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Existence of positive periodic solutions for certain differential equations

A Doctoral Thesis, By Farid Nouioua

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Dedication

This work is dedicated to

My dear children,

Hani Ilyas Haythem,

Med el Amine,

Yahia Islem,

Adem Nazim ...

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Abstract

In this thesis we study qualitative properties of broad classes of nonlinear delay differential equations. We start by giving some fixed point theorems and results for delay differential equations. Second we study the periodicity and positivity of solutions for a class of nonlinear differential equations with functional delay by using the Krasnoselskii's fixed point theorem, the contraction mapping principle and the Green's function.

Keywords: Delay differential equations, Fixed point theory, Periodicity, Positivity.

Mathematics Subject Classification: 34K13, 34K20, 45D05, 45J05, 47H10.

Résumé

Cette thèse est consacrée à l'étude de propriétés qualitatives de larges classes d'équations différentielles non linéaires à retard fonctionnel. On commence par donner les théorèmes de point fixe et des résultats sur les équations différentielles à retard. Deuxièmement, on étudie la périodicité et la positivité des solutions pour une classe d'équations différentielles non linéaires avec un retard fonctionnel en utilisant le théorème de point fixe de Krasnoselskii, le principe de la contraction et la fonction de Green.

Mots-clés: Equations différentielles à retard, Théorèmes des points fixes, Périodicité, Positivité.

Mathematics Subject Classification: 34K13, 34K20, 45D05, 45J05, 47H10.

Contents

Chapter 1

Introduction

The theory of fixed point is one of the most powerful tools of modern mathematics. Theorem concerning the existence and properties of fixed points are known as fixed point theorem. Fixed point theory is a beautiful mixture of analysis, topology and geometry. In particular fixed point theorem has been applied in such field as mathematics engineering, physics, economics, game theory, biology and chemistry etc. Classical and major results in these areas are Banach's fixed point theorem, Schauder's fixed point theorem and Krasnoselskii's fixed point theorem (see [8], [11], [47], [64], [70]).

In 1886, Poincare [64] was the first to work in this field. Then Brouwer [11] in 1912, proved fixed point theorem for the solution of the equation $f(x) = x$. He also proved fixed point theorem for a square, a sphere and their n-dimensional counter parts which was further extended by Kakutani [47]. Meanwhile Banach principle came into existence which was considered as one of the fundamental principles in the field of functional analysis. In 1922, Banach [8] proved that a contraction mapping in the field of a complete metric space possesses a unique fixed point.

An important generalization of Brouwer's theorem was discovered in 1930 by Schauder it may be stated as follows: any non empty, compact convex subset of a Banach space has the topological fixed point property. The compactness condition on subset is a stronger one. It is natural to modify the theorem by relaxing the condition of compactness. Schauder also proved a theorem for a compact map which is known as second form of the above stated theorem. Second fixed point theorem of Schauder stated that, every compact self mapping of a closed bounded convex subset of a Banach space has at least one fixed point.

In 1932, Krasnoselskii studied a paper of Schauder on partial differential equations and formulated the working hypothesis principle: the inversion of a perturbed differential operator yields the sum of a contraction and a compact map. Accordingly, he formulated an hybrid theorem known under its name. The reader is referred to the classical textbook on fixed point [70].

Delay differential equations are differential equations in which the unknown function and its derivatives enter, generally speaking, under different values of the argument (see [1]-[7], [9], [10], [12]-[27], [29]-[46], [49]-[63], [65]-[69], [71]-[77]). For example, $x'(t) = f(t, x(t), x(t-\tau))$ with $\tau > 0$ is an example of a such equation.

Delay differential equations describe many processes with an aftereffect or delayed response phenomenon. Such equations appear, for example, any time when in physics or technology we consider a problem of a force, acting on a material point, that depends on the velocity and position of the point not only at the given moment but at some moment preceding the given moment.

We have been interested in the use of fixed point theory to problem of periodicity and positivity. We have studied and contributed to it and have obtained interesting results. In this thesis we present a collection of results to some problems of delay differential equations by using fixed point theory.

This thesis contains five chapters which are briefly presented below. Chapter two is essentially an introduction to the fixed point theory, delay differential equations, where we fix notations, terminology to be used. It is a survey aimed at recalling some basic definitions and theory. While some of the classical and recent results about fixed point theory, delay differential equations are also presented in this chapter. Fixed point theorems frequently call for compact sets in Banach spaces which may be subsets of continuous functions. For that purpose, we give topologies which will provide many of those compact sets.

In the chapter three, we present and correct two papers of Raffoul [65, 66]. Raffoul considered the first order nonlinear neutral differential equation of the form

$$x'(t) = -a(t)x(t) + c(t)x'(t-\tau(t)) + f\left(t, x(t), x(t-\tau(t))\right),$$

which arises in a population model. By using the Krasnoselskii's fixed point theorem, sufficient conditions are presented for the existence of periodic and positive periodic solutions. Also, By employing the contraction mapping principle, Raffoul showed that the periodic solution is unique.

In the chapter four, we establish sufficient conditions for the periodicity and positivity of solutions for second order nonlinear differential equation with variable delay

$$x''(t) + p(t)x'(t) + q(t)x(t) = c(t)x'(t-\tau(t)) + f(t, x(t), x(t-\tau(t))),$$

by appealing the Krasnoselskii's fixed point theorem, the contraction mapping principle and Green's function. The results obtained in this chapter extend and improve the work of Wang, Lian and Ge [75] and the work of Yankson [77].

Finally in the chapter five, we discuss the periodicity and positivity of solutions for the third order delay differential equation with periodic coefficients

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) = f\left(t, x\left(t\right), x(t-\tau(t))\right) + c(t)x'(t-\tau(t)).$$

By converting the delay differential equation to an integral equation, our main results are obtained via the Krasnoselskii's fixed point theorem, the contraction mapping principle and Green's function in a Banach space, which surely provides a new way to the periodicity and positivity analysis (see [62, 63]).

Chapter 2

Preliminaries

2.1 Functional analysis

Questions concerning the existence and uniqueness of periodic solutions of delay differential equations can be well formulated in terms of fixed points of mappings. In fact, fixed-point theory was developed, in large measure, as a means of answering such questions. All but one of the fixed point theorems which we consider here require a setting in a compact subset of a metric space. In this section we discus compact sets [12, 78].

Definition 2.1 A pair (E, ρ) is a metric space if E is a set and $\rho : E \times E \to [0, \infty)$ such that when y, z and u are in E then

(a) $\rho(y, z) \geq 0$, $\rho(y, y) = 0$ and $\rho(y, z) = 0$ implies $y = z$,

(b) $\rho(y, z) = \rho(z, y)$,

(c) $\rho(y, z) \leq \rho(y, u) + \rho(u, z)$.

The metric space is complete if every Cauchy sequence in (E, ρ) has a limit in that space. A sequence $\{x_n\} \subset E$ is a Cauchy sequence if for each $\varepsilon > 0$ there exists N such that $n, m > N$ imply $\rho(x_n, x_m) < \varepsilon$.

Definition 2.2 1) A set $\mathbb{M}$ in a metric space (E, ρ) is compact if each sequence $\{x_n\} \subset \mathbb{M}$ has a subsequence with limit in $\mathbb{M}$.

2) A set $\mathbb{M}$ in a metric space (E, ρ) is relatively compact if its closure is compact, i.e., $\overline{\mathbb{M}}$ is compact.

Definition 2.3 Let $\{f_n\}$ be a sequence of functions with $f_n : [a, b] \to \mathbb{R}$.

(a) $\{f_n\}$ is uniformly bounded on $[a, b]$ if there exists $M > 0$ such that $|f_n(t)| \leq M$ for all n and all $t \in [a, b]$.

(b) $\{f_n\}$ is equicontinuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in [a, b]$ and $|t_1 - t_2| < \delta$ imply $|f_n(t_1) - f_n(t_2)| < \varepsilon$, for all n.

The following result gives the main method of proving compactness in the spaces in which we are interested.

Theorem 2.1 (Ascoli-Arzela [12]) *If $\{f_n(t)\}$ is a uniformly bounded and equicontinuous sequence of real functions on an interval $[a, b]$, then there is a subsequence which converges uniformly on $[a, b]$ to a continuous function.*

Definition 2.4 A linear space $(\mathbb{B}, +, .)$ is a normed space if for each $x \in \mathbb{B}$ there is a nonnegative real number $\|x\|$, called the norm of x, such that

(1) $\|x\| = 0$ if and only if $x = 0$,

(2) $\|\lambda x\| = |\lambda| \, \|x\|$ for each $\lambda \in \mathbb{R}$, and

(3) $\|x + y\| \leq \|x\| + \|y\|$.

Note a normed space is a vector space and it is a metric space with $\rho(x, y) = \|x - y\|$. But a vector space with a metric is not always a normed space.

Definition 2.5 A Banach space is a complete normed space.

We often say a Banach space is a complete normed vector space.

Example 2.1 (a) The space $\mathbb{R}^n$ over the field $\mathbb{R}$ is a vector space and there are many suitable norms for it. For example, if $x = (x_1, ..., x_n)$ then

(1) $\|x\| = \max_i |x_i|$,

(2) $\|x\| = \left[\sum_{i=1}^n x_i^2\right]^{1/2}$, or

(3) $\|x\| = \sum_{i=1}^n |x_i|$,

are all suitable norms. Norm (2) is the Euclidean norm. Notice that the square root is required in order that $\|\lambda x\| = |\lambda| \, \|x\|$.

(b) With any of these norms, $(\mathbb{R}^n, \|.\|)$ is a Banach space. It is complete because the real numbers are complete.

(c) A set $\mathbb{M}$ in $(\mathbb{R}^n, \|.\|)$ is compact if and only if it is closed and bounded, as is seen in any text on advanced calculus.

Example 2.2 (a) The space $C([a,b],\mathbb{R}^n)$ consisting of all continuous functions $f : [a,b] \to \mathbb{R}^n$ is a vector space over the real.

(b) If $\|f\| = \max_{a\le t\le b} |f(t)|$, where $|.|$ is a norm in $\mathbb{R}^n$, then it is a Banach space.

(c) For a given pair of positive constants M and K, the set $\mathbb{M} = \{f \in C([a,b],\mathbb{R}^n) | \ \|f\| \le M, \ |f(u) - f(v)| \le K|u-v|\}$ is compact. To see this, note first that Ascoli-Arzela theorem is also true for vector sequences; apply it to each component successively. If $\{f_n\}$ is any sequence in $\mathbb{M}$, then it is uniformly bounded and equicontinuous. By Ascoli-Arzela theorem it has a subsequence converging uniformly to a continuous function $f : [a,b] \to \mathbb{R}^n$. But $|f_n(t)| \le M$ for any fixed t, so $\|f\| \le M$. Moreover, if we denote the subsequence by $\{f_n\}$ again, then for fixed u and v there exist $\varepsilon_n > 0$ and $\delta_n > 0$ with

$$
\begin{aligned}
|f(u) - f(v)| &\le |f(u) - f_n(u)| + |f_n(u) - f_n(v)| + |f_n(v) - f(v)| \\
&\overset{def}{=} \varepsilon_n + |f_n(u) - f_n(v)| + \delta_n \\
&\le \varepsilon_n + \delta_n + K|u-v| \to K|u-v|,
\end{aligned}
$$

as $n \to \infty$. Hence, $f \in \mathbb{M}$ and $\mathbb{M}$ is compact.

Example 2.3 (a) Let $\phi : [a,b] \to \mathbb{R}^n$ be continuous and let E be the set of continuous functions $f : [a,c] \to \mathbb{R}^n$ with $c > b$ and with $\phi(t) = f(t)$ for $a \le t \le b$. Define $\rho(f,g) = \sup_{a\le t\le c} |f(t) - g(t)|$ for $f, g \in E$.

(b) Then (E,ρ) is a complete metric space but not a Banach space because $f + g$ is not in E.

2.2 Fixed point theory

2.2.1 Banach fixed point theorem

Recall that an initial value problem

$$
x' = f(t,x), \ x(t_0) = x_0, \tag{2.1}
$$

can be expressed as an integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \tag{2.2}$$

from which a sequence of functions $\{x_n\}$ may be inductively defined by

$$x_0(t) = x_0,\ x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0)ds,$$

and, in general

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_n(s))ds. \tag{2.3}$$

This is called Picard method of successive approximations and, under liberal conditions on f, one can show that $\{x_n\}$ converges uniformly on some interval $|t - t_0| \leq k$ to some continuous function, say x. Taking the limit in the equation defining x_{n+1}, we pass the limit through the integral and have

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds,$$

so that $x(t_0) = x_0$ and, upon differentiation, we obtain $x'(t) = f(t, x(t))$. Thus, x is a solution of the initial value problem.

Banach realized that this was actually a fixed-point theorem with wide application. For if we define an operator H on a complete metric space $C([t_0, t_0 + k], \mathbb{R})$ with the supremum norm $\|.\|$ (see Example 2.2) by $x \in C$ implies

$$(Hx)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \tag{2.4}$$

then a fixed point of H, say $H\phi = \phi$, is a solution of the initial value problem. The idea had two outstanding features. First, it had application to problems in every area of mathematics which used complete metric spaces. And it was clean. For example, the standard muddy and shaky proofs of implicit function theorems became clear and solid using the fixed-point theory. We will use it here to prove existence of solutions of various kinds of differential equations.

Definition 2.6 Let (E, ρ) be a complete metric space and $H : E \to E$. The operator H is a contraction operator if there is an $\lambda \in (0, 1)$ such that $x, y \in E$ imply

$$\rho(Hx, Hy) \leq \lambda\rho(x, y).$$

Theorem 2.2 (Contraction Mapping Principle [70]) *Let (E, ρ) be a complete metric space and $H : E \to E$ a contraction operator. Then there is a unique $x \in E$ with $Hx = x$. Furthermore, if $y \in E$ and if $\{y_n\}$ is defined inductively by $y_1 = Hy$ and $y_{n+1} = Hy_n$, then $y_n \to x$, the unique fixed point. In particular, the equation $Hx = x$ has one and only one solution.*

In applying this result to (2.1), a distressing event occurred which we now briefly describe. Assume that f is continuous and satisfies a global Lipschitz condition in x, say

$$|f(t, x_1) - f(t, x_2)| \leq L |x_1 - x_2|,$$

for $t \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}^n$. Then by (2.4) we obtain (for $t \geq t_0$)

$$\begin{aligned} |Hx_1(t) - Hx_2(t)| &= \left| \int_{t_0}^{t} [f(s, x_1(s)) - f(s, x_2(s))] \, ds \right| \\ &\leq \int_{t_0}^{t} L |x_1(s) - x_2(s)| \, ds, \end{aligned}$$

so that if $\|.\|$ is the sup norm on continuous functions on $[t_0, t_0 + k]$, then

$$\|Hx_1 - Hx_2\| \leq Lk \|x_1 - x_2\|.$$

This is a contraction if $Lk = \lambda < 1$. Now L is fixed and we take k small enough that $Lk < 1$. This gives a fixed point which is a solution of (2.1) on $[t_0, t_0 + k]$.

