definition II.3.1: The relative degree r of the system with respect to the output $y(t)$ is the minimum number of derivatives of $y(t)$ required before the control input u explicitly appears.

Assuming that the relative degree r is constant, well-defined, and time-invariant, and that the internal (zero) dynamics are asymptotically stable (Dinuzzo & Fererra, 2009),

we obtain:

$$y^{(r)}(t) = \tilde{\varphi}(x(t), t) + \tilde{\gamma}(x(t), t)u(t), \quad (2.16)$$

where $\tilde{\varphi}$ and $\tilde{\gamma}$ are smooth functions satisfying the bounds:

$$0 < \gamma_m \leq \tilde{\gamma}(x(t), t) \leq \gamma_M, \quad |\tilde{\varphi}(x(t), t)| \leq \bar{\varphi} \quad (2.17)$$

Define $s^{(i)} = d^i y / dt^i$. The higher-order sliding manifold of order r is given by:

$$S^{(r)} = \{x \in \mathbb{R}^n \mid s^{(0)} = s^{(1)} = \dots = s^{(r-1)} = 0\}. \quad (2.18)$$

Definition II.3.2: A higher-order sliding mode of order r exists if there exists a finite time T_s such that for all $t > T_s$, the system trajectories lie on $S^{(r)}$.

To formalize this, let $z = [z_1 \ z_2 \ \dots \ z_r]^T = [s \ \dot{s} \ \dots \ s^{(r-1)}]^T$, transforming the problem into the following system:

$$\begin{aligned} \dot{z}_i &= z_{i+1}, \quad i = 1, \dots, r-1 \\ \dot{z}_r &= \varphi(t) + \gamma(t)u \end{aligned} \tag{2.19}$$

where $\varphi(t) \in [-\bar{\varphi}, \bar{\varphi}]$ and $\gamma(t) \in [\gamma_m, \gamma_M]$. This yields the differential inclusion:

$$z_1^{(r)} \in I_\varphi + uI_\gamma. \tag{2.20}$$

The primary objective of HOSMC is to stabilize the system to the origin in finite time using discontinuous feedback $U(z)$. Due to the discontinuity in u , the solution is understood in the Filippov sense .

Definition II.3.3 [36]: At points of discontinuity, the right-hand side vector set is expanded to the convex hull of all directional limits of velocity vectors approaching the discontinuity, excluding zero-measure sets.

A crucial notion is the Real Sliding Mode, defined by [10] as follows:

Definition II.3.4 [37]: A control algorithm establishes a real sliding mode of order r if, for any initial conditions and any time interval $[t_1, t_2]$, there exist constants $\delta_1, \dots, \delta_r$ such that for all $t > t_1$:

$$|s| \leq \Delta_1, |\dot{s}| \leq \Delta_2, \dots, |s^{(r-1)}| \leq \Delta_r. \tag{2.21}$$

II.3.2 Examples of Second Order Sliding Mode Controllers:

Following the introduction of Higher Order Sliding Mode (HOSM) control, this section explores several well-known Second Order Sliding Mode (SOSM) control algorithms. These include the supertwisting, twisting, and sub-optimal algorithms. In SOSM control, both the sliding variable and its first time derivative are required to converge to zero within a finite time horizon.

II .3.2.1 Supertwisting Algorithm [38, 39]:

The Supertwisting algorithm is a continuous SOSM control strategy applicable to systems with relative degree one. It assumes the boundedness of the time derivative of system perturbations. Specifically, for a system defined by $\dot{z}_1 = \varphi(t) + \gamma(t)u$, the condition $|\varphi| \leq \bar{\varphi}$ must hold. Letting $\varphi = z_2$, the control objective is to stabilize the following differential inclusion:

$$\begin{aligned} \dot{z}_1 &\in z_2 + [\gamma_m, \gamma_M]u, \\ \dot{z}_2 &\in [-\bar{\varphi}, \bar{\varphi}]. \end{aligned} \tag{2.22}$$

The Supertwisting control law is given by:

$$u = -K_p |z_1|^{(1)} \text{sign}(z_1) - \int_0^t k_i \text{sign}(z_1) dt, \quad (2.23)$$

where K_p and K_i are positive gains determined as follows:

$$\begin{aligned} K_p &= k_p \sqrt{L}, K_i = k_i L \\ L &= \frac{\bar{\phi}}{\gamma_m}, k_i > 1, k_p > \sqrt{-2k_i + 2\sqrt{k_i^2 + 2k_i + 2}} \end{aligned} \quad (2.24)$$

Structurally, this algorithm can be interpreted as a nonlinear finite-time analogue of the classical PI controller.

II .3.2.2 Twisting Algorithm [40]:

The Twisting algorithm is a discontinuous SOSM controller generally employed for systems with relative degree two. The control action alternates between two values depending on the sign of the product $z_1 z_2$. From the differential inclusion:

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 \in [-\bar{\phi}, \bar{\phi}] + [\gamma_m, \gamma_M]u \end{cases} \quad (2.25)$$

the control law is defined as:

$$u = \begin{cases} -K_m \text{sign}(z_1) & \text{if } z_1 z_2 < 0, \\ -K_M \text{sign}(z_1) & \text{if } z_1 z_2 \geq 0, \end{cases} \quad (2.26)$$

Alternatively, it can be expressed in a compact form as:

$$u = -l_1 \text{sign}(Z_1) - l_2 \text{sign}(Z_2), \quad (2.27)$$

subject to the constraints:

$$\begin{aligned} \gamma_m(l_1 + l_2) - \bar{\phi} &> \gamma_M(l_1 - l_2) + \bar{\phi}, \\ \gamma_m(l^1 - l^2) &> \bar{\phi}. \end{aligned} \quad (2.28)$$

This control scheme is often regarded as a robust, nonlinear analogue of the conventional PD controller.

II.3.2.3 Sub-optimal Algorithm [40]:

The Sub-optimal algorithm, like the Twisting controller, is a discontinuous SOSM strategy designed for systems with relative degree two. Its formulation closely mirrors that of the Twisting algorithm. The control input is given by:

$$u = \lambda(t)u_M \text{sign}\left(z_1(t) - \frac{z_1(t_M)}{2}\right) \quad (2.29)$$

$$\lambda(t) = \begin{cases} 1 & \text{if } z_1(t) \geq z_1(t_M), \\ \lambda^* & \text{if } z_1(t) < z_1(t_M), \end{cases}$$

Here, t_M denotes the most recent time at which $z_2 = 0$. For finite-time convergence, the gains must satisfy:

$$\lambda^* \in (0,1] \cap \left(0, \frac{3\gamma_m}{\gamma_M}\right) \quad (2.30)$$

$$u_M > \max\left(\frac{\bar{\varphi}}{\lambda^* \gamma_m}, \frac{4\bar{\varphi}}{3\gamma_m \lambda^* \gamma_M}\right)$$

This algorithm is similarly interpreted as a robust, nonlinear version of the classical PD controller.

II.3.3 Arbitrary Order Sliding Mode Controllers

In addition to Second Order Sliding Mode (SOSM) controllers, various High Order Sliding Mode (HOSM) controllers have been developed to achieve sliding modes of arbitrary order. For instance, Laghrouche et al. [41] introduced a two-component integral sliding mode control method to separately address the problems of finite-time stabilization and uncertainty rejection. Similarly, Dinuzzo et al. [42] approached the HOSM control problem as a Robust Fuller's problem, offering a novel perspective. Another significant contribution is by Defoort et al. [43], who proposed a robust Multi Input Multi Output (MIMO) HOSM controller using a constructive algorithm based on geometric homogeneity to ensure finite-time stabilization of integrator chains. Furthermore, Levant [44] designed a method for arbitrary-order sliding mode controllers tailored for Single Input Single Output (SISO) systems, as elaborated in the following theorem.

Theorem 1.3.5: Let l_1, \dots, l_{r-1} , $M > 0$ be chosen sufficiently large in sequence. Then, the bounded controller $u = -M \text{sign}(\phi_{r-1})$ ensures finite-time stability for System, where ϕ_{r-1} is defined inductively as:

$$N_1 = |z_1|^{r-1/r}$$

$$N_i = \left(|z_1|^{d/r} + |z_2|^{d/(r-1)} + \dots + |z_i|^{d/(r-i+1)}\right)^{(r-i)/d}, i = 2, \dots, r-1 \quad (2.31)$$

$$\phi_0 = z_1$$

$$\phi_i = z_{i+1} + l_i N_i \text{sign}(\phi_{i-1}), i = 2, \dots, r-1$$

where $d > r$ is an arbitrary positive constant.

