PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA

MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH
MOHAMED BOUDIAF UNIVERSITY - M’SILA

Faculty : Mathematics and Computer Science

Department : Mathematics

N° :………………………………

1985
جامعة محمد بوضياف - المسيلة
Université Mohamed Boudiaf - M'sila

THESIS

In order to obtain

Doctorate degree in science

SPECIALTY : Mathematics

OPTION : Functional and Numerical Analysis

Presented by:

BENYOUSSEF. Soufiane

Theme:

Numerical treatment of integro-differential equations by spectral collocation methods

Presented in …/.../... in view of the composed jury

Mostefa NADIR	Prof. University of M'sila	President
Azedine RAHMOUNE	Prof . University of B. Bou Arreridj	Supervisor
Abdelkader GASMI	Prof. University of M'sila	Examiner
Abdelbaki MEROUANI	Prof. University of Setif 1	Examiner
Toufik SERRAR	Prof. University of Setif 1	Examiner
Bachir GAGUI	MCA. University of M'sila	Examiner

Dedications

⋆ ***To my Parents Aicha and Saad*** ⋆
⋆ ***My Wife and my Children Najoua, Maissa, Youcef and Yaakoub*** ⋆
⋆ ***To my Brothers and Sisters*** ⋆

acknowledgment

My first thanks go to Mr. prof. Azedine Rahmoune from the University of Bordj Bou Arreridj for having accepted to supervise the work of this thesis, for his kindness, his goodwill, his availability and his patience as well as these wise guidance and guidance.

Special thanks to Mr. prof. Mostefa Nadire, who kindly accepted to chair the Jury, I thus express my warm thanks to Mrs. Profs Abdelkader Gasmi, Abdelbaki Merouani, Toufik Serrar and Bachir Gagui for having accepted be members of the Jury. I also want to thank my comrades Mrs. Abdelhake Mortari and Rachide Lamri.

Finally, I would also like to thank my comrades and my friends who supported me morally and to anyone who participated directly or indirectly in the realization of this work.

Beyond all of them, a special thought goes to my family who have supported me during all these years to them I dedicate this work.

ملخص

في السنوات الأخيرة، لعبت طرق الحل التقريبي للمعادلات التكاملية التفاضلية دورا مهما للغاية في العديد من المجالات العلمية مثل الكيمياء و البيولوجيا و الفيزياء الرياضية.

مع ميزة آلات الحوسبة الرقمية، بما في ذلك أجهزة الكمبيوتر، أصبحت هذه الطرق الآن أداة أساسية للبحث في مختلف المسائل المهمة لاستيعابنا للظواهر العلمية التي كان من الصعب أو من المستحيل حلها في الماضي، لهذا سنهتم في هذا العمل بطرق التجميع الطيفي لحل المعادلات التكاملية التفاضلية .

الكلمات المفتاحية : طرق التجميع الطيفية، كثيرات الحدود المتعامدة، المعادلات التفاضلية التكاملية، تقدير الخطأ.

Abstract

In recent years, the approximate solution methods for integro-differential equations play a very important role in many scientific fields such as chemistry, biology and mathematical physics. With the advantage of digital computing machines, including computers, these methods have now become an essential tool for investigation in various fundamental problems of our assimilation of scientific phenomena that are difficult or impossible to solve in the past . In this sense, our focus will be on the spectral collocation methods for solving integro-differential equations.

Keywords : Spectral collocation methods, orthogonal polynomials, integro-differential equations, errors estimation.

Résumé

Durant ces dernières années, les méthodes de résolution approchée des équations intégro-diffrentielles jouent un rôle très important dans plusieurs domaines scientifiques tels que la chimie, la biologie et la physique mathématique. Avec l'avantage des machines de calcul numérique, notamment les ordinateurs, ces méthodes sont devenues aujourd'hui un outil essentiel pour l'investigation dans les différents problèmes fondamentaux de notre assimilation des phénomènes scientifiques qui sont difficiles, voir impossible résoudre dans le passé. Dans ce sens, nous allons s'intéresser aux méthodes spectrales de collocation pour la résolution des équations intégro-différentielles.

Mots clés: Méthodes spectrales de collocation, polynômes orthogonaux, quations intégro-différentielles, estimation des erreurs.

Contents

List of Figures

List of Tables

Notations

IDEs: Integro-differential equations

ODE: Ordinary differential equation

PDE: Partial differential equation

$\mathbf{R}_N$: Residual functions

$\langle .,. \rangle$: Scalar product

$\chi = L^2(I)$: Hilbert space provided with a scalar product $\langle .,. \rangle$

χ_N: Sequence of finite dimensional subspaces of a Hilbert space χ

Φ: Orthogonal polynomials basis vector

$\mathcal{P}_N$: Projection operator

$H^m(-1,1)$: Sobolev space

$P_n(x)$: Legendre polynomials

$T_n(x)$: Tchybechev polynomials of the first kind

$\omega(x)$: Weight function

$\delta_{n,m}$: Kronecker function

$\theta(y)$: Mapping function on the interval from interval $[-1,1]$ to $[a,b]$

$u(x)$: Exact solution of IDEs

$u_N(x)$: Approximate solution of $u(x)$

$\mathcal{I}_N$: Legendre–Gauss interpolation operator

σ_i: Legendre-Gauss points

ω_i: Christoffel numbers

Λ : Interval $[0, \infty)$

$\varphi_s(y)$: Mapping from the interval I to Λ with the scaled s

$\varphi_s^{-1}(y)$: Inverse Mapping

$L_{n,s}(x)$: Legendre functions

ω_s: Weight Legendre functions

$R_n(x)$: Rational Legendre functions

$\mathcal{L}_n(x)$: Logarithmic Legendre functions

$E_n(x)$: Exponential Legendre functions

$\omega_r(x)$, $\omega_l(x)$ and $\omega_e(x)$: Weight functions

$f(x)$: Free term in the IDEs

$L^2_{\rho_s}(\Lambda)$: Hilbert space provided in the interval Λ

$\xi_{N,j}$ and $\omega_{N,j}$: Legendre-Gauss nodes and weights on the interval $[-1, 1]$

$\varsigma_{s,N,j}$ and $\rho_{s,N,j}$: Legendre-Gauss nodes and weights on the interval $[0, \infty)$

$\mathcal{I}_{s,N}$: Mapped Legendre-Gauss interpolation operator

$\mathcal{C}(I)$: Set of continues functions on the interval I

RLC: Rational Legendre collocation

LLC: Logarithmic Legendre collocation

ELC: Exponential Legendre collocation

$L_n^{(\alpha)}(x)$: Generalized Laguerre polynomials

Introduction

This work deals with one of the most applied problems in the engineering sciences. It is concerned with the integro-differential equations (IDEs) where both differential and integral operators will appear in the same equation. This type of equations was introduced by Volterra for the first time in the early 1900. Integro-differential equations appear in many fields of scientific and engineering, such as in heat and mass transfer theory, electric circuit problems, electromagnetic theory, radiative transfer, fluid dynamics, neuron transport theory, neutron diffusion, and biological species coexisting together with increasing and decreasing rates of generating. Applications of the integro-differential equations in electromagnetic theory and dispersive waves and ocean circulations are enormous ([63] [17] [1] [19] [14][64]).

However, in practice, the IDEs are too difficult to solve analytically, so this is why many numerical methods have been used to solve the IDEs such problems [[10],[67] and [51]] where these methods play a very important role in various scientific fields. With the advantage of digital computing machines, especially computers, and these methods have become today an essential tool for the investigation of the various fundamental problems of our assimilation of scientific phenomena which are difficult, namely impossible to solve in the past. These methods include: Adomian decomposition [23], variational iteration method [52], homotopy analysis method [20], Chebyshev and Taylor collocation [62], Haar wavelet [35], sinc-collocation method [3], Tau method [61], Taylor's series expansion [55],[53], hybrid Legendre and Block Pulse functions [34], integral mean value [68],Legendre-Galerkin, Chebyshev-Galerkin method [51] [24], and indirect RBF method [31].

Integro-differential equations have been widely treated in the bounded interval while in the unbounded interval they have been treated very narrowly that's why our main objective of the present work is to extend the Legendre spectral method to a class of singular Fredholm integro-differential equations on the half-line our thesis is divided into four chapters .

The first chapter is an introduction to the terminology and classification of integro-differential equations, which aims to familiarize the reader of this thesis with the concept of integro-differential equation. Thus, we expose some typical models to see where such equations come from, and mainly illustrate their link with differential equations, we also present some spectral methods namely the collocation method, Galerkin and Tau.

The second chapter is devoted to the Fredholm IDEs defined on bounded intrerval, where we apply the two spectral collocation methods and Galerkin methods and the orthogonal polynomial used is Legendre polynomial, so we compare the results obtained with some famous methods.

in the third chapter we introduce mapped Legendre functions which are orthogonal on the positive semi infinite interval, to obtain these last functions we use three mappings namely algebric, logarithmic and exponenetiel which helps us to pass from the interval $[0, \infty)$ to $[-1, 1]$ and vice versa, so we also study the properties of each mapping.

In the last chapter matrix method is presented to approximate the solution of Fredholm integro-differential equation on the semi infinite interval $[0, \infty)$ using Laguerre polynomials, after that all the results obtained in chapter two and three is used for transform the last equation into a singular equation on the finite interval $[-1, 1]$, by means of a suitable family of one-to-one mappings, and then apply the Legendre spectral collocation method to solve the resulting equation, finally some numerical results are given to clarify and demonstrate the feasibility of the proposed method .

Chapter 1

Integro-differential equations and spectral methods

1.1 Classification of integro-differential equations

Our goal in this first chapter is to familiarize the concept of integro-differential equation, we will therefore present the form and classification of integro-differential. we will also discuss the origin and usefulness of such equations, we will briefly expose some typical models, representative of a more general variety, and we will focus on spectral methods such as Galerkin methods, tau and collocation or pseudospectral methods .

An IDE is an equation composed of two integral and differential operations that involve the unknown function u, where the unknown is generally a function of one or more variables.

We are going to be much more interested in the linear IDEs the most famous of the form

$$\sum_{k=0}^{m} \nu_k(x)u^{(k)}(x) - \lambda \int_a^b k(x,t)u(t)dt = f(x), \tag{1.1}$$

Under mixed conditions:

$$\sum_{k=0}^{m-1} a_{lk}u^{(k)}(a) + b_{lk}u^{(k)}(b) = \alpha_l, \quad l = 0, 1, ..., m-1 \tag{1.2}$$

In these equations the function u is the unknown, the function $k(x,t)$ which is called kernel with the free term f are given.

λ is generally complex, it plays a crucial role in practical applications, is usually composed of physical quantities.

1.1.1 Classification and terminology

Before going into details it is useful to mention the four basic characteristics describe their overall structure

1. **Limits of integration**
   There are three major types of IDEs

   - **a)** If the limits of integration are fixed, then the IDEs is called Fredholm IDEs as we have in (1.1)
   - **b)** If the limits of integration are undefined, then the IDEs is called Volterra IDEs
   - **c)** If the two operators of the integration of Fredholm and Volterra consist then the IDEs is called of Fredholm-Volterra IDEs

2. **Order of IDEs**
   The order of a IDE is the highest derivative order that appears in the differential operator

3. **Linear or nonlinear**
   The IDEs (1.1) is linear, if the one or both integral or differential operator are nonlinear the IDEs is to be nonlinear

4. **Number of variable of unknown function u**
   An IDEs is said to be ordinary if the unknown function depends on a unique independent variable, then if it depends on two or more independent variables the IDEs is said to be partial

5. **Singular**
   An IDEs is said to be singular if one or both of the following hypotheses consist

   - **a)** One or both of the limits of integration are infinite
   - **b)** The kernel becomes infinite in the neighborhood of one or more points of the integration interval

   The weakly singular kernel of the form

   $$k(x,t) = \frac{g(x,t)}{|x-t|^{\alpha}}, \quad 0 < \alpha < 1 \tag{1.3}$$

   Where α is given and $g(x,t)$ a bounded function, is an example of an unbounded kernel, which requires special treatment. However, an IDEs defined with a kernel of type (1.3) is called a weakly singular IDEs, due to the place it occupies in different fields of application. Also, it is imperative to note that the kernel with logarithmic singularity

   $$k(x,t) = g(x,t) \ln |x-t|$$

   Where g is bounded, can be considered as a weakly singular kernel, since it can be written in the form

   $$k(x,t) = \frac{g(x,t)|x-t|^{\epsilon} \ln |x-t|}{|x-t|^{\epsilon}}$$

Whose numerator is bounded for all $\epsilon > 0$.
As an example of a kernel that has a strong singularity, we consider the Cauchy kernel, it is characterized by

$$k(x,t) = \frac{g(x,t)}{x-t}$$

Where g is always bounded

1.2 Genesis and formulation of integro-differential equations

Many problems in applied science lead to IDEs in a natural way. As a result, these equations emerge as proficient mathematical tools in modeling phenomena and processes arising in these areas of research. We briefly see some real mathematical models and physical that such equations are essues.
IDEs can be clearly seen when we convert the differential equation to an integral equation by using Leibnitz rule. The IDEs can be viewed in this case as an intermediate stage when finding an equivalent Volterra integral equation to the given differential equation.

1.2.1 Hear and mass transfer theory

The generalized linear integro-differential heat equation

$$C\frac{\partial^2 T(x,t)}{\partial t^2} + \beta(0)\frac{\partial T(x,t)}{\partial t} + \int_0^{+\infty} \beta^{'}(s)\frac{\partial T(x,t-s)}{\partial t}ds$$
$$- k(0).\Delta T(\vec{x},t) - \int_0^{+\infty} k^{'}(s)\Delta T(x,t-s)ds = \dot{Q}(x,t) \tag{1.4}$$

Equation 1.4 is referred to as the Gurtin–Pipkin equation Where:
$\vec{x}$: Vector of space variable, and t: time
$\dot{Q}(x,t)$: External heat source
$T(x,t)$: Temperature
$\beta(s)$: Internal energy relaxation function
$k(s)$: Heat flux relaxation function If $\beta^{'}(s) \equiv 0$, $k^{'}(s) \equiv 0$, and $\dot{Q}(x,t) \equiv 0$ in equation 1.4 then we obtain the hyperbolic Cattaneo–Maxwell equation

$$\frac{\partial T(x,t)}{\partial t} + \tau_r\frac{\partial^2 T(x,t)}{\partial t^t} = a(0).\Delta T(x,t)$$

which characterizes a wave process of heat propagation. Here $\tau_r = C/\beta(0)$ is the Cattaneo–Maxwell relaxation time, and $a(0) = k(0)/\beta(0)$.
If $\beta(s) \equiv 0$, then the equation 1.4 becomes

$$C\frac{\partial^2 T(x,t)}{\partial t^2} - k(0).\Delta T(\vec{x},t) - \int_0^{+\infty} k^{'}(s).\Delta T(x,t-s)ds = \dot{Q}(x,t) \tag{1.5}$$

The resulting Eq. 1.5 also describes one-dimensional longitudinal vibrations of a viscoelastic rod .
By integrating Eq. 1.5 with respect to t, we obtain the first-order integro-differential equation

$$C\frac{\partial T(x,t)}{\partial t} - \int_0^{+\infty} k(s).\Delta T(x,t-s)ds = \dot{Q}(x,t)$$

which is also called the Gurtin–Pipkin equation

1.2.2 Models for spreading and infectious disease

Invasion of disease into new territory is a worldwide problem, Traditionally, contact and dispersal have been modeled with local operators, resulting in reaction–diffusion equations. In ecology, reaction–diffusion models may underestimate speeds of invasion. One solution to underestimating speeds of invasion has been the use of integral operators instead of diffusion operators, resulting in integro-differential or integro-difference equations. These models incorporate detailed information about small-scale contact or the dispersal process to predict large-scale effects more accurately.

1.2.2.1 Distributed-contacts model

The governing equations (Kendall, 1957, 1965; Mollison, 1972):

$$\frac{\partial I}{\partial t} = \beta(\int_\Omega k(x-y)I(y,t)dy)(N-I)$$

and $S(x,t) + I(x,t) = N$.
x: location in some domain Ω, t: time
$S(x,t)$: densities of susceptible individuals
$I(x,t)$: densities of infectious individuals
β: infection rate
$k(x,y)$: density function for the proportion of infective at y that contact susceptible at x with $k(x,y) \geq 0, \quad \int_\Omega k(x,y)dx = 1$
N: supposed constant because there is no birth on death

1.2.2.2 Distributed-infective model

$$\frac{\partial I}{\partial t} = \beta I(N-I) - \int_\Omega D(x)k(y,x)I(x,t)dy + \int_\Omega D(y)k(x,y)I(y,t)dy \tag{1.6}$$

$k(x,y)$: density function that prescribes the proportion of infective leaving y that go to x
$D(x)$: rate at which infective individuals move from x to some new location in Ω.
For convenience, we take $D \geq 0$ to be constant, integro-differential equation 1.6 now simplifies to

$$\frac{\partial I}{\partial t} = \beta I(N-I) - DI + D\int_\Omega k(x,y)I(y,t)dy$$

1.2.3 Radiative transfer problems

Radiative transfer problems typically involve scattering, which implies a source function that itself depends on the radiation field. This leads mathematically to an integro-differential equation of transfer. For these cases of scattering, the formal solution does not provide an explicit solution, although it may be used to re-formulate the problem as an integral equation .

Much of the character of general radiative transfer problems already appears in what is perhaps the simplest example, the case of unpolarized radiation with isotropic scattering in plane-parallel geometry with axial symmetry. In this case the radiative transfer equation is

$$\mu\frac{\partial I(\tau,\mu)}{\partial\tau} = I(\tau,\mu) - \frac{\varpi_0}{2}\int_{-1}^{1} I(\tau,\mu^{'})d\mu^{'}$$

I: the specific intensity

τ: optical depth

$\mu = \cos\theta$: θ is the angle of ray with respect to the outward normal

ϖ_0: is called the single scattering albedo

For the more general case of anisotropic scattering, the transfer equation takes the form

$$\mu\frac{\partial I(\tau,\mu,\varphi)}{\partial\tau} = I(\tau,\mu,\varphi) - \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} p(\mu,\varphi;\mu^{'},\varphi^{'})I(\tau,\mu^{'},\varphi^{'})d\mu^{'}d\varphi^{'}$$

φ: azimuthal angle.