2.2.2 Krasnoselskii's fixed point theorem

Definition 2.7 Let $\mathbb{M}$ be a subset of a Banach space and $H_1 : \mathbb{M} \to \mathbb{B}$ application. If H_1 is continuous and $H_1\mathbb{M}$ is contained in a compact set in $\mathbb{B}$, then we say that H_1 is a compact application "we also say that H_1 is completely continuous".

Theorem 2.3 (Schauder [12, 70, 78]) *Let $\mathbb{M}$ be a convex set in a Banach space $\mathbb{B}$ and $H_1 : \mathbb{M} \to \mathbb{M}$ a compact application. Then H_1 has a fixed point.*

In 1955 Krasnoselskii's (see [12], [70]) observed that in a good number of problems, the integration of a perturbed differential operator gives rise to a sum of two applications, a contraction and a compact application. It declares then,

Principle: the integration of a perturbed differential operator can produce a sum of two applications, a contraction and a compact operator.

For better understanding this observation of Krasnoselskii's, consider the following differential equation.

$$x'(t) = -a(t)\,x(t) - g(t,x). \tag{2.5}$$

We can transform this equation in another form while writing, formally

$$x'(t)\,e^{-\int_0^t a(s)ds} = -a(t)\,e^{-\int_0^t a(s)ds}x(t) - g(t,x)\,e^{-\int_0^t a(s)ds},$$

thus

$$x'(t)\,e^{-\int_0^t a(s)ds} + a(t)\,e^{-\int_0^t a(s)ds}x(t) = -g(t,x)\,e^{-\int_0^t a(s)ds},$$

or

$$\left(x(t)\,e^{-\int_0^t a(s)ds}\right)' = -g(t,x)\,e^{-\int_0^t a(s)ds},$$

then integrating from $t-T$ to t, we obtain

$$\int_{t-T}^{t}\left(x(u)\,e^{-\int_0^u a(s)ds}\right)' du = -\int_{t-T}^{t} g(u,x)\,e^{-\int_0^u a(s)ds}du,$$

that gives

$$x(t)\,e^{-\int_0^t a(s)ds} - x(T-t)\,e^{-\int_0^{T-t} a(s)ds} = -\int_{t-T}^{t} g(u,x)\,e^{-\int_0^u a(s)ds}du,$$

or

$$x(t) = x(T-t)\,e^{-\int_{T-t}^t a(s)ds} - \int_{t-T}^{t} g(u,x)\,e^{-\int_t^u a(s)ds}du. \tag{2.6}$$

If we suppose that $e^{-\int_{T-t}^t a(s)ds} := \lambda$ and if $(\mathbb{B}, \|.\|)$ is the Banach space of functions $\varphi : \mathbb{R} \to \mathbb{B}$ continuous, then the Equation (2.6) can be written as

$$\varphi(t) = (H_2\varphi)(t) + (H_1\varphi)(t) := (H\varphi)(t).$$

where H_2 is contraction provides that the constant $\lambda < 1$ and H_1 is compact mapping.

This example shows the birth of the mapping $H\varphi := H_2\varphi + H_1\varphi$ which is identified with a sum of a contraction and a compact mapping.

Thus, the search of the solution for Equation (2.6) requires an adequate theorem which applies to this hybrid operator H and which can conclude the existence for a fixed point which will be, in its turn, solution of the initial Equation (2.5). Krasnoselskii's found the solution by combining the two theorems of Banach and that of Schauder in one hybrid theorem which bears its name. In light, it establishes the following result [70].

Theorem 2.4 (Krasnoselskii) *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that H_1 and H_2 map $\mathbb{M}$ into $\mathbb{B}$ such that*

(i) $x, y \in \mathbb{M}$, implies $H_1 x + H_2 y \in \mathbb{M}$,

(ii) H_1 is compact and continuous,

(iii) H_2 is a contraction mapping.

Then there exists $z \in \mathbb{M}$ with $z = H_1 z + H_2 z$.

Remark 2.1 Note that if $H_1 = 0$, the theorem becomes the theorem of Banach. If $H_2 = 0$, then the theorem is not other than the theorem of Schauder.

2.3 Retarded functional differential equations

As has been asked by many students in many classrooms, "Why study this subject?" Why study differential equations with time delays when so much is known about equations without delays, and they are so much easier? The answer is because so many of the processes, both natural and manmade, in biology, medicine, chemistry, physics, engineering, economics, etc., involve time delays. Like it or not, time delays occur so often, in almost every situation, that to ignore them is to ignore reality see [31, 36, 37, 45].

2.3.1 Delay differential equations

Suppose $\tau \geq 0$ is a given real number, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^n$ is an n-dimensional linear vector space over the reals with norm $|.|$, $C([a, b], \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[a, b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. If $[a, b] = [-\tau, 0]$ we let $C = C([-\tau, 0], \mathbb{R}^n)$ and designate the norm of an element ϕ in C by $|\phi| = \sup_{-\tau<\theta<0} |\phi(\theta)|$. Even though single bars are used for norms in different spaces, no confusion should arise. If

$$t_0 \in \mathbb{R}, A \geq 0 \text{ and } x \in C\left(\left[t_0 - \tau, t_0 + A\right], \mathbb{R}^n\right),$$

then for any $t \in [t_0, t_0 + A]$, we let $x_t \in C$ be defined by $x_t(\theta) = x(t + \theta), -\tau \leq \theta \leq 0$.

Definition 2.8 If Ω is a subset of $\mathbb{R} \times C$, $f : \Omega \to \mathbb{R}^n$ is a given function and $'$ represents the right-hand derivative, we say that the relation

$$x'(t) = f(t, x_t), \tag{2.7}$$

is a retarded functional differential equation on Ω and will denote this equation by $RFDE$. If we wish to emphasize that the equation is defined by f, we write the $RFDE\,(f)$. A function x is said to be a solution of Equation (2.7) on $[t_0 - \tau, t_0 + A)$ if there are $t_0 \in \mathbb{R}$ and $A > 0$ such that $x \in C\left([t_0 - \tau, t_0 + A], \mathbb{R}^n\right), (t, x_t) \in \Omega$ and x satisfies Equation (2.7) for $t \in [t_0, t_0 + A)$. For given $t_0 \in \mathbb{R}, \phi \in C$, we say $x(t, t_0, \phi)$ is a solution of Equation (2.7) with initial value ϕ at t_0 or simply a solution through (t_0, ϕ) if there is an $A > 0$ such that $x(t, t_0, \phi)$ is a solution of Equation (2.7) on $[t_0 - \tau, t_0 + A)$ and $x_{t_0}(t, t_0, \phi) = \phi$.

Equation (2.7) is a very general type of equation and includes ordinary differential equations ($\tau = 0$).

We say Equation (2.7) is linear if $f(t, \phi) = L(t, \phi) + h(t)$ where $L(t, \phi)$ is linear in ϕ; is homogeneous if $h \equiv 0$ and nonhomogeneous $h \neq 0$. We claim Equation (2.7) is autonomous if $f(t, \phi) = g(\phi)$ where g does not depend on t.

For example, the following equations are delay differential equations

$$x'(t) = 2x(t) + 5x(t - 1), \tag{2.8}$$

$$x'(t) = a(t)x(t) + b(t)x'(t - \tau(t)) + h(t), \tag{2.9}$$

$$x'(t) = \int_{-\tau}^{0} x(t + s)ds. \tag{2.10}$$

a, b, τ are continuous functions. Equation (2.8) is an linear autonomous delay differential equation with constant $\tau = 1$, Equation (2.9) is nonhomogeneous, linear nonautonomous delay functional differential equations and Equation (2.10) is a delay linear integro-differential equation.

If $t_0 \in \mathbb{R},\ \phi \in C$ are given and $f(t,\phi)$ is continuous, then finding a solution of Equation (2.7) through (t_0, ϕ) is equivalent to solving the integral equation

$$\begin{aligned} x_{t_0} &= \phi, \\ x(t) &= \phi(0) + \int_{t_0}^{t} f(s, x_s)ds,\ t \geq t_0. \end{aligned} \tag{2.11}$$

We define Tx by

$$\begin{aligned} Tx(t) &= \phi(0) + \int_{t_0}^{t} f(s, x_s)ds,\ t \geq t_0, \\ x_{t_0} &= \phi. \end{aligned}$$

To prove the existence of the solution through a point $(t_0, \phi) \in \mathbb{R} \times C$, we consider an $\eta > 0$ and all functions x on $[t_0 - \tau, t_0 + A]$ which are continuous and coincide with ϕ on $[t_0 - \tau, t_0]$; that is, $x_{t_0} = \phi$. The values of these functions on $[t_0, t_0 + \eta]$ are restricted to the class of x such that $|x(t) - \phi(0)| < \delta$ for $t \in [t_0, t_0 + \eta]$. The usual mapping T obtained from the corresponding integral equation is defined and it is then shown that η and δ can be so chosen that T maps this class into itself and is completely continuous. Thus, Schauder's fixed-point theorem implies existence (for examples details see the books [36, 37, 45]).

Theorem 2.5 (Existence) *In (2.7), suppose Ω is an open subset in $\mathbb{R} \times C$ and f is continuous on Ω. If $(t_0, \phi) \in \Omega$, then there is a solution of (2.7) passing through (t_0, ϕ).*

Definition 2.9 We say $f(t, \phi)$ is Lipschitz in ϕ in a compact set K of $\mathbb{R} \times C$ if there is a constant $k > 0$ such that, for any $(t, \phi_i) \in K$, $i = 1, 2$,

$$|f(t, \phi_1) - f(t, \phi_2)| \leq k\,|\phi_1 - \phi_2|\,. \tag{2.12}$$

Theorem 2.6 (Uniqueness) *Suppose Ω is an open set in $\mathbb{R} \times C$, $f : \Omega \to \mathbb{R}^n$ is continuous, and $f(t, \phi)$ is Lipschitz in ϕ in each compact set in Ω. If $(t_0, \phi) \in \Omega$, then there is a unique solution of Eq. (2.7) through (t_0, ϕ).*

2.3.2 Neutral delay differential equations

In order to define a general class of neutral delay differential equations ($NDDEs$) (or neutral functional differential equations ($NFDEs$)), we need the definition of atomic.

Definition 2.10 Suppose $\Omega \subseteq \mathbb{R}\times C$ is open with elements (t,ϕ). A function $\Psi : \Omega \to \mathbb{R}^n$ is said to be atomic at β on Ω if Ψ is continuous together with its first and second Fréchet derivatives with respect to ϕ and Ψ_ϕ, the derivative with respect to ϕ, is atomic at β on Ω.

Definition 2.11 Suppose $\Omega \subseteq \mathbb{R}\times C$ is open, $f : \Omega \to \mathbb{R}^n$, $\Psi : \Omega \to \mathbb{R}^n$ are given continuous functions with Ψ atomic at zero. The equation

$$\frac{d}{dt}\Psi(t,x_t) = f(t,x_t), \tag{2.13}$$

is called the neutral delay differential equation $NDDE(\Psi, f)$.

Definition 2.12 A function x is said to be a solution of the $NDDE(\Psi, f)$ or Equation (2.13), if there are $t_0 \in \mathbb{R}$, $A > 0$, such that $x \in C([t_0-\tau, t_0+A), \mathbb{R}^n)$, $(t, x_t) \in \Omega$, $t \in [t_0, t_0+A)$, $\Psi(t,x_t)$ is continuously differentiable and satisfies Eq. (2.13) on $[t_0, t_0+A)$. For a given $t_0 \in \mathbb{R}$, $\phi \in C$, and $(t_0,\phi) \in \Omega$, we say $x(t_0,\phi)$ is a solution of Eq. (2.13) with initial value ϕ at t_0, or simply a solution through (t_0,ϕ) ,if there is an $A > 0$ such that $x(t_0,\phi)$, is a solution of (2.13) on $[t_0-\tau, t_0+A)$ and $x_{t_0}(t_0,\phi) = \phi$.

Theorem 2.7 *(Existence) if Ω is an open set in $\mathbb{R}\times C$ and $(t_0,\phi) \in \Omega$, then there exists a solution of the $NDDE(\Psi, f)$ through (t_0,ϕ).*

Theorem 2.8 *(Uniqueness). If $\Omega \subseteq \mathbb{R}\times C$ is open and $f : \Omega \to \mathbb{R}^n$ as Lipschitz in ϕ on compact sets of Ω, then, for any $(t_0,\phi) \in \Omega$, there exists a unique solution of the $NDDE(\Psi, f)$ through (t_0,ϕ).*

For example

$$\begin{aligned} x'(t) &= -x'(t-1), \\ x'(t) &= x(t-1) + [x'(t-3)+1]^3, \\ x''(t) &= x(\frac{t}{2}) + x'(t-1) - x'(t-3), \end{aligned}$$

are neutral delay differential equations.

2.3.3 Method of Steps

The method of steps is an elementary method that can be used to solve some $DDEs$ analytically. This method is usually discarded as being too tedious, but in some cases the tedium can be removed by using computer algebra see [38]. Consider the following general DDE :

$$y'(t) = a_0 y(t) + a_1 y(t - w_1) + ... + a_m y(t - w_m), \tag{2.14}$$

where $y(t) = H(t)$ on the initial interval $-\max(w_i) \leq t \leq 0$. Let $b = \min(w_i)$. Then it is clear that the values of $y(t - w_m)$ are known in the interval $0 \leq t \leq b$. These values are $H(t - w_m)$. Thus, for the interval $0 \leq t \leq b$ we have

$$y'(t) = a_0 y(t) + a_1 H(t - w_1) + ... + a_m H(t - w_m),$$

and so

$$y(t) = \int_0^t \left(a_0 y(v) + a_1 H(v - w_1) + ... + a_m H(v - w_m)\right) dv + y(0).$$

Now that we know $y(t)$ on $[0, b]$ we can repeat this procedure to obtain $y(t)$ on the interval $b \leq t \leq 2b$. This is given by:

$$y(t) = \int_b^t \left(a_0 y(v) + a_1 H(v - w_1) + ...\right) dv + y(b). \tag{2.15}$$

This process can be continued indefinitely, so long as the integrals that occur can be evaluated without too much effort. It is this last restriction that usually causes people to give up on this method, because the tedium and length of the method quickly overwhelms a human computer. However, it turns out that for certain classes of problems, where the phenomenon of "expression swell" is not too serious, we can take the method quite far, with a computer algebra system to automate the solution of the tedious sub-problems.