Despite the guaranteed finite-time convergence of this algorithm, the selection of parameters $l_1, l_2, l_r - 1$, and M remains challenging due to the absence of necessary or sufficient conditions based on Lyapunov stability analysis.

II.3.3.1 Homogeneity:

The concept of homogeneity plays a crucial role in facilitating the design of finite-time controllers by establishing an equivalence between local asymptotic convergence and global finite-time convergence. Consider the time-invariant system:

$$\dot{x} = f(z), f(0) = 0 \quad z \in \mathbb{R}^r.$$

II.3.6 Definition [45]:

A family of dilations ζ_ε^p , with $\varepsilon > 0$, is defined on \mathbb{R}^r as:

$$\zeta_\varepsilon^p(z_1, \dots, z_r) = (\varepsilon^{p_1} z_1, \dots, \varepsilon^{p_r} z_r), \quad (2.32)$$

where $p = (p_1, \dots, p_r)$ and $p_i > 0$ for all $i=1, \dots, r$.

Definition II.3.7 [45]:

A vector field $f(z) = [f_1(z), \dots, f_r(z)]^T$ is homogeneous of degree $\kappa \in \mathbb{R}$ with respect to ζ_ε^p if:

$$f_i(\varepsilon^{p_1} z_1, \dots, \varepsilon^{p_r} z_r) = \varepsilon^{p_i + \kappa} f_i(z_1, \dots, z_r), \quad i = 1, \dots, r, \varepsilon > 0. \quad (2.33)$$

system (2.33) is termed homogeneous if its vector field $f(z)$ satisfies this homogeneity condition.

Lemma II.3.8 [46]: If f is homogeneous under the dilation family δ_ε^p and the origin is an attractive equilibrium, then it is also globally asymptotically stable.

Lemma 1 II 3.9 [45]: If system (1.33) is homogeneous of degree $\kappa < 0$ under δ_ε^p , and $f(z)$ is continuous with $z = 0$ as an asymptotically stable equilibrium, then the equilibrium is globally finite-time stable.

Remark II.3.10:

The concept of homogeneity extends to differential inclusions of the form $\dot{x} \in F(z)$. In this context, finite-time convergence remains valid [48].

These lemmas confirm that for closed-loop systems designed with homogeneous controllers of negative degree, establishing local asymptotic stability suffices to ensure global finite-time stability. Many HOSM controllers, such as the super-twisting algorithm, exhibit homogeneity with a degree of $\kappa = -0.5$ under the dilation family $(1, 0.5)$. Other algorithms, including the twisting and sub-optimal schemes, are also negatively homogeneous. Levant's recent works [47,

48] have reinforced the use of homogeneity in finite-time stabilization, further elaborated in Chapter 3 of this thesis.

II.4 Adaptive Sliding Mode Controllers

Sliding Mode Control (SMC) and Higher Order Sliding Mode Control (HOSMC) have been widely recognized for their robustness in the presence of system uncertainties. However, a common prerequisite for their implementation is the knowledge of bounded uncertainties within the system. Determining these bounds can be practically infeasible as it often demands extensive testing under worst-case conditions. In cases where such bounds exist but are unknown, conventional approaches rely on conservative gain selection, which can introduce high control amplitudes and consequently severe chattering, making them unsuitable for physical applications. An effective alternative is adaptive sliding mode control (ASMC), where the control gains are adjusted dynamically. This method ensures that the control effort remains sufficient to counteract parametric variations and external disturbances without resorting to overly conservative gain estimations. The pioneering work in this area was by Huang et al. [21], who proposed a gain adaptation law for first-order SMC based solely on the sliding variable. Although this strategy did not require prior knowledge of uncertainty bounds, it did not address the issue of gain overestimation.

Subsequent research has proposed various improvements to ASMC, aiming for efficient adaptation laws that maintain robustness while minimizing chattering. Among the notable contributions are:

II.4.1 Adaptive First Order Sliding Mode Controller [49]:

Designed for systems with relative degree one, this controller modifies the control gain based on the deviation from the sliding manifold. The control law is given by:

$$u = K(t) \text{sign}(s), \tag{2.34}$$

where the gain $K(t)$ is adapted as follows:

$$\dot{K} = \begin{cases} \bar{K}|s|\text{sign}(|s| - \epsilon) & , \quad K > 0 \text{ or } \text{sign}(|s| - \epsilon) > 0 \\ 0 & , \quad K = 0 \text{ and } \text{sign}(|s| - \epsilon) < 0 \end{cases} \tag{2.35}$$

where \bar{K} is a positive constant. This controller guarantees finite-time convergence of the sliding variable s to a domain $|s| < \epsilon$, establishing a real sliding mode. However, the state trajectories may overshoot and reside within a bounded neighborhood $|s| \leq \mu$ [49].

II.4.2 Adaptive Supertwisting Controller [50]:

This controller is based on the supertwisting algorithm, offering continuous control output:

$$u = -k_p(t)|z_l|^{\{1/2\}}\text{sign}(z_l) - \int_0^t k_i(t)\text{sign}(z_l)dt \quad (2.36)$$

The adaptation dynamics are:

$$\begin{aligned} \dot{K}_p &= \begin{cases} \omega_1 \sqrt{\frac{\gamma_1}{2}} \text{sign}(|s| - \epsilon) & , K_p > K_{pm} \\ \eta & , K_p \leq K_{pm} \end{cases} \\ \dot{K}_i &= 2aK_p \end{aligned} \quad (2.37)$$

with ω_1 , γ_1 , K_{pm} , η , and a as positive constants. As in [23], this controller ensures convergence to $|s| < \epsilon$ but does not ensure invariance within this neighborhood, and μ is not explicitly defined [24].

II.4.3 Adaptive Twisting Controller [51]:

Targeting systems with relative degree two, the control law is:

$$u = -K(t) (\text{sign}(z_1) + 0.5 \text{sign}(z_2)),$$

with adaptation law:

$$\dot{K} = \begin{cases} \frac{\frac{\omega_1}{\sqrt{2\gamma_1}}}{\frac{1}{\gamma_1} - \frac{2Kz_1^2 + |z_1|z_2^2}{|K-K^*|^3}} \text{sign}(N(z_1, z_2) - \epsilon) & , K \geq K_m \\ \xi & , K < K_m \end{cases} \quad (2.38)$$

defined by :

$$N(z_1, z_2) = \frac{z_1^2}{a} + \frac{z_2^2}{b}$$

The domain of convergence is defined by $N(z_1, z_2) < \epsilon$

Remark II.4.1:

A notable limitation across all adaptive SMC algorithms is that although real sliding mode is achieved, reducing control gains may lead to abrupt system perturbations, causing the sliding variable to depart from zero. Hence, convergence can be guaranteed only within a neighborhood of the origin.

II.5 Summary:

This chapter has provided a comprehensive examination of Sliding Mode Control (SMC) and Higher Order Sliding Mode Control (HOSMC) as effective strategies for the robust control of uncertain nonlinear systems. SMC was highlighted for its simplicity and strong robustness properties, particularly its ability to force system trajectories to a sliding manifold, making the system's behavior insensitive to matched uncertainties. However, the challenge of chattering was identified as a critical limitation.

To address this, the chapter introduced HOSMC, which offers smoother control actions by acting on higher derivatives of the sliding variable, thereby mitigating chattering without compromising robustness. Several HOSMC strategies, including the supertwisting, twisting, and sub-optimal algorithms, were explored in depth, each with specific design advantages and constraints. Furthermore, the concept of **homogeneity** was presented as a theoretical foundation for ensuring finite-time stability in the design of these controllers.

Lastly, the chapter addressed the limitations of fixed-gain SMC approaches and introduced **adaptive sliding mode control** as a viable solution, particularly in scenarios where uncertainty bounds are unknown. Adaptive strategies allow for dynamic gain adjustment, offering improved performance with reduced chattering, though challenges remain in achieving high-order adaptive schemes.

