RÉPUBLIQUE DÉMOCRATIQUE POPULAIRE ALGÉRIENNE
MINISTÈRE D'ENSEIGNEMENT SUPÉRIEUR ET DE RECHERCHE SCIENTIFIQUE
UNIVERSITÉ DE M'SILA
FACULTÉ DE MATHÉMATIQUES ET DE L'INFORMATIQUE
DÉPARTEMENT DE MATHÉMATIQUES

THÈSE

Présenté pour l'obtention de grade de docteur en sciences
Spécialité:
Mathématiques
Option:
Mathématiques Appliquées

Par:
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Intitulée:

Sur le Comportement Asymptotique des Solutions de Certaines Équations aux Dérivées Partielles avec un Paramètre

Soutenu publiquement le : 03 / 06 / 2012, devant le jury :

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Année universitaire: 2011 / 2012

Thesis presented to obtain the doctoral degree in sciences
Speciality: Mathematics
Option: Applied Mathematics

By:

Abdelmouhcene Sengouga

Titled:

On the Asymptotic Behaviour of the Solutions of Some Partial Differential Equations with a Parameter

Supervisors: Senoussi Guesmia & Nabil Merazga

University of M’sila, 2012.

ACKNOWLEDGEMENTS

First of all, I would like to thank Dr. Senoussi Guesmia for his advice, support and patience during the realization of this work. Especially his careful reading of the final version and for insightful discussions and constructive comments, which were of great value.

Furthermore, my special thanks to Professor Nabil Merazga who kindly accepted to be the supervisor of this thesis.

I wish to express my sincere thanks to Professors N. Benhamidouche, A. Aibeche, M. S. Moulay and M. Moussai who kindly agreed to serve on my thesis committee.

My scientific work benefited significantly from three very enjoyable and fruitful research stays and visits funded by the university of M'sila. This support is gratefully acknowledged. The first training visit was to the Laboratory of Mathematics, Informatics and Applications (L.I.M.A) at the university of Haute Alsace, Mulhouse, France. I am grateful to Professor Bernard Brighi who made this visit possible. The second stay was at the institute of mathematics, university of Zurich, Switzerland. Many thanks go to Professor Michel Chipot and Dr. S. Guesmia for there invitation and welcome. The third stay was at the Abdus Salam International Center of Theoretical Physics (I.C.T.P) in Trieste, Italy, where I had the opportunity to finalize some results, together with Prof. M. Chipot and Dr. S. Guesmia.

This thesis is typeset using LaTeX2$_{\varepsilon}$ and is based on a version of the *ClassicThesis* template, with many modifications.

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ABSTRACT

In this thesis we deal with anisotropic singular perturbations of linear as well as nonlinear problems, depending on a small parameter $\varepsilon > 0$. First we consider an abstract approach to some singular perturbations of variational inequalities, involving nonlinear operators defined on Banach spaces, then we describe the asymptotic behaviour of the solution when $\varepsilon \to 0$. Next, the obtained abstract results are applied to some boundary value problems. Moreover, we consider a variational inequality involving the $p-$Laplacian operator with a perturbation that also includes the convex set. In the same framework of anisotropic perturbations, we investigate some linear and semilinear evolution problems of hyperbolic type.

Keywords and phrases:

Anisotropic singular perturbations, asymptotic behaviour, hyperbolic problems, variational inequalities, qualitative properties of solutions, monotone operators, $p-$Laplacian, quasilinear problems.

العنوان: حول السّلوك المقارب لحلول بعض المعادلات التفاضليّة الجزئيّة ذات معامل

ملخص:

هذه الرّسالة تتناول بالدّراسة الإضطرابات المتباينة الشاذّة، لمسائل خطّية وغير خطّية، تتعلّق بمعامل صغير $\varepsilon > 0$. أولا ننتهج مقاربة مجرّدة لبعض الإضطرابات المتباينة الشاذّة لمتراجحات تغايريّة، تنطوي على مؤثّرات غير خطّية معرّفة على فضاءات باناخ، ثمّ نقوم بوصف السّلوك المقارب للحلّ لمّا $\varepsilon \to 0$. بعد ذلك نقوم بتطبيق النّتائج المجرّدة المتحصّل عليها على بعض المسائل ذات القيم الحدّية. علاوة على ذلك، فإننا ندرس متراجحة تغايريّة تنطوي على المؤثر p-لابلاس مع اضطراب يشمل أيضا المجموعة المحدّبة. دائما في إطار دراسة الإضطرابات المتباينة، نتناول بعض المسائل الخطّية ونصف الخطّية المتعلّقة بالزّمن من النّوع الزّائدي.

كلمات مفتاحية :

الإضطرابات المتباينة الشاذّة، السّلوك المقارب، المسائل من النّوع الزّائدي، المتراجحات التغايريّة، الخصائص النّوعيّة للحلول، المؤثّرات الرّتيبة، المؤثّر p-لابلاس، المسائل شبه الخطّية.

Titre: **Sur le Comportement Asymptotique des Solutions de Certaines Équations aux Dérivées Partielles avec un Paramètre**

RÉSUMÉ:

Dans cette thèse on traite des perturbations singulières anisotropes de problèmes linéaires et non linéaires qui dépendent d'un petit paramètre $\varepsilon > 0$. D'abord nous considérons une approche abstraite de certaines inégalités variationnelles, faisant intervenir des opérateurs non linéaires définis sur des espaces de Banach, puis nous décrivons le comportement asymptotique de la solution lorsque $\varepsilon \to 0$. Ensuite, ces résultats abstraits sont appliqués à quelques problèmes aux limites. En outre, nous considérons une inégalité variationnelle faisant intervenir l'opérateur $p-$Laplacien avec une perturbation qui comprend également l'ensemble convexe. Dans le même cadre de perturbations anisotropes, on considère quelques problèmes d'évolution, linéaires et semi-linéaires, de type hyperbolique.

Mots Clés:

Perturbations singulières anisotropes, comportement asymptotique, problèmes hyperboliques, inégalités variationnelles, propriétés qualitatives des solutions, opérateurs monotones, $p-$Laplacien, problèmes quasi linéaires.

CONTENTS

INTRODUCTION

SINGULAR PERTURBATIONS

The subject of applied mathematics is to study natural processes by constructing mathematical models that describe all the essential features of these processes. In many cases, the model involves some parameters that can be considered as ″small″ ones. In order to simplify such models, and their analysis, we may neglect theses small parameters. However, some questions will come up. For instance, we wonder if neglecting these small parameters, in the model, will have also a small effect on its accuracy, or may causes total modifications in the characters of the model and its behaviour. Answering to this question is one of the main issues in asymptotic analysis.

The theory of singular perturbations is an important and active topic in asymptotic analysis and applied mathematics, it has its beginnings in the treatment of fluid dynamics problems towards the end of 19th century (see for instance Van Dyke [67]), then its use spread to other areas of mathematical physics, engineering and natural sciences. A wealth of techniques and results of this theory can be found described in the books of Lions [48], Nayfeh [58], O'Malley [59], Vasil'eva, Butuzov, and Kalachev [68]. Surveys on singularly perturbed problems and there numerical solutions are given in Kadalbajoo and Gupta [40], Kadalbajoo and Patidar [41], Mo and Ni [53]. See also the references cited therein.

This thesis deals with mathematical models that are formulated as partial differential equations, depending on one small parameter $\varepsilon > 0$. Let us consider a model formally described by the equation

$$\begin{cases} L_\varepsilon\left(u_\varepsilon\right) = 0, \\ + \text{ some conditions,} \end{cases} \tag{1}$$

where L_ε is a partial differential operator that involves ε, and u_ε is a solution to this problem, which is called the perturbed problem. Omitting the small parameter, i.e. setting $\varepsilon = 0$ in (1), we obtain

$$\begin{cases} L_0\left(u_0\right) = 0, \\ + \text{ some conditions.} \end{cases} \tag{2}$$

which is called the unperturbed (or the limit) problem. In general, solutions of Problem (2) will not be as smooth as the solutions of the perturbed Problem (1), and do not satisfy the same conditions, i.e. some conditions in Problem (1) will be lost in the passage to the limit. So it is not a priori sure whether the solution u_0 can be a good approximation of u_ε when $\varepsilon \to 0$.

ANISOTROPIC SINGULAR PERTURBATION

There are many different types of perturbations treated in the literature. Considering a PDE defined on a bounded domain $\Omega \subset \mathbb{R}^n$ (n is an integer), the perturbations can occur in the coefficients of the differential operator, in the source term, in some parts of the domain or its boundary,... In this thesis we have more particularly in mind anisotropic cases of singular perturbations of PDE, where the parameter of perturbation ε only acts on some of the directions of the domain variables. To be more specific we can take, as a model, the diffusion problem defined on the unit square $\Omega := (0,1) \times (0,1)$

$$\begin{cases} -\varepsilon\, \partial^2_{x_1} u_\varepsilon - \partial^2_{x_2} u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\varepsilon > 0$ and f represents the source term. We assume that the diffusion in the x_1-direction is negligible with respect to the other direction when $\varepsilon \to 0$. This is what we call an anisotropic singular perturbation. Formally the natural limit of u_ε is a function u_0 defined on the sections $\{x_1\} \times (0,1)$ for a.e. $x_1 \in (0,1)$ as a solution of

$$\begin{cases} -\partial^2_{x_2} u_0\left(x_1, \cdot\right) = f\left(x_1, \cdot\right) & \text{in } \left(0,1\right), \\ u_0\left(x_1, \cdot\right) = 0 & \text{on } \left\{0,1\right\}. \end{cases} \tag{4}$$

Note that the variable x_1 plays a role of a parameter. In this case it is clear that if f (not identically equal to 0) is independent of x_1, i.e. $f = f\left(x_2\right)$, then $u_0 \notin H^1_0\left(\Omega\right)$ which prevents the convergence $u_\varepsilon \to u_0$ to occur in Sobolev space $H^1\left(\Omega\right)$, and a boundary layer occurs at the lateral boundary of cylindrical domains. So, from this remark we may discuss many issues concerning this convergence.

Unlike the isotropic case, the image is far to be complete for the anisotropic singular perturbations theory. Linear elliptic, parabolic and hyperbolic problems defined on arbitrary domains are analyzed in different contexts and the convergence $u_\varepsilon \to u_0$ is obtained in different norms. Even we do not have in general the convergence on $H^1(\Omega)$, the convergence in Sobolev spaces may be shown in regions far from this lateral boundary. We can see this clearly when our perturbed problem satisfies some cylindrical symmetries. This means that $f = f(x_2)$ in the above example. In this case u_ε converges towards u_0 at an exponential rate. For more details we refer to the works Brighi and Guesmia [7], Chipot [11, 13], Chipot and Guesmia [15, 16, 17, 18], Guesmia [31, 33]. See also Laevsky [44] where the author investigated, in a simple two-dimensional elliptic problem, the influence of the shape of the domain on the rate of convergence.

A number of papers concerning numerical methods for anisotropic singularly perturbed problems, defined on cylindrical domains in 2 and 3 dimensions, has been published in recent years. For instance, Apel, Nicaise, and Sirch [1], Li [45], Li and Wheeler [46], Zhu and Chen [70] used finite elements and achieve convergence, of the numeric approximation, uniformly in the perturbation parameter ε. See also Braianov and Vulkov [4], Degond, Deluzet, and Negulescu [24] and related works.

SINGULAR PERTURBATIONS OF VARIATIONAL INEQUALITIES

In order to give a general abstract approach to singular perturbation problems some authors, see for instance Sanchez-Palencia [61], Zhang [69], considered the following operator equation

$$\varepsilon A u_\varepsilon + B u_\varepsilon = f, \tag{5}$$

where A and B are linear operators defined on Hilbert spaces. This approach covers diagonal structure problems as problem (3). As in the case of partial differential equations, it is shown that u_ε converges towards u_0 solution to

$$B u_0 = f,$$

when $\varepsilon \to 0$. There are also some previous works on singular perturbations of variational inequalities, i.e. when (5) is replaced by

$$(\varepsilon A u_\varepsilon, v - u_\varepsilon) + (B u_\varepsilon, v - u_\varepsilon) \geq (f, v - u_\varepsilon), \quad \forall v \in K, \tag{6}$$

where K is some nonempty closed convex set and $(\cdot, \cdot)$ is the scalar product. The study of (isotropic) singular perturbations of variational inequalities, was initiated in Lions and

Stampacchia [50], for A and B linear then in Browder [8] for some nonlinear operators, see also Lions [48], Rodrigues [60], Stampacchia [63] and related works.

CONTRIBUTIONS OF THE PRESENT THESIS

This thesis offers an asymptotic analysis of some important classes of problems with anisotropic singular perturbations. In particular, we are interested in nonlinear problems since most phenomena are originally described by nonlinear models. Their anisotropic singular perturbations is very interesting and requires some special tools of functional analysis.

We start by dealing with the variational inequality (6) when A and B are both nonlinear operators defined on different Banach spaces $\mathcal{V}$ and $\mathcal{W}$ respectively. As far as we know, the abstract approach presented in this thesis is different from all the previous works. It covers both isotropic and anisotropic cases. This is emphasized by many applications and examples introduced in order to illustrate some points of the theory as, for instance, the lack of compactness which prevents dealing with nonlinear terms directly.

In the same framework of anisotropic singular perturbations, asymptotic behaviour of the solutions of linear as well as semilinear evolution problems, of hyperbolic type, is investigated in the second part of this thesis. To be more clear, let us consider the hyperbolic evolution version of problem (3), i.e. a wave problem where the propagation speed of the wave is very small in the x_1 direction, defined as

$$\begin{cases} u_\varepsilon'' - \varepsilon\, \partial_{x_1}^2 u_\varepsilon - \partial_{x_2}^2 u_\varepsilon = f & \text{in } (0,T) \times (0,1)^2, \\ u_\varepsilon = 0 & \text{on } (0,T) \times \partial (0,1)^2, \\ u_\varepsilon(0) = u^0, u_\varepsilon'(0) = u^1 & \text{in } (0,1)^2. \end{cases} \tag{7}$$

The primes stand for derivatives in time, $T > 0$, u^0 and u^1 are initial conditions and f represents the source term (for instance f represents the force driving a wave on a membrane, the charge and the current density in the Lorenz gauge of electromagnetism). As in the above stationary problem, we are interested in the limit behaviour of u_ε, when $\varepsilon \to 0$. The natural candidate is u_0 defined on the sections $(0,T) \times \{x_1\} \times (0,1)$, for a.e. $x_1 \in (0,1)$, as a solution of

$$\begin{cases} u_0''(\cdot; x_1, \cdot) - \partial_{x_2}^2 u_0(\cdot; x_1, \cdot) = f(\cdot; x_1, \cdot) & \text{in } (0,T) \times (0,1), \\ u_0(\cdot; x_1, \cdot) = 0 & \text{on } (0,T) \times \{0,1\}, \\ u_0(0; x_1, \cdot) = u^0(x_1), u_0'(0; x_1, \cdot) = u^1(x_1) & \text{in } (0,1). \end{cases}$$

In Brighi and Guesmia [7], Guesmia [33] a polynomial rate of the convergence $u_\varepsilon \to u_0$ is shown but for a quite restricted class of problems. The argument is more detailed in Guesmia [32]. More general results for elliptic and parabolic problems, defined on arbitrary domains, have been shown in Chipot and Guesmia [15, 16, 17, 18]. In these latter papers, as well as here, when the domain and the data have some cylindrical symmetries the rate of convergence can be improved, whereas the dependence of the limit solution on all directions negatively effects the rate of convergence. Furthermore, some difficulties arise, especially here for the hyperbolic problems.

Using some weak convergence methods, combined with a monotonicity argument, we can investigate the same issues for some semilinear hyperbolic problems, for instance the following semilinear version of problem (7) defined as

$$\begin{cases} u_\varepsilon'' - \varepsilon\, \partial_{x_1}^2 u_\varepsilon - \partial_{x_2}^2 u_\varepsilon + \beta\left(u_\varepsilon'\right) = f & \text{in } (0,T) \times (0,1)^2, \\ u_\varepsilon = 0 & \text{on } (0,T) \times \partial\,(0,1)^2, \\ u_\varepsilon(0) = u^0, u_\varepsilon'(0) = u^1 & \text{in } (0,1)^2, \end{cases}$$

where β is a nonlinear operator. For regions located far from the lateral boundary of cylindrical domains, a convergence in a Sobolev space and some improvements of the rate of convergence are shown. Moreover, using an iterative technique, introduced in Chipot and Rougirel [19] and improved in Chipot and Yeressian [20], an exponential rate of convergence is carried out instead of the polynomial one in Brighi and Guesmia [7], Guesmia [33].

STRUCTURE OF THESIS

After the present introduction, the rest of the material of this thesis is divided into two parts. The first part includes three chapters and the second one two chapters. The main definitions, lemmas, theorems, corollaries, propositions, remarks and equations are labelled by two digits where the first one indicates the chapter.

The first part is titled *Perturbations of Abstract and Stationary Problems*. In Chapter 1, we introduce the classes of perturbed abstract problems we intend to study, i.e. variational inequalities involving nonlinear operators defined on different abstract Banach spaces. General convergence results are established in an abstract setting, then improved in some cases. In Chapter 2, we give some direct applications of the abstract results obtained in the precedent chapter. To emphasize the power of our analysis, the considered cases involve an obstacle problem and quasilinear elliptic problems. Some interesting linear elliptic cases with mixed boundary conditions are brought to attention. In Chapter 3, we consider a variational inequality, that involves a perturbed

$p-$Laplacian, which can not be treated in the abstract setting of Chapter 1 since it is not possible to write the operator as the sum of two operators, i.e. on the form $\varepsilon A + B$, and of course since ε appears also in the convex set.

The second part, titled *Perturbations of Evolution Problems of Hyperbolic Type*, is concerned with perturbed problems depending on time. In Chapter 4, which is the first chapter of this part, we discuss in details the asymptotic behaviour of the solution of linear hyperbolic problems, with an anisotropic singular perturbation, and obtain some convergence results in arbitrary domains. Also, when the domain is cylindrical, we establish a regularity result for the solutions and improve the rate of convergence far from the lateral boundaries of the domain. In chapter 5, we study the case of semilinear hyperbolic problems and obtain the convergences, and improve them under some considerations.

The notations and definitions used in this thesis are standard and commonly used. The reader can find them, for instance, in the textbooks Chipot [13], Dautray and Lions [22]. However, to make this thesis consistent and to avoid ambiguity, a list of notations and definitions is included in the appendix, as well as some elementary inequalities frequently used in through this work.

We assume that the reader is familiar with partial differential equations theory and basic functional analysis (Sobolev spaces, weak topologies, reflexive spaces, differential operators,...). Some of the relevant definitions and concepts will be recalled in the sequel, but the reader who wants a more thorough information may consult Brezis [6], Chipot [13], Dautray and Lions [22], Evans [26].

Part I

Perturbations of Abstract and Stationary Problems

1

AN ABSTRACT APPROACH TO ANISOTROPIC SINGULAR PERTURBATIONS

In this chapter we deal with some perturbations of nonlinear operators, defined on different abstract Banach spaces, which in particular will be applied to anisotropic singular perturbations problems. In order to make our approach as general as possible, we investigate the asymptotic behaviour of perturbed variational inequalities for nonlinear operators. We shall give some general convergence theorems, then we improve these results when some assumptions are satisfied.

1.1 PROBLEM SETTING

Let $\mathcal{V}$ and $\mathcal{W}$ be two reflexive separable Banach spaces equipped with the norms $|\cdot|_{\mathcal{V}}$ and $|\cdot|_{\mathcal{W}}$ respectively. We suppose that the space $\mathcal{V}\cap\mathcal{W}$ is dense with continuous inclusion in $\mathcal{V}$ and $\mathcal{W}$, and is equipped with the norm

$$|\cdot|_{\mathcal{V}\cap\mathcal{W}} := |\cdot|_{\mathcal{V}} + |\cdot|_{\mathcal{W}}.$$

It is clear that

$$\mathcal{V}\cap\mathcal{W} \subset \mathcal{V},\mathcal{W} \ \text{ and } \ \mathcal{V}',\mathcal{W}' \subset (\mathcal{V}\cap\mathcal{W})'.$$

Moreover one can check that (see [3, 66])

$$(\mathcal{V}\cap\mathcal{W})' = \mathcal{V}' + \mathcal{W}'.$$

We consider two nonlinear operators A and B such that

$$A : \mathcal{V} \to \mathcal{V}', \quad B : \mathcal{W} \to \mathcal{W}'.$$

We suppose that A, B are *monotone*, that is to say

$$\langle Au - Av, u - v \rangle_{\mathcal{V}} \geq 0, \qquad \forall u, v \in \mathcal{V}, \tag{1.1}$$

$$\langle Bu - Bv, u - v \rangle_{\mathcal{W}} \geq 0, \qquad \forall u, v \in \mathcal{W}, \tag{1.2}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{W}}$ stands for the dual products of $\mathcal{V}$ and $\mathcal{W}$ respectively. We denote by $K \neq \varnothing$ a closed convex set of $\mathcal{V} \cap \mathcal{W}$ and for A, B we make the following *coerciveness* assumption. We suppose that for some $v_0 \in K$ one has

$$\frac{\langle Au - Av_0, u - v_0 \rangle_{\mathcal{V}}}{|u - v_0|_{\mathcal{V}}} \to +\infty \quad \text{when } |u|_{\mathcal{V}} \to +\infty, \ \ u \in K, \tag{1.3}$$

$$\frac{\langle Bu - Bv_0, u - v_0 \rangle_{\mathcal{W}}}{|u - v_0|_{\mathcal{W}}} \to +\infty \quad \text{when } |u|_{\mathcal{W}} \to +\infty, \ \ u \in K. \tag{1.4}$$

Remark 1.1 *If K is bounded in $\mathcal{V}$ (resp. in $\mathcal{W}$) we will not need the assumption (1.3) (resp. (1.4)). Note also that for some $v_0 \in K$, (1.3) and (1.4) are equivalent to*

$$\frac{\langle Au, u - v_0 \rangle_{\mathcal{V}}}{|u - v_0|_{\mathcal{V}}} \to +\infty \quad \textit{when } |u|_{\mathcal{V}} \to +\infty, \ \ u \in K, \tag{1.5}$$

$$\frac{\langle Bu, u - v_0 \rangle_{\mathcal{W}}}{|u - v_0|_{\mathcal{W}}} \to +\infty \quad \textit{when } |u|_{\mathcal{W}} \to +\infty, \ \ u \in K. \tag{1.6}$$

In addition we assume that A and B are *bounded*, i.e.

A sends bounded sets of $\mathcal{V}$ in bounded sets of $\mathcal{V}'$,

B sends bounded sets of $\mathcal{W}$ in bounded sets of $\mathcal{W}'$,

and A, B are *hemicontinuous* on $\mathcal{V}$ and $\mathcal{W}$ respectively, i.e.

$$t \mapsto \langle A(u + tv), w \rangle_{\mathcal{V}} \quad \text{is continuous on } \mathbb{R}, \ \ \forall u, v, w \in \mathcal{V}.$$

$$t \mapsto \langle B(u + tv), w \rangle_{\mathcal{W}} \quad \text{is continuous on } \mathbb{R}, \ \ \forall u, v, w \in \mathcal{W}.$$

Under the assumptions above, we have the following assertion.

Theorem 1.1 *For $f \in (\mathcal{V} \cap \mathcal{W})'$ and $\varepsilon > 0$ there exists u_ε solution to*

$$\begin{cases} \varepsilon \langle Au_\varepsilon, v - u_\varepsilon \rangle_{\mathcal{V}} + \langle Bu_\varepsilon, v - u_\varepsilon \rangle_{\mathcal{W}} \geq \langle f, v - u_\varepsilon \rangle_{\mathcal{V} \cap \mathcal{W}}, & \forall v \in K, \\ u_\varepsilon \in K. \end{cases} \tag{1.7}$$

Moreover if A or B is strictly monotone (i.e. if one of the inequalities (1.1), (1.2) is strict for $u \neq v$) the solution is unique.

Proof. One consider the operator $\mathcal{A}_\varepsilon$ defined by

$$\begin{aligned}\mathcal{A}_\varepsilon : \mathcal{V}\cap\mathcal{W} &\to (\mathcal{V}\cap\mathcal{W})' = \mathcal{V}'+\mathcal{W}', \\ v &\mapsto \varepsilon Av + Bv,\end{aligned}$$

then the existence of u_ε will follows from the classical theory of variational inequalities (cf. [42, 47]), since this operator is monotone, hemicontinuous and coercive. To clarify this last point[1] one notice that

$$\begin{aligned}\frac{\langle \varepsilon Au + Bu, u-v_0\rangle_{\mathcal{V}\cap\mathcal{W}}}{|u-v_0|_{\mathcal{V}\cap\mathcal{W}}} &= \frac{|u-v_0|_{\mathcal{V}}}{|u-v_0|_{\mathcal{V}}+|u-v_0|_{\mathcal{W}}}\cdot\frac{\langle \varepsilon Au, u-v_0\rangle_{\mathcal{V}}}{|u-v_0|_{\mathcal{V}}} \\ &\quad + \frac{|u-v_0|_{\mathcal{W}}}{|u-v_0|_{\mathcal{V}}+|u-v_0|_{\mathcal{W}}}\cdot\frac{\langle Bu, u-v_0\rangle_{\mathcal{W}}}{|u-v_0|_{\mathcal{W}}}. \qquad (1.8)\end{aligned}$$

Let $(u_n)_n$ be a sequence such that

$$\lim_{|u_n|_{\mathcal{V}\cap\mathcal{W}}\to+\infty} \frac{\langle \varepsilon Au_n + Bu_n, u_n-v_0\rangle_{\mathcal{V}\cap\mathcal{W}}}{|u_n-v_0|_{\mathcal{V}\cap\mathcal{W}}} = \liminf_{|u|_{\mathcal{V}\cap\mathcal{W}}\to+\infty} \frac{\langle \varepsilon Au + Bu, u-v_0\rangle_{\mathcal{V}\cap\mathcal{W}}}{|u-v_0|_{\mathcal{V}\cap\mathcal{W}}},$$

then, having $|u_n|_{\mathcal{V}}+|u_n|_{\mathcal{W}}\to+\infty$, at least one of the sequences $|u_n|_{\mathcal{V}}$, $|u_n|_{\mathcal{W}}$ is unbounded. For instance if $|u_n|_{\mathcal{V}}\to+\infty$, or eventually for a subsequence. **If** $|u_n|_{\mathcal{W}}$ is also unbounded then -up to a subsequence- $|u_n|_{\mathcal{W}}\to+\infty$ and, for an arbitrary M, one has

$$\frac{\langle \varepsilon Au_n, u_n-v_0\rangle_{\mathcal{V}}}{|u_n-v_0|_{\mathcal{V}}}, \quad \frac{\langle Bu_n, u_n-v_0\rangle_{\mathcal{W}}}{|u_n-v_0|_{\mathcal{W}}} \geq M$$

for n large enough. Thus it follows from (1.8) that

$$\frac{\langle \varepsilon Au_n + Bu_n, u_n-v_0\rangle_{\mathcal{V}\cap\mathcal{W}}}{|u_n-v_0|_{\mathcal{V}\cap\mathcal{W}}} \geq M,$$

which means that the limit is $+\infty$. **Else,** i.e. if $|u_n|_{\mathcal{W}}$ is bounded, so $\langle Bu_n, u_n-v_0\rangle_{\mathcal{W}}$ is also bounded, then for large n the second term in the right hand side of (1.8), written for u_n, goes to 0 when $n\to+\infty$ whereas the first term goes to $+\infty$.

In all cases we have

$$\lim_{|u_n|_{\mathcal{V}\cap\mathcal{W}}\to+\infty} \frac{\langle \varepsilon Au_n + Bu, u_n-v_0\rangle_{\mathcal{V}\cap\mathcal{W}}}{|u_n-v_0|_{\mathcal{V}\cap\mathcal{W}}} = +\infty.$$

The same argument can be applied when $|u_n|_{\mathcal{V}}$ is bounded and $|u_n|_{\mathcal{W}}$ is unbounded, and the coercivity of $\mathcal{A}_\varepsilon$ on $\mathcal{V}\cap\mathcal{W}$ follows.

To check the uniqueness when A or B is strictly monotone, let u_1 and u_2 be two possible solutions, then it comes

$$\begin{aligned}\varepsilon\langle Au_1, u_1-u_2\rangle_{\mathcal{V}} + \langle Bu_1, u_1-u_2\rangle_{\mathcal{W}} &\leq \langle f, u_1-u_2\rangle_{\mathcal{W}}, \\ -\varepsilon\langle Au_2, u_1-u_2\rangle_{\mathcal{V}} - \langle Bu_2, u_1-u_2\rangle_{\mathcal{W}} &\leq -\langle f, u_1-u_2\rangle_{\mathcal{W}}\end{aligned}$$

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[1] See [21] for another elementary proof.

then adding we get

$$\varepsilon \langle Au_1 - Au_2, u_1 - u_2 \rangle_{\mathcal{V}} + \langle Bu_1 - Bu_2, u_1 - u_2 \rangle_{\mathcal{W}} \leq 0.$$

By the strict monotonicity of A or B we must have $u_1 = u_2$. This end the proof of the theorem. ■

Remark 1.2 *When $K = \mathcal{V} \cap \mathcal{W}$ one can see by taking $v = u_\varepsilon \pm w, w \in K$, that u_ε is solution to the abstract equation*

$$\begin{cases} \varepsilon A u_\varepsilon + B u_\varepsilon = f, \\ u_\varepsilon \in \mathcal{V} \cap \mathcal{W}. \end{cases} \tag{1.9}$$

1.2 CONVERGENCE THEOREMS

We are now interested in studying the behaviour of u_ε when $\varepsilon \to 0$. Note that u_ε may have no limit. Indeed, taking for instance $\mathcal{V}$ a Hilbert space, $A =$ the identity, $B = 0, f \in \mathcal{V}' = \mathcal{V}$ we can see that the solution of (1.9) is given by $u_\varepsilon = \frac{1}{\varepsilon} f$ which has no limit.

In what follows we will assume that

$$f \in \mathcal{W}'.$$

The essential convergences are given as follows.

Theorem 1.2 *Let u_ε be solution to (1.7). Then we have, when $\varepsilon \to 0$,*

$$(i)\ u_\varepsilon \text{ is bounded in } \mathcal{W} \text{ independently of } \varepsilon, \tag{1.10}$$

$$(ii)\ \varepsilon u_\varepsilon \to 0 \text{ in } \mathcal{V}, \tag{1.11}$$

$$(iii)\ \varepsilon A u_\varepsilon \to 0 \text{ in } \mathcal{V}', \tag{1.12}$$

$$(iv)\ \langle \varepsilon A u_\varepsilon, u_\varepsilon \rangle_{\mathcal{V}} \to 0. \tag{1.13}$$

Proof. *Proof of (i)*. Choose $v_0 \in K$, such that (1.5) and (1.6) hold. Suppose that $|u_\varepsilon|_{\mathcal{W}}$ is unbounded. For some sequence $\varepsilon_k \to 0$ one then has $|u_{\varepsilon_k} - v_0|_{\mathcal{W}} \to +\infty$. Taking $v = v_0$ in (1.7) we derive

$$\begin{aligned} \varepsilon_k \langle A u_{\varepsilon_k}, u_{\varepsilon_k} - v_0 \rangle_{\mathcal{V}} + \langle B u_{\varepsilon_k}, u_{\varepsilon_k} - v_0 \rangle_{\mathcal{W}} &\leq \langle f, u_{\varepsilon_k} - v_0 \rangle_{\mathcal{W}} \\ &\leq |f|_{\mathcal{W}'} |u_{\varepsilon_k} - v_0|_{\mathcal{W}}. \end{aligned}$$

It follows that

$$\frac{\langle B u_{\varepsilon_k}, u_{\varepsilon_k} - v_0 \rangle_{\mathcal{W}}}{|u_{\varepsilon_k} - v_0|_{\mathcal{W}}} \leq |f|_{\mathcal{W}'} - \frac{\varepsilon_k \langle A u_{\varepsilon_k}, u_{\varepsilon_k} - v_0 \rangle_{\mathcal{V}}}{|u_{\varepsilon_k} - v_0|_{\mathcal{W}}}. \tag{1.14}$$

Due to the coerciveness of B, we obtain a contradiction by showing that the right hand side is bounded from above. Indeed, if $|u_{\varepsilon_k}|_{\mathcal{V}}$ is bounded, then so is $|Au_{\varepsilon_k}|_{\mathcal{V}'}$, we have

$$\frac{\left|\varepsilon_k \left\langle Au_{\varepsilon_k}, u_{\varepsilon_k} - v_0\right\rangle_{\mathcal{V}}\right|}{|u_{\varepsilon_k} - v_0|_{\mathcal{W}}} \le |Au_{\varepsilon_k}|_{\mathcal{V}'} \left(|u_{\varepsilon_k}|_{\mathcal{V}} + |v_0|_{\mathcal{V}}\right) \frac{\varepsilon_k}{|u_{\varepsilon_k} - v_0|_{\mathcal{W}}} \longrightarrow 0,$$

else, i.e. if $|u_{\varepsilon_k} - v_0|_{\mathcal{V}} \to +\infty$, by the coerciveness of A one has

$$\varepsilon_k \left\langle Au_{\varepsilon_k}, u_{\varepsilon_k} - v_0\right\rangle_{\mathcal{V}} \ge 0,$$

for some k large enough. In both cases the contradiction holds in (1.14), which proves (1.10).

Proof of (ii). We derive from (1.7), written for $v = v_0$, that

$$\varepsilon \left\langle Au_\varepsilon, u_\varepsilon - v_0\right\rangle_{\mathcal{V}} \le \left(|Bu_\varepsilon|_{\mathcal{W}'} + |f|_{\mathcal{W}'}\right) |u_{\varepsilon_k} - v_0|_{\mathcal{W}}.$$

Since u_ε is bounded in $\mathcal{W}$, and so is $|Bu_\varepsilon|_{\mathcal{W}'}$, we get

$$\varepsilon \left\langle Au_\varepsilon, u_\varepsilon - v_0\right\rangle_{\mathcal{V}} \le C, \tag{1.15}$$

where here and in the following C is a positive constant independent of ε that can take different values. If u_ε is bounded in $\mathcal{V}$ then (1.11) is obvious. Else we have from (1.5), (1.15) -up to a subsequence-

$$\varepsilon |u_\varepsilon|_{\mathcal{V}} \le \varepsilon |u_\varepsilon - v_0|_{\mathcal{V}} + \varepsilon |v_0|_{\mathcal{V}} \le \frac{C\, |u_\varepsilon - v_0|_{\mathcal{V}}}{\left\langle Au_\varepsilon, u_\varepsilon - v_0\right\rangle_{\mathcal{V}}} + \varepsilon |v_0|_{\mathcal{V}} \longrightarrow 0$$

and the result follows.

Proof of the weak convergence $\varepsilon Au_\varepsilon \rightharpoonup 0$ in $\mathcal{V}'$. Let $v \in \mathcal{V}$, then by the monotonicity of A we have

$$\begin{aligned} \varepsilon \left\langle Au_\varepsilon, v - v_0\right\rangle_{\mathcal{V}} &= \varepsilon \left\langle Au_\varepsilon, u_\varepsilon - v_0\right\rangle_{\mathcal{V}} + \varepsilon \left\langle Au_\varepsilon, v - u_\varepsilon\right\rangle_{\mathcal{V}} \\ &\le \varepsilon \left\langle Au_\varepsilon, u_\varepsilon - v_0\right\rangle_{\mathcal{V}} + \varepsilon \left\langle Av, v - u_\varepsilon\right\rangle_{\mathcal{V}} \end{aligned}$$

then using (1.15), we get

$$\begin{aligned} \varepsilon \left\langle Au_\varepsilon, v - v_0\right\rangle_{\mathcal{V}} &\le C + \left\langle Av, \varepsilon\left(v - u_\varepsilon\right)\right\rangle_{\mathcal{V}}, \\ &\le C + \varepsilon \left\langle Av, v\right\rangle_{\mathcal{V}} - \left\langle Av, \varepsilon u_\varepsilon\right\rangle_{\mathcal{V}}. \end{aligned} \tag{1.16}$$

Choosing $v \in v_0 + \mathcal{B}_1$, where $\mathcal{B}_1$ is the unit ball in $\mathcal{V}$, we arrive to

$$\varepsilon \left\langle Au_\varepsilon, v_1\right\rangle_{\mathcal{V}} \le C', \qquad \forall v_1 \in \mathcal{B}_1,$$

where C' is independent of ε. Thus $\varepsilon Au_\varepsilon$ is bounded in the reflexive Banach space $\mathcal{V}'$ and one can suppose that, for some subsequence ε_k, we have $\varepsilon_k Au_{\varepsilon_k} \rightharpoonup \psi$ in $\mathcal{V}'$. Passing to the limit in (1.16) we derive

$$\left\langle \psi, v - v_0\right\rangle_{\mathcal{V}} \le C, \quad \forall v \in \mathcal{V}.$$

Changing v by $v_0 \pm v$ it comes that $\psi = 0$. By the uniqueness of the possible limits we showed that for the whole sequence we have

$$\varepsilon A u_\varepsilon \rightharpoonup 0 \text{ in } \mathcal{V}'.$$

Proof of (iv). For any $v \in K$ we have by (1.7) and the monotonicity of B

$$\begin{aligned} \varepsilon \langle A u_\varepsilon, u_\varepsilon \rangle_{\mathcal{V}} &\leq \langle \varepsilon A u_\varepsilon, v \rangle_{\mathcal{V}} + \langle f, u_\varepsilon - v \rangle_{\mathcal{W}} + \langle B u_\varepsilon, v - u_\varepsilon \rangle_{\mathcal{W}} \\ &\leq \langle \varepsilon A u_\varepsilon, v \rangle_{\mathcal{V}} + \langle f, u_\varepsilon - v \rangle_{\mathcal{W}} + \langle B v, v - u_\varepsilon \rangle_{\mathcal{W}} . \end{aligned} \tag{1.17}$$

Let $(\varepsilon_k)_k$ be a sequence such that

$$\varepsilon_k \langle A u_{\varepsilon_k}, u_{\varepsilon_k} \rangle_{\mathcal{V}} \to \limsup_{\varepsilon \to 0} \varepsilon \langle A u_\varepsilon, u_\varepsilon \rangle_{\mathcal{V}} .$$

Since u_{ε_k} is bounded in $\mathcal{W}$ one can suppose that - extracting if necessary a new subsequence -

$$u_{\varepsilon_k} \rightharpoonup \tilde{u} \text{ in } \mathcal{W}.$$

Then passing to the limit in (1.17), written for ε_k, we get

$$\limsup_{\varepsilon \to 0} \varepsilon \langle A u_\varepsilon, u_\varepsilon \rangle_{\mathcal{V}} \leq \langle f, \tilde{u} - v \rangle_{\mathcal{W}} + \langle B v, v - \tilde{u} \rangle_{\mathcal{W}} , \quad \forall v \in K. \tag{1.18}$$

It is clear that $\tilde{u}$ belongs to $\bar{K}^{\mathcal{W}}$, the weak closure of K in $\mathcal{W}$ which coincides with its strong closure since K is convex (see [6]). Thus, there exists a sequence $v_n \in K$ such that $v_n \to \tilde{u}$ in $\mathcal{W}$. Taking $v = v_n$ in (1.18) and passing to the limit we derive

$$\limsup_{\varepsilon \to 0} \varepsilon \langle A u_\varepsilon, u_\varepsilon \rangle_{\mathcal{V}} \leq 0. \tag{1.19}$$

By the monotonicity of A one has

$$\varepsilon \langle A u_\varepsilon, v \rangle_{\mathcal{V}} \leq \varepsilon \langle A u_\varepsilon, u_\varepsilon \rangle_{\mathcal{V}} + \langle A v, \varepsilon (v - u_\varepsilon) \rangle_{\mathcal{V}} , \tag{1.20}$$

then passing to the limit, we obtain

$$\liminf_{\varepsilon \to 0} \varepsilon \langle A u_\varepsilon, u_\varepsilon \rangle_{\mathcal{V}} \geq 0.$$

This, with (1.19), prove (*iv*).