Example 2.4 For an example of this method we look first at a very simple DDE

$$y'(t) = -y(t - 1),$$

with $y(t) = H(t) = 1$ for $-1 \leq t \leq 0$. The solution in the interval $0 \leq t \leq 1$ is given by:

$$y(t) = \int_0^t -H(x - 1) dx + y(0) = 1 - t.$$

Now we can solve for the solution in the interval $1 \leq t \leq 2$. This solution is given by:

$$y(t) = \int_1^t -H(t-1)dx + y(1) = \frac{t^2}{2} - 2t + \frac{3}{2}.$$

This method can be programmed in Maple using a simple for loop.

2.3.4 Problems with a delay

In this subsection we introduce a large number of problems, both old and new, which are treated using the general theory of differential equations. We attempt to give sufficient description concerning the derivation, solution, and properties of solutions so that the reader will be able to appreciate some of the flavor of the problem. In none of the cases do we give a complete treatment of the problem, but offer references for further study.

Economics models

The following problem is copied from an elementary text on differential equations by Boyce and DiPrima [10]: "A young person with no initial capital invests k dollars per year at an annual interest rate τ. Assume that investments are made continuously and that interest is compounded continuously. If $\tau = 7.5\%$, determine k so that one million dollars will be available at the end of forty years."

It is solved by writing

$$S' = 0.075S + k, \; S(0) = 0,$$

and solving for $S(40)$. Several things are idealized in the problem, but still it is a fair model. It is noted there that in certain contexts continuous investment yields roughly the same as daily investment and it allows the student the opportunity to see the power of differential equations in giving a simple solution to an otherwise tedious problem.

Now the forty years is up and for computational convenience instead of the one million dollars let us say that the person has $\$900,000$ to invest and to live off the proceeds. During times of low interest rates a financial advisor may recommend bank certificates of deposit of 90-day maturity, automatically renewed at the existing interest rate, but laddered so that $\$10,000$ of the total matures every day and both principal and interest are reinvested. This enables the investor to quickly take advantage of rising rates and to lock in high interest long-term instruments if they become available. We imagine that this is changed to continuous reinvestment, just as the elementary problem imagined continuous investment of k dollars per year. If the total value is again $S(t)$, then from just the investment we would have

$$S'(t) = b(t)S(t - (1/4)).$$

The $b(t)$ represents a product. One factor is the fraction of the total amount of $S(t-1/4)$ which was invested three months earlier and matured today. The other factor is the interest being offered at that time. In addition, the person withdraws a percentage of the total $S(t)$ continuously for living expenses, resulting in an equation

$$S'(t) = -a(t)S(t) + b(t)S(t - 1/4), \; S(t) = \psi(t) \text{ for } -1/4 \leq t \leq t_0.$$

Here, the initial condition is an initial function $\psi : [-1/4, 0] \to \mathbb{R}$ with $\psi(t)$ being exactly that amount $S(t)$ which was invested at time t.

We can draw several conclusions of the following type. First, if the solutions are bounded, then times are likely to become difficult since inflation will eat away at the value and medical bills will increase with time; at this time, some studies have shown that those retiring with income sufficient to meet three times their current need approach desperate conditions within fifteen years. Next, we can ask if solutions will tend to zero. If they do, the person will be destined for the poor farm. At a minimum, the retiree must adjust the withdrawals so that the conditions of our theorem are not met.

Clearly, in this example it will make sense for both $a(t)$ and $b(t)$ to vary; $a(t)$ can be negative the day the income tax refund check arrives, and $b(t)$ can be negative when the bank fails and the FDIC assumes control see [13].

Controlling a ship

Minorsky (1962) designed an automatic steering device for the battleship New Mexico. The following is a sketch of the problem see [12].

Let the rudder of the ship have angular position $x(t)$ and suppose there is a friction force proportional to the velocity, say $-cx'(t)$. There is a direction indicating instrument which points in the actual direction of motion and there is an instrument pointing in the desired direction. These two are connected by a device which activates an electric motor producing a certain force to move the rudder so as to bring the ship onto the desired course. There is a time lag of amount $h > 0$ between the time the ship gets off course and the time the electric motor activates the restoring force. The equation for $x(t)$ is

$$x''(t) + cx'(t) + g(x(t-h)) = 0, \tag{2.16}$$

where $xg(x) > 0$ if $x \neq 0$ and c is a positive constant. The object is to give conditions ensuring that $x(t)$ will stay near zero so that the ship closely follows its proper course.

Epidemics (Cooke and Yorke)

In the work of Cooke and Yorke (1973) the Lotka assumption is changed so that the number of births per unit time is a function only of the population size, not of the age distribution see [12]. Under this assumption, we let $x(t)$ be the population size and let the number of births be $B(t) = g(x(t))$. Assume each individual has life span L so that the number of deaths per unit time is $g(x(t-L))$. Then the population size is described by

$$x'(t) = g(x(t)) - g(x(t-L)), \tag{2.17}$$

where g is some differentiable function. We note that every constant function is a solution of (2.17).

The following model for the spread of gonorrhea is considered by Cooke and Yorke (1973). The population is divided into two classes:

(a) $S(t)$ =the number of susceptibles, and

(b) $x(t)$ =the number of infectious.

The rate of new infection depends only on contacts between susceptible and infectious individuals. Since Set) equals the constant total population minus $x(t)$, the rate is some function $g(x(t))$. Assume that an exposed individual is immediately infectious and stays infectious for a period L (the time for treatment and cure). Then x also satisfies (2.17) holds.Now, at any time t, $x(t)$ equals the sum of capital produced over the period $[t-L, t]$

plus a constant c denoting the value of nondepreciating assets. Thus,

$$\begin{aligned} x(t) &= \int_0^L P(s)g[x(t-s)]ds + c \\ &= \int_{t-L}^t P(t-u)g[x(u)]du + c. \end{aligned} \tag{2.18}$$

Some models of war and peace

L. F. Richardson (1881-1953, see [12]), a British Quaker, observed two world wars and was concerned about them (cf. Richardson, 1960; Jacobson, 1984). He speculated that wars begin where arms races end and he felt that international dynamics could be modeled mathematically because of human motivations. He claimed that men are guided by "their traditions, which are fixed, and their instincts which are mechanical"; thus, on a grand scale they are incapable of good and evil. He sought to develop a theory of international dynamics to guide statesmen with domestic and foreign policy, much as dynamics guides machine design.

Let X and Y be nations suspicious of each other. Suppose X and Y create stocks of arms x and y, respectively; more generally, x and y represent "threats minus cooperation" so that negative values have meaning. At least three things affect the arms buildup of X;

(a) economic burden;

(b) terror at the sight of $y(t)$ (or national pride);

(c) grievances and suspicions of y.

The same will, of course, apply to Y.

Richardson assumed that each side had complete and instantaneous knowledge of the arms of the other side and that each side could react instantaneously. He reasoned from (a) that

$$dx/dt = -a_1 x,$$

because the burden is proportional to the size x, and he argued from (b) that

$$dx/dt = -a_1 x + b_1 y,$$

because the terror is proportional to the size y. Finally, Richardson assumed constant standing grievances, say g_t so that the complete system is

$$\begin{aligned} x' &= -a_1x + b_1y + g_1, \\ y' &= -a_2y + b_2x + g_2, \end{aligned} \tag{2.19}$$

with a_i, b_i, and g_i , $i = 1, 2$ being positive constants. Domestic and foreign policy will set the a_i and b_i, although Richardson maintained a more mechanical view.

Hill (1978) recognized deficiencies in Richardson's model. He reasoned that it takes time to respond to an observed situation and, therefore, proposed the model

$$\begin{aligned} x' &= -a_1x(t-T) + b_1y(t-T) + g_1, \\ y' &= -a_2y(t-T) + b_2x(t-T) + g_2, \end{aligned}$$

where T is a positive constant.

Prey-predator population models (Lotka-Voltera)

Let $x(t)$ be the population at time t of some species of animal called prey and let $y(t)$ be the population of a predator species which lives off these prey. We assume that $x(t)$ would increase at a rate proportional to $x(t)$ if the prey were left alone, i.e., we would have $x'(t) = a_1x(t)$, where $a_1 > 0$. However the predators are hungry, and the rate at which each of them eats prey is limited only by his ability to find prey. (This seems like a reasonable assumption as long as there are not too many prey available.) Thus we shall assume that the activities of the predators reduce the growth rate of $x(t)$ by an amount proportional to the product $x(t)y(t)$, i.e.,

$$x'(t) = a_1x(t) - b_1x(t)y(t),$$

where b_1 is another positive constant.

Now let us also assume that the predators are completely dependent on the prey as their food supply. If there were no prey, we assume $y'(t) = -a_2y(t)$, where $a_2 > 0$, i.e., the predator species would die out exponentially. However, given food the predators breed at a rate proportional to their number and to the amount of food available to them. Thus we consider the pair of equations

$$\begin{aligned} x'(t) &= a_1x(t) - b_1x(t)y(t), \\ y'(t) &= -a_2y(t) + b_2x(t)y(t), \end{aligned} \tag{2.20}$$

where a_1, a_2, b_1, and b_2 are positive constants. This well-known model was invented and studied by Lotka [1920], [1925] and Volterra [1928], [1931].

Vito Volterra was trying to understand the observed fluctuations in the sizes of populations $x(t)$ of commercially desirable fish and $y(t)$ of larger fish which fed on the smaller ones in the Adriatic Sea in the decade from 1914 to 1923 see [31].

The sunflower equation

Somolinos (1978) has considered the equation

$$x'' + (a/r)x' + (b/r)\sin x(t-r) = 0,$$

and has obtained interesting results on the existence of periodic solutions. The study of this problem goes back to the early 1800s and has attracted much attention. It involves the motion of a sunflower plant see [12].

Chapter 3

Periodic and positive periodic solutions for first order neutral differential equations

In this chapter, we study the existence of periodic and positive periodic solutions of the nonlinear neutral differential equation

$$x'(t) = -a(t)x(t) + c(t)x'(t-\tau(t)) + f(t, x(t), x(t-\tau(t))). \qquad (3.1)$$

We invert this equation to construct a sum of a completely continuous map and a contraction which is suitable for applying the Krasnoselskii's theorem. Also, by transforming the problem to an integral equation we are able, using the contraction mapping principle, to show that the periodic solution is unique see [65, 66].

3.1 Introduction and inversion of the equation

Theory of functional differential equations with delay has undergone a rapid development in the previous fifty years. We refer the readers to [1]-[7], [9], [10], [12]-[27], [29]-[46], [49]-[63], [65]-[69], [71]-[77] and references therein for a wealth of reference materials on the subject. More recently researchers have given special attention to the study of equations in which the delay argument occurs in the derivative of the state variable as well as in the independent variable, so-called neutral differential equations. In particular, qualitative analysis such as periodicity and positivity of solutions of neutral differential equations has been studied extensively by many authors.

The purpose of this chapter is to transform (3.1) to an integral equation and then use Krasnosleskii's fixed point theorem to show the existence of periodic and positive periodic solutions. The obtained integral equation is the sum of two mappings, one is a contraction and the other is completely continuous. Transforming (3.1) to an integral equation enables us to show the uniqueness of the periodic solution by appealing to the contraction mapping principle.

For $T > 0$ define $P_T = \{\phi : C(\mathbb{R}, \mathbb{R}),\ \phi(t+T) = \phi(t)\}$ where $C(\mathbb{R}, \mathbb{R})$ is the space of all real valued continuous functions. Then P_T is a Banach space when it is endowed with the supremum norm

$$\|x\| = \max_{t\in[0,T]} |x(t)|.$$

In this chapter, we assume that

$$a(t+T) = a(t),\ c(t+T) = c(t),\ \tau(t+T) = \tau(t),\ \tau(t) \geq \tau^* > 0, \tag{3.2}$$

with c continuously differentiable, τ twice continuously differentiable and τ^* is constant. In [33], the author made the assumption that a is positive, while here we only ask that

$$\int_0^T a(s)\, ds > 0. \tag{3.3}$$

It is interesting to note that equation (3.1) becomes of advanced type when $\tau(t) < 0$. Since we are searching for periodic solutions, it is natural to ask that $f(t, x, y)$ is continuous and periodic in t and Lipschitz continuous in x and y. That is

$$f(t+T, x, y) = f(t, x, y), \tag{3.4}$$

and some positive constants k_1 and k_2,

$$|f(t, x, y) - f(t, z, w)| \leq k_1 \|x - z\| + k_2 \|y - w\|. \tag{3.5}$$

Also, we assume that for all t, $0 \leq t \leq T$,

$$\tau'(t) \neq 1. \tag{3.6}$$

Since τ is periodic, condition (3.6) implies that $\tau'(t) < 1$.