$p(\mu,\varphi;\mu^{'},\varphi^{'})$: the phase function describes the scattering from direction $(\mu^{'},\varphi^{'})$ into direction (μ,φ).

1.2.4 The LRC series circuit

Inductance, L, resistance, R, and capacitance, C, are the building blocks of basic electrical systems studied in first-year undergraduate engineering. A voltage (the input) applied to such a circuit containing these elements will result in a current flow (the output, or response) and a change in the charge on the capacitor.

A current $i(t)$ will flow in the circuit when a voltage $e(t)$ is applied, Using $i(t)$ and $e(t)$ indicates that current and voltage vary with time (i.e. they are timedependent variables). Of course this doesn't preclude the possibility that the voltage, for example, could be constant (e.g. the voltage source is a battery).

The drop in voltage (i.e. potential drop) across the resistance is iR (from Ohm's Law), across L it is $L\frac{di}{dt}$ and across C it is $\frac{1}{C}\int_0^t idt$.

Kirchhoff's 2nd Law says that the sum (i.e. addition) of potential drops across all of the non-supply elements in the circuit equals the applied voltage of the supply. So,

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t idt = e(t)$$

and this an integro-differential equation.

it is possible to relate i and q using $i = \frac{dq}{dt}$ where q is the charge on the capacitor

Upon integration, this gives $q = \int_0^t i dt$ and upon differentiation, $\frac{di}{dt} = \frac{d^2q}{dt^2}$. So rewriting the above integro-differential equation in terms of q rather than i gives

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = e(t)$$

This is now a linear second-order differential equation with constant coefficients, which can be solved by a variety of methods to find the charge q, in terms of time t

1.2.5 Initial value problem of the third order

Suppose u satisfied

$$\begin{cases} u^{'''}(x) = \Psi(x, u(x)), \quad 0 < x < 1 \\ u(0) = u_0, u^{'}(0) = u_1, u^{''}(0) = u_2 \end{cases} \tag{1.7}$$

a first integration gives

$$u^{''}(x) = \int_0^x \Psi(t, u(t))dt + u_0^{''}, \quad 0 < x < 1$$

a second integration gives

$$u^{''}(x) = \int_0^x ds \int_0^s \Psi(t, u(t))dt + u_0^{''}x + u_0^{'}, \quad 0 < x < 1 \tag{1.8}$$

The simplification of the double integral in (1.8) follows from the relation if F is a continuous function for the two variables

$$\int_0^x ds \int_0^s F(s,t)dt = \int_0^x dt \int_t^x F(s,t)ds \tag{1.9}$$

If we assume that ψ is a continuous function with respect to the two variables, then (1.9) gives

$$\int_0^x \int_0^s \psi(t, u(t)) = \int_0^x (x-t)\psi(t, u(t))dt$$

and the integro-differential equation corresponding to (1.7) in its simple form is given by

$$u^{'}(x) = \int_0^x (x-t)\psi(t, u(t))dt + u_0^{''}x + u_0^{'}$$

it is a first order volterra integro-differential equation

1.2.6 Boundary Value Problems

We consider the boundary Value Problems of second order

$$\begin{cases} u^{''}(x) = \Psi(x, u(x)), \quad 0 < x < 1 \\ u(0) = u_0, u(1) = u_1 \end{cases} \tag{1.10}$$

We proceed in the same way as (1.7), we obtain

$$u^{'}(x) = \int_0^x \psi(t, u(t))dt + C \quad 0 \le x \le 1 \tag{1.11}$$

The difference between this and the previous calculation, being that the value $u^{'}(0) = C$ is not given and C must be determined by imposing the condition' $u(1) = 1$, which implies

$$C = u_1 - u_0 - \int_0^1 (1-t)\psi(t, u(t))dt$$

And therefore (1.11) can be written in the form

$$u^{'}(x) + \int_0^1 (1-t)\psi(t, u(t))dt = f(x)$$

Witch is Fredholm integro-differential equation of first order with

$$f(x) = u_1 - u_0 + \int_0^x \psi(t, u(t))dt$$

1.3 Spectral methods

Spectral methods are recent techniques to discretize EDP and EDO. Unlike traditional methods such as finite element and finite difference methods, spectral methods use a complete family (base) of global smooth functions and called trial functions are often orthogonal polynomials. The idea is to approximate the solution $u(x)$ by $u_N(x) = \sum_{k=0}^{N} c_k\phi_k$ with ϕ_k a basis of orthogonal polynomials, and then choose the coefficients of the sum in order to satisfy the deviation equation as well as possible. Depending on the boundary condition satisfied by the trial function and its relationship with the test function, we can distinguish three spectral methods: Galerkin method, Tau method and collocation method.

1.3.1 Collocation method

We consider the linear problem (1.1) and we assume that operator integral $\mathcal{K}$ where $(\mathcal{K}u)(x) = \int_a^b k(x,t)u(t)dt$ is compact on a Banach space χ into it self, To solve approximately the latter problem we choose a sequence of finite-dimensional subspaces $\chi_N \subset \chi$, with

$$\chi_N = span\{\phi_0, \phi_2, ..., \phi_N\} \Longrightarrow dim\chi_N = N+1$$

We expand the approximate solution as

$$u_N(x) = \sum_{j=0}^{N} c_j\phi_j(x) \in \chi_N \tag{1.12}$$

The collocation method forces the residual to vanish point wisely at a set of preassigned points. More precisely, let $\{x_0, x_1, ..., x_N\}$ be a set of distinct node points in the interval $I = [a, b]$. The collocation method for Eq (1.1) amounts to finding $u_N \in \chi_N$ such that the residual

$$\begin{aligned}\mathbf{R}_N(x) &= \sum_{k=0}^{m} \nu_k(x)u_N^{(k)}(x) - \lambda\int_a^b k(x,t)u_N(t)dt - f(x) \\ &= \sum_{j=0}^{N} c_j\Big\{\sum_{k=0}^{m} \nu_k(x)\phi_j^{(k)}(x) - \lambda\int_a^b k(x,t)\phi_j(t)dt\Big\} - f(x), \quad x \in I\end{aligned} \tag{1.13}$$

equal to zero at the collocation points, namely,

$$\mathbf{R}_N(x_i) = 0, \quad 0 \leq i \leq N \tag{1.14}$$

This leads to determining $\{c_j\}_{j=0}^N$ as the solution of the linear system

$$\sum_{j=0}^{N} c_j \Big\{ \sum_{k=0}^{m} \nu_k(x_i)\phi_j^{(k)}(x_i) - \lambda \int_a^b k(x_i,t)\phi_j(t)dt \Big\} = f(x_i), \quad 0 \leq i \leq N \tag{1.15}$$

With the mixed conditions

$$\sum_{j=0}^{N} c_j \Big\{ \sum_{k=0}^{m-1} a_{lk}(x_i)\phi_j^{(k)}(a) + b_{lk}(x_i)\phi_j^{(k)}(b) \Big\} = \alpha_l, \quad 0 \leq i \leq N \tag{1.16}$$

by adding the m equations of mixed conditions to the $(N+1)$ equations of the I-DE equation we get a system of $(N+m+1)$ linear equations which can be readily solved by using least square method.

1.3.2 Galerkin's method

Let $\chi = L^2(I)$ be a Hilbert space provided with a scalar product $\langle .,. \rangle$, we consider a sequence of subsequence $\chi_N \subset \chi$ of finite dimension, let $\{\phi_0, ..., \phi_N\}$ an orthonormal basis of χ_N, we search for a function $u_N \in \chi_N$ of the form (2.15) close to the exact solution of the original problem.
The idea is to minimize the error $\mathbf{R}_N$ to satisfy

$$\langle \mathbf{R}_N, \phi_i \rangle = 0, \quad 0 \leq i \leq N \tag{1.17}$$

This yields the linear system

$$\sum_{j=0}^{N} c_j \Big\{ \sum_{k=0}^{m} \langle \nu_k \phi_j^{(k)}, \phi_i \rangle - \lambda \langle \mathcal{K}\phi_j, \phi_i \rangle \Big\} = \langle f, \phi_i \rangle, \quad 0 \leq i \leq N \tag{1.18}$$

1.3.3 Tau method

Matrix of differential part

Let $\Phi = (\phi_0(x), \phi_1(x), ..., \phi_N(x))$ be an orthogonal polynomials basis vector given by

$$\Phi = \Pi^T X$$

With $X = (1, x, x^2, ..., x^N)$ and Π is a non singular upper triangular matrix, from (2.15) we can write

$$u_N(x) = C\Phi$$

$C = (c_0, c_1, ...c_N)$ the coefficients vector to be founded, the matrix of the differential part will be

$$D = CS\Phi \tag{1.19}$$

With

$$S = (\Pi^T)M(\Pi^T)^{-1}, \quad and \quad M = \sum_{k=0}^{m}(B^T)^k \nu_k(\mu)$$

$$\mu = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 2 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 3 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & N-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & N \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \tag{1.20}$$

Matrix of integral part

Consider the expansion of the kernel $k(x,t)$

$$k(x,t) = \sum_{i=0}^{N}\sum_{l=0}^{N} k_{il}\phi_i(x)\phi_l(t)$$

$$\begin{aligned} \int_a^b k(x,t)u(t)dt &= \sum_{j=0}^{N}\sum_{i=0}^{N}\sum_{l=0}^{N} k_{il}C_j\phi_i(x)\int_a^b \phi_l(t)\phi_j(t)dt \\ &= CK\Phi \end{aligned} \tag{1.21}$$

Where

$$K = \begin{pmatrix} \sum_{l=0}^{N} k_{0l}\gamma_{l0} & \dots & \dots & \sum_{l=0}^{N} k_{Nl}\gamma_{l0} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \sum_{l=0}^{N} k_{0l}\gamma_{lN} & \dots & \dots & \sum_{l=0}^{N} k_{Nl}\gamma_{lN} \end{pmatrix}$$

$$\gamma_{lj} = \int_a^b \phi_l(t)\phi_j(t)dt, \quad l,j = 0,1,...,N$$

Matrix of second member

Let the function of second member $f(x)$

$$\begin{aligned} f(x) &= \sum_{j=0}^{N} f_j\phi_j(x) \\ &= f\Phi \end{aligned} \tag{1.22}$$

The coefficient vector $f = (f_0, f_1, ..., f_N)$.

Matrix of mixed conditions

Replacing (2.15) in the left hand side of (1.2) it can be written as

$$\begin{aligned}\sum_{k=0}^{m-1}\left[a_{lk}u^{(k)}(a)+b_{lk}u^{(k)}(b)\right]=\alpha_l &= \sum_{j=0}^{N}c_j\sum_{k=0}^{m-1}\left[a_{lk}\phi^{(k)}(a)+b_{lk}\phi^{(k)}(b)\right]\\ &= C\mathcal{B}_l, \quad l=0,1,...,m-1 \end{aligned} \tag{1.23}$$

$$\mathcal{B}_l = \begin{pmatrix} a_{l0}\phi_0(a)+b_{l0}\phi_0(b) \\ \sum_{k=0}^{1}\left[a_{lk}\phi_1^{(k)}(a)+b_{lk}\phi_1^{(k)}(b)\right] \\ \vdots \\ \sum_{k=0}^{m-1}\left[a_{lk}\phi_{m-1}^{(k)}(a)+b_{lk}\phi_{m-1}^{(k)}(b)\right] \end{pmatrix}$$

We refer to η as the matrix representation of mixed conditions and $\mathcal{B}_l$ its lth column. The following relations for computing the elements of the matrix η can be deduced from

$$b_{il} = \sum_{k=0}^{m-1}\left[a_{lk}\phi_{i-1}^{(k)}(a)+b_{lk}\phi_{i-1}^{(k)}(b)\right], \quad i,l=0,1,...,m-1$$

We introduce $d = (\alpha_0, \alpha_1, ..., \alpha_{m-1})$, the vector that contains right hand sides of conditions

$$C\eta = d \tag{1.24}$$

It follow from (1.19), (1.21) and (1.22)

$$C(S-\lambda K) = f$$

We put

$$W = S - \lambda K$$

The coefficient of exact solution $u_N(x) = C\Phi$ of the problem (1.1), (1.2) satisfies the following algebraic system

$$\begin{cases} CW_i = f_i, & i=0,1,...,N \\ C\mathcal{B}_l = \alpha_l, & l=0,1,...,m-1 \end{cases} \tag{1.25}$$

Setting

$$\widetilde{W} = (\mathcal{B}_0, \mathcal{B}_1, ..., \mathcal{B}_{m-1}, K_0, K_1, ..., K_N)$$

And

$$\widetilde{f} = (\alpha_0, \alpha_1, ..., \alpha_{m-1}, f_0, f_1, ..., f_N)$$

We can write instead of (1.25)

$$C\widetilde{W} = \widetilde{f}$$

C can be founded and $u_N(x)$ as well .

1.4 Orthogonal projection and discussion of convergence

Let χ_N be a sequence of finite dimensional subspaces of a Hilbert space χ, and let u be an element of χ, of the form

$$u = \sum_{j=0}^{\infty} c_j \phi_j \tag{1.26}$$

Where, c_j are the coefficients of this expansion, and ϕ_j are the elements of an orthogonal system. Our goal is to find an approximate solution in χ_N using a sequence of projectors $\mathcal{P}_N : \chi \longrightarrow \chi_N$, in the form of the truncated series

$$\mathcal{P}_N\Big(\sum_{j=0}^{\infty} c_j \phi_j\Big) = \sum_{j=0}^{N} c_j \phi_j \tag{1.27}$$

By the weight function $w = w(x)$ on G (measurable set in Jordan's sense), the orthogonality is defined by

$$\int_G \phi_i(x)\phi_j(x)w(x)dx = \delta_{ij}$$

Then, the coefficients c_j in (1.26) are given by

$$c_j = \frac{1}{\|\phi_j\|_w^2} \int_G w(x)u(x)\phi_j(x)dx \tag{1.28}$$

With

$$\|\phi_j\|_w = \Big(\int_G \phi_j(x)\phi_j(x)w(x)\Big)^{\frac{1}{2}} \tag{1.29}$$

Let I be a bounded interval of $\mathbb{R}$, $\chi = L_w^2(I)$ and $\chi_N = \mathbb{P}_N$, the subspace of polynomials of degree at most N.

Theorem 1.1 *Let $u \in L_w^2(I)$ and $N \in \mathbb{N}$. Then $\mathcal{P}_N u$ is the best approximation within the meaning of (1.29), in another way*

$$\|u - \mathcal{P}_N u\|_{L_w^2} = \inf_{\psi \in \mathbb{P}_N} \|u - \psi\|_{L_w^2} \tag{1.30}$$

Proof — As $\psi \in \mathbb{P}_N$, there are coefficients c_j, $0 \le j \le N$ such that $\psi = \sum_{j=0}^{N} c_j \psi_j$. To minimize $\|u - \psi\|_{L_w^2}$ is equivalent to minimize $\|u - \psi\|_{L_w^2}^2$, we have

$$\begin{aligned} \frac{\partial}{\partial c_k}\|u - \psi\|_{L_w^2}^2 &= \frac{\partial}{\partial c_k}(\|u\|_{L_w^2}^2 - 2\sum_{j=0}^{N} c_j \langle u, \psi_j\rangle_{L_w^2} + \sum_{j=0}^{N} c_j^2 \|u\|_{L_w^2}^2) \\ &= -2\langle u - \psi_k\rangle_{L_w^2} + 2c_k\|\psi_k\|_{L_w^2}^2, \quad 0 \le k \le N \end{aligned}$$

The minimum is reached at the point where the derivative vanishes, so

$$c_k = \frac{\langle u, \psi_k\rangle_{L_w^2}}{\|\psi_k\|_{L_w^2}^2}, \quad 0 \le j \le N$$

which completes the proof ■

Theorem 1.2 *For all* $u \in L^2_w(I)$,

$$\lim_{N \longrightarrow \infty} \|u - \mathcal{P}_N u\| = 0 \tag{1.31}$$

Proof — See [16] ■

The development (1.26) is the basis of all projection methods. So in practice, note that several orthogonal polynomials can be used in an analogous way such as polynomials of Legendre and chebyshev, or others .

1.4.1 Legendre polynomials

we can use Legendre polynomials defined by the recurrence relation

$$\begin{cases} P_0(x) = 1 \\ P_1(x) = x \\ (n+1)P_{n+1}(x) = (2n+1)P_n(x) - nP_{n-1}(x), \quad n \geq 1 \end{cases} \tag{1.32}$$

$$\int_{-1}^{1} P_n(x)P_m(x)w(x)dx = \gamma_n \delta_{nm} \tag{1.33}$$

Which are orthogonal on $[-1,1]$, with $w(x) = 1$, and

$$\gamma_n = \|P_n\|_w^2 = \frac{2}{2n+1} \tag{1.34}$$

Let

$$H^m(-1,1) = \Big\{ u \in L^2(-1,1) : for \quad 0 \leq k \leq m, \quad \frac{\partial^k u}{\partial x^k} \in L^2(-1,1) \Big\} \tag{1.35}$$

$H^m(-1,1)$ provided with the scalar product

$$(u,\psi)_m = \sum_{k=0}^{m} \int_{-1}^{1} \frac{\partial^k u}{\partial x^k}(x) \frac{\partial^k \psi}{\partial x^k}(x)dx$$

is a Hilbert, called Sobolev space. The norm associated is

$$\|u\|_{H^m(-1,1)} = \Big(\sum_{k=0}^{m} \|\frac{\partial^k u}{\partial x^k}\|^2_{L^2(-1,1)} \Big)^{\frac{1}{2}}$$

Theorem 1.3 *Let* $u \in H^m(-1,1)$. *Then, the truncated Legendre series,* $\mathcal{P}_N(u(x)) = \sum_{j=0}^{N} c_j P_j(x)$ *is the best polynomial approximation of* $u(x)$ *within the meaning of the* L^2 *norm. In addition,* $\exists C > 0$, *such as*

$$\|u - \mathcal{P}_N u\|_{L^2(-1,1)} \leq CN^{-m} \|u\|_{H^m(-1,1)} \tag{1.36}$$

Proof — See[12] ■

1.4.2 Chebyshev polynomials

In a similar way to Legendre polynomials, we will use in this section the Chebyshev polynomials $T_N(x)$ of the first kind, defined by the recurrence relation

$$\begin{cases} T_0(x) = 1 \\ T_1(x) = x \\ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1 \end{cases} \tag{1.37}$$

Which are orthogonal on $[-1, 1]$, with $w(x) = (1 - x^2)^{-\frac{1}{2}}$. Likewise, Let

$$H_w^m(-1,1) = \Big\{u \in L_w^2(-1,1) : for \quad 0 \leq k \leq m, \quad \frac{\partial^k u}{\partial x^k} \in L_w^2(-1,1)\Big\}$$

It is recalled that the derivative $\frac{\partial^k u}{\partial x^k}$ is always taken within the meaning of the distributions.
The space $H_w^m(-1,1)$ provided with the scalar product

$$(u, \psi)_{m,w} = \sum_{k=0}^{m} \int_{-1}^{1} \frac{\partial^k u}{\partial x^k}(x) \frac{\partial^k \psi}{\partial x^k}(x) \frac{dx}{\sqrt{1-x^2}}$$

is in turn a Hilbert. The associated norm is given by

$$\|u\|_{H_w^m(-1,1)} = \Big(\sum_{k=0}^{m} \|\frac{\partial^k u}{\partial x^k}\|^2_{L_w^2(-1,1)}\Big)^{\frac{1}{2}}$$

Theorem 1.4 *Let* $u \in H_w^m(-1,1)$ *and* $\mathcal{P}_N(u(x)) = \sum_{j=0}^{N} c_j T_j(x)$ *the truncated Chebyshev series of* u*. So we have the estimate*

$$\|u - \mathcal{P}_N u\|_{L_w^2(-1,1)} \leq CN^{-m}\Big(\sum_{k=\min(m,N+1)}^{m} \|u^{(k)}\|^2_{L_w^2(-1,1)}\Big)^{\frac{1}{2}} \tag{1.38}$$

Proof — See[12] ■

Chapter 2

Fredholm IDEs on bounded interval

2.1 Legendre spectral collocation method

In this chapter we apply the two spectral collocation methods and Galerkin methods where the orthogonal polynomial used is Legendre polynomial, and we compare the results obtained with some famous methods.