Proof of (iii). It remains to show the strong convergence $\varepsilon A u_\varepsilon \to 0$. To this end we set $v = v_1 \in \mathcal{B}_1$ in (1.20), it comes

$$\begin{aligned} \varepsilon \langle A u_\varepsilon, v_1 \rangle_{\mathcal{V}} &\leq \varepsilon \langle A u_\varepsilon, u_\varepsilon \rangle_{\mathcal{V}} + |A v_1|_{\mathcal{V}'} (\varepsilon + |\varepsilon u_\varepsilon|_{\mathcal{V}}) \\ &\leq \varepsilon \langle A u_\varepsilon, u_\varepsilon \rangle_{\mathcal{V}} + C (\varepsilon + |\varepsilon u_\varepsilon|_{\mathcal{V}}) \to 0 \end{aligned}$$

where C is independent of v_1 and ε. Thus

$$|\varepsilon A u_\varepsilon|_{\mathcal{V}'} = \sup_{v_1 \in \mathcal{B}_1} \langle \varepsilon A u_\varepsilon, v_1 \rangle_{\mathcal{V}} \longrightarrow 0,$$

which completes the proof of the theorem. ■

A careful reading, of the above proof, enable us to give another variant of assumptions. In one hand we assume that

$$\mathcal{A}_\varepsilon = \varepsilon A + B \text{ is coercive on } \mathcal{V} \cap \mathcal{W}, \tag{1.21}$$

which ensure the existence of a solution u_ε to the perturbed problem (1.7). On the other hand we drop the coerciveness assumption on A, i.e. (1.5), and we only assume that

$$\liminf \frac{\langle Au, u - v_0 \rangle_{\mathcal{V}}}{|u - v_0|_{\mathcal{V}}} > 0 \quad \text{when } |u|_{\mathcal{V}} \to +\infty, \;\; u \in K. \tag{1.22}$$

for some $v_0 \in K$. Without changing the other assumptions of Theorem 1.2, we still have some convergence results in this case.

Theorem 1.3 *Under the assumption* (1.21)*, and changing* (1.5) *by* (1.22)*, in the assumptions of Theorem* 1.2*, we have when* $\varepsilon \to 0$

$$|u_\varepsilon|_{\mathcal{W}} \text{ and } \varepsilon\, |u_\varepsilon|_{\mathcal{V}} \text{ are bounded independently of } \varepsilon. \tag{1.23}$$

Moreover, if there exists some subsequence u_{ε_k} *such that*

$$\varepsilon_k u_{\varepsilon_k} \rightharpoonup 0 \text{ in } \mathcal{V}, \tag{1.24}$$

then it holds

$$(i)\ \varepsilon_k A u_{\varepsilon_k} \rightharpoonup 0 \ \text{ in } \mathcal{V}', \tag{1.25}$$

$$(ii)\ \langle \varepsilon_k A u_{\varepsilon_k}, u_{\varepsilon_k} \rangle_{\mathcal{V}} \to 0. \tag{1.26}$$

Proof. The assumption (1.22) is sufficient to establish the boundness of $|u_\varepsilon|_{\mathcal{W}}$ as in the proof of Theorem 1.2. Taking this into account, (1.15) still holds and due to (1.22) we have

$$\varepsilon\, |u_\varepsilon|_{\mathcal{V}} \le \varepsilon\, |u_\varepsilon - v_0|_{\mathcal{V}} + \varepsilon\, |v_0|_{\mathcal{V}} \le \frac{C\, |u_\varepsilon - v_0|_{\mathcal{V}}}{\langle A u_\varepsilon, u_\varepsilon - v_0 \rangle_{\mathcal{V}}} + \varepsilon\, |v_0|_{\mathcal{V}} < +\infty$$

when $|u_\varepsilon|_{\mathcal{V}} \to +\infty$. If the convergence (1.24) holds, then (i) and (ii) follows again as in the proof precedent theorem. ■

Remark 1.3 ***i)*** *In fact, $|u_\varepsilon|_{\mathcal{W}}$ is bounded if we assume only that A is positive for large $|u|_{\mathcal{V}}$, i.e.*

$$\liminf \langle Au, u - v_0\rangle_{\mathcal{V}} \geq 0 \quad \textit{when } |u|_{\mathcal{V}} \to +\infty,\ u \in K. \tag{1.27}$$

ii) *When A is a partial differential operator, the assumption (1.24) is, in many cases, a consequence of the continuity of derivation operator on the distributions space, see for instance Section 2.4.*

Remark 1.4 *In the case $K = \mathcal{V} \cap \mathcal{W}$, it follows from the abstract equation (1.9) that*

$$Bu_\varepsilon - f \to 0,\ \ (\text{resp.} \rightharpoonup 0,\ \textit{up to a subsequence}) \text{ in } \mathcal{W}',$$

under the assumptions of Theorem 1.2 (resp. Theorem 1.3).

In both cases, of the above convergence theorems, we have in addition

Theorem 1.4 *Suppose that for some sequence $\varepsilon_k \to 0$ one has*

$$u_{\varepsilon_k} \rightharpoonup \tilde{u} \text{ in } \mathcal{W}. \tag{1.28}$$

Then $\tilde{u}$ is a solution to the variational inequality

$$\begin{cases} \langle B\tilde{u}, v - \tilde{u}\rangle_{\mathcal{W}} \geq \langle f, v - \tilde{u}\rangle_{\mathcal{W}}, & \forall v \in \bar{K}^{\mathcal{W}}, \\ \tilde{u} \in \bar{K}^{\mathcal{W}}. \end{cases} \tag{1.29}$$

Moreover one has

$$Bu_{\varepsilon_k} \rightharpoonup B\tilde{u} \ \textit{ in } \mathcal{W}', \qquad \langle Bu_{\varepsilon_k}, u_{\varepsilon_k}\rangle_{\mathcal{W}} \to \langle B\tilde{u}, \tilde{u}\rangle_{\mathcal{W}}. \tag{1.30}$$

Proof. Up to a subsequence - still labelled ε_k - one can assume that $Bu_{\varepsilon_k} \rightharpoonup \chi$ in $\mathcal{W}'$. Passing to the limit in (1.7) written for ε_k we obtain (see Theorem 1.2)

$$\limsup_{\varepsilon_k \to 0} \langle Bu_{\varepsilon_k}, u_{\varepsilon_k}\rangle_{\mathcal{W}} \leq \langle \chi, v\rangle_{\mathcal{W}} + \langle f, \tilde{u} - v\rangle_{\mathcal{W}}, \qquad \forall v \in K. \tag{1.31}$$

Considering a sequence $v = v_n \to \tilde{u}$ in $\mathcal{W}$ as above, we obtain

$$\limsup_{\varepsilon_k \to 0} \langle Bu_{\varepsilon_k}, u_{\varepsilon_k}\rangle_{\mathcal{W}} \leq \langle \chi, \tilde{u}\rangle_{\mathcal{W}}.$$

From the monotonicity of B we have

$$\langle Bu_{\varepsilon_k}, u_{\varepsilon_k}\rangle_{\mathcal{W}} \geq \langle Bu_{\varepsilon_k}, v\rangle_{\mathcal{W}} + \langle Bv, u_{\varepsilon_k} - v\rangle_{\mathcal{W}}, \qquad \forall v \in \mathcal{W}.$$

Then

$$\liminf_{\varepsilon_k \to 0} \langle Bu_{\varepsilon_k}, u_{\varepsilon_k}\rangle_{\mathcal{W}} \geq \langle \chi, v\rangle_{\mathcal{W}} + \langle Bv, \tilde{u} - v\rangle_{\mathcal{W}}, \qquad \forall v \in \mathcal{W}. \tag{1.32}$$

It follows - taking $v = \tilde{u}$ - that

$$\lim_{\varepsilon_k \to 0} \langle Bu_{\varepsilon_k}, u_{\varepsilon_k} \rangle_{\mathcal{W}} = \langle \chi, \tilde{u} \rangle_{\mathcal{W}} .$$

From this and (1.32) we derive

$$\langle \chi - Bv, \tilde{u} - v \rangle_{\mathcal{W}} \geq 0, \quad \forall v \in \mathcal{W}.$$

Replacing v by $\tilde{u} + tw$ and, due to the hemicontinuity of B, letting $t \to 0$ we obtain

$$\langle \chi - B\tilde{u}, w \rangle_{\mathcal{W}} \geq 0, \quad \forall w \in \mathcal{W},$$

i.e. $\chi = B\tilde{u}$. It follows that the whole sequence Bu_{ε_k} converges toward $B\tilde{u}$. Moreover (1.31) becomes

$$\langle B\tilde{u}, v - \tilde{u} \rangle_{\mathcal{W}} \geq \langle f, v - \tilde{u} \rangle_{\mathcal{W}} , \quad \forall v \in K.$$

Since $\bar{K}^{\mathcal{W}}$ is closed - weakly closed - one has $\tilde{u} \in \bar{K}^{\mathcal{W}}$ and the above inequality holds also for every $v \in \bar{K}^{\mathcal{W}}$. This completes the proof of the theorem. ■

Remark 1.5 ***i)*** *We proved that the only possible limits for the subsequences of* $(u_\varepsilon)_\varepsilon$ *in* $\mathcal{W}$ *are solutions of the variational inequality* (1.29)*. In particular if the solution is unique, for instance when B is strictly monotone one has, for the whole sequence,*

$$u_\varepsilon \rightharpoonup \tilde{u} \text{ in } \mathcal{W}, \qquad Bu_\varepsilon \rightharpoonup B\tilde{u} \text{ in } \mathcal{W}'.$$

ii) *If* $K = \mathcal{V} \cap \mathcal{W}$ *then* $\bar{K}^{\mathcal{W}} = \mathcal{W}$ *and* $\tilde{u}$ *is a solution to the equation* $B\tilde{u} = f$.

1.3 MORE CONVERGENCES

The precedent convergences can be improved if the operators A or B satisfy some new assumptions.

Corollary 1.1 (i) *Suppose that* A *is strongly coercive in the sense that*

$$\langle Av, v \rangle_{\mathcal{V}} \geq \lambda \left| v \right|_{\mathcal{V}}^{\alpha} + g\left(v \right), \quad \forall v \in \mathcal{V}, \tag{1.33}$$

for some constants $\lambda > 0, \alpha > 1$ *and a bounded function* $g : \mathcal{V} \to \mathbb{R}$*, then one has*

$$\varepsilon^{1/\alpha} u_\varepsilon \to 0 \text{ in } \mathcal{V}. \tag{1.34}$$

(ii) *If B is strongly monotone in the sense that for some $\delta > 0$ and $\beta > 1$*

$$\langle Bu - Bv, u - v\rangle_{\mathcal{W}} \geq \delta \left|u - v\right|_{\mathcal{W}}^{\beta}, \quad \forall v, u \in \mathcal{W}' \tag{1.35}$$

then the solution $\tilde{u}$ of (1.29) is unique and one has

$$u_\varepsilon \to \tilde{u} \ \text{ in } \mathcal{W}.$$

Proof. *(i)* follows directly from

$$\lambda \left|\varepsilon^{1/\alpha} u_\varepsilon\right|_{\mathcal{V}}^{\alpha} = \lambda\varepsilon \left|u_\varepsilon\right|_{\mathcal{V}}^{\alpha} \leq \varepsilon \langle Au_\varepsilon, u_\varepsilon\rangle_{\mathcal{V}} - \varepsilon g\left(u_\varepsilon\right) \longrightarrow 0$$

by Theorem 1.2-(*iv*). For *(ii)* one has

$$\delta \left|\tilde{u} - u_\varepsilon\right|_{\mathcal{W}}^{\beta} \leq \langle B\tilde{u} - Bu_\varepsilon, \tilde{u} - u_\varepsilon\rangle_{\mathcal{W}} = \langle B\tilde{u}, \tilde{u} - u_\varepsilon\rangle_{\mathcal{W}} + \langle Bu_\varepsilon, u_\varepsilon\rangle_{\mathcal{W}} - \langle Bu_\varepsilon, \tilde{u}\rangle_{\mathcal{W}} \longrightarrow 0$$

by the convergences (1.30). ■

Remark 1.6 *Assuming only the basic coerciveness (1.3) of A, the convergence result (1.11), in Theorem 1.2, is sharp since the exponent of ε tends to 1 when $\alpha \to 1$ in (1.34).*

In the following corollary some monotonicity property of $\left(u_\varepsilon\right)_\varepsilon$ is shown.

Corollary 1.2 *Let $\varepsilon > \varepsilon' > 0$, then*

$$\langle Au_\varepsilon, u_\varepsilon\rangle_{\mathcal{V}} \leq \langle Au_\varepsilon, u_{\varepsilon'}\rangle_{\mathcal{V}}. \tag{1.36}$$

Proof. Set $v = u_\varepsilon$ (resp. $v = u_{\varepsilon'}$) in (1.7), written for ε (resp. ε'), we get

$$\varepsilon \langle Au_\varepsilon, u_\varepsilon - u_{\varepsilon'}\rangle_{\mathcal{V}} - \varepsilon' \langle Au_{\varepsilon'}, u_\varepsilon - u_{\varepsilon'}\rangle_{\mathcal{V}} + \langle Bu_\varepsilon - Bu_{\varepsilon'}, u_\varepsilon - u_{\varepsilon'}\rangle_{\mathcal{W}} \leq 0.$$

The third term is positive due to the monotonicity of B, hence

$$\varepsilon \langle Au_\varepsilon, u_\varepsilon - u_{\varepsilon'}\rangle_{\mathcal{V}} \leq \varepsilon' \langle Au_{\varepsilon'}, u_\varepsilon - u_{\varepsilon'}\rangle_{\mathcal{V}} \leq \varepsilon' \langle Au_\varepsilon, u_\varepsilon - u_{\varepsilon'}\rangle_{\mathcal{V}},$$

by the monotonicity of A. It follows that

$$\left(\varepsilon - \varepsilon'\right) \langle Au_\varepsilon, u_\varepsilon - u_{\varepsilon'}\rangle_{\mathcal{V}} \leq 0$$

and (1.36) follows, since $\varepsilon > \varepsilon'$. ■

Remark 1.7 *The above characterization is more clear if A is linear. For instance if $\mathcal{V}$ is a Hilbert space and $A = I_d$ then (1.36) yields*

$$\left|u_\varepsilon\right|_{\mathcal{V}} \leq \left|u_{\varepsilon'}\right|_{\mathcal{V}}, \quad \text{for } \varepsilon' < \varepsilon.$$

Next we pay attention to more regular problems, i.e. when one of the solutions of the limit problem (1.29) is in $\mathcal{V}$.

Corollary 1.3 (i) *If the variational inequality (1.29) has a solution $\hat{u} \in K$ such that*

$$\liminf \langle Au, u-\hat{u}\rangle_{\mathcal{V}} > 0 \ \text{ when } \ |u|_{\mathcal{V}} \to +\infty, \ u \in K, \tag{1.37}$$

then there exists always a sequence u_{ε_k} such that

$$u_{\varepsilon_k} \rightharpoonup \tilde{u} \ \text{ in } \mathcal{V} \text{ and } \mathcal{W}, \tag{1.38}$$

where $\tilde{u} \in K$ is solution to (1.29), and the accumulation points of $(u_\varepsilon)_\varepsilon$ are all in K.

(ii) *If B satisfies (1.35), one has*

$$|u_\varepsilon - \tilde{u}|_{\mathcal{W}} = o(\varepsilon^{1/\beta}). \tag{1.39}$$

Proof. Taking $v = \hat{u}$ in (1.7) we derive

$$\varepsilon \langle Au_\varepsilon, u_\varepsilon - \hat{u}\rangle_{\mathcal{V}} \le \langle f, u_\varepsilon - \hat{u}\rangle_{\mathcal{W}} - \langle Bu_\varepsilon, u_\varepsilon - \hat{u}\rangle_{\mathcal{W}} \le -\langle Bu_\varepsilon - B\hat{u}, u_\varepsilon - \hat{u}\rangle_{\mathcal{W}} \le 0. \tag{1.40}$$

Thus $\langle Au_\varepsilon, u_\varepsilon - \hat{u}\rangle_{\mathcal{V}} \le 0$ for all $\varepsilon > 0$, and

$$\limsup_{\varepsilon\to 0} \langle Au_\varepsilon, u_\varepsilon - \hat{u}\rangle_{\mathcal{V}} \le 0.$$

By (1.37), u_ε must be bounded in $\mathcal{V}$ and one can find a sequence ε_k such that

$$u_{\varepsilon_k} \rightharpoonup \tilde{u} \quad \text{in } \mathcal{W}, \mathcal{V} \text{ and } \mathcal{V}\cap\mathcal{W},$$

i.e., we have the same limit in the three spaces. Indeed, since u_{ε_k} is bounded in $\mathcal{V}, \mathcal{W}$ and $\mathcal{W}\cap\mathcal{V}$ one can assume that -up to a subsequence-

$$u_{\varepsilon_k} \rightharpoonup \tilde{v} \text{ in } \mathcal{V}, \qquad u_{\varepsilon_k} \rightharpoonup \tilde{w} \text{ in } \mathcal{W}, \qquad u_{\varepsilon_k} \rightharpoonup \tilde{u} \text{ in } \mathcal{V}\cap\mathcal{W}.$$

We have to check that these limits are the same. If $h \in \mathcal{V}' \subset (\mathcal{V}\cap\mathcal{W})'$ one has

$$\langle h, u_{\varepsilon_k}\rangle_{\mathcal{V}\cap\mathcal{W}} \to \langle h, \tilde{v}\rangle_{\mathcal{V}\cap\mathcal{W}}, \ \ \langle h, u_{\varepsilon_k}\rangle_{\mathcal{V}\cap\mathcal{W}} \to \langle h, \tilde{u}\rangle_{\mathcal{V}\cap\mathcal{W}}$$

whence

$$\langle h, \tilde{v}\rangle_{\mathcal{V}\cap\mathcal{W}} = \langle h, \tilde{u}\rangle_{\mathcal{V}\cap\mathcal{W}}, \ \ \forall h \in \mathcal{V}'.$$

Similarly one can show that

$$\langle h, \tilde{w}\rangle_{\mathcal{V}\cap\mathcal{W}} = \langle h, \tilde{u}\rangle_{\mathcal{V}\cap\mathcal{W}}, \ \ \forall h \in \mathcal{W}'.$$

It follows that $\tilde{v} = \tilde{w} = \tilde{u}$, and $\tilde{u}$ is necessarily a solution to Problem (1.29).

For *(ii)*, since by the uniqueness of the solution of (1.29) $\hat{u} = \tilde{u}$, then from (1.40) one has

$$\delta \left|\tilde{u} - u_\varepsilon\right|_{\mathcal{W}}^{\beta} \leq \langle B\tilde{u} - Bu_\varepsilon, \tilde{u} - u_\varepsilon\rangle_{\mathcal{W}} \leq \varepsilon \langle Au_\varepsilon, \tilde{u} - u_\varepsilon\rangle_{\mathcal{V}} \leq \varepsilon \langle A\tilde{u}, \tilde{u} - u_\varepsilon\rangle_{\mathcal{V}} = o\left(\varepsilon\right)$$

and the result follows. ■

Remark 1.8 *We can obtain a strong convergence of u_ε in $\mathcal{W}$ for the Saint-Venant problem. Assume that $f = 0$, $B\left(0\right) = 0, 0 \in K$ and B satisfies a hypothesis as* (1.35) *then*

$$u_\varepsilon \to 0 \quad \textit{in } \mathcal{W}.$$

Indeed, taking $v = 0$ in (1.7) we obtain $\varepsilon \langle Au_\varepsilon, u_\varepsilon\rangle_{\mathcal{V}} + \langle Bu_\varepsilon, u_\varepsilon\rangle_{\mathcal{W}} \leq 0$, *and by the monotonicity of A we have*

$$\delta \left|u_\varepsilon\right|_{\mathcal{W}}^{\beta} \leq \varepsilon \langle Au_\varepsilon - A\left(0\right), u_\varepsilon\rangle_{\mathcal{V}} + \langle Bu_\varepsilon, u_\varepsilon\rangle_{\mathcal{W}} \leq -\varepsilon \langle A\left(0\right), u_\varepsilon\rangle_{\mathcal{V}}.$$

The convergence follows by Theorem 1.2.

COMMENTS

Variational inequalities was introduced to study elliptic problems with unilateral conditions at the boundary, the obstacle problem, the elastic plastic problem, and other similar problems of mathematical physics. The pioneer works in this field are due to Lions and Stampacchia [50], Stampacchia [63]. The interested reader can see, for instance, [5, 9, 14, 42, 47, 60] and the literature cited therein.

The study of (isotropic) perturbations of variational inequalities, was initiated in the 60's by Lions and Stampacchia [50], for linear operators, and Browder [8] for some nonlinear operators. see also [48, 60, 61, 63, 69]. In these works, the asymptotic behaviour of solutions was investigated when there is no perturbation of the convex set. Other types of perturbations were considered, see the literature cited in Chapter 3 of this thesis.

The results of this chapter (except for theorem 1.3) has appeared in [21].

2

SOME APPLICATIONS AND EXAMPLES

This chapter is devoted to some applications of the abstract theory considered in the precedent chapter. First, to see the power of our abstract analysis in general, we consider a very simple (isotropic) case of nonlinear obstacle problems. Then we apply the theory to some anisotropic singular perturbations problems in the next examples. In the last section we consider a situation that does not fits with our abstract theory.

In all sections, Ω is a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary. We denote by u_ε the solutions of the perturbed problems and by $\tilde{u}$ the solutions of the limit problems.

2.1 NONLINEAR OBSTACLE PROBLEMS

We denote by $a \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ a continuous vector field on $\mathbb{R}^n$. We suppose that a is such that for some $\lambda, \Lambda > 0$ and $c \in \mathbb{R}$

$$a(\xi) \cdot \xi \geq \lambda |\xi|^2 + c, \qquad |a(\xi)| \leq \Lambda |\xi|, \qquad \forall \xi \in \mathbb{R}^n, \tag{2.1}$$

in addition it satisfies the monotonicity condition

$$(a(\xi) - a(\zeta)) \cdot (\xi - \zeta) \geq 0, \qquad \forall \xi, \zeta \in \mathbb{R}^n.$$

Then, for $f \in L^2(\Omega)$ there exists a unique u_ε solution to (see [14, 60])

$$\begin{cases} \varepsilon \int_\Omega a(\nabla u_\varepsilon) \cdot \nabla (v - u_\varepsilon)\, dx + \int_\Omega u_\varepsilon (v - u_\varepsilon)\, dx \geq \int_\Omega f(v - u_\varepsilon)\, dx, \qquad \forall v \in K_0, \\ u_\varepsilon \in K_0 := \left\{ v \in H_0^1(\Omega) \,|\, v(x) \geq 0 \text{ a.e. } x \in \Omega \right\}. \end{cases} \tag{2.2}$$

This problem is a mathematical formulation of a membrane occupying a domain Ω of the plane, loaded by normal force f. The membrane is attached to $\Gamma = \partial\Omega$ and the obstacle here is "0", i.e. the deformation of the membrane can not go in the negative direction as it is considered in the definition of the convex set K_0. The variational inequality gives the equilibrium position and it is obtained as a consequence of the principle of energy minimization (see [60]).

We are interested in the asymptotic behaviour of the solution u_ε when $\varepsilon \to 0$. Setting

$$\mathcal{V} = H_0^1(\Omega), \qquad \mathcal{W} = L^2(\Omega), \qquad Au = -div(a(\nabla u)), \qquad B = I_d,$$

our abstract results can be applied and we get

$$u_\varepsilon \to f^+ \text{ in } L^2(\Omega)$$

where $f^+ := \max\{0, f\}$ denotes the positive part of f. Indeed, thanks to Theorems 1.2, 1.4 and Corollary 1.1, we see that $u_\varepsilon \to \tilde{u}$ in $L^2(\Omega)$ where $\tilde{u}$ satisfies

$$\begin{cases} \int_\Omega \tilde{u}(v - \tilde{u})\, dx \geq \int_\Omega f(v - \tilde{u})\, dx, \qquad \forall v \in \bar{K}_0, \\ \tilde{u} \in \bar{K}_0 = \left\{ v \in L^2(\Omega) \,|\, v(x) \geq 0, \ \text{a.e. } x \in \Omega \right\}. \end{cases} \tag{2.3}$$

Denoting the negative part of f by $f^- := \max\{0, -f\}$, we have clearly

$$\begin{aligned} \int_\Omega f^+(v - f^+)\, dx &= \int_\Omega (f + f^-)(v - f^+)\, dx \\ &= \int_\Omega f(v - f^+)\, dx + \int_\Omega f^- v dx \geq \int_\Omega f(v - f^+)\, dx, \qquad \forall v \in \bar{K}_0 \end{aligned}$$

since $f^- v \geq 0$. Thus f^+ is a solution to (2.3) and by uniqueness we deduce $\tilde{u} = f^+$.

Now we can state the following result.

Theorem 2.1 *When $\varepsilon \to 0$, we have*

$$\begin{aligned} u_\varepsilon &\to f^+ \ \text{ in } L^2(\Omega), \\ -\varepsilon \partial_{x_i}(a(\nabla u_\varepsilon)) &\to 0 \ \text{ in } H^{-1}(\Omega), \quad i = 1, \cdots, n, \\ \varepsilon \int_\Omega a(\nabla u_\varepsilon) \cdot \nabla u_\varepsilon dx &\to 0, \\ \sqrt{\varepsilon} \nabla u_\varepsilon &\to 0 \ \text{ in } L^2(\Omega). \end{aligned}$$

In the above convergences the vectorial convergence means the convergence component by component.

Remark 2.1 *See* Eckhaus and Moet [25] *for other results on this obstacle problem.*

2.2 SEMILINEAR ELLIPTIC PROBLEMS

Most of the notations of this section will be used in the rest of this thesis.

Some notations and assumptions

Let Ω be a bounded open subset of $\mathbb{R}^n$ with sufficiently smooth boundary. We split the components of a point $x \in \mathbb{R}^n$ into the q first components and the $n-q$ last ones, i.e.

$$X_1 := (x_1, \dots, x_q) \qquad \text{and} \qquad X_2 := (x_{q+1}, \dots, x_n),$$

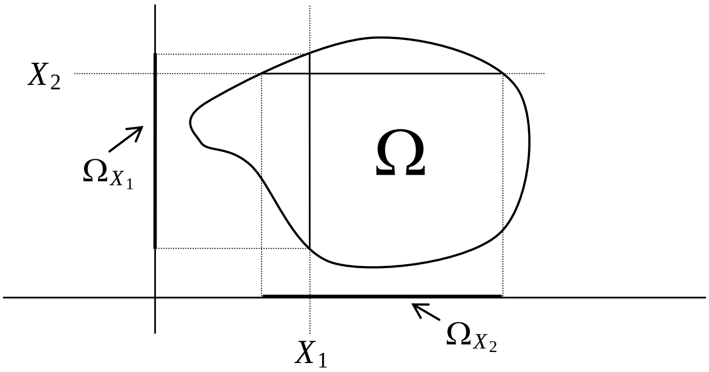
where q is a positive integer such that $q < n$. With this notation we set

$$\nabla u := (\partial_{x_1} u, \dots, \partial_{x_n} u)^T = \begin{pmatrix} (\partial_{x_1} u, \dots, \partial_{x_q} u)^T \\ (\partial_{x_{q+1}} u, \dots, \partial_{x_n} u)^T \end{pmatrix} = \begin{pmatrix} \nabla_{X_1} u \\ \nabla_{X_2} u \end{pmatrix}.$$

We denote by Π_{X_1} (resp. Π_{X_2}) the orthogonal projection from $\mathbb{R}^n$ onto the space $X_2 = 0$ (resp. $X_1 = 0$) and we set

$$\Pi_1 := \Pi_{X_1}(\Omega) = \{\, X_1 \mid \exists X_2 \text{ such that } (X_1, X_2) \in \Omega \,\},$$
$$\Pi_2 := \Pi_{X_2}(\Omega) = \{\, X_2 \mid \exists X_1 \text{ such that } (X_1, X_2) \in \Omega \,\}.$$

For any $X_1 \in \Pi_1$ and $X_2 \in \Pi_2$, we denote by Ω_{X_1} (resp. Ω_{X_2}) the section[1] of Ω corresponding to X_1. (resp. X_2).



$$\Omega_{X_1} := \{\, X_2 \mid (X_1, X_2) \in \Omega \,\} \quad \text{and} \quad \Omega_{X_2} := \{\, X_1 \mid (X_1, X_2) \in \Omega \,\}.$$

For a real valued function u defined on $\mathbb{R}^n$, we set

$$u(X_1, \cdot) : \mathbb{R}^{n-q} \to \mathbb{R}, \quad X_2 \mapsto u(X_1, X_2), \quad \forall X_1 \in \Pi_1,$$
$$u(\cdot, X_2) : \mathbb{R}^q \to \mathbb{R}, \quad X_1 \mapsto u(X_1, X_2), \quad \forall X_2 \in \Pi_2.$$

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[1] Some authors use the term *"slice of* Ω *"*.

We consider the following semilinear elliptic problem

$$\begin{cases} -\varepsilon\Delta_{X_1} u_\varepsilon - \Delta_{X_2} u_\varepsilon + g\left(x, u_\varepsilon\right) = f \quad \text{in } \Omega, \\ u_\varepsilon \in H_0^1\left(\Omega\right) \cap L^p\left(\Omega\right), \end{cases} \tag{2.4}$$

where $\varepsilon > 0$, $p > 1$,

$$\Delta_{X_1} := \sum_{i=1}^{i=q} \frac{\partial^2}{\partial^2 x_i}, \quad \Delta_{X_2} := \sum_{i=q+1}^{i=n} \frac{\partial^2}{\partial^2 x_i}, \quad f \in L^2\left(\Omega\right) + L^{p'}\left(\Omega\right)$$

and $p' := p/\left(p-1\right)$ is the conjugate number of p. In contrast with the precedent example, the perturbation in this case is anisotropic in the sense that the parameter ε effects only the X_1-directions of the domain variables.

In order to apply the abstract approach, we assume that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and is nondecreasing in the second variable, i.e.

$$x \quad \mapsto \quad g(x,s) \text{ is measurable on } \Omega, \quad \forall s \in \mathbb{R}, \tag{2.5}$$

$$s \quad \mapsto \quad g(x,s) \text{ is continuous and nondecreasing on } \mathbb{R}, \quad \text{for a.e. } x \in \Omega \tag{2.6}$$

and there exist $c, c' \geq 0$, such that

$$\begin{aligned} &\left|g\left(x,s\right)\right| \leq c\left|s\right|^{p-1} + c', && \forall s \in \mathbb{R}, \;\; \text{a.e. } x \in \Omega, \\ &g\left(x,s\right) s \geq \left|s\right|^p, && \forall s \in \mathbb{R}, \;\; \text{a.e. } x \in \Omega. \end{aligned}$$

For $u \in L^p\left(\Omega\right)$ we have

$$\left|g\left(\cdot, u\left(\cdot\right)\right)\right|_{L^{p'}(\Omega)} \leq \left(\int_\Omega \left(c\left|u\right|^{(p-1)'} + c'\right)^p dx\right)^{1/p'} \leq C\left|u\right|_{L^p(\Omega)}^{p/p'} + C',$$

where C and C' are positive constants. Thus g defines a bounded operator -still labelled by g- from $L^p\left(\Omega\right)$ into $L^{p'}\left(\Omega\right)$ by

$$u \mapsto g\left(\cdot, u\left(\cdot\right)\right) \tag{2.7}$$

which is hemicontinuous, and monotone since g is nondecreasing in the second variable.

Choosing spaces and operators

The choice of suitable spaces and operators that fit with our approach is more delicate in this case. We consider the space[2]

$$\mathcal{V}_1\left(\Omega\right) = \left\{u \in L^2\left(\Omega\right) | \nabla_{X_1} u \in \left[L^2\left(\Omega\right)\right]^q, \;\; u(\cdot, X_2) \in H_0^1\left(\Omega_{X_2}\right), \quad \text{a.e. } X_2 \in \Pi_2\right\}, \tag{2.8}$$

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[2] The spaces $\mathcal{V}_i\left(\Omega\right)$ and $\mathcal{W}_i\left(\Omega\right)$ may be changed from a section to another.

equipped with the norm

$$|v|^2_{\mathcal{V}_1(\Omega)} = |v|^2_{L^2(\Omega)} + |\nabla_{X_1} v|^2_{L^2(\Omega)} . \tag{2.9}$$

It is clear that $\mathcal{V}_1(\Omega)$ is a Hilbert space and the mapping

$$v \mapsto |\nabla_{X_1} v|_{L^2(\Omega)} \tag{2.10}$$

is a norm equivalent to (2.9). In fact, since Ω is bounded we can choose a Poincaré constant in

$$|v(\cdot, X_2)|_{L^2\left(\Omega_{X_2}\right)} \leq C\,|\nabla_{X_1} v(\cdot, X_2)|_{L^2\left(\Omega_{X_2}\right)}, \qquad \forall v \in \mathcal{V}_1(\Omega), \text{ a.e. } X_2$$

independent of X_1 (see (A.10)). Then extending the elements of $\mathcal{V}_1(\Omega)$ by zero outside of Ω and integrating on Π_1, then it comes

$$|v|_{\mathcal{V}_1(\Omega)} \leq (1+C)\,|\nabla_{X_1} v|^2_{L^2(\Omega)} \leq (1+C)\,|v|_{\mathcal{V}_1(\Omega)} . \tag{2.11}$$

Remark 2.2 *We can easily check that $\mathcal{D}(\Omega)$ is dense in $\mathcal{V}_1(\Omega)$, hence it holds that*

$$H^1_0(\Omega) \subset \mathcal{V}_1(\Omega) \subset L^2(\Omega) \subset \mathcal{V}_1'(\Omega) \subset H^{-1}(\Omega)$$

with continuous injections. Note that in general the embedding $\mathcal{V}_1(\Omega) \subset L^2(\Omega)$ is not compact as it is shown in the following example: Let $\Omega = (0,1)^2$ and consider

$$v_n(x_1, x_2) := \sqrt{2} \sin \pi x_1 \ \sin n\pi x_2,$$

then it holds that

$$|\partial_{x_1} v_n|_{L^2(\Omega)} = \frac{1}{\sqrt{2}\pi} \quad \text{and} \quad |v_n - v_m|_{L^2(\Omega)} = 1 \quad (\text{for } m \neq n).$$

Thus v_n is bounded in $\mathcal{V}_1\left((0,1)^2\right)$ but there is no Cauchy - converging - subsequences. This lack of compactness is the source of many difficulties when nonlinear problems are considered.

On the other hand we choose

$$\mathcal{W}_2(\Omega) = \left\{ u \in L^2(\Omega) \cap L^p(\Omega) \;\middle|\; \begin{array}{l} \nabla_{X_2} u \in \left[L^2(\Omega)\right]^{n-q}, \\ u(X_1, \cdot) \in H^1_0(\Omega_{X_1}), \text{ a.e. } X_1 \in \Pi_1 \end{array} \right\}. \tag{2.12}$$

equipped with the norm

$$|v|_{\mathcal{W}_2(\Omega)} = |\nabla_{X_2} v|_{L^2(\Omega)} + |v|_{L^p(\Omega)} .$$

We can easily check that $\mathcal{V}_1(\Omega)$ and $\mathcal{W}_2(\Omega)$ are separable Banach spaces. The reflexivity of $\mathcal{V}_1(\Omega)$ is obvious (it is a Hilbert space). Concerning $\mathcal{W}_2(\Omega)$, one notes that $\mathcal{W}_2(\Omega)$ itself is the intersection of two reflexive spaces, $L^p(\Omega)$ and the Hilbert space

$$\mathcal{V}_2(\Omega) = \left\{ u \in L^2(\Omega) \,|\, \nabla_{X_2} u \in \left[L^2(\Omega)\right]^{n-q}, \quad u(X_1,\cdot) \in H_0^1(\Omega_{X_1}), \text{ a.e. } X_1 \in \Pi_1 \right\}.$$

Thus the reflexivity of $\mathcal{W}_2(\Omega)$ follows since we can write (see [3, 66])

$$\mathcal{W}_2''(\Omega) = (\mathcal{V}_2(\Omega) \cap L^p(\Omega))'' = \left(\mathcal{V}_2'(\Omega) + L^{p'}(\Omega)\right)' = \mathcal{V}_2(\Omega) \cap L^p(\Omega) = \mathcal{W}_2(\Omega).$$

Next we set

$$A = -\Delta_{X_1} \text{ and } B = -\Delta_{X_2} + g(x,\cdot).$$

Then the operator $A : \mathcal{V}_1(\Omega) \to \mathcal{V}_1'(\Omega)$ is linear, bounded and coercive. Since the operator $B : \mathcal{W}_2(\Omega) \to \mathcal{W}_2'(\Omega)$ is a sum of a linear operator, satisfying the same properties as A, and the operator defined in (2.7), then it is bounded, monotone and coercive.

The intersection space $\mathcal{V}_1(\Omega) \cap \mathcal{W}_2(\Omega)$

The following proposition precise the intersection space $\mathcal{V}_1(\Omega) \cap \mathcal{W}_2(\Omega)$.

Proposition 2.2 *Let $\mathcal{V}_1(\Omega)$ and $\mathcal{W}_2(\Omega)$ be the spaces defined in (2.8) and (2.12) respectively, then if the boundary of Ω is smooth we have*

$$\mathcal{V}_1(\Omega) \cap \mathcal{W}_2(\Omega) = H_0^1(\Omega) \cap L^p(\Omega).$$

Proof. The first inclusion $H_0^1(\Omega) \cap L^p(\Omega) \subset \mathcal{V}_1(\Omega) \cap \mathcal{W}_2(\Omega)$ is easy. For $u \in H_0^1(\Omega) \cap L^p(\Omega)$ there exists a sequence $(u_n)_n \subset \mathcal{D}(\Omega)$ such that $u_n \to u$ in $H_0^1(\Omega) \cap L^p(\Omega)$. In particular we have

$$|u_n - u|_{L^2(\Omega)} \to 0, \quad |\nabla_{X_1}(u_n - u)|_{L^2(\Omega)} \to 0, \quad |\nabla_{X_2}(u_n - u)|_{L^2(\Omega)} \to 0.$$

As a standard result of integration theory (see [6]), we get -up to a subsequence-

$$\begin{aligned} |\nabla_{X_2}(u_n(X_1,\cdot) - u(X_1,\cdot))|_{L^2(\Omega_{X_1})} &\to 0, \quad \text{for a.e. } X_1 \in \Pi_1, \\ |\nabla_{X_1}(u_n(\cdot,X_2) - u(\cdot,X_2))|_{L^2(\Omega_{X_2})} &\to 0, \quad \text{for a.e. } X_2 \in \Pi_2. \end{aligned}$$

This means that $u \in \mathcal{V}_1(\Omega)$ and $u \in \mathcal{W}_2(\Omega)$ since these spaces are complete.