II.6 Conclusion:

Sliding Mode Control (SMC) offers a robust approach for controlling nonlinear systems affected by uncertainties. Despite its effectiveness, its main limitation lies in chattering, which can degrade performance. To overcome this, Higher Order Sliding Mode Control (HOSMC) was introduced, significantly reducing chattering by acting on higher-order derivatives. Furthermore, adaptive versions of SMC and HOSMC provide flexibility by adjusting control gains in real-time, enhancing performance without prior knowledge of uncertainty bounds.

Chapitre III:
Higher Order Sliding Mode
Controllers With Optimal
Reaching

III.1 Introduction:

Higher-order sliding mode (HOSM) control design is considered for systems with a known constant relative degree. In this paper, we introduce the Robust Fuller’s Problem, a robust generalization of the classical Fuller’s problem, which serves as a standard benchmark in optimal control. Solving the Robust Fuller’s Problem enables the derivation of feedback laws that correspond to generic-order HOSM algorithms, while also ensuring optimal finite-time convergence to the sliding manifold.

A common challenge in applying existing HOSM algorithms lies in the tuning of design parameters. Our methodology offers a systematic approach for parameter tuning to guarantee the desired performance and prevent instability. The convergence and stability properties of the proposed controller family are analyzed theoretically, and simulation results confirm their effectiveness[54].

III.2 RELIMINARIES P

Consider a SISO dynamic system affine in the control vari- able

$$\begin{cases} \dot{z} = a(z, t) + b(z, t)u(t), \\ y(t) = \sigma(z, t), \end{cases} \tag{3.1}$$

Where $z \in \mathbb{R}^n, u \in \mathbb{R}, \sigma : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ is a sufficiently smooth output function. We refer to system (3,1) as an uncertain system if the system order nand the functions f and g are unknown[55]. Nevertheless, we assume that the relative degree of the system [56], [57] is globally well-defined, uniform, time-invariant, and equal to r. The relative degree of the nonlinear system (3,1) plays a crucial role in the subsequent analysis.For autonomous systemst,is the minimum orderi of the time derivatives $\sigma^{(i)}$ inwhich the control appears explicitly. For suitable functions h(z,t)and g(z,t) , one has

$$y^{(r)}(t) = h(\mathbf{z}(t), t) + g(\mathbf{z}(t), t)u(t). \tag{3.2}$$

Functions h(z(t),t) and g(z(t),t) are assumed to be bounded.More precisely, it is assumed that there exist positive constants

C, k_m, k_M such that

$$\text{Let } \begin{aligned} -C &\leq h(\mathbf{z}(t), t) \leq C, \\ 0 < K_m &\leq g(\mathbf{z}(t), t) \leq K_M. \end{aligned} \quad (3.3)$$

$$\sigma^{(i)} := \frac{d^i}{dt^i} y$$

The goal of any r th order sliding mode control is to attain and keep the manifold

$$\sigma^{(0)}(\mathbf{z}, t) = \sigma^{(1)}(\mathbf{z}, t) = \dots = \sigma^{(r-1)}(\mathbf{z}, t) = 0 \quad (3.4)$$

in finite time. By introducing

$$\boldsymbol{\sigma} := (\sigma^{(0)}, \sigma^{(1)}, \dots, \sigma^{(r-1)})^T \quad (3.5)$$

condition (3.4) can be rewritten as $\boldsymbol{\sigma} = 0$. Since the only available information about $h(\mathbf{z}, t)$ and $g(\mathbf{z}, t)$

are the bounds (3.3), the original dynamical system (3.1) implies the differential inclusion

$$\dot{\boldsymbol{\sigma}} = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix} \boldsymbol{\sigma} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f_1 + f_2 u \end{pmatrix} \quad (3.6)$$

$f_1(t) \in [-C, C], f_2(t) \in [K_m, K_M], \text{ a.e.}$

In the context of sliding mode control, we consider control inputs u defined by discontinuous feedback laws $u = U(\boldsymbol{\sigma})$. The resulting system dynamics are modeled using differential inclusions, interpreted in the Filippov sense—meaning that solutions are absolutely continuous functions whose derivatives satisfy the inclusion almost everywhere. At points of discontinuity, the right-hand side of the differential inclusion is replaced by the convex hull of all possible limiting velocity vectors, excluding zero-measure directions [58], [59].

The design of higher-order sliding mode (HOSM) controllers can be reduced to the finite-time stabilization of an auxiliary system—specifically, a chain of integrators—to the origin. Once the origin is reached, the system is said to be in an r -th order sliding mode, where the order corresponds to the number of derivatives (plus one) of the sliding variable driven to zero. This auxiliary stabilization problem has been extensively studied in the literature [61], [62].

In this work, we focus on controller designs capable of robust finite-time stabilization in the presence of bounded perturbations. All considered functions are defined on R^+ , and the control

inputs are elements of L^∞ , the Banach space of essentially bounded functions, equipped with the corresponding norm.[62]

The subsequent theorem pertains to a general class of optimal control problems, referred to as *Problem 1*, which is formulated for an ideal (unperturbed) chain of integrators.

$$\dot{\mathbf{z}} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \mathbf{z} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix}, \mathbf{z} \in \mathbb{R}^r \quad (3.7)$$

with bounded control $\|v\|_\infty \leq 1$. Let

$$c_1 = 1, c_i \geq 0, i = 2, \dots, r$$

$$L(\mathbf{z}) := \sum_{i=1}^r c_i |z_i|^{\frac{vr}{r+1-i}}, v > 0. \quad (3.8)$$

Problem 1:

$$\min_{\|v\|_\infty \leq 1} \int_0^{+\infty} L(\mathbf{z}(t)) dt \quad (3.9)$$

subject to (6) and $\mathbf{z}(0) = \mathbf{z}_0$.

In the following, the pair (v^*, \mathbf{z}^*) denotes a generic optimal solution (control, state trajectory).

Theorem 1: For all the initial conditions $\mathbf{z}_0 \in \mathbb{R}^T$, Problem 1 admits at least one optimal solution.

Proof: The proof of this Theorem can be found in [63]

There are two limiting cases for Problem 1. For $v \rightarrow +\infty$ we

Obtain

$$\min_{\|v\|_\infty \leq 1} \sum_{i=1}^r c_i \|z_i(t)\|_\infty \quad (3.10)$$

subject to (6), $\mathbf{z}(0) = \mathbf{z}_0$. For $v \rightarrow +0$, it can be shown that the solution of Problem 1 converges to the solution of the minimal time problem defined as

Problem 2:

$$\min_{\|v\|_\infty \leq 1} \int_0^T 1 dt \quad (3.11)$$

subject to (6), $\mathbf{z}(0) = \mathbf{z}_0$, and $\mathbf{z}(T) = \mathbf{0}$.

Problem 3 (Fuller's Problem):

$$\min_{\|v\|_{\infty} \leq 1} \int_0^{+\infty} |z_1(t)|^v dt \quad (3.12)$$

subject to (6) and $\mathbf{z}(0) = \mathbf{z}_0$.

Problem 4 (Robust Fuller's Problem):

$$\min_{\|u\|_{\infty} \leq \alpha} \max_{\substack{\|f_1\|_{\infty} \leq C \\ K_m \leq f_2(t) \leq K_M}} \int_0^{+\infty} |\sigma^{(0)}(t)|^v dt \quad (3.13)$$

subject to (5) and $\sigma(0) = \sigma_0$.