3.1. Introduction and inversion of the equation

Lemma 3.1 *Suppose (3.2), (3.3) and (3.6) hold. If* $x \in P_T$*, then* x *is a solution of equation (3.1) if and only if*

$$x(t) = \frac{c(t)}{1-\tau'(t)} x(t-\tau(t)) + \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \times \int_{t-T}^{t} \left[-r(u)x(u-\tau(u)) + f(u, x(u), x(u-\tau(u)))\right] e^{-\int_u^t a(s)ds} du, \tag{3.7}$$

where

$$r(u) = \frac{(c'(u) - c(u)a(u))(1-\tau'(u)) + \tau''(u)c(u)}{(1-\tau'(u))^2}. \tag{3.8}$$

Proof. Let $x \in P_T$ be a solution of (3.1). Multiply both sides of (3.1) by $e^{\int_0^t a(s)ds}$ and then integrate from $t-T$ to t we obtain

$$\int_{t-T}^{t} \left[x(u) e^{\int_0^u a(s)ds}\right]' du = \int_{t-T}^{t} \left[c(u)x'(u-\tau(u)) + f(u, x(u), x(u-\tau(u)))\right] e^{\int_0^u a(s)ds} du.$$

As a consequence, we arrive at

$$x(t) e^{\int_0^t a(s)ds} - x(t-T) e^{\int_0^{t-T} a(s)ds} = \int_{t-T}^{t} \left[c(u)x'(u-\tau(u)) + f(u, x(u), x(u-\tau(u)))\right] e^{\int_0^u a(s)ds} du.$$

By dividing both sides of the above equation by $e^{\int_0^t a(s)ds}$ and the fact that $x(t) = x(t-T)$, we obtain

$$x(t) = \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \times \int_{t-T}^{t} \left[c(u)x'(u-\tau(u)) + f(u, x(u), x(u-\tau(u)))\right] e^{-\int_u^t a(s)ds} du. \tag{3.9}$$

Rewrite

$$\int_{t-T}^{t} c(u)x'(u-\tau(u)) e^{-\int_u^t a(s)ds} du = \int_{t-T}^{t} \frac{c(u)x'(u-\tau(u))(1-\tau'(u))}{(1-\tau'(u))} e^{-\int_u^t a(s)ds} du.$$

Integration by parts on the above integral with

$$U = \frac{c(u)}{1-\tau'(u)} e^{-\int_u^t a(s)ds} \text{ and } dV = x'(u-\tau(u))(1-\tau'(u)) du,$$

we obtain

$$\int_{t-T}^{t} c(u)\, x'(u-\tau(u))\, e^{-\int_u^t a(s)ds} du$$
$$= \frac{c(t)}{1-\tau'(t)} x(t-\tau(t)) \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right) - \int_{t-T}^{t} r(u)\, e^{-\int_u^t a(s)ds} x(u-\tau(u))\, du, \quad (3.10)$$

where r is given by (3.8). Then substituting (3.10) into (3.9) completes the proof. ■

3.2 Study of periodic solutions

In this section, we study the existence and the uniqueness of periodic solutions of (3.1). Define the mapping $H : P_T \longrightarrow P_T$ by

$$(H\varphi)(t) = \frac{c(t)}{1-\tau'(t)} \varphi(t-\tau(t)) + \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1}$$
$$\times \int_{t-T}^{t} \left[-r(u)\, \varphi(u-\tau(u)) + f(u, \varphi(u), \varphi(u-\tau(u)))\right] e^{-\int_u^t a(s)ds} du. \quad (3.11)$$

Note that to apply the Krasnosleskii's theorem we need to construct two mappings, one is a contraction and the other is completely continuous. Therefore, we express (3.11) as

$$(H\varphi)(t) = (H_2\varphi)(t) + (H_1\varphi)(t),$$

where $H_1, H_2 : P_T \longrightarrow P_T$ are given by

$$(H_2\varphi)(t) = \frac{c(t)}{1-\tau'(t)} \varphi(t-\tau(t)), \quad (3.12)$$

and

$$(H_1\varphi)(t) = \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1}$$
$$\times \int_{t-T}^{t} \left[-r(u)\, \varphi(u-\tau(u)) + f(u, \varphi(u), \varphi(u-\tau(u)))\right] e^{-\int_u^t a(s)ds} du. \quad (3.13)$$

Lemma 3.2 *Suppose (3.2)-(3.6) hold. Then $H_1 : P_T \to P_T$, as defined by (3.13), is completely continuous.*

Proof. A change of variable in (3.13) shows that $(H_1\varphi)(t+T) = (H_1\varphi)(t)$. To see that H_1 is continuous, we let $\varphi, \psi \in P_T$ with $\|\varphi\| \le C$ and $\|\psi\| \le C$. Let

$$\eta = \left| \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \right|, \ \beta = \max_{t\in[0,T]} |r(t)|, \ \gamma = \max_{u\in[t-T,t]} e^{-\int_u^t a(s)ds}. \quad (3.14)$$

Given $\epsilon > 0$, take $\delta = \epsilon/N$ such that $\|\varphi - \psi\| < \delta$. By making use of (3.5) into (3.13) we get

$$\begin{aligned} |(H_1\varphi)(t) - (H_1\psi)(t)| &\leq \gamma\eta \int_{t-T}^{t} [k_1 \|\varphi - \psi\| + k_2 \|\varphi - \psi\| + \beta \|\varphi - \psi\|] \, du \\ &\leq N \|\varphi - \psi\| < \epsilon, \end{aligned}$$

where k_1 and k_2 are given by (3.5) and $N = T\gamma\eta [\beta + k_1 + k_2]$. This proves H_1 is continuous.

To show H_1 is compact, we let $\varphi_n \in P_T$ with $\|\varphi_n\| \leq R$, where n is a positive integer and $R > 0$. Observe that in view of (3.5) we arrive at

$$\begin{aligned} |f(t, x, y)| &= |f(t, x, y) - f(t, 0, 0) + f(t, 0, 0)| \\ &\leq |f(t, x, y) - f(t, 0, 0)| + |f(t, 0, 0)| \\ &\leq k_1 \|x\| + k_2 \|y\| + \alpha \end{aligned}$$

where $\alpha = \max_{t\in[0,T]} |f(t, 0, 0)|$. Hence, if H_1 is given by (3.13) we obtain that

$$\|H_1\varphi_n\| \leq D,$$

for some positive constant D. Now, it can be easily checked that

$$(H_1\varphi_n)'(t) = -a(t)(H_1\varphi_n)(t) - r(u)\varphi_n(t - \tau(t)) + f(t, \varphi_n(t), \varphi_n(t - \tau(t))).$$

Hence we obtain $\|(H_1\varphi_n)'\| \leq F$, for some positive constant F. Thus the sequence $(H_1\varphi_n)$ is uniformly bounded and equicontinuous. The Ascoli-Arzela theorem implies that $(H_1\varphi_{n_k})$ uniformly converges to a continuous $T-$periodic function φ^*. Thus H_1 is compact. ■

Lemma 3.3 *Let H_2 be defined by (3.12) and*

$$\left| \frac{c(t)}{1 - \tau'(t)} \right| \leq \zeta < 1. \tag{3.15}$$

Then H_2 is a contraction.

Proof. For $\varphi, \psi \in P_T$, we have

$$\begin{aligned}
\|H_2(\varphi) - H_2(\psi)\| &= \max_{t\in[0,T]} |(H_2\varphi)(t) - (H_2\psi)(t)| \\
&= \max_{t\in[0,T]} \left\{ \left| \frac{c(t)}{1-\tau'(t)} \right| |\varphi(t-\tau(t)) - \psi(t-\tau(t))| \right\} \\
&\leq \zeta \|\varphi - \psi\|.
\end{aligned}$$

Hence H_2 defines a contraction mapping with contraction constant ζ. ■

Theorem 3.1 *Let* $\alpha = \max_{t\in[0,T]} |f(t,0,0)|$. *Let* η, β *and* γ *be given by (3.14). Suppose (3.2)–(3.6) and (3.15) hold. Suppose there is a positive constant* J *such that all solutions* x *of equation (3.1),* $x \in P_T$ *satisfy* $|x(t)| \leq J$, *the inequality*

$$\{\zeta + \eta\gamma T(\beta + k_1 + k_2)\} J + \eta\gamma\alpha T \leq J, \tag{3.16}$$

holds. Then equation (3.1) has a $T-$*periodic solution.*

Proof. Define $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leq J\}$. Then lemma **3.2** implies $H_1 : P_T \longrightarrow P_T$ and H_1 is compact and continuous. Also, from Lemma **3.3**, the mapping H_2 is a contraction and it is clear that $H_2 : P_T \longrightarrow P_T$. Next, we show that if $\varphi, \psi \in \mathbb{M}$, we have $\|H_1\varphi + H_2\psi\| \leq J$. Let $\varphi, \psi \in \mathbb{M}$ with $\|\varphi\|, \|\psi\| \leq J$. Then from (3.12), (3.13) and the fact that $|f(t,x,y)| \leq k_1\|x\| + k_2\|y\| + \alpha$, we have

$$\begin{aligned}
&|(H_1\varphi)(t) + (H_2\psi)(t)| \\
&\leq \gamma\eta \int_{t-T}^{t} [k_1\|\varphi\| + k_2\|\psi\| + \beta\|\varphi\| + \alpha] \, du + \zeta\|\psi\| \\
&\leq \{\zeta + \eta\gamma T(\beta + k_1 + k_2)\} J + \eta\gamma\alpha T \leq J.
\end{aligned}$$

We see that all the conditions of Krasnoselskii's theorem are satisfied on the set $\mathbb{M}$. Thus there exists a fixed point z in $\mathbb{M}$ such that $z = H_1 z + H_2 z$. By Lemma **3.1**, this fixed point is a solution of (3.1). Hence (3.1) has a $T-$periodic solution. ■

Theorem 3.2 *Suppose (3.2)-(3.6) and (3.15) hold. Let* η, β *and* γ *be given by (3.14) If*

$$\zeta + \eta\gamma T(\beta + k_1 + k_2) < 1,$$

the equation (3.1) has a unique $T-$*periodic solution.*

Proof. Let the mapping H be given by (3.11). For $\varphi, \psi \in P_T$, in view of (3.11), we have

$$\begin{aligned}
&\left|(H\varphi)(t) - (H\psi)(t)\right| \\
&\leq \zeta \left\|\varphi - \psi\right\| + \gamma\eta \int_{t-T}^{t} \left[k_1 \left\|\varphi - \psi\right\| + k_2 \left\|\varphi - \psi\right\| + \beta \left\|\varphi - \psi\right\|\right] du \\
&\leq \left[\zeta + \eta\gamma T \left(\beta + k_1 + k_2\right)\right] \left\|\varphi - \psi\right\|.
\end{aligned}$$

This completes the proof. ■

3.3 Study of positive periodic solutions

In this section, we study the existence of positive periodic solutions by considering the two cases

$$0 \leq \frac{c(t)}{1 - \tau'(t)} < 1, \quad -1 \leq \frac{c(t)}{1 - \tau'(t)} \leq 0.$$

To simplify notation, we let

$$M = \frac{e^{\int_0^{2T} |a(s)| ds}}{1 - e^{-\int_0^T a(s) ds}}, \quad m = \frac{e^{-\int_0^{2T} |a(s)| ds}}{1 - e^{-\int_0^T a(s) ds}}, \tag{3.17}$$

and

$$G(t,u) = \frac{e^{-\int_u^t a(s) ds}}{1 - e^{\int_0^T a(s) ds}}. \tag{3.18}$$

It is easy to see that, for all $(t,u) \in [0,T] \times [-T,T]$,

$$m \leq G(t,u) \leq M,$$

and, for all $t, u \in \mathbb{R}$, we have

$$G(t+T, u+T) = G(t,u).$$

For some non-negative constant $\mathbf{K}$ and a positive constant $\mathbf{L}$, we define the set

$$\mathbb{M} = \{\phi \in P_T : \mathbf{K} \leq \phi \leq \mathbf{L}\},$$

which is a closed convex and bounded subset of the Banach space P_T. In addition, we assume that there are constants c_1 and c_2 such that

$$0 \leq c_1 \leq \frac{c(t)}{1 - \tau'(t)} \leq c_2 < 1, \tag{3.19}$$

and for all $u \in \mathbb{R}$, $x, y \in \mathbb{M}$

$$\frac{(1-c_1)\,\mathbf{K}}{mT} \leq f(u,x,y) - r(u)\,y \leq \frac{(1-c_2)\,\mathbf{L}}{MT}, \tag{3.20}$$

where M and m are defined by (3.17). It is clear from condition (3.19) that H_2 defines a contraction mapping under the supremum norm.

Lemma 3.4 *If (3.2)-(3.4), (3.6), (3.19) and (3.20) hold, then the operator H_1 is completely continuous on $\mathbb{M}$.*

Proof. For $t \in [0,T]$ which implies that $u \in [t-T, t] \subseteq [-T, T]$, and for $\varphi \in \mathbb{M}$, we have

$$\begin{aligned} |(H_1\varphi)(t)| &\leq \left| \int_{t-T}^{t} G(t,u)\,[f(u,\varphi(u),\varphi(u-\tau(u))) - r(u)\,\varphi(u-\tau(u))]\,du \right| \\ &\leq TM\frac{(1-c_2)\,\mathbf{L}}{MT}. \end{aligned}$$

From the estimation of $|(H_1\varphi)(t)|$, it follows that

$$\|H_1\varphi\| \leq (1-c_2)\,\mathbf{L} \leq Q_1,$$

for some positive constant Q_1. This shows that $H_1(\mathbb{M})$ is uniformly bounded. It is left to show that $H_1(\mathbb{M})$ is equicontinuous. Let $\varphi \in \mathbb{M}$. Then a differentiation of (3.13) with respect to t yields

$$(H_1\varphi)'(t) = -a(t)\,(H_1\varphi)(t) + f(t,\varphi(t),\varphi(t-\tau(t))) - r(t)\,\varphi(t-\tau(t)).$$

Hence, by taking the supremum norm in the above expression, we have

$$\left\|(H_1\varphi)'\right\| \leq \left(\|a\| + \frac{1}{MT}\right) Q_1.$$

Thus the estimation on $\left\|(H_1\varphi)'\right\|$ implies that $H_1(\mathbb{M})$ is equicontinuous. Then, using the Ascoli-Arzela theorem, we obtain that H_1 is a compact map. Due to the continuity of all terms in (3.13) for $t \in [0,T]$, we have that H_1 is continuous. This completes the proof of Lemma **3.4** ■

Theorem 3.3 *If (3.2)-(3.4), (3.6), (3.19) and (3.20) hold, then the equation (3.1) has a positive periodic solution z satisfying $\mathbf{K} \leq z \leq \mathbf{L}$.*

Proof. Let $\varphi, \psi \in \mathbb{M}$. Then, by (3.12) and (3.13) we have that

$$\begin{aligned}
&(H_2\varphi)(t) + (H_1\psi)(t) \\
&= \frac{c(t)}{1-\tau'(t)}\varphi(t-\tau(t)) \\
&+ \int_{t-T}^{t} G(t,u)\left[f(u,\psi(u),\psi(u-\tau(u))) - r(u)\psi(u-\tau(u))\right] du \\
&\leq c_2\mathbf{L} + MT\frac{(1-c_2)\mathbf{L}}{MT} = \mathbf{L}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(H_2\varphi)(t) + (H_1\psi)(t) \\
&= \frac{c(t)}{1-\tau'(t)}\varphi(t-\tau(t)) \\
&+ \int_{t}^{t+T} G(t,u)\left[f(u,\psi(u),\psi(u-\tau(u))) - r(u)\psi(u-\tau(u))\right] du \\
&\geq c_1\mathbf{K} + m\int_{t-T}^{t} \left[f(u,\psi(u),\psi(u-\tau(u))) - r(u)\psi(u-\tau(u))\right] du \\
&\geq c_1\mathbf{K} + mT\frac{(1-c_1)\mathbf{K}}{mT} = \mathbf{K}.
\end{aligned}$$

This shows that $H_2\varphi + H_1\psi \in \mathbb{M}$. All the hypotheses of Krasnoselskii's theorem are satisfied, and therefore equation (3.1) has a periodic solution, say z, residing in $\mathbb{M}$. This completes the proof. ■

For the next theorem, we assume that there are negative constants c_3 and c_4 with $-1 < c_3 \leq c_4 \leq 0$, such that

$$-1 < c_3 \leq \frac{c(t)}{1-\tau'(t)} \leq c_4 \leq 0, \tag{3.21}$$

and, for all $u \in \mathbb{R}$, $x, y \in \mathbb{M}$,

$$\frac{(\mathbf{K} - c_3\mathbf{L})}{mT} \leq f(u,x,y) - r(u)y \leq \frac{(\mathbf{L} - c_4\mathbf{K})}{\mathbb{M}T}, \tag{3.22}$$

where M and m are defined by (3.17).