For the converse inclusion we use a singular perturbation argument. Let $u \in \mathcal{V}_1(\Omega) \cap \mathcal{W}_2(\Omega)$, for $\epsilon > 0$ consider the following elliptic problem

$$\begin{cases} -\epsilon \Delta v_\epsilon + v_\epsilon = u & \text{in } \Omega, \\ v_\epsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.13}$$

Since Ω is sufficiently regular and of course $u \in \mathcal{V}_1(\Omega) \cap \mathcal{W}_2(\Omega) \subset H^1(\Omega)$, we have $v_\epsilon \in H^2(\Omega) \cap H_0^1(\Omega)$, (see Evans [26]). According to Corollary 1.1-(ii), we derive

$$v_\epsilon \to u \text{ in } L^2(\Omega), \quad \text{as } \epsilon \to 0. \tag{2.14}$$

Next we show that v_ϵ is also bounded in $H_0^1(\Omega)$. To this end, we apply the Laplace operator to the first equation in (2.13) and taking $-v_\epsilon$ as a test function, we obtain

$$\epsilon \left\langle \Delta^2 v_\epsilon, v_\epsilon \right\rangle_{H_0^1(\Omega)} - \int_\Omega \Delta v_\epsilon v_\epsilon dx = - \langle \Delta u, v_\epsilon \rangle_{H_0^1(\Omega)}.$$

Note that $\Delta u \in H^{-1}(\Omega)$ and $\Delta^2 v_\epsilon \in H^{-1}(\Omega)$ since

$$-\Delta v_\epsilon = \frac{u - v_\epsilon}{\epsilon} \in H^1(\Omega). \tag{2.15}$$

It follows that

$$-\epsilon \int_\Omega \nabla(\Delta v_\epsilon) \cdot \nabla v_\epsilon dx + |\nabla v_\epsilon|^2_{L^2(\Omega)} = \int_\Omega \nabla u \cdot \nabla v_\epsilon dx$$

and extending v_ϵ by 0 outside of Ω, we can write

$$\begin{aligned} -\epsilon \int_{\Pi_1} \int_{\Omega_{X_2}} \nabla_{X_1}(\Delta v_\epsilon) \cdot \nabla_{X_1} v_\epsilon dX_1 dX_2 - \epsilon \int_{\Pi_2} \int_{\Omega_{X_1}} \nabla_{X_2}(\Delta v_\epsilon) \cdot \nabla_{X_2} v_\epsilon dX_2 dX_1 \\ + |\nabla v_\epsilon|^2_{L^2(\Omega)} \leq \frac{1}{2} |\nabla u|^2_{L^2(\Omega)} + \frac{1}{2} |\nabla v_\epsilon|^2_{L^2(\Omega)}. \end{aligned} \tag{2.16}$$

Since $v_\epsilon \in H_0^1(\Omega) \subset \mathcal{V}_1(\Omega) \cap \mathcal{W}_2(\Omega)$ and $u \in \mathcal{V}_1(\Omega) \cap \mathcal{W}_2(\Omega)$ in (2.15), we deduce that $\Delta v_\epsilon \in \mathcal{V}_1(\Omega) \cap \mathcal{W}_2(\Omega)$. In particular we have (see [16])

$$\begin{aligned} \Delta v_\epsilon(\cdot, X_2) &\in H_0^1(\Omega_{X_2}), \quad \text{for a.e. } X_2 \in \Pi_2, \\ \Delta v_\epsilon(X_1, \cdot) &\in H_0^1(\Omega_{X_1}), \quad \text{for a.e. } X_1 \in \Pi_1. \end{aligned}$$

Thus we can rewrite (2.16) as

$$\epsilon \int_{\Pi_1} \int_{\Omega_{X_2}} \Delta v_\epsilon \Delta_{X_1} v_\epsilon dX_1 dX_2 + \epsilon \int_{\Pi_2} \int_{\Omega_{X_1}} \Delta v_\epsilon \Delta_{X_2} v_\epsilon dX_2 dX_1 + \frac{1}{2} |\nabla v_\epsilon|^2_{L^2(\Omega)} \leq \frac{1}{2} |\nabla u|^2_{L^2(\Omega)},$$

whence

$$2\epsilon |\Delta v_\epsilon|^2_{L^2(\Omega)} + |\nabla v_\epsilon|^2_{L^2(\Omega)} \leq |\nabla u|^2_{L^2(\Omega)}.$$

It follows that v_ϵ is bounded in $H_0^1(\Omega)$, then -up to a subsequence- its weak limit, which is necessarily u due to (2.14), is in $H_0^1(\Omega)$. This ends the proof of the proposition. ■

Remark 2.3 *To obtain $v_\epsilon \in H^2(\Omega)$, in Problem (2.13), it is sufficient to have $\partial\Omega$ of class $C^{1,1}$, i.e. continuously differentiable and its derivatives are Lipschitz continuous. No regularity is imposed if Ω is convex (see* Grisvard [30]).

Convergences

In order to obtain some convergences for u_ε when $\varepsilon \to 0$, we can establish some a priori estimates, by testing the first equation in (2.4) by u_ε, and obtain

$$|\nabla_{X_2} u_\varepsilon|_{L^2(\Omega)}, \quad |u_\varepsilon|_{L^p(\Omega)} \text{ are bounded.}$$

Then – up to a subsequence – we have only weak convergences, particularly in $\mathcal{W}_2(\Omega)$. Since the embedding $\mathcal{W}_2(\Omega) \subset L^2(\Omega)$ is not compact (see Remark 2.2), these estimates are not sufficient to pass to the limit in the nonlinear term $g(\cdot, u_\varepsilon)$. In this case, the monotonicity assumption is very important to overcome this difficulty.

As an obvious consequence of Theorems 1.2, 1.4 and Corollary 1.1, the limit problem in this case is defined as

$$\begin{cases} -\Delta_{X_2}\tilde{u} + g(x, \tilde{u}) = f \quad \text{in } \Omega, \\ \tilde{u} \in \mathcal{W}_2(\Omega), \end{cases} \tag{2.17}$$

and we have the following convergence results.

Theorem 2.3 *When $\varepsilon \to 0$, we have*

$$u_\varepsilon \to \tilde{u}, \quad \nabla_{X_2} u_\varepsilon \to \nabla_{X_2}\tilde{u} \quad \text{and} \quad \sqrt{\varepsilon}\nabla_{X_1} u_\varepsilon \to 0 \quad \text{in } L^2(\Omega).$$

Moreover if g is strongly monotone, in the sense of (1.35)*, then we have*

$$u_\varepsilon \to \tilde{u} \quad \text{in } L^p(\Omega).$$

Remark 2.4 *Although the operator B in this case is not strongly monotone, the first two convergences hold strongly. This is due to the following monotone type inequality*

$$-\langle \Delta_{X_2} v - \Delta_{X_2} u, v - u\rangle_{\mathcal{W}_2(\Omega)} + \int_\Omega (g(x,v) - g(x,u))(v-u)\,dx \geq |\nabla_{X_2}(v-u)|^2_{L^2(\Omega)}, \qquad \forall u, v \in \mathcal{W}_2(\Omega).$$

Working on the sections Ω_{X_1}

The limit $\tilde{u}$, solution to (2.17), can be defined on the sections Ω_{X_1} for a.e. $X_1 \in \Pi_1$. In fact, since Δ_{X_1} is missed in the limit problem (2.17), it will be natural to work on the sections Ω_{X_1}, $X_1 \in \Pi_1$, by considering, for a.e. $X_1 \in \Pi_1$, the problem

$$\begin{cases} -\Delta_{X_2} u\left(X_1, \cdot\right) + g\left(\left(X_1, \cdot\right), u\left(X_1, \cdot\right)\right) = f\left(X_1, \cdot\right) & \text{in } \Omega_{X_1}, \\ u\left(X_1, \cdot\right) = 0 & \text{on } \partial\Omega_{X_1}, \end{cases} \tag{2.18}$$

which is from the same type of problem (2.4), but defined in a lower dimensional space, and X_1 plays the role of a parameter. Due to Fubini theorem it holds that for a.e. $X_1 \in \Pi_1$

$$f\left(X_1, \cdot\right) \in L^2\left(\Omega_{X_1}\right) + L^{p'}\left(\Omega_{X_1}\right),$$

and $g\left(\left(X_1, \cdot\right), \cdot\right) : \Omega_{X_1} \times \mathbb{R} \to \mathbb{R}$ is still a Carathéodory function, defining a bounded monotone hemicontinuous operator from $L^p\left(\Omega_{X_1}\right)$ into $L^{p'}\left(\Omega_{X_1}\right)$. Then the Problem (2.18) has a unique solution for a.e. $X_1 \in \Pi_1$. This solution is identical to $\tilde{u}$ and to check this one can use the argument introduced by Chipot and Guesmia [16], see also the proof of Theorem 4.2.

2.3 $p-$LAPLACIAN TYPE PROBLEM

The second example of the abstract theory, in the anisotropic case, is the following quasilinear elliptic equation

$$\begin{cases} -\varepsilon\Delta_{p_1} u_\varepsilon - \Delta_{p_2} u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{2.19}$$

where $p_1, p_2 > 1$ are real constants, and

$$\Delta_{p_1} \cdot := \nabla_{X_1} \cdot \left(\left|\nabla_{X_1} \cdot\right|^{p_1 - 2} \nabla_{X_1} \cdot\right), \qquad \Delta_{p_2} \cdot := \nabla_{X_2} \cdot \left(\left|\nabla_{X_2} \cdot\right|^{p_2 - 2} \nabla_{X_2} \cdot\right).$$

Assuming that $f \in L^{p_2'}\left(\Omega\right)$, the above problem has a unique solution that belongs to the space[3]

$$W_0^{1;p_1,p_2}\left(\Omega\right) := \left\{ u \in L^m\left(\Omega\right) \left| \begin{array}{l} \nabla_{X_1} u \in \left[L^{p_1}\left(\Omega\right)\right]^q, \\ \nabla_{X_2} u \in \left[L^{p_2}\left(\Omega\right)\right]^{n-q}, \quad u|_{\partial\Omega} = 0 \end{array} \right. \right\},$$

where $m = \max\left(p_1, p_2\right)$.

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[3] We follow the notations of Fučik and Kufner [27, page 147].

Remark 2.5 *The space* $W_0^{1;p_1,p_2}(\Omega)$ *is nothing but the closure of* $\mathcal{D}(\Omega)$ *with respect to the norm*

$$|v|_{L^m(\Omega)} + |\nabla_{X_1} v|_{L^{p_1}(\Omega)} + |\nabla_{X_2} v|_{L^{p_2}(\Omega)} .$$

Clearly $W_0^{1;p_1,p_2}(\Omega)$ *is included in the usual Sobolev space* $W_0^{1,r}(\Omega)$*, with* $r := \min(p_1, p_2)$*, which gives a sense to the boundary conditions in* (2.19).

In order to apply our abstract approach we argue as in the precedent section. First we set

$$\mathcal{V}_1(\Omega) = \left\{ u \in L^{p_1}(\Omega) \;\middle|\; \begin{array}{l} \nabla_{X_1} u \in [L^{p_1}(\Omega)]^q , \\ u(\cdot, X_2) \in W_0^{1,p_1}(\Omega_{X_2}) , \text{ a.e. } X_2 \in \Pi_2 \end{array} \right\}, \tag{2.20}$$

Thanks to Poincaré's inequality (see (A.10) and Section 2.2), this space is equipped with the norm

$$|v|_{\mathcal{V}_1(\Omega)} = |\nabla_{X_1} v|_{L^{p_1}(\Omega)} .$$

For the other space we set

$$\mathcal{W}_2(\Omega) = \left\{ u \in L^{p_2}(\Omega) \;\middle|\; \begin{array}{l} \nabla_{X_2} u \in [L^{p_2}(\Omega)]^{n-q} , \\ u(X_1, \cdot) \in W_0^{1,p_2}(\Omega_{X_1}) , \text{ a.e. } X_1 \in \Pi_1 \end{array} \right\}, \tag{2.21}$$

equipped with the norm

$$|v|_{\mathcal{W}_2(\Omega)} = |\nabla_{X_2} v|_{L^{p_2}(\Omega)} .$$

We can easily show that $\mathcal{V}_1(\Omega)$ and $\mathcal{W}_2(\Omega)$ are separable reflexive Banach spaces. Then we define the operators $A : \mathcal{V}_1(\Omega) \to \mathcal{V}_1'(\Omega)$ and $B : \mathcal{W}_2(\Omega) \to \mathcal{W}_2'(\Omega)$ as

$$A = -\Delta_{p_1}, \qquad B = -\Delta_{p_2}.$$

One can check that A and B are coercive, bounded and hemicontinuous. Thanks to the inequalities (A.7)–(A.9), the operator A (resp. B) is strictly monotone for all $p_1 > 1$ (resp. $p_2 > 1$) and strongly monotone, in the sense of (1.35), if $p_1 \geq 2$ (resp. $p_2 \geq 2$).

The limit problem in this case is defined as

$$\begin{cases} -\Delta_{p_2} u = f & \text{in } \Omega, \\ u \in \mathcal{W}_2(\Omega) . \end{cases} \tag{2.22}$$

Then by Theorems 1.2, 1.4, Corollary 1.1 and when

$$\mathcal{V}_1(\Omega) \cap \mathcal{W}_2(\Omega) = W_0^{1;p_1,p_2}(\Omega) ,$$

we have the following convergences.

Theorem 2.4 *For all* $p_1, p_2 > 1$*, we have*

$$\begin{aligned} u_\varepsilon &\rightharpoonup \tilde{u} \quad \text{in} \quad \mathcal{W}_2(\Omega), & \varepsilon \nabla_{X_1} u_\varepsilon &\to 0 \quad \text{in} \quad L^{p_1}(\Omega), \\ \varepsilon \Delta_{p_1} u_\varepsilon &\to 0 \quad \text{in} \quad \mathcal{V}_1'(\Omega), & \Delta_{p_2} u_\varepsilon &\rightharpoonup f \quad \text{in} \quad \mathcal{W}_2'(\Omega), \end{aligned}$$

where u_ε *and* $\tilde{u}$ *are the solutions of* (2.19) *and* (2.22) *respectively. Moreover, if* $p_1 \geq 2$ *then*

$$\varepsilon^{1/p_1} \nabla_{X_1} u_\varepsilon \to 0 \quad \text{in } L^{p_1}(\Omega),$$

and if $p_2 \geq 2$ *then*

$$u_\varepsilon \to \tilde{u}, \quad \nabla_{X_2} u_\varepsilon \to \nabla_{X_2} \tilde{u} \quad \text{in } L^{p_2}(\Omega).$$

Remark 2.6 *As above, the limit problem can be defined, for a.e.* $X_1 \in \Pi_1$*, as*

$$\begin{cases} -\Delta_{p_2} u(X_1, \cdot) = f(X_1, \cdot) & \text{in } \Omega_{X_1}, \\ \tilde{u}(X_1, \cdot) = 0 & \text{on } \partial\Omega_{X_1}. \end{cases}$$

2.4 AN APPLICATION OF THEOREM 1.3

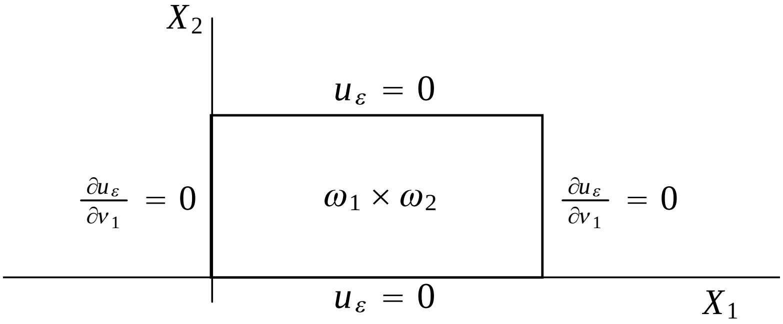
In this section we give a simple case where the assumption of Theorem 1.3 are satisfied, i.e. the perturbed operator $\varepsilon A + B$ is coercive whereas the operator A is not coercive. We set

$$\Omega := \omega_1 \times \omega_2$$

where ω_2 and ω_1 are two bounded smooth domains of $\mathbb{R}^{n-q}$ and $\mathbb{R}^q$ respectively. In this cylindrical case, the boundary is given by

$$\partial\Omega = (\partial\omega_1 \times \omega_2) \cup (\omega_1 \times \partial\omega_2).$$

For $\varepsilon > 0$ and $f \in L^2(\Omega)$, we consider the following elliptic problem, with mixed



Neumann-Dirichlet boundary conditions,

$$\begin{cases} -\varepsilon\Delta_{X_1}u_\varepsilon - \Delta_{X_2}u_\varepsilon = f & \text{in} \quad \Omega, \\ \dfrac{\partial u_\varepsilon}{\partial \nu_1} = 0 & \text{on} \quad \partial\omega_1 \times \omega_2, \\ u_\varepsilon = 0 & \text{on} \quad \omega_1 \times \partial\omega_2, \end{cases} \tag{2.23}$$

where ν_1 is the outward unit normal vector to $\partial\omega_1 \times \omega_2$. This problem has a unique solution

$$u_\varepsilon \in H_0^1(\Omega, \omega_1 \times \partial\omega_2) := \left\{ v \in H^1(\Omega) \,|u = 0 \text{ on } \omega_1 \times \partial\omega_2 \right\}.$$

In order to treat this problem, as an example of Theorem 1.3, we consider the space

$$\mathcal{V}_1(\Omega) = \left\{ u \in L^2(\Omega) \,|\nabla_{X_1}u \in \left[L^2(\Omega)\right]^q \right\},$$

equipped with the norm

$$|v|^2_{\mathcal{V}_1(\Omega)} = |v|^2_{L^2(\Omega)} + |\nabla_{X_1}v|^2_{L^2(\Omega)},$$

then it holds that

$$H_0^1(\Omega) \subset \mathcal{V}_1(\Omega) \subset L^2(\Omega) \subset \mathcal{V}_1'(\Omega) \subset H^{-1}(\Omega)$$

with continuous injections. For the second space we set

$$\mathcal{W}_2(\Omega) = \left\{ u \in L^2(\Omega) \,|\nabla_{X_2}u \in \left[L^2(\Omega)\right]^{n-q}, \ \ u = 0 \ \text{ on } \omega_1 \times \partial\omega_2 \right\},$$

equipped with the norm, (see Section 2.2),

$$|v|_{\mathcal{W}_2(\Omega)} = |\nabla_{X_2}v|_{L^2(\Omega)}.$$

We can easily check that $\mathcal{V}_1(\Omega)$ and $\mathcal{W}_2(\Omega)$ are separable Hilbert spaces. Next we consider $A : \mathcal{V}_1(\Omega) \to \mathcal{V}_1'(\Omega)$ as the linear operator $-\Delta_{X_1}$ associated to a zero Neumann condition on $\partial\omega_1 \times \omega_2$. This operator is linear, bounded but not coercive, since $|\nabla_{X_1}v|_{L^2(\Omega)}$ is not a norm for $\mathcal{V}_1(\Omega)$. However it is positive since

$$\int_\Omega (\Delta_{X_1}v) \ v \, dx = |\nabla_{X_1}v|^2_{L^2(\Omega)} - \int_\Omega \frac{\partial v}{\partial \nu_1} \ v \, dx = |\nabla_{X_1}v|^2_{L^2(\Omega)} \geq 0. \tag{2.24}$$

For the second operator we set $B = -\Delta_{X_2} : \mathcal{W}_2(\Omega) \to \mathcal{W}_2'(\Omega)$ which is linear, bounded and coercive on $\mathcal{W}_2(\Omega)$. In this cylindrical case we have clearly

$$H_0^1(\Omega, \omega_1 \times \partial\omega_2) = \mathcal{V}_1(\Omega) \cap \mathcal{W}_2(\Omega).$$

On one hand, the positivity of A implies that, (see Remark 1.3-*i*),

$$|\nabla_{X_2} u_\varepsilon|_{L^2(\Omega)} \text{ is bounded} \tag{2.25}$$

and, by Poincaré's inequality in the X_2-direction, it follows that

$$|u_\varepsilon|^2_{L^2(\Omega)} < C_0, \tag{2.26}$$

for some positive constant C_0 independent of ε. On the other hand, **if** $|\nabla_{X_1} u_\varepsilon|_{L^2(\Omega)}$ is bounded then obviously $\sqrt{\varepsilon}\nabla_{X_1} u_\varepsilon \to 0$ in $L^2(\Omega)$. **Else**, i.e. if $|\nabla_{X_1} u_\varepsilon|_{L^2(\Omega)} \to +\infty$ (eventually up to a subsequence), then the assumption (1.22) holds since

$$\frac{\int_\Omega (-\Delta_{X_1} u_\varepsilon)\, u_\varepsilon dx}{|u_\varepsilon|_{\mathcal{V}_1(\Omega)}} > \frac{|\nabla_{X_1} u_\varepsilon|^2_{L^2(\Omega)}}{\left(C_0 + |\nabla_{X_1} u_\varepsilon|^2_{L^2(\Omega)}\right)^{1/2}} \simeq |\nabla_{X_1} u_\varepsilon|_{L^2(\Omega)} \to +\infty.$$

when $|u_\varepsilon|_{\mathcal{V}_1(\Omega)} \to \infty$ (i.e. when $|\nabla_{X_1} u_\varepsilon|_{L^2(\Omega)} \to \infty$). Whence by Theorems 1.3 we get, in addition to (2.25) and (2.26),

$$\varepsilon\, |\nabla_{X_1} u_\varepsilon|_{L^2(\Omega)} \text{ is bounded.}$$

The convergence (1.24) also holds in this case. In fact, for some $\zeta_0, \zeta_1 \in L^2(\Omega)$, we have -up to a subsequence-

$$u_\varepsilon \rightharpoonup \zeta_0 \text{ and } \varepsilon\nabla_{X_1} u_\varepsilon \rightharpoonup \zeta_1 \text{ in } L^2(\Omega)$$

and, by the continuous injection $L^2(\Omega) \subset \mathcal{D}'(\Omega)$ and the continuity of the derivative operator on $\mathcal{D}'(\Omega)$, it comes that $\nabla_{X_1} u_\varepsilon \rightharpoonup \nabla_{X_1}\zeta_0$, i.e.

$$\varepsilon\nabla_{X_1} u_\varepsilon \rightharpoonup 0 \text{ in } \mathcal{D}'(\Omega)$$

and (1.24) follows by the uniqueness of the limit in $\mathcal{D}'(\Omega)$.

The limit problem is given by, when $\varepsilon \to 0$,

$$\begin{cases} -\Delta_{X_2}\tilde{u} = f & \text{in } \Omega, \\ u_\varepsilon \in \mathcal{W}_2(\Omega) \end{cases} \tag{2.27}$$

and, as a consequence of Theorems 1.3, 1.4 and Corollary 1.1, we have

Theorem 2.5 *When $\varepsilon \to 0$, we have*

$$u_\varepsilon \to \tilde{u}, \quad \nabla_{X_2} u_\varepsilon \to \nabla_{X_2}\tilde{u} \quad \text{and} \quad \sqrt{\varepsilon}\nabla_{X_1} u_\varepsilon \to 0 \quad \text{in } L^2(\Omega)$$

where $\tilde{u}$ is the unique solution of (2.27).

Remark 2.7 ***i)*** *As above, the limit problem can also be defined, for a.e.* $X_1 \in \omega_1$*, as*

$$\begin{cases} -\Delta_{X_2} u\left(X_1, \cdot\right) = f\left(X_1, \cdot\right) & \text{in } \omega_2, \\ \tilde{u}\left(X_1, \cdot\right) = 0 & \text{on } \partial\omega_2. \end{cases} \tag{2.28}$$

ii) *Arguing as in* Chipot and Guesmia [16] *we can show that, under the assumption* $\nabla_{X_1} f \in \left[L^2\left(\Omega\right)\right]^q$*, we have*

$$\begin{aligned} \left|u_\varepsilon - \tilde{u}\right|_{L^2(\Omega)}, \quad \left|\nabla_{X_2} u_\varepsilon - \nabla_{X_2}\tilde{u}\right|_{L^2(\Omega)} &\leq C\varepsilon \left|\nabla_{X_1} f\right|_{L^2(\Omega)}, \\ \left|\nabla_{X_1} u_\varepsilon - \nabla_{X_1}\tilde{u}\right|_{L^2(\Omega)} &\leq C\sqrt{\varepsilon} \left|\nabla_{X_1} f\right|_{L^2(\Omega)}. \end{aligned}$$

Obviously u_ε *and* $\tilde{u}$ *are identical when* f *depend only on* X_2*, i.e.* $\nabla_{X_1} f = 0$.

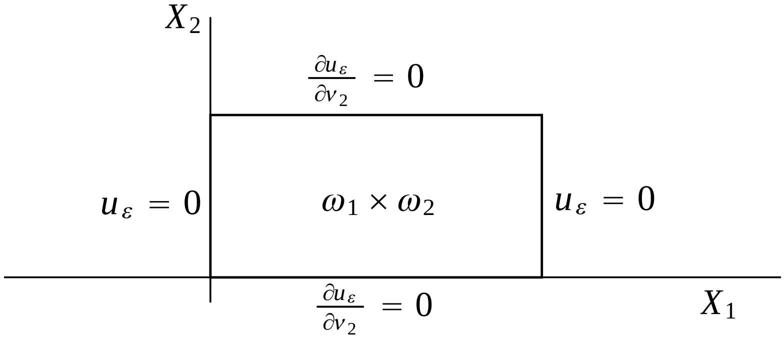
2.5 ANOTHER TYPE OF PROBLEMS

Switching the vertical and the horizontal boundary conditions, in the precedent perturbed problem, an interesting situation arises. For instance, when we pass formally to the limit problem, we loose the uniqueness of solution. In this example we can see how the process of perturbation choose the limit solution among an infinity of possible solutions.

Using the notations of the precedent section, let $\Omega = \omega_1 \times \omega_2$ be a bounded domain in $\mathbb{R}^n$. For $\varepsilon > 0$ and $f \in L^2\left(\Omega\right)$, we consider the following elliptic problem

$$\begin{cases} -\varepsilon\Delta_{X_1} u_\varepsilon - \Delta_{X_2} u_\varepsilon = f & \text{in} \quad \Omega, \\ u_\varepsilon = 0 & \text{on} \quad \partial\omega_1 \times \omega_2 \\ \dfrac{\partial u_\varepsilon}{\partial\nu_2} = 0 & \text{on} \quad \omega_1 \times \partial\omega_2. \end{cases} \tag{2.29}$$

where ν_2 is the outward unit normal vector to $\omega_1 \times \partial\omega_2$.



This problem has a unique solution

$$u_\varepsilon \in H_0^1\left(\Omega, \partial\omega_1 \times \omega_2\right) := \left\{u \in H^1\left(\Omega\right) \middle| u = 0 \text{ on } \partial\omega_1 \times \omega_2\right\}.$$

Since we can not apply the abstract theory of the precedent chapter, a direct approach is necessary. Passing formally to the limit in (2.29), one expects that the limit of u_ε is a solution to the problem

$$\begin{cases} -\Delta_{X_2} u = f & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu_2} = 0 & \text{on} \quad \omega_1 \times \partial\omega_2. \end{cases} \tag{2.30}$$

The solutions of problem (2.30) are only determined up to a function depending only on X_1, since $u + \bar{u}$ is another solution for any $\bar{u} \in H^1(\omega)$.

In order to characterize the solutions of (2.30) which are limits of the perturbation process, among the other solutions, we go back to problem (2.29) and, assuming that u_ε is sufficiently regular, we perform an integration of the first equation over ω_2, we get

$$-\varepsilon \int_{\omega_2} \Delta_{X_1} u_\varepsilon dX_2 - \int_{\omega_2} \Delta_{X_2} u_\varepsilon dX_2 = -\varepsilon \Delta_{X_1} \left(\int_{\omega_2} u_\varepsilon dX_2 \right) - \int_{\partial\omega_2} \frac{\partial u_\varepsilon}{\partial \nu_2} dX_2 = \int_{\omega_2} f dX_2. \tag{2.31}$$

Let us set

$$U_\varepsilon := \int_{\omega_2} u_\varepsilon dX_2, \qquad F := \int_{\omega_2} f dX_2$$

and, of course, both U_ε and F depend only on X_1. Then, since $\dfrac{\partial u_\varepsilon}{\partial \nu_2} = 0$ on $\omega_1 \times \partial\omega_2$, it follows from (2.31) that U_ε is the unique solution of the elliptic problem

$$\begin{cases} -\Delta_{X_1} U_\varepsilon = \dfrac{1}{\varepsilon} F & \text{in} \quad \omega_1, \\ U_\varepsilon = 0 & \text{on} \quad \partial\omega_1. \end{cases}$$

In particular, $|U_\varepsilon|_{L^2(\omega_1)}$ is bounded independently of ε if and only if $F = 0$, i.e. when

$$\int_{\omega_2} f(X_1, \cdot)\, dX_2 = 0 \quad \text{for a.e. } X_1 \in \omega_1. \tag{2.32}$$

In this case it holds that $U_\varepsilon = 0$, i.e.

$$\int_{\omega_2} u_\varepsilon(X_1, \cdot)\, dX_2 = 0 \quad \text{for a.e. } X_1 \in \omega_1.$$

Having this in mind, one expects that any u, limit to u_ε, is characterized by the condition

$$\int_{\omega_2} u(X_1, \cdot)\, dX_2 = 0 \text{ for a.e. } X_1 \in \omega_1.$$

More precisely, we claim that the limit problem in this case is the following elliptic problem, with mixed Neumann and integral conditions,

$$\begin{cases} -\Delta_{X_2} u = f & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu_2} = 0 & \text{on} \quad \omega_1 \times \partial\omega_2, \\ \displaystyle\int_{\omega_2} u(X_1, \cdot)\, dX_2 = 0 & \text{for} \quad a.e.\ X_1 \in \omega_1. \end{cases} \tag{2.33}$$

Remark 2.8 *Of course, for a function* $v \in L^2(\Omega)$, *we have*

$$\left(\int_{\omega_2} v(X_1,\cdot)\,dX_2 = 0 \ \textit{for a.e. } X_1 \in \omega_1\right) \Rightarrow \int_\Omega v(x)\,dx = 0.$$

and the converse implication does not hold in general.

We assume that ω_1 is sufficiently regular to ensure that the injection

$$H^1(\omega_1) \subset L^2(\omega_1) \ \text{ is compact,} \tag{2.34}$$

then the last integral condition in (2.33) guarantee the coerciveness of $-\Delta_{X_2}$ on the space

$$\mathbb{W}_2(\Omega) = \left\{ v \in L^2(\Omega)\,|\nabla_{X_2} v \in \left[L^2(\Omega)\right]^{n-q} \text{ and } \int_{\omega_2} u(X_1,\cdot)\,dX_2 = 0 \text{ for a.e. } X_1 \in \omega_1 \right\}.$$

In fact, by Poincaré's inequality for $H^1(\omega_2)$, (see (A.12)), we have

$$|v(X_1,\cdot)|_{L^2(\omega_2)} \le c|\nabla_{X_2} v(X_1,\cdot)|_{L^2(\omega_2)}, \quad \forall v \in \mathbb{W}_2(\Omega).$$

for some constant c independent of X_1. Then integrating on ω_1, it comes

$$|v|_{L^2(\Omega)} \le C|\nabla_{X_2} v|_{L^2(\Omega)}, \quad \forall v \in \mathbb{W}_2(\Omega). \tag{2.35}$$

Thus the mapping $v \mapsto |\nabla_{X_2} v|_{L^2(\Omega)}$ define a norm on $\mathbb{W}_2(\Omega)$, equivalent to the norm

$$|v|_{L^2(\Omega)} + |\nabla_{X_2} v|_{L^2(\Omega)},$$

and by consequence Problem (2.33) has a unique solution $\tilde{u} \in \mathbb{W}_2(\Omega)$.

In the following we give some convergence results.

Theorem 2.6 *Under the assumptions* (2.34) *and* (2.32) *we have, when* $\varepsilon \to 0$,

$$u_\varepsilon \to \tilde{u}, \quad \nabla_{X_2} u_\varepsilon \to \nabla_{X_2}\tilde{u} \quad \textit{and} \quad \sqrt{\varepsilon}\nabla_{X_1} u_\varepsilon \to 0 \quad \textit{in } L^2(\Omega),$$

where $\tilde{u}$ *is the unique solution of* (2.33).

Proof. Testing the first equation in (2.29) by $v \in \mathbb{W}_2(\Omega)$, yields

$$\int_\Omega \varepsilon \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} v dx + \int_\Omega \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} v\,dx - \int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial \nu} u_\varepsilon d\Gamma = \int_\Omega f v dx \quad \forall v \in \mathbb{W}_2(\Omega)$$

and, taking into account the boundary conditions in (2.29), we derive

$$\int_\Omega \varepsilon \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} v dx + \int_\Omega \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} v\,dx = \int_\Omega f v dx \quad \forall v \in \mathbb{W}_2(\Omega). \tag{2.36}$$

Let us take $v = u_\varepsilon$, it comes

$$\int_\Omega \varepsilon|\nabla_{X_1}u_\varepsilon|^2 + |\nabla_{X_2}u_\varepsilon|^2\,dx \le |f|_{L^2(\Omega)}|u_\varepsilon|_{L^2(\Omega)} \le C|f|_{L^2(\Omega)}|\nabla_{X_2}u_\varepsilon|_{L^2(\Omega)},$$

thanks to Poincaré's inequality (2.35). Then using Young's inequality $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$ yields

$$\int_\Omega \varepsilon|\nabla_{X_1}u_\varepsilon|^2 + \frac{1}{2}|\nabla_{X_2}u_\varepsilon|^2\,dx \le C|f|^2_{L^2(\Omega)}.$$

We deduce that

$$u_\varepsilon, \quad \sqrt{\varepsilon}\nabla_{X_1}u_\varepsilon, \quad \nabla_{X_2}u_\varepsilon \quad \text{are bounded in } L^2(\Omega)$$

where the first estimate also follows by Poincaré's inequality (2.35). Thus there exists u_0, u_1, u_2 such that – up to a subsequence –

$$u_\varepsilon \rightharpoonup u_0, \quad \sqrt{\varepsilon}\nabla_{X_1}u_\varepsilon \rightharpoonup u_1, \quad \nabla_{X_2}u_\varepsilon \rightharpoonup u_2 \quad \text{in } L^2(\Omega)$$

and, by the continuity of the derivative operator on $\mathcal{D}'(\Omega)$, one can check that

$$u_\varepsilon \rightharpoonup u_0, \quad \sqrt{\varepsilon}\nabla_{X_1}u_\varepsilon \rightharpoonup 0, \quad \nabla_{X_2}u_\varepsilon \rightharpoonup \nabla_{X_2}u_0 \quad \text{in } L^2(\Omega). \tag{2.37}$$

Passing to the limit in (2.36) we get

$$\int_\Omega \nabla_{X_2}u_0 \cdot \nabla_{X_2}v\,dx = \int_\Omega fv\,dx \quad \forall v \in W_2(\Omega). \tag{2.38}$$

Taking $v = u_\varepsilon$, in the last equation, and passing to the limit we get

$$|\nabla_{X_2}u_0|^2_{L^2(\Omega)} = \int_\Omega fu_0\,dx. \tag{2.39}$$

Next, one compute

$$\begin{aligned} I_\varepsilon &:= \varepsilon\,|\nabla_{X_1}u_\varepsilon|^2_{L^2(\Omega)} + |\nabla_{X_2}(u_\varepsilon - u_0)|^2_{L^2(\Omega)} \\ &= \varepsilon\,|\nabla_{X_1}u_\varepsilon|^2_{L^2(\Omega)} + |\nabla_{X_2}u_\varepsilon|^2_{L^2(\Omega)} - 2\int_\Omega \nabla_{X_2}u_\varepsilon \cdot \nabla_{X_2}u_0 dx + |\nabla_{X_2}u_0|^2_{L^2(\Omega)}. \end{aligned}$$

Using (2.29), (2.37) and (2.39) we get

$$I_\varepsilon = \int_\Omega fu_\varepsilon\,dx - 2\int_\Omega \nabla_{X_2}u_\varepsilon \cdot \nabla_{X_2}u_0 dx + |\nabla_{X_2}u_0|^2_{L^2(\Omega)} \longrightarrow \int_\Omega fu_0\,dx - |\nabla_{X_2}u_0|^2_{L^2(\Omega)} = 0.$$

It follows that

$$\sqrt{\varepsilon}\nabla_{X_1}u_\varepsilon \to 0, \quad \nabla_{X_2}u_\varepsilon \to \nabla_{X_2}u_0 \quad \text{in } L^2(\Omega)$$

and by the Poincaré inequality, in the X_2-*direction*, it follows

$$u_\varepsilon \to u_0 \quad \text{in } L^2(\Omega).$$

By completeness of $W_2(\Omega)$ for the norm $v \mapsto |\nabla_{X_2}v|_{L^2(\Omega)}$, it follows that $u_0 \in W_2(\Omega)$ and taking into account (2.38) we deduce that $u_0 = \tilde{u}$, i.e. u_0 is the unique solution of (2.33). This ends the proof of the theorem. ■

Remark 2.9 ***i)*** *As above, the limit problem can also be defined, for a.e.* $X_1 \in \omega_1$, *as*

$$\begin{cases} -\Delta_{X_2} u\left(X_1, \cdot\right) = f\left(X_1, \cdot\right) & \text{in} \quad \omega_2, \\ \dfrac{\partial u}{\partial \nu_2}\left(X_1, \cdot\right) = 0 & \text{on} \quad \partial\omega_2, \\ \displaystyle\int_{\omega_2} u\left(X_1, \cdot\right) dX_2 = 0. \end{cases} \tag{2.40}$$

ii) *In the framework of* ℓ *goes to infinity, an analogous condition to* (2.32) *was considered in* Chipot [12]. *See also* [24] *and related works.*

COMMENTS

The anisotropic perturbations of linear elliptic equations is investigated in Chipot [11], Chipot and Guesmia [16, 17], Guesmia [31, 33] and related works. See also Laevsky [44]. Other type of equations are considered in Chipot and Guesmia [15, 18], Guesmia and Sengouga [34, 35].

The first three sections of this chapter has appeared in [21].

3

ANISOTROPIC SINGULAR PERTURBATIONS OF THE $P-$LAPLACIAN

In this chapter we deal with a variational inequality, that involves the $p-$Laplacian operator, perturbed in some directions. In contrast with the $p-$Laplacian type problem considered in Section 2.3, the problem treated here is not covered by the abstract approach given in the first chapter. Moreover the convex set, considered in the variational inequality, may depends on ε and converges (in some suitable sense) when $\varepsilon \to 0$.

3.1 PROBLEM SETTING

Let Ω be a bounded open subset of $\mathbb{R}^n$ and for $\varepsilon > 0$, we consider the following perturbed variational inequality

$$\begin{cases} \displaystyle\int_\Omega \left| \begin{matrix} \varepsilon\nabla_{X_1} u_\varepsilon \\ \nabla_{X_2} u_\varepsilon \end{matrix} \right|^{p-2} \begin{pmatrix} \varepsilon\nabla_{X_1} u_\varepsilon \\ \nabla_{X_2} u_\varepsilon \end{pmatrix} \cdot \begin{pmatrix} \varepsilon\nabla_{X_1} (v_\varepsilon - u_\varepsilon) \\ \nabla_{X_2} (v_\varepsilon - u_\varepsilon) \end{pmatrix} dx \geq \langle f, v_\varepsilon - u_\varepsilon \rangle_{W_0^{1,p}(\Omega)}, \quad \forall v_\varepsilon \in K_\varepsilon, \\ \\ u_\varepsilon \in K_\varepsilon \end{cases}$$

where $K_\varepsilon \neq \varnothing, \forall \varepsilon > 0$, is a closed convex subset of $W_0^{1,p}(\Omega)$. To simplify the notations we denote

$$\nabla^\varepsilon v := \begin{pmatrix} \varepsilon\nabla_{X_1} v \\ \nabla_{X_2} v \end{pmatrix}$$

and we rewrite the above problem as follows

$$\begin{cases} \displaystyle\int_\Omega |\nabla^\varepsilon u_\varepsilon|^{p-2} \nabla^\varepsilon u_\varepsilon \cdot \nabla^\varepsilon (v_\varepsilon - u_\varepsilon)\, dx \geq \langle f, v_\varepsilon - u_\varepsilon \rangle_{W_0^{1,p}(\Omega)}, \quad \forall v_\varepsilon \in K_\varepsilon, \\ u_\varepsilon \in K_\varepsilon. \end{cases} \tag{3.1}$$

Assuming $f \in W^{-1,p'}(\Omega)$, the above problem has a unique solution $u_\varepsilon \in K_\varepsilon$ (see [9, 13]).