III.3 Second Order Sliding Mode Algorithms:

Consider Problem 3 in a two dimensional state space. It can be shown [64] that an optimal feedback law with $V > 0$ is given by:

$$V(x_1, x_2) = -\text{sgn}(x_1 + \beta(v)x_2|x_2|), \quad \beta(v) \in \left(\frac{1}{4}, \frac{1}{2}\right). \quad (3.14)$$

Different values for the continuous function $\beta(v)$ are tabulated in [65] where it can be seen that the curve $\beta(v)$ is decreasing from 1/2 to 1/4 as v tends to $+\infty$. For $v \rightarrow +\infty$, we have

DINUZZO AND FERRARA: HIGHER ORDER SLIDING MODE CONTROLLERS

$\beta(v) \rightarrow 1$, while for $v \rightarrow 0^+$ (minimal time problem) the optimal feedback law is given by

$$V(x_1, x_2) = - \begin{cases} 0, & (x_1, x_2) \in M_0 \\ \text{sgn}(x_2), & (x_1, x_2) \in M_1 \setminus M_0 \\ \text{sgn}\left(x_1 + \frac{1}{2}x_2|x_2|\right), & \text{else} \end{cases} \quad (3.15)$$

where

$$M_0 := \{(x_1, x_2) \in \mathbb{R}^2: x_1 = x_2 = 0\}$$

$$M_1 := \{(x_1, x_2) \in \mathbb{R}^2: x_1 + \frac{1}{2}x_2|x_2| = 0\}.$$

By Corollary 1, an optimal family of second order switching curves can be derived from the solution to the Robust Fuller's Problem (Problem 4):

$$U(\sigma, \dot{\sigma}) = -\alpha \text{sgn}\left(\sigma + \frac{\beta(v)}{\alpha_r} \dot{\sigma}|\dot{\sigma}|\right), \beta(v) \in \left[\frac{1}{4}, \frac{1}{2}\right] \quad (3.16)$$

Theorem 2 guarantees that the proposed control laws qualify as second-order sliding mode algorithms. It is important to note that if the value of the derivative of the sliding variable is not directly measurable, Levant's exact differentiator [66] can be employed to estimate it in finite time. Consequently, the proposed second-order sliding mode control law can be effectively implemented using only measurements of the sliding variable itself.

III.4third-order sliding mode algorithm:

addresses the general Robust Fuller Problem, which, in this case $v \rightarrow 0^+$, leads to a highly complex solution structure. Therefore, the analysis is limited to the specific case of minimal-time control, representing the worst-case trajectory scenario. This approach aligns with the time-optimal control strategy for third-order systems, as also discussed in [67]. While it is feasible to extend the analysis to the general case $v > 0$ by adapting the methodologies presented in [68] and [69], the minimal-time formulation is particularly significant. It not only establishes an upper bound on the worst-case reaching time but also ensures finite-time stabilization of the associated auxiliary system.

The solution to Problem 3 with $v \rightarrow 0^+$ for a chain of threeintegrators is known since [70]. By applying Corollary 1, the solution to the Robust Fuller Problem 4 with $v \rightarrow 0^+$ in the third order case is obtained.

$$U(\sigma) = -\alpha \begin{cases} u_0 := 0, & \sigma \in M_0 \\ u_1 := \text{sgn}(\ddot{\sigma}), & \sigma \in M_1 \setminus M_0 \\ u_2 := \text{sgn}\left(\dot{\sigma} + \frac{\ddot{\sigma}^2 u_1}{2\alpha_r}\right), & \sigma \in M_2 \setminus M_1 \\ u_3 := \text{sgn}(\text{sg}(\sigma, \dot{\sigma}, \ddot{\sigma})), & \text{else} \end{cases} \quad (3.17)$$

Where

$$s(\sigma, \dot{\sigma}, \ddot{\sigma}) := \sigma + \frac{\ddot{\sigma}^3}{3\alpha_r^2} + u_2 \left[\frac{1}{\sqrt{\alpha_r}} \left(u_2 \dot{\sigma} + \frac{\ddot{\sigma}^2}{2\alpha_r} \right)^{\frac{3}{2}} + \frac{\dot{\sigma} \ddot{\sigma}}{\alpha_r} \right]$$

and the manifolds M_1, M_2, M_3 are defined by

$$\begin{aligned} M_0 &:= \{(\sigma, \dot{\sigma}, \ddot{\sigma}) \in \mathbb{R}^3 : \sigma = \dot{\sigma} = \ddot{\sigma} = 0\} \\ M_1 &:= \left\{ (\sigma, \dot{\sigma}, \ddot{\sigma}) \in \mathbb{R}^3 : \sigma - \frac{\ddot{\sigma}^3}{6\alpha_r^2} = 0, \dot{\sigma} + \frac{\ddot{\sigma}|\ddot{\sigma}|}{2\alpha_r} = 0 \right\} \\ M_2 &:= \{(\sigma, \dot{\sigma}, \ddot{\sigma}) \in \mathbb{R}^3 : s(\sigma, \dot{\sigma}, \ddot{\sigma}) = 0\}. \end{aligned} \quad (3.18)$$

III.5Generic Order Sliding Mode Algorithms

The methodology presented in the previous sections can be extended to develop a general r th order Higher-Order Sliding Mode (HOSM) control algorithm with optimal reaching properties. Solving the Robust Fuller Problem for $v \rightarrow 0^+$, an arbitrary order n requires the determination of switching hypersurfaces associated with the standard time-optimal control. This time-optimal control exhibits a nested structure similar to that described in equation (3.2).

For practical implementation, it is typically sufficient to compute only the "outer" switching surface. This can be achieved through a standard procedure involving the backward-time integration of the system equations (3.6), with the control sign altered at specific time instants, beginning from either +1 or -1. By parameterizing the state trajectory with these switching instants, two systems of polynomial equations emerge: one corresponding to the control sequence (+1,-1,+1,...), and the other to (-1,+1,-1,...)

If the system state does not lie exactly on the switching surface, only one of these systems will yield a solution that satisfies the required conditions $t_i \leq 0, i = 0$ for all iii. Determining which system has a solution within the real negative orthant enables the identification of the correct sign for the optimal control input.

By applying an appropriate change of variables outlined in the Appendix it is shown that the solution to the Robust Fuller Problem for $v \rightarrow 0^+$ order nnn leads to the formulation of a corresponding rth order HOSM algorithm.

$$U(\sigma) = \alpha(-1)^{r+1}\tilde{U}(\sigma) \tag{3.19}$$

where $U \in \{-1, +1\}$ is such that the system of equation and inequalities

$$z_1^{r-i+1} + 2 \sum_{n=2}^r (-1)^{n+1} z_n^{r-i+1} = \tilde{U} \frac{(r-i+1)!}{\alpha_r} \sigma^{(i-1)} \tag{3.20}$$

$$i = 1, \dots, r$$

$$z_1 \leq \dots \leq z_r \leq 0$$

The complexity of the switching surface increases rapidly with the system order nnn. An intriguing research direction is to explore whether exact tests for identifying the switching hypersurface can be developed using recent advancements in computational algebra [71]. As an alternative, numerical techniques may be utilized to verify the existence of solutions to equations (3.20). However, the development of efficient methods—either numerical or algebraic—for addressing this problem remains an open and unresolved challenge.

III.6 System Model:

The benchmark system is a nonholonomic car motion planning problem, described by the following dynamics:

$$\begin{cases} \dot{x} = v \cos \varphi, \\ \dot{y} = v \sin \varphi, \\ \dot{\varphi} = \frac{v}{l} \tan \theta, \\ \dot{\theta} = u, \end{cases} \quad (3.21)$$

Variables:

x, y : Coordinates of the vehicle.

φ : Heading angle of the vehicle.

θ : Steering angle.

u : Control input.

$v=10$ m/s, $l=5$, $mv=10$ m/s, $l=5$ m : Constants.

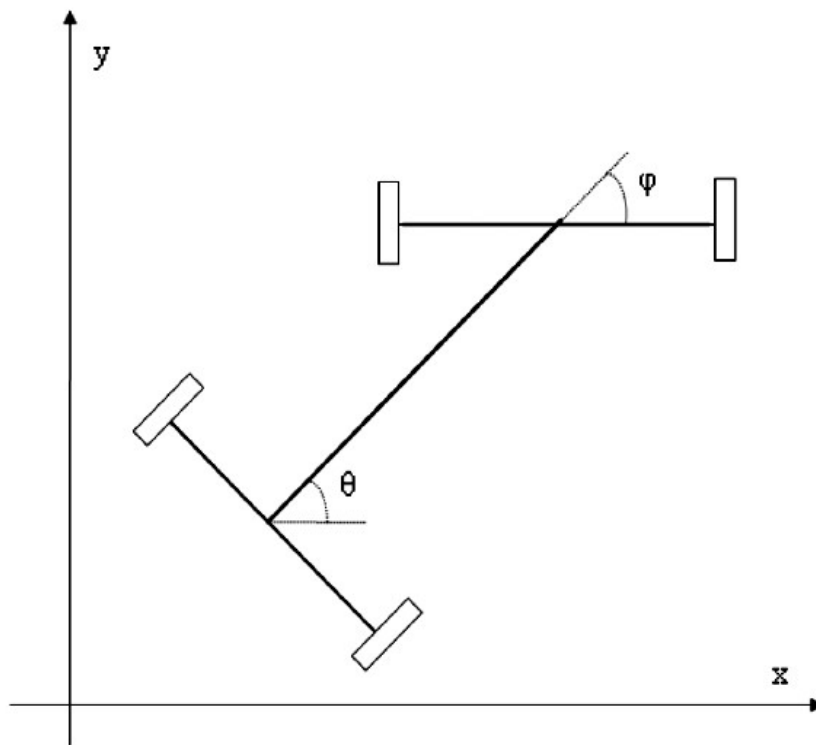


Figure5: Kinematic car model.