Theorem 3.4 *If (3.2)-(3.4), (3.6), (3.21) and (3.22) hold, then equation (3.1) has a positive periodic solution z satisfying $\mathbf{K} \leq z \leq \mathbf{L}$.*

The proof follows along the lines of Theorem 3.4, and hence we omit it here.

Example 3.1 The neutral differential equation

$$x'(t) = -\frac{1}{2}\sin^2(t)\,x(t) + \frac{1}{50}x'(t-\pi) + \frac{\cos^2(t)}{x^2(t-\pi)+100} + \frac{1}{25}, \tag{3.23}$$

has a positive $\pi-$periodic solution x satisfying

$$\frac{1}{10} \le x \le 2.$$

To see this, we let

$$f(t,\rho_1,\rho_2) = \frac{\cos^2(t)}{\rho_2^2+100} + \frac{1}{25},\ \ r(t) = \frac{1}{2}\sin^2(t) \ \text{ and } T = \tau(t) = \pi.$$

Then

$$\frac{c(t)}{1-\tau'(t)} = \frac{1}{50} < 1,$$

and

$$r(t) = \frac{1}{100}\sin^2(t).$$

A simple calculation yields

$$4.030 < M < 4.032 \text{ and } 1.74 < m < 1.75.$$

Let $\mathbf{L} = 2$ and $\mathbf{K} = 1/10$, and define the set $\mathbb{M} = \{1/10 \le v \le 2\}$. Then, for $x, y \in [1/10, 2]$, we have

$$\begin{aligned} f(u,x,y) - r(u)\,y &= \frac{\cos^2(u)}{y^2+100} - \frac{1}{2}\sin^2(u)\,y + \frac{1}{25} \\ &\le \frac{1}{100} + \frac{1}{50} + \frac{1}{25} \\ &= 0.07 < \frac{(1-c_2)\,\mathbf{L}}{MT}. \end{aligned}$$

On the other hand

$$\begin{aligned} f(u,\rho_1,\rho_2) - r(u)\,\rho_2 &= \frac{\cos^2(u)}{\rho_2^2+100} + \frac{1}{2}\sin^2(u)\,\rho_2 + \frac{1}{25} \\ &> \frac{1}{25} > \frac{(1-c_1)\,\mathbf{K}}{mT}. \end{aligned}$$

We see that all the conditions of Theorem 3.3 are satisfied, and hence equation (3.23) has a positive $\pi-$periodic solution x satisfying $1/10 \le x \le 2$.

3.3. Study of positive periodic solutions

Chapter 4

Periodicity and positivity of second order delay differential equations

In this chapter, we study the periodicity and positivity for the second order nonlinear delay differential equation with a periodic coefficients

$$x''(t) + p(t) x'(t) + q(t) x(t) = c(t) x'(t - \tau(t)) + f(t, x(t), x(t - \tau(t))), \ t \in \mathbb{R}.$$

By using Krasnoselskii's fixed point theorem, we establish some criteria for the existence of periodic and positive periodic solutions to the delay differential equation. Also, by applying contraction mapping principle, we show that the periodic solution is unique. The results obtained here extend and improve previous results due to Wang, Lian and Ge [75] and Yankson [77].

4.1 Introduction

The first order delay differential equation

$$x'(t) = -a(t) x(t) + \lambda h(t) f(x(t - \tau(t))),$$

and its generalizations have attracted much attention, for details see [1]-[7], [9], [10], [12]-[27], [29]-[46], [49]-[63], [65]-[69], [71]-[77] and the references therein. Liu and Ge [56] investigated the following nonlinear Duffing equation with delay and variable coefficients

$$x''(t) + p(t) x'(t) + q(t) x(t) = \lambda h(t) f(t, x(t - \tau(t))) + r(t).$$

The existence and nonexistence of positive periodic solutions are obtained with suitable conditions imposed on f by using the fixed point theorem in cones.

In this chapter, we aim to discuss the periodicity and positivity of solutions for the second order delay differential equation with periodic coefficients

$$x''(t)+p(t)x'(t)+q(t)x(t)=c(t)x'(t-\tau(t))+f(t,x(t),x(t-\tau(t))),\ t\in\mathbb{R}. \tag{4.1}$$

We assume that

(A_1) $p,q:\mathbb{R}\longrightarrow\mathbb{R}^+$, $c,\tau:\mathbb{R}\longrightarrow\mathbb{R}$ are all continuous T-periodic functions, $\int_0^T p(s)ds>0$, $\int_0^T q(s)ds>0$, and $\tau'(t)\neq 1$ for all $t\in[0,T]$.

(A_2) $f:\mathbb{R}^3\longrightarrow\mathbb{R}$ is continuous for any $(t,x,y)\in\mathbb{R}^3$ and is T-periodic in t for all $(x,y)\in\mathbb{R}^2$.

(A_3) There exist positive constants k_1 and k_2 such that

$$|f(t,x,y)-f(t,z,w)|\le k_1\|x-z\|+k_2\|y-w\|.$$

4.2 Green's function and inversion of the equation

Let T be a positive constant. Consider the space $P_T=\{\varphi: C(\mathbb{R},\mathbb{R}),\ \phi(t+T)=\phi(t)\}$ with the maximum norm $\|x\|=\max_{t\in[0,T]}|x(t)|$. Obviously, P_T is a Banach space.

Lemma 4.1 ([56]) *Suppose that* (A_1) *holds and*

$$\frac{R_1\left(e^{\int_0^T p(u)du}-1\right)}{Q_1T}\ge 1, \tag{4.2}$$

where

$$R_1=\max_{t\in[0,T]}\left|\int_t^{t+T}\frac{e^{\int_t^s p(u)du}}{e^{\int_0^T p(u)du}-1}q(s)\,ds\right|,\ Q_1=\left(1+e^{\int_0^T p(u)du}\right)^2R_1^2.$$

Then there are continuous $T-$*periodic functions* a *and* b *such* $b(t)>0$, $\int_0^T a(u)\,du>0$ *and*

$$a(t)+b(t)=p(t),\ b'(t)+a(t)b(t)=q(t),\ \textit{for } t\in\mathbb{R}.$$

Lemma 4.2 ([75]) *Suppose the conditions of Lemma 4.1 hold and* $\phi\in P_T$. *Then the equation*

$$x''(t)+p(t)x'(t)+q(t)x(t)=\phi(t), \tag{4.3}$$

has a $T-$periodic solution. Moreover, the periodic solution can be expressed by

$$x(t) = \int_t^{t+T} G(t,s)\,\phi(s)\,ds, \tag{4.4}$$

where

$$G(t,s) = \frac{\int_t^s e^{\int_t^u b(v)dv + \int_u^s a(v)dv} du + \int_t^{t+T} e^{\int_t^u b(v)dv + \int_u^{s+T} a(v)dv} du}{\left[e^{\int_0^T a(u)du} - 1\right]\left[e^{\int_0^T b(u)du} - 1\right]}.$$

Proof. Define

$$E_a = e^{\int_0^T a(u)du} - 1,\ E_b = e^{\int_0^T b(u)du} - 1.$$

By direct calculation, we can see that (4.4) is a $T-$periodic solution of (4.3). Suppose x is a $T-$periodic solution of (4.3), from Lemma **4.1**, we have

$$x''(t) + a(t)x'(t) + b'(t)x(t) + b(t)x'(t) + a(t)b(t)x(t) = \phi(t),$$

which is equivalent to

$$\left(x'(t)\,e^{\int_0^t a(u)du}\right)' + \left(b(t)x(t)e^{\int_0^t a(u)du}\right)' = \phi(t)\,e^{\int_0^t a(u)du}.$$

Integrating it from t to $t+T$, we obtain

$$x'(t) + b(t)x(t) = \int_t^{t+T} \frac{e^{\int_t^s a(u)du}}{e^{\int_0^T a(u)du} - 1}\phi(s)\,ds.$$

Therefore,

$$\begin{aligned}
x(t) &= \int_t^{t+T} \frac{e^{\int_t^s b(u)du}}{e^{\int_0^T b(u)du} - 1}\left[\int_t^{s+T} \frac{e^{\int_s^v a(u)du}}{e^{\int_0^T a(u)du} - 1}\phi(v)\,dv\right] ds \\
&= \frac{1}{E_aE_b}\int_t^{t+T} e^{\int_t^s b(u)du}\int_s^{s+T} e^{\int_s^v a(u)du}\phi(v)\,dvds \\
&= \frac{1}{E_aE_b}\int_t^{t+T} dv \int_t^s e^{\int_t^s b(u)du} e^{\int_s^v a(u)du}\phi(v)\,ds \\
&\quad + \frac{1}{E_aE_b}\int_{t+T}^{t+2T} dv \int_{v-T}^{t+T} e^{\int_t^s b(u)du} e^{\int_s^v a(u)du}\phi(v)\,ds \\
&= \frac{1}{E_aE_b}\int_t^{t+T} \phi(s)\,ds \int_t^s e^{\int_t^u b(v)dv + \int_u^s a(v)dv} du \\
&\quad + \frac{1}{E_aE_b}\int_t^{t+T} \phi(s)\,ds \int_t^{t+T} e^{\int_t^u b(v)dv + \int_u^{s+T} a(v)dv} du \\
&= \int_t^{t+T} G(t,s)\,\phi(s)\,ds.
\end{aligned}$$

■

4.2. Green's function and inversion of the equation

Corollary 4.1 *Green's function $G(t,s)$ satisfies the following properties*

$$\begin{aligned} G(t,t+T) &= G(t,t),\ G(t+T,s+T)=G(t,s), \\ \frac{\partial}{\partial s}G(t,s) &= a(s)G(t,s)-\frac{e^{\int_t^s b(v)dv}}{e^{\int_0^T b(u)du}-1}, \\ \frac{\partial}{\partial t}G(t,s) &= -b(t)G(t,s)+\frac{e^{\int_t^s a(v)dv}}{e^{\int_0^T a(v)dv}-1}. \end{aligned}$$

Lemma 4.3 ([75]) *The function $x\in P_T$ is a solution of (4.1) if and only if*

$$x(t)=\int_t^{t+T} E(t,s)\,x(s-\tau(s))+G(t,s)\left[f(s,x(s),x(s-\tau(s)))-R(s)\,x(s-\tau(s))\right]ds, \tag{4.5}$$

where

$$\begin{aligned} E(t,s) &= \frac{c(s)}{1-\tau'(s)}\frac{e^{\int_t^s b(v)dv}}{e^{\int_0^T b(v)dv}-1}, \\ R(s) &= \frac{(c'(s)+a(s)c(s))(1-\tau'(s))+r(s)\tau''(s)}{(1-\tau'(s))^2}. \end{aligned}$$

Proof. Easily, we can see that (4.5) is a $T-$periodic solution of (4.1). On the other hand, if x is a $T-$periodic solution of (4.1) we have

$$x(t)=\int_t^{t+T} G(t,s)\left[c(s)\,x'(s-\tau(s))+f(s,x(s),x(s-\tau(s)))\right]ds.$$

Meanwhile,

$$\begin{aligned} \int_t^{t+T} G(t,s)\,c(s)\,x'(s-\tau(s))\,ds &= \int_t^{t+T}\frac{c(s)\,x'(s-\tau(s))\,(1-\tau'(s))}{1-\tau'(s)}G(t,s)\,ds \\ &= \int_t^{t+T}\frac{c(s)}{1-\tau'(s)}G(t,s)\,dx(s-\tau(s)) \\ &= \int_t^{t+T} x(s-\tau(s))\left[E(t,s)-R(s)\,G(t,s)\right]ds. \end{aligned}$$

So, the conclusion is obvious. ■

Lemma 4.4 ([56]) *Let $A=\int_0^T p(u)\,du$, $B=T^2e^{\frac{1}{T}\int_0^T \ln q(u)du}$. If*

$$A^2\geq 4B, \tag{4.6}$$

then we have

$$\min\left\{\int_0^T a(u)\,du, \int_0^T b(u)\,du\right\} \geq \frac{1}{2}\left(A-\sqrt{A^2-4B}\right) := l$$
$$\max\left\{\int_0^T a(u)\,du, \int_0^T b(u)\,du\right\} \leq \frac{1}{2}\left(A+\sqrt{A^2-4B}\right) := L.$$

Proof. As $a(t)+b(t)=p(t)$, $b'(t)+a(t)b(t)=q(t)$, we have

$$\int_0^T a(u)\,du+\int_0^T b(u)\,du=\int_0^T p(u)\,du \text{ and } \int_0^T a(u)\,du=\int_0^T \frac{q(u)}{b(u)}du.$$

Applying the inequality

$$\frac{\int_a^b p(x)f(x)\,dx}{\int_a^b p(x)\,dx} \geq e^{\frac{\int_a^b p(x)\ln f(x)dx}{\int_a^b p(x)dx}},$$

where $p(x)\geq 0$, $\int_a^b p(x)\,dx>0$, $\inf f(x)>0$. We have

$$\frac{1}{T}\int_0^T b(u)\,du \geq e^{\frac{1}{T}\int_0^T \ln b(u)du}, \ \frac{1}{T}\int_0^T \frac{q(u)}{b(u)}du \geq e^{\frac{1}{T}\int_0^T \ln\frac{q(u)}{b(u)}du}.$$

So, we can easily get

$$\begin{aligned}\int_0^T a(u)\,du\int_0^T b(u)\,du &= \int_0^T b(u)\,du\int_0^T \frac{q(u)}{b(u)}du\\ &\geq T^2 e^{\frac{1}{T}\int_0^T \ln q(u)du}.\end{aligned}$$

Notice that $\int_0^T a(u)\,du+\int_0^T b(u)\,du=\int_0^T q(u)\,du$. So the conclusion is obvious. ■

Corollary 4.2 *Functions $G(t,s)$ and $E(t,s)$ satisfy*

$$m\leq G(t,s)\leq M, \ |E(t,s)|\leq\left|\frac{c(s)}{1-\tau'(s)}\right|\frac{e^L}{e^l-1},$$

where

$$m=\frac{T}{\left(e^L-1\right)^2} \text{ and } M=\frac{Te^{\frac{1}{T}\int_0^T p(u)du}}{\left(e^l-1\right)^2}. \tag{4.7}$$

4.3 Existence and uniqueness of periodic solutions

In this section, we study the existence and the uniqueness of periodic solutions of (4.1). Define the mapping $H : P_T \longrightarrow P_T$ by

$$(H\varphi)(t) = \int_t^{t+T} \{E(t,s)\varphi(s-\tau(s)) + G(t,s)[f(s,\varphi(s),\varphi(s-\tau(s))) - R(s)\varphi(s-\tau(s))]\} ds. \quad (4.8)$$

Note that to apply Krasnoselskii's fixed point theorem we need to construct two mappings: one is a contraction and the other is completely continuous. Therefore, we express (4.8) as

$$(H\varphi)(t) = (H_2\varphi)(t) + (H_1\varphi)(t),$$

where $H_1, H_2 : P_T \longrightarrow P_T$ are given by

$$(H_2\varphi)(t) = \int_t^{t+T} E(t,s)\varphi(s-\tau(s))\, ds, \quad (4.9)$$

$$(H_1\varphi)(t) = \int_t^{t+T} G(t,s)[f(s,\varphi(s),\varphi(s-\tau(s))) - R(s)\varphi(s-\tau(s))]\, ds. \quad (4.10)$$

Note that in our consideration the definition of H_1 and H_2 is different from that in [75]. As a consequence, our results improve the results of Wang, Lian and Ge [75].