Let us consider a family of one-to-one mappings between $y \in I = [-1, 1] \quad and \quad x \in [a, b]$ of the form

$$x = \theta(y) = \frac{b-a}{2} y + \frac{b+a}{2}, \quad y \in I \tag{2.1}$$

Such that

$$\frac{dx}{dy} = \theta^{'}(y) > 0, \quad with \quad , \theta(a) = -1, \quad \theta(b) = 1, \tag{2.2}$$

And we have

$$y = \theta^{-1}(x), \quad x \in [a, b] \tag{2.3}$$

For a given mapping $y = \theta^{-1}(x)$, we define the so-called mapped Legendre functions by

$$L_n(x) = P_n(\theta^{-1}(x)), \quad n = 0, 1, 2, ... \tag{2.4}$$

They are orthogonal in the interval $[a, b]$ with respect to the weight function

$$w_{a,b}(x) = \frac{dy}{dx} = \frac{2}{b-a}$$

With the orthogonality property

$$\int_a^b L_n(x) L_m(x) w_{a,b}(x) dx = \frac{2}{2n+1} \delta_{n,m} \tag{2.5}$$

Let us defined

$$L^2_{w_{a,b}}(a, b) = \Big\{ u : [a, b] \longrightarrow \mathbb{R}/u \quad is \quad measurable \quad and \quad \|u\|_{w_{a,b}} < \infty \Big\}$$

Where

$$\|u\|_{w_{a,b}}^2 = \int_a^b |u(x)|^2 w_{a,b} dx$$

is the norm induced by the inner product of the space $L^2_{w_{a,b}}$

$$\langle u, v \rangle_{w_{a,b}} = \int_a^b u(x)v(x)w_{a,b}(x)dx \tag{2.6}$$

Thus, $\{L_j\}_{j=0}^{\infty}$ denotes a system which is mutually orthogonal under the inner product given by (2.6), i.e.,

$$\langle L_n, L_m \rangle_{w_{a,b}} = \frac{2}{2n+1}\delta_{n,m}.$$

It is not hard to show that $\{L_j\}_{j=0}^{\infty}$ forms a complete basis in $L^2_{w_{a,b}}([a,b])$. Thus, for any function $u \in L^2_{w_{a,b}}([a,b])$, the following expansion holds

$$u(x) = \sum_{j=0}^{\infty} \widehat{u}_j L_j(x) \quad with \quad \widehat{u}_j = \frac{2j+1}{2}\int_a^b u(x)L_j(x)w_{a,b}dx \tag{2.7}$$

If $u(x)$ is truncated up to the $N+1$ terms, then it can be written as

$$u(x) \simeq u_N(x) = \sum_{j=0}^{N} \widehat{u}_j L_j(x) \tag{2.8}$$

Let the linear Fredholm IDE of second order on the bounded interval

$$\sum_{k=0}^{2} \nu_k(x)u^{(k)}(x) - \lambda \int_a^b k(x,t)u(t)dt = f(x) \tag{2.9}$$

Under the boundary conditions

$$u(a) = \alpha, \quad u(b) = \beta \tag{2.10}$$

Applying the transformation

$$x = \theta(y), \quad and \quad t = \theta(z)$$

We get

$$\sum_{k=0}^{2} H_k(y)U^{(k)}(y) - \lambda \int_{-1}^{1} K(y,z)U(z)\theta'(z)dz = F(y), \quad -1 \le y \le 1 \tag{2.11}$$

With

$$U(-1) = \alpha, \quad U(1) = \beta \tag{2.12}$$

Where $K(y,z) = k(\theta(y),\theta(z))$, $F(y) = f(\theta(y))$ *and* $H_k(y)$ are functions defined in the interval $-1 \le y, z \le 1$, with

$$\begin{cases} H_0(y) = \nu_0(\theta(y)) \\ H_1(y) = \frac{\nu_1(\theta(y))}{\theta'(y)} - \frac{\theta''(y)}{(\theta'(y))^3}\nu_2(\theta(y)) \\ H_2(y) = \frac{\nu_2(\theta(y))}{(\theta'(y))^2} \end{cases} \tag{2.13}$$

Let $\mathcal{I}_N : \mathcal{C}(I) \longrightarrow \mathcal{P}_N$ be the Legendre–Gauss interpolation operator. The Legendre spectral method consists in finding $U_N \in \mathcal{P}_N$

$$\sum_{k=0}^{2} H_k(y)U_N^{(k)}(y) - \lambda \int_{-1}^{1} K(y,z)U_N(z)\theta^{'}(z)dz = \mathcal{I}_N F(y), \quad -1 \leq y \leq 1 \tag{2.14}$$

If $U_N(y) = \sum_{j=0}^{N} c_j P_j(y)$ is an approximate solution of Eq.(2.11) using Legendre spectral method, then $u_N(x) = U_N(\theta^{-1}(x))$ an approximate solution of Eq.(2.9) using mapped Legendre spectral method. Indeed, it holds that

$$\widehat{u}_j = \frac{2j+1}{2}\int_a^b u(x)L_j(x)w_{a,b}(x)dx = \frac{2j+1}{j}\int_{-1}^{1} U(y)P_j(y)dy = c_j$$

Hence using (2.4), we obtain

$$u_N(x) = \sum_{j=0}^{N} \widehat{u}_j L_j(x) = \sum_{j=0}^{N} c_j P_j(\theta^{-1}(x)) = U_N(y) \tag{2.15}$$

2.1.1 Fundamental matrix relations

Let write the Eq. (2.14) in the form

$$D_N(y) - \lambda I_N(y) = F_N(y) \tag{2.16}$$

Where

$$D_N(y) = \sum_{k=0}^{2} H_k(y)U_N^{(k)}(y), \quad I_N(y) = \int_{-1}^{1} K(y,z)U_N(z)\theta^{'}(z)dz, \quad F_N(y) = \mathcal{I}_N F(y) \tag{2.17}$$

And

$$U_N(y) = \sum_{j=0}^{N} c_j P_j(y) = P(y)C, \quad -1 \leq y \leq 1 \tag{2.18}$$

With

$$C = [c_0, c_1, ..., c_N]^T, \quad P(y) = [P_0(y), P_1(y), ..., P_N(y)].$$

The derivative of the solution expressed by (2.18) can be written in the matrix form as follows

$$[U_N^k(y)] = P^{(k)}(y)C. \tag{2.19}$$

We can write $P(y)$ in the matrix form as follows

$$P(y) = Y(y)\Pi$$

Where

$$Y(y) = [1, y, y^2, ..., y^N],$$

And for odd values of N,

$$\Pi = \begin{pmatrix} \frac{1}{2^0 0!0!} & 0 & \frac{-2!}{2^2 1!1!} & \cdots & \cdots & 0 & \frac{(-1)^{\frac{N-1}{2}}(N-1)!}{2^{N-1}(\frac{N-1}{2})!(\frac{N-1}{2})!} & 0 \\ 0 & \frac{2!}{2^1 0!1!} & 0 & & & \frac{(-1)^{\frac{N-3}{2}}(N-1)!}{2^{N-2}(\frac{N-3}{2})!(\frac{N-1}{2})!} & 0 & \frac{(-1)^{\frac{N-1}{2}}(N+1)!}{2^{N}(\frac{N-1}{2})!(\frac{N+1}{2})!} \\ 0 & 0 & \frac{4!}{2^2 0!2!} & & & & \frac{(-1)^{\frac{N-3}{2}}(N+1)!}{2^{N-1}(\frac{N-3}{2})!(\frac{N+1}{2})!} & 0 \\ 0 & 0 & 0 & \ddots & & & & \frac{(-1)^{\frac{N-3}{2}}(N+3)!}{2^{N}(\frac{N-3}{2})!(\frac{N+3}{2})!} \\ \vdots & \vdots & & & & & & \\ \vdots & \vdots & & & & & & \vdots \\ \vdots & 0 & & & & & 0 & \vdots \\ 0 & 0 & & & & \ddots & \frac{-(2N-4)!}{2^{N-1}1!(N-2)!} & \\ 0 & 0 & & & & & 0 & \frac{-(2N-2)!}{2^{N}(1)!(N-1)!} \\ 0 & 0 & & & \cdots & 0 & \frac{(2N-2)!}{2^{N-1}0!(N-1)!} & 0 \\ 0 & 0 & & & \cdots & 0 & 0 & \frac{(2N)!}{2^{N}0!N!} \end{pmatrix} \tag{2.20}$$

For even value of N

$$\Pi = \begin{pmatrix} \frac{1}{2^0 0!0!} & 0 & \frac{-2!}{2^2 1!1!} & \cdots & \cdots & 0 & \frac{(-1)^{\frac{N}{2}-1}(N-2)!}{2^{N-2}(\frac{N}{2}-1)!(\frac{N}{2}-1)!} & 0 & \frac{(-1)^{\frac{N}{2}}N!}{2^{N}\frac{N}{2}!\frac{N}{2}!} \\ 0 & \frac{2!}{2^1 0!1!} & 0 & & & & 0 & \frac{(-1)^{\frac{N}{2}-1}(N)!}{2^{N-1}(\frac{N}{2}-1)!(\frac{N}{2})!} & 0 \\ 0 & 0 & \frac{4!}{2^2 0!2!} & & & & & 0 & \frac{(-1)^{\frac{N}{2}-1}(N+2)!}{2^{N}(\frac{N}{2}-1)!(\frac{N}{2}+1)!} \\ 0 & 0 & 0 & \ddots & & & & & 0 \\ \vdots & \vdots & & & & & & & \\ \vdots & \vdots & & & & & & & \vdots \\ \vdots & 0 & & & & & & 0 & \vdots \\ 0 & 0 & & & & & \ddots & \frac{-(2N-4)!}{2^{N-1}1!(N-2)!} & \\ 0 & 0 & & & & & & 0 & \frac{-(2N-2)!}{2^{N}(1)!(N-1)!} \\ 0 & 0 & & & & \cdots & 0 & \frac{(2N-2)!}{2^{N-1}0!(N-1)!} & 0 \\ 0 & 0 & & & & \cdots & 0 & 0 & \frac{(2N)!}{2^{N}0!N!} \end{pmatrix} \tag{2.21}$$

On the other hand, we can write

$$P^{(k)}(y) = Y^{(k)}(y)\Pi, \tag{2.22}$$

And

$$Y^{(1)}(y) = Y(y)B \tag{2.23}$$

The matrix B is given in (1.20)

Also, from (2.23), we obtain

$$\begin{aligned} Y^{(1)}(y) &= Y(y)B \\ Y^{(2)}(y) &= Y^{(1)}(y)B = Y(y)B^2 \\ &\vdots \\ Y^{(k)}(y) &= Y^{(k-1)}(y)B = Y(y)B^k \end{aligned}$$

Then

$$P^{(k)}(y) = Y(y)B^k\Pi \tag{2.24}$$

Consequently, by substituting the matrix relation (2.24) into (2.19), we obtain the matrix relation for $U_N^{(k)}(y)$ as follows

$$[U_N^{(k)}(y)] = Y(y)B^k\Pi C \tag{2.25}$$

Next, let $\{\sigma_i\}_{i=0}^N$,be the Legendre-Gauss points, which are zeros of Legendre polynomial $P_{N+1}(y)$. By [18], there exists a unique set of Christoffel numbers $\{\omega_i\}_{i=0}^N$ such that

$$\int_{-1}^{1} \phi(y)dy = \sum_{i=0}^{N} \phi(\sigma_i)\omega_i, \quad \forall\phi \in \mathcal{P}_{2N+1} \tag{2.26}$$

In fact, we have

$$\omega_i = \frac{2}{(1-\sigma_i^2)[P'_{N+1}(\sigma_i)]^2}, \quad 0 \le i \le N \tag{2.27}$$

The fundamental matrix relation for differential part $D_N(y)$ based on collocation points is given by

$$D = \sum_{k=0}^{2} H_k Y B^k \Pi C \tag{2.28}$$

Where

$$H_k = diag(H_k(\sigma_j))_{0\le j\le N} = \begin{pmatrix} H_k(\sigma_0) & 0 & \dots & 0 & 0 \\ 0 & H_k(\sigma_1) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & H_k(\sigma_N) \end{pmatrix}$$

$$\mathbf{Y} = \begin{pmatrix} Y(\sigma_0) \\ Y(\sigma_1) \\ Y(\sigma_2) \\ \vdots \\ Y(\sigma_N) \end{pmatrix} = \begin{pmatrix} 1 & \sigma_0 & \sigma_0^2 & \dots & \sigma_0^N \\ 1 & \sigma_1 & \sigma_1^2 & \dots & \sigma_1^N \\ 1 & \sigma_2 & \sigma_2^2 & \dots & \sigma_2^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \sigma_N & \sigma_N^2 & \dots & \sigma_N^N \end{pmatrix} \tag{2.29}$$

For the Fredholm integral part, we have

$$I_N(y) = \sum_{j=0}^{N} c_j \int_{-1}^{1} K(y,z)P_j(z)\theta'(z)dz \simeq \sum_{j=0}^{N} c_j \sum_{i=0}^{N} K(y,\sigma_i)P_j(\sigma_i)\theta'(\sigma_i)\omega_i. \tag{2.30}$$

By setting

$$\upsilon_i = \theta^{'}(\sigma_i)\omega_i,$$

The fundamental matrix relation for the integral part $I_N(y)$ based on collocation points is given by

$$I_f = M\Phi Y\Pi C \tag{2.31}$$

With

$$M = (K(\sigma_j, \sigma_i))_{0\leq j,i\leq N}, \quad \Phi = diag((\nu_i)_{0\leq i\leq N})$$

Using relation (2.28) and (2.30), Eq. (2.14) is reduced to the following system

$$WC = F \tag{2.32}$$

Where

$$W = [\omega_{ij}] = \sum_{k=0}^{2} H_k Y B^k \Pi - M\Phi Y\Pi \tag{2.33}$$

$$F = [F(\sigma_0), F(\sigma_1), ..., F(\sigma_N)]^T$$

On the other hand, the fundamental matrix for the boundary conditions can be written as

$$V_0 C = \alpha \tag{2.34}$$

$$V_1 C = \beta \tag{2.35}$$

With

$$V_0 = [\nu_{00}, \nu_{01}, ..., \nu_{0N}] = [1, -1, ..., (-1)^N] \tag{2.36}$$

$$V_1 = [\nu_{10}, \nu_{11}, ..., \nu_{1N}] = [1, 1, ..., 1] \tag{2.37}$$

In order to satisfy the boundary conditions in the collocation method we add the above equations to the final system given in (2.32), then we obtain

$$\widetilde{W}C = \widetilde{F} \tag{2.38}$$

So that the new augmented matrix is of the form

$$[\widetilde{W}|\widetilde{F}] = \left[\begin{array}{ccccc|c} w_{00} & w_{01} & w_{02} & \dots & w_{0N} & F(\sigma_0) \\ w_{10} & w_{11} & w_{12} & \dots & w_{1N} & F(\sigma_1) \\ w_{20} & w_{21} & w_{22} & \dots & w_{2N} & F(\sigma_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{N-1,0} & w_{N-1,1} & w_{N-1,2} & \dots & w_{N-1,N} & F(\sigma_{N-1}) \\ v_{00} & v_{01} & v_{02} & \dots & v_{0N} & \alpha \\ v_{10} & v_{11} & v_{12} & \dots & v_{1N} & \beta \end{array}\right] \tag{2.39}$$

Finally, we have an over-determined system with $(N+3)$ linear equations which can be readily solved by using least square method.