In this chapter we shall investigate the behaviour of u_ε when $\varepsilon \to 0$. To this end we need the following lemma (see [42]).

Lemma 3.1 (Minty's Lemma) *Let T be a monotone hemicontinuous operator from a closed convex K to X' and $f \in X'$, then $u_0 \in K$ satisfies*

$$\langle Tu_0, v - u_0\rangle_X \geq \langle f, v - u_0\rangle_X, \quad \forall v \in K,$$

if and only if

$$\langle Tv, v - u_0\rangle_X \geq \langle f, v - u_0\rangle_X, \quad \forall v \in K.$$

3.2 A PRIORI ESTIMATES

In what follows we assume that

$$f \in \mathcal{W}_2'(\Omega) \tag{3.2}$$

where $\mathcal{W}_2(\Omega)$ is the Banach space defined as

$$\mathcal{W}_2(\Omega) := \left\{ u \in L^p(\Omega) \,\middle|\, \nabla_{X_2} u \in [L^p(\Omega)]^{n-q}, \;\; u(X_1, \cdot) \in W_0^{1,p}(\Omega_{X_1}), \quad \text{a.e. } X_1 \in \Pi_1 \right\}.$$

This is a reflexive Banach space equipped, due to Poincaré's inequality in the $X_2 -$ *direction*, with the norm $|v|_{\mathcal{W}_2(\Omega)} := |\nabla_{X_2} v|_{L^p(\Omega)}$.

Remark 3.1 *We can check that*

$$\begin{aligned} W_0^{1,p}(\Omega) &\subset \mathcal{W}_2(\Omega) \subset L^p(\Omega), \\ L^{p'}(\Omega) &\subset \mathcal{W}_2'(\Omega) \subset W^{-1,p'}(\Omega). \end{aligned}$$

The following theorem gives some a priori estimates of u_ε.

Theorem 3.1 *Let u_ε be a solution to (3.1), for $p > 1$. Under the assumption (3.2), in addition we assume that there exists a sequence $(w_\varepsilon)_\varepsilon \subset W_0^{1,p}(\Omega)$, $w_\varepsilon \in K_\varepsilon$ for all $\varepsilon > 0$, s.t.*

$$\varepsilon \nabla_{X_1} w_\varepsilon \quad \textit{and} \quad \nabla_{X_2} w_\varepsilon \quad \textit{are bounded in } L^p(\Omega), \tag{3.3}$$

then we have

$$u_\varepsilon, \quad \varepsilon \nabla_{X_1} u_\varepsilon \quad \textit{and} \quad \nabla_{X_2} u_\varepsilon \quad \textit{are bounded in } L^p(\Omega)$$

and

$$|\nabla^\varepsilon u_\varepsilon|^{p-2} \nabla^\varepsilon u_\varepsilon \quad \textit{is bounded in } L^{p'}(\Omega).$$

Proof. Taking $v_\varepsilon = w_\varepsilon$ in (3.1), it follows that

$$\begin{aligned}\int_\Omega \left|\nabla^\varepsilon u_\varepsilon\right|^{p-2} \nabla^\varepsilon u_\varepsilon \cdot \nabla^\varepsilon \left(u_\varepsilon - w_\varepsilon\right) dx &\leq \langle f, u_\varepsilon - w_\varepsilon \rangle_{\mathcal{W}_2(\Omega)} \\ &\leq |f|_{\mathcal{W}_2'(\Omega)} \left|\nabla_{X_2}\left(u_\varepsilon - w_\varepsilon\right)\right|_{L^p(\Omega)} \\ &\leq |f|_{\mathcal{W}_2'(\Omega)} \left|\nabla^\varepsilon\left(u_\varepsilon - w_\varepsilon\right)\right|_{L^p(\Omega)},\end{aligned}$$

then

$$\left|\nabla^\varepsilon u_\varepsilon\right|^p_{L^p(\Omega)} \leq |f|_{\mathcal{W}_2'(\Omega)} \left(\left|\nabla^\varepsilon u_\varepsilon\right|_{L^p(\Omega)} + \left|\nabla^\varepsilon w_\varepsilon\right|_{L^p(\Omega)}\right) + \int_\Omega \left|\nabla^\varepsilon u_\varepsilon\right|^{p-2} \nabla^\varepsilon u_\varepsilon \cdot \nabla^\varepsilon w_\varepsilon dx. \tag{3.4}$$

Using Hölder's inequality, the last integral term can be majorated as follows

$$\int_\Omega \left|\nabla^\varepsilon u_\varepsilon\right|^{p-2} \nabla^\varepsilon u_\varepsilon \cdot \nabla^\varepsilon w_\varepsilon dx \leq \left|\left|\nabla^\varepsilon u_\varepsilon\right|^{(p-1)}\right|_{L^{p'}(\Omega)} \left|\nabla^\varepsilon w_\varepsilon\right|_{L^p(\Omega)} = \left|\nabla^\varepsilon u_\varepsilon\right|^{p/p'}_{L^p(\Omega)} \left|\nabla^\varepsilon w_\varepsilon\right|_{L^p(\Omega)},$$

since $(p-1)\,p' = p$. Then by Young's inequality $ab \leq \alpha a^p + C_\alpha b^{p'}$ (with $\alpha = 1/4$, see (A.2)), it comes

$$\int_\Omega \left|\nabla^\varepsilon u_\varepsilon\right|^{p-2} \nabla^\varepsilon u_\varepsilon \cdot \nabla^\varepsilon w_\varepsilon dx \leq \frac{1}{4} \left|\nabla^\varepsilon u_\varepsilon\right|^p_{L^p(\Omega)} + C \left|\nabla^\varepsilon w_\varepsilon\right|^p_{L^p(\Omega)}.$$

The same inequality yields

$$|f|_{\mathcal{W}_2'(\Omega)} \left(\left|\nabla^\varepsilon u_\varepsilon\right|_{L^p(\Omega)} + \left|\nabla^\varepsilon w_\varepsilon\right|_{L^p(\Omega)}\right) \leq \frac{1}{4} \left|\nabla^\varepsilon u_\varepsilon\right|^p_{L^p(\Omega)} + C\left(|f|^{p'}_{\mathcal{W}_2'(\Omega)} + \left|\nabla^\varepsilon w_\varepsilon\right|^p_{L^p(\Omega)}\right).$$

Going back to (3.4), we get

$$\frac{1}{2} \left|\nabla^\varepsilon u_\varepsilon\right|^p_{L^p(\Omega)} \leq C\left(|f|^{p'}_{\mathcal{W}_2'(\Omega)} + \left|\nabla^\varepsilon w_\varepsilon\right|^p_{L^p(\Omega)}\right).$$

It follows that $\left|\nabla^\varepsilon u_\varepsilon\right|_{L^p(\Omega)}$ is bounded since $\left|\nabla^\varepsilon w_\varepsilon\right|_{L^p(\Omega)}$ is bounded, i.e.

$$\varepsilon \nabla_{X_1} u_\varepsilon, \quad \nabla_{X_2} u_\varepsilon \quad \text{and} \quad u_\varepsilon \quad \text{are bounded in } L^p(\Omega). \tag{3.5}$$

The boundness of $(u_\varepsilon)_\varepsilon$ follows from Poincaré's inequality in the $X_2 -$ *direction*. For the last estimate in the theorem, one has

$$\left|\left|\nabla^\varepsilon u_\varepsilon\right|^{p-2} \nabla^\varepsilon u_\varepsilon\right|^{p'}_{L^{p'}(\Omega)} = \int_\Omega \left|\nabla^\varepsilon u_\varepsilon\right|^{(p-1)p'} dx = \left|\nabla^\varepsilon u_\varepsilon\right|^p_{L^p(\Omega)},$$

which ends the proof of the theorem. ■

Remark 3.2 ***i)*** *We may also check that (see the proof of the next theorem)*

$$\varepsilon \nabla_{X_1} u_\varepsilon \to 0 \quad \text{in } L^p(\Omega).$$

ii) *In particular (3.3) holds if* $\cap_{\varepsilon<\varepsilon_0} K_\varepsilon \neq \varnothing$, *for some* $\varepsilon_0 > 0$. *For instance, this is the case for monotone sequence (in the inclusion sense) of sets* K_ε, *with a nonempty intersection if the sequence is decreasing. In this case it suffices to fix* $w_\varepsilon = w_0$, *for some* $w_0 \in \cap_{\varepsilon<\varepsilon_0} K_\varepsilon$.

3.3 A CONVERGENCE THEOREM

With an anisotropic perturbation in the variational inequality (3.1), we are led to introduce a definition of convergence, for convex sets, that fit with this problem. More precisely, we claim that the candidate limit of u_ε, when $\varepsilon \to 0$, is the solution of the following problem

$$\begin{cases} \int_\Omega |\nabla_{X_2} u|^{p-2} \nabla_{X_2} u \cdot \nabla_{X_2} (v-u)\, dx \geq \int_\Omega f (v-u)\, dx, \qquad \forall v \in \bar{\mathcal{K}}^{\mathcal{W}_2}, \\ \tilde{u} \in \bar{\mathcal{K}}^{\mathcal{W}_2} \end{cases} \tag{3.6}$$

where $\bar{\mathcal{K}}^{\mathcal{W}_2} \subset \mathcal{W}_2(\Omega)$ is the closure (weak closure) in $\mathcal{W}_2(\Omega)$, of the non-empty set $\mathcal{K}$ defined as a limit of the sequence $(K_\varepsilon)_\varepsilon$ in the following sense:

$$\left.\begin{array}{l} i) \text{ For any } w \in \mathcal{K}, \text{ there exist a sequence} \\ \qquad (w_\varepsilon)_\varepsilon \subset W_0^{1,p}(\Omega), \;\; w_\varepsilon \in K_\varepsilon \;\text{ for all } \varepsilon > 0, \text{ s.t.} \\ \qquad \varepsilon \nabla_{X_1} w_\varepsilon \to 0 \quad \textit{and} \quad \nabla_{X_2} w_\varepsilon \to \nabla_{X_2} w \quad \textit{in } L^p(\Omega). \\ ii) \text{ For any } (w_\varepsilon)_\varepsilon \subset W_0^{1,p}(\Omega), \;\; w_\varepsilon \in K_\varepsilon \text{ for all } \varepsilon > 0, \text{ s.t. -up to a subsequence-} \\ \qquad \varepsilon \nabla_{X_1} w_\varepsilon \rightharpoonup 0 \quad \textit{and} \quad \nabla_{X_2} w_\varepsilon \rightharpoonup \nabla_{X_2} \tilde{w} \quad \textit{in } L^p(\Omega) \\ \qquad \text{then the limit } \tilde{w} \in \bar{\mathcal{K}}^{\mathcal{W}_2}. \end{array}\right\} \tag{3.7}$$

Remark 3.3 *The convergences considered in* (3.7)*, replace the* $s - \liminf K_\varepsilon$ *and* $w - \limsup K_\varepsilon$ *in the definition of the so-called Mosco convergence, introduced in* Mosco [54, 55] *to study* (*isotropic*) *singular perturbations of variational inequalities.*

Remark 3.4 *For* $p > 1$*, it follows from* (3.7−*i*) *that*

$$|\nabla^\varepsilon w_\varepsilon|^{p-2} \nabla^\varepsilon w_\varepsilon \to |\nabla_{X_2} w|^{p-2} \begin{pmatrix} 0 \\ \nabla_{X_2} w \end{pmatrix}, \quad \textit{in } L^{p'}(\Omega). \tag{3.8}$$

In fact, if $p \geq 2$ *then using inequality* (A.6) *and Hölder's inequality for the couple* $\left(\frac{1}{p-1} + \frac{p-2}{p-1} = 1\right)$*, yields*

$$\begin{aligned} &\left| |\nabla_\varepsilon w_\varepsilon|^{p-2} \nabla_\varepsilon w_\varepsilon - |\nabla_{X_2} w|^{p-2} \begin{pmatrix} 0 \\ \nabla_{X_2} w \end{pmatrix} \right|_{L^{p'}(\Omega)} \\ &\leq C \left(\int_\Omega \left| \nabla_\varepsilon w_\varepsilon - \begin{pmatrix} 0 \\ \nabla_{X_2} w \end{pmatrix} \right|^{p'} \{ |\nabla_\varepsilon w_\varepsilon| + |\nabla_{X_2} w| \}^{(p-2)p'} dx \right)^{1/p'} \\ &\qquad \leq C \left(\int_\Omega \left| \begin{matrix} \varepsilon \nabla_{X_1} w_\varepsilon \\ \nabla_{X_2} (w_\varepsilon - w) \end{matrix} \right|^{p'(p-1)} dx \right)^{\frac{1}{p'(p-1)}} \times \left(\int_\Omega \{ |\nabla_\varepsilon w_\varepsilon| + |\nabla_{X_2} w| \}^{p'(p-1)} dx \right)^{\frac{(p-2)}{p'(p-1)}}. \end{aligned}$$

Since $p'(p-1) = p$ *and* $|\nabla_\varepsilon w_\varepsilon|_{L^p(\Omega)}$ *is bounded, it comes that*

$$\left| |\nabla_\varepsilon w_\varepsilon|^{p-2} \nabla_\varepsilon w_\varepsilon - |\nabla_{X_2} w|^{p-2} \begin{pmatrix} 0 \\ \nabla_{X_2} w \end{pmatrix} \right|_{L^{p'}(\Omega)} \leq C \left| \begin{matrix} \varepsilon \nabla_{X_1} w_\varepsilon \\ \nabla_{X_2}(w_\varepsilon - w) \end{matrix} \right|_{L^p(\Omega)} .$$

If $1 < p < 2$, *then by inequality* (A.8) *we obtain*

$$\left| |\nabla_\varepsilon w_\varepsilon|^{p-2} \nabla_\varepsilon w_\varepsilon - |\nabla_{X_2} w|^{p-2} \begin{pmatrix} 0 \\ \nabla_{X_2} w \end{pmatrix} \right|_{L^{p'}(\Omega)} \leq C \left(\int_\Omega \left| \nabla_\varepsilon w_\varepsilon - \begin{pmatrix} 0 \\ \nabla_{X_2} w \end{pmatrix} \right|^{(p-1)p'} dx \right)^{1/p'}$$
$$= C \left| \begin{matrix} \varepsilon \nabla_{X_1} w_\varepsilon \\ \nabla_{X_2}(w_\varepsilon - w) \end{matrix} \right|_{L^p(\Omega)}^{p/p'} .$$

Thus (3.8) *holds for all* $p > 1$.

The operator $\nabla_{X_2} \cdot \left(|\nabla_{X_2} u|^{p-2} \nabla_{X_2} \cdot \right)$, appearing in the variational inequality (3.6) and already considered in Section 2.3, is coercive on $\mathcal{W}_2(\Omega)$, bounded, hemicontinuous and strictly monotone for all $p > 1$. Thus the above problem has a unique solution for all $p > 1$.

Now we can show the following convergence results.

Theorem 3.2 *Under the assumptions of Theorem* 3.1, *assume in addition that* K_ε *converges to the nonempty subset* $\mathcal{K}$ *in the sense of* (3.7), *then we have*

$$u_\varepsilon \to \tilde{u}, \quad \varepsilon \nabla_{X_1} u_\varepsilon \to 0 \quad \textit{and} \quad \nabla_{X_2} u_\varepsilon \to \nabla_{X_2} \tilde{u} \quad \textit{in } L^p(\Omega), \tag{3.9}$$

where $\tilde{u}$ *is the solution to the variational inequality* (3.6).

Proof. Thanks to Theorem 3.1, there exists a subsequence of u_ε -still labelled u_ε- such that

$$u_\varepsilon \rightharpoonup \tilde{u}, \quad \nabla_{X_2} u_\varepsilon \rightharpoonup \nabla_{X_2} \tilde{u}, \quad \varepsilon \nabla_{X_1} u_\varepsilon \rightharpoonup 0 \quad \text{in } L^p(\Omega). \tag{3.10}$$

Such $\tilde{u}$ is necessarily in $\bar{\mathcal{K}}^{\mathcal{W}_2}$ by ($3.7 - ii$). Next we choose an arbitrary $w \in \mathcal{K}$, and let $(w_\varepsilon)_\varepsilon$ be a sequence satisfying ($3.7 - i$). By the monotonicity of the $p-$Laplacian operator we rewrite (3.1) as

$$\int_\Omega |\nabla^\varepsilon w_\varepsilon|^{p-2} \nabla^\varepsilon w_\varepsilon \cdot \nabla^\varepsilon (w_\varepsilon - u_\varepsilon)\, dx \geq \int_\Omega f(w_\varepsilon - u_\varepsilon)\, dx, \quad \forall w_\varepsilon \in K_\varepsilon. \tag{3.11}$$

Then using assumption ($3.7 - i$), (3.8) and (3.10) and passing to the limit, it comes

$$\int_\Omega \left| \nabla_{X_2} w \right|^{p-2} \nabla_{X_2} w \cdot \nabla_{X_2}(w - \tilde{u}) \geq \int_\Omega f(w - \tilde{u})\, dx, \quad \forall w \in \mathcal{K}$$

and by density this inequality holds also for $w \in \bar{\mathcal{K}}^{\mathcal{W}_2}$. Thanks to Minty's lemma, it follows that

$$\int_\Omega \left| \nabla_{X_2} \tilde{u} \right|^{p-2} \nabla_{X_2} \tilde{u} \cdot \nabla_{X_2}(w - \tilde{u})\, dx \geq \int_\Omega f(w - \tilde{u})\, dx, \quad \forall w \in \bar{\mathcal{K}}^{\mathcal{W}_2},$$

i.e. $\tilde{u}$ is the solution to (3.6). In order to show the strong convergence, we note that

$$\begin{aligned}
&\int_\Omega \left\{ |\nabla^\varepsilon u_\varepsilon|^{p-2} \nabla^\varepsilon u_\varepsilon - |\nabla^\varepsilon w_\varepsilon|^{p-2} \nabla^\varepsilon w_\varepsilon \right\} \cdot \nabla^\varepsilon (u_\varepsilon - w_\varepsilon)\, dx \\
&= |\nabla^\varepsilon u_\varepsilon|^p_{L^p(\Omega)} + |\nabla^\varepsilon w_\varepsilon|^p_{L^p(\Omega)} - \int_\Omega |\nabla^\varepsilon u_\varepsilon|^{p-2} \nabla^\varepsilon u_\varepsilon \cdot \nabla^\varepsilon w_\varepsilon dx - \int_\Omega |\nabla^\varepsilon w_\varepsilon|^{p-2} \nabla^\varepsilon w_\varepsilon \cdot \nabla^\varepsilon u_\varepsilon dx \\
&\geq |\nabla^\varepsilon u_\varepsilon|^p_{L^p(\Omega)} + |\nabla^\varepsilon w_\varepsilon|^p_{L^p(\Omega)} - |\nabla^\varepsilon u_\varepsilon|^{p-1}_{L^p(\Omega)} \times |\nabla^\varepsilon w_\varepsilon|_{L^p(\Omega)} - |\nabla^\varepsilon w_\varepsilon|^{p-1}_{L^p(\Omega)} \times |\nabla^\varepsilon u_\varepsilon|_{L^p(\Omega)}\,.
\end{aligned}$$

Hölder's inequality is used to obtain the two last terms. Since $\tilde{u} \in \bar{\mathcal{K}}^{\mathcal{W}_2}$ we can take $w = \tilde{u}$ in (3.7$-i$), i.e.

$$\varepsilon \nabla_{X_1} w_\varepsilon \to 0 \quad \text{and} \quad \nabla_{X_2} w_\varepsilon \to \nabla_{X_2} \tilde{u} \quad \text{in } L^p(\Omega)\,,$$

and it follows that

$$\begin{aligned}
0 &\leq \left\{ |\nabla^\varepsilon u_\varepsilon|^{p-1}_{L^p(\Omega)} - |\nabla^\varepsilon w_\varepsilon|^{p-1}_{L^p(\Omega)} \right\}_{L^p(\Omega)} \times \left\{ |\nabla^\varepsilon u_\varepsilon|_{L^p(\Omega)} - |\nabla^\varepsilon w_\varepsilon|_{L^p(\Omega)} \right\} \\
&\leq \int_\Omega \left\{ |\nabla^\varepsilon u_\varepsilon|^{p-2} \nabla^\varepsilon u_\varepsilon - |\nabla^\varepsilon w_\varepsilon|^{p-2} \nabla^\varepsilon w_\varepsilon \right\} \cdot \nabla^\varepsilon (u_\varepsilon - w_\varepsilon)\, dx \\
&\leq \int_\Omega f\,(u_\varepsilon - w_\varepsilon)\, dx - \int_\Omega |\nabla^\varepsilon w_\varepsilon|^{p-2} \nabla^\varepsilon w_\varepsilon \cdot \nabla^\varepsilon (u_\varepsilon - w_\varepsilon)\, dx \longrightarrow 0.
\end{aligned}$$

Whence

$$|\nabla^\varepsilon u_\varepsilon|_{L^p(\Omega)} \to |\nabla_{X_2} \tilde{u}|_{L^p(\Omega)}\,, \quad \text{as } \varepsilon \to 0.$$

Thanks to (3.10) we have simultaneously

$$\begin{pmatrix} \varepsilon \nabla_{X_1} u_\varepsilon \\ \nabla_{X_2} u_\varepsilon \end{pmatrix} \rightharpoonup \begin{pmatrix} 0 \\ \nabla_{X_2} \tilde{u} \end{pmatrix} \text{ in } L^p(\Omega) \quad \text{and} \quad \left| \begin{matrix} \varepsilon \nabla_{X_1} u_\varepsilon \\ \nabla_{X_2} u_\varepsilon \end{matrix} \right|_{L^p(\Omega)} \longrightarrow \left| \begin{matrix} 0 \\ \nabla_{X_2} \tilde{u} \end{matrix} \right|_{L^p(\Omega)}.$$

Then the strong convergence follows since $L^p(\Omega)$ is uniformly convex (cf. [6]), for all $p > 1$. The strong convergence $u_\varepsilon \to \tilde{u}$ in $L^p(\Omega)$ follows by Poincaré's inequality in the X_2-direction. Of course the convergence holds for the whole sequence since the limit is always the unique solution of (3.6). This ends the proof. ∎

COMMENTS

The study of (isotropic) perturbations of nonlinear variational inequalities, when the perturbation involves the convex set, was first considered by Mosco [54, 55], where the so-called *Mosco convergence* was introduced. Numerous papers have appeared since then, for instance, see Nashed and Liu [56, 57] and the further references therein.

Part II

Perturbations of Evolution Problems of Hyperbolic Type

4

ANISOTROPIC SINGULAR PERTURBATIONS OF LINEAR HYPERBOLIC PROBLEMS

In this chapter, we consider the general linear hyperbolic problems, with coefficients depending in time, defined on an arbitrary domain Ω. We give a convergence theorem in general domains. In the case of cylindrical domains we show a regularity result, then we improve the rate of convergence far from the lateral boundaries.

4.1 NOTATIONS AND PROBLEM SETTING

Let Ω be a bounded open subset of $\mathbb{R}^n$. For a positive constant T, we set

$$Q := (0, T) \times \Omega.$$

Let us denote by $A = (a_{ij}(t, x))$ a $n \times n$ matrix such that

$$a_{ij} \in C^1(\bar{Q}), \quad a_{ij} = a_{ji}, \qquad \forall\, i, j = 1, \cdots, n \tag{4.1}$$

and for some $\lambda > 0$ we have the hyperbolicity hypothesis

$$A\xi \cdot \xi \geq \lambda |\xi|^2 \quad \forall\, \xi \in \mathbb{R}^n, \quad \text{a.e. } x \in \Omega, \ \ \forall t \in (0, T)\,. \tag{4.2}$$

We decompose A into four blocks by writing

$$A := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A' := \frac{\partial}{\partial t} A = \begin{pmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{pmatrix}, \tag{4.3}$$

where A_{11} and A_{22} are respectively $q \times q$ and $(n-q) \times (n-q)$ matrices. We then set for every $0 < \varepsilon < 1$

$$A_\varepsilon := A_\varepsilon(t,x) = \begin{pmatrix} \varepsilon^2 A_{11} & \varepsilon A_{12} \\ \varepsilon A_{21} & A_{22} \end{pmatrix}. \tag{4.4}$$

Using (4.1) we can see that $\exists C > 0$,

$$A\xi \cdot \eta \leq C\,|\eta|\,|\xi|\,, \qquad A'\xi \cdot \eta \leq C\,|\eta|\,|\xi|\,. \tag{4.5}$$

We also have from (4.2), for a.e. $x \in \Omega$ and $\forall t \in [0,T]$

$$A_\varepsilon \xi \cdot \xi \geq \lambda \left(\varepsilon^2 |\bar{\xi}_1|^2 + |\bar{\xi}_2|^2 \right), \quad \forall \xi \in \mathbb{R}^n, \tag{4.6}$$

$$A_{22}\bar{\xi}_2 \cdot \bar{\xi}_2 \geq \lambda |\bar{\xi}_2|^2, \quad \forall \bar{\xi}_2 \in \mathbb{R}^{n-q}, \tag{4.7}$$

where $\xi := (\bar{\xi}_1, \bar{\xi}_2)^T$ with $\bar{\xi}_1 := (\xi_1, \dots, \xi_q)^T$ and $\bar{\xi}_2 := (\xi_{q+1}, \dots, \xi_n)^T$. This means that A_ε and A_{22} are positive definite.

Existence and uniqueness

In this section we would like to consider the following problem

$$\begin{cases} u'' - \nabla \cdot (A_\varepsilon \nabla u) = f & \text{in} \quad Q, \\ u = 0 & \text{on} \quad (0,T) \times \partial\Omega, \\ u\,(0) = u^0_\varepsilon, \;\; u'\,(0) = u^1_\varepsilon & \text{in} \quad \Omega \end{cases} \tag{4.8}$$

where u', u'' stands for the derivatives in time of u and the superscript, in the initial conditions, should not be confused with the power notation.

We make the following assumptions on the initial data and the source term

$$u^0_\varepsilon \in H^1_0(\Omega), \quad u^1_\varepsilon \in L^2(\Omega), \quad f \in L^2(0,T; L^2\,(\Omega)) = L^2(Q). \tag{4.9}$$

Under the assumptions (4.1), (4.2) and (4.9) there exists a weak solution u_ε satisfying

$$\begin{cases} u_\varepsilon \in C([0,T]; H^1_0\,(\Omega)), \;\; u'_\varepsilon \in C([0,T]; L^2\,(\Omega)), \;\; u''_\varepsilon \in L^2(0,T; H^{-1}\,(\Omega)), \\ \langle u''_\varepsilon(t,x), v \rangle_{H^1_0(\Omega)} + \displaystyle\int_\Omega A_\varepsilon(t,x) \nabla u_\varepsilon(t,x) \cdot \nabla v dx \\ \qquad\qquad = \displaystyle\int_\Omega f(t,x) v dx, \;\; \forall t \in (0,T)\,, \;\; \forall v \in H^1_0\,(\Omega)\,, \\ u_\varepsilon\,(0) = u^0_\varepsilon, \quad u'_\varepsilon\,(0) = u^1_\varepsilon \quad \text{in } \Omega. \end{cases} \tag{4.10}$$

We recall, for the sequel of this chapter, that problem (4.10) is a particular case of the general abstract setting of existence and uniqueness of a weak solution of hyperbolic problems, see [22, 49]. Let H be a Hilbert space with scalar product $(\cdot,\cdot)_H$. The dual space H is identified with H. Let V be another Hilbert space. We assume that $V \subset H$ with dense and continuous injection[1], so that

$$V \subset H \subset V'. \tag{4.11}$$

For each $t \in [0,T]$ we are given a symmetric continuous bilinear form $a(t;u,v) : V \times V \to \mathbb{R}$ such that

(i) the function $t \to a\,(t;u,v)$ is of class $C^1 \quad \forall u,v \in V$,

(ii) $a\,(t;v,v) \geq \alpha\,|v|_V^2 - C|v|_H^2, \quad \forall t \in [0,T], \forall v \in V, \alpha > 0.$

Theorem 4.1 ([49]) *Given $f \in L^2(0,T;H)$, $u^0 \in V$, and $u^1 \in H$, there exists a unique function u satisfying*

$$\begin{cases} u \in C\left([0,T]\,;V\right), u' \in C\left([0,T]\,;H\right), u'' \in L^2\left(0,T;V'\right), \\ \langle u'',v\rangle_V + a\left(t;u(t),v\right) = \langle f(t),v\rangle_V\,, \quad \textit{for a.e. } t \in (0,T), \quad \forall v \in V, \\ u(0) = u^0 \textit{ and } u'(0) = u^1. \end{cases}$$

and the following energy equality holds

$$\begin{aligned} \left|u_\varepsilon'(t)\right|_H^2 + a\left(t;u(t),u(t)\right) &= \left|u_\varepsilon^1\right|_H^2 + a\left(0;u^0,u^0\right) \\ &\quad + \int_0^t a'\left(t;u(t),u(t)\right) ds + 2\int_0^t \left(f,u'\right) ds, \quad \forall t \in [0,T]\,. \end{aligned}$$

The limit problem

We assume that there exists $u^0 \in H_0^1(\Omega)$ and $u^1 \in L^2\left(\Omega\right)$ s.t.

$$u_\varepsilon^0 \to u^0 \text{ in } H_0^1(\Omega), \qquad u_\varepsilon^1 \to u^1 \text{ in } L^2(\Omega), \tag{4.12}$$

as $\varepsilon \to 0$. Then, due to the structure of the matrix A_ε, the limit problem does not contain derivatives in the $X_1 -$ *direction* and it will be natural to consider as candidate limit of u_ε a function $\tilde{u}$, defined for a.e. $X_1 \in \Pi_1$, as a solution to

$$\begin{cases} \tilde{u}''(\cdot\,;X_1,\cdot) - \nabla_{X_2}\cdot\left(A_{22}(\cdot\,;X_1,\cdot)\nabla_{X_2}\tilde{u}(\cdot\,;X_1,\cdot)\right) = f(\cdot\,;X_1,\cdot) & \text{in } Q_{X_1} = (0,T)\times\Omega_{X_1}, \\ \tilde{u}(\cdot\,;X_1,\cdot) = 0 & \text{on } (0,T)\times\partial\Omega_{X_1}, \\ \tilde{u}\left(0;X_1,\cdot\right) = u^0\left(X_1,\cdot\right) \text{ and } \tilde{u}'\left(0;X_1,\cdot\right) = u^1\left(X_1,\cdot\right) & \text{in } \Omega_{X_1}. \end{cases} \tag{4.13}$$

---

[1] The injection $V \subset H$ is not assumed compact.

In fact it holds that (see Chipot and Guesmia [16])

$$u^0(X_1,\cdot)\in H_0^1(\Omega_{X_1}),\quad u^1(X_1,\cdot)\in L^2(\Omega_{X_1})\quad\text{and}\quad f(\cdot;X_1,\cdot)\in L^2(0,T;L^2(\Omega_{X_1}))$$

for a.e. $X_1\in\Pi_1$. Then problem (4.13) has a unique solution in the sense that

$$\begin{cases} \tilde{u}(\cdot;X_1,\cdot)\in C([0,T];H_0^1(\Omega_{X_1})),\ \ \tilde{u}'(\cdot;X_1,\cdot)\in C([0,T];L^2(\Omega_{X_1})),\\ \tilde{u}''(\cdot;X_1,\cdot)\in L^2(0,T;H^{-1}(\Omega_{X_1})),\\ \langle\tilde{u}''(t;X_1,X_2),v\rangle_{H_0^1(\Omega_{X_1})}+\int_{\Omega_{X_1}}A_{22}(t;X_1,X_2)\nabla_{X_2}\tilde{u}(t;X_1,X_2)\cdot\nabla_{X_2}v\,dX_2\\ \qquad\qquad=\int_{\Omega_{X_1}}f(t;X_1,X_2)v\,dX_2,\qquad\forall t\in(0,T),\quad\forall v\in H_0^1(\Omega_{X_1}),\\ \tilde{u}(0,X_1,\cdot)=u^0(X_1,\cdot),\ \ \tilde{u}'(0,X_1,\cdot)=u^1(X_1,\cdot)\quad\text{in }\ \Omega_{X_1}, \end{cases}\tag{4.14}$$

for a.e. $X_1\in\Pi_1$. Moreover, as in Section 2.2, if we consider the space

$$\mathcal{V}_2(\Omega):=\left\{v\in L^2(\Omega)\,|\nabla_{X_2}v\in L^2(\Omega),\ v(X_1,\cdot)\in H_0^1(\Omega_{X_1})\text{ a.e. }X_1\in\Pi_1\right\}\tag{4.15}$$

equipped with the norm $|\nabla_{X_2}v|_{L^2(\Omega)}$, then we show, in the next theorem, that $\tilde{u}$ is also solution to the following problem

$$\begin{cases} u\in C([0,T];\mathcal{V}_2(\Omega)),\ \ u'\in C([0,T];L^2(\Omega)),\ \ u''\in L^2(0,T;\mathcal{V}_2'(\Omega)),\\ \langle u''(t,x),v\rangle_{\mathcal{V}_2(\Omega)}+\int_\Omega A_{22}(t,x)\nabla_{X_2}u(t,x)\cdot\nabla_{X_2}v\,dx\\ \qquad\qquad=\int_\Omega f(t,x)v\,dx,\quad\forall t\in(0,T),\quad\forall v\in\mathcal{V}_2(\Omega),\\ u(0)=u^0,\quad u'(0)=u^1\qquad\text{in }\ \Omega. \end{cases}\tag{4.16}$$

which has a unique solution. To see that problem (4.16) has a unique solution, it suffices to take the triple $\mathcal{V}_2(\Omega)\subset L^2(\Omega)\subset\mathcal{V}_2'(\Omega)$ in (4.11) and apply Theorem 4.1. The solution also satisfies, for every $t\in[0,T]$, the energy equality

$$\begin{aligned} |u'(t)|^2_{L^2(\Omega)}&+\int_\Omega A_{22}(t)\nabla_{X_2}u(t)\cdot\nabla_{X_2}u(t)\,dx\\ &=\left|u^1\right|^2_{L^2(\Omega)}+\int_\Omega A_{22}(0)\nabla_{X_2}u^0\cdot\nabla_{X_2}u^0dx\\ &\quad+2\int_0^t\int_\Omega fu'dxds+\int_0^t\int_\Omega A_{22}'\nabla_{X_2}u\cdot\nabla_{X_2}udxds. \end{aligned}\tag{4.17}$$

4.2 CONVERGENCES IN ARBITRARY DOMAINS

We are now ready to state the main result in this section.

Theorem 4.2 *Under the assumptions above, we have for every* $t \in [0,T]$

$$\begin{aligned} u_\varepsilon(t) \to \tilde{u}(t) \quad &in \quad \mathcal{V}_2(\Omega), \\ \varepsilon\nabla_{X_1} u_\varepsilon(t) \to 0, \quad u'_\varepsilon(t) \to \tilde{u}'(t) \quad &in \quad L^2(\Omega), \end{aligned} \tag{4.18}$$

where $\tilde{u}$ *(resp.* u_ε*) is the solution of* (4.14) *(resp.* (4.10)*). Moreover,* $\tilde{u}$ *is also the solution of problem* (4.16).

Proof. We first show that u_ε converges to the solution of (4.16), then we check that the solution of problems (4.14) and (4.16) is the same. The proof is given in several steps.