III.6.1 the Sliding Variable Definition:

The objective of the system is to follow the desired path:

$$y = 10 \sin(x/20) + 5 \tag{3.22}$$

Accordingly, the sliding variable is defined as:

$$\sigma = y - y_{ref} = y - 10\sin(20x) - 5 \tag{3.23}$$

The purpose of the control is to drive $\sigma \rightarrow 0$ in finite time.

III.7 Relative Degree Analysis

The relative degree is defined as the number of time derivatives required for the sliding variable σ to explicitly depend on the control input u . We begin by computing the derivatives of σ :

III.7.1 First derivative:

$$\sigma' = v \left(\sin\varphi - \frac{1}{2\cos\left(\frac{x}{20}\right)\cos\varphi} \right) \tag{3.24}$$

This equation represents the rate of change of the sliding variable σ with respect to time.

It depends on:

v : The vehicle speed.

φ : The heading angle.

x : The horizontal position.

For small angles ($\varphi \approx 0$):

* $\sin\varphi \approx \varphi$

* $\cos\varphi \approx 1$

* $\cos(20x)$ can be considered approximately constant if x is small.

III.7.2. Second Derivative Equation:

$$\sigma'' = \frac{v^2}{l} \left[\tan\theta \left(\cos\varphi + \frac{1}{2\cos\left(\frac{x}{20}\right)\sin\varphi} \right) + \frac{l}{40} \sin\left(\frac{x}{20}\right) \cos\varphi^2 \right] \tag{3.25}$$

his equation represents the acceleration (i.e., the second derivative of σ).

It depends on:

The steering angle φ ,

The wheelbase length l .

For small angles (i.e., $\varphi \approx 0$, $\dot{\varphi} \approx 0$):

$$\tan \varphi \approx \varphi,$$

$$\sin \varphi \approx \varphi,$$

$$\cos \varphi \approx 1.$$

III.7.3. Third Derivative Equation

$$\begin{aligned} \sigma = & u \frac{v^2}{2} \cdot \sec^2 \theta (\cos \varphi \\ & + \frac{1}{2} \cos \left(\frac{x}{20} \right) \sin \varphi) \\ & + \frac{v^3}{l^2} \tan^2 \theta (-\sin \varphi \\ & + \frac{1}{2} \cos \left(\frac{x}{20} \right) \cos \varphi) - \frac{v^3}{40l} \sin \left(\frac{x}{20} \right) \cos \varphi \sin \varphi (2 \tan \theta + 1) + \frac{v^3}{20} \cos \left(\frac{x}{20} \right) \cos^3 \varphi. \end{aligned}$$

III.7.4 Feasibility Condition:

$$\alpha r = \alpha Km - C > 0$$

$$\alpha r = \alpha Km - C > 0$$

Where:

$Km=5$ (minimum system gain),

$C=90$ (maximum disturbance bound).

For the control to be stable, the following condition must be satisfied:

$$\alpha > CKm \Rightarrow \alpha > 18$$

$$\alpha > KmC \Rightarrow \alpha > 18$$

In the simulation, a value of $\alpha=20$ was selected to ensure system stability.

III.7.5. Controller Comparison:

(a) Levant Controller ($U_L^3(\sigma)$):

$$U_L^3(\sigma) = -\alpha \text{sgn}[\ddot{\sigma} + \beta_2(|\dot{\sigma}|^3 + \sigma^2)^{\frac{1}{6}} \times \text{sgn}(\dot{\sigma} + \beta_1|\sigma|^{\frac{2}{3}}\text{sgn}(\sigma))] \quad (3.27)$$

(b) Quasi-Continuous Controller ($U_{QC}^3(\sigma)$):

$$U_{QC}^3(\sigma) = -\alpha \frac{\left(\ddot{\sigma} + \beta_2 \left(|\dot{\sigma}| + b_1 \times |\sigma|^{\frac{3}{2}}\right)^{\frac{1}{3}}\right) \left(\dot{\sigma} + \beta_1 |\sigma|^{\frac{2}{3}} \text{sgn}(\sigma)\right)}{|\ddot{\sigma} + \beta_2 \left(|\dot{\sigma}| + \beta_1 |\sigma|^{\frac{3}{2}}\right)^{\frac{1}{3}}} \quad (3.28)$$

(c) Optimal Reaching Controller ($U_{OR}^3(\sigma)$):

$$U_{OR}^3(\sigma) = -\alpha \text{sgn} \left[\sigma + \frac{\dot{\sigma}^3}{3\alpha_r^2} + \frac{u_2}{\sqrt{\alpha_r}} \left(u_2 \dot{\sigma} + \frac{\dot{\sigma}^2}{2\alpha_r} \right)^{\frac{3}{2}} + \frac{u_2 \dot{\sigma} \ddot{\sigma}}{\alpha_r} \right]$$

III.8. Simulation results:

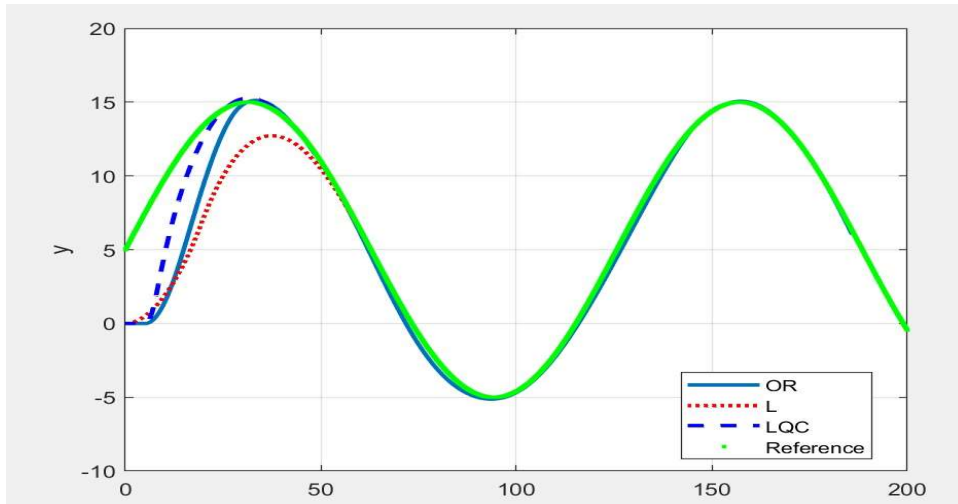


Figure6:Car trajectory.