2.2 Legendre-Galerkin method

We assume the solution of (2.11) is approximated by the finite expansion of Legendre basis (2.18)
The unknown coefficients c_j in (2.18) are determined by orthogonalizing the residual with respect to the basis functions $P_i(x)$. This yields the discrete system

$$\begin{aligned}\langle H_2(y)U^{''}(y), P_i(y)\rangle + \langle H_1(y)U^{'}(y), P_i(y)\rangle + \langle H_0(y)U(y), P_i(y)\rangle \\ - \lambda\langle\int_{-1}^{1} K(y,z)\theta^{'}(z)U(z), P_i(y)\rangle = \langle F(y), P_i(y)\rangle\end{aligned} \tag{2.40}$$

The weighted inner product $\langle , \rangle$ is taken to be

$$\langle u, v\rangle = \int_{-1}^{1} u(x)v(x)dx$$

We need now to present some useful relations

1. $\int_{-1}^{1} yP_n(y)P_m(y)dy = 0$
2. $\int_{-1}^{1} P_n^{'}(y)P_m(y)dy = \begin{cases} 2, & if \quad n = m + l \\ 0, & if \quad n \neq m + l \quad or \quad m \geq n \end{cases}$
3. $\int_{-1}^{1} yP_n^{'}(y)P_m(y)dy = \begin{cases} \frac{2n}{2n+1}, & if \quad n = m \\ 0, & if \quad n = m + l \quad or \quad m > n \\ 1, & if \quad n \neq m + l \end{cases}$
4. $\int_{-1}^{1} P_n^{''}(y)P_m(y)dy = \begin{cases} n(n+1) - m(m+1), & if \quad n \neq m + l \\ 0, & if \quad n = m + l \quad or \quad m \geq n \end{cases}$
5. $\int_{-1}^{1} yP_n^{''}(y)P_m(y)dy = \begin{cases} n(n+1) - m(m+1) - 2, & if \quad n = m + l \\ 0, & if \quad n \neq m + l \quad or \quad m \geq n \end{cases}$
6.
7. $\int_{-1}^{1} y^2P_n^{''}(y)P_m(y)dy = \begin{cases} \frac{2n(n-1)}{2n+1}, & if \quad n = m \\ n(n+1) - m(m+1) - 4, & if \quad n = m + l + 1 \\ 0, & if \quad n \neq m + l + 1 \quad or \quad m > n \end{cases}$

Where $l = 1, 3, 5, ..., 2k + 1 \leq N - m$.
The method of approximating the integrals in (2.40) begins by integrating by parts to transfer all derivatives from U to P_j. Therefore, the following relations is needed.

$$\langle H_1(y)U^{'}(y), P_j(y)\rangle = -\int_{-1}^{1}(H_1(y)P_j(y))^{'}U(y)dy \tag{2.41}$$

$$\langle H_2(y)U^{''}(y), P_j(y)\rangle = \Big[U^{'}(y)H_2(y)P_j(y)\Big]_{-1}^{1} + \int_{-1}^{1}(H_2(y)P_j(y))^{''}U(y)dy \tag{2.42}$$

$$\langle F(y), P_j(y)\rangle \simeq \sum_{k=0}^{N} \omega_k F(\sigma_k)P_j(\sigma_k) \tag{2.43}$$

$$\langle \int_{-1}^{1} k(y,z)U(z)\theta^{'}(z)dz, P_j(y)\rangle \simeq \sum_{k=0}^{N}\sum_{l=0}^{N} \omega_k\omega_l k(\sigma_k,\sigma_l)U(\sigma_l)\theta^{'}(\sigma_l)P_j(\sigma_k) \tag{2.44}$$

Where

$$\omega_k\omega_l = \frac{4}{\left[(1-\sigma_k^2)(1-\sigma_l^2)(P_N^{'}(\sigma_k)P_N^{'}(\sigma_l))^2\right]}$$

Replacing each term of (2.40) with the approximation defined in (2.41)–(2.44) respectively, then the discrete Galerkin–Legendre system for the determination of the unknown coefficients $\{c_j\}_{j=0}^{N}$ is given by

$$\sum_{j=0}^{N} c_i \Big[\int_{-1}^{1} H_0(y)P_i(y)P_j(y)dy - \int_{-1}^{1} (H_1(y)P_j(y))^{'}P_i(y)dy + [H_2(y)P_j(y)U^{'}(y)]_{-1}^{1} -$$

$$\int_{-1}^{1} (H_2(y)P_j(y))^{''}P_i(y)dy - \lambda \sum_{k=0}^{N}\sum_{l=0}^{N} \omega_k\omega_l k(\sigma_k,\sigma_l)P_i(\sigma_l)P_j(\sigma_k)\theta^{'}(\sigma_l)\Big] = \sum_{k=0}^{N} \omega_k F(\sigma_k)P_j(\sigma_k) \tag{2.45}$$

The system (2.45) takes the matrix form

$$\mathbf{Ac} = \mathbf{b} \tag{2.46}$$

Where

$$\mathbf{A} = \begin{pmatrix} A_{0,0} & A_{0,1} & \dots & A_{0,N} \\ A_{1,0} & A_{1,1} & \dots & A_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N,0} & A_{N,1} & \dots & A_{N,N} \end{pmatrix} \tag{2.47}$$

With

$$A_{i,j} = d_{i,j} - \lambda h_{i,j}$$

$$d_{i,j} = \int_{-1}^{1} H_0(y)P_i(y)P_j(y)dy - \int_{-1}^{1} (H_1(y)P_j(y))^{'}P_i(y)dy + [H_2(y)P_j(y)U^{'}(y)]_{-1}^{1} -$$

$$\int_{-1}^{1} (H_2(y)P_j(y))^{''}P_i(y)dy$$

$$h_{i,j} = \sum_{k=0}^{N}\sum_{l=0}^{N} \omega_k\omega_l k(\sigma_k,\sigma_l)P_i(\sigma_l)P_j(\sigma_k)\theta^{'}(\sigma_l)$$

2.2.1 Treatment of boundary condition

If the boundary conditions are nonhomogeneous

$$U(-1) = \alpha, \quad U(1) = \beta. \tag{2.48}$$

Then these conditions need be converted to homogeneous conditions via an interpolation by a known function. Applying the transformation

$$\widehat{U}(y) = U(y) - \frac{1-y}{2}\alpha - \frac{1+y}{2}\beta$$

to the problem (2.11),(2.12) yields

$$\sum_{k=0}^{2} H_k(y)\widehat{U}^k(y) - \lambda \int_{-1}^{1} k(y,z)\widehat{U}(z)dz = \widehat{f}(y)$$

With boundary conditions

$$\widehat{U}(-1) = 0, \quad \widehat{U}(1) = 0$$

Where

$$\widehat{f}(y) = f(y) - \frac{\beta-\alpha}{2}H_1(y) - \left[\frac{(\beta-\alpha)y+\beta+\alpha}{2}\right]H_0(y) - \lambda\int_{-1}^{1} k(y,z)\left(\frac{(1+z)\beta+(1-z)\alpha}{2}\right)dz$$

The resulting discrete system for the coefficients $N+1$ in the approximate Legendre–Galerkin solution

$$U(y) = \sum_{i=0}^{N} c_i P_i(y) + \frac{1-y}{2}\alpha + \frac{1+y}{2}\beta$$

2.3 Numerical examples

Example 1

$$\begin{cases} u^{'}(x) - u(x) - \int_0^1 \sin(4\pi x + 2\pi t)u(t)dt = -\cos(2\pi x) - 2\pi\sin(2\pi x) - \frac{1}{2}\sin(4\pi x) \\ u(0) = 1 \end{cases}$$

Whose exact solution is $u(x) = \cos(2\pi x)$

N	Legendre collocation method	Legendre-Galerkin method
5	1.584e-01	5.165e-02
10	1.023e-06	2.371e-04
15	1.648e-10	1.697e-10
20	3.483e-15	1.278e-13
25	6.607e-16	1.165e-14

Table 2.1: Maximum absolute errors for Example 1 at different N

Method	Error
Legendre collocation	3.483e-15
Legendre–Galerkin	1.278e-13
Sinc basis functions [25]	5.275e-05
Homotopy analysis [20]	2.231e-06
CAS wavelet [21]	1.746e-02

Table 2.2: Comparison of maximum absolute errors for Example 1 for $N = 20$

Example 2

$$\begin{cases} u'(x) - u(x) - \frac{1}{(\ln(2))^2}\int_0^1 \frac{x}{1+t}u(t)dt = -\frac{1}{2}x + \frac{1}{1+x} - \ln(1+x) \\ u(0) = 0 \end{cases}$$

Whose exact solution is $u(x) = \ln(1+x)$

N	Legendre collocation	Legendre-Galerkin	Homotopy Analysis [20]	Sinc basis functions [25]
10	1.033e-09	1.301e-05	7.816e-04	1.740e-04
20	4.175e-16	7.108e-13	6.240e-07	3.185e-06

Table 2.3: Comparison of maximum absolute errors for Example 2 at different N

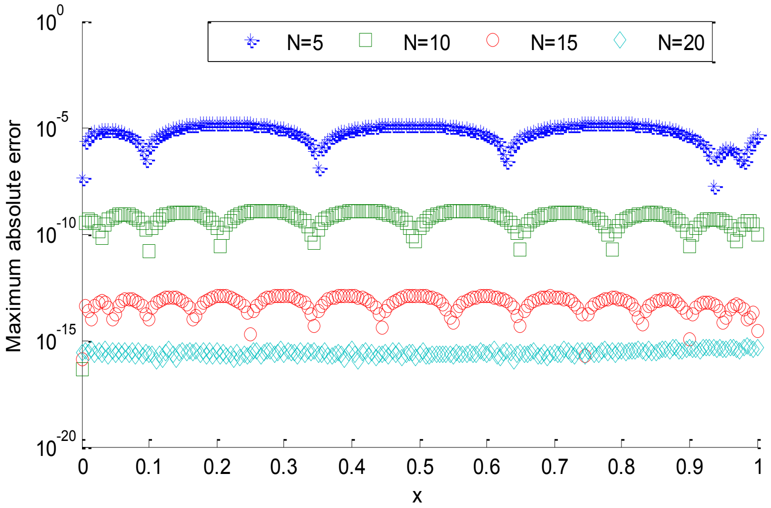


Figure 2.1: Comparison of error with $N = 5, 10, 15, 20$ by Legendre collocation method for example 2

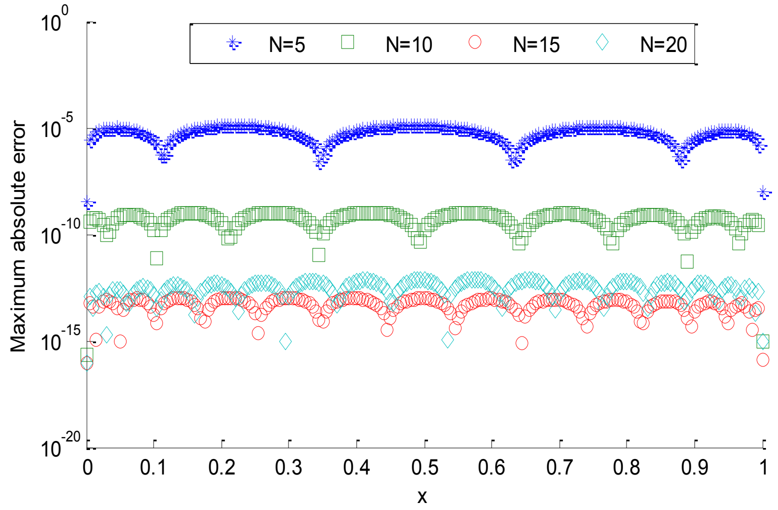


Figure 2.2: Comparison of error with $N = 5, 10, 15, 20$ by Galerkin-Legendre method for example 2

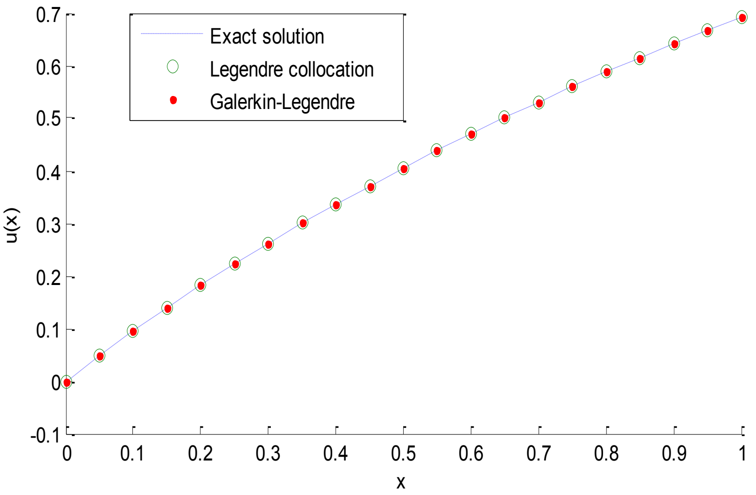


Figure 2.3: Legendre collocation, Galerkin-Legendre and exact solutions for example 2, $N = 10$

Example 3

$$\begin{cases} u''(x) + xu'(x) - xu(x) - \int_{-1}^{1} \sin(x)e^{-t}u(t)dt = e^{x} - 2\sin(x) \\ u(-1) = e^{-1}, \quad u(1) = e \end{cases}$$

Whose exact solution is $u(x) = e^{x}$

Method	Error for :	$N = 3$	$N = 7$	$N = 9$
Legendre collocation		6.977e-02	1.230e-06	2.032e-09
Legendre–Galerkin		3.460e-02	7.555e-07	1.967e-09
Legendre-collocation matrix [59]		1.832e-01	5.751e-05	3.288e-05

Table 2.4: Comparison of maximum absolute errors for Example 3

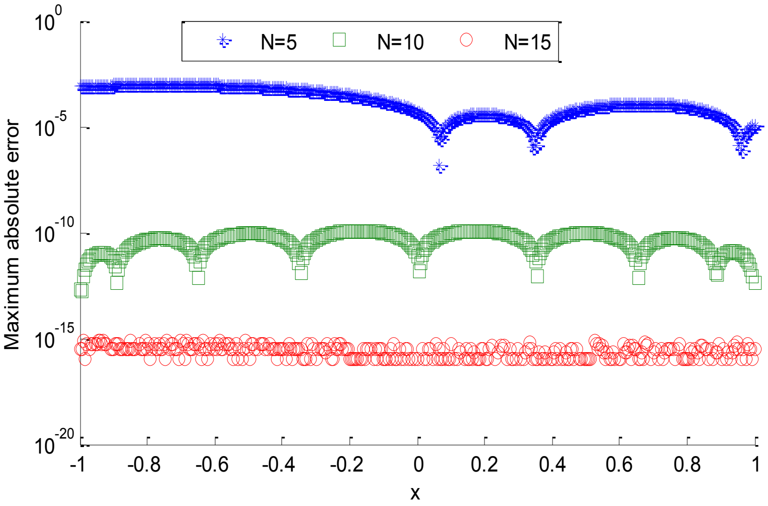


Figure 2.4: Comparison of error with $N = 5, 10, 15$ by Legendre collocation method for example 3

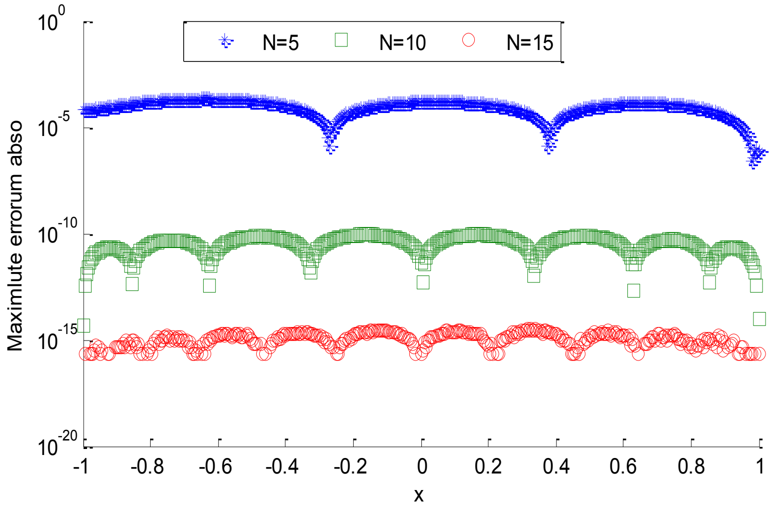


Figure 2.5: Comparison of error with $N = 5, 10, 15$ by Galerkin-Legendre method for example 3

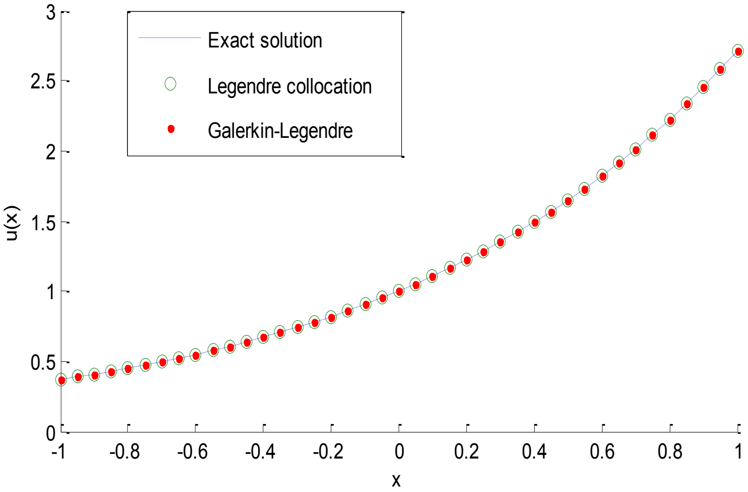


Figure 2.6: Legendre collocation, Galerkin-Legendre and exact solutions for example 3, $N = 10$

Chapter 3

Mapping Legendre functions on the semi-infinite interval

Thanks to three mappings namely algebric, Logarithmic and exponenetiel for passing from the interval $[0,\infty)$ to $[-1,1]$ and vice versa, where help us to define a new mapped Legendre functions which are orthogonal on the positive semi infinite interval, we will also study the properties of each mapping.

Let us consider a family of mapping of the form

$$x=\varphi(y,s),\quad s>0,\quad y\in[-1,1],\quad x\in[0,+\infty) \tag{3.1}$$

Such that

$$\begin{aligned}&\frac{dx}{dy}=\varphi^{'}(y,s),\quad s>0,\quad y\in I\\&\varphi(-1,s)=0,\quad \varphi(1,s)=+\infty\end{aligned} \tag{3.2}$$

In this one-to-one transform, the parameter s is a positive scaling factor, without loss of generality, we further assume that the mapping is explicitly invertible and denotes its inverse mapping by

$$y=\varphi^{-1}(x,s)=h(x,s),\quad x\in\Lambda,\quad y\in I,\quad s>0 \tag{3.3}$$

Several typical mappings that have been proposed between $x\in[0,\infty[$ and $y\in[-1,1]$

1. **Algebraic mapping**

$$x=\frac{s(1+y)}{1-y},\quad y=\frac{x-s}{x+s}$$

2. **Logarithmic mapping**

$$x=sarctan(\frac{y+1}{2})=\frac{s}{2}\ln(\frac{3+y}{1-y})\quad ,y=1-2\tanh(\frac{x}{s})$$

3. **Exponential mapping**

$$x=\sinh(\frac{s}{2}(1+y)),\quad y=\frac{2}{s}\ln(x+\sqrt{x^2+1})-1$$

Given mapping $x=\varphi(y,s)$ satisfying (3.1), (3.2) and a family of orthogonal polynomials $\{P_k(y)\}$ with $y\in I=(-1,1)$ $\{P_k(\varphi(x;s))\}$ forms a new family of orthogonal functions in $\Lambda=(0,+\infty)$.
For the sake of generality, we consider the mapping Legendre approximation. Let $P_k(y)$ be the $k-th$ degree classical Legendre polynomials .
We define the mapped Legendre functions as

$$L_{s,n}(x)=P_n(\varphi^{-1}(x;s)),x\in\Lambda,y\in I \tag{3.4}$$

We infer from (1.33) that (3.4) defines a new family of orthogonal functions $\{L_{s,n}\}$ in $L^2_{w_s}(\Lambda)$

$$\int_\Lambda L_{s,n}(x)L_{s,m}(x)w_s=\gamma_n\delta_{nm} \tag{3.5}$$

Where the constant γ_n is given in (1.34), and the weight function

$$w_s=w(y)\frac{dy}{dx}=(\varphi^{-1}(x;s))' \tag{3.6}$$

With $y=\varphi^{-1}(x,s)$ and $w(y)=1$.