I) *A priori estimates.* The energy equality for problem (4.10) is given by

$$\begin{aligned} |u'_\varepsilon(t)|^2_{L^2(\Omega)} + \int_\Omega A_\varepsilon \nabla u_\varepsilon(t)\cdot\nabla u_\varepsilon(t)dx = \left|u^1_\varepsilon\right|^2_{L^2(\Omega)} + \int_\Omega A_\varepsilon(0)\nabla u^0_\varepsilon \cdot \nabla u^0_\varepsilon dx \\ + \int_0^t \int_\Omega A'_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon dxds + 2\int_0^t\int_\Omega f u'_\varepsilon dxds, \end{aligned} \tag{4.19}$$

for every $t \in [0,T]$. Using (4.1), (4.5) and (4.6), we get

$$\begin{aligned} |u'_\varepsilon(t)|^2_{L^2(\Omega)} &+\lambda\left(\varepsilon^2 |\nabla_{X_1} u_\varepsilon(t)|^2_{L^2(\Omega)} + |\nabla_{X_2} u_\varepsilon(t)|^2_{L^2(\Omega)}\right) \\ &\leq \left|u^1_\varepsilon\right|^2_{L^2(\Omega)} + C\varepsilon^2 |\nabla_{X_1} u^0_\varepsilon|^2_{L^2(\Omega)} + C|\nabla_{X_2} u^0_\varepsilon|^2_{L^2(\Omega)} \\ &\quad + 2\int_0^t \int_\Omega |f u'_\varepsilon| dxds + C\int_0^t \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2_{L^2(\Omega)} + |\nabla_{X_2} u_\varepsilon|^2_{L^2(\Omega)} ds, \end{aligned}$$

where C is a positive constant independent of ε. For ε small enough, and thanks to (4.12), we derive

$$\left|u^1_\varepsilon\right|^2_{L^2(\Omega)} \leq 2\left|u^1\right|^2_{L^2(\Omega)} \quad \text{and} \quad \varepsilon^2 |\nabla_{X_1} u^0_\varepsilon|^2_{L^2(\Omega)} + |\nabla_{X_2} u^0_\varepsilon|^2_{L^2(\Omega)} \leq 2|\nabla_{X_2} u^0|^2_{L^2(\Omega)}. \tag{4.20}$$

Of course if $u^1 = 0$ or $u^0 = 0$ the right hand side in the above inequalities can be replaced by any positive constant. This implies

$$\begin{aligned} |u'_\varepsilon(t)|^2_{L^2(\Omega)} &+\lambda\left(\varepsilon^2 |\nabla_{X_1} u_\varepsilon(t)|^2_{L^2(\Omega)} + |\nabla_{X_2} u_\varepsilon(t)|^2_{L^2(\Omega)}\right) \\ &\leq 2\left|u^1\right|^2_{L^2(\Omega)} + C|\nabla_{X_2} u^0|^2_{L^2(\Omega)} + C|f|^2_{L^2(Q)} \\ &\quad + C\int_0^t |u'_\varepsilon|^2_{L^2(\Omega)} + \lambda\left(\varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2_{L^2(\Omega)} + |\nabla_{X_2} u_\varepsilon|^2_{L^2(\Omega)}\right) ds. \end{aligned}$$

Using Gronwall's inequality (see A.13) we get for every $t \in [0,T]$

$$|u'_\varepsilon(t)|^2_{L^2(\Omega)} + \lambda\left(\varepsilon^2 |\nabla_{X_1} u_\varepsilon(t)|^2_{L^2(\Omega)} + |\nabla_{X_2} u_\varepsilon(t)|^2_{L^2(\Omega)}\right) \leq C. \tag{4.21}$$

Thus

$$\begin{aligned} (u_\varepsilon)_\varepsilon \quad &\text{is bounded in} \quad L^\infty(0,T;\mathcal{V}_2(\Omega)), \\ (u'_\varepsilon)_\varepsilon \ \text{ and } \ (\varepsilon\nabla_{X_1} u_\varepsilon)_\varepsilon \quad &\text{are bounded in} \quad L^\infty(0,T;L^2(\Omega)). \end{aligned} \tag{4.22}$$

Going back to (4.10) and expanding A_ε to its different blocks we get

$$\begin{aligned}\left|\left\langle u_\varepsilon''(t), v\right\rangle_{H_0^1(\Omega)}\right| &\leq \left|\int_\Omega f(t)\, v dx\right| + \left|\int_\Omega A_\varepsilon(t)\nabla u_\varepsilon(t)\cdot\nabla v dx\right| \\ &\leq |f(t)|_{L^2(\Omega)}|v|_{L^2(\Omega)} + C\varepsilon\left|\nabla_{X_1}u_\varepsilon(t)\right|_{L^2(\Omega)}\left(\varepsilon\left|\nabla_{X_1}v\right|_{L^2(\Omega)} + \left|\nabla_{X_2}v\right|_{L^2(\Omega)}\right) \\ &\quad + C\left|\nabla_{X_2}u_\varepsilon(t)\right|_{L^2(\Omega)}\left(\varepsilon\left|\nabla_{X_1}v\right|_{L^2(\Omega)} + \left|\nabla_{X_2}v\right|_{L^2(\Omega)}\right).\end{aligned}$$

Then we apply Poincaré's inequality and regroup similar terms, it comes

$$\left|\left\langle u_\varepsilon''(t), v\right\rangle_{H_0^1(\Omega)}\right| \leq C\left\{|f(t)|_{L^2(\Omega)} + \varepsilon\left|\nabla_{X_1}u_\varepsilon(t)\right|_{L^2(\Omega)} + \left|\nabla_{X_2}u_\varepsilon(t)\right|_{L^2(\Omega)}\right\}|\nabla v|_{L^2(\Omega)},$$

for every $v \in H_0^1(\Omega)$ and a.e. $t \in [0, T]$, hence

$$\left|u_\varepsilon''\right|_{H^{-1}(\Omega)} \leq C\left(|f|_{L^2(\Omega)} + \varepsilon\left|\nabla_{X_1}u_\varepsilon\right|_{L^2(\Omega)} + \left|\nabla_{X_2}u_\varepsilon\right|_{L^2(\Omega)}\right). \tag{4.23}$$

Then taking the square in both sides and integrating over $(0, T)$, we obtain

$$\left|u_\varepsilon''\right|^2_{L^2(0,T;H^{-1}(\Omega))} \leq C\left(|f|^2_{L^2(Q)} + \varepsilon^2\left|\nabla_{X_1}u_\varepsilon\right|^2_{L^2(Q)} + \left|\nabla_{X_2}u_\varepsilon\right|^2_{L^2(Q)}\right).$$

By (4.22) the second member is bounded, then

$$\left(u_\varepsilon''\right)_\varepsilon \quad \text{is bounded in } L^2(0,T;H^{-1}(\Omega)). \tag{4.24}$$

II) *Weak convergences.* In this step we study the weak convergence of u_ε in Q. Thanks to (4.22) and (4.24), we can extract from $(u_\varepsilon)_\varepsilon$ a weak star converging subsequence -still labeled u_ε- such that

$$\begin{aligned} & u_\varepsilon \overset{*}{\rightharpoonup} z \quad in \quad L^\infty(0,T;\mathcal{V}_2(\Omega)), \\ \varepsilon\nabla_{X_1}u_\varepsilon \overset{*}{\rightharpoonup} 0, \quad & u_\varepsilon' \overset{*}{\rightharpoonup} z' \quad in \quad L^\infty(0,T;L^2(\Omega)), \\ & u_\varepsilon'' \rightharpoonup z'' \quad in \quad L^2(0,T;H^{-1}(\Omega)). \end{aligned} \tag{4.25}$$

To verify that the limits are as stated, we use the continuous injections $L^2(Q) \subset \mathcal{D}'(Q)$ and $L^2(0,T;H^{-1}(\Omega)) \subset \mathcal{D}'(Q)$ with the continuity of the derivative operator in $\mathcal{D}'(Q)$. In order to show that z satisfies the equation in (4.16), we multiply (4.10) by $\phi \in C^\infty([0,T])$ and integrate over $[0,t]$, we get

$$\int_0^t \left\langle u_\varepsilon'', v\phi(s)\right\rangle_{H_0^1(\Omega)} ds + \int_0^t\int_\Omega A_\varepsilon\nabla u_\varepsilon\cdot\nabla v\phi(s)\, dxds = \int_0^t\int_\Omega fv\phi(s)\, dxds$$

for every $t \in [0, T]$. Expanding this identity, using the different blocks of A, we derive

$$\begin{aligned}&\int_0^t \left\langle u_\varepsilon'', v\phi\right\rangle_{H_0^1(\Omega)} ds + \int_0^t\int_\Omega \varepsilon^2 A_{11}\nabla_{X_1}u_\varepsilon\cdot\nabla_{X_1}v\phi dxds + \int_0^t\int_\Omega \varepsilon A_{12}\nabla_{X_2}u_\varepsilon\cdot\nabla_{X_1}v\phi dxds \\ &\quad + \int_0^t\int_\Omega \varepsilon A_{21}\nabla_{X_1}u_\varepsilon\cdot\nabla_{X_2}v\phi dxds + \int_0^t\int_\Omega A_{22}\nabla_{X_2}u_\varepsilon\cdot\nabla_{X_2}v\phi dxds = \int_0^t\int_\Omega fv\phi dxds.\end{aligned}$$

Then passing to the limit using (4.25), it follows that

$$\int_0^t \langle z'', v\rangle_{H_0^1(\Omega)} \phi ds + \int_0^t \int_\Omega A_{22}\nabla_{X_2} z \cdot \nabla_{X_2} v\phi dxds = \int_0^t \int_\Omega f v\phi dxds,$$

for every $v \in \mathcal{D}(\Omega)$ and $\phi \in C^\infty([0,T])$. Thus

$$z'' = (f - \nabla_{X_2} \cdot A_{22}\nabla_{X_2} z) \in L^2(0,T; L^2(\Omega)) + L^2(0,T; \mathcal{V}_2'(\Omega)) \subset L^2(0,T; \mathcal{V}_2'(\Omega)),$$

and by consequence z satisfies the equation in (4.16), since, for a.e. $t \in (0,T)$,

$$\langle z''(t), v\rangle_{H_0^1(\Omega)} = \langle z''(t), v\rangle_{\mathcal{V}_2(\Omega)}, \quad \forall \in \mathcal{D}(\Omega).$$

Now, in order to check the initial conditions, we first derive from (4.25) that[2] $z \in C([0,T]; L^2(\Omega))$ and $z' \in C([0,T]; \mathcal{V}_2'(\Omega))$ since we have (see [22, 26])

$$\begin{aligned} \{v | v \in L^2(0,T; \mathcal{V}_2(\Omega)), v \in L^2(0,T; L^2(\Omega))\} &\subset C([0,T]; L^2(\Omega)), \\ \{v | v \in L^2(0,T; L^2(\Omega)), v \in L^2(0,T; \mathcal{V}_2'(\Omega))\} &\subset C([0,T]; \mathcal{V}_2'(\Omega)). \end{aligned}$$

So $z(0)$ and $z'(0)$ make sense, at least in $L^2(\Omega)$ and $\mathcal{V}_2'(\Omega)$ respectively. To show that $z(0) = u^0$ and $z'(0) = u^1$, we use the identities

$$\begin{aligned} \int_0^t \langle z'', v\phi\rangle_{\mathcal{V}_2(\Omega)} ds &= \langle z'(t), v\phi(t)\rangle_{\mathcal{V}_2(\Omega)} - \langle z'(0), v\phi(0)\rangle_{\mathcal{V}_2(\Omega)} \\ &\quad - \int_\Omega z(t) v\phi'(t)dx + \int_\Omega z(0) v\phi'(0)dx + \int_0^t \int_\Omega z v\phi'' dxds \end{aligned} \tag{4.26}$$

and

$$\begin{aligned} \int_0^t \langle u_\varepsilon'', v\phi\rangle_{H_0^1(\Omega)} ds &= \int_\Omega u_\varepsilon'(t) v\phi(t)dx - \int_\Omega u_\varepsilon^1 v\phi(0)dx \\ &\quad - \int_\Omega u_\varepsilon(t) v\phi'(t)dx + \int_\Omega u_\varepsilon^0 v\phi'(0)dx + \int_0^t \int_\Omega u_\varepsilon v\phi'' dxds. \end{aligned} \tag{4.27}$$

(We integrated by parts twice with respect to t in (4.26) and (4.27)). Then, passing to the limit in (4.27), using (4.12), (4.25) and taking $\phi(t) = \phi'(t) = 0$, we get

$$\begin{aligned} \int_0^t \langle z'', v\phi\rangle_{H_0^1(\Omega)} ds &= \int_0^t \langle z'', v\phi\rangle_{\mathcal{V}_2(\Omega)} ds \\ &= -\int_\Omega u^1 v\phi(0)dx + \int_\Omega u^0 v\phi'(0)dx + \int_0^t \int_\Omega z v\phi'' dxds. \end{aligned} \tag{4.28}$$

By comparing (4.26) and (4.28), we deduce that for every $v \in \mathcal{D}(\Omega)$

$$-\langle z'(0), v\phi(0)\rangle_{\mathcal{V}_2(\Omega)} + \int_\Omega z(0) v\phi'(0)dx = -\int_\Omega u^1 \phi(0) v dx + \int_\Omega u^0 v\phi'(0)dx.$$

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[2] after possibly being redefined on a set of measure zero.

Then we conclude that $z(0) = u^0$ by choosing $\phi(0) = 0$, $\phi'(0) = 1$ and that $z'(0) = u^1$ by choosing $\phi(0) = 1$, $\phi'(0) = 0$. Thus we have shown that the unique limit of u_ε is the solution of (4.16) and the convergences (4.25) hold for the whole sequence.

Let us fix some $t \in [0, T]$, since the estimate (4.21) holds for every $t \in [0, T]$, there exist ζ_t^0, ζ_t^1 and ζ_t^2 such that -up to a subsequence-

$$\varepsilon \nabla_{X_1} u_\varepsilon(t) \rightharpoonup \zeta_t^2, \quad u_\varepsilon'(t) \rightharpoonup \zeta_t^1 \text{ in } L^2(\Omega) \text{ and } u_\varepsilon(t) \rightharpoonup \zeta_t^0 \text{ in } \mathcal{V}_2(\Omega).$$

Passing to the limit in (4.27) for $\phi(0) = \phi'(0) = 0$, comparing with (4.26) and arguing as above, in the initial conditions, for different choices of $\phi(t)$ and $\phi'(t)$ yield

$$\zeta_t^0 = z(t), \quad \zeta_t^1 = z'(t)$$

and by consequence $\zeta_t^2 = 0$. Then, by the uniqueness of the limit, the following convergences hold for the whole sequence and for every $t \in [0, T]$

$$\varepsilon \nabla_{X_1} u_\varepsilon(t) \rightharpoonup 0, \quad u_\varepsilon'(t) \rightharpoonup z'(t) \text{ in } L^2(\Omega) \text{ and } u_\varepsilon(t) \rightharpoonup z(t) \text{ in } \mathcal{V}_2(\Omega). \tag{4.29}$$

III) *Strong convergences.* Now we show the strong convergence of u_ε. We set for every $t \in [0, T]$

$$\begin{aligned} J_\varepsilon(t) &:= \left|(u_\varepsilon - z)'(t)\right|^2_{L^2(\Omega)} \\ &+ \int_\Omega A_\varepsilon(t) \begin{pmatrix} \nabla_{X_1} u_\varepsilon(t) \\ \nabla_{X_2}(u_\varepsilon - z)(t) \end{pmatrix} \cdot \begin{pmatrix} \nabla_{X_1} u_\varepsilon(t) \\ \nabla_{X_2}(u_\varepsilon - z)(t) \end{pmatrix} dx \\ &- \int_0^t \int_\Omega A_\varepsilon' \begin{pmatrix} \nabla_{X_1} u_\varepsilon \\ \nabla_{X_2}(u_\varepsilon - z) \end{pmatrix} \cdot \begin{pmatrix} \nabla_{X_1} u_\varepsilon \\ \nabla_{X_2}(u_\varepsilon - z) \end{pmatrix} dxds. \end{aligned} \tag{4.30}$$

Developing J_ε, by using the different blocks of A, we get

$$\begin{aligned} J_\varepsilon(t) = {} & \left|u_\varepsilon'(t)\right|^2_{L^2(\Omega)} + \left|z'(t)\right|^2_{L^2(\Omega)} - 2\int_\Omega u_\varepsilon'(t)\, z'(t)\, dx \\ & + \int_\Omega A_\varepsilon(t) \nabla u_\varepsilon(t) \cdot \nabla u_\varepsilon(t)\, dx - \int_0^t \int_\Omega A_\varepsilon' \nabla u_\varepsilon \cdot \nabla u_\varepsilon dxds \\ & - \int_\Omega \varepsilon A_{12}(t) \nabla_{X_2} u_\varepsilon(t) \cdot \nabla_{X_1} z(t)\, dx - \int_\Omega \varepsilon A_{21}(t) \nabla_{X_1} u_\varepsilon(t) \cdot \nabla_{X_2} z(t)\, dx \\ & - \int_\Omega A_{22}(t) \nabla_{X_2} u_\varepsilon(t) \cdot \nabla_{X_2} z(t)\, dx + \int_\Omega A_{22}(t) \nabla_{X_2} z(t) \cdot \nabla_{X_2}(z - u_\varepsilon)(t)\, dx \\ & - \int_0^t \int_\Omega \varepsilon A_{12}' \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_1} z dxds - \int_0^t \int_\Omega \varepsilon A_{21}' \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_2} z dxds \\ & - \int_0^t \int_\Omega A_{22}' \nabla_{X_2} z \cdot \nabla_{X_2} u_\varepsilon dxds + \int_0^t \int_\Omega A_{22}' \nabla_{X_2} z \cdot \nabla_{X_2}(z - u_\varepsilon)\, dxds. \end{aligned}$$

Using the energy equality (4.19) we derive

$$\begin{aligned} J_\varepsilon(t) = {} & \left|u_\varepsilon^1\right|^2_{L^2(\Omega)} + \int_\Omega A_\varepsilon(0)\nabla u_\varepsilon^0\cdot\nabla u_\varepsilon^0 dx - 2\int_\Omega u_\varepsilon'(t)\, z'(t)\, dx + 2\int_0^t\int_\Omega f u_\varepsilon' dxds \\ & - \int_\Omega \varepsilon A_{12}(t)\nabla_{X_2}u_\varepsilon(t)\cdot\nabla_{X_1}z(t)\,dx - \int_\Omega \varepsilon A_{21}(t)\nabla_{X_1}u_\varepsilon(t)\cdot\nabla_{X_2}z(t)\,dx \\ & - \int_\Omega A_{22}(t)\nabla_{X_2}z(t)\cdot\nabla_{X_2}u_\varepsilon(t)\,dx + \int_\Omega A_{22}(t)\nabla_{X_2}z(t)\cdot\nabla_{X_2}(z-u_\varepsilon)(t)\,dx \\ & - \int_0^t\int_\Omega \varepsilon A_{12}'\nabla_{X_2}u_\varepsilon\cdot\nabla_{X_1}z dxds - \int_0^t\int_\Omega \varepsilon A_{21}'\nabla_{X_1}u_\varepsilon\cdot\nabla_{X_2}z dxds \\ & - \int_0^t\int_\Omega A_{22}'\nabla_{X_2}z\cdot\nabla_{X_2}u_\varepsilon dxds + \int_0^t\int_\Omega A_{22}'\nabla_{X_2}z\cdot\nabla_{X_2}(z-u_\varepsilon)\,dxds. \end{aligned}$$

Then, by the weak limits in (4.25) and (4.29), it follows that

$$\begin{aligned} \lim_{\varepsilon\to 0} J_\varepsilon(t) = {} & -\left|z'(t)\right|^2_{L^2(\Omega)} + \left|u^1\right|^2_{L^2(\Omega)} - \int_\Omega A_{22}\nabla_{X_2}z(t)\cdot\nabla_{X_2}z(t)\,dx \\ & + \int_\Omega A_{22}(0)\nabla_{X_2}u^0\cdot\nabla_{X_2}u^0 dx + \int_0^t\int_\Omega A_{22}'\nabla_{X_2}z\cdot\nabla_{X_2}z dxds + 2\int_0^t\int_\Omega f z' dxds = 0, \end{aligned}$$

since we already have the equality (4.17). Then using (4.1) and (4.6), we get from (4.30)

$$\begin{aligned} & \left|(u_\varepsilon - z)'(t)\right|^2_{L^2(\Omega)} + \lambda\left(\varepsilon^2\left|\nabla_{X_1}u_\varepsilon(t)\right|^2_{L^2(\Omega)} + \left|\nabla_{X_2}(u_\varepsilon - z)(t)\right|^2_{L^2(\Omega)}\right) \\ & \qquad \le \left|J_\varepsilon(t)\right| + C\int_0^t \varepsilon^2\left|\nabla_{X_1}u_\varepsilon(s)\right|^2_{L^2(\Omega)} + \left|\nabla_{X_2}(u_\varepsilon - z)(s)\right|^2_{L^2(\Omega)} ds, \end{aligned} \tag{4.31}$$

for every $t\in[0,T]$. We set

$$\Phi_\varepsilon(t) := \left|(u_\varepsilon - z)'(t)\right|^2_{L^2(\Omega)} + \varepsilon^2\left|\nabla_{X_1}u_\varepsilon(t)\right|^2_{L^2(\Omega)} + \left|\nabla_{X_2}(u_\varepsilon - z)(t)\right|^2_{L^2(\Omega)}.$$

Then we can rewrite (4.31) as

$$\Phi_\varepsilon(t) \le C\left|J_\varepsilon(t)\right| + C\int_0^t \Phi_\varepsilon(s)\,ds, \quad \text{where} \quad \left|J_\varepsilon(t)\right| \to 0.$$

Since Φ_ε is bounded in $C([0,T])$, independently of ε, there exists $\overline{\Phi} \in L^1(0,T)$ such that $\overline{\Phi}(t) := \lim_{\varepsilon\to 0}\sup \Phi_\varepsilon(t)$. This implies

$$\overline{\Phi}(t) \le 0 + C\limsup_{\varepsilon\to 0}\int_0^t \Phi_\varepsilon(s)\,ds \le C\int_0^t \limsup_{\varepsilon\to 0}\Phi_\varepsilon(s)\,ds = C\int_0^t \overline{\Phi}(s)\,ds. \tag{4.32}$$

By the Gronwall's inequality, we derive

$$\overline{\Phi}(t) = 0 \quad \text{a.e. } t\in[0,T].$$

On the other hand $\lim_{\varepsilon\to 0}\inf\Phi_\varepsilon(t) \ge 0$ since $\Phi_\varepsilon(t)\ge 0, \forall t\in[0,T]$, hence

$$\lim_{\varepsilon\to 0}\Phi_\varepsilon(t) = 0, \quad \forall t\in[0,T].$$

This means that

$$\left.\begin{array}{rl} u_\varepsilon(t) \to z(t) & \textit{in} \quad \mathcal{V}_2(\Omega), \\ \varepsilon \nabla_{X_1} u_\varepsilon(t) \to 0, \quad u'_\varepsilon(t) \to z'(t) & \textit{in} \quad L^2(\Omega), \end{array}\right. \quad \forall t \in [0, T].$$

IV) *The limit* $z(\cdot\,; X_1, \cdot)$ *is the solution of* (4.14) *for a.e.* $X_1 \in \Pi_1$. To check that

$$z(\cdot\,; X_1, \cdot) = \tilde{u}(\cdot\,; X_1, \cdot) \text{ on } Q_{X_1} = (0, T) \times \Omega_{X_1}, \text{ for a.e. } X_1 \in \Pi_1,$$

we use the argument introduced in Chipot and Guesmia [16] for elliptic problems. For convenience, and since we have an evolution case, we write it here.

First we cover Ω by a countable family of open sets of the form $U_i \times V_i \subset \Omega$, $i \in \mathbb{N}$ where U_i, V_i are open hypercubes of $\mathbb{R}^q, \mathbb{R}^{n-q}$ respectively. Choosing $\varphi \in \mathcal{D}(V_i), \eta \in \mathcal{D}(U_i)$ and taking into account that

$$\mathcal{V}_2(U_i \times V_i) = L^2(U_i) \times H_0^1(V_i),$$

it comes that

$$\langle z''(t), \eta(X_1)\varphi(X_2)\rangle_{\mathcal{V}_2(U_i \times V_i)} = \int_{U_i} \eta(X_1) \langle z''(t), \varphi(X_2)\rangle_{H_0^1(V_i)} \, dX_1.$$

Since $\eta\varphi \in \mathcal{D}(\Omega) \subset \mathcal{V}_2(\Omega)$, we derive from (4.16) that for every $t \in (0, T)$,

$$\int_{U_i} \eta(X_1) \left\{ \langle z''(t), \varphi(X_2)\rangle_{H_0^1(V_i)} + \int_{V_i} A_{22} \nabla_{X_2} z(t) \cdot \nabla_{X_2} \varphi(X_2) \, dX_2 \right\} dX_1$$
$$= \int_{U_i} \eta(X_1) \int_{V_i} f(t) \varphi(X_2) \, dX_2 \, dX_1, \qquad \forall \eta \in \mathcal{D}(U_i).$$

Then, there exists a set of measure zero $N(\varphi)$, such that

$$\langle z''(t), \varphi(X_2)\rangle_{H_0^1(V_i)} + \int_{V_i} A_{22} \nabla_{X_2} z(t) \cdot \nabla_{X_2} \varphi(X_2) \, dX_2 = \int_{V_i} f(t)\varphi(X_2) \, dX_2, \tag{4.33}$$

for all $X_1 \in U_i \backslash N(\varphi)$. Denote by $(\varphi_n)_n$ a Hilbert basis of $H_0^1(V_i)$, then (4.33) holds for all X_1 such that

$$X_1 \in U_i \backslash \cup_n N_i(\varphi_n),$$

where $\cup_n N_i(\varphi_n)$ is of course a set of zero measure (a countable union of zero measure sets). Thus (4.33) holds for any $\varphi \in H_0^1(V_i)$ and for all $X_1 \in U_i \backslash \cup_n N_i(\varphi_n)$. This easily follows from the density in $H_0^1(V_i)$ of the linear combinations of the basis $(\varphi_n)_n$.

Let us fix some $X_1 \in \Pi_\Omega \backslash N$ where N is the set (of measure zero)

$$N := \Pi_\Omega \backslash \cup_i \cup_n N_i(\varphi_n),$$

and let $\varphi \in \mathcal{D}\left(\Omega_{X_1}\right)$, if K_{X_1} denotes the support of φ we have clearly

$$K_{X_1} \subset \left(\Omega_{X_1} \cap \left(\cup_i V_i\right)\right) \subset \Omega_{X_1},$$

and thus K_{X_1} can be covered by a finite number of V_i that, for simplicity, will be denoted by $V_1, \cdots, V_k$. Using a partition of unity, there exist $\psi_i \in \mathcal{D}(V_i)$ such that

$$\textstyle\sum_{i=1}^{k} \psi_i = 1, \quad \text{on } K_{X_1}.$$

By (4.33) we derive

$$\begin{aligned}
&\left\langle z''\left(t\right), \varphi\left(X_2\right)\right\rangle_{H_0^1\left(\Omega_{X_1}\right)} + \int_{\Omega_{X_1}} A_{22} \nabla_{X_2} z\left(t\right) \cdot \nabla_{X_2} \varphi\left(X_2\right) dX_2 \\
&\qquad = \left\langle z''(t), \textstyle\sum_{i=1}^{k} \psi_i \varphi\left(X_2\right)\right\rangle_{H_0^1\left(\Omega_{X_1}\right)} + \int_{\Omega_{X_1}} A_{22} \nabla_{X_2} z(t) \cdot \nabla_{X_2}\left(\textstyle\sum_{i=1}^{k} \psi_i \varphi\left(X_2\right)\right) dX_2 \\
&\qquad = \textstyle\sum_{i=1}^{k} \left\{ \left\langle z''(t), \psi_i \varphi\left(X_2\right)\right\rangle_{H_0^1(V_i)} + \int_{V_i} A_{22} \nabla_{X_2} z(t) \cdot \nabla_{X_2}\left(\psi_i \varphi\left(X_2\right)\right) dX_2 \right\} \\
&\qquad = \textstyle\sum_{i=1}^{k} \int_{V_i} f(t) \psi_i \varphi\left(X_2\right) dX_2 \\
&\qquad = \int_{\Omega_{X_1}} f(t) \varphi\left(X_2\right) dX_2.
\end{aligned}$$

and it follows that for a.e. $X_1 \in \Pi_1$,

$$\begin{aligned}
&\left\langle z''(t; X_1, X_2), \varphi\right\rangle_{H_0^1(\Omega_{X_1})} + \int_{\Omega_{X_1}} A_{22}(t; X_1, X_2) \nabla_{X_2} z(t; X_1, X_2) \cdot \nabla_{X_2} \varphi\left(X_2\right) dX_2 \\
&\qquad\qquad = \int_{\Omega_{X_1}} f(t; X_1, X_2) \varphi\left(X_2\right) dX_2, \quad \forall \varphi \in \mathcal{D}(\Omega_{X_1}), \ \forall t \in [0, T].
\end{aligned}$$

Taking into account the fact that problems (4.14) and (4.16) have the same initial conditions, we derive that $z\left(\cdot; X_1, \cdot\right)$ is the unique solution of (4.14), for a.e. $X_1 \in \Pi_1$. This completes the proof of the theorem. ∎

Remark 4.1 *We can replace the condition (4.12) by the following weaker one*

$$u_\varepsilon^0 \rightarrow u^0 \quad \text{in } \mathcal{V}_2\left(\Omega\right) \ \text{ and } \ \varepsilon \nabla_{X_1} u_\varepsilon^0 \rightarrow 0, \quad u_\varepsilon^1 \rightarrow u^1 \ \text{ in } L^2(\Omega).$$

In this case we only have to suppose that $u^0 \in \mathcal{V}_2\left(\Omega\right).$

As a consequence of Theorem 4.2 and (4.22) we have

Corollary 4.1 *For every* $1 \leq q < \infty$ *we have*

$$\begin{aligned}
&& u_\varepsilon \rightarrow \tilde{u} \quad &\text{in} \quad L^q\left(0, T; \mathcal{V}_2\left(\Omega\right)\right), \\
\varepsilon \nabla_{X_1} u_\varepsilon \rightarrow 0, && u_\varepsilon' \rightarrow \tilde{u}' \quad &\text{in} \quad L^q\left(0, T; L^2\left(\Omega\right)\right), \\
&& \left(u_\varepsilon'' - \tilde{u}''\right) \rightarrow 0 \quad &\text{in} \quad L^q(0, T; H^{-1}(\Omega)).
\end{aligned} \tag{4.34}$$

Proof. The first two lines of (4.34) follow from Lebesgue's theorem, (4.22) and the pointwise convergences in t given by Theorem 4.2. For the last limit, we subtract (4.16) from (4.10) and estimate the second time derivative of $u_\varepsilon - \tilde{u}$ by

$$\begin{aligned}\langle (u_\varepsilon - \tilde{u})''(t), v\rangle_{H^{-1}(\Omega)} =& -\int_\Omega \varepsilon^2 A_{11}(t)\nabla_{X_1}u_\varepsilon(t)\cdot\nabla_{X_1}v - \varepsilon A_{12}(t)\nabla_{X_2}u_\varepsilon(t)\cdot\nabla_{X_1}v dx\\ &-\int_\Omega \varepsilon A_{21}(t)\nabla_{X_1}u_\varepsilon(t)\cdot\nabla_{X_2}v - A_{22}(t)\nabla_{X_2}(u_\varepsilon-\tilde{u})(t)\cdot\nabla_{X_2}v dx\\ \leq& C\left(\varepsilon\left|\nabla_{X_1}u_\varepsilon(t)\right|_{L^2(\Omega)} + \varepsilon\left|\nabla_{X_2}u_\varepsilon(t)\right|_{L^2(\Omega)} + \left|\nabla_{X_2}(u_\varepsilon-\tilde{u})(t)\right|_{L^2(\Omega)}\right)\left|\nabla v\right|_{L^2(\Omega)},\end{aligned}$$

for every $v \in H_0^1(\Omega)$ and every $t \in (0,T)$. Then deduce that

$$\left|(u_\varepsilon - \tilde{u})''(t)\right|_{H^{-1}(\Omega)} \leq C\left(\varepsilon\left|\nabla_{X_1}u_\varepsilon(t)\right|_{L^2(\Omega)} + \varepsilon\left|\nabla_{X_2}u_\varepsilon(t)\right|_{L^2(\Omega)} + \left|\nabla_{X_2}(u_\varepsilon-\tilde{u})(t)\right|_{L^2(\Omega)}\right).$$

Taking both sides to power q, applying the inequality $(a+b)^q \leq 2^{q-1}(a^q+b^q)$ and integrating on $(0,T)$, yield

$$\left|u_\varepsilon'' - \tilde{u}''\right|^q_{L^q(0,T;H^{-1}(\Omega))} \leq C\int_0^t \varepsilon^q\left|\nabla_{X_1}u_\varepsilon\right|^q_{L^2(\Omega)} + \varepsilon^q\left|\nabla_{X_2}u_\varepsilon\right|^q_{L^2(\Omega)} + \left|\nabla_{X_2}(u_\varepsilon-\tilde{u})\right|^q_{L^2(\Omega)} ds.$$

Using the first convergences in this corollary the right hand side of the inequality above tends to zero. This completes the proof. ■

Remark 4.2 *In particular, since* $u_\varepsilon'', \tilde{u}'' \in L^2\left(0,T;H^{-1}(\Omega)\right)$, *we have* $u_\varepsilon'' \to \tilde{u}''$ *in* $L^2(0,T;H^{-1}(\Omega))$.

4.3 CONVERGENCES IN CYLINDRICAL DOMAINS

One notes that the missing of the convergence of the gradient $\nabla_{X_1}u_\varepsilon$, in Theorem 4.2, prevents the convergence to be in Sobolev space $H^1(\Omega)$. This was expected since u_ε and its limit $\tilde{u}$ do not necessarily belong to the same space $H_0^1(\Omega)$. In particular, on cylinders a boundary layer may appears near to the lateral boundary. In this section, we show that the above convergences can be improved far from this lateral boundary.

In this section we assume that the domain is cylindrical, i.e.

$$\Omega := \Delta \times \omega, \tag{4.35}$$

where Δ (resp. ω) is an open bounded subset of $\mathbb{R}^q$ (resp. $\mathbb{R}^{n-q}$). Then it holds that

$$\mathcal{V}_2(\Omega) = L^2\left(\Delta, H_0^1(\omega)\right) \text{ and } \mathcal{V}_2'(\Omega) = L^2\left(\Delta, H^{-1}(\omega)\right).$$

In addition we assume that the matrix A is independent of time, i.e.

$$A(t,x) = A(x). \tag{4.36}$$

For a.e. $X_1 \in \Delta$, Problem (4.14) can be stated as

$$\begin{cases} \tilde{u}\left(\cdot\,; X_1, \cdot\right) \in C([0,T]\,; H_0^1\left(\omega\right)),\ \tilde{u}'\left(\cdot\,; X_1, \cdot\right) \in C([0,T]\,; L^2\left(\omega\right)), \\ \tilde{u}''\left(\cdot\,; X_1, \cdot\right) \in L^2(0,T; H^{-1}\left(\omega\right)), \\ \left\langle \tilde{u}''(t; X_1, X_2), v \right\rangle_{H_0^1(\omega)} + \displaystyle\int_\omega A_{22}(X_1, X_2) \nabla_{X_2} \tilde{u}(t; X_1, X_2) \cdot \nabla_{X_2} v dX_2 \\ \qquad\qquad = \displaystyle\int_\omega f(t; X_1, X_2) v dX_2, \quad \forall t \in \left(0, T\right), \quad \forall v \in H_0^1\left(\omega\right), \\ \tilde{u}\left(0, X_1, \cdot\right) = u^0\left(X_1, \cdot\right), \quad \tilde{u}'\left(0, X_1, \cdot\right) = u^1\left(X_1, \cdot\right) \quad \text{in } \omega. \end{cases} \tag{4.37}$$

Our aim in this section is to improve the convergence $u_\varepsilon \to \tilde{u}$, in particular the convergence of $\nabla_{X_1} u_\varepsilon$. For this reason, as a necessary condition, we need to assume more regularity hypothesis on the solution of the limit problem, i.e. $\tilde{u}\left(t\right) \in H^1\left(\Omega\right)$ for a.e. $t \in \left(0, T\right)$. Thus we start by studying the regularity of $\tilde{u}$.

4.3.1 *Regularity Results*

The following proposition shows that the regularity of $\tilde{u}$ in the $X_1 -$ *direction* depends on the regularity of the data in the same direction.

Proposition 4.3 *Under the assumptions of Theorem 4.2, the additional assumptions (4.35), (4.36) and*

$$\partial_{x_i} u^1 \in L^2\left(\Delta; H^{-1}\left(\omega\right)\right),\ \partial_{x_i} f \in L^2\left(\left(0,T\right) \times \Delta; H^{-1}\left(\omega\right)\right),\ i = 1, \cdots, q, \tag{4.38}$$

we have

$$\tilde{u} \in L^\infty\left(0, T; H^1\left(\Omega\right)\right),\ \partial_{x_i} \tilde{u}' \in L^\infty\left(0, T; L^2\left(\Delta; H^{-1}\left(\omega\right)\right)\right),\ i = 1, \cdots, q. \tag{4.39}$$

Proof. Let Δ' be an open subset such that $\Delta' \subset\subset \Delta$ (i.e. the closure of Δ' is a subset of Δ). We set $h_0 := dist\left(\Delta', \partial\Delta\right)$ and for $0 < h < h_0$ we denote

$$\tau_h^i v\left(X_1, X_2\right) := v\left(X_1 + he_i, X_2\right), \quad \text{for } X_1 \in \Delta', \quad i = 1, \cdots, q,$$

where e_i is the unit vector in the $i^{th} -$ *direction*. Subtracting the identity in (4.37) from itself written for $X_1 + he_i$, we derive

$$\begin{aligned} &\langle \tau_h^i \tilde{u}'' - \tilde{u}'', v \rangle_{H_0^1(\omega)} + \int_\omega \tau_h^i A_{22} \nabla_{X_2} \left(\tau_h^i \tilde{u} - \tilde{u}\right) \cdot \nabla_{X_2} v dX_2 \\ &\qquad = \int_\omega \left(\tau_h^i f - f\right) v dX_2 - \int_\omega \left(\tau_h^i A_{22} - A_{22}\right) \nabla_{X_2} \tilde{u} \cdot \nabla_{X_2} v dX_2, \ \forall v \in H_0^1\left(\omega\right). \end{aligned} \tag{4.40}$$

For $s \in \left]0,T\right]$, we set

$$\tilde{U}(t) := \int_0^t \tilde{u}(\sigma)\, d\sigma, \qquad \psi(t) := \begin{cases} -\displaystyle\int_t^s \tilde{u}(\sigma)\, d\sigma & t \leq s, \\ 0 & t > s, \end{cases}$$

then it holds that

$$\psi(\sigma) = \tilde{U}(\sigma) - \tilde{U}(s) \quad \text{if} \quad \sigma \leq s. \tag{4.41}$$

Taking $v = \left(\tau_h^i \psi - \psi\right)(\sigma) \in H_0^1(\omega)$ in (4.40) and integrating over $(0,s)$, we get

$$\begin{aligned} &\int_0^s \langle \tau_h^i \tilde{u}'' - \tilde{u}'', \tau_h^i \psi - \psi \rangle_{H_0^1(\omega)} d\sigma + \int_0^s \int_\omega \tau_h^i A_{22} \nabla_{X_2} \left(\tau_h^i \tilde{u} - \tilde{u}\right) \cdot \nabla_{X_2} \left(\tau_h^i \psi - \psi\right) dX_2 d\sigma \\ &\qquad = \int_0^s \int_\omega \left(\tau_h^i f - f\right) \left(\tau_h^i \psi - \psi\right) dX_2 d\sigma \\ &\qquad - \int_0^s \int_\omega \left(\tau_h^i A_{22} - A_{22}\right) \nabla_{X_2} \tilde{u} \cdot \nabla_{X_2} \left(\tau_h^i \psi - \psi\right) dX_2 d\sigma. \end{aligned} \tag{4.42}$$

Next, integrating by parts in the first integral, we obtain

$$\begin{aligned} \int_0^s \left\langle \tau_h^i \tilde{u}'' - \tilde{u}'', \tau_h^i \psi - \psi \right\rangle_{H_0^1(\omega)} d\sigma &= \int_\omega \left(\tau_h^i \tilde{u}'(s) - \tilde{u}'(s)\right) \left(\tau_h^i \psi(s) - \psi(s)\right) dX_2 \\ &\quad - \int_\omega \left(\tau_h^i u^1 - u^1\right) \left(\tau_h^i \psi(0) - \psi(0)\right) dX_2 \\ &\quad - \int_0^s \int_\omega \left(\tau_h^i \tilde{u}' - \tilde{u}'\right) \left(\tau_h^i \psi' - \psi'\right) dX_2 d\sigma, \\ &= \int_\omega \left(\tau_h^i u^1 - u^1\right) \left(\tau_h^i \tilde{U}(s) - \tilde{U}(s)\right) dX_2 \\ &\quad + \int_0^s \int_\omega \left(\tau_h^i \tilde{u}' - \tilde{u}'\right) \left(\tau_h^i \tilde{u} - \tilde{u}\right) dX_2 d\sigma \end{aligned}$$

since $\psi(s) = 0, \psi(0) = -U(s)$ and $\psi' = \tilde{u}$. Integrating the last term yields

$$\begin{aligned} \int_0^s \left\langle \tau_h^i \tilde{u}'' - \tilde{u}'', \tau_h^i \psi - \psi \right\rangle_{H_0^1(\omega)} d\sigma &= \int_\omega \left(\tau_h^i u^1 - u^1\right) \left(\tau_h^i \tilde{U}(s) - \tilde{U}(s)\right) dX_2 \\ &\quad - \frac{1}{2} \left| \left(\tau_h^i \tilde{u} - \tilde{u}\right)(s) \right|_{L^2(\omega)}^2 + \frac{1}{2} \left| \tau_h^i u^0 - u^0 \right|_{L^2(\omega)}^2. \end{aligned} \tag{4.43}$$