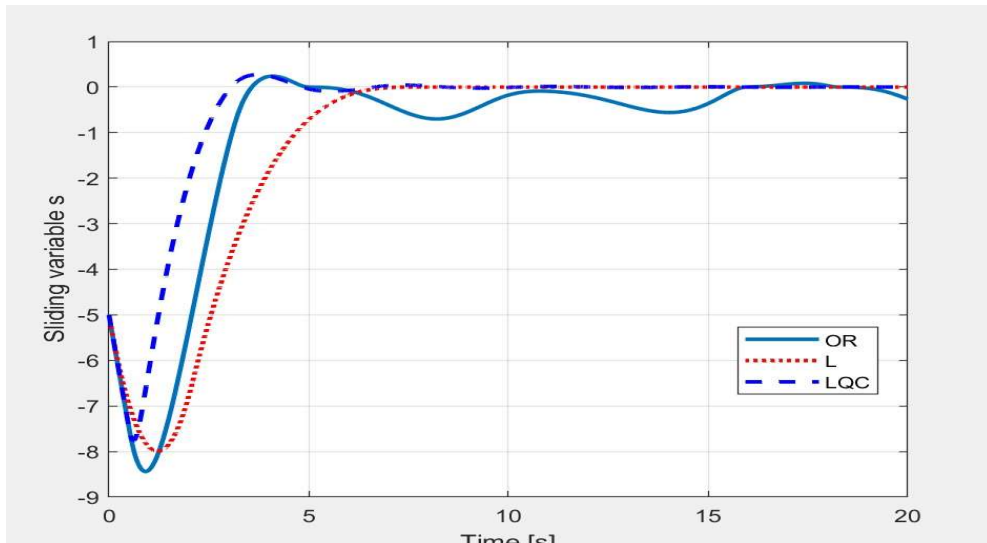


Figure7:Sliding variable

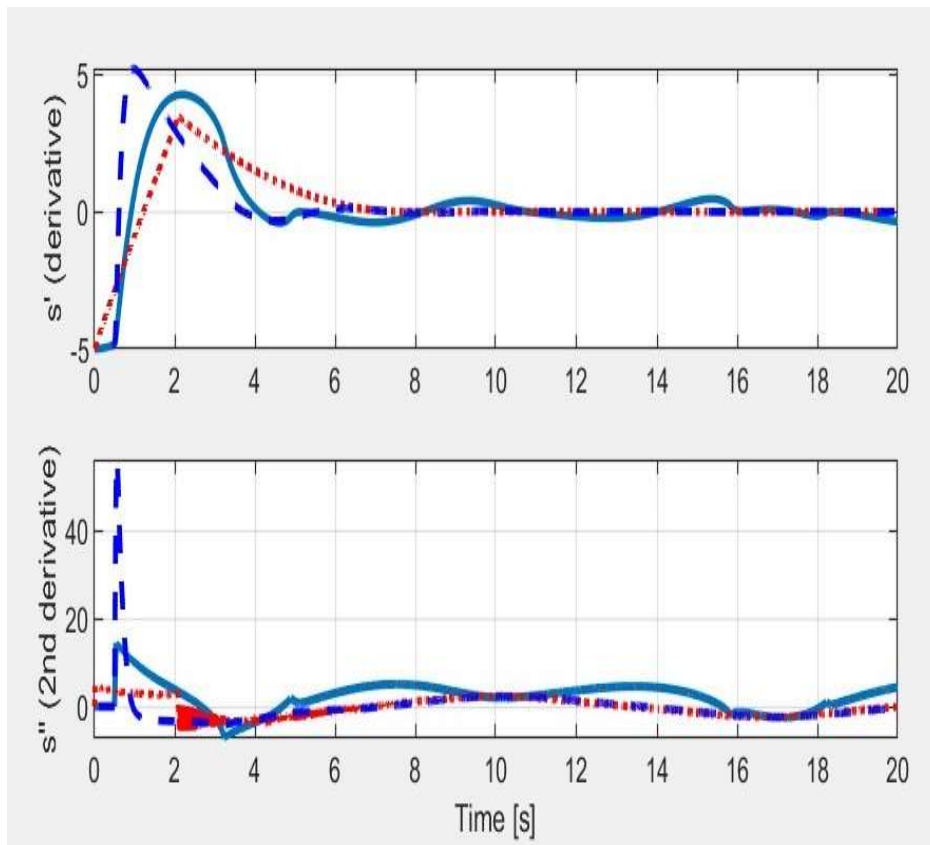


Figure8:Sliding derivatives versus time [s].

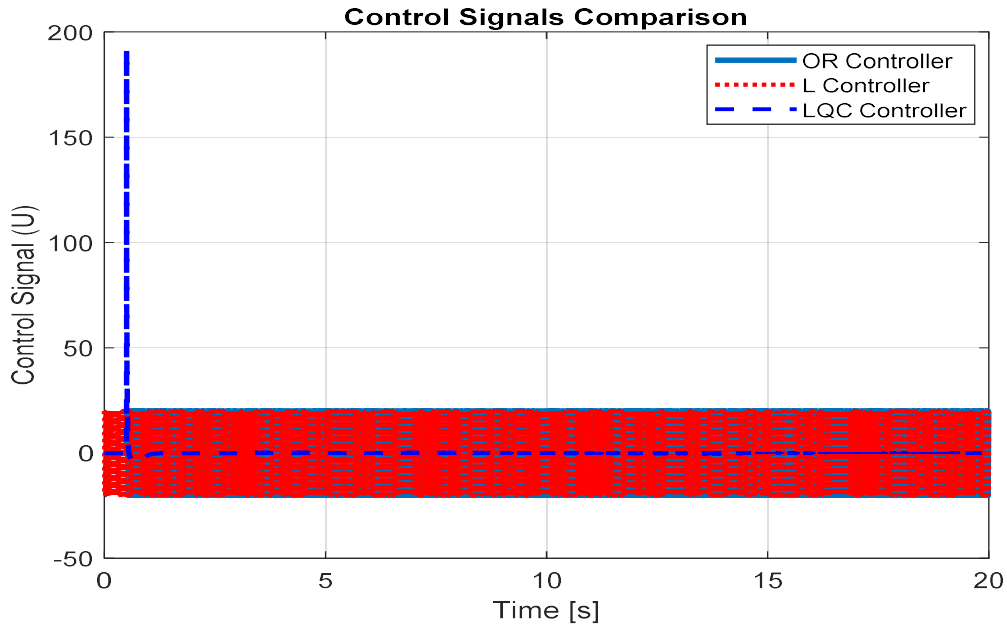


Figure9:Comparison of Control Signals for High-Order Sliding Mode Controllers

III.8.1 Results discussion:

The results clearly demonstrate that the Optimal Reaching (OR) controller outperforms the other two controllers. It significantly excels in terms of settling time and achieves accurate trajectory tracking while producing smooth and efficient steering angles free from oscillations. An important characteristic of the OR controller is that its convergence speed is not strongly dependent on the maximum control amplitude.

In contrast, the LQC controller shows good performance in terms of smoothness and vibration avoidance, but it is slower than the OR controller. Meanwhile, the L controller exhibits the slowest response and may lead to undesirable oscillations.

III.9 Conclusion:

Based on this comparison, the OR controller is considered the optimal choice for achieving high-efficiency and dynamic control performance in this scenario, outperforming both the L and LQC controllers in most key performance aspects.

In addition to its superior tracking capabilities, the OR controller demonstrates enhanced robustness against disturbances and modeling uncertainties. It also exhibits faster convergence

and reduced steady-state error compared to the other controllers. These advantages make it a highly suitable solution for applications requiring precise and responsive control under varying operating conditions.

Conclusion générale

Conclusion générale

This paper has established a clear connection between the design of Higher Order Sliding Mode (HOSM) controllers and optimal control theory by introducing and solving the Robust Fuller's Problem. By extending the classical Fuller's problem to incorporate robustness with respect to bounded uncertainties, the proposed framework enables the derivation of HOSM controllers of arbitrary order that ensure finite-time convergence to the sliding manifold while satisfying optimality criteria.

The developed approach provides several important advantages. Most notably, it offers a systematic and theoretically grounded method for designing and tuning HOSM controllers without relying on heuristic or trial-and-error procedures. The resulting control laws guarantee not only robustness and finite-time stabilization of the auxiliary system but also allow for the specification of reaching dynamics in an optimal manner. Furthermore, the approach unifies several existing HOSM algorithms under a general framework, demonstrating its versatility and coherence.

Special attention was given to the derivation of second- and third-order controllers, for which explicit control laws and switching surfaces were obtained. These controllers were shown to provide desirable properties such as reduced chattering, finite-time convergence, and simplicity in parameter selection. In particular, the third-order controller designed using the Robust Fuller's formulation demonstrated superior performance in simulations, outperforming existing methods in terms of convergence speed and robustness.

The theoretical contributions were validated through simulation results on a nonlinear benchmark problem, confirming the effectiveness and practical relevance of the proposed methodology. Overall, this work represents a significant step forward in the development of high-performance HOSM controllers, combining robustness, optimality, and simplicity in a unified control design framework.

Future research may explore the implementation of these controllers in real-time applications, the extension to multi-variable systems with coupled dynamics, and the development of efficient numerical methods for higher-order switching surface computation.