Lemma 4.5 *Suppose* $(A_1)-(A_3)$ *and conditions (4.2) and (4.6) hold. Then* $H_1 : P_T \longrightarrow P_T$ *is completely continuous.*

Proof. We divide our proof into two steps.

Step 1. The mapping $H_1 : P_T \longrightarrow P_T$ is continuous.

It is easy to show that $(H_1\varphi)(t+T) = (H_1\varphi)(t)$. To see that H_1 is continuous, we let $\varphi, \psi \in P_T$. Let $\beta = \max_{t\in[0,T]} |R(t)|$. Given $\epsilon > 0$, take $\delta = \epsilon/N$ such that $\|\varphi - \psi\| < \delta$. By making use of (A_3) we get

$$\begin{aligned} \|H_1\varphi - H_1\psi\| &\leq \int_t^{t+T} M(k_1\|\varphi-\psi\| + k_2\|\varphi-\psi\| + \beta\|\varphi-\psi\|)\, ds \\ &\leq N\|\varphi-\psi\| < \epsilon, \end{aligned}$$

where $N = TM(k_1 + k_2 + \beta)$. So H_1 is continuous.

Step 2. The mapping $H_1 : P_T \longrightarrow P_T$ is compact.

To show that H_1 is compact, we let $\varphi_n \in P_T$ with $\|\varphi_n\| \leq K$, where n is a positive integer and $K > 0$. In view of (A_3) we arrive at

$$\begin{aligned} |f(t,x,y)| &= |f(t,x,y) - f(t,0,0) + f(t,0,0)| \\ &\leq |f(t,x,y) - f(t,0,0)| + |f(t,0,0)| \\ &\leq k_1 \|x\| + k_2 \|y\| + F, \end{aligned}$$

where $F = \max_{t\in[0,T]} |f(t,0,0)|$. Hence, we obtain that

$$\|H_1\varphi_n\| \leq D,$$

for some positive constant D. Now, it can be easily checked that

$$\begin{aligned} &(H_1\varphi_n)'(t) \\ &= \int_t^{t+T} \left(-b(t)\,G(t,s) + \frac{e^{\int_t^s b(v)dv}}{e^{\int_0^T b(v)dv} - 1} \right) \\ &\times [f(s,\varphi_n(s),\varphi_n(s-\tau(s))) - R(s)\,\varphi_n(s-\tau(s))]\,ds, \end{aligned}$$

so

$$\left\|(H_1\varphi_n)'\right\| \leq T\left(M\|b\| + \frac{e^L}{e^l - 1} \right)(Kk_1 + Kk_2 + F + K\beta).$$

Thus the sequence $(H_1\varphi_n)$ is uniformly bounded and equicontinuous.The Ascoli-Arzela theorem implies that $(H_1\varphi_n)$ is relatively compact. So H_1 is a compact operator. ■

Lemma 4.6 *Suppose* $(A_1) - (A_2)$ *and conditions (4.2) and (4.6) hold. Then* $H_2 : P_T \longrightarrow P_T$ *is a contraction provided that*

$$T\gamma < 1, \tag{4.11}$$

where

$$\gamma = \max_{t\in[0,T]} \left| \frac{c(t)}{1 - \tau'(t)} \right| \frac{e^L}{e^l - 1}. \tag{4.12}$$

Proof. It is easy to check, so we omit the details. ■

Theorem 4.1 *Suppose* $(A_1) - (A_3)$ *and conditions (4.2), (4.6) and (4.11) hold. Suppose* $J > 0$ *satisfies*

$$T(\gamma + M(k_1 + k_2 + \beta))\,J + MTF \leq J.$$

Then (4.1) has at least one $T-$periodic solution.

Proof. Define $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leq J\}$. Then Lemma 4.5 implies $H_1 : P_T \longrightarrow P_T$ is compact and continuous. Also, from Lemma 4.6, the mapping H_2 is contraction and it is clear that $H_1 : P_T \longrightarrow P_T$. Next, we show that if $\varphi, \psi \in \mathbb{M}$, we have $\|H_1\varphi + H_2\psi\| \leq J$. Let $\varphi, \psi \in \mathbb{M}$ with $\|\varphi\|, \|\psi\| \leq J$. Then from (4.9) and (4.10) and the fact that

$$|f(t, x, y)| \leq k_1 \|x\| + k_2 \|y\| + F,$$

we have

$$\begin{aligned} \|H_1\varphi + H_2\psi\| &\leq \gamma \int_t^{t+T} \|\varphi\| \, ds + M \int_t^{t+T} \left[(k_1 + k_2 + \beta) \|\psi\| + F\right] ds \\ &\leq T(\gamma + M(k_1 + k_2 + \beta)) J + MTF \leq J. \end{aligned}$$

Krasnoselskii's fixed point theorem implies that there exists x in $\mathbb{M}$ such that $x = H_1x + H_2x$. By Lemma 4.3, this fixed point is a solution of (4.1). Hence (4.1) has a $T-$periodic solution. ∎

Theorem 4.2 *Suppose $(A_1) - (A_3)$ and conditions (4.2) and (4.6) hold. If*

$$T(\gamma + M(k_1 + k_2 + \beta)) < 1,$$

then (4.1) has a unique $T-$periodic solution.

Proof. Let the mapping H be given by (4.8). For $\varphi, \psi \in P_T$, we have

$$\begin{aligned} \|H\varphi - H\psi\| &\leq \int_t^{t+T} \left[\gamma \|\varphi - \psi\| + \alpha (k_1 \|\varphi - \psi\| + k_2 \|\varphi - \psi\| + \beta \|\varphi - \psi\|)\right] ds \\ &\leq T(\gamma + M(k_1 + k_2 + \beta)) \|\varphi - \psi\|. \end{aligned}$$

This completes the proof. ∎

4.4 Existence of positive periodic solutions

In this section, we prove the existence of positive periodic solutions of (4.1). Our contributions comparing with the existing results are taken variable coefficient $c(t)$ instead of constant c. We present our existence results by considering two cases, $E(t, s) \geq 0$ and

$E(t,s) \leq 0$. For some non-negative constant $\mathbf{K}$ and a positive constant $\mathbf{L}$ we define the set

$$\mathbb{D} = \{\varphi \in P_T : \mathbf{K} \leq \varphi \leq \mathbf{L}\},$$

which is a closed convex and bounded subset of the Banach space P_T. In addition we assume that there are constants σ_1 and σ_2 such that

$$0 \leq \sigma_1 T \leq E(t,s)\, T \leq \sigma_2 T < 1 \text{ for all } (t,s) \in [0,T] \times [0,T], \tag{4.13}$$

and for all $s \in \mathbb{R}$, $x, y \in \mathbb{D}$

$$\frac{\mathbf{K}(1-\sigma_1 T)}{mT} \leq f(s,x,y) - R(s)\, y \leq \frac{\mathbf{L}(1-\sigma_2 T)}{MT}. \tag{4.14}$$

where M and m are defined by (4.7). It is clear from condition (4.13) that H_2 defines a contraction mapping under the supremum norm.

Lemma 4.7 *Suppose* (A_1) *and* (A_2) *and conditions (4.2), (4.6), (4.13) and (4.14) hold. Then* $H_1 : \mathbb{D} \to P_T$ *is completely continuous.*

Proof. Let H_1 be defined by (4.10). It is easy to see that $(H_1\varphi)(t+T) = (H_1\varphi)(t)$. For $t \in [0,T]$ and for $\varphi \in \mathbb{D}$ we have that

$$\begin{aligned} |(H_1\varphi)(t)| &\leq \left| \int_t^{t+T} G(t,s)\left[f(s,\varphi(s),\varphi(s-\tau((s)))) - R(s)\varphi(s-\tau((s)))\right] ds \right| \\ &\leq TM\frac{\mathbf{L}(1-\sigma_2 T)}{MT} \leq \mathbf{L}(1-\sigma_2 T). \end{aligned}$$

Thus from the estimation of $|(H_1\varphi)(t)|$ we have

$$\|H_1\varphi\| \leq \mathbf{L}(1-\sigma_2 T).$$

This shows that $H_1(\mathbb{D})$ is uniformly bounded. We next show that $H_1(\mathbb{D})$ is equicontinuous. Let $\varphi_n \in \mathbb{D}$. By using (A_1) and (A_2), we obtain by taking the derivative in (4.10) that

$$\begin{aligned} \frac{d}{dt}(H_1\varphi_n)(t) &= \int_t^{t+T} \left(-b(t)\, G(t,s) + \frac{e^{\int_t^s a(v)dv}}{e^{\int_0^T a(v)dv} - 1} \right) \\ &\quad \times \left[f(s,\varphi_n(s),\varphi_n(s-\tau(s))) - R(s)\varphi_n(s-\tau(s))\right] ds. \end{aligned}$$

Consequently, by invoking (4.14), we obtain

$$\left|\frac{d}{dt}(H_1\varphi_n)(t)\right| \leq T\left(\|b\|M + \frac{e^L}{e^l - 1}\right)\frac{\mathbf{L}(1-\sigma_2 T)}{MT} \leq D,$$

for some positive constant D. Hence $(H_1\varphi_n)$ is equicontinuous. Then by the Ascoli-Arzela theorem we obtain that H_1 is a compact map. Due to the continuity of all the terms in (4.10), we have that H_1 is continuous. This completes the proof. ∎

Theorem 4.3 *Suppose (A_1) and (A_2) and conditions (4.2), (4.6), (4.13) and (4.14) hold. Then (4.1) has a positive periodic solution z satisfying $\mathbf{K} \leq z \leq \mathbf{L}$.*

Proof. Let $\varphi, \psi \in \mathbb{D}$. Using (4.9) and (4.10) we obtain

$$\begin{aligned}
&(H_2\psi)(t) + (H_1\varphi)(t) \\
&= \int_t^{t+T} E(t,s)\varphi(s-\tau((s)))\,ds \\
&+ \int_t^{t+T} G(t,s)\left[f(s,\psi(s),\psi(s-\tau(s))) - R(s)\psi(s-\tau(s))\right]ds \\
&\leq \sigma_2\mathbf{L}T + M\int_t^{t+T}\left[f(s,\psi(s),\psi(s-\tau(s))) - R(s)\psi(s-\tau(s))\right]ds \\
&\leq \sigma_2\mathbf{L}T + MT\frac{\mathbf{L}(1-\sigma_2 T)}{MT} = \mathbf{L}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
&(H_2\psi)(t) + (H_1\varphi)(t) \\
&= \int_t^{t+T} E(t,s)\varphi(s-\tau((s)))\,ds+ \\
&+ \int_t^{t+T} G(t,s)\left[f(s,\psi(s),\psi(s-\tau(s))) - R(s)\psi(s-\tau(s))\right]ds \\
&\geq \sigma_1\mathbf{K}T + m\int_t^{t+T}\left[f(s,\psi(s),\psi(s-\tau(s))) - R(s)\psi(s-\tau(s))\right]ds \\
&\geq \sigma_1 KT + mT\frac{\mathbf{K}(1-\sigma_1 T)}{mT} = \mathbf{K}.
\end{aligned}$$

This shows that $H_2\psi + H_1\varphi \in \mathbb{D}$. Thus all the Hypotheses of Krasnoselskii theorem are satisfied and therefore (4.1) has a periodic solution in $\mathbb{D}$. This completes the proof. ∎

We next consider the case when $E(t,s) \leq 0$. To this end we substitute conditions (4.13) and (4.14) with the following conditions respectively. We assume that there are constants σ_3 and σ_4 such that

$$-1 < \sigma_3 T \leq E(t,s)\,T \leq \sigma_4 T \leq 0, \tag{4.15}$$

and for all $s \in \mathbb{R}$, $x, y \in \mathbb{D}$

$$\frac{\mathbf{K} - \sigma_3 \mathbf{L} T}{mT} \leq f(s,x,y) - R(s)\,y \leq \frac{\mathbf{L} - \sigma_4 \mathbf{K} T}{MT}. \tag{4.16}$$

Theorem 4.4 *Suppose* (A_1) *and* (A_2) *and conditions (4.2), (4.6), (4.15) and (4.16) hold. Then (4.1) has a positive periodic solution* z *satisfying* $\mathbf{K} \leq x \leq \mathbf{L}$.

The proof follows along the lines of Theorem 4.3, and hence we omit it.

Chapter 5

Periodic and positive periodic solutions for third-order delay differential equations

The goal of this chapter is to present very recent works published in [62, 63], namely,
F. Nouioua, A. Ardjouni, A. Djoudi, Periodic solutions for a third-order delay differential equation, Applied Mathematics E-Notes, 16 (2016), 210–221.
F. Nouioua, A. Ardjouni, A. Merzougui, A. Djoudi, Existence of positive periodic solutions for a third-order delay differential equation, International Journal of Analysis and Applications, International Journal of Analysis and Applications, Volume 13, Number 2 (2017), 136-143.

In this chapter, the following third-order nonlinear delay differential equation with periodic coefficients

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) = f\left(t, x\left(t\right), x(t - \tau(t))\right) + c(t)x'(t - \tau(t)),$$

is considered. By employing Green's function and Krasnoselskii's fixed point theorem, we state and prove the existence of periodic and positive periodic solutions to the third-order delay differential equation. Also, by using the contraction mapping principle, we show that the periodic solution is unique.

5.1 Introduction

Third order differential equations arise from in a variety of different areas of applied mathematics and physics, as the deflection of a curved beam having a constant or varying cross section, three layer beam, electromagnetic waves or gravity driven flows and so on [34, 69].

Delay differential equations have received increasing attention during recent years since these equations have been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering, see [1]-[7], [9], [10], [12]-[27], [29]-[46], [49]-[63], [65]-[69], [71]-[77] and the references therein.