The second integral in (4.42) can be written as

$$\begin{aligned} &\int_0^s \int_\omega \tau_h^i A_{22} \nabla_{X_2} \left(\tau_h^i \tilde{u} - \tilde{u}\right) \cdot \nabla_{X_2} \left(\tau_h^i \psi - \psi\right) dX_2 d\sigma \\ &\qquad = \int_0^s \int_\omega \tau_h^i A_{22} \nabla_{X_2} \left(\tau_h^i \psi' - \psi'\right) \cdot \nabla_{X_2} \left(\tau_h^i \psi - \psi\right) dX_2 d\sigma \\ &\qquad = \int_\omega \tau_h^i A_{22} \nabla_{X_2} \left(\tau_h^i \psi - \psi\right)(s) \cdot \nabla_{X_2} \left(\tau_h^i \psi - \psi\right)(s)\, dX_2 \\ &\qquad - \int_\omega \tau_h^i A_{22} \nabla_{X_2} \left(\tau_h^i \psi - \psi\right)(0) \cdot \nabla_{X_2} \left(\tau_h^i \psi - \psi\right)(0)\, dX_2 \\ &\qquad = -\frac{1}{2} \int_\omega \tau_h^i A_{22} \nabla_{X_2} \left(\tau_h^i \tilde{U} - \tilde{U}\right)(s) \cdot \nabla_{X_2} \left(\tau_h^i \tilde{U} - \tilde{U}\right)(s)\, dX_2. \end{aligned} \tag{4.44}$$

We then use (4.43) and (4.44) in (4.42), it comes

$$\begin{aligned}\frac{1}{2}\left|\left(\tau_h^i\tilde{u}-\tilde{u}\right)(s)\right|^2_{L^2(\omega)} &+\frac{1}{2}\int_\omega \tau_h^i A_{22}\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(s)\cdot\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(s)\,dX_2\\ &=\frac{1}{2}\left|\tau_h^i u^0-u^0\right|^2_{L^2(\omega)}+\int_\omega\left(\tau_h^i u^1-u^1\right)\left(\tau_h^i\tilde{U}-\tilde{U}\right)(s)\,dX_2\\ &\quad-\int_0^s\int_\omega\left(\tau_h^i f-f\right)\left(\tau_h^i\psi-\psi\right)dX_2 d\sigma\\ &\quad+\int_0^s\int_\omega\left(\tau_h^i A_{22}-A_{22}\right)\nabla_{X_2}\tilde{u}\cdot\nabla_{X_2}\left(\tau_h^i\psi-\psi\right)dX_2 d\sigma.\end{aligned}$$

Applying (4.7), the Cauchy-Schwarz and Poincaré's inequalities, we obtain

$$\begin{aligned}\frac{1}{2}\left|\left(\tau_h^i\tilde{u}-\tilde{u}\right)(s)\right|^2_{L^2(\omega)}&+\frac{\lambda}{2}\left|\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(s)\right|^2_{L^2(\omega)}\\ &\le\frac{1}{2}\left|\tau_h^i u^0-u^0\right|^2_{L^2(\omega)}+C\left|\tau_h^i u^1-u^1\right|_{H^{-1}(\omega)}\left|\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(s)\right|_{L^2(\omega)}\\ &+C\left|\tau_h^i f-f\right|_{L^2(0,T;H^{-1}(\omega))}\left|\nabla_{X_2}\left(\tau_h^i\psi-\psi\right)\right|_{L^2((0,s)\times\omega)}\\ &+\left|\left(\tau_h^i A_{22}-A_{22}\right)\nabla_{X_2}\tilde{u}\right|_{L^2((0,T)\times\omega)}\left|\nabla_{X_2}\left(\tau_h^i\psi-\psi\right)\right|_{L^2((0,s)\times\omega)}.\end{aligned}$$

(In the second and the third terms of the right hand side, we considered the duality product). Then by Young's inequality $2ab\le\alpha a^2+\left(b^2/\alpha\right)$ with convenient values[3] of α, we derive

$$\begin{aligned}\left|\left(\tau_h^i\tilde{u}-\tilde{u}\right)(s)\right|^2_{L^2(\omega)}&+\frac{\lambda}{2}\left|\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(s)\right|^2_{L^2(\omega)}\\ &\le\left|\tau_h^i u^0-u^0\right|^2_{L^2(\omega)}+C\left|\tau_h^i u^1-u^1\right|^2_{H^{-1}(\omega)}+C\left|\tau_h^i f-f\right|^2_{L^2(0,T;H^{-1}(\omega))}\\ &+C\left|\left|\tau_h^i A_{22}-A_{22}\right|_*\right|^2_{L^\infty((0,T)\times\omega)}\left|\nabla_{X_2}\tilde{u}\right|^2_{L^2((0,T)\times\omega)}\\ &+\frac{\lambda}{8T}\int_0^s\left|\nabla_{X_2}\left(\tau_h^i\psi-\psi\right)\right|^2_{L^2(\omega)}d\sigma\end{aligned}\tag{4.45}$$

where $|\cdot|_*$ denotes a matrix norm. We estimate the last term above, using (4.41), as follows

$$\begin{aligned}\int_0^s\left|\nabla_{X_2}\left(\tau_h^i\psi-\psi\right)(\sigma)\right|^2_{L^2(\omega)}d\sigma&=\int_0^s\left|\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(\sigma)-\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(s)\right|^2_{L^2(\omega)}d\sigma\\ &\le 2\int_0^s\left|\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(\sigma)\right|^2_{L^2(\omega)}d\sigma+2T\left|\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(s)\right|^2_{L^2(\omega)}.\end{aligned}$$

Using this in (4.45), we get

$$\begin{aligned}\left|\left(\tau_h^i\tilde{u}-\tilde{u}\right)(s)\right|^2_{L^2(\omega)}&+\frac{\lambda}{4}\left|\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(s)\right|^2_{L^2(\omega)}\\ &\le\left|\tau_h^i u^0-u^0\right|^2_{L^2(\omega)}+C\left|\tau_h^i u^1-u^1\right|^2_{H^{-1}(\omega)}+C\left|\tau_h^i f-f\right|^2_{L^2(0,T;H^{-1}(\omega))}\\ &+C\left|\left|\tau_h^i A_{22}-A_{22}\right|_*\right|^2_{L^\infty(\omega)}\left|\nabla_{X_2}\tilde{u}\right|^2_{L^2((0,T)\times\omega)}\\ &+C\int_0^s\left|\left(\tau_h^i\tilde{u}-\tilde{u}\right)(\sigma)\right|^2_{L^2(\omega)}+\left|\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(\sigma)\right|^2_{L^2(\omega)}d\sigma.\end{aligned}$$

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[3] for instance, $\alpha=\frac{2C}{\lambda}$ in the second term of the right hand side

For $s \in \left]0,T\right]$, we apply Gronwall's inequality, integrate on Δ' and divide on h^2, we obtain

$$\begin{aligned}\left|\frac{\left(\tau_h^i\tilde{u}-\tilde{u}\right)(s)}{h}\right|^2_{L^2(\Delta'\times\omega)} &+ \left|\frac{\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(s)}{h}\right|^2_{L^2(\Delta'\times\omega)}\\ &\le C\left|\frac{\tau_h^iu^0-u^0}{h}\right|^2_{L^2(\Delta'\times\omega)} + C\left|\frac{\tau_h^iu^1-u^1}{h}\right|^2_{L^2(\Delta';H^{-1}(\omega))}\\ &\quad + C\left|\frac{\tau_h^if-f}{h}\right|^2_{L^2((0,T)\times\Delta';H^{-1}(\omega))} + C\left|\left\|\frac{\tau_h^iA_{22}-A_{22}}{h}\right\|_*\right|^2_{L^\infty(\Delta'\times\omega)}.\end{aligned}$$

According to (4.38) and using the standard method as in Gilbarg and Trudinger [29, Lemma 7.23], all the terms in the right hand side are bounded independently of h, i.e.

$$\left|\frac{\left(\tau_h^i\tilde{u}-\tilde{u}\right)(s)}{h}\right|^2_{L^2(\Delta'\times\omega)} + \left|\frac{\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(s)}{h}\right|^2_{L^2(\Delta'\times\omega)} \le C, \quad \forall s\in[0,T],$$

for every $h>0$ and $\Delta' \subset\subset \Delta$. We deduce that

$$\partial_{x_i}\tilde{u}(s),\ \partial_{x_i}\left(\nabla_{X_2}\tilde{U}\right)(s) \in L^2(\Omega), \quad i=1,\cdots,q, \quad \forall s\in\left]0,T\right]. \tag{4.46}$$

Since we already have $\partial_{x_i}\tilde{u}(s)\in L^2(\Omega)$ for $i=q+1,\cdots,n$ and $\tilde{u}(0)=u^0\in H^1(\Omega)$, it follows that

$$\tilde{u}(s)\in H^1(\Omega), \quad \forall s\in[0,T]. \tag{4.47}$$

Now we look to the regularity of $\tilde{u}'$. Recall that, for every $v\in\mathcal{D}(\omega)$,

$$\left\langle \tilde{u}''(\sigma), v\right\rangle_{H_0^1(\omega)} + \int_\omega A_{22}\nabla_{X_2}\tilde{u}(\sigma)\cdot\nabla_{X_2}v\,dX_2 = \int_\omega f(\sigma)\,v\,dX_2, \quad \forall\sigma\in[0,T].$$

Integrating on $[0,s]\subset[0,T]$ and taking into account the fact that $\tilde{U}'=\tilde{u}$ and $\tilde{U}(0)=0$ yield

$$\int_\omega \tilde{u}'(s)\,v\,dX_2 = \int_\omega u^1 v\,dX_2 + \int_0^s\int_\omega f v\,dX_2\,d\sigma - \int_\omega A_{22}\nabla_{X_2}\tilde{U}(s)\cdot\nabla_{X_2}v\,dX_2, \quad \forall v\in\mathcal{D}(\omega).$$

Applying the derivative operator ∂_{x_i}, $(i=1,\cdots,q)$, on both sides of the identity above, we obtain

$$\begin{aligned}\left\langle\partial_{x_i}\tilde{u}'(s), v\right\rangle_{H_0^1(\omega)} &= \left\langle\partial_{x_i}u^1, v\right\rangle_{H_0^1(\omega)} + \int_0^s \left\langle\partial_{x_i}f(\sigma), v\right\rangle_{H_0^1(\omega)} d\sigma\\ &- \int_\omega\left(\partial_{x_i}A_{22}\right)\nabla_{X_2}\tilde{U}(s)\cdot\nabla_{X_2}v\,dX_2 - \int_\omega A_{22}\partial_{x_i}\left(\nabla_{X_2}\tilde{U}(s)\right)\cdot\nabla_{X_2}v\,dX_2, \quad \forall v\in\mathcal{D}(\omega).\end{aligned}$$

Next, we apply the Cauchy-Schwarz inequality in each term to get

$$\begin{aligned}\left\langle\partial_{x_i}\tilde{u}'(s), v\right\rangle_{H_0^1(\omega)} \le C\Big\{&\left|\partial_{x_i}u^1\right|_{H^{-1}(\omega)} + \left|\partial_{x_i}f\right|_{L^2(0,T;H^{-1}(\omega))} + \left|\nabla_{X_2}\tilde{U}(s)\right|_{L^2(\omega)}\\ &+ \left|\nabla_{X_2}\tilde{U}(s)\right|_{L^2(\omega)} + \left|\partial_{x_i}\nabla_{X_2}\tilde{U}(s)\right|_{L^2(\omega)}\Big\}\left|\nabla_{X_2}v\right|_{L^2(\omega)}, \quad \forall v\in\mathcal{D}(\omega),\end{aligned}$$

hence

$$\left|\partial_{x_i}\tilde{u}'(s)\right|_{H^{-1}(\omega)} \leq C\left\{\left|\partial_{x_i}u^1\right|_{H^{-1}(\omega)} + \left|\partial_{x_i}f\right|_{L^2(0,T;H^{-1}(\omega))} + \left|\nabla_{X_2}\tilde{U}(s)\right|_{L^2(\omega)} \right.$$
$$\left. + \left|\nabla_{X_2}\tilde{U}(s)\right|_{L^2(\omega)} + \left|\partial_{x_i}\nabla_{X_2}\tilde{U}(s)\right|_{L^2(\omega)}\right\}.$$

Taking both sides to power 2 and integrating on Δ, we derive

$$\int_\Delta \left|\partial_{x_i}\tilde{u}'(s)\right|^2_{H^{-1}(\omega)} dX_1 \leq C\left\{\left|\partial_{x_i}u^1\right|^2_{L^2(\Delta,H^{-1}(\omega))} + \left|\nabla_{X_2}\tilde{U}(s)\right|^2_{L^2(\Omega)} \right.$$
$$\left. + \left|\partial_{x_i}\nabla_{X_2}\tilde{U}(s)\right|^2_{L^2(\Omega)} + \left|\partial_{x_i}f\right|^2_{L^2((0,T)\times\Delta,H^{-1}(\omega))}\right\}.$$

Due to (4.38) and (4.46), we deduce that

$$\partial_{x_i}\tilde{u}'(s) \in L^2\left(\Delta; H^{-1}(\omega)\right), \quad i = 1,\cdots,q, \quad \forall s \in [0,T]. \tag{4.48}$$

This ends the proof of the proposition. ∎

Remark 4.3 ***i)*** *The proof also shows that*

$$\partial_{x_i}\tilde{U} \in L^\infty\left(0,T; L^2\left(\Delta, H^1_0(\omega)\right)\right), \quad i = 1,\cdots,q.$$

ii) *Note that although we ignore if* $\tilde{u} \in C\left(0,T;H^1(\Omega)\right)$ *or not, we have*

$$\tilde{u}(t), \tilde{U}(t) \in H^1_0(\Omega, \Delta\times\partial\omega) := \left\{v \in H^1(\Omega) \,|u = 0 \text{ on } \Delta\times\partial\omega\right\}, \quad \forall t \in [0,T]. \tag{4.49}$$

Indeed, having $\tilde{u}(t;X_1,\cdot), \tilde{U}(t) \in H^1_0(\omega)$ *for a.e.* $X_1 \in \Delta$ *and* $\tilde{u}(t), \tilde{U}(t) \in H^1(\Omega), \forall t \in [0,T]$, *then according to* Chipot and Guesmia [16] *it follows that* $\tilde{u}(t)$ *and* $\tilde{U}(t)$ *vanish, in the trace sense, on* $\Delta\times\partial\omega$, *i.e.* $\tilde{u}(t), \tilde{U}(t) \in H^1_0(\Omega,\Delta\times\partial\omega)$. *It follows by the above proposition that*

$$\tilde{u}, \tilde{U}(t) \in L^\infty\left(0,T; H^1_0(\Omega,\Delta\times\partial\omega)\right) \cap C\left(0,T;\mathcal{V}_2(\Omega)\right). \tag{4.50}$$

By the density of $H^1_0(\Omega,\Delta\times\partial\omega)$ *in* $\mathcal{V}_2(\Omega)$, *and arguing as in* Strauss [64, Theorem 2.1], *we derive that* $\tilde{u}(t) \in H^1_0(\Omega,\Delta\times\partial\omega), \forall t \in [0,T]$. *By consequence we also have*

$$\tilde{U} \in C\left(0,T; H^1_0(\Omega,\Delta\times\partial\omega)\right). \tag{4.51}$$

Remark 4.4 *If the coefficients of* A_{22} *are time dependent, i.e.* $A_{22} = A_{22}(t,x)$, *Proposition 4.3 still holds under the assumption*

$$\partial_{x_k}a'_{ij} \in L^\infty(Q), \qquad 1 \leq k \leq q \text{ and } q+1 \leq i,j \leq n.$$

In this case, some estimates of the above proof holds only in a subinterval $[0,\tau]$, *for some* τ *sufficiently small* $(0 < \tau \leq T)$. *Thus we establish the regularity on* $[0,\tau]$ *as above and taking into account that* $\partial_{x_i}u(\tau)$ *and* $\partial_{x_i}u'(\tau)$ *are as regular as* $\partial_{x_i}u(0)$ *and* $\partial_{x_i}u'(0)$ *respectively, for* $i = 1,\cdots,q$, *then we repeat the same arguments in next interval* $[\tau,2\tau]$, *and so on. We obtain the regularity on* $[0,T]$ *after a finite number of steps.*

4.3.2 *Convergence Results*

Let Δ_0 and Δ_1 be two open subsets of $\mathbb{R}^q$ satisfying $\Delta_0 \subset\subset \Delta_1 \subset\subset \Delta$. With this notation we set

$$\Omega_i := \Delta_i \times \omega \text{ and } Q_i := (0,T) \times \Omega_i, \qquad i = 0,1.$$

Consider a smooth cut-off function $\varrho = \varrho\,(X_1)$ satisfying

$$\operatorname{supp}(\varrho) \subset \Delta_1,\ \ \varrho = 1 \text{ on } \Delta_0,\ \ 0 \leq \varrho \leq 1 \ \text{ and } \ |\nabla_{X_1}\varrho| \leq C.$$

For every $s \in \,]0,T]$, we define the functions

$$\begin{aligned} \psi_\varepsilon(t) &:= \begin{cases} -\displaystyle\int_t^s w_\varepsilon(\sigma)\,d\sigma & t \leq s, \\ 0 & t > s, \end{cases} \\ W_\varepsilon(t) &:= \int_0^t w_\varepsilon(\sigma)\,d\sigma \ \text{ and } \ U_\varepsilon(t) := \int_0^t u_\varepsilon(\sigma)\,d\sigma, \end{aligned}$$

where $w_\varepsilon := u_\varepsilon - \tilde{u}$. Then we have

Theorem 4.4 *Under the assumptions of Proposition 4.3, we assume in addition that*

$$\left|u_\varepsilon^0 - u^0\right|_{L^2(\Omega_1)} = O(\varepsilon) \ \text{ and } \ \left|u_\varepsilon^1 - u^1\right|_{L^2(\Delta_1;H^{-1}(\omega))} = O(\varepsilon), \tag{4.52}$$

then we have

$$\begin{aligned} \sup_{t\in[0,T]} \left|\left(u'_\varepsilon - \tilde{u}'\right)(t)\right|_{H^{-1}(\Omega_0)}, \sup_{t\in[0,T]} \left|\left(u_\varepsilon - \tilde{u}\right)(t)\right|_{L^2(\Omega_0)}, &= O(\varepsilon), \\ \sup_{t\in[0,T]} \left|\nabla_{X_2}\left(U_\varepsilon - \tilde{U}\right)(t)\right|_{L^2(\Omega_0)} &= O(\varepsilon), \end{aligned}$$

and the weak convergence

$$\nabla_{X_1}U_\varepsilon(t) \rightharpoonup \nabla_{X_1}\tilde{U}(t) \quad \text{in } L^2(\Omega_0), \qquad \forall t \in [0,T],$$

where $\tilde{u}$ (resp. u_ε) is the solution of problem (4.37) (resp. (4.10)).

Proof. By comparing (4.10) and (4.16), we deduce that

$$\begin{aligned} \left\langle w''_\varepsilon, v\right\rangle + \int_\Omega A_\varepsilon \nabla w_\varepsilon \cdot \nabla v dx = -\int_\Omega \varepsilon^2 A_{11} \nabla_{X_1}\tilde{u} \cdot \nabla_{X_1} v dx \\ -\int_\Omega \varepsilon A_{12}\nabla_{X_2}\tilde{u} \cdot \nabla_{X_1} v dx - \int_\Omega \varepsilon A_{21} \nabla_{X_1}\tilde{u} \cdot \nabla_{X_2} v dx, \ \ \forall v \in H_0^1(\Omega). \end{aligned} \tag{4.53}$$

According to (4.49), we have

$$\psi_\varepsilon(\sigma)\,\varrho^2 = 0 \ \text{ on } \partial\Omega, \qquad \forall \sigma \in [0,s]$$

which allows us to test (4.53) with $\psi_\varepsilon(\sigma)\varrho^2 \in H_0^1(\Omega)$. Then integrating over $[0,s]$ we get

$$\begin{aligned}
&\int_\Omega w_\varepsilon'(s)\,\psi_\varepsilon(s)\,\varrho^2 dx - \int_\Omega w_\varepsilon^1 \psi_\varepsilon(0)\,\varrho^2 dx - \int_0^s \int_\Omega w_\varepsilon' \psi_\varepsilon' \varrho^2 + A_\varepsilon \nabla w_\varepsilon \cdot \nabla\left(\psi_\varepsilon \varrho^2\right) dx d\sigma \\
&\quad = - \int_0^s \int_\Omega \varepsilon^2 A_{11} \nabla_{X_1} \tilde{u} \cdot \nabla_{X_1}\left(\psi_\varepsilon \varrho^2\right) dx d\sigma - \int_0^s \int_\Omega \varepsilon A_{12} \nabla_{X_2} \tilde{u} \cdot \nabla_{X_1}\left(\psi_\varepsilon \varrho^2\right) dx d\sigma \\
&\qquad - \int_0^s \int_\Omega \varepsilon A_{21} \nabla_{X_1} \tilde{u} \cdot \nabla_{X_2}\left(\psi_\varepsilon \varrho^2\right) dx d\sigma,
\end{aligned} \tag{4.54}$$

where $w_\varepsilon^0 := u_\varepsilon^0 - u^0$ and $w_\varepsilon^1 := u_\varepsilon^1 - u^1$. The A_ε integral term can be written as

$$\begin{aligned}
\int_0^s \int_\Omega A_\varepsilon \nabla w_\varepsilon \cdot \nabla\left(\psi_\varepsilon \varrho^2\right) dx d\sigma &= \int_0^s \int_\Omega A_\varepsilon \nabla w_\varepsilon \cdot \nabla \psi_\varepsilon \varrho^2 + 2\psi_\varepsilon \varrho A_\varepsilon \nabla w_\varepsilon \cdot \nabla \varrho dx d\sigma \\
&= \frac{1}{2} \int_\Omega A_\varepsilon \nabla \psi_\varepsilon(s) \cdot \nabla \psi_\varepsilon(s)\, \varrho^2 dx - \frac{1}{2} \int_\Omega A_\varepsilon \nabla \psi_\varepsilon(0) \cdot \nabla \psi_\varepsilon(0)\, \varrho^2 dx \\
&\quad + 2 \int_0^s \int_\Omega \psi_\varepsilon \varrho A_\varepsilon \nabla w_\varepsilon \cdot \nabla \varrho dx d\sigma,
\end{aligned}$$

and, since $\psi_\varepsilon(s) = 0$, $\psi_\varepsilon(0) = -W_\varepsilon(s)$ and $w_\varepsilon \in H^1(\Omega)$, we have

$$\begin{aligned}
\int_0^s \int_\Omega A_\varepsilon \nabla w_\varepsilon \cdot \nabla\left(\psi_\varepsilon \varrho^2\right) dx d\sigma = -\frac{1}{2} \int_\Omega A_\varepsilon \nabla W_\varepsilon(s) \cdot \nabla W_\varepsilon(s)\, \varrho^2 dx \\
+ 2 \int_0^s \int_\Omega \psi_\varepsilon \varrho A_\varepsilon \nabla w_\varepsilon \cdot \nabla \varrho dx d\sigma.
\end{aligned}$$

Considering this in (4.54) and noting that ϱ is independent of X_2, we obtain

$$\begin{aligned}
&\frac{1}{2} \left|w_\varepsilon(s)\,\varrho\right|_{L^2(\Omega)}^2 + \frac{1}{2} \int_\Omega A_\varepsilon \nabla W_\varepsilon(s) \cdot \nabla W_\varepsilon(s)\, \varrho^2 dx \\
&\quad = \frac{1}{2} \left|w_\varepsilon^0 \varrho\right|_{L^2(\Omega)}^2 + \int_\Omega w_\varepsilon^1 W_\varepsilon(s)\, \varrho^2 dx + \varepsilon^2 \int_0^s \int_\Omega A_{11} \nabla_{X_1} \tilde{u} \cdot \nabla_{X_1} \psi_\varepsilon \varrho^2 dx d\sigma \\
&\quad + \varepsilon \int_0^s \int_\Omega A_{12} \nabla_{X_2} \tilde{u} \cdot \nabla_{X_1}\left(\psi_\varepsilon \varrho^2\right) dx d\sigma + \varepsilon \int_0^s \int_\Omega A_{21} \nabla_{X_1} \tilde{u} \cdot \nabla_{X_2} \psi_\varepsilon \varrho^2 dx d\sigma \\
&\quad + 2\varepsilon^2 \int_0^s \int_\Omega \varrho \psi_\varepsilon A_{11} \nabla_{X_1} \tilde{u} \cdot \nabla_{X_1} \varrho dx d\sigma + 2\varepsilon^2 \int_0^s \int_\Omega \psi_\varepsilon \varrho A_{11} \nabla_{X_1} w_\varepsilon \cdot \nabla_{X_1} \varrho dx d\sigma \\
&\quad + 2\varepsilon \int_0^s \int_\Omega \psi_\varepsilon \varrho A_{12} \nabla_{X_2} w_\varepsilon \cdot \nabla_{X_1} \varrho dx d\sigma.
\end{aligned} \tag{4.55}$$

Using the Cauchy-Schwarz, Poincaré's inequalities and Young's inequality $2ab \leq \alpha a^2 + \left(b^2/\alpha\right)$ with convenient choices of α, it follows that

$$\begin{aligned}
&\left|w_\varepsilon(s)\,\varrho\right|_{L^2(\Omega)}^2 + \lambda\varepsilon^2 \left|\nabla_{X_1} W_\varepsilon(s)\,\varrho\right|_{L^2(\Omega)}^2 + \lambda \left|\nabla_{X_2} W_\varepsilon(s)\,\varrho\right|_{L^2(\Omega)}^2 \\
&\quad \leq \left|w_\varepsilon^0 \varrho\right|_{L^2(\Omega)}^2 + C \left|w_\varepsilon^1 \varrho\right|_{L^2(\Delta; H^{-1}(\omega))}^2 + \frac{\lambda}{4} \left|\nabla_{X_2} W_\varepsilon(s)\,\varrho\right|_{L^2(\Omega)}^2 + C\varepsilon^2 \left|\nabla_{X_1} \tilde{u}\right|_{L^2(Q)}^2 \\
&\quad + C\varepsilon^4 \left|\nabla_{X_1} w_\varepsilon\right|_{L^2(Q)}^2 + C\varepsilon^2 \left|\nabla_{X_2} w_\varepsilon\right|_{L^2(Q)}^2 + \frac{\lambda}{8T} \int_0^s \varepsilon^2 \left|\nabla_{X_1} \psi_\varepsilon \varrho\right|_{L^2(\Omega)}^2 + \left|\nabla_{X_2} \psi_\varepsilon \varrho\right|_{L^2(\Omega)}^2 d\sigma \\
&\quad + 2\varepsilon \int_0^s \left| \int_\Omega A_{12} \nabla_{X_2} \tilde{u} \cdot \nabla_{X_1}\left(\psi_\varepsilon \varrho^2\right) dx \right| d\sigma,
\end{aligned} \tag{4.56}$$

Then using the density of $\mathcal{D}\left(\Delta_1 \times \omega\right)$ in $H_0^1\left(\Delta_1 \times \omega\right)$ and the fact that $A_{12} = A_{21}^T$, we can rewrite the last integral as

$$\begin{aligned}
\int_\Omega A_{12}\nabla_{X_2}\tilde{u}\cdot\nabla_{X_1}(\psi_\varepsilon\left(\sigma\right)\varrho^2)dx &= \sum_{i=1}^{q}\sum_{j=q+1}^{n}\int_\Omega a_{ij}\partial_{x_j}\tilde{u}\partial_{x_i}\left(\psi_\varepsilon\left(\sigma\right)\varrho^2\right)dx \\
&= \sum_{i=1}^{q}\sum_{j=q+1}^{n}\int_\Omega \partial_{x_i}\left(\varrho^2 a_{ij}\psi_\varepsilon\left(\sigma\right)\right)\partial_{x_j}\tilde{u} - \partial_{x_i}a_{ij}\partial_{x_j}\tilde{u}\varrho^2\psi_\varepsilon\left(\sigma\right)dx \\
&= \sum_{i=1}^{q}\sum_{j=q+1}^{n}\int_\Omega \varrho^2 a_{ij}\partial_{x_j}\psi_\varepsilon\left(\sigma\right)\partial_{x_i}\tilde{u} + \varrho^2\partial_{x_j}a_{ij}\psi_\varepsilon\left(\sigma\right)\partial_{x_i}\tilde{u}dx \\
&\quad - \sum_{i=1}^{q}\sum_{j=q+1}^{n}\int_\Omega \partial_{x_i}a_{ij}\partial_{x_j}\tilde{u}\varrho^2\psi_\varepsilon\left(\sigma\right)dx,
\end{aligned}$$

hence

$$\begin{aligned}
\int_\Omega A_{12}\nabla_{X_2}\tilde{u}\cdot\nabla_{X_1}(\psi_\varepsilon\left(\sigma\right)\varrho^2)dx &= \int_\Omega A_{12}\nabla_{X_2}\psi_\varepsilon\left(\sigma\right)\cdot\nabla_{X_1}\tilde{u}\varrho^2 dx \\
&+ \int_\Omega\left(\nabla_{X_2}\cdot A_{21}\right)\cdot\nabla_{X_1}\tilde{u}\psi_\varepsilon\left(\sigma\right)\varrho^2 dx - \int_\Omega\left(\nabla_{X_1}\cdot A_{12}\right)\cdot\nabla_{X_2}\tilde{u}\psi_\varepsilon\left(\sigma\right)\varrho^2 dx.
\end{aligned}$$

The same techniques above yield

$$\begin{aligned}
2\varepsilon\int_0^s\left|\int_\Omega A_{12}\nabla_{X_2}\tilde{u}\cdot\nabla_{X_1}\left(\psi_\varepsilon\varrho^2\right)dx\right|d\sigma &\le C\varepsilon^2\left(\left|\nabla_{X_1}\tilde{u}\right|^2_{L^2(Q_1)} + \left|\nabla_{X_2}\tilde{u}\right|^2_{L^2(Q_1)}\right) \\
&+ \frac{\lambda}{8T}\int_0^s\left|\nabla_{X_2}\psi_\varepsilon\varrho\right|^2_{L^2(\Omega)}d\sigma.
\end{aligned}$$

Going back to (4.56) we derive

$$\begin{aligned}
&\left|w_\varepsilon\left(s\right)\varrho\right|^2_{L^2(\Omega)} + \lambda\varepsilon^2\left|\nabla_{X_1}W_\varepsilon\left(s\right)\varrho\right|^2_{L^2(\Omega)} + \frac{3\lambda}{4}\left|\nabla_{X_2}W\left(s\right)\varrho\right|^2_{L^2(\Omega)} \\
&\quad\le \left|w_\varepsilon^0\right|^2_{L^2(\Omega_1)} + C\left|w_\varepsilon^1\right|^2_{L^2(\Delta_1;H^{-1}(\omega))} \\
&\quad + C\varepsilon^2\left\{\left|\nabla\tilde{u}\right|^2_{L^2(Q_1)} + \varepsilon^2\left|\nabla_{X_1}w_\varepsilon\right|^2_{L^2(Q_1)} + \left|\nabla_{X_2}w_\varepsilon\right|^2_{L^2(Q_1)}\right\} \\
&\quad + \frac{\lambda}{4T}\int_0^s\varepsilon^2\left|\nabla_{X_1}\psi_\varepsilon\left(\sigma\right)\varrho\right|^2_{L^2(\Omega)} + \left|\nabla_{X_2}\psi_\varepsilon\left(\sigma\right)\varrho\right|^2_{L^2(\Omega)}d\sigma.
\end{aligned} \tag{4.57}$$

Using the inequality $(a \pm b)^2 \le 2a^2 + b^2$, the last integral term of the right hand side can be estimated as

$$\begin{aligned}
&\frac{\lambda}{4T}\int_0^s\varepsilon^2\left|\nabla_{X_1}\psi_\varepsilon\left(\sigma\right)\varrho\right|^2_{L^2(\Omega)} + \left|\nabla_{X_2}\psi_\varepsilon\left(\sigma\right)\varrho\right|^2_{L^2(\Omega)}d\sigma \\
&\qquad\le \frac{\lambda}{2T}\int_0^s\varepsilon^2\left|\nabla_{X_1}W_\varepsilon\left(\sigma\right)\varrho\right|^2_{L^2(\Omega)} + \left|\nabla_{X_2}W_\varepsilon\left(\sigma\right)\varrho\right|^2_{L^2(\Omega)}d\sigma \\
&\qquad + \frac{\lambda}{2}\left(\varepsilon^2\left|\nabla_{X_1}W_\varepsilon\left(s\right)\varrho\right|^2_{L^2(\Omega)} + \left|\nabla_{X_2}W_\varepsilon\left(s\right)\varrho\right|^2_{L^2(\Omega)}\right).
\end{aligned}$$

Considering this in (4.57) we get

$$\begin{aligned}&|w_\varepsilon(s)\varrho|^2_{L^2(\Omega)}+\frac{\lambda}{2}\varepsilon^2|\nabla_{X_1}W_\varepsilon(s)\varrho|^2_{L^2(\Omega)}+\frac{\lambda}{4}|\nabla_{X_2}W_\varepsilon(s)\varrho|^2_{L^2(\Omega)}\\&\quad\le|w^0_\varepsilon|^2_{L^2(\Omega_1)}+C\left|w^1_\varepsilon\right|^2_{L^2(\Delta_1;H^{-1}(\omega))}+C\varepsilon^2\left(|\nabla\tilde u|^2_{L^2(Q_1)}+|\nabla_{X_2}w_\varepsilon|^2_{L^2(Q_1)}+\varepsilon^2|\nabla_{X_1}w_\varepsilon|^2_{L^2(Q_1)}\right)\\&\quad+\frac{\lambda}{2T}\int_0^s\varepsilon^2|\nabla_{X_1}W_\varepsilon(\sigma)\varrho|^2_{L^2(\Omega)}+|\nabla_{X_2}W_\varepsilon(\sigma)\varrho|^2_{L^2(\Omega)}\,d\sigma.\end{aligned}$$

For $s\in[0,T]$, using the Gronwall inequality we obtain

$$\begin{aligned}&|w_\varepsilon(s)\varrho|^2_{L^2(\Omega)}+\varepsilon^2|\nabla_{X_1}W_\varepsilon(s)\varrho|^2_{L^2(\Omega)}+|\nabla_{X_2}W_\varepsilon(s)\varrho|^2_{L^2(\Omega)}\\&\qquad\le C\left(|w^0_\varepsilon|^2_{L^2(\Omega_1)}+\left|w^1_\varepsilon\right|^2_{L^2(\Delta_1;H^{-1}(\omega))}\right)\\&\qquad+C\varepsilon^2\left(|\nabla\tilde u|^2_{L^2(Q_1)}+|\nabla_{X_2}w_\varepsilon|^2_{L^2(Q_1)}+\varepsilon^2|\nabla_{X_1}w_\varepsilon|^2_{L^2(Q_1)}\right),\end{aligned}\tag{4.58}$$

where all the terms in the right hand side are independent of s. Applying (4.21) and (4.52), we derive

$$\sup_{s\in[0,T]}|w_\varepsilon(s)|_{L^2(\Omega_0)},\ \sup_{s\in[0,T]}|\nabla_{X_2}W_\varepsilon(s)|_{L^2(\Omega_0)}=O(\varepsilon),\tag{4.59}$$

and by consequence we have

$$|w_\varepsilon|_{L^2(Q_0)},|\nabla_{X_2}W_\varepsilon|_{L^2(Q_0)}=O(\varepsilon).\tag{4.60}$$

We also derive from (4.58) that

$$\nabla_{X_1}W_\varepsilon(s)\quad\text{is bounded in } L^2(\Omega_0),\tag{4.61}$$

independently of $s\in[0,T]$. So for a fixed $s\in[0,T]$, there exist χ such that -up to a subsequence- $\nabla_{X_1}W_\varepsilon(s)\rightharpoonup\chi(s)$, in $L^2(\Omega_0)$, and we may verify as above (see (4.25)) that $\chi(s)=0$, i.e.

$$\nabla_{X_1}W_\varepsilon(s)\rightharpoonup 0\quad\text{in } L^2(\Omega_0).$$

The convergence holds for the whole sequence, since the limit is unique, and for every $s\in[0,T]$. By Lebesgue's theorem and the pointwise convergence in s, we also obtain

$$\nabla_{X_1}W_\varepsilon\rightharpoonup 0,\quad\text{in } L^2(Q_0).\tag{4.62}$$

Integrating (4.53) over $(0,s)$, it comes

$$\begin{aligned}&\int_\Omega w'_\varepsilon(s)\,v\,dx=\int_\Omega w^1_\varepsilon v\,dx-\int_\Omega A_\varepsilon\nabla W_\varepsilon(s)\cdot\nabla v\,dx\\&-\int_0^s\int_\Omega\varepsilon^2A_{11}\nabla_{X_1}\tilde u\cdot\nabla_{X_1}v+\varepsilon A_{12}\nabla_{X_2}\tilde u\cdot\nabla_{X_1}v+\varepsilon A_{21}\nabla_{X_1}\tilde u\cdot\nabla_{X_2}v\,dxds,\quad\forall v\in H^1_0(\Omega_0).\end{aligned}$$

Applying the Cauchy-Schwarz inequality to each term, we find

$$\int_{\Omega_0} w'_\varepsilon (s)\, v dx \leq C \left\{ \left| w^1_\varepsilon \right|_{L^2(\Delta_0;H^{-1}(\omega))} + \varepsilon \left| \nabla_{X_1} W_\varepsilon (s) \right|_{L^2(\Omega_0)} \right.$$
$$\left. + \left| \nabla_{X_2} W_\varepsilon (s) \right|_{L^2(\Omega_0)} + \varepsilon \left| \nabla \tilde{u} \right|_{L^2(Q_0)} \right\} \left| \nabla v \right|_{L^2(\Omega_0)}, \quad \forall v \in H^1_0 (\Omega_0).$$

Then we conclude, after dividing by ε, that

$$\frac{1}{\varepsilon} \left| w'_\varepsilon (s) \right|_{H^{-1}(\Omega_0)} \leq C \left\{ \frac{1}{\varepsilon} \left| w^1_\varepsilon \right|_{L^2(\Delta_0;H^{-1}(\omega))} \right.$$
$$\left. + \left| \nabla_{X_1} W_\varepsilon (s) \right|_{L^2(\Omega_0)} + \frac{1}{\varepsilon} \left| \nabla_{X_2} W_\varepsilon (s) \right|_{L^2(\Omega_0)} + \left| \nabla \tilde{u} \right|_{L^2(Q_0)} \right\}. \tag{4.63}$$

Due to (4.52) and (4.59)–(4.61), the right hand side terms of the above inequality are bounded independently of $s \in [0, T]$. Thus we get

$$\sup_{s\in[0,T]} \left| w'_\varepsilon (s) \right|_{H^{-1}(\Omega_0)} = O(\varepsilon).$$

This ends the proof of the theorem. ■

Matrix with diagonal structure

When the matrix A has a diagonal structure we can improve the above convergences.