Bibliographic References

BibliographicReferences

Bibliographic References

- [1] V. I. Utkin, *Sliding Modes in Optimization and Control Problems*, Springer, 1992.
- [2] C. Edwards and S. K. Spurgeon, *Sliding Mode Control: Theory and Applications*, Taylor & Francis, 1998.
- [3] V. I. Utkin et al., *Sliding Mode Control in Electromechanical Systems*, Taylor & Francis, 1999.
- [4] L. M. Fridman, "Singularly perturbed analysis of chattering in relay control systems," *IEEE TAC*, vol. 47, no. 12, pp. 2079–2084, 2002.
- [5] I. Boiko and L. M. Fridman, "Analysis of chattering in continuous sliding-mode controllers," *IEEE TAC*, vol. 50, no. 9, 2005.
- [6] A. Levant, "Sliding order and sliding accuracy in sliding mode control," *Int. J. Control*, vol. 58, no. 6, 1993.
- [7] G. Bartolini et al., "Chattering avoidance by second order sliding mode control," *IEEE TAC*, vol. 43, no. 2, 1998.
- [8] G. Bartolini et al., "Multi-input chattering-free second order sliding mode control," *IEEE TAC*, vol. 45, no. 9, 2000.
- [9] G. Bartolini et al., "Time-optimal stabilization for a third-order integrator," *Lecture Notes in Control and Information Sciences*, vol. 273, Springer, 2002.
- [10] A. Levant, "Higher-order sliding modes, differentiation and output-feedback control," *Int. J. Control*, vol. 76, no. 9/10, 2003.
- [11] A. Levant, "Quasi-continuous high-order sliding-mode controllers," *IEEE TAC*, vol. 41, no. 5, 2005.
- [12] A. Laghrouche et al., "Higher order sliding mode control based on integral sliding mode," *Automatica*, vol. 43, no. 3, 2007.
- [13] G. Bartolini et al., "On second order sliding mode controllers," in *Lecture Notes in Control and Information Sciences*, Springer, 1999.
- [14] J. Guldner and V.I. Utkin, "Tracking the gradient of artificial potentialfields: sliding mode control for mobile robots", *Int. J. Control*, Vol. 63(1996) No. 3, pp. 417-432.
- [15] A.S.I. Zinober, O.M.E. El-Ghezawi and S.A. Billings, "Multivariablestructureadaptive model-following control systems", *Proceedings ofIEE*, Vol. 129, pp. 6-12, 1982.

BibliographicReferences

- [16] J.J.E. Slotine, J.K. Hedrick and E.A. Misawa, "On sliding observers for nonlinear systems", *Transactions of the ASME: Journal of Dynamic Systems Measurement and Control*, Vol. 109, pp. 245-252, 1987.
- [17] K. Furuta, "Sliding mode control of a discrete system", *Systems and Control Letters*, Vol. 14, pp. 145-152, 1990.
- [18] H. Sira-Ramirez, "Differential geometric methods in variable-structure control", *Int. J. Control*, Vol. 48, No. 4, pp. 1359-1390, 1988.
- [19] R.A. De Carlo, S.H. Zak, G.P. Matthews, "Variable structure control of nonlinear variable systems: a tutorial", *Proceedings of IEEE*, Vol. 76, pp. 212-232, 1988.
- [20] C. Edwards and S. Spurgeon, *Sliding mode control: theory and applications*, Taylor and Francis, 1998.
- [21] U. Itkis, *Control systems of variable structure*, Wiley, New-York, 1976.
- [22] V.I. Utkin, *Sliding Modes in Control Optimization, Communication and Control Engineering Series*, Springer-Verlag, 1992.
- [23] Fossard, A. J., & Floquet, T. (2002). *Introduction and historical account*. In W. Perruquetti & J. P. Barbot (Eds.), *Sliding mode control in engineering* (pp. 1–44). Marcel Dekker.
- [24] B. Hamel, "Contribution a l'etude mathematique des systemes derelage par tout ou rien", (in French) 1949.
- [25] Y. Z. Tzypkin, *Theory of control relay systems*, Moscow: Gostekhizdat, 1955 (in Russian).
- [26] S.V. Emelyanov, "On peculiarities of variable structure control systems with discontinuous switching functions", *Doklady ANSSR*, Vol. 153, pp. 776-778, 1963.
- [27] Fossard, A. J., & Floquet, T. (2002). *Introduction: An Overview of Classical Sliding Mode Control*. In W. Perruquetti & J.-P. Barbot (Eds.), *Sliding Mode Control in Engineering* (pp. 1–5). New York: Marcel Dekker Inc.
- [28] [Utkin and Shi, 1996; Guldner and Utkin, 2000].
- [33] Emel'yanov, S.V., Korovin, S.K., & Levant, A. (1996). High-order sliding modes in control systems. *Computational Mathematics and Modeling*, 7(3), 294–318.
- [30] Slotine, J.J. (1984). Sliding controller design for non-linear systems. *International Journal of Control*, 40(2), 421–434.
- [29] Utkin, V.I. (1992). *Sliding Mode Control and Optimization*. Springer-Verlag, Berlin.

BibliographicReferences

- [32] Utkin, V.I., Guldner, J., & Shi, J. (1999). Sliding Mode in Control in Electromechanical Systems. Taylor and Francis, London.
- [34] Young, K.D., Utkin, V.I., & Ozguner, U. (1999). A control engineer's guide to sliding mode control. *IEEE Transactions on Control System Technology*, 7(3), 328–342.
- [36] Filippov, A. F. (1988). *Differential Equations with Discontinuous Right-Hand Side*. Kluwer Academic Publishers.
- [37] Levant, A. (1993). Sliding order and sliding accuracy in sliding mode control. *International Journal of Control*, 58(6), 1247–1263.
- [38] A. Levant. Sliding order and sliding accuracy in sliding mode control. *International Journal of Control*, 58(6):1247–1263, 1993. [cited at p. v, 17, 18, 19, 32]
- [39] Bregeault, V. *Quelques contributions à la théorie de la commande par modes glissants*. PhD thesis, Ecole Centrale de Nantes.
- [40] Levant, A. (1993). Sliding order and sliding accuracy in sliding mode control. *International Journal of Control*, 58(6), 1247–1263.
- [43] M. Defoort et al., **Systems and Control Letters**, 58(2):102–108, 2009.
- [42] F. Dinuzzo and A. Fererra, **IEEE Transactions on Automatic Control**, 54(9):2126–2136, 2009.
- [45] Y. Hong, **Systems and Control Letters**, 46(4):231–236, 2002.
- [41] S. Laghrouche et al., **Automatica**, 43(3):531–537, 2007.
- [44] A. Levant, **IEEE Transactions on Automatic Control**, 46(9):1447–1451, 2001.
- [47] A. Levant, **International Journal of Control**, 76(9/10):924–941, 2003.
- [48] A. Levant, **Automatica**, 41(5):823–830, 2005.
- [46] S.P. Bhat and D.S. Bernstein, **Mathematics of Control, Signals and Systems**, 17(2):101–127, 2005.
- [51] J. Kochalummoottil, Y.B. Shtessel, J.A. Moreno, and L. Fridman. Adaptive twist sliding mode control: A Lyapunov design. In *50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, 2011.
- [49] F. Plestan, Y. Shtessel, V. Brégeault, and A. Poznyak. New methodologies for adaptive sliding mode control. *International Journal of Control*, 83(9):1907 – 1919, 2010.