3.1 Properties of Rational, Logarithmic and Exponnetial Legendre functions

In this section we will introduce Rational, Logarithmic and Exponnetial Legendre functions that all of them are defined on the semi-infinite interval .

3.1.1 Rational Legendre functions

The Legendre polynomials are orthogonal in the interval $[-1,1]$, with respect to the weight function $\rho(y)=1$. They can be determined by the following recurrence formula :

$$P_0(y)=1,\quad P_1(y)=y$$

$$P_{n+1}(y)=(\frac{2n+1}{n+1})yP_n(y)-(\frac{n}{n+1})P_{n-1}(y),\qquad n\geq 1$$

Legendre polynomials can also acheive the following relation :

$$\int_{-1}^{1}y^mP_n(y)dy=\begin{cases}0, & n\neq m\\ \frac{2^{n+1}(n!)^2}{(2n+1)!}, & n=m\end{cases} \tag{3.7}$$

The Rational Legendre functions is denoted by $R_n(x)=P_n(\frac{x-s}{x+s})$, where s is a constant parameter and sets the lenght scale of the mapping

$$R_{n+1}(x)=(\frac{2n+1}{n+1})(\frac{x-s}{x+s})R_n(x)-(\frac{n}{n+1})R_{n-1}(x),\qquad n\geq 1$$

By using above recursive formula the first four rational Legendre function are obtained as below :

$$\begin{cases} R_0(x) = 1 \\ R_1(x) = \frac{x-s}{x+s} \\ R_2(x) = \frac{3}{2}(\frac{x-s}{x+s})^2 - \frac{1}{2} \\ R_3(x) = \frac{5}{2}(\frac{x-s}{x+s})^3 - \frac{3}{2}(\frac{x-s}{x+s}) \end{cases}$$

The behavior of these four functions for $s = 1$ are plotted in Figure.1
Rational Legendre function (RL) are orthogonal with respect to the weight function $w_r(x) = \frac{2s}{(x+s)^2}$ in the interval $[0, +\infty[$, with the orthogonality property .

$$\int_0^{+\infty} R_n(x)R_m(x)w_r(x)dx = \frac{2}{2n+1}\delta_{nm}$$

Where δ_{nm} is the Kronecker function

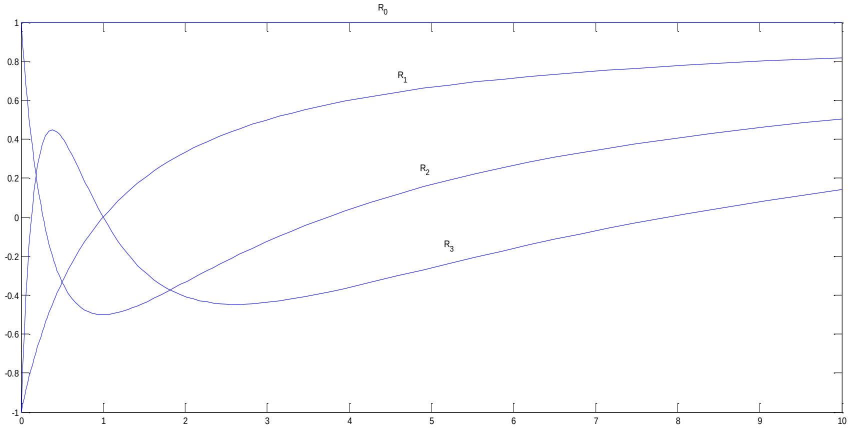


Figure 3.1: Graph of four rational Legendre functions for s=1

3.1.2 Logarithmic Legendre functions

The Logarithmic Legendre functions is denoted by $\mathcal{L}_n(x) = P_n(2\tanh(x/s) - 1)$

$$\mathcal{L}_{n+1}(x) = (\frac{2n+1}{n+1})(2\tanh(x/s) - 1)\mathcal{L}_n(x) - (\frac{n}{n+1})\mathcal{L}_{n-1}(x), \qquad n \geq 1$$

By using above recursive formula the first four logarithmic Legendre function are obtained as below :

$$\begin{cases} \mathcal{L}_0(x) = 1 \\ \mathcal{L}_1(x) = 2\tanh(x/s) - 1 \\ \mathcal{L}_2(x) = \frac{3}{2}(2\tanh(x/s) - 1)^2 - \frac{1}{2} \\ \mathcal{L}_3(x) = \frac{5}{2}(2\tanh(x/s) - 1)^3 - \frac{3}{2}(2\tanh(x/s) - 1) \end{cases}$$

The behavior of these four functions for $s = 1$ are plotted in Figure.2
Logarithmic Legendre function (LL) are orthogonal with respect to the weight function $w_l(x) = \frac{2}{s}(1 - \tanh^2(x/s))$ in the interval $[0, +\infty[$, with the orthogonality property .

$$\int_0^{+\infty} \mathcal{L}_n(x)\mathcal{L}_m(x)w_l(x)dx = \frac{2}{2n+1}\delta_{nm}$$

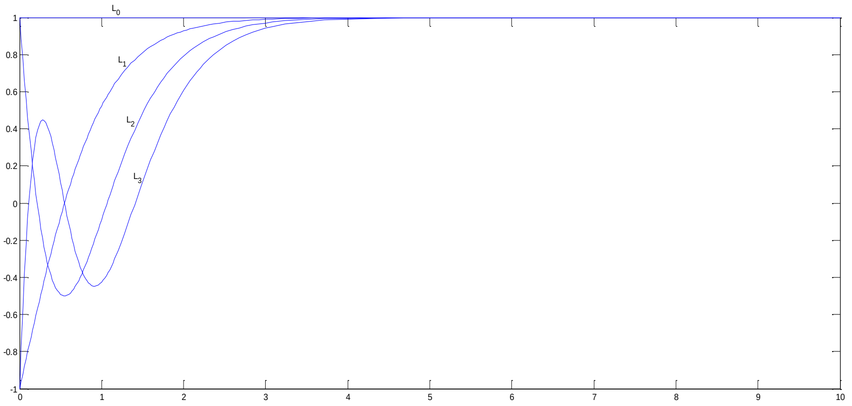


Figure 3.2: Graph of four logarithmic Legendre functions for s=1

3.1.3 Exponential Legendre functions

The Exponnential Legendre functions is denoted by $E_n(x) = P_n(1 - 2\exp(x/s))$

$$E_{n+1}(x) = (\frac{2n+1}{n+1})(1 - 2\exp(x/s))E_n(x) - (\frac{n}{n+1})E_{n-1}(x), \qquad n \geq 1$$

By using above recursive formula the first four exponnential Legendre function are obtained as below :

$$\begin{cases} E_0(x) = 1 \\ E_1(x) = 1 - 2\exp(-x/s) \\ E_2(x) = \frac{3}{2}(1 - 2\exp(-x/s)) - \frac{1}{2} \\ E_3(x) = \frac{5}{2}(1 - 2\exp(-x/s)) - \frac{3}{2}(1 - 2\exp(-x/s)) \end{cases}$$

The behavior of these four functions for $s = 1$ are plotted in Figure.3
Exponnential Legendre function (EL) are orthogonal with respect to the weight function $w_e(x) = \frac{2}{s}\exp(-x/s)$ in the interval $[0, +\infty[$, with the orthogonality property .

$$\int_0^{+\infty} E_n(x)E_m(x)w_e(x)dx = \frac{2}{2n+1}\delta_{nm}$$

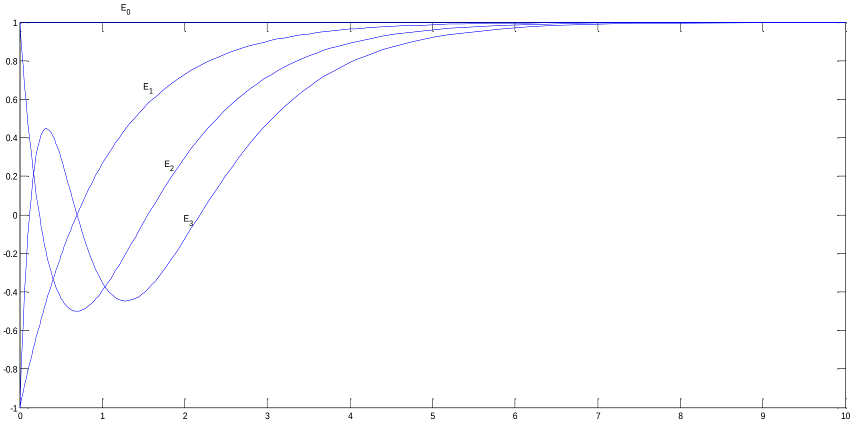


Figure 3.3: Graph of four exponential Legendre functions for s=1

3.2 Function approximation

Let $\rho_s(x)$ is $w_r(x), w_l(x)$ or $w_e(x)$ denotes a non-negative, integrable real valued function over the interval $\Lambda = [0, \infty[$, we define.

$$L^2_w(\Lambda) = \{\nu : \Lambda \longrightarrow \mathbb{R} \quad and \quad \|\nu\|_w < \infty\}$$

Where

$$\|\nu\|^2_w = \int_0^{+\infty} \nu^2(x)w(x)dx$$

is the norm induced by the inner product of the space $L^2_w(\Lambda)$

$$\langle \mu, \nu \rangle_w = \int_0^{+\infty} \mu(x)\nu(x)\rho_s(x)dx \tag{3.8}$$

The $\{\phi_j(x)\}_{j\geq 0}$, are considered $\{R_j(x)\}_{j\geq 0}$, $\{\mathcal{L}_j(x)\}_{j\geq 0}$ or $\{E_j(x)\}_{j\geq 0}$, denotes a system which is mutually orthogonal under (3.8)

$$\langle \phi_n(x), \phi_m(x) \rangle_{\rho_s} = \frac{2}{2n+1}\delta_{nm}$$

The classical Weierstrass theorem implies that such a system is complete in the space $L^2_{\rho_s}(\Lambda)$.
Thus, for any function $f(x) \in L^2_{\rho_s}(\Lambda)$ the following expansion holds

$$f(x) = \sum_{j=0}^{\infty} a_j \phi_j(x) \tag{3.9}$$

Where

$$a_j = \frac{2j+1}{2} \int_0^{\infty} \phi_j(x) f(x) \rho_s(x) dx \tag{3.10}$$

If $f(x)$ in (3.9) is truncated up to the Nth terms, then it can be written as

$$f(x) \simeq f_N(x) = \sum_{j=0}^{N-1} a_j \phi_j(x) = A^T \phi(x) \tag{3.11}$$

With

$$A = [a_0, a_1, ..., a_{N-1}]^T$$

$$\phi(x) = R(x) = [R_0, R_1, ..., R_{N-1}]^T \tag{3.12}$$

$$\phi(x) = \mathcal{L}(x) = [\mathcal{L}_0, \mathcal{L}_1, ..., \mathcal{L}_{N-1}]^T \tag{3.13}$$

$$\phi(x) = E(x) = [E_0, E_1, ..., E_{N-1}]^T \tag{3.14}$$

Proposition 3.2.1 *The orthogonal mapped Legendre functions $\{L_{s,j}\}$ form a complete orthogonal basis in $L^2_{\rho_s}(\Lambda)$*

Proof — Let $u \in L^2_{\rho_s}(\Lambda)$, we have from the orthogonality and the completeness of Legendre polynomials that $U(y) = \sum_{j=0}^{+\infty} C_j P_j(y)$, for all $U \in L^2_w(I)$.

$$C_j = \frac{2j+1}{2} \int_{-1}^{1} U(y) P_j(y) dy = \frac{2j+1}{2} \int_0^{\infty} u(x) L_{s,j}(x) \rho_s(x) dx$$

We can write

$$u(x) = \sum_{j=0}^{+\infty} C_j L_{s,j}(x)$$

This shows that $\{L_{s,j}\}$ forms a complete basis in $L^2_{\rho_s}(\Lambda)$ ■

Now, we can estimate an upper bound for function approximation in a spectral case. In first, the error may be defined in the following form .

$$e_n = \|f(x) - f_n(x)\|^2_{\rho_s} \tag{3.15}$$

The completeness of the system $\{\phi_i\}_{i\geq 0}$ is equivalent to the following property

$$f_n(x) \longrightarrow f(x), \quad e_n \longrightarrow 0 \quad an \quad n \longrightarrow +\infty$$

Lemma 3.2.1 *The error which is defined in (3.15) can be rewritten as*

$$e_n = \sum_{i=n}^{\infty} \frac{2i+1}{2} \langle f(x), \phi_i(x) \rangle^2_{\rho_s} \tag{3.16}$$

Proof — The completeness of the system $\{\phi_i\}_{i\geq 0}$ helped us to consider the error as $e_n = \| \sum_{i=n}^{\infty} a_i \phi_i(x)\|^2_{\rho_s}$.
Using the definition of $\|.\|_w$, one has

$$e_n = \sum_{i=n}^{\infty}\sum_{j=n}^{\infty} a_i a_j \langle \phi_i(x), \phi_j(x) \rangle_w = \sum_{i=n}^{\infty}\sum_{j=n}^{\infty} a_i a_j \frac{2}{2i+1}\delta_{ij} = \sum_{i=n}^{+\infty} \frac{2a_i^2}{2i+1}$$

and using Eq.(3.10) the lemma can be proved .
This lemma shows that the convergence rate is involved with function $f(x)$. Now, by knowing that the function $f(x) \in L^2_{\rho_s}(\Lambda)$ have some good properties, we could present an upper bound for estimating the error of function approximation by this basis function ■

Theorem 3.1 *Let $f_n(x)$ is function approximation of $f(x) \in L^2_{\rho_s}(\Lambda)$ obtained by (3.11) and $\mathcal{F}(y) = f(\psi(y))$ is analytic on $[-1, 1]$, then an error bound of this approximation can be presented as follows :*

$$e_n \leq \sum_{i=n}^{\infty} \frac{4^{i+1} M_i^2 i!^2}{2(2i)!(2i+1)!}$$

Where :

$$\psi(y) = s(\frac{1+y}{1-y}); \quad \frac{s}{2}\ln(\frac{3+y}{1-y}) \quad or \quad s\ln(\frac{2}{1-y})$$

And : $M_i = \max|\mathcal{F}^i(y)|, \quad y \in (-1, 1)$

Proof — We have the following properties for each case of $\psi(x)$.

- **Case1:**

$$\psi(y) = s\frac{1+y}{1-y}, \quad R_n(\psi(y)) = P_n(y)$$

$$w_r(\psi(y)) = \frac{(1-y)^2}{2s}, \quad dx = \frac{2s}{(1-y)^2}dy$$

- **Case2:**

$$\psi(y) = \frac{s}{2}\ln(\frac{3+y}{1-y}), \quad \mathcal{L}_n(\psi(y)) = P_n(y)$$

$$w_l(\psi(y)) = \frac{(1-y)(3+y)}{2s}, \quad dx = \frac{2s}{(1-y)(3+y)}dy$$

- **Case3:**

$$\psi(y) = s\ln(\frac{2}{1-y}), \quad E_n(\psi(y)) = P_n(y)$$

$$w_e(\psi(y)) = \frac{(1-y)}{s}, \quad dx = \frac{s}{(1-y)}dy$$

By substituting each case in $\langle f(x), \phi_i(x)\rangle_{\rho_s} = \int_{-1}^{1} \mathcal{F}(y)P_i(y)dy$. Also knowing that $\mathcal{F}(y)$ is analytic, we have

$$\langle f(x), \phi_i(x)\rangle_{\rho_s} = \sum_{j=0}^{i-1} \frac{\mathcal{F}^{(j)}(0)}{j!}\int_{-1}^{1} y^j P_i(y)dy + \frac{\mathcal{F}^{(i)}(\xi_i)}{i!}\int_{-1}^{1} y^i P_i(y)dy \qquad \xi_i \in]-1,1[$$

Using the Legendre polynomials property mentioned in equation (3.7), the above equation can be rewritten in the following form .

$$\langle f(x), \phi_i(x)\rangle_{\rho_s} = \frac{\mathcal{F}^i(\xi_i)}{i!}\frac{2^{i+1}i!^2}{(2i+1)!} \leq \frac{2^{i+1}i!M_i}{(2i+1)!}$$

The theorem can be proved by substituting above inequality in equation (3.16). ■
The next theorem would show that the error defined in Eq.(3.15) have superlinear convergence. Firstly , we define the order of convergence is superlinear

Definition 3.2.1 *$x_n \longrightarrow \overline{x}$ with superlinear convergence if there is a positive sequence $\lambda_n \longrightarrow 0$ and an integer number N such that*

$$|x_{n+1} - \overline{x}| \leq \lambda_n |x_n - \overline{x}|, \quad n \geq N$$

Theorem 3.2 *Let $M \geq M_i$ in the theorem (3.1) then the error is superlinear convergence to zero*

Proof — Using theorem (3.1) we have

$$e_n \leq M^2 \sum_{i=n}^{\infty} \frac{4^{i+1}i!^2}{2(2i)!(2i+1)!} = M^2 \sum_{i=n}^{\infty} \frac{4^{i+1}i!^2}{2(2i)!!(2i-1)!!(2i+1)!!(2i)!!}$$