The second order nonlinear delay differential equation with periodic coefficients

$$x''(t) + p(t)x'(t) + q(t)x(t) = c(t)x'(t - \tau(t)) + f(t, x(t), x(t - \tau(t))),$$

has been investigated in [75]. By using Krasnoselskii's fixed point theorem and the contraction mapping principle, Wang, Lian and Ge obtained existence and uniqueness of periodic solutions.

In [69], Ren, Siegmund and Chen discussed the existence of positive periodic solutions for the third-order differential equation

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + c(t)x(t) = g(t, x(t)).$$

By employing the fixed point index, the authors obtained existence results for positive periodic solutions.

Inspired and motivated by the works mentioned above, we concentrate on the existence of periodic solutions for the third-order nonlinear delay differential equation

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) = f(t, x(t), x(t - \tau(t))) + c(t)x'(t - \tau(t)), \quad (5.1)$$

where p, q, r are continuous real-valued functions. The function $c : \mathbb{R} \longrightarrow \mathbb{R}$ is continuously differentiable, $\tau : \mathbb{R} \longrightarrow \mathbb{R}^+$ is twice continuously differentiable and $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous in their respective arguments. To show the existence of periodic and positive periodic solutions, we transform (5.1) into an integral equation and then use Krasnoselskii's fixed point theorem. The obtained integral equation splits in the sum of two mappings, one is a contraction and the other is compact. We also obtain the existence of a unique periodic solution of (5.1) by employing the contraction mapping principle as the basic mathematical tool.

The organization of this chapter is as follows. In section 2, we introduce some notations and lemmas, and state some preliminary results needed in later section, then we give the Green's function of (5.1), which plays an important role in this chapter. In section 3 and 4, we present our main results on existence and uniqueness.

In this chapter, we give the assumptions as follows that will be used in the main results.

(h1) There exist differentiable positive T-periodic functions a_1 and a_2 and a positive real constant ρ such that

$$\begin{cases} a_1(t) + \rho = p(t), \\ a_1'(t) + a_2(t) + \rho a_1(t) = q(t), \\ a_2'(t) + \rho a_2(t) = r(t). \end{cases}$$

(h2) $p, q, r, c \in C(\mathbb{R}, \mathbb{R}^+)$ are T-periodic functions with $\tau(t) \geq \tau^* > 0$, $\tau'(t) \neq 1$ for all $t \in [0, T]$ and

$$\int_0^T p(s)ds > \rho T, \quad \int_0^T q(s)ds > 0.$$

(h3) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is continuous for any $(t, x, y) \in \mathbb{R}^3$ and is T-periodic in t for all $(x, y) \in \mathbb{R}^2$.

(h4) There exist positive constants k_1 and k_2 such that

$$|f(t, x, y) - f(t, z, w)| \leq k_1 \|x - z\| + k_2 \|y - w\|.$$

5.2 Green's function of third-order differential equation

For $T > 0$, let P_T be the set of all continuous scalar functions x, periodic in t of period T. Then $(P_T, \|.\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|.$$

We consider

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) = h(t), \tag{5.2}$$

where h is a continuous T-periodic function. Obviously, by the condition $(h1)$, (5.2) is transformed into

$$\begin{cases} y'(t) + \rho y(t) = h(t), \\ x''(t) + a_1(t)x'(t) + a_2(t)x(t) = y(t). \end{cases}$$

Lemma 5.1 ([2]) *If $y, h \in P_T$, then y is a solution of equation*

$$y'(t) + \rho y(t) = h(t),$$

if only if

$$y(t) = \int_t^{t+T} G_1(t,s)h(s)ds, \tag{5.3}$$

where

$$G_1(t,s) = \frac{e^{\rho(s-t)}}{e^{\rho T} - 1}. \tag{5.4}$$

Corollary 5.1 *Green function G_1 satisfies the following properties*

$$G_1(t+T, s+T) = G_1(t,s),\ G_1(t,t+T) = G_1(t,t)e^{\rho T},$$
$$G_1(t+T,s) = G_1(t,s)e^{-\rho T},\ G_1(t,s+T) = G_1(t,s)e^{\rho T},$$
$$\frac{\partial}{\partial t}G_1(t,s) = -\rho G_1(t,s),$$
$$\frac{\partial}{\partial s}G_1(t,s) = \rho G_1(t,s),$$

and

$$m_1 \le G_1(t,s) \le M_1,$$

where

$$m_1 = \frac{1}{e^{\rho T} - 1},\ M_1 = \frac{e^{\rho T}}{e^{\rho T} - 1}.$$

Lemma 5.2 ([56]) *Suppose that $(h1)$ and $(h2)$ hold and*

$$\frac{R_1\left[e^{\int_0^T a_1(v)dv} - 1\right]}{Q_1 T} \ge 1, \tag{5.5}$$

where

$$R_1 = \max_{t\in[0,T]} \left| \int_t^{t+T} \frac{e^{\int_0^T a_1(v)dv}}{e^{\int_0^T a_1(v)dv} - 1} a_2(s)\, ds \right|,$$
$$Q_1 = \left(1 + e^{\int_0^T a_1(v)dv}\right)^2 R_1^2.$$

Then there are continuous T-periodic functions a and b such that

$$b(t) > 0, \quad \int_0^T a(v)dv > 0,$$

and

$$a(t) + b(t) = a_1(t), \; b'(t) + a(t)b(t) = a_2(t), \; \textit{for } t \in \mathbb{R}.$$

Lemma 5.3 ([75]) *Suppose the conditions of Lemma 5.2 hold and $y \in P_T$. Then the equation*

$$x''(t) + a_1(t)x'(t) + a_2(t)x(t) = y(t),$$

has a T periodic solution. Moreover, the periodic solution can be expressed by

$$x(t) = \int_t^{t+T} G_2(t,s)y(s)ds, \tag{5.6}$$

where

$$G_2(t,s) = \frac{\int_t^s e^{\int_t^v b(u)du + \int_v^s a(u)du} dv + \int_s^{t+T} e^{\int_t^v b(u)du + \int_v^{s+T} a(u)du} dv}{\left[e^{\int_0^T a(v)dv} - 1\right]\left[e^{\int_0^T b(v)dv} - 1\right]}. \tag{5.7}$$

Corollary 5.2 *Green's function G_2 satisfies the following proprieties*

$$G_2(t+T, s+T) = G_2(t,s), \; G_2(t, t+T) = G_2(t,t),$$
$$G_2(t+T, s) = e^{-\int_0^T b(v)dv}\left[G_2(t,s) + \int_t^{t+T} E(t,u)\, F(u,s)\, du\right],$$
$$\frac{\partial}{\partial t} G_2(t,s) = -b(t)G_2(t,s) + F(t,s),$$
$$\frac{\partial}{\partial s} G_2(t,s) = a(t)G_2(t,s) - E(t,s),$$

where

$$E(t,s) = \frac{e^{\int_t^s b(v)dv}}{e^{\int_0^T b(v)dv} - 1}, \quad F(t,s) = \frac{e^{\int_t^s a(v)dv}}{e^{\int_0^T a(v)dv} - 1}.$$

5.2. Green's function of third-order differential equation

Lemma 5.4 ([56]) *Let* $A = \int_0^T a_1(v)dv$ *and* $B = T^2 e^{\frac{1}{T}\int_0^T \ln(a_2(v))dv}$. *If*

$$A^2 \geq 4B, \tag{5.8}$$

then

$$\min\left\{\int_0^T a(v)dv, \int_0^T b(v)dv\right\} \geq \frac{1}{2}\left(A - \sqrt{A^2 - 4B}\right) = l,$$
$$\max\left\{\int_0^T a(v)dv, \int_0^T b(v)dv\right\} \leq \frac{1}{2}\left(A + \sqrt{A^2 - 4B}\right) = L.$$

Corollary 5.3 *Functions* G_2, E *and* F *satisfy*

$$m_2 \leq G_2(t,s) \leq M_2,\ E(t,s) \leq \frac{e^L}{e^l - 1},\ F(t,s) \leq e^L,$$

where

$$m_2 = \frac{T}{(e^L - 1)^2},\ M_2 = \frac{Te^{\int_0^T a_1(v)dv}}{(e^l - 1)^2}.$$

Lemma 5.5 ([26]) *Suppose the conditions of Lemma 5.2 hold and* $h \in P_T$. *Then the equation*

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) = h(t),$$

has a T*-periodic solution. Moreover, the periodic solution can be expressed by*

$$x(t) = \int_t^{t+T} G(t,s)h(s)ds, \tag{5.9}$$

where

$$G(t,s) = \int_t^{t+T} G_2(t,\sigma) G_1(\sigma,s) d\sigma. \tag{5.10}$$

Corollary 5.4 *Green's function* G *satisfies the following properties*

$$G(t+T, s+T) = G(t,s),\ G(t, t+T) = G(t,t)e^{\rho T},$$
$$\frac{\partial}{\partial t}G(t,s) = \left(e^{-\rho T} - 1\right) G_1(t,t) G_2(t,s) - b(t) G(t,s) + \int_t^{t+T} F(t,\sigma) G_1(\sigma,s) d\sigma,$$
$$\frac{\partial}{\partial s}G(t,s) = \rho G(t,s),$$

and

$$m \leq G(t,s) \leq M,$$

where

$$m = \frac{T^2}{(e^l - 1)^2 (e^{\rho T} - 1)},\ M = \frac{T^2 e^{\rho T + \int_0^T a(v)dv}}{(e^l - 1)^2 (e^{\rho T} - 1)}.$$

5.3 Periodic solutions

In this section we will study the existence and uniqueness of periodic solutions of (5.1).

Lemma 5.6 *Suppose* $(h1)-(h3)$ *and (5.5) hold. The function* $x\in P_T$ *is a solution of (5.1) if and only if*

$$\begin{aligned} x(t) &= Z(t)\left(e^{\rho T}-1\right)G(t,t)\,x(t-\tau(t)) \\ &\quad+\int_t^{t+T} G(t,s)\{-R(s)\,x(s-\tau(s))+f(s,x(s),x(s-\tau(s)))\}\,ds, \end{aligned} \tag{5.11}$$

where

$$R(s)=\frac{(c'(s)+c(s)\rho)(1-\tau'(s))+c(s)\tau''(s)}{(1-\tau'(s))^2}, \tag{5.12}$$

$$Z(t)=\frac{c(t)}{1-\tau'(t)}. \tag{5.13}$$

Proof. Let $x\in P_T$ be a solution of (5.1). From Lemma 5.5, we have

$$\begin{aligned} x(t) &= \int_t^{t+T} G(t,s)\left[f(s,x(s),x(s-\tau(s)))+c(s)\,x'(s-\tau(s))\right]ds \\ &= \int_t^{t+T} G(t,s)\,f(s,x(s),x(s-\tau(s)))\,ds+\int_t^{t+T} G(t,s)\,c(s)\,x'(s-\tau(s))\,ds. \end{aligned} \tag{5.14}$$

Performing an integration by parts, we get

$$\begin{aligned} &\int_t^{t+T} G(t,s)\,c(s)\,x'(s-\tau(s))\,ds \\ &= \int_t^{t+T} \frac{c(s)(1-\tau'(s))\,x'(s-\tau(s))}{1-\tau'(s)}G(t,s)\,ds \\ &= \int_t^{t+T} \frac{c(s)}{1-\tau'(s)}G(t,s)\,dx(s-\tau(s)) \\ &= \frac{c(s)}{1-\tau'(s)}G(t,s)\,x(s-\tau(s))\Big|_t^{t+T}-\int_t^{t+T}\frac{\partial}{\partial s}\left[\frac{c(s)}{1-\tau'(s)}G(t,s)\right]x(s-\tau(s))\,ds \\ &= Z(t)\left(e^{\rho T}-1\right)x(t-\tau(t))\,G(t,t)-\int_t^{t+T} R(s)\,G(t,s)\,x(s-\tau(s))\,ds, \end{aligned} \tag{5.15}$$

where R and Z are given by (5.12) and (5.13), respectively. We obtain (5.11) by substituting (5.15) in (5.14). Since each step is reversible, the converse follows easily. This completes the proof. ■

Define the mapping $H : P_T \to P_T$ by

$$\begin{aligned}&(H\varphi)(t)\\&= \int_t^{t+T} G(t,s)\{-R(s)\varphi(s-\tau(s)) + f(s,\varphi(s),\varphi(s-\tau(s)))\}\,ds\\&+ Z(t)\left(e^{\rho T}-1\right)G(t,t)\varphi(t-\tau(t)). \qquad (5.16)\end{aligned}$$

Note that to apply Krasnoselskii's fixed point theorem we need to construct two mappings, one is a contraction and the other is compact. Therefore, we express (5.16) as

$$(H\varphi)(t) = (H_1\varphi)(t) + (H_2\varphi)(t).$$

where $H_1, H_2 : P_T \to P_T$ are given by

$$(H_1\varphi)(t) = \int_t^{t+T} G(t,s)\{-R(s)\varphi(s-\tau(s)) + f(s,\varphi(s),\varphi(s-\tau(s)))\}\,ds, \quad (5.17)$$

and

$$(H_2\varphi)(t) = Z(t)\left(e^{\rho T}-1\right)G(t,t)\varphi(t-\tau(t)). \quad (5.18)$$

To simplify notation, we introduce the constants

$$\alpha = \max_{t\in[0,T]}|Z(t)|,\ \beta = \max_{t\in[0,T]}\{b(t)\},\ \delta = \frac{e^L}{e^l-1},\ \gamma = \max_{t\in[0,T]}|R(s)|,\ \upsilon = e^{\rho T}-1. \quad (5.19)$$

Lemma 5.7 *Suppose* $(h1)-(h4)$*, (5.5) and (5.8) hold. Then* $H_1 : P_T \to P_T$ *is continuous and compact.*

Proof. Let H_1 be defined by (5.17). Obviously, $H_1\varphi$ is continuous and it is easy to show that $(H_1\varphi)(t+T) = (H_1\varphi)(t)$. To see that H_1 is continuous, we let $\varphi, \psi \in P_T$. Given $\varepsilon > 0$, take $\theta = \varepsilon/N$ with $N = MT(\gamma + k_1 + k_2)$ where k_1 and k_2 are given by $(h4)$. Now, for $\|\varphi - \psi\| < \theta$, we obtain

$$\|H_1\varphi - H_1\psi\| \le M\int_t^{t+T}[\gamma\|\varphi-\psi\| + (k_1+k_2)\|\varphi-\psi\|]\,ds \le N\|\varphi-\psi\| < \varepsilon.$$

This proves that H_1 is continuous. To show that the image of H_1 is contained in a compact set, we consider $\mathbb{S} = \{\varphi \in P_T : \|\varphi\| \le \mathfrak{L}\}$, where $\mathfrak{L}$ is a fixed positive constant.