BibliographicReferences

- [50] Y. Shtessel, M. Taleb, and F. Plestan. A novel adaptive gain supertwisting sliding mode controller: methodology and application. *Automatica*, 48(5):759–769, 2011.
- [52] Harmouche, M. (2013). *Contribution to the theory of higher order sliding mode control and the control of underactuated mechanical systems* (Doctoral dissertation, Université de Technologie de Belfort-Montbéliard). Chapter 1, Section 1.1. Retrieved from <https://theses.hal.science/tel-01503945v1>
- [53] Harmouche, M. (2013). *Contribution to the theory of higher order sliding mode control and the control of underactuated mechanical systems* (Doctoral dissertation, Université de Technologie de Belfort-Montbéliard). Chapter 1, Section 1.2.
- [54] Francesco Dinuzzo and Antonella Ferrara, *Higher Order Sliding Mode Controllers With Optimal Reaching*, *IEEE Transactions on Automatic Control*, vol. 54, no. 9, pp. 2126–2136, Sept. 2009.
- [55] Francesco Dinuzzo and Antonella Ferrara, *Higher Order Sliding Mode Controllers With Optimal Reaching*, *IEEE Transactions on Automatic Control*, vol. 54, no. 9, pp. 2127–2136, Sept. 2009.
- [56] A. Isidori, *Nonlinear Control Systems*, 3rd ed. Berlin, Germany:Springer Verlag, 1995.
- [57] D. Liberzon, A. S. Morse, and E. D. Sontag, “Output-input stability and minimum-phase nonlinear systems,” *IEEE Trans. Automat. Control*, vol. 47, no. 3, pp. 422–436, Mar. 2002.
- [58] J. P. Aubin and A. Cellina, *Differential Inclusions, Set-Valued Maps and Viability Theory*. Berlin, Germany: Springer Verlag, 1984.
- [59] A. F. Filippov, *Differential Equations With Discontinuous Right-Hand Side*. Dordrecht, The Netherlands: Kluwer, 1988.
- [60] H. J. Sussman and Y. Yang, “On the stabilizability of multiple integrators by means of bounded feedback,” in *Proc. IEEE Conf. Decision Control*, 1991, vol. 1, pp. 70–73.
- [61] A. R. Teel, “Global stabilization and restricted tracking for multiple integrators with bounded control,” *IEEE Trans. Automat. Control*, vol. 43, no. 9, pp. 1365–1371, 1998.
- [62] A. S. I. Zinober, Ed., *Variable Structure and Lyapunov Control*. Berlin, Germany: Springer Verlag, 1994.
- [63] M. I. Zelikin and V. F. Borisov, *Theory of Chattering Control With Applications to Astronautics, Robotics, Economics, and Engineering*, ser. Systems and Control: Foundations and Application. Boston, MA: Birkhäuser, 1994.

BibliographicReferences

- [64] A. T. Fuller, “Relay control systems optimized for various performancecriteria,” in *Proc. 1st World Congress IFAC*, London, U.K., 1961, vol.1, pp. 510–519.
- [65] A. T. Fuller, “Minimization of various performance indices for asystem with bounded control,” *Int. J. Control*, vol. 41, no. 1, pp. 1–37,1985.
- [66] A. Levant, “Robust exact differentiation via sliding mode technique,”*Automatica*, vol. 34, no. 3, pp. 379–384, 1998.
- [67] G. Bartolini, S. Pilloso, A. Pisano, and E. Usai, “Time-optimal stabilizationfor a third-order integrator: A robust state-feedback implementation,”in *Dynamics, Bifurcations and Control*, ser. Lecture Notes inControl and Information Sciences, F. Colonius and L. Gruene, Eds. New York: Springer Verlag, 2002, vol. 273, pp. 131–144.
- [68] P. E. Grensted and A. T. Fuller, “Minimization of integral-square errorfor non-linear control system of third and higher order,” *Int. J. Control*,vol. 2, no. 1, pp. 33–73, 1965.
- [69] E. P. Ryan and C. M. Dorling, “Minimization of a non-quadratic costfunctional for third-order saturating control systems,” *Int. J. Control*,vol. 34, no. 2, pp. 231–258, 1981.
- [70] A. A. Feldbaum, “On synthesis of optimal systems with the help ofphase space,” *Avtomatika i Telemekhanika*, vol. 16, no. 2, pp. 129–149,1955.
- [71] U. Walther, T. T. Georgiou, and A. Tannenbaum, “On the computationof switching surfaces in optimal control: A Gröbner basis approach,”*IEEE Trans. Automat. Control*, vol. 46, no. 4, pp. 534–540, Apr. 2001

Utenuen 2015

[1] :V.M.L.Fadiga«Commande H^∞ àbasedemodèlesnonentiers».ThèsedeDoctorat.Universitédebordeaux. Soutenule 12juillet 2014

References (IEEE Style):

- [1] V. I. Utkin, *Sliding Modes in Optimization and Control Problems*, Springer, 1992.
- [2] C. Edwards and S. K. Spurgeon, *Sliding Mode Control: Theory and Applications*, Taylor & Francis, 1998.
- [3] V. I. Utkin et al., *Sliding Mode Control in Electromechanical Systems*, Taylor & Francis, 1999.
- [4] L. M. Fridman, “Singularly perturbed analysis of chattering in relay control systems,” *IEEE TAC*, vol. 47, no. 12, pp. 2079–2084, 2002.

BibliographicReferences

- [5] I. Boiko and L. M. Fridman, “Analysis of chattering in continuous sliding-mode controllers,” *IEEE TAC*, vol. 50, no. 9, 2005.
- [6] A. Levant, “Sliding order and sliding accuracy in sliding mode control,” *Int. J. Control*, vol. 58, no. 6, 1993.
- [7] G. Bartolini et al., “Chattering avoidance by second order sliding mode control,” *IEEE TAC*, vol. 43, no. 2, 1998.
- [8] G. Bartolini et al., “Multi-input chattering-free second order sliding mode control,” *IEEE TAC*, vol. 45, no. 9, 2000.
- [9] G. Bartolini et al., “Time-optimal stabilization for a third-order integrator,” *Lecture Notes in Control and Information Sciences*, vol. 273, Springer, 2002.
- [10] A. Levant, “Higher-order sliding modes, differentiation and output-feedback control,” *Int. J. Control*, vol. 76, no. 9/10, 2003.
- [11] A. Levant, “Quasi-continuous high-order sliding-mode controllers,” *IEEE TAC*, vol. 41, no. 5, 2005.
- [12] A. Laghrouche et al., “Higher order sliding mode control based on integral sliding mode,” *Automatica*, vol. 43, no. 3, 2007.
- [13] G. Bartolini et al., “On second order sliding mode controllers,” in *Lecture Notes in Control and Information Sciences*, Springer, 1999.

Titre :

Improved Reaching Time for Higher-Order Sliding Mode Controllers

الملخص :

تعالج هذه المذكرة موضوع تصميم المتحكمات بالانزلاق من الرتبة العليا (Higher Order Sliding Mode Controllers) للوصول الأمثل (Optimal Reaching) في الأنظمة غير الخطية ذات الدرجة النسبية المعروفة. تم اقتراح تعميم قوي لمشكلة فولر الكلاسيكية يُعرف بمشكلة "فولر القوية"، والتي تُمكن من اشتقاق قوانين تحكم تضمن الوصول في زمن نهائي إلى سطح الانزلاق مع تحقيق متطلبات الاستقرار والصلابة. يعرض العمل طريقة منهجية لضبط معالم التحكم ويؤسس إطاراً نظرياً موحداً يشمل العديد من خوارزميات HOSM المعروفة. تم التحقق من فعالية الطريقة المقترحة من خلال نتائج محاكاة تؤكد تفوق الأداء من حيث الاستجابة، تقليل الاهتزازات (Chattering) ، وزيادة مقاومة الاضطرابات.

Résumé :

Ce mémoire traite de la conception des contrôleurs en mode glissant d'ordre supérieur avec un objectif d'atteinte optimale dans des systèmes non linéaires à degré relatif constant. Un nouveau cadre théorique basé sur une généralisation robuste du problème classique de Fuller est proposé, permettant de dériver des lois de commande assurant une convergence en temps fini vers la surface de glissement, tout en garantissant robustesse et stabilité. Cette approche offre une méthode systématique pour le réglage des paramètres du contrôleur, évitant les ajustements empiriques classiques. Des simulations numériques démontrent la supériorité des contrôleurs développés, notamment en termes de rapidité de convergence, réduction du chattering, et résistance aux perturbations

Abstract :

This thesis addresses the design of Higher Order Sliding Mode (HOSM) controllers with optimal reaching properties for nonlinear systems with known relative degree. A robust extension of the classical Fuller's problem—termed the Robust Fuller's Problem—is introduced to derive control laws that ensure finite-time convergence to the sliding manifold while maintaining robustness against uncertainties. The proposed approach provides a systematic parameter tuning method and unifies various existing HOSM algorithms within a generalized theoretical framework. Simulation results validate the superior performance of the developed controllers in terms of convergence speed, chattering reduction, and disturbance rejection.