Where $(2i-1)!! = (2i-1)(2i-3)\times\ldots\times 3\times 1$ and $(2i)!! = (2i)(2i-2)\times\ldots\times 4\times 2 = 2^i i!$. Then one has

$$e_n \leq 2M^2 \sum_{i=n}^{\infty} \frac{1}{(2i-1)!!(2i+1)!!} \leq 2M^2 \sum_{i=n}^{\infty} \frac{1}{(2i)!}$$

We define $x_n = \sum_{i=n}^{\infty} \frac{1}{(2i)!}$, and then, there is a positive sequence

$$\lambda_n = 1 - \frac{1}{(2n)! \sum_{i=n}^{\infty} \frac{1}{(2i)!}} \longrightarrow 0$$

That $|x_{n+1}| \leq \lambda_n |x_n|$. Therefore, x_n and subsequently e_n are superlinear convergence to zero .
According to theorem (3.2), any function defined in $L^2_{\rho_s}([0, +\infty[)$, which their mapping under transformation $s\frac{1+y}{1-y}, \frac{s}{2}\ln(\frac{3+y}{1-y}) \quad or \quad s\ln(\frac{2}{1-y})$ are analytic a series solution in the form (3.11) with the superlinear convergence ■

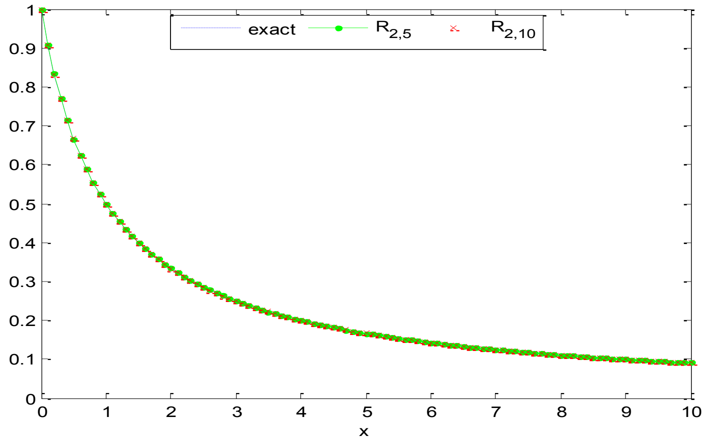


Figure 3.4: Convergence of the Algebraic mapping of Legendre series of function $\frac{1}{1+x}$

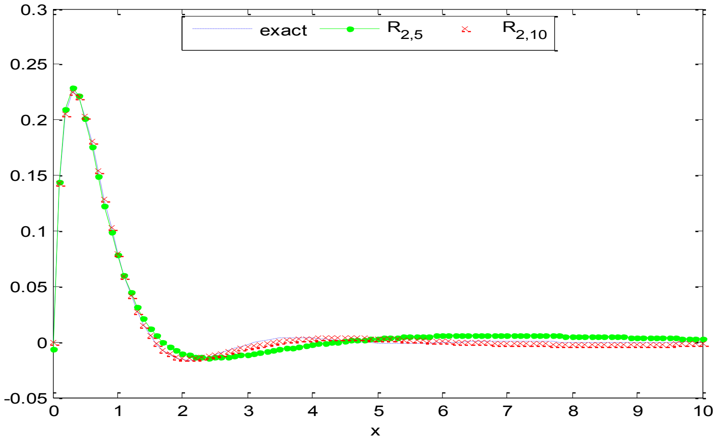


Figure 3.5: Convergence of the Algebraic mapping of Legendre series of function $\frac{\sin(2x)}{(1+x)^{7/2}}$

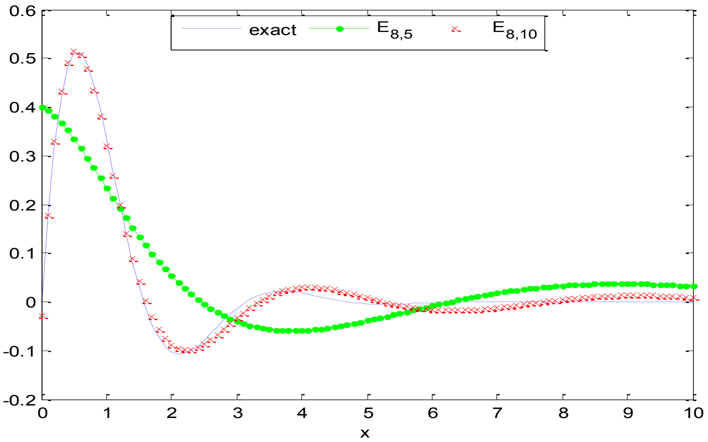


Figure 3.6: Convergence of the exponential mapping of Legendre series of function $\sin(2x)e^{-x}$

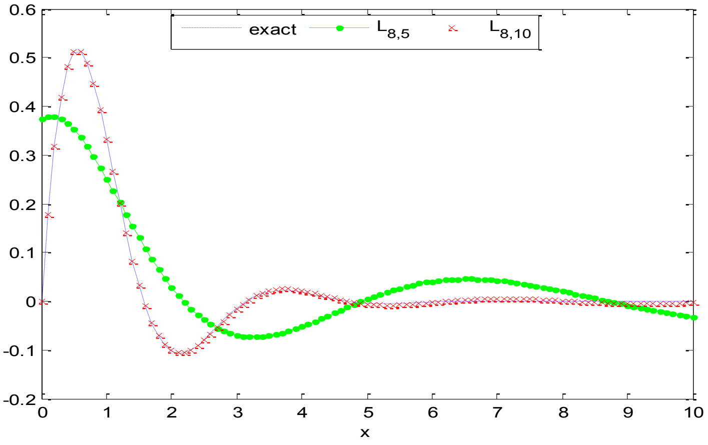


Figure 3.7: Convergence of the Logarithmic mapping of Legendre series of function $\sin(2x)e^{-x}$

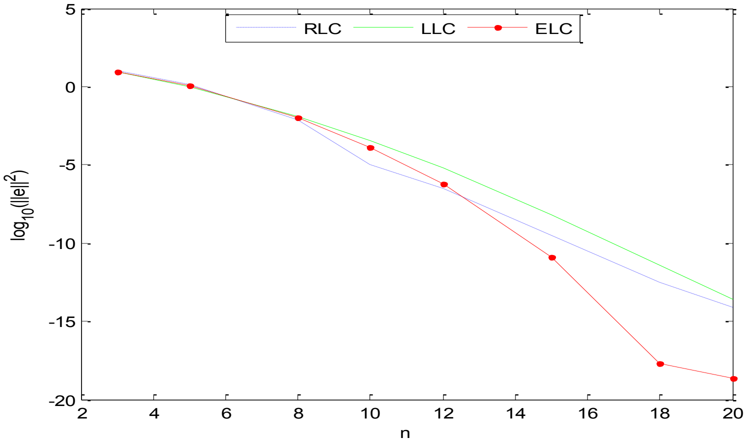


Figure 3.8: Spectral convergence: projection error in $L^2_{\rho_{20}}$ norm between the function e^{-x} and their truncated algebraic, Logarithmic and Exponential mapping Legendre series of order n

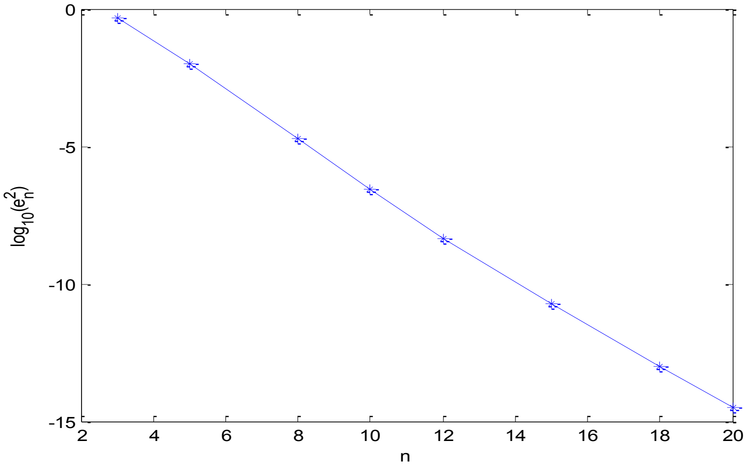


Figure 3.9: Spectral convergence: projection error in $L^2_{\rho_5}$ norm between the function $\frac{1}{1+x}$ and their truncated algebraic mapping Legendre series of order n

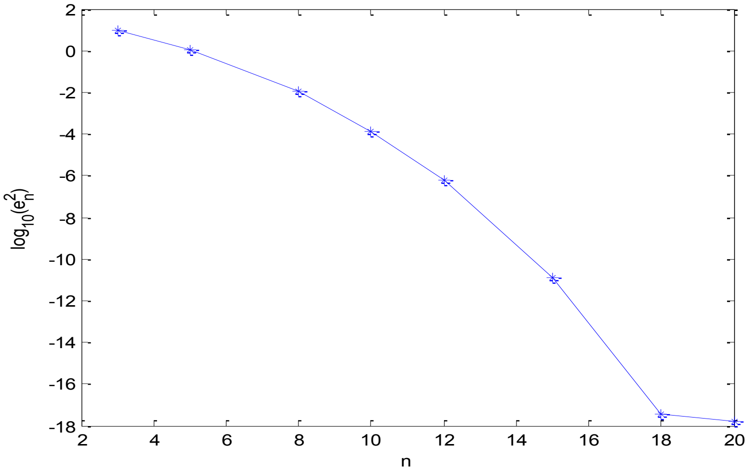


Figure 3.10: Spectral convergence: projection error in $L^2_{\rho_{20}}$ norm between the function $2 - e^{-x}$ and their truncated exponential mapping Legendre series of order n

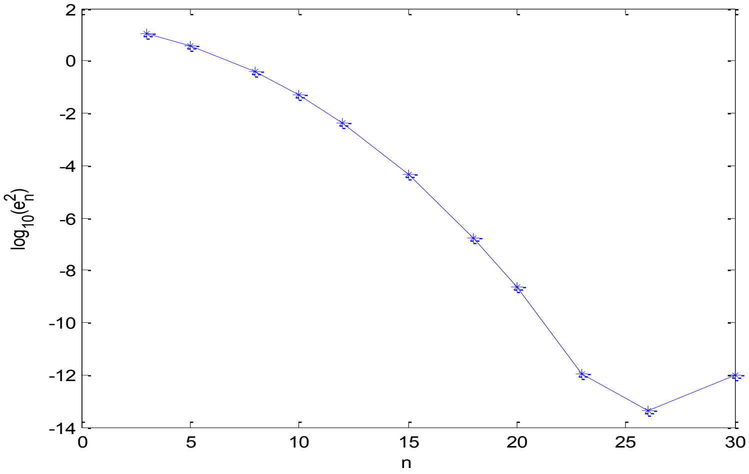


Figure 3.11: Spectral convergence: projection error in $L^2_{\rho_{20}}$ norm between the function e^{-2x} and their truncated exponential mapping Legendre series of order n

3.2.1 Mapped Legendre interpolation approximations

We now consider the Gauss quadrature formulas on unbounded domains based on mapped Legendre polynomials. To fix the idea, we only consider the Gauss quadrature.

Let $\{\xi_{N,j}, \omega_{N,j}\}_{j=0}^{N}$ be the Legendre-Gauss nodes and weights, and there holds

$$\int_{-1}^{1} \phi(y)w(y)dy = \sum_{j=0}^{N} \phi(\xi_{N,j})\omega_{N,j}, \quad \forall \quad \phi \in P_{2N+1} \tag{3.17}$$

Applying a mapping (3.1) to the above leads to the mapped Legendre-Gauss quadrature

$$\int_{\Lambda} u(x)\rho_s(x)dx = \sum_{j=0}^{N} u(\zeta_{s,N,j})\rho_{s,N,j}, \quad \forall \quad u \in V_{s,2N+1} \tag{3.18}$$

Where

$$\zeta_{s,N,j} = \varphi(\xi_{N,j}; s), \quad \rho_{s,N,j} = \omega_{N,j}; \quad 0 \le j \le N$$

Are the mapped Legendre-Gauss nodes and weights .

Accordingly, we can define the discrete inner product and discrete norm

$$\langle u, v \rangle_{\rho_{s,N}} = \sum_{j=0}^{N} u(\zeta_{s,N,j})v(\zeta_{s,N,j})\rho_{s,N,j}$$

$$\|u\|_{\rho_{s,N}} = \langle u, u \rangle^{\frac{1}{2}}_{\rho_{s,N}}, \quad \forall u, v \in C(\Lambda)$$

The mapped Legendre-Gauss interpolation operator $\mathcal{I}_{s,N} : C(\Lambda) \longrightarrow V_{s,N}$, is defined by $\mathcal{I}_{s,N} \in V_{s,N}$ such that

$$(\mathcal{I}_{s,N}u)(\zeta_{s,N,j}) = u(\zeta_{s,N,j}), \quad j = 0, 1, ..., N \tag{3.19}$$

Let $\mathcal{I}_N$ be the Legendre-gauss interpolation operator, by definition, we have

$$\mathcal{I}_{s,N}u(x) = (\mathcal{I}_N U_s)(y) = (\mathcal{I}_N U_s)(\varphi^{-1}(x; s))$$

Chapter 4

Fredholm IDEs on semi infinite interval

4.1 Matrix method using Laguerre polynomials

The purpose of this section is to present a matrix method for solving linear Fredholm integro-differential equations (FIDEs) on semi finite domain

$$\sum_{k=0}^{2} \nu_k(x)u^{(k)}(x) - \int_0^{\infty} k(x,t)u(t)dt = f(x), \quad x \in [0,\infty[\tag{4.1}$$

Under the boundary conditions

$$u(0) = \alpha, \quad u(\infty) = \beta \tag{4.2}$$

The method is based on the approximation of the truncated generalized Laguerre series. Then the system of (FIDEs) are transformed into the matrix equations, which corresponds to a system of linear algebraic equations with the unknown generalized Laguerre coefficients. Combining these matrix equations and then solving the system yields the generalized Laguerre coefficients of the solution function. our goal is to approximate the exact solution $u(x)$ as

$$u(x) \simeq u_N(x) = \sum_{n=0}^{N} c_n L_n^{(\alpha)}(x) \tag{4.3}$$

Such that : $c_n, n = 0, 1, 2, ..., N$ are the coefficients to be determined et $L_n^{(\alpha)}(x)$ are Laguerre polynomials

4.1.1 fundamentals matrix

We write $L_n^{(\alpha)}(x)$ under matrix form as :

$$L^{\alpha}(x) = \Pi_L X^T(x) \Longleftrightarrow L^{(\alpha)T}(x) = X(x)\Pi_L^T. \tag{4.4}$$

Where :

$$L^{(\alpha)}(x) = [L_0^{(\alpha)}(x) \quad L_1^{(\alpha)}(x) \quad ... \quad L_N^{(\alpha)}(x)] \quad and \quad X(x) = [1 \quad x^1 \quad x^2 \quad ... \quad x^N]$$

$$\Pi_L = \begin{pmatrix} \frac{(0!)^2}{(0!)^2 0!} & 0 & 0 & \cdots & 0 & 0 \\ \frac{(1!)^2}{(0!)^2 1!} & -\frac{(1!)^2}{(1!)^2 0!} & 0 & \cdots & 0 & 0 \\ \frac{(2!)^2}{(0!)^2 2!} & -\frac{(2!)^2}{(1!)^2 1!} & \frac{(2!)^2}{(2!)^2 0!} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{[(N-1)!]^2}{(0!)^2 (N-1)!} & -\frac{[(N-1)!]^2}{(1!)^2 (N-2)!} & \frac{[(N-1)!]^2}{(2!)^2 (N-3)!} & \cdots & (-1)^{N-1}\frac{[(N-1)!]^2}{[(N-1)!]^2 0!} & 0 \\ \frac{(N!)^2}{(0!)^2 N!} & -\frac{(N!)^2}{(1!)^2 (N-1)!} & \frac{(N!)^2}{(2!)^2 (N-2)!} & \cdots & (-1)^{N-1}\frac{(N!)^2}{[(N-1)!]^2 1!} & (-1)^N \frac{(N!)^2}{(N!)^2 0!} \end{pmatrix}_{(N+1)\times(N+1)}$$

We put (4.1) under form :

$$D(x) - I(x) = f(x) \tag{4.5}$$

With:

$$D(x) = \sum_{k=0}^{2} \nu_k(x) u^k(x), I(x) = \int_0^{\infty} k(x,t) u(t) dt$$

Differential part matrix

$$[u(x)] = L^{(\alpha)}(x) C; \quad C = [c_0 \quad c_1 \quad ... \quad c_N]^T \tag{4.6}$$

From (4.64) we have :

$$[u(x)] = X(x) \Pi_L^T C. \tag{4.7}$$

We have also the relation between $X(x)$ and its derivatives $X^{(1)}(x)$ as :

$$X^{(1)}(x) = X(x) B \tag{4.8}$$

Where B is given in (4.48), from (4.8) we write .

$$\begin{cases} X^{(0)}(x) = X(x) \\ X^{(1)}(x) = X(x) B \\ X^{(2)}(x) = X^{(1)}(x) B^T = X(x) B^2 \\ \vdots \\ X^{(k)}(x) = X^{(k-1)}(x) B^T = X(x) B^k. \end{cases} \tag{4.9}$$

Using (4.7) and (4.55) we obtain

$$u^{(k)}(x) = X^{(k)}(x) \Pi_L^T C = X(x) B^k \Pi_L^T C, \quad k = 0, 1, 2. \tag{4.10}$$

Replacing (4.10) into (4.53) we get :

$$[D(x)] = \sum_{k=0}^{2} \nu_k(x) X(x) B^k \Pi_L^T C. \tag{4.11}$$

The kernel $K(x,t)$ can be approximated by Maclaurin truncated series and Laguerre truncated series respectively

$$K(x,t)=\sum_{m=0}^{N}\sum_{n=0}^{N}k_{mn}^{t}x^{m}t^{n} \quad and \quad K(x,t)=\sum_{m=0}^{N}\sum_{n=0}^{N}k_{mn}^{b}L_{m}^{(\alpha)}(x)L_{n}^{(\alpha)}(t), \tag{4.12}$$

$$k_{mn}^{t}=\frac{1}{m!n!}\frac{\partial^{m+n}k(0,0)}{\partial x^{m}\partial t^{n}}; \quad m,n=0,1,2,...,N.$$