Corollary 4.2 (Diagonal matrix) *Under the assumptions of Theorem 4.4, in addition we suppose that* $A_{12}, A_{21} = 0$ *and*

$$\left| u^0_\varepsilon - u^0 \right|_{L^2(\Omega)} = o(\varepsilon) \quad \textit{and} \quad \left| u^1_\varepsilon - u^1 \right|_{L^2(\Delta;H^{-1}(\omega))} = o(\varepsilon), \tag{4.64}$$

then

$$\sup_{t\in[0,T]} \left| \left(u'_\varepsilon - \tilde{u}' \right)(t) \right|_{H^{-1}(\Omega_0)}, \quad \sup_{t\in[0,T]} \left| \left(u_\varepsilon - \tilde{u} \right)(t) \right|_{L^2(\Omega_0)} = o(\varepsilon),$$
$$\sup_{t\in[0,T]} \left| \nabla_{X_1} \left(U_\varepsilon - \tilde{U} \right)(t) \right|_{L^2(\Omega_0)} \to 0,$$
$$\sup_{t\in[0,T]} \left| \nabla_{X_2} \left(U_\varepsilon - \tilde{U} \right)(t) \right|_{L^2(\Omega_0)} = o(\varepsilon).$$

In particular we have the strong convergence

$$\sup_{t\in[0,T]} \left| \left(U_\varepsilon - \tilde{U} \right)(t) \right|_{H^1(\Omega_0)} \to 0.$$

Proof. Taking into account the fact that $A_{12}, A_{21} = 0$ in the proof of Theorem 4.4, then (4.55) becomes

$$\begin{aligned}
&\frac{1}{2}\left|w_\varepsilon(s)\varrho\right|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega \varepsilon^2 A_{11}\nabla_{X_1}W_\varepsilon(s)\cdot\nabla_{X_1}W_\varepsilon(s)\varrho^2 + A_{22}\nabla_{X_2}W_\varepsilon(s)\cdot\nabla_{X_2}W_\varepsilon(s)\varrho^2dx \\
&\qquad = \frac{1}{2}\left|w^0_\varepsilon\varrho\right|^2_{L^2(\Omega)} + \int_\Omega w^1_\varepsilon W_\varepsilon(s)\varrho^2dx + \varepsilon^2\int_0^s\int_\Omega A_{11}\nabla_{X_1}\tilde{u}\cdot\nabla_{X_1}\psi_\varepsilon\varrho^2dxd\sigma \\
&\qquad + 2\varepsilon^2\int_0^s\int_\Omega \varrho\psi_\varepsilon A_{11}\nabla_{X_1}\tilde{u}\cdot\nabla_{X_1}\varrho dxd\sigma + 2\varepsilon^2\int_0^s\int_\Omega \psi_\varepsilon\varrho A_{11}\nabla_{X_1}w_\varepsilon\cdot\nabla_{X_1}\varrho dxd\sigma,
\end{aligned}$$

hence

$$\begin{aligned}
&\left|w_\varepsilon(s)\varrho\right|^2_{L^2(\Omega)} + \varepsilon^2\left|\nabla_{X_1}W_\varepsilon(s)\varrho\right|^2_{L^2(\Omega)} + \left|\nabla_{X_2}W_\varepsilon(s)\varrho\right|^2_{L^2(\Omega)} \\
&\qquad \leq C\left(\left|w^0_\varepsilon\right|^2_{L^2(\Omega)} + \left|w^1_\varepsilon\right|^2_{L^2(\Delta;H^{-1}(\omega))}\right) + C\varepsilon^4\left(\left|\nabla_{X_1}\tilde{u}\right|^2_{L^2(Q_1)} + \left|\nabla_{X_1}w_\varepsilon\right|^2_{L^2(Q_1)}\right) \\
&\qquad + C\varepsilon^2\int_0^s\left|\int_\Omega A_{11}\nabla_{X_1}\tilde{u}\cdot\nabla_{X_1}\psi_\varepsilon\varrho^2dx\right|d\sigma + \frac{1}{4}\int_0^s\left|\nabla_{X_2}\psi_\varepsilon\varrho\right|^2_{L^2(\Omega)}dxd\sigma
\end{aligned}$$

Estimating as above and dividing on ε^2, we end up with

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\left|w_\varepsilon(s)\varrho\right|^2_{L^2(\Omega)} + \left|\nabla_{X_1}W_\varepsilon(s)\varrho\right|^2_{L^2(\Omega)} + \frac{1}{\varepsilon^2}\left|\nabla_{X_2}W_\varepsilon(s)\varrho\right|^2_{L^2(\Omega)} \\
&\quad \leq \frac{C}{\varepsilon^2}\left(\left|w^0_\varepsilon\right|^2_{L^2(\Omega)} + \left|w^1_\varepsilon\right|^2_{L^2(\Delta;H^{-1}(\omega))}\right) + C\varepsilon^2\left(\left|\nabla_{X_1}\tilde{u}\right|^2_{L^2(Q_1)} + \left|\nabla_{X_1}w_\varepsilon\right|^2_{L^2(Q_1)}\right) \\
&\quad + C\left|\int_0^s\int_\Omega A_{11}\nabla_{X_1}\tilde{u}\cdot\nabla_{X_1}\psi_\varepsilon\varrho^2dxd\sigma\right|,
\end{aligned}$$

for every $s \in [0,T]$. We can always change Ω_0 by Ω_1 in Theorem 4.4 and use the weak convergence of $\nabla_{X_1}W_\varepsilon$, (see (4.62)), in the remaining A_{11} integral above, it comes

$$\begin{aligned}
\int_0^s\int_\Omega A_{11}\nabla_{X_1}\tilde{u}\cdot\nabla_{X_1}\psi_\varepsilon(\sigma)\varrho^2dx = \Bigg\{&\int_0^s\int_\Omega A_{11}\nabla_{X_1}\tilde{u}\cdot\nabla_{X_1}W_\varepsilon(\sigma)\varrho^2dxds \\
&- \int_0^s\int_\Omega A_{11}\nabla_{X_1}\tilde{u}\cdot\nabla_{X_1}W_\varepsilon(s)\varrho^2dxds\Bigg\} \longrightarrow 0.
\end{aligned}$$

Thus by (4.64) we obtain

$$\sup_{s\in[0,T]}\left|w_\varepsilon(s)\right|_{L^2(\Omega_0)}, \quad \sup_{s\in[0,T]}\left|\nabla_{X_2}W_\varepsilon(s)\right|_{L^2(\Omega_0)} = o(\varepsilon), \quad \sup_{s\in[0,T]}\left|\nabla_{X_1}W_\varepsilon(s)\right|_{L^2(\Omega_0)} \to 0. \tag{4.65}$$

Next, the equivalent of (4.63), in this case ($A_{12}, A^T_{21} = 0$), is given by

$$\begin{aligned}
\frac{1}{\varepsilon}\left|w'_\varepsilon(s)\right|_{H^{-1}(\Omega_0)} \leq C\Bigg\{&\frac{1}{\varepsilon}\left|w^1_\varepsilon\right|_{L^2(\Delta_0;H^{-1}(\omega))} + \left|\nabla_{X_1}W_\varepsilon(s)\right|_{L^2(\Omega_0)} \\
&+ \frac{1}{\varepsilon}\left|\nabla_{X_2}W_\varepsilon(s)\right|_{L^2(\Omega_0)} + \varepsilon\left|\nabla_{X_1}\tilde{u}\right|_{L^2(Q_0)}\Bigg\}.
\end{aligned}$$

It follows, by (4.64) and (4.65), that

$$\sup_{s\in[0,T]}\left|w'_\varepsilon(s)\right|_{H^{-1}(\Omega_0)} = o(\varepsilon).$$

This completes the proof of the corollary. ■

COMMENTS

Existence, uniqueness and regularity results for solutions of linear hyperbolic problems can be found in the classic books Evans [26], Dautray and Lions [22], Ladyzhenskaya [43], Lions and Magenes [49]. There exist many works on the (isotropic) singular perturbations of hyperbolic problems, see for instance Lions [48, Chapter 4] and de Jager and Furu [23].

The results of this chapter has appeared in Guesmia and Sengouga [34, 35], Sengouga and Guesmia [62].

5

ANISOTROPIC SINGULAR PERTURBATIONS OF SEMILINEAR HYPERBOLIC PROBLEMS

In this chapter we deal with anisotropic singular perturbations of some semilinear hyperbolic problems. Beyond the nonlinearity of the problem, the improvements obtained here include some results of the precedent chapter. Moreover an exponential rate of convergence is obtained, far from the lateral boundaries, when the problem satisfies some cylindrical symmetries.

5.1 PROBLEM SETTING

We keep the same notations as in the precedent chapter and we assume that the matrix A is independent of time, i.e. $A = (a_{ij}(x))$ and

$$a_{ij} = a_{ij}(x) \in C^1(\bar{\Omega}), \quad \forall\, i, j = 1, \cdots, n, \tag{5.1}$$

In addition we make the following hypotheses (see [51]). Let W be a reflexive separable Banach space satisfying $W \subset L^2(\Omega)$ (continuous inclusion) such that

$$H_0^1(\Omega) \cap W \ \text{ is dense in } H_0^1(\Omega) \ \text{ and in } W. \tag{5.2}$$

Thus we have

$$H_0^1(\Omega) \cap W \subset H_0^1(\Omega), \quad W \subset L^2(\Omega) \subset H^{-1}(\Omega),$$

$$W' \subset \left(H_0^1(\Omega) \cap W\right)' = H^{-1}(\Omega) + W'.$$

We consider a family of (nonlinear) operators $\beta(t) : W \to W'$ satisfying

$$\beta v \in L^{p'}(0,T;W'), \quad \forall v \in L^p(0,T;W) \tag{5.3}$$

where $1 \le p < \infty$. Moreover we assume that

$$\left.\begin{array}{ll} i) & \text{For a.e. } t \in [0,T],\ \beta(t) \text{ is continuous from finite dimensional} \\ & \text{subspaces of } W \text{ to the weak topology of } W', \\ ii) & \exists b > 0 \text{ such that } \langle \beta(t) v, v\rangle_W \ge b\,|v|_W^p,\ \ \forall v \in W,\ \text{a.e. } t \in [0,T], \\ iii) & \langle \beta(t) v_1 - \beta(t) v_2, v_1 - v_2\rangle_W \ge 0,\ \ \forall v_1, v_2 \in W,\ \text{a.e. } t \in [0,T], \\ iv) & \beta \text{ is hemicontinuous and send bounded sets of } L^p(0,T;W) \\ & \text{to bounded sets in } L^{p'}(0,T;W'). \end{array}\right\} \tag{5.4}$$

We also make the following assumptions on the source term and the initial data

$$f \in L^2(0,T;L^2(\Omega)) = L^2(Q), \tag{5.5}$$

$$u_\varepsilon^0 \in H_0^1(\Omega), \quad u_\varepsilon^1 \in L^2(\Omega). \tag{5.6}$$

Then we consider the semilinear problem defined by

$$\begin{cases} u'' - \nabla \cdot (A_\varepsilon \nabla u) + \beta u' = f & \text{in} \quad Q, \\ u = 0 & \text{on} \quad (0,T) \times \partial\Omega, \\ u(0) = u_\varepsilon^0,\ \ u'(0) = u_\varepsilon^1 & \text{in} \quad \Omega. \end{cases} \tag{5.7}$$

This problem represents a vibrating membrane with resistance proportional to the velocity. The hypotheses above ensure the existence of a unique weak solution u_ε satisfying (see [51, 64])

$$\begin{cases} u_\varepsilon \in C([0,T];H_0^1(\Omega)),\ u_\varepsilon' \in C([0,T];L^2(\Omega)) \cap L^p(0,T;W), \\ \langle u_\varepsilon''(t), v\rangle_{H_0^1(\Omega)} + \displaystyle\int_\Omega A_\varepsilon \nabla u_\varepsilon(t) \cdot \nabla v dx + \langle \beta(t) u_\varepsilon'(t), v\rangle_W \\ \qquad\qquad = \displaystyle\int_\Omega f(t) v dx,\ \ \forall v \in H_0^1(\Omega), \\ u_\varepsilon(0) = u_\varepsilon^0,\ \ u_\varepsilon'(0) = u_\varepsilon^1, \quad \text{in } \ \Omega. \end{cases} \tag{5.8}$$

In this chapter we investigate the asymptotic behaviour of u_ε, when the propagation speed of the wave is very small in the X_1 direction, i.e. when $\varepsilon \to 0$. We assume that there exist

$$u^0 \in \mathcal{V}_2(\Omega), \quad u^1 \in L^2(\Omega), \tag{5.9}$$

where $\mathcal{V}_2(\Omega)$ is defined as in (4.15), such that

$$\varepsilon \nabla_{X_1} u_\varepsilon^0 \to 0, \quad \nabla_{X_2} u_\varepsilon^0 \to \nabla_{X_2} u^0 \quad \text{and} \quad u_\varepsilon^1 \to u^1 \;\text{ in } L^2(\Omega). \tag{5.10}$$

Following the precedent chapter, the expected limit of u_ε is $\tilde{u}$, defined as a solution to

$$\begin{cases} \tilde{u} \in C([0,T]; \mathcal{V}_2(\Omega)),\; \tilde{u}' \in C([0,T]; L^2(\Omega)) \cap L^p(0,T;W), \\ \langle \tilde{u}''(t), v\rangle_{\mathcal{V}_2(\Omega)} + \int_\Omega A_{22} \nabla_{X_2} \tilde{u}(t) \cdot \nabla_{X_2} v dx + \langle \beta(t) \tilde{u}'(t), v\rangle_W \\ \qquad\qquad = \int_\Omega f(t) v dx, \qquad \forall v \in \mathcal{V}_2(\Omega), \\ \tilde{u}(0) = u^0, \;\; \tilde{u}'(0) = u^1 \quad \text{in } \;\Omega. \end{cases} \tag{5.11}$$

Even if this problem is not classic, the abstract approach established in Lions and Struass [51] can be applied. Indeed:

Theorem 5.1 *Under the assumptions (4.7), (5.1)–(5.5) and (5.9), the problem (5.11) has a unique solution $\tilde{u}$ and the following energy equality holds*

$$\begin{aligned} &|\tilde{u}'(t)|_\Omega^2 + \int_\Omega A_{22} \nabla_{X_2} \tilde{u}(t) \cdot \nabla_{X_2} \tilde{u}(t)\, dx + 2\int_0^t \langle \beta \tilde{u}', \tilde{u}'\rangle_W ds \\ &\qquad = \left|u^1\right|_\Omega^2 + \int_\Omega A_{22} \nabla_{X_2} u^0 \cdot \nabla_{X_2} u^0 dx + 2\int_0^t \int_\Omega f \tilde{u}' dx ds, \qquad \forall t \in [0,T]. \end{aligned} \tag{5.12}$$

Proof. Since $\mathcal{V}_2(\Omega) \cap W$ is dense in $L^2(\Omega)$ and that from (5.2) one has

$$\mathcal{V}_2(\Omega) \cap W \subset L^2(\Omega) \subset (\mathcal{V}_2(\Omega) \cap W)',$$

the existence and the uniqueness of the solution of (5.11) follow by choosing $\mathcal{V}_2(\Omega)$ as a basic space in [51, Theorem 2.1]. The energy equality and the continuity in time follow from [64, Theorems 4.1, 4.2]. ■

5.2 CONVERGENCES IN ARBITRARY DOMAINS

In this section, we study the convergence of u_ε towards u_0 in $\mathcal{V}_2(\Omega)$ using some weak convergence methods combined with a monotonicity argument.

Theorem 5.2 *Under the assumptions (4.2), (5.1)–(5.6) and (5.10) we have, when $\varepsilon \to 0$,*

$$\begin{gathered} u_\varepsilon(t) \to \tilde{u}(t) \;\; \text{in } \mathcal{V}_2(\Omega), \\ u_\varepsilon'(t) \to \tilde{u}'(t), \quad \varepsilon \nabla_{X_1} u_\varepsilon(t) \to 0 \;\; \text{in } L^2(\Omega), \end{gathered} \tag{5.13}$$

for every $t \in [0,T]$, *and*

$$\begin{aligned} u_\varepsilon' &\rightharpoonup \tilde{u}' \ \text{ in } L^p(0,T;W), \\ \beta u_\varepsilon' &\rightharpoonup \beta\tilde{u}' \ \text{ in } L^{p'}(0,T;W'), \\ \langle \beta u_\varepsilon' - \beta\tilde{u}', u_\varepsilon' - \tilde{u}'\rangle_W &\to 0 \ \text{ in } L^1(0,T) \end{aligned} \tag{5.14}$$

where $\tilde{u}$ *(resp.* u_ε*) is the solution of problem* (5.11) *(resp.* (5.8)*).*

We shall prove this theorem in the following lemmas.

Lemma 5.1 *Under the assumptions of Theorem* 5.2 *we have*

$$\begin{aligned} (u_\varepsilon)_\varepsilon \ &\text{is bounded in } \ L^\infty(0,T;\mathcal{V}_2(\Omega)), \\ (\varepsilon\nabla_{X_1}u_\varepsilon)_\varepsilon \ &\text{is bounded in } \ L^\infty(0,T;L^2(\Omega)), \\ (u_\varepsilon')_\varepsilon \ &\text{is bounded in } \ L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W), \\ (\beta u_\varepsilon')_\varepsilon \ &\text{is bounded in } \ L^{p'}(0,T;W'). \end{aligned} \tag{5.15}$$

Proof. The energy equality for the problem (5.8) is given by (see [64])

$$\begin{aligned} &|u_\varepsilon'(t)|_\Omega^2 + \int_\Omega A_\varepsilon \nabla u_\varepsilon(t)\cdot\nabla u_\varepsilon(t)\,dx + 2\int_0^t \langle \beta u_\varepsilon', u_\varepsilon'\rangle_W\,ds \\ &\qquad = \left|u_\varepsilon^1\right|_\Omega^2 + \int_\Omega A_\varepsilon\nabla u_\varepsilon^0\cdot\nabla u_\varepsilon^0 dx + 2\int_0^t\int_\Omega f u_\varepsilon' dx ds, \qquad \forall t\in[0,T]. \end{aligned} \tag{5.16}$$

Using (4.2), (5.1), (5.4-*ii*) and (5.10), it follows that

$$\begin{aligned} &|u_\varepsilon'(t)|_\Omega^2 + \lambda\left(\varepsilon^2|\nabla_{X_1}u_\varepsilon(t)|_\Omega^2 + |\nabla_{X_2}u_\varepsilon(t)|_\Omega^2\right) + 2b\int_0^t |u_\varepsilon'|_W^p\,ds \\ &\qquad\qquad \le C + C\int_0^t |u_\varepsilon'|_\Omega^2\,ds, \quad \forall t\in[0,T] \end{aligned}$$

where C is a positive constant, independent of ε and t. Applying Gronwall's inequality we get

$$|u_\varepsilon'(t)|_\Omega, \ \ \varepsilon|\nabla_{X_1}u_\varepsilon(t)|_\Omega, \ \ |\nabla_{X_2}u_\varepsilon(t)|_\Omega, \ \ \int_0^t |u_\varepsilon'|_W^p\,ds \le C, \qquad \forall t\in[0,T]. \tag{5.17}$$

The last estimate of the lemma follows by $(5.4 - iv)$. ■

Lemma 5.2 *If there exist* $z \in \mathcal{D}'(Q)$ *and a subsequence of* $(u_\varepsilon)_\varepsilon$ *-still labelled* $(u_\varepsilon)_\varepsilon$*- such that* $u_\varepsilon \to z$ *in* $\mathcal{D}'(Q)$ *then*

$$z \in C([0,T];\mathcal{V}_2(\Omega)), \quad z' \in C([0,T];L^2(\Omega)) \cap L^p(0,T;W) \tag{5.18}$$

and z *satisfies the initial conditions*

$$z(0) = u^0, \quad z'(0) = u^1. \tag{5.19}$$

Moreover, for every $t \in [0,T]$ we have

$$\begin{aligned} u_\varepsilon(t) \rightharpoonup z(t) \quad &in \quad \mathcal{V}_2(\Omega), \\ u'_\varepsilon(t) \rightharpoonup z'(t), \quad \varepsilon \nabla_{X_1} u_\varepsilon(t) \rightharpoonup 0 \quad &in \quad L^2(\Omega). \end{aligned} \tag{5.20}$$

Proof. Due to Lemma 5.1, we can extract a weak converging subsequence -still labeled $(u_\varepsilon)_\varepsilon$- and there exists $\chi \in L^{p'}(0,T;W')$ such that

$$\begin{aligned} u_\varepsilon \overset{*}{\rightharpoonup} z \quad &in \quad L^\infty(0,T;\mathcal{V}_2(\Omega)), \\ \varepsilon \nabla_{X_1} u_\varepsilon \overset{*}{\rightharpoonup} 0, \quad u'_\varepsilon \overset{*}{\rightharpoonup} z' \quad &in \quad L^\infty(0,T;L^2(\Omega)), \\ u'_\varepsilon \rightharpoonup z' \quad &in \quad L^p(0,T;W), \\ \beta u'_\varepsilon \rightharpoonup \chi \quad &in \quad L^{p'}(0,T;W'). \end{aligned} \tag{5.21}$$

To verify that the limits are as stated we argue as in Remark 1.3. Then multiplying the equation in (5.8) by $\phi \in \mathcal{D}(]0,T[)$ and integrating over $(0,T)$, we get

$$\int_0^T \langle u_\varepsilon, v\rangle_{H_0^1(\Omega)\cap W} \phi'' ds + \int_0^T \int_\Omega A_\varepsilon \nabla u_\varepsilon \cdot \nabla v \phi dx ds + \int_0^T \langle \beta u'_\varepsilon, v\rangle_W \phi ds = \int_0^T \int_\Omega f v \phi dx ds,$$

where the first integral term is obtained after integrating by parts on $(0,T)$. Expending A_ε to different blocks and passing to the limit using (5.21), it follows that

$$\begin{aligned} \int_0^T \langle z, v\rangle_{H_0^1(\Omega)\cap W} \phi'' ds + \int_0^T \int_\Omega A_{22} \nabla_{X_2} z \cdot \nabla_{X_2} v \phi dx ds \\ + \int_0^T \langle \chi, v\rangle_W \phi ds = \int_0^T \int_\Omega f v \phi dx ds, \end{aligned} \tag{5.22}$$

for every $v \in \mathcal{D}(\Omega)$ and every $\phi \in \mathcal{D}(]0,T[)$. Note that we have, from (5.21),

$$z \in L^\infty(0,T;\mathcal{V}_2(\Omega)), \qquad z' \in L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W)$$

and we have, from (5.22),

$$z'' - \nabla_{X_2} \cdot (A_{22} \nabla_{X_2} z) = (f - \chi) \in L^2(0,T;L^2(\Omega)) + L^{p'}(0,T;W'). \tag{5.23}$$

Then, according to Strauss [64, Theorems 4.1, 4.2], z satisfies the continuity in time (5.18) and the following energy equality

$$\begin{aligned} |z'(t)|_\Omega^2 + \int_\Omega A_{22} \nabla_{X_2} z(t) \cdot \nabla_{X_2} z(t) dx + 2\int_0^t \langle \chi, z'\rangle_W ds \\ = |z'(0)|_\Omega^2 + \int_\Omega A_{22} \nabla_{X_2} z(0) \cdot \nabla_{X_2} z(0) dx + 2\int_0^t \int_\Omega f z' dx ds, \quad \forall t \in [0,T]. \end{aligned} \tag{5.24}$$

To show that the initial conditions are satisfied, we use on the one hand the following identities, established using an integration by parts twice on $[0,t]$,

$$\begin{aligned} \int_0^t \langle z'', v\rangle_{H_0^1(\Omega)\cap W} \phi ds = \int_\Omega z'(t) v \phi(t) dx - \int_\Omega z'(0) v \phi(0) dx \\ - \int_\Omega z(t) v \phi'(t) dx + \int_\Omega z(0) v \phi'(0) dx + \int_0^t \int_\Omega z v \phi'' dx ds \end{aligned} \tag{5.25}$$

and

$$\int_0^t \langle u_\varepsilon'', v\rangle_{H_0^1(\Omega)\cap W} \phi ds = \int_\Omega u_\varepsilon'(t)\, v\phi(t)dx - \int_\Omega u_\varepsilon^1 v\phi(0)dx \\ - \int_\Omega u_\varepsilon(t)\, v\phi'(t)dx + \int_\Omega u_\varepsilon^0 v\phi'(0)dx + \int_0^t \int_\Omega u_\varepsilon v\phi'' dxds \tag{5.26}$$

for every $\phi \in C^\infty([0,T])$ and every $v \in \mathcal{D}(\Omega)$. On the other hand using (5.8), (5.21) and (5.23), we get

$$\int_0^t \langle u_\varepsilon'', v\rangle_{H_0^1(\Omega)\cap W} \phi ds = \int_0^t \int_\Omega f v\phi dxds - \int_0^t \langle \beta u_\varepsilon', v\rangle_W \phi ds - \int_0^t \int_\Omega A_\varepsilon \nabla u_\varepsilon \cdot \nabla v\phi dxds \\ \longrightarrow \int_0^t \int_\Omega f v\phi dxds - \int_0^t \langle \chi, v\rangle_W \phi ds - \int_0^t \int_\Omega A_{22} \nabla_{X_2} z \cdot \nabla_{X_2} v\phi dxds \\ = \int_0^t \langle z'', v\rangle_{H_0^1(\Omega)\cap W} \phi ds.$$

Then passing to the limit in (5.26) for $\phi(t) = \phi'(t) = 0$ using (5.10) and (5.21), it comes

$$\int_0^t \langle z'', v\rangle_{H_0^1(\Omega)\cap W} \phi ds = -\int_\Omega u^1 v\phi(0)dx + \int_\Omega u^0 v\phi'(0)dx + \int_0^t \int_\Omega z v\phi'' dxds. \tag{5.27}$$

Comparing (5.25) and (5.27), we derive

$$-\langle z'(0), v\rangle_{H_0^1(\Omega)\cap W} \phi(0) + \int_\Omega z(0)\, v\phi'(0)dx = -\int_\Omega u^1 \phi(0) v dx + \int_\Omega u^0 v\phi'(0)dx, \ \forall v \in \mathcal{D}(\Omega).$$

This shows the first identity in (5.19) by choosing $\phi(0) = 1$ and $\phi'(0) = 0$. The second identity is shown by choosing $\phi(0) = 0$ and $\phi'(0) = 1$.

Next, since (5.17) holds for every $t \in [0,T]$, there exist ζ_t^0, ζ_t^1 and ζ_t^2 such that -up to a subsequence-

$$\varepsilon \nabla_{X_1} u_\varepsilon(t) \rightharpoonup \zeta_t^2, \quad u_\varepsilon'(t) \rightharpoonup \zeta_t^1 \quad \text{in } L^2(\Omega) \quad \text{and} \quad u_\varepsilon(t) \rightharpoonup \zeta_t^0 \ \text{in } \mathcal{V}_2(\Omega).$$

Passing to the limit in (5.26) for $\phi(0) = \phi'(0) = 0$, comparing with (5.25) and arguing as above for different choices of $\phi(t)$ and $\phi'(t)$ yields

$$\zeta_t^0 = z(t), \quad \zeta_t^1 = z'(t) \quad \text{and} \quad \zeta_t^2 = 0.$$

This completes the proof of the lemma. ■

Note that we did not use the monotonicity of β yet. This assumption plays an important role to overcome the lack of compactness and to determine clearly the limit of the nonlinear term, as we will see in the following lemma.

Lemma 5.3 *Under the assumptions of Lemma 5.2 we have*

$$\beta u_\varepsilon' \rightharpoonup \beta z' \quad in \ L^{p'}(0,T;W') \tag{5.28}$$

and z is the solution of (5.11).

Proof. For $v \in L^\infty\left(0,T;L^2\left(\Omega\right)\right) \cap L^p\left(0,T;W\right)$, let us set

$$\begin{aligned} M_\varepsilon\left(t,v\right) &:= \frac{1}{2}\left|\left(u_\varepsilon - z\right)'\left(t\right)\right|_\Omega^2 + \int_0^t \left\langle \beta u_\varepsilon' - \beta v, u_\varepsilon' - v\right\rangle_W ds \\ &+ \frac{1}{2}\int_\Omega A_\varepsilon \begin{pmatrix} \nabla_{X_1} u_\varepsilon\left(t\right) \\ \nabla_{X_2}\left(u_\varepsilon - z\right)\left(t\right) \end{pmatrix} \cdot \begin{pmatrix} \nabla_{X_1} u_\varepsilon\left(t\right) \\ \nabla_{X_2}\left(u_\varepsilon - z\right)\left(t\right) \end{pmatrix} dx \end{aligned}$$

for every $t \in \left[0,T\right]$. Thanks to the monotonicity of β and (4.2), it is clear that $M_\varepsilon\left(t,v\right) \geq 0$. Developing M_ε, we obtain

$$\begin{aligned} M_\varepsilon\left(t,v\right) &= \frac{1}{2}\left|u_\varepsilon'\left(t\right)\right|_\Omega^2 + \frac{1}{2}\left|z'\left(t\right)\right|_\Omega^2 - \int_\Omega u_\varepsilon'\left(t\right) z'\left(t\right) dx + \frac{1}{2}\int_\Omega A_\varepsilon \nabla u_\varepsilon\left(t\right) \cdot \nabla u_\varepsilon\left(t\right) dx \\ &+ \int_0^t \left\langle \beta u_\varepsilon', u_\varepsilon'\right\rangle_W ds - \int_0^t \left\langle \beta u_\varepsilon', v\right\rangle_W ds + \int_0^t \left\langle \beta v, v - u_\varepsilon'\right\rangle_W ds \\ &- \frac{1}{2}\int_\Omega \varepsilon A_{12} \nabla_{X_2} z\left(t\right) \cdot \nabla_{X_1} u_\varepsilon\left(t\right) dx - \frac{1}{2}\int_\Omega \varepsilon A_{21} \nabla_{X_1} u_\varepsilon\left(t\right) \cdot \nabla_{X_2} z\left(t\right) dx \\ &- \frac{1}{2}\int_\Omega A_{22} \nabla_{X_2} z\left(t\right) \cdot \nabla_{X_2}\left(u_\varepsilon - z\right)\left(t\right) dx - \frac{1}{2}\int_\Omega A_{22} \nabla_{X_2} u_\varepsilon\left(t\right) \cdot \nabla_{X_2} z\left(t\right) dx. \end{aligned}$$

Taking into account (5.16), we get

$$\begin{aligned} M_\varepsilon\left(t,v\right) &= \int_0^t \int_\Omega f u_\varepsilon' dx ds + \frac{1}{2}\left|u_\varepsilon^1\right|_\Omega^2 + \frac{1}{2}\int_\Omega A_\varepsilon \nabla u_\varepsilon^0 \cdot \nabla u_\varepsilon^0 dx \\ &+ \frac{1}{2}\left|z'\left(t\right)\right|_\Omega^2 - \int_\Omega u_\varepsilon'\left(t\right) z'\left(t\right) dx - \int_0^t \left\langle \beta u_\varepsilon', v\right\rangle_W ds + \int_0^t \left\langle \beta v, v - u_\varepsilon'\right\rangle_W ds \\ &- \frac{1}{2}\int_\Omega \varepsilon A_{12} \nabla_{X_2} z\left(t\right) \cdot \nabla_{X_1} u_\varepsilon\left(t\right) dx - \frac{1}{2}\int_\Omega \varepsilon A_{21} \nabla_{X_1} u_\varepsilon\left(t\right) \cdot \nabla_{X_2} z\left(t\right) dx \\ &- \frac{1}{2}\int_\Omega A_{22} \nabla_{X_2} z\left(t\right) \cdot \nabla_{X_2}\left(u_\varepsilon - z\right)\left(t\right) dx - \frac{1}{2}\int_\Omega A_{22} \nabla_{X_2} u_\varepsilon\left(t\right) \cdot \nabla_{X_2} z\left(t\right) dx. \end{aligned}$$

Passing to the limit yields

$$\begin{aligned} \lim_{\varepsilon \to 0} M_\varepsilon\left(t,v\right) &= \int_0^t \int_\Omega f z' dx ds + \frac{1}{2}\left|u^1\right|_\Omega^2 + \frac{1}{2}\int_\Omega A_{22} \nabla_{X_2} u^0 \cdot \nabla_{X_2} u^0 dx - \frac{1}{2}\left|z'\left(t\right)\right|_\Omega^2 \\ &- \int_0^t \left\langle \chi, v\right\rangle_W ds + \int_0^t \left\langle \beta v, v - z'\right\rangle_W ds - \frac{1}{2}\int_\Omega A_{22} \nabla_{X_2} z\left(t\right) \cdot \nabla_{X_2} z\left(t\right) dx. \end{aligned}$$

Then, by (5.19) and (5.24), it comes

$$\begin{aligned} \lim_{\varepsilon \to 0} M_\varepsilon\left(t,v\right) &= \int_0^t \left\langle \chi, z'\right\rangle_W ds - \int_0^t \left\langle \chi, v\right\rangle_W ds + \int_0^t \left\langle \beta v, v - z'\right\rangle_W ds \\ &= \int_0^t \left\langle \beta v - \chi, v - z'\right\rangle_W ds \geq 0, \quad \forall t \in \left[0,T\right]. \end{aligned} \tag{5.29}$$

Choosing $v = z' - \theta\varphi$, $\varphi \in L^\infty\left(0,T;L^2\left(\Omega\right)\right) \cap L^p\left(0,T;W\right)$ and $\theta > 0$ we get

$$\int_0^T \left\langle \beta\left(z' - \theta\varphi\right) - \chi, \varphi\right\rangle_W ds \geq 0.$$

Letting $\theta \to 0$ and using (5.4$-iv$) we derive

$$\int_0^T \left\langle \beta z' - \chi, \varphi\right\rangle_W ds \geq 0, \quad \forall \varphi \in L^\infty\left(0,T;L^2\left(\Omega\right)\right) \cap L^p\left(0,T;W\right).$$

This implies that $\chi = \beta z'$ a.e. in Q. Considering this in (5.22) with (5.19) we conclude that z is the unique solution of (5.11), i.e. $z = \tilde{u}$. ∎

Proof of Theorem 5.2. Thanks to Lemmas 5.2 the existence of the subsequence of Lemma 5.3 is ensured. Thus, since the limit of each subsequence of u_ε is unique, the convergences (5.20), (5.21) and (5.28) hold for the whole sequence .
Finally, taking $v = \tilde{u}'$ in (5.29), we get

$$\lim_{\varepsilon \to 0} M_\varepsilon \left(t, \tilde{u}'\right) = \int_0^t \left\langle \beta\tilde{u}' - \beta\tilde{u}', \tilde{u}' - \tilde{u}'\right\rangle_W ds = 0.$$

Since, by (4.2) and (5.4$-iii$), we have

$$\frac{1}{2}\left|\left(u_\varepsilon - \tilde{u}\right)'(t)\right|_\Omega^2 + \frac{\lambda}{2}\left(\varepsilon^2 \left|\nabla_{X_1} u_\varepsilon (t)\right|_\Omega^2 + \left|\nabla_{X_2}\left(u_\varepsilon - \tilde{u}\right)(t)\right|_\Omega^2\right) \\ + \int_0^t \left\langle \beta u_\varepsilon' - \beta\tilde{u}', u_\varepsilon' - \tilde{u}'\right\rangle_W ds \leq M_\varepsilon\left(t, \tilde{u}'\right) \to 0,$$

the strong convergences in (5.13) and (5.14) follow. This ends the proof of the theorem. ∎

Remark 5.1 *If β is uniformly monotone, i.e. for some constant $\delta > 0$ we have*

$$\left\langle \beta(t)\, u - \beta(t)\, v, u - v\right\rangle_W \geq \delta \left|u - v\right|_W^p, \quad \forall u, v \in V, \text{ a.e. } t \in [0, T],$$

then we obtain the strong convergence

$$u_\varepsilon' \to \tilde{u}' \quad \text{in } L^p(0, T; W). \tag{5.30}$$

Remark 5.2 *If A depends on t, i.e. $A = A(t, x)$, with coefficients in $C^1(\bar{Q})$, we can show that the convergences of Theorem 5.2 hold, at least, when*

$$\int_\Omega A_\varepsilon'(t)\, \nabla v \cdot \nabla v dx \leq 0, \qquad \forall v \in H_0^1(\Omega), \;\; \forall t \in [0, T].$$

Example As a usual example, we take $\beta v := |v|^{p-2}\, v$, for $p \geq 2$. Then β satisfies the required conditions in (5.4) with $W = L^p(\Omega)$, and β is uniformly monotone since we have

$$\int_\Omega \left(|u|^{p-2}\, u - |v|^{p-2}\, v\right)(u - v)\, dx \geq |u - v|_{L^p(\Omega)}^p.$$

Thus the convergences (5.13), (5.14) and (5.30) hold in this case. Moreover, taking into account (5.3), (5.5) and (5.9) we can easily see that

$$u^0(X_1, \cdot) \in H_0^1(\Omega_{X_1}), \quad u^1(X_1, \cdot) \in L^2(\Omega_{X_1}) \\ \beta\tilde{u}'(\cdot; X_1, \cdot) \in L^{p'}\left(0, T; L^{p'}(\Omega_{X_1})\right), \quad f(\cdot; X_1, \cdot) \in L^2(0, T; L^2(\Omega_{X_1})),$$

for a.e. $X_1 \in \Pi_1$. Then by the same technique used in proof of Theorem 4.2, which is independent of the linearity of the problem, it is easy to show that, for a.e. $X_1 \in \Pi_1$, $\tilde{u}\left(X_1, \cdot\right)$ is the unique solution to the problem

$$\begin{cases} \tilde{u}\left(\cdot\,; X_1, \cdot\right) \in C([0,T]\,; H_0^1\left(\Omega_{X_1}\right)), \tilde{u}'\left(\cdot\,; X_1, \cdot\right) \in C([0,T]\,; L^2\left(\Omega_{X_1}\right)) \cap L^p\left(0,T; L^p\left(\Omega_{X_1}\right)\right), \\ \left\langle \tilde{u}''\left(t; X_1, X_2\right), v\right\rangle + \displaystyle\int_{\Omega_{X_1}} A_{22}(t; X_1, X_2) \nabla_{X_2} \tilde{u}(t; X_1, X_2) \cdot \nabla_{X_2} v dX_2 \\ \quad + \displaystyle\int_{\Omega_{X_1}} \left|\tilde{u}'\left(t; X_1, X_2\right)\right|^{p-2} \tilde{u}'\left(t; X_1, X_2\right) v dX_2 = \displaystyle\int_{\Omega_{X_1}} f(t; X_1, X_2) v dX_2, \qquad \forall v \in H_0^1\left(\Omega_{X_1}\right), \\ \tilde{u}\left(0, X_1, \cdot\right) = u^0\left(X_1, \cdot\right), \ \ \tilde{u}'\left(0, X_1, \cdot\right) = u^1\left(X_1, \cdot\right) \quad \text{in } \ \Omega_{X_1}, \end{cases}$$

which is from the same type of (5.8), defined in a lower dimension.

5.3 CONVERGENCES IN CYLINDRICAL DOMAINS

Note that, in general the convergence of u_ε does not hold in $H^1\left(\Omega\right)$. For regions located far from the lateral boundary of the cylindrical domains, a convergence in Sobolev space and some improvements of the rate of convergence are shown at the end of this section.

We consider a cylindrical domain

$$\Omega := \Delta \times \omega,$$

where Δ (resp. ω) is an open bounded subset of $\mathbb{R}^q$ (resp. $\mathbb{R}^{n-q}$). Let Δ_0 and Δ' be two open subsets of $\mathbb{R}^q$ satisfying

$$\Delta_0 \subset\subset \Delta' \subset\subset \Delta,$$

and set

$$\Omega_0 := \Delta_0 \times \omega, \quad Q_0 := (0,T) \times \Omega_0,$$
$$\Omega' := \Delta' \times \omega, \quad Q' := (0,T) \times \Omega'.$$

Consider a smooth cut-off function $\varrho = \varrho\left(X_1\right)$, satisfying

$$\operatorname{supp}\left(\varrho\right) \subset \Delta', \ \ \varrho = 1 \text{ on } \Delta_0, \quad 0 \leq \varrho \leq 1 \ \text{ and } \ \left|\nabla_{X_1} \varrho\right| \leq C.$$

We slightly modify the assumption (5.4-*iii*) by assuming that the monotonicity of β holds when we multiply by a cut-off function, i.e.

$$\left\langle \beta\left(t\right) v_1 - \beta\left(t\right) v_2, \left(v_1 - v_2\right) \eta \right\rangle_W \geq 0, \quad \forall v_1, v_2 \in W, \ \text{a.e. } t \in [0,T], \tag{5.31}$$

where $\eta = \eta\left(X_1\right)$ is any positive function with $\operatorname{supp}(\eta) \subset \Delta$. In fact, we only need to take η of the type $\varrho^2\left(X_1\right)$ in the proofs of the next theorems, see also Remark 5.4 bellow.