Let $\varphi_n \in \mathbb{S}$, where n is a positive integer. Observe that in view of $(h4)$ we have

$$\begin{aligned} |f(t,x,y)| &= |f(t,x,y) - f(t,0,0) + f(t,0,0)| \\ &\leq |f(t,x,y) - f(t,0,0)| + |f(t,0,0)| \\ &\leq k_1 \|x\| + k_2 \|y\| + \mu, \end{aligned}$$

where $\mu = \max_{t\in[0,T]} |f(t,0,0)|$. Hence if H_1 is given by (5.17) we obtain

$$\|H_1\varphi_n\| \leq D,$$

for some positive D. Next we calculate $\frac{d}{dt}(H_1\varphi_n)(t)$ and show that it is uniformly bounded. By making use of $(h1)$, $(h2)$ and $(h3)$ we obtain by taking the derivative in (5.17) that

$$\begin{aligned} &\frac{d}{dt}(H_1\varphi_n)(t) \\ &= \int_t^{t+T} \left[\left(e^{-\rho T} - 1\right) G_1(t,t)\, G_2(t,s) - b(t)\, G(t,s) + \int_t^{t+T} F(t,\sigma) G_1(\sigma,s)\, d\sigma \right] \\ &\times \left[-R(s)\,\varphi(s-\tau(s)) + f(s,\varphi(s),\varphi(s-\tau(s)))\right] ds. \end{aligned}$$

Consequently, by invoking $(h3)$ and (5.19), we obtain

$$\begin{aligned} \left|\frac{d}{dt}(H_1\varphi_n)(t)\right| &\leq \left[\left(1-e^{-\rho T}\right) M_1 M_2 + M\beta + M_1 \delta T\right] \left(\gamma \mathfrak{L} + (k_1+k_2)\,\mathfrak{L} + \mu\right) T \\ &\leq K, \end{aligned}$$

for some positive K. Hence the sequence $(H_1\varphi_n)$ is uniformly bounded and equicontinuous. The Ascoli-Arzela theorem implies that a subsequence $\left(H_1\varphi_{n_k}\right)$ of $(H_1\varphi_n)$ converges uniformly to continuous T-periodic function. Thus H_1 is continuous and $H_1(\mathbb{S})$ is contained in a compact subset of P_T. ■

Lemma 5.8 *If H_2 is given by (5.18) with*

$$\alpha \upsilon M < 1, \tag{5.20}$$

then $H_2 : P_T \to P_T$ is a contraction.

Proof. Let H_2 be defined by (5.18). It is easy to show that $(H_2\varphi)(t+T) = (H_2\varphi)(t)$. To see that H_2 is a contraction. Let $\varphi, \psi \in P_T$ we have

$$\|H_2\varphi - H_2\psi\| = \sup_{t\in[0,T]} |(H_2\varphi)(t) - (H_2\psi)(t)| \leq \alpha v M \|\varphi - \psi\|.$$

Hence $H_2 : P_T \to P_T$ is a contraction. ■

Theorem 5.1 *Let α and γ be given by (5.19). Suppose that conditions $(h1)-(h4)$, (5.5), (5.8) and (5.20) hold. Suppose there exist a positive constant J satisfying the inequality*

$$\alpha v M J + (\gamma J + (k_1 + k_2) J + \mu) MT \leq J.$$

Then (5.1) has a solution $x \in P_T$ such that $\|x\| \leq J$.

Proof. Define $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leq J\}$. By Lemma 5.7, the operator $H_1 : \mathbb{M} \to P_T$ is compact and continuous. Also, from Lemma 5.8, the operator $H_2 : \mathbb{M} \to P_T$ is a contraction. Conditions (ii) and (iii) of Krasnoselskii theorem are satisfied. We need to show that condition (i) is fulfilled. To this end, let $\varphi, \psi \in \mathbb{M}$. Then

$$\begin{aligned}
&|(H_1\varphi)(t) + (H_2\psi)(t)| \\
&\leq M \int_t^{t+T} [\gamma \|\varphi\| + (k_1 + k_2)\|\varphi\| + \mu]\, ds + \alpha v M \|\psi\| \\
&\leq \alpha v M J + (\gamma J + (k_1 + k_2) J + \mu) MT \leq J.
\end{aligned}$$

Thus $\|H_1\varphi + H_2\psi\| \leq J$ and so $H_1\varphi + H_2\psi \in \mathbb{M}$. All the conditions of Krasnoselskii theorem are satisfied and consequently the operator H defined in (5.16) has a fixed point in $\mathbb{M}$. By Lemma 5.6 this fixed point is a solution of (5.1) and the proof is complete. ■

Theorem 5.2 *Let α and γ be given by (5.19). Suppose that conditions $(h1)-(h4)$, (5.5) and (5.8) hold. If*

$$\alpha v M + (\gamma + (k_1 + k_2)) MT < 1,$$

then (5.1) has a unique T-periodic solution.

Proof. Let the mapping H be given by (5.16). For $\varphi, \psi \in P_T$, we have

$$\begin{aligned}
&|(H\varphi)(t) - (H\psi)(t)| \\
&\leq M \int_t^{t+T} [\gamma \|\varphi - \psi\| + (k_1 + k_2)\|\varphi - \psi\|]\, ds + \alpha v M \|\varphi - \psi\|.
\end{aligned}$$

Hence

$$\|H\varphi - H\psi\| \leq [\alpha\upsilon M + (\gamma + (k_1 + k_2)) MT] \|\varphi - \psi\| .$$

By the contraction mapping principle, H has a fixed point in P_T and by Lemma 5.6, this fixed point is a solution of (5.1). The proof is complete. ■

Example 5.1 Consider the third-order nonlinear delay differential equation

$$\begin{aligned} & x'''(t) + 10.125x''(t) + 25.25x'(t) + 3x(t) \\ & = \frac{1}{5}\sin t + \frac{1}{20}\sin(x(t)) + \frac{1}{40}\cos(x(t-2\pi)) + 0.01\sin(t)\,x'(t-2\pi). \end{aligned} \tag{5.21}$$

Then

$$T = 2\pi,\ p(t) = 10.125,\ q(t) = 25.25,\ r(t) = 3,\ \tau(t) = 2\pi,\ c(t) = 0.01\sin t,$$
$$f(t, x, y) = \frac{1}{5}\sin t + \frac{1}{20}\sin(x) + \frac{1}{40}\cos(y).$$

Doing straightforward computations, we obtain

$$a(t) = 4,\ b(t) = 6,\ a_1(t) = 10,\ a_2(t) = 24,\ R(t) = 0.01(\cos t + 4\sin t),$$
$$Z(t) = 0.01\sin t,\ \rho = 0.125,\ \alpha = 1,\ \beta = 4,\ \delta \simeq 2.868 \times 10^5,\ \gamma \simeq 0.041,$$
$$k_1 = 0.05,\ k_2 = 0.025,\ \mu = 0.2,\ m \simeq 4.893 \times 10^{-21},\ M \simeq 8.825 \times 10^{-10},\ J = 5.$$

All hypotheses of Theorem 5.1 are fulfilled and so the equation (5.21) has a 2π-periodic solution. Also, we have

$$\alpha\upsilon M + (\gamma + (k_1 + k_2)) MT \simeq 0.73 < 1,$$

then by Theorem 5.2, the equation (5.21) has a unique 2π-periodic solution.

5.4 Existence of positive periodic solutions

In this section we obtain the existence of a positive periodic solution of (5.1) by considering the two cases; $c(t) \geq 0$ and $c(t) \leq 0$ for all $t \in \mathbb{R}$. For a non-negative constant $\mathbf{K}$ and a positive constant $\mathbf{L}$ we define the set

$$\mathbb{D} = \{\varphi \in P_T : \mathbf{K} \leq \varphi \leq \mathbf{L}\},$$

which is a closed convex and bounded subset of the Banach space P_T.

In case $c(t) \geq 0$, we assume that there exists a positive constant η such that

$$\eta \leq Z(t), \text{ for all } t \in [0, T], \tag{5.22}$$

and for all $s \in [0, T]$, $x, y \in \mathbb{D}$

$$\frac{\mathbf{K}(1 - \eta m\upsilon)}{mT} \leq f(s, x, y) - R(s)y \leq \frac{\mathbf{L}(1 - \alpha M\upsilon)}{MT}. \tag{5.23}$$

Lemma 5.9 *Suppose* $(h1) - (h3)$, *(5.5), (5.8) and (5.20), (5.22) and (5.23) hold. Then* $H_1 : \mathbb{D} \to P_T$ *is compact.*

Proof. Let H_1 be defined by (5.17). Obviously, $H_1\varphi$ is continuous and it is easy to show that $(H_1\varphi)(t + T) = (H_1\varphi)(t)$. For $t \in [0, T]$ and for $\varphi \in \mathbb{D}$, we have

$$\begin{aligned}|(H_1\varphi)(t)| &= \left| \int_t^{t+T} G(t, s) \{f(s, \varphi(s), \varphi(s - \tau(s))) - R(s)\varphi(s - \tau(s))\} ds \right| \\ &\leq MT \frac{\mathbf{L}(1 - \alpha M\upsilon)}{MT} = \mathbf{L}(1 - \alpha M\upsilon).\end{aligned}$$

Thus from the estimation of $|(H_1\varphi)(t)|$ we have

$$\|H_1\varphi\| \leq \mathbf{L}(1 - \alpha M\upsilon).$$

This shows that $H_1(\mathbb{D})$ is uniformly bounded.

To show that $H_1(\mathbb{D})$ is equicontinuous, let $\varphi_n \in \mathbb{D}$, where n is a positive integer. Next we calculate $\frac{d}{dt}(H_1\varphi_n)(t)$ and show that it is uniformly bounded. By using $(h1)$, $(h2)$ and $(h3)$ we obtain by taking the derivative in (5.17) that

$$\begin{aligned}&\frac{d}{dt}(H_1\varphi_n)(t) \\ &= \int_t^{t+T} \left[\left(e^{-\rho T} - 1\right) G_1(t, t) G_2(t, s) - b(t) G(t, s) + \int_t^{t+T} F(t, \sigma) G_1(\sigma, s)\, d\sigma \right] \\ &\times [f(s, \varphi_n(s), \varphi_n(s - \tau(s))) - R(s)\varphi_n(s - \tau(s))]\, ds.\end{aligned}$$

Consequently, by invoking (5.19) and (5.23), we obtain

$$\left| \frac{d}{dt}(H_1\varphi_n)(t) \right| \leq \left[\left(1 - e^{-\rho T}\right) M_1 M_2 + M\beta + M_1 \delta T \right] \frac{\mathbf{L}(1 - \alpha M\upsilon)}{M} \leq D,$$

for some positive constant D. Hence the sequence $(H_1\varphi_n)$ is equicontinuous. The Ascoli-Arzela theorem implies that a subsequence $\left(H_1\varphi_{n_k}\right)$ of $(H_1\varphi_n)$ converges uniformly to a continuous T-periodic function. Thus H_1 is continuous and $H_1(\mathbb{D})$ is contained in a compact subset of $\mathbb{D}$. ■

Lemma 5.10 *Suppose that (5.20) holds. If H_2 is given by (5.18), then $H_2 : \mathbb{D} \to P_T$ is a contraction.*

Proof. Let H_2 be defined by (5.18). It is easy to show that $(H_2\varphi)(t+T) = (H_2\varphi)(t)$. Let $\varphi, \psi \in \mathbb{D}$, we have

$$\|H_2\varphi - H_2\psi\| = \sup_{t\in[0,T]} |(H_2\varphi)(t) - (H_2\psi)(t)| \leq \alpha\upsilon M \|\varphi - \psi\|.$$

Hence $H_2 : \mathbb{D} \to P_T$ is a contraction by (5.20). ■

Theorem 5.3 *Suppose that conditions $(h1)-(h3)$, (5.5), (5.8), (5.20), (5.22) and (5.23) hold. Then equation (5.1) has a positive T-periodic solution x in the subset $\mathbb{D}$.*

Proof. By Lemma 5.9, the operator $H_1 : \mathbb{D} \to P_T$ is compact and continuous. Also, from Lemma 5.10, the operator $H_2 : \mathbb{D} \to P_T$ is a contraction. Moreover, if $\varphi, \psi \in \mathbb{D}$, we see that

$$\begin{aligned}
&(H_2\psi)(t) + (H_1\varphi)(t) \\
&= \upsilon Z(t) G(t,t) \varphi(t-\tau(t)) \\
&+ \int_t^{t+T} G(t,s) \{f(s,\varphi(s),\varphi(s-\tau(s))) - R(s)\varphi(s-\tau(s))\} ds \\
&\leq \upsilon\alpha M\mathbf{L} + \mathbf{L}(1-\alpha M\upsilon) = \mathbf{L}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
&(H_2\psi)(t) + (H_1\varphi)(t) \\
&= \upsilon Z(t) G(t,t) \varphi(t-\tau(t)) \\
&+ \int_t^{t+T} G(t,s) \{f(s,\varphi(s),\varphi(s-\tau(s))) - R(s)\varphi(s-\tau(s))\} ds \\
&\geq \upsilon\alpha m\mathbf{K} + \mathbf{K}(1-\alpha m\upsilon) = \mathbf{K}.
\end{aligned}$$

This shows that $H_2\psi + H_1\varphi \in \mathbb{D}$. Clearly, all the Hypotheses of Theorem 2.4, are satisfied. Thus there exists a fixed point $x \in \mathbb{D}$ such that $x = H_1\psi + H_2\varphi$. By Lemma 5.6 this fixed point is a solution of (5.1) and the proof is complete. ■

In the case $c(t) \leq 0$, we substitute conditions (5.22), (5.20) and (5.23) with the following conditions respectively. We assume that there exist a negative constant z_1 and a non-positive constant z_2 such that

$$z_1 \leq Z(t) \leq z_2, \text{ for all } t \in [0, T], \tag{5.24}$$

$$-z_1 M\upsilon < 1, \tag{5.25}$$

and for all $s \in [0, T]$, $x, y \in \mathbb{D}$

$$\frac{\mathbf{K} - z_1 M\upsilon \mathbf{L}}{mT} \leq f(s, x, y) - R(s)\, y \leq \frac{\mathbf{L} - z_2 m\upsilon \mathbf{K}}{MT}. \tag{5.26}$$

Theorem 5.4 *Suppose that conditions* $(h1) - (h3)$*, (5.5), (5.8) and (5.24)-(5.26) hold. Then equation (5.1) has a positive* T*-periodic solution* x *in the subset* $\mathbb{D}$*.*

The proof follows along the lines of Theorem 5.3, and hence we omit it.

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