We can put (4.12) in the matrix form :

$$K(x,t)=X(x)K_{t}X^{T}(t), \quad K_{t}=\left[k_{mn}^{t}\right], \quad m,n=0,1,...,N. \tag{4.13}$$

And

$$K(x,t)=L^{(\alpha)}(x)K_{L}L^{(\alpha)T}(t), \quad K_{L}=\left[k_{mn}^{L}\right], \quad m,n=0,1,...,N. \tag{4.14}$$

We write from (4.13) and (4.14)

$$X(x)K_{t}X^{T}(t)=L^{(\alpha)}(x)K_{L}L^{(\alpha)T}(t)\Longrightarrow X(x)K_{t}X^{T}(t)=X(x)\Pi_{L}^{T}K_{L}\Pi_{L}X^{T}(t),$$

Where

$$K_{t}=\Pi_{L}^{T}K_{L}\Pi_{L} \quad or \quad K_{L}=(\Pi_{L}^{T})^{-1}K_{t}\Pi_{L}^{-1}. \tag{4.15}$$

Replacing 4.6 and 4.14 in the integral part :

$$\begin{aligned}\left[I(x)\right] &= \int_{a}^{b}L^{(\alpha)}(x)K_{L}L^{(\alpha)T}(t)L^{(\alpha)}(t)Cdt\\ &= L^{(\alpha)}(x)K_{L}QC\end{aligned} \tag{4.16}$$

Such that

$$\begin{aligned}Q &= \int_{a}^{b}L^{(\alpha)T}(t)L^{(\alpha)}(t)dt,\\ &= \int_{a}^{b}\Pi_{L}X^{T}(t)X(t)\Pi_{L}^{T}dt\\ &= \Pi_{L}H\Pi_{L}^{T};\end{aligned}$$

With

$$H=\int_{a}^{b}X^{T}(t)X(t)dt=[h_{ij}]$$

$$h_{ij}=\frac{b^{i+j+1}-a^{i+j+1}}{i+j+1} \quad ,i,j=0,1,2,...,N.$$

Replacing (4.64) in (4.16) we get

$$\left[I(x)\right]=X(x)\Pi_{L}^{T}K_{L}QC \tag{4.17}$$

Boundary conditions matrix

From (4.61) and (4.62) we can write the matrix of boundary conditions as

$$V_a = [\nu_{00}, \nu_{01}, ..., \nu_{0N}] = [1, a, ..., a^N] \tag{4.18}$$

$$V_b = [\nu_{10}, \nu_{11}, ..., \nu_{1N}] = [1, b, ..., b^N] \tag{4.19}$$

Replacing (4.11) et (4.17) in (4.53) we get

$$\sum_{k=0}^{2} \nu_k(x)X(x)B^k\Pi_L^T C - X(x)\Pi_L^T K_L Q C = f(x). \tag{4.20}$$

Using the collocation points:

$$x_i = a + \frac{b-a}{N}i, \quad i = 0, 1, ..., N.$$

We get the matrix system for (4.20)

$$\sum_{k=0}^{2} \nu_k(x_i)X(x_i)B^k\Pi_L^T C = f(x_i) + X(x_i)\Pi_L^T K_L Q C, \quad i = 0, 1, 2, ..., N.$$

Abbreviated

$$\Big\{\sum_{k=0}^{2} \nu_k X B^k \Pi_L^T - X\Pi_L^T K_L Q\Big\}C = F, \tag{4.21}$$

$$\nu_k = \begin{pmatrix} \nu_k(x_0) & 0 & \dots & 0 & 0 \\ 0 & \nu_k(x_1) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \nu_k(x_N) \end{pmatrix}, \quad F = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{pmatrix},$$

And

$$X = \begin{pmatrix} X(x_0) \\ X(x_1) \\ \vdots \\ X(x_N) \end{pmatrix} = \begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^N \\ 1 & x_1 & x_1^2 & \dots & x_1^N \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & x_N & x_N^2 & \dots & x_N^N \end{pmatrix}$$

The fundamental matrix can be written

$$WC = F \quad \mathit{and} \quad [W; F]; \quad W = \sum_{k=0}^{2} \nu_k X B^k \Pi_L^T - X\Pi_L^T K_L Q \tag{4.22}$$

$$= \Big(\sum_{k=0}^{2} \nu_k X B^k - X K_t H\Big)\Pi_L^T$$

The augmented matrix is of the form

$$[\widetilde{W}|\widetilde{F}] = \left[\begin{array}{ccccc|c} w_{00} & w_{01} & w_{02} & \dots & w_{0N} & F(x_0) \\ w_{10} & w_{11} & w_{12} & \dots & w_{1N} & F(x_1) \\ w_{20} & w_{21} & w_{22} & \dots & w_{2N} & F(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{N,0} & w_{N,1} & w_{N,2} & \dots & w_{N,N} & F(x_N) \\ v_{00} & v_{01} & v_{02} & \dots & v_{0N} & \alpha \\ v_{10} & v_{11} & v_{12} & \dots & v_{1N} & \beta \end{array}\right] \tag{4.23}$$

Finally, we have an over-determined system with $(N+3)$ linear equations which can be readily solved by using least square method.

4.1.2 Numerical illustrations

In this section we present some results of maximum absolute error at 1000 equidistant points on the interval $[0, L]$ for two examples (4.2) and (4.3) at different value N.

Example 4.2

$$u^{'}(x) - \int_0^\infty xte^{-2t}\sin(t)u(t)dt = -\frac{3}{50}x - e^{-x}, \quad x \in [0,\infty) \tag{4.24}$$

With $u(0) = 1$, whose exact solution is $u(x) = e^{-x}$.

N	Maximum absolute error
5	9.55e-02
10	2.4210e-004
15	2.3489e-004

Table 4.1: The maximum absolute errors for the Example (4.2) with L=10.

Example 4.3 *[58] Consider the problem*

$$u^{''}(x) - 2u^{'}(x) - 8u(x) - \int_0^\infty x(t^2+1)u(t)dt = -\frac{3}{4}x, \quad x \in [0,\infty) \tag{4.25}$$

With $u(0) = 1, u(\infty) = 0$, whose exact solution is $u(x) = e^{-2x}$.

N	Maximum absolute error
5	2.927e-01
10	1.4923e-004
15	4.5407e-005

Table 4.2: The maximum absolute errors for the Example (4.3) with L=5.

Conclusion

As we have seen in the previous examples, the matrix method using Laguerre does not give good results, because the Laguerre polynomials increase when the value of N increases, and they are unstable while the approximate solution in the examples decreases. This is why we used another basis more stable at infinity which is mapped Legendre functions.

4.4 Numerical methods using mapped Legendre functions

The main objective in this section is to extend the Legendre spectral method to a class of Fredholm integro-differential equations on the half-line (4.1), (4.2). The key idea is to transform integro-differential equation on the half-line into a singular equation on the finite interval $[-1, 1]$, by means of a suitable family of one-to-one mappings, and then apply the Legendre spectral collocation method to solve the resulting equation. We introduce some properties of mapped Legendre functions, the Legendre collocation method is presented for the resulting equation in which the differential and integral parts are replaced by their operational matrix representations with collocation points, finally some numerical results are given to clarify and demonstrate the feasibility of the proposed method. The following abbreviations are used throughout the present work, whenever convenient: ELC-Exponential Legendre collocation. RLC-Rational Legendre collocation. LLC-Logarithmic Legendre collocation.

In Eq.(4.1), $k(x, t), f(x)$ and $\nu_k(x), k = 0, 1, 2$ are given continuous functions and $u(x)$ is unknown function. Furthermore, the integral part is assumed to be an improper Riemann integral and its value is to exist.
Various spectral methods are available in the literature to solve problems in semi-infinite domains, $x \in [0, \infty)$.They can be essentially classified in four approaches. The classical approach is to expand the underling solution of the problem in Laguerre polynomials or functions [12]. The second approach consists to solve the problem on a large but finite

4.4.1 Mapped Legendre functions for the semi-infinite interval

In this section, we introduce mapped Legendre functions and express some of their basic properties. Moreover, we present function approximations using orthogonal mapped Legendre functions basis in some weighted $L^2_{\rho_s}[0, \infty)$ space

Mapped Legendre functions

let us consider a family of one-to-one mappings as we said before (3.4),(3.5) and (3.6) between $y \in [-1,1]$ and $x \in [0,\infty)$ of the form

$$x = \varphi_s(y), y \in I, s > 0 \tag{4.26}$$

Such that

$$\frac{dx}{dy} = \varphi_s^{'}(y) > 0, \quad \varphi(-1) = 0, \varphi(1) = \infty \tag{4.27}$$

And s is a positive scaling factor. In an adaptive procedure, s would be chosen to minimize some measure of the error.
Without loss of generality, we assume that the mapping (4.26) is explicitly invertible and denote

$$y = \varphi_s^{-1}(x), \quad x \in \Lambda, \quad s > 0 \tag{4.28}$$

Several interesting mappings have been proposed and implemented in practice. However, the most frequently used are exponential, rational and logarithmic functions as we had shown in chapter 3.
For a given mapping $y = \varphi_s^{-1}(x)$,we define the so-called mapped Legendre functions by

$$L_{s,n}(x) = P_n(\varphi_s^{-1}(x)), \quad n = 0, 1, 2, ... \tag{4.29}$$

They are orthogonal in the interval $[0,\infty)$ with respect to the weight function

$$\rho_s(x) = \frac{dy}{dx} \tag{4.30}$$

With the orthogonality property:

$$\int_0^{\infty} L_{s,n}(x)L_{s,m}(x)\rho_s(x)dx = \frac{2}{2n+1}\delta_{n,m} \tag{4.31}$$

For any function $u \in L^2_{\rho_s}(\Lambda)$, the following expansion holds

$$u(x) = \sum_{j=0}^{+\infty} \widehat{u}_{s,j} L_{s,j}(x) \quad with \quad \widehat{u}_{s,j} = \frac{2j+1}{2} \int_0^{+\infty} u(x)L_{s,j}(x)\rho_s(x)dx \tag{4.32}$$

If $u(x)$ is truncated up to the $N+1$ terms, then it can be written as

$$u(x) \simeq u_N(x) = \sum_{j=0}^{+\infty} \widehat{u}_{s,j} L_{s,j}(x) \tag{4.33}$$

4.4.2 The solution method

By applying a mapping (4.26), Eq. (4.1) can be reformed as

$$\sum_{k=0}^{2} H_{s,k} U_s^{(k)}(y) - \int_{-1}^{1} K_s(y,z)U_s(y)\varphi_s^{'}(z)dz = F_s(y) \quad s > 0, \quad -1 \le y \le 1 \tag{4.34}$$

Under the boundary conditions

$$U_s(-1)=\alpha, \quad U_s(1)=\beta \tag{4.35}$$

Where $K_s(y,z)=k(\varphi_s(y),\varphi_s(z))$, $F_s(y)=f(\varphi_s(y))$ and $H_{s,k}(y)$ are functions defined in the interval $-1 \leq y,z \leq 1$ with

$$H_{s,0}(y)=\nu_0(\varphi_s(y)), \quad H_{s,1}(y)=\frac{\nu_1(\varphi_s(y))}{\varphi_s^{'}(y)}-\frac{\varphi_s^{''}(y)}{(\varphi_s^{'}(y))^3}\nu_2(\varphi_s(y)), \quad H_{s,2}(y)=\frac{\nu_2(\varphi_s(y))}{(\varphi_s^{'}(y))^2} \tag{4.36}$$

Let $\mathcal{I}_N : \mathcal{C}(I) \longrightarrow \mathcal{P}_{\mathcal{N}}$ be the Legendre–Gauss interpolation operator. The Legendre spectral method consists in finding $U_{s,N} \in \mathcal{P}_{\mathcal{N}}$ such that

$$\sum_{k=0}^{2} H_{s,k}U_{s,N}^{(k)}(y)-\int_{-1}^{1} k_s(y,z)U_{s,N}(y)\varphi_s^{'}(z)dz=\mathcal{I}_{\mathcal{N}}F_s(y) \qquad -1 \leq y \leq 1 \tag{4.37}$$

If $U_{s,N}(y)=\sum_{j=0}^{N} c_{s,j}P_j(y)$ is an approximate solution of Eq. (4.34) using Legendre spectral method, then $u_N(x)=U_{s,N}(y)$ with $y=\varphi_s^{-1}(x)$ an approximate solution of Eq.(4.1)using mapped Legendre spectral method. Indeed, it holds that

$$\widehat{u}_{s,j}=\frac{2j+1}{2}\int_0^{\infty} u(x)L_{s,j}(x)\rho_s(x)dx=\frac{2j+1}{j}\int_{-1}^{1} U_s(y)P_j(y)dy=c_{s,j}$$

Hence using (4.29) , we obtain

$$u_N(x)=\sum_{j=0}^{N}\widehat{u}_{s,j}L_{s,j}(x)=\sum_{j=0}^{N}c_{s,j}P_j(\varphi_s^{-1}(x))=U_{s,N}(y) \tag{4.38}$$

Clearly, $u_N(x)$ is a combination of the mapping $y=\varphi_s^{-1}(y)$ for $y \in I$,with $U_{s,N}(y)$.

4.4.3 Fundamental matrix relations

First, let us show Eq. (4.37) in the form

$$D_{s,N}(y)-I_{s,N}(y)=F_{s,N}(y) \tag{4.39}$$

Where

$$D_{s,N}(y)=\sum_{k=0}^{2} H_{s,k}(y)U_{s,N}^{(k)}(y), \quad I_{s,N}(y)=\int_{-1}^{1} K_s(y,z)U_{s,N}(z)\varphi_s^{'}(z)dz \quad F_{s,N}(y)=\mathcal{I}_N F_s(y), \tag{4.40}$$

And

$$U_{s,N}(y)=\sum_{j=0}^{N} c_{s,j}P_j(y)=P(y)C, \quad -1 \leq y \leq 1 \tag{4.41}$$

With

$$C=[c_{s,0},c_{s,1},...,c_{s,N}]^T$$

$$P(y) = [P_0(y), P_1(y), ..., P_N(y)] \tag{4.42}$$

The derivative of the solution expressed by (4.41) can be written in the matrix forms as follows

$$[U^{(k)}_{s,N}(y)] = P^{(k)}(y)C. \tag{4.43}$$

First, we can write $P(y)$ in the matrix form as follows

$$P(y) = Y(y)\Pi, \tag{4.44}$$

Where

$$Y(y) = [1, y, y^2, ..., y^N], \tag{4.45}$$

And Π is given in (2.20) and (2.21), on the other hand, we can write

$$P^{(k)}(y) = Y^{(k)}(y)\Pi \tag{4.46}$$

And

$$Y^{(1)}(y) = Y(y)B, \tag{4.47}$$

With

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 2 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 3 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & N-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & N \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \tag{4.48}$$

Also, from (4.47), we obtain

$$\begin{array}{rcl} Y^{(1)}(y) & = & Y(y)B \\ Y^{(2)}(y) & = & Y^{(1)}(y)B = Y(y)B^2 \\ & \vdots & \\ Y^{(k)}(y) & = & Y^{(k-1)}(y)B = Y(y)B^k \end{array}$$

Then

$$P^{(k)}(y) = Y(y)B^k\Pi \tag{4.49}$$

Consequently, by substituting the matrix relation (4.49) into Eq. (4.43), we obtain the matrix relation for $U^{(k)}_{s,N}(y)$ as follows

$$[U^{(k)}_{s,N}(y)] = Y(y)B^k\Pi C \tag{4.50}$$

Next, let $\{\sigma_i\}_{i=0}^N$,be the Legendre-Gauss points, which are zeros of Legendre polynomial $P_{N+1}(y)$. By [18], there exists a unique set of Christoffel numbers $\{\omega_i\}_{i=0}^N$ such that

$$\int_{-1}^{1} \phi(y)dy = \sum_{i=0}^{N} \phi(\sigma_i)\omega_i, \quad \forall \phi \in \mathcal{P}_{2N+1} \tag{4.51}$$

In fact, we have

$$\omega_i = \frac{2}{(1-\sigma_i^2)[P'_{N+1}(\sigma_i)]^2}, \quad 0 \le i \le N \tag{4.52}$$

The fundamental matrix relation for differential part $D_{s,N}(y)$ based on collocation points is given by

$$D = \sum_{k=0}^{2} H_k Y B^k \Pi C \tag{4.53}$$

Where

$$H_k = diag(H_{s,k}(\sigma_j))_{0\le j\le N} = \begin{pmatrix} H_{s,k}(\sigma_0) & 0 & \dots & 0 & 0 \\ 0 & H_{s,k}(\sigma_1) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & H_{s,k}(\sigma_N) \end{pmatrix}$$

$$\mathbf{Y} = \begin{pmatrix} Y(\sigma_0) \\ Y(\sigma_1) \\ Y(\sigma_2) \\ \vdots \\ Y(\sigma_N) \end{pmatrix} = \begin{pmatrix} 1 & \sigma_0 & \sigma_0^2 & \dots & \sigma_0^N \\ 1 & \sigma_1 & \sigma_1^2 & \dots & \sigma_1^N \\ 1 & \sigma_2 & \sigma_2^2 & \dots & \sigma_2^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \sigma_N & \sigma_N^2 & \dots & \sigma_N^N \end{pmatrix} \tag{4.54}$$

For the Fredholm integral part, we have

$$I_{s,N}(y) = \sum_{j=0}^{N} c_{s,j} \int_{-1}^{1} K_s(y,z) P_j(z) \varphi_s^{'}(z) dz \simeq \sum_{j=0}^{N} c_{s,j} \sum_{i=0}^{N} K_s(y,\sigma_i) P_j(\sigma_i) \varphi_s^{'}(\sigma_i)\omega_i. \tag{4.55}$$

By setting

$$\upsilon_i = \varphi_s^{'}(\sigma_i)\omega_i,$$

the fundamental matrix relation for the integral part $I_{s,N}(y)$ based on collocation points is given by

$$I_f = M\Phi Y \Pi C \tag{4.56}$$

With

$$M = (K_s(\sigma_j,\sigma_i))_{0\le j,i\le N}, \quad \Phi = diag((\nu_i)_{0\le i\le N})$$

Using relation (4.53) and (4.56), Eq. (4.34) is reduced to the following system

$$WC = F \tag{4.57}$$

Where

$$W = [\omega_{ij}] = \sum_{k=0}^{2} H_k Y B^k \Pi - M\Phi Y \Pi \tag{4.58}$$

$$F = [F_s(\sigma_0), F_s(\sigma_1), ..., F_s(\sigma_N)]^T$$

On the other hand, the fundamental matrix for the boundary conditions can be written as

$$V_0 C = \alpha \tag{4.59}$$

$$V_1 C = \beta \tag{4.60}$$

With

$$V_0 = [\nu_{00}, \nu_{01}, ..., \nu_{0N}] = [1, -1, ..., (-1)^N] \tag{4.61}$$

$$V_1 = [\nu_{10}, \nu_{11}, ..., \nu_{1N}] = [1, 1, ..., 1] \tag{4.62}$$

Name	Function $x = \varphi_s(y)$	Inverse $y = \varphi_s^{-1}(x)$	Derivative $\varphi_s'(y) = \frac{dx}{dy}$	Deriv. Mapped $\rho_s(x) = \frac{dy}{dx}$
Exponential	$x = s\ln(\frac{2}{1-y})$	$1 - 2\exp(-x/s)$	$\frac{s}{1-y}$	$\frac{2}{s}\exp(-x/s)$
Rational	$\frac{s(1+y)}{1-y}$	$\frac{x-s}{x+s}$	$\frac{2s}{(1-y)^2}$	$\frac{2s}{(x+s)^2}$
Logarithmic	$\frac{s}{2}\ln(\frac{3+y}{1-y})$	$2\tanh(x/s) - 1$	$\frac{2s}{(1-y)(3+y)}$	$\frac{2}{s}(1 - \tanh^2(x/s))$

Table 4.3: Exponential, rational and logarithmic mapping functions, their inverse, and derivative.