5.3.1 *Convergences in Sobolev Space H^1*

When the limit solution $\tilde{u}$ satisfies some regularity assumptions, we will improve the rate of convergence in regions far from the lateral boundary.

Theorem 5.3 *Under the assumptions of Theorem 5.2, in addition we assume that (5.31) holds and*

$$\partial_{x_i}\partial_{x_j}\tilde{u} \in L^2(Q) \quad \textit{for } i = 1, \cdots, n, \;\; j = 1, \cdots, q, \tag{5.32}$$

$$\left|u_\varepsilon^1 - u^1\right|_{\Omega'}, \quad \left|\nabla_{X_2}\left(u_\varepsilon^0 - u^0\right)\right|_{\Omega'} = O(\varepsilon), \; \left|\nabla_{X_1} u_\varepsilon^0\right|_{\Omega'} = O(1), \tag{5.33}$$

then

$$\begin{gathered} \sup_{t\in[0,T]} \left|(u_\varepsilon - \tilde{u})(t)\right|_{\Omega_0}, \quad \sup_{t\in[0,T]} \left|\left(u_\varepsilon' - \tilde{u}'\right)(t)\right|_{\Omega_0} = O(\varepsilon), \\ \sup_{t\in[0,T]} \left|\nabla_{X_2}(u_\varepsilon - \tilde{u})(t)\right|_{\Omega_0} = O(\varepsilon) \\ \int_0^T \left\langle \beta u_\varepsilon' - \beta\tilde{u}', \left(u_\varepsilon' - \tilde{u}'\right)\varrho^2 \right\rangle_W ds = O\left(\varepsilon^2\right), \end{gathered} \tag{5.34}$$

and we have the weak convergence

$$\nabla_{X_1} u_\varepsilon(t) \rightharpoonup \nabla_{X_1}\tilde{u}(t) \;\; \textit{in } L^2(\Omega_0), \quad \forall t \in [0, T]. \tag{5.35}$$

Proof. Setting $w_\varepsilon := u_\varepsilon - \tilde{u}$ then comparing (5.8) and (5.11) yields

$$\begin{aligned} &\left\langle w_\varepsilon'', v\right\rangle + \int_\Omega A_\varepsilon \nabla w_\varepsilon \cdot \nabla v dx + \int_0^t \left\langle \beta u_\varepsilon' - \beta\tilde{u}', v\right\rangle_W ds \\ &\qquad = \int_\Omega \left\{\varepsilon^2 \nabla_{X_1} \cdot \left(A_{11}\nabla_{X_1}\tilde{u}\right) + \varepsilon\nabla_{X_2} \cdot \left(A_{21}\nabla_{X_1}\tilde{u}\right) + \varepsilon\nabla_{X_1} \cdot \left(A_{12}\nabla_{X_2}\tilde{u}\right)\right\} v dx \end{aligned}$$

for every $v \in H_0^1(\Omega)$. Testing formally this identity with $v = w_\varepsilon'\varrho^2$ and performing the integration on $(0, t)$, we obtain

$$\begin{aligned} &\frac{1}{2}\left|w_\varepsilon'(t)\varrho\right|_\Omega^2 + \frac{1}{2}\int_\Omega A_\varepsilon \nabla w_\varepsilon(t) \cdot \nabla w_\varepsilon(t)\varrho^2 dx + \int_0^t \left\langle \beta u_\varepsilon' - \beta\tilde{u}', w_\varepsilon'\varrho^2\right\rangle_W ds \\ &\quad = \frac{1}{2}\left|w_\varepsilon^1\varrho\right|_\Omega^2 + \frac{1}{2}\int_\Omega A_\varepsilon \nabla w_\varepsilon^0 \cdot \nabla w_\varepsilon^0 \varrho^2 dx \\ &\quad + \int_0^t \int_\Omega \varepsilon\left\{\varepsilon\nabla_{X_1} \cdot \left(A_{11}\nabla_{X_1}\tilde{u}\right) + \nabla_{X_2} \cdot \left(A_{21}\nabla_{X_1}\tilde{u}\right) + \nabla_{X_1} \cdot \left(A_{12}\nabla_{X_2}\tilde{u}\right)\right\} w_\varepsilon'\varrho^2 dxds \\ &\quad - 2\varepsilon^2 \int_0^t \int_\Omega A_{11}\nabla_{X_1} w_\varepsilon \cdot \nabla_{X_1}\varrho w_\varepsilon'\varrho dxds - 2\varepsilon \int_0^t \int_\Omega A_{12}\nabla_{X_2} w_\varepsilon \cdot \nabla_{X_1}\varrho w_\varepsilon'\varrho dxds. \end{aligned} \tag{5.36}$$

We can easily see this identity when the regularity of the solution allows to do the multiplications (see [32]). For the general case, and since (5.1) and (5.32) implies that

$$\left\{\varepsilon\nabla_{X_1} \cdot \left(A_{11}\nabla_{X_1}\tilde{u}\right) + \nabla_{X_2} \cdot \left(A_{21}\nabla_{X_1}\tilde{u}\right) + \nabla_{X_1} \cdot \left(A_{12}\nabla_{X_2}\tilde{u}\right)\right\} \in L^2(Q),$$

one can proceed by regularization as in [51, 65], see also [7, 32]. Then using (4.1), (4.2), Cauchy-Schwarz and Young's inequalities, we get

$$\begin{aligned}
&\left|w_\varepsilon'(t)\varrho\right|_\Omega^2 + \lambda\varepsilon^2\left|\nabla_{X_1} w_\varepsilon(t)\varrho\right|_\Omega^2 + \lambda\left|\nabla_{X_2} w_\varepsilon(t)\varrho\right|_\Omega^2 + \int_0^t \left\langle \beta u_\varepsilon' - \beta\tilde{u}', w_\varepsilon'\varrho^2\right\rangle ds \\
&\quad \leq C\left(\left|w_\varepsilon^1\varrho\right|_\Omega^2 + \varepsilon^2\left|\nabla_{X_1} w_\varepsilon^0\varrho\right|_\Omega^2 + \left|\nabla_{X_2} w_\varepsilon^0\varrho\right|_\Omega^2\right) \\
&\quad + C\varepsilon^2\int_0^t \left|\varepsilon\nabla_{X_1}\cdot\left(A_{11}\nabla_{X_1}\tilde{u}\right)\varrho + \nabla_{X_2}\cdot\left(A_{21}\nabla_{X_1}\tilde{u}\right)\varrho + \nabla_{X_1}\cdot\left(A_{12}\nabla_{X_2}\tilde{u}\right)\varrho\right|_\Omega^2 ds \\
&\quad + C\varepsilon^2\int_0^t \left|w_\varepsilon'\nabla_{X_1}\varrho\right|_\Omega^2 ds + C\int_0^t \left|w_\varepsilon'\varrho\right|_\Omega^2 + \lambda\varepsilon^2\left|\nabla_{X_1} w_\varepsilon\varrho\right|_\Omega^2 + \lambda\left|\nabla_{X_2} w_\varepsilon\varrho\right|_\Omega^2 ds.
\end{aligned}$$

Thanks to Lemma 5.1, $\left|w_\varepsilon'\nabla_{X_1}\varrho\right|_{L^2(Q)}$ is bounded, then using (5.33) it comes

$$\begin{aligned}
&\left|w_\varepsilon'(t)\varrho\right|_\Omega^2 + \lambda\varepsilon^2\left|\nabla_{X_1} w_\varepsilon(t)\varrho\right|_\Omega^2 + \lambda\left|\nabla_{X_2} w_\varepsilon(t)\varrho\right|_\Omega^2 + \int_0^t \left\langle \beta u_\varepsilon' - \beta\tilde{u}', w_\varepsilon'\varrho^2\right\rangle ds \\
&\quad \leq O\left(\varepsilon^2\right) + C\int_0^t \left|w_\varepsilon'\varrho\right|_\Omega^2 + \lambda\varepsilon^2\left|\nabla_{X_1} w_\varepsilon\varrho\right|_\Omega^2 + \lambda\left|\nabla_{X_2} w_\varepsilon\varrho\right|_\Omega^2 ds.
\end{aligned}$$

Applying the Gronwall's inequality we obtain

$$\begin{aligned}
\left|w_\varepsilon'(t)\varrho\right|_\Omega^2 + \left|\nabla_{X_2} w_\varepsilon(t)\varrho\right|_\Omega^2 + \int_0^t \left\langle \beta\tilde{u}_\varepsilon' - \beta\tilde{u}', w_\varepsilon'\varrho^2\right\rangle ds &= O\left(\varepsilon^2\right), &\quad (5.37)\\
\left|\nabla_{X_1} w_\varepsilon(t)\varrho\right|_\Omega^2 &\leq C. &\quad (5.38)
\end{aligned}$$

Thus we derive (5.34) since $\varrho = 1$ on Ω_0. Finally, thanks to (5.38), for any $t \in [0, T]$ we can extract a weakly converging subsequence of $\left(\nabla_{X_1} u_\varepsilon(t)\right)_\varepsilon$ to the only possible limit $\nabla_{X_1}\tilde{u}(t)$ in $L^2\left(\Omega_0\right)$. This implies the convergence of the whole sequence since the limit is unique. ∎

Remark 5.3 *Of course the second assumption in* (5.33) *means that* $\nabla_{X_1} u_\varepsilon^0$ *is bounded in* $L^2\left(\Omega_0\right)$*, then by the first assumption in the same line we may see that* $\nabla_{X_1} u_\varepsilon^0$ *converges weakly to* $\nabla_{X_1} u^0$ *in* $L^2\left(\Omega_0\right)$*, as what we exactly got in the theorem for* $\nabla_{X_1} u_\varepsilon(t)$*, for every* $t \in [0, T]$*. Note that, as it is shown in* Guesmia [32]*, this type of conditions is necessary and sufficient to deduce the convergence of* u_ε*.*

Matrix with diagonal structure

As in the linear case, when the matrix A has a diagonal structure, we can improve the above convergences.

Corollary 5.1 (Diagonal matrix) *Under the assumptions of Theorem 5.3, in addition we suppose that* $A_{12}, A_{21} = 0$ *and*

$$\left|u_\varepsilon^1 - u^1\right|_{\Omega'}, \ \left|\nabla_{X_2}\left(u_\varepsilon^0 - u^0\right)\right|_{\Omega'} = o(\varepsilon) \text{ and } \left|\nabla_{X_1}\left(u_\varepsilon^0 - u^0\right)\right|_{\Omega'} \to 0, \qquad (5.39)$$

then

$$\sup_{t\in[0,T]} \left|(u_\varepsilon-\tilde{u})(t)\right|_{\Omega_0}, \quad \sup_{t\in[0,T]} \left|\left(u'_\varepsilon-\tilde{u}'\right)(t)\right|_{\Omega_0} = o(\varepsilon),$$

$$\sup_{t\in[0,T]} \left|\nabla_{X_1}(u_\varepsilon-\tilde{u})(t)\right|_{\Omega_0} \to 0,$$

$$\sup_{t\in[0,T]} \left|\nabla_{X_2}(u_\varepsilon-\tilde{u})(t)\right|_{\Omega_0} = o(\varepsilon)$$

$$\int_0^T \left\langle \beta u'_\varepsilon-\beta\tilde{u}', \left(u'_\varepsilon-\tilde{u}'\right)\varrho^2\right\rangle_W ds = o\left(\varepsilon^2\right).$$

In particular, we have the strong convergence

$$\sup_{t\in[0,T]} \left|(u_\varepsilon-\tilde{u})(t)\right|_{H^1(\Omega_0)} \to 0.$$

Proof. Taking into account the fact that A_{12}, $A_{21}=0$ in (5.36), we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\left|w'_\varepsilon(t)\varrho\right|^2_\Omega + \left|\nabla_{X_1}w_\varepsilon(t)\varrho\right|^2_\Omega + \frac{1}{\varepsilon^2}\left|\nabla_{X_2}w_\varepsilon(t)\varrho\right|^2_\Omega + \frac{1}{\varepsilon^2}\int_0^t \left\langle \beta u'_\varepsilon-\beta\tilde{u}', w'_\varepsilon\varrho^2\right\rangle_W ds \\
&\quad\le \frac{C}{\varepsilon^2}\left\{\left|w^1_\varepsilon\varrho\right|^2_\Omega + \varepsilon^2\left|\nabla_{X_1}\left(w^0_\varepsilon\right)\varrho\right|^2_\Omega + \left|\nabla_{X_2}\left(w^0_\varepsilon\right)\varrho\right|^2_\Omega\right\} \\
&\quad\quad + C\left|\int_0^t\int_\Omega \nabla_{X_1}\cdot\left(A_{11}\nabla_{X_1}\tilde{u}\right)w'_\varepsilon\varrho^2 dxds + \int_0^t\int_\Omega A_{11}\nabla_{X_1}w_\varepsilon\cdot\nabla_{X_1}\varrho w'_\varepsilon\varrho dxds\right|,
\end{aligned}$$

for every $t\in[0,T]$. We can always change Ω_0 by Ω' in Theorem 5.3, then use the convergences (5.34) and (5.35) in the A_{11} integrals of the above inequality, we deduce that the last two integrals tend to zero. Thus by (5.39), the corollary follows. ∎

Remark 5.4 *The assumption* (5.31) *is frequent, and sometimes it is a simple consequence of the monotonicity as in the example above. Moreover in the context of this example, we use the inequality*

$$\begin{aligned}
\int_\Omega \left(|u|^{p-2}u-|v|^{p-2}v\right)(u-v)\varrho^2 dx &\ge \int_{\Omega_0}\left(|u|^{p-2}u-|v|^{p-2}v\right)(u-v)\,dx \\
&\ge |u-v|^p_{L^p(\Omega_0)}
\end{aligned}$$

to deduce that

$$\left|u'_\varepsilon-\tilde{u}'\right|_{L^p(Q_0)} = O(\varepsilon^{2/p}), \qquad \left(= o(\varepsilon^{2/p}) \text{ in the diagonal structure case}\right).$$

5.3.2 *Exponential Convergences*

As for the elliptic problems (see [11, 20]), an exponential rate of the convergence $u_\varepsilon \to \tilde{u}$ can be shown if the hyperbolic boundary value problems are approximately invariant under arbitrary translations in the X_1-directions, i.e. we assume that

$$\begin{aligned} &A_{12}(X_1, X_2) = A_{12}(X_2), \quad A_{22}(X_1, X_2) = A_{22}(X_2), \\ &u^0(X_1, X_2) = u^0(X_2), \quad u^1(X_1, X_2) = u^1(X_2), \quad f(t; X_1, X_2) = f(t; X_2). \end{aligned} \tag{5.40}$$

Of course $\tilde{u}$ is independent of X_1 in this case. Then we have the following theorem.

Theorem 5.4 *Under the assumptions of Theorem 5.2, in addition assume that* (5.31), (5.40) *hold and*

$$\left|u_\varepsilon^1 - u^1\right|_{\Omega'}, \quad \left|u_\varepsilon^0 - u^0\right|_{H^1(\Omega')} \leq K_0 e^{-\frac{\gamma_0}{2\varepsilon}} \tag{5.41}$$

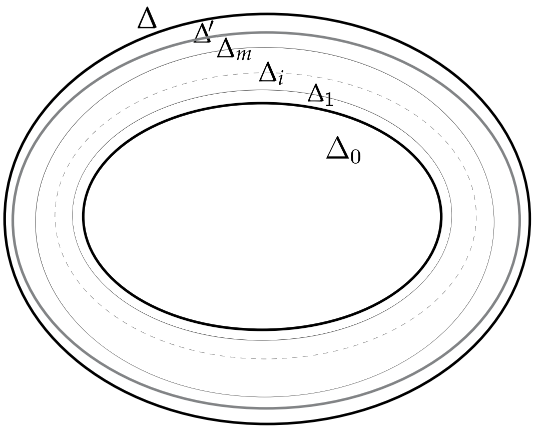
for some constants $K_0, \gamma_0 > 0$, *then*

$$\sup_{t\in[0,T]} \left|\left(u_\varepsilon' - \tilde{u}'\right)(t)\right|_{\Omega_0}, \quad \sup_{t\in[0,T]} \left|\left(u_\varepsilon - \tilde{u}\right)(t)\right|_{H^1(\Omega_0)} \leq K e^{-\frac{\gamma}{\varepsilon}}.$$

for some constants $K, \gamma > 0$.

Proof. The proof is based on the iteration technique introduced in Chipot and Yeressian [20]. Without loss of generality, we assume that $dist\,(\Delta_0, \Delta\backslash\Delta') > 1$ and set $m = \left[\frac{1}{\varepsilon}\right]$ for a fixed $\varepsilon > 0$ ($[\cdot]$ denotes the integer part). For ε small enough we can always construct a sequence $(\Delta_i)_{0\leq i\leq m+1}$ of strictly increasing sets such that

$$\begin{aligned} &\Delta_0 \subset\subset \Delta_1 \subset\subset \cdots \subset\subset \Delta_m \subset\subset \Delta_{m+1} := \Delta', \\ &dist\,(\Delta_i, \Delta\backslash\Delta_{i+1}) \geq \varepsilon, \qquad i = 0, \cdots, m. \end{aligned}$$



Let $(\varrho_i)_{1\leq i\leq m+1}$ be a family of smooth functions depending only on X_1 such that

$$\left.\begin{aligned}&\operatorname{supp}(\varrho_i)\subset\Delta_i,\quad \varrho_i=1 \text{ on } \Delta_{i-1},\\ &0\leq\varrho_i\leq 1 \quad\text{and}\quad |\nabla_{X_1}\varrho_i|\leq\frac{C}{\varepsilon},\end{aligned}\right.\qquad i=1,\cdots,m+1. \tag{5.42}$$

where C is a constant independent of ε. Next, using (5.40), we rewrite (5.36) as

$$\begin{aligned}\frac{1}{2}\left|w_\varepsilon'(t)\varrho_i\right|_\Omega^2&+\frac{1}{2}\int_\Omega A_\varepsilon\nabla w_\varepsilon\cdot\nabla w_\varepsilon\varrho_i^2dx+\int_0^t\left\langle\beta u_\varepsilon'-\beta\tilde{u}',w_\varepsilon'\varrho_i^2\right\rangle_W ds\\ &=\frac{1}{2}\left|w_\varepsilon^1\varrho_i\right|_{\Omega'}+\frac{1}{2}\int_\Omega A_\varepsilon\nabla w_\varepsilon^0\cdot\nabla w_\varepsilon^0\varrho_i^2dx\\ &\quad-2\varepsilon^2\int_0^t\int_\Omega A_{11}\nabla_{X_1}w_\varepsilon\cdot\nabla_{X_1}\varrho_iw_\varepsilon'\varrho_idxds-2\varepsilon\int_0^t\int_\Omega A_{12}\nabla_{X_2}w_\varepsilon\cdot\nabla_{X_1}\varrho_iw_\varepsilon'\varrho_idxds\end{aligned}$$

since $\tilde{u}$ and $A_{12}=A_{21}^T$ are independent of X_1. Estimating as above it comes that

$$\begin{aligned}\left|w_\varepsilon'(t)\varrho\right|_\Omega^2&+\lambda\varepsilon^2\left|\nabla_{X_1}w_\varepsilon(t)\varrho\right|_\Omega^2+\lambda\left|\nabla_{X_2}w_\varepsilon(t)\varrho\right|_\Omega^2+\int_0^t\left\langle\beta u_\varepsilon'-\beta\tilde{u}',w_\varepsilon'\varrho^2\right\rangle ds\\ &\leq C\left(\left|w_\varepsilon^1\varrho\right|_\Omega^2+\varepsilon^2\left|\nabla_{X_1}w_\varepsilon^0\varrho\right|_\Omega^2+\left|\nabla_{X_2}w_\varepsilon^0\varrho\right|_\Omega^2\right)\\ &\quad+C\varepsilon^2\int_0^t\left|w_\varepsilon'\nabla_{X_1}\varrho\right|_\Omega^2ds+C\int_0^t\lambda\varepsilon^2\left|\nabla_{X_1}w_\varepsilon\varrho\right|_\Omega^2+\lambda\left|\nabla_{X_2}w_\varepsilon\varrho\right|_\Omega^2ds.\end{aligned}$$

For simplicity let us denote

$$\begin{aligned}&\Omega_i:=\Delta_i\times\omega,\quad i=0,\cdots,m\\ &E_\varepsilon^0:=\left|w_\varepsilon^1\right|_{\Omega'}^2+\varepsilon^2\left|\nabla_{X_1}w_\varepsilon^0\right|_{\Omega'}^2+\left|\nabla_{X_2}w_\varepsilon^0\right|_{\Omega'}^2.\end{aligned}$$

Then, applying Gronwall's inequality we get

$$\begin{aligned}\left|w_\varepsilon'(t)\right|_{\Omega_{i-1}}^2&+\varepsilon^2\left|\nabla_{X_1}w_\varepsilon(t)\right|_{\Omega_{i-1}}^2+\left|\nabla_{X_2}w_\varepsilon(t)\right|_{\Omega_{i-1}}^2\\ &+\int_0^t\left\langle\beta u_\varepsilon'-\beta\tilde{u}',w_\varepsilon'\varrho_i^2\right\rangle_W ds\leq CE_\varepsilon^0+C\varepsilon^2\int_0^t\left|w_\varepsilon'\nabla_{X_1}\varrho_i\right|_\Omega^2ds.\end{aligned}\tag{5.43}$$

Taking into account the fact that $supp(\nabla_{X_1}\varrho_i)\subset\Delta_i\backslash\Delta_{i-1}$ and $|\nabla_{X_1}\varrho_i|\leq\frac{C}{\varepsilon}$, we have in particular

$$\left|w_\varepsilon'(t)\right|_{\Omega_{i-1}}^2\leq CE_\varepsilon^0+C\int_0^T\left|w_\varepsilon'\right|_{\Omega_i\backslash\Omega_{i-1}}^2ds.$$

Integrating both sides on $[0,T]$ we obtain

$$\begin{aligned}\int_0^T\left|w_\varepsilon'\right|_{\Omega_{i-1}}^2dt&\leq CTE_\varepsilon^0+CT\int_0^T\left|w_\varepsilon'\right|_{\Omega_i\backslash\Omega_{i-1}}^2dt\\ &=CTE_\varepsilon^0+CT\int_0^T\left|w_\varepsilon'\right|_{\Omega_i}^2dt-CT\int_0^T\left|w_\varepsilon'\right|_{\Omega_{i-1}}^2dt.\end{aligned}$$

Thus

$$\int_0^T\left|w_\varepsilon'\right|_{\Omega_{i-1}}^2dt\leq kE_\varepsilon^0+k\int_0^T\left|w_\varepsilon'\right|_{\Omega_i}^2dt,$$

where $0 < k = \frac{CT}{1+CT} < 1$. Starting by $i = 2$ we have

$$\begin{aligned}\int_0^T |w_\varepsilon'|_{\Omega_1}^2 \, dt &\leq kE_\varepsilon^0 + k\int_0^T |w_\varepsilon'|_{\Omega_2}^2 \, dt \\ &\leq kE_\varepsilon^0 + k\left(kE_\varepsilon^0 + k\int_0^T |w_\varepsilon'|_{\Omega_3}^2 \, dt\right) = E_\varepsilon^0 \left(k + k^2\right) + k^2 \int_0^T |w_\varepsilon'|_{\Omega_3}^2 \, dt.\end{aligned}$$

For $i = m + 1$, we obtain

$$\int_0^T |w_\varepsilon'|_{\Omega_1}^2 \, dt \leq E_\varepsilon^0 \sum_{l=1}^m k^l + k^m \int_0^T |w_\varepsilon'|_{\Omega_{m+1}}^2 \, dt \leq E_\varepsilon^0 \left(\frac{k}{1-k}\right) + k^{\frac{1}{\varepsilon}-1} |w_\varepsilon'|_Q^2,$$

since $\frac{1}{\varepsilon} - 1 \leq \left[\frac{1}{\varepsilon}\right]$.

Thanks to Lemma 5.1, $|w_\varepsilon'|_Q^2$ is bounded, then using (5.41) and taking $i = 1$ in (5.43) yields, for every $t \in [0, T]$,

$$\begin{aligned}|w_\varepsilon'(t)|_{\Omega_0}^2 + |\nabla_{X_2} w_\varepsilon(t)|_{\Omega_0}^2 + \int_0^T \langle \beta u_\varepsilon' - \beta \tilde{u}', w_\varepsilon' \varrho_1^2 \rangle_W \, ds &\leq CE_\varepsilon^0 + C\int_0^T |w_\varepsilon'|_{\Omega_1}^2 \, ds \\ &\leq \left(C + \frac{k}{1-k}\right) E_\varepsilon^0 + \left(\frac{C}{k} |w_\varepsilon'|_Q^2\right) k^{\frac{1}{\varepsilon}} \\ &\leq Ce^{-\frac{\gamma_0}{\varepsilon}} + Ce^{-\frac{1}{\varepsilon}(-\ln k)} \\ &\leq K_1 e^{-\frac{\gamma_1}{\varepsilon}},\end{aligned}$$

where $\gamma_1 := \min\{-\ln k, \gamma_0\} > 0$ and K_1 is a positive constant independent of ε and t. In addition we have

$$|\nabla_{X_1} w_\varepsilon(t)|_{\Omega_0}^2 \leq \frac{K_1}{\varepsilon^2} e^{-\frac{\gamma_1}{\varepsilon}} \leq K_1 e^{-\frac{\gamma_2}{\varepsilon}},$$

for any γ_2 satisfying $0 < \gamma_2 < \gamma_1$. Taking the sup on $[0, T]$ in the right hand side of the above inequalities, it comes

$$\sup_{t\in[0,T]} |w_\varepsilon'(t)|_{\Omega_0}^2, \quad \sup_{t\in[0,T]} |\nabla_{X_2} w_\varepsilon(t)|_{\Omega_0}^2, \quad \sup_{t\in[0,T]} |\nabla_{X_1} w_\varepsilon(t)|_{\Omega_0}^2 \leq K_1 e^{-\frac{\gamma_2}{\varepsilon}}.$$

To end the proof, we take $K = \sqrt{K_1}$ and $\gamma \leq \frac{\gamma_2}{2}$. ■

Remark 5.5 *The above proof also gives*

$$\int_0^T \langle \beta u_\varepsilon' - \beta \tilde{u}', (u_\varepsilon' - \tilde{u}') \varrho_1^2 \rangle_W \, ds \leq K^2 e^{-\frac{2\gamma}{\varepsilon}}.$$

Note that ϱ_1 depends on ε. In the case $\beta v = |v|^{p-2} v$, $p \geq 2$, as in the precedent example (page 76)*, it follows that*

$$|u_\varepsilon' - \tilde{u}'|_{L^p(Q_0)} \leq K^{\frac{2}{p}} e^{-\left(\frac{2}{p}\right)\frac{\gamma}{\varepsilon}}.$$

COMMENTS

The existence, uniqueness and regularity of solution of semilinear hyperbolic problems are teated in Lions [47], Lions and Struass [51], Strauss [64, 65]. More results on semilinear hyperbolic problems can be found in Brezis [5], Cazenave and Haraux [10], Haraux [38, 39]. There is an extensive literature on (isotropic) singular perturbation of nonlinear hyperbolic problems, for instance, see Genet and Madaune [28], Hajouj [37], Madaune [52] and related works.

The iteration technique, used in the last section, introduced in Chipot and Rougirel [19] and improved in Chipot and Yeressian [20]. A polynomial rate of convergence, in the framework of cylindrical domains becoming large, was obtained in Brighi and Guesmia [7], Guesmia [33] for quasilinear problems.

The results of this chapter has appeared in Guesmia and Sengouga [36].

A

APPENDICES

This appendix comprehends notations, some inequalities and definitions of spaces used in the thesis. Proofs and more advanced results can be found in the standard books of functional analysis and partial differential equations, see for instance Brezis [6], Chipot [13], Dautray and Lions [22], Evans [26].

A.1 NOTATIONS

General Notations

$:=$ equal by definition

$E' =$ dual of space E

$p' =$ conjugate exponent of p, *i.e.* $p' = \dfrac{p}{p-1}$, for $1 \leq p \leq \infty$

$|\cdot|_E =$ norm of space E

$\langle \cdot,\cdot \rangle_E =$ duality product of a space E', E

$|\cdot|_\Omega =$ norm of $L^2(\Omega)$,

$x = (X_1, X_2) \in \mathbb{R}^n$ where $X_1 = (x_1, \ldots, x_q) \in \mathbb{R}^q$ and $X_2 = (x_{q+1}, \ldots, x_n) \in \mathbb{R}^{n-q}$

$x \cdot y = \sum_{i=1}^{n} x_i y_i$, for $x, y \in \mathbb{R}^n$ (Euclidean scalar product)

$|x| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$, for $x, y \in \mathbb{R}^n$

Ω open domain in $\mathbb{R}^n$

$\bar{\Omega} =$ closure of Ω

$\Omega' \subset\subset \Omega$ Ω' strongly included in Ω, *i.e.*, $\bar{\Omega}' \subset \Omega$

$\partial\Omega = \Gamma$ boundary of Ω

$\Pi_i =$ orthogonal projection of Ω onto the space $X_i = 0,\ i = 1, 2$

$$\begin{aligned}
\Omega_{X_1} &= \{X_2 \mid (X_1, X_2) \in \Omega\}, \text{ for any } X_1 \in \Pi_1 \\
\Omega_{X_2} &= \{X_1 \mid (X_1, X_2) \in \Omega\}, \text{ for any } X_2 \in \Pi_2
\end{aligned}$$

Functions and Differential Operators

For a function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, we note

$$\begin{aligned}
u(x) &= u(x_1, \dots, x_n), \ x \in \Omega \\
u(t; x) &= u(t; x_1, \dots, x_n), \ t \in [0, T], \text{ for } T > 0 \text{ and } x \in \Omega \\
u^+ &= \max\{u, 0\} \\
u^- &= \max\{-u, 0\} \\
\operatorname{supp}(u) &= \text{The closure of } \{x \in \Omega | u(x) \neq 0\} \\
A^T &= \text{Transposed matrix} \\
\nabla u &= (\partial_{x_1} u, \dots, \partial_{x_n} u)^T \ \text{ gradient of } u \\
\nabla_{X_1} u &= (\partial_{x_1} u, \dots, \partial_{x_q} u)^T, \ q \text{ and } n \text{ are integers, } 0 < q < n \\
\nabla_{X_2} u &= (\partial_{x_{q+1}} u, \dots, \partial_{x_n} u)^T \\
\Delta u &= \sum_{i=1}^{i=n} \frac{\partial^2 u}{\partial^2 x_i}, \ \text{ Laplacian of } u \\
\Delta_{X_1} u &= \sum_{i=1}^{i=q} \frac{\partial^2 u}{\partial^2 x_i} \\
\Delta_{X_2} u &= \sum_{i=q+1}^{i=n} \frac{\partial^2 u}{\partial^2 x_i} \\
\Delta_p u &= \sum_{i=1}^{i=n} \frac{\partial}{\partial x_i} \left(|\nabla u|^{p-2} \frac{\partial}{\partial x_i} u \right), \ \ p-\text{Laplacian of } u, \ \ 1 < p < \infty \\
\Delta_{p, X_1} u &= \sum_{i=1}^{i=q} \frac{\partial}{\partial x_i} \left(|\nabla_{X_1} u|^{p-2} \frac{\partial}{\partial x_i} u \right) \\
\Delta_{p, X_2} u &= \sum_{i=q+1}^{i=n} \frac{\partial}{\partial x_i} \left(|\nabla_{X_2} u|^{p-2} \frac{\partial}{\partial x_i} u \right)
\end{aligned}$$

Function Spaces

$$\begin{aligned}
C^k(\Omega) &= \{u : \Omega \to \mathbb{R} | u \text{ is } k \text{ times continuously differentiable}\}, k \in \mathbb{N} \\
\mathcal{D}(\Omega) &= \{u : \Omega \to \mathbb{R} | u \text{ is infinitely differentiable with compact support in } \Omega\} \\
\mathcal{D}'(\Omega) &= \text{space of distributions on } \Omega \\
L^p(\Omega) &= \{u : \Omega \to \mathbb{R} | u \text{ is measurable and } \int_\Omega |u|^p dx < \infty\}, \ 1 \leq p < \infty \\
L^\infty(\Omega) &= \{u : \Omega \to \mathbb{R} | u \text{ is measurable and } |u(x)| \leq C \text{ a.e. in } \Omega \text{ for some constant } C\}
\end{aligned}$$

$$
\begin{aligned}
H^k(\Omega) &= \left\{u \in L^2(\Omega) | D^\alpha u \in L^2(\Omega), \forall \alpha \in \mathbb{N}^n, |\alpha| \le k\right\}, k \in \mathbb{N} \\
H_0^k(\Omega) &= \text{the closure of } \mathcal{D}(\Omega) \text{ in } H^k(\Omega) \\
W^{k,p}(\Omega) &= \{u \in L^p(\Omega) | D^\alpha u \in L^p(\Omega), \forall \alpha \in \mathbb{N}^n, |\alpha| \le k\}, k \in \mathbb{N}, 1 \le p \le \infty \\
W_0^{k,p}(\Omega) &= \text{the closure of } \mathcal{D}(\Omega) \text{ in } W^{k,p}(\Omega) \\
C^k(I;E) &= \text{space of } k \text{ times continuously differentiable functions} \\
&\quad\ \text{from an interval } I \text{ to the Banach space } E \\
L^p(I;E) &= \text{space of measurable functions } u \text{ on } I \text{ with values in } E \\
&\quad\ \text{and such that } |u|_E^p \text{ is integrable, } 1 \le p < \infty \\
L^\infty(I;E) &= \text{space of measurable functions } u \text{ on } I \text{ such that there exists a constant } C \\
&\quad \text{such that } |u|_E^p < C \text{ for almost every } x \in I
\end{aligned}
$$

Notations for Estimates

In the process of estimates, we use C to denote various constants that can be explicitly computed in terms of known quantities. The value of C may change from line to line in a given computation.

$\rightharpoonup$ weak convergence

$\overset{*}{\rightharpoonup}$ weak star convergence

$\to$ strong convergence

In particular $\nabla u_\varepsilon \to \nabla u$ (resp. $\nabla u_\varepsilon \rightharpoonup \nabla u$) denotes the strong (resp. weak) vectorial convergences, i.e. component by component convergence.

$$
\begin{aligned}
&(|u_\varepsilon - u| = O(\varepsilon) \text{ as } \varepsilon \to 0) \Leftrightarrow \exists C \ge 0, |u_\varepsilon - u| \le C\varepsilon \text{ for } \varepsilon \text{ sufficiently close to } 0 \\
&(|u_\varepsilon - u| = o(\varepsilon) \text{ as } \varepsilon \to 0) \Leftrightarrow \frac{|u_\varepsilon - u|}{\varepsilon} \to 0 \text{ as } \varepsilon \to 0.
\end{aligned}
$$

A.2 SOME USEFUL INEQUALITIES

The following inequalities are often used to derive estimates in Analysis.

A polynomial inequality

Let $1 < p < +\infty$ and $a, b > 0$, then

$$
a^p + b^p \le (a+b)^p \le 2^{p-1}(a^p + b^p) \tag{A.1}
$$

Young's inequality

Assume $1 < p, p' < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. Then for any $a, b > 0$, it holds

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

It is sometimes convenient to use the form

$$ab \leq \alpha a^p + C_\alpha b^{p'}, \quad C_\alpha = \alpha^{-1/(p-1)}. \tag{A.2}$$

The Cauchy-Schwarz inequality

$$|\langle x \cdot y \rangle_{\mathbb{R}^n}| \leq |x| \, |y| \, , \forall x, y \in \mathbb{R}^n \tag{A.3}$$

Hölder's inequality

Let Ω be a domain in $\mathbb{R}^n$. Assume that $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$ with $1 \leq p \leq \infty$. Then

$$\int_\Omega |uv| \, dx \leq |u|_{L^p(\Omega)} \, |v|_{L^{p'}(\Omega)} \, . \tag{A.4}$$

Minkowski's inequality

Assume $1 \leq p \leq \infty$. Then for any $u, v \in L^p(\Omega)$

$$|u + v|_{L^p(\Omega)} \leq |u|_{L^p(\Omega)} + |u|_{L^p(\Omega)} \, . \tag{A.5}$$

Some inequalities for the $p-$Laplacian

For all $p > 1$, it holds that for some constants $C, c > 0$, depending on p, (see Barrett and Liu [2])

$$\left| |\xi|^{p-2} \xi - |\eta|^{p-2} \eta \right| \leq C \, |\xi - \eta| \, \{|\xi| + |\eta|\}^{p-2} \, , \tag{A.6}$$

$$\left(|\xi|^{p-2} \xi - |\eta|^{p-2} \eta \right) \cdot (\xi - \eta) \geq c \, \{|\xi| + |\eta|\}^{p-2} \, |\xi - \eta|^2 \, . \tag{A.7}$$

for all $\xi, \eta \in \mathbb{R}^n$. In particular, if $1 < p < 2$ then

$$\left| |\xi|^{p-2} \xi - |\eta|^{p-2} \eta \right| \leq C \, |\xi - \eta|^{p-1} \, , \quad \forall \xi, \eta \in \mathbb{R}^n, \tag{A.8}$$

and if $p \geq 2$ then

$$\left(|\xi|^{p-2} \xi - |\eta|^{p-2} \eta \right) \cdot (\xi - \eta) \geq c \, |\xi - \eta|^p \, . \tag{A.9}$$

Here $|\cdot|$ is the usual euclidean norm and $" \cdot "$ is the scalar product in $\mathbb{R}^n$.

Poincaré's inequality

Let ν be a unit vector in $\mathbb{R}^n, a > 0$ and $1 < p < \infty$. Suppose that Ω is bounded in one direction more precisely

$$\Omega \subset \{x \in \mathbb{R}^n \| x \cdot \nu| \leq a\},$$

then we have

$$|u|_{L^p(\Omega)} \leq \frac{2a}{p^{\frac{1}{p}}} |\frac{\partial u}{\partial \nu}|_{L^p(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega) \tag{A.10}$$

where $\frac{\partial u}{\partial \nu}$ is the derivative in the $\nu-$*direction*. In particular if $p = 2$, then

$$|u|_{L^2(\Omega)} \leq \sqrt{2}a |\frac{\partial u}{\partial \nu}|_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega). \tag{A.11}$$

Poincaré's inequality for $H^1(\Omega)$

Let Ω be a bounded connected open set in $\mathbb{R}^n$ sufficiently smooth to insure that the injection

$$H^1(\Omega) \subset L^2(\Omega) \text{ is compact.}$$

Then we have, for some constant C depending on Ω,

$$|u|^2_{L^2(\Omega)} \leq C \sum_{i=1}^{n} |\frac{\partial u}{\partial x_i}|^2_{L^2(\Omega)}, \tag{A.12}$$

for any $u \in H^1(\Omega)$ such that $\int_\Omega u(x)dx = 0$.

Gronwall's inequality

Let ϕ be a function in $L^\infty(0, T)$, $\phi(t) \geq 0$, *a.e.* $t \in [0, T]$. Assume that

$$\phi(t) \leq a + b \int_0^t \phi(s)\, ds, \quad \text{a.e. } t \in [0, T], \qquad a, b \text{ constants,} \tag{A.13}$$

then

$$\phi(t) \leq ae^t, \quad a.e.\ t \in [0, T].$$

In particular if $a = 0$, then $\phi(t) = 0$, a.e. $t \in [0, T]$.

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