In order to satisfy the boundary conditions in the collocation method we add the above equations to the final system given in (4.57), then we obtain

$$\widetilde{W}C = \widetilde{F} \tag{4.63}$$

So that the new augmented matrix is of the form

$$[\widetilde{W}|\widetilde{F}] = \left[\begin{array}{ccccc|c} w_{00} & w_{01} & w_{02} & \dots & w_{0N} & F_s(\sigma_0) \\ w_{10} & w_{11} & w_{12} & \dots & w_{1N} & F_s(\sigma_1) \\ w_{20} & w_{21} & w_{22} & \dots & w_{2N} & F_s(\sigma_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{N,0} & w_{N,1} & w_{N,2} & \dots & w_{N,N} & F_s(\sigma_N) \\ v_{00} & v_{01} & v_{02} & \dots & v_{0N} & \alpha \\ v_{10} & v_{11} & v_{12} & \dots & v_{1N} & \beta \end{array}\right] \tag{4.64}$$

Finally, we have an over-determined system with $(N+3)$ linear equations which can be readily solved by using least square method.

4.4.4 Numerical illustrations

In this section, the numerical results of the following examples are obtained by considering three practical mapping functions, that are given with some of their properties in Table (4.3). All computations were carried out by MATLAB R2009b. For the most part, the error of the approximations is measured as the discrete L^2-norm with respect to the mapped Legendre weight $\rho_s(x)$, of the difference between approximate and exact solutions, given by

$$\mathbf{e}_N = \Big(\sum_{i=0}^{N}[u(\zeta_i) - u_N(\zeta_i)]^2 w_i\Big)^{\frac{1}{2}} \tag{4.65}$$

Where

$$\zeta_i = \varphi_s(\sigma_i), \quad w_i = \omega_i, \quad 0 \le i \le N \tag{4.66}$$

Are the mapped Legendre–Gauss points and weights.

Remark 4.4.1 *Note that, according to Theorem (3.1), any function defined in $L^2_{\rho_s}(\Lambda)$, whose transformations under aforementioned mappings (see Table (4.3)) are analytic on $[-1, 1]$, can be approximated by a truncated series of the form (4.33) with the superlinear convergence.*

We will focus on the first Example (4.2) mentioned in the previous section . By applying exponential mapping transformation with $s = 2$, we obtain

$$U_2(y) = u(\varphi_2(y)) = e^{-2\ln(\frac{2}{1-y})} = 0.25 - 0.5y + 0.25y^2, \quad y \in [-1, 1] \tag{4.67}$$

Clearly, $U_2(y)$ is a quadratic, analytic function on $[-1, 1]$. Then by Theorem (3.1), $u(x)$ can be approximated by a truncated series of the form (4.33) with superlinear convergence. Now, it may be acceptable to seek an approximate solution for the given problem. Let $N = 5$, then ELC method consists in finding $u_5(x) \in L^2_{\rho_2}(\Lambda)$ such that

$$u_5(x) = U_{2,5}(y) = \sum_{j=0}^{5} c_{2,j} P_j(y), \quad y = 1 - 2\exp(-x/2), \quad y \in I, x \in \Lambda.$$

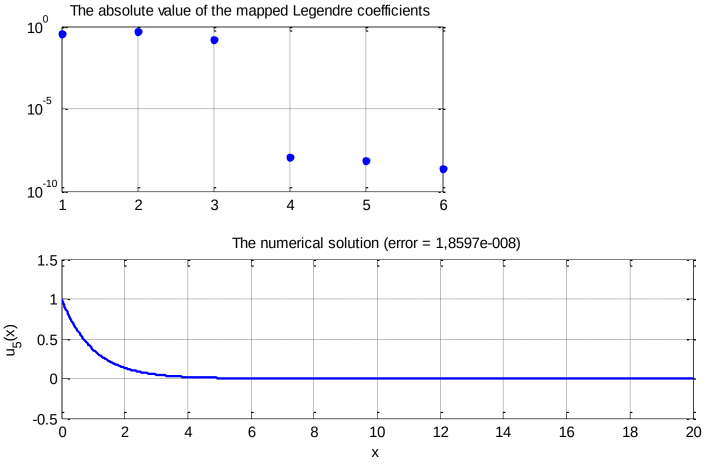


Figure 4.1: Numerical results of ELC method for Example (4.2).

The augmented matrix based on the conditions from Eq.(4.63) is calculated as

$$[\widetilde{W}|\widetilde{F}] = \left[\begin{array}{cccccc|c} -0.0110 & 0.9683 & -2.6995 & 4.8503 & -6.9531 & 8.5218 & -0.9377 \\ -0.0594 & 0.8419 & -1.6287 & 1.4699 & -0.0877 & -1.7219 & -0.7122 \\ -0.1532 & 0.6484 & -0.3945 & -0.6845 & 0.9487 & 0.3192 & -0.4410 \\ -0.3089 & 0.4393 & 0.3709 & -0.4487 & -0.6158 & 0.2026 & -0.2608 \\ -0.5678 & 0.2772 & 0.5168 & 0.2274 & -0.0291 & -0.3346 & -0.2418 \\ -1.0836 & 0.2396 & 0.4395 & 0.0284 & 0.1551 & 0.3301 & -0.4077 \\ 1.0000 & -1.0000 & 1.0000 & -1.0000 & 1.0000 & -1.0000 & 1.0000 \end{array}\right]$$

By solving this system, the Legendre coefficients matrix is gained as

$$C = [1/3 - 1/21/6 - 1/87540636 - 1/145575192 - 1/418815286]^T,$$

and the approximate solution of the integro-differential equation (4.34) is given by

$$U_{2,5}(y) = 0.333333334481310P_0(y) - 0.500000012393087P_1(y) + 0.166666651635238P_2(y)$$

$$-0.000000011423266P_3(y) - 0.000000006869302P_4(y) - 0.000000002387687P_5(y),$$

Or in the form

$$U_{2,5}(y) == 0.25 - 0.5y + 0.25y^2 - (7.6659e-09)y^3 - (3.0053e-08)y^4 - (1.8803e-08)y^5.$$

From (4.67), it follows

$$U_2(y) = U_{2,5}(y) + E(y), \qquad E(y) = (7.6659e-09)y^3 + (3.0053e-08)y^4 + (1.8803e-08)y^5.$$

Thus, we get

$$\|u - u_2\|_{\rho_2}^2 = \int_0^\infty |u(x) - u_5(x)|^2 \rho_s(x) dx = \int_{-1}^1 |E(y)|^2 dy = 3.4584e - 16.$$

Figure (4.1) shows the obtained numerical results. From this figure, it is clear that with an appropriate scaling factor $s > 0$, the ELC method gives us results with higher degree of accuracy with small values of N.

Example 4.5 *Consider the problem*

$$u^{'}(x) - \int_0^\infty \sqrt{x} t e^{-t} u(t) dt = e^{-x} - \frac{7}{4}\sqrt{x}, \quad x \in [0, \infty) \tag{4.68}$$

with $u(0) = 1$, whose exact solution is $u(x) = 2 - e^{-x}$. Table (4.4) shows the numerical errors obtained by using ELC, RLC and LLC methods with $s = 8$ and different values of N.

N	ELC	RLC	LLC
6	7.8662e-03	7.6029e-03	2.6636e-02
8	8.6465e-10	6.8591e-04	2.4486e-03
10	3.6415e-11	8.3579e-05	1.7688e-04
12	2.6318e-12	1.5909e-05	1.0957e-05
14	2.7790e-13	4.1988e-06	6.1064e-07
16	1.3974e-14	1.1358e-06	3.2104e-08
18	5.2350e-16	2.7400e-07	1.9096e-09
20	2.5363e-15	5.1813e-08	2.6364e-10
24	5.4519e-15	1.4717e-09	5.1830e-11
32	7.3918e-14	1.4672e-11	6.0497e-12
40	2.3759e-14	5.3125e-12	1.1109e-12

Table 4.4: Discrete L^2 errors by ELC, RLC and LLC methods for Example (4.5).

N	ELC		RLC		LLC		[58] $m = 0, m^{'} = 1$
	$s = 2$	$s = 3$	$s = 2$	$s = 7/2$	$s = 3/2$	$s = 2$	
4	2.09e-07	1.96e-02	1.55e-02	1.44e-02	1.92e-02	4.45e-02	9.27e-02
6	1.42e-08	6.47e-11	2.46e-03	2.46e-03	2.32e-04	4.35e-04	2.30e-02
8	1.86e-09	2.92e-12	4.41e-04	4.09e-04	1.90e-05	7.18e-05	3.20e-03

Table 4.5: Comparison of the maximum absolute errors for Example (4.3).

N	$s=1$	$s=2$	$s=3$
08	5.4381e-05	2.0199e-05	9.0576e-04
10	9.6498e-07	1.4783e-06	1.1360e-04
12	1.1506e-06	1.2708e-07	1.2084e-05
14	2.8205e-07	1.3798e-08	1.1902e-06
16	5.3692e-08	2.5737e-09	1.1103e-07
18	8.3869e-09	2.9531e-10	1.0021e-08
24	1.0922e-10	1.8303e-13	6.3295e-12
30	2.4630e-12	5.3664e-15	1.0460e-14
36	1.5953e-14	5.3071e-14	5.5598e-14
42	6.6118e-15	3.2543e-14	7.4442e-14

Table 4.6: Discrete L^2 errors by RLC method for Example (4.6).

N	RLC	[58]
4	5.7945e-04	6.5671e-04
6	1.4138e-04	7.1255e-04
8	2.1271e-06	2.8510e-04

Table 4.7: Comparison of the maximum absolute errors for Example (4.6).

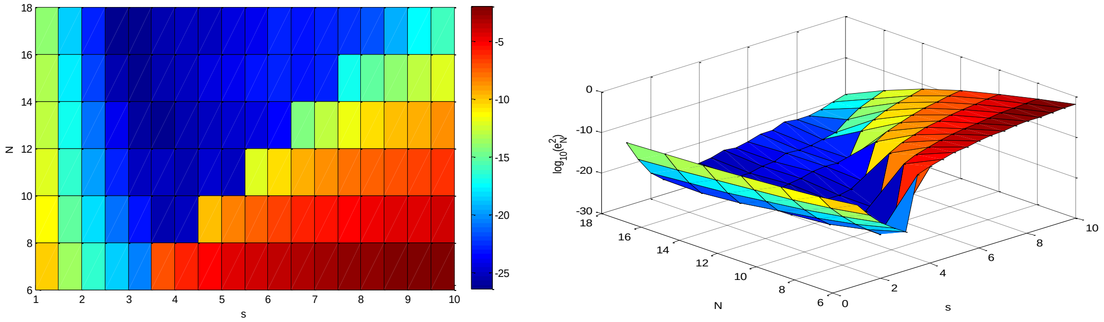
Sub-figures (a), (b), (c), (d), (e) and (f) of Fig. 2 show the 2D contour plots and corresponding 3D surface plots of the Logarithmic $\mathbf{e}_N^2$ for ELC, RLC and LLC methods with different values of s and N. These sub-figures illustrate the interval that we can choose the s-parameter to provide a good estimate of the truncation error for the given problem. As we can see, using ELC method gives more accurate results and faster convergence compared to RLC and LLC methods. However, LLC method is less sensitive to the scale factor s. In Table 3, comparison between the values of $\|U_s - U_{s,N}\|_\infty$ (computed as the maximum of the absolute difference between the exact solution of the resulting Eq. (18) and the approximated ones) shows that the numerical results based on all the three methods are in good agreement with those obtained from [58].

Example 4.6 *Consider the problem*

$$(1+x^2)u''(x)-(1+x)u'(x)+u(x)-\int_0^\infty x(t^2-1)e^{-3t}u(t)dt = \frac{4}{1+x}+\frac{2}{9}x, \quad x\in[0,\infty) \tag{4.69}$$

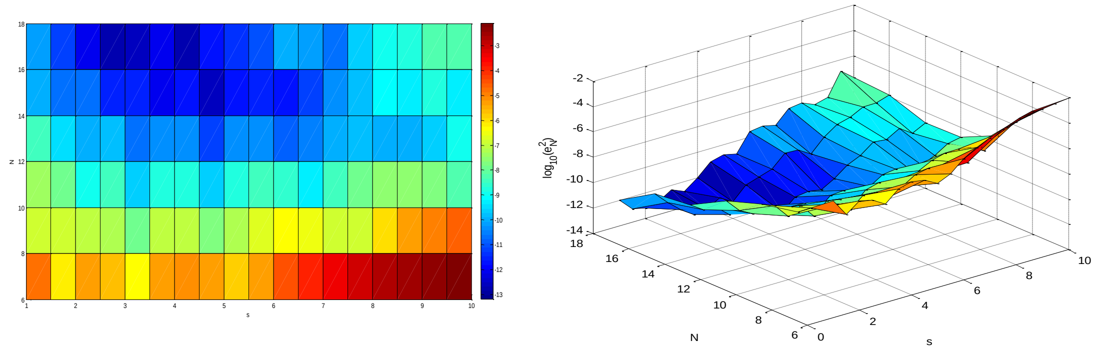
With $u(0)=1, u(\infty)=0$.The exact solution of this problem is $u(x)=\frac{1}{1+x}$, which is a smooth function and decays very slowly at infinity. Numerical errors obtained by using RLC method with different values of N and s are reported in Table (4.6).

Once again,Table (4.7) shows that RLC method with $s = 3/2$ performs significantly better than [58].



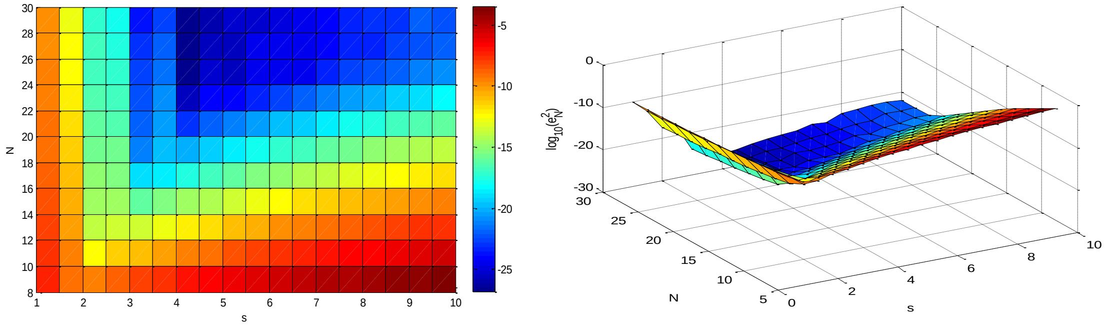
(a) ELC:2D view.

(b) ELC:3D view.



(c) RLC:2D view.

(d) RLC:3D view.



(e) LLC:2D view.

(f) LLC:3D view.

Figure 4.2: $2D$ contour plots and corresponding $3D$ surface plots of the Logarithmic e_N^2 by ELC, RLC and LLC methods for Example (4.3)

General conclusion and future prospects

This work has been concerned with the numerical solution of Fredholm IDEs over the real half-line, where finding analytical solutions are usually difficult, we have focused on some recent numerical methods such that spectral collocation method for approximate the exact solution by truncated series of Legendre functions as basis. Firstly, we needed to test spectral collocation and Galerkin-legendre methods on Fredholm IDEs on bounded interval $[a, b]$ and the obtained results it turned out to be so much better than the famous classic methods such that Sinc basis functions, Homotopy analysis, CAS wavelet, Legendre-collocation matrix and the approximate solution is convergent provided that the accuracy is increased sufficiently.

For mapping between the interval $[0, \infty[$ and $[-1, 1]$ we have used three mappings namely algebraic, logarithmic, and exponential, it resulted from that three orthogonal family functions on the semi-infinite interval namely respectively Legendre algebraic functions, Legendre logarithmic functions and Legendre exponential functions, without loss of generality, we further assume that the mapping is explicitly invertible, and the exact solution is is analytic on $[-1, 1]$. The behavior of each functions have been plotted for $s = 1$, moreover the parameter $s > 0$ is a scaling/stretching factor which can be used to fine tune the spacing of collocation points.

A suitable family of variable transformations to reform Fredholm IDE on the half-line, and then fit the solution of the resulting singular equation on the finite interval $[-1, 1]$ by the Legendre polynomials. One of the main advantages of this method is that the numerical solution of the problem can be converted into a system of algebraic equations using the operational matrix. Superlinear convergence rates of the mapped Legendre function approximations can be achieved under sufficient conditions Therefore, a careful choice of the mappings is required to obtain a superlinear convergence rate of the proposed scheme. This has been clearly illustrated.

Among the typical numerical tests discussed , where the superlinear convergence was achieved for our three proposed methods. Moreover, it is worth noticing that the convergence can be greatly accelerated by using ELC method with an appropriate scaling parameter s and using ELC method gives more accurate results and faster convergence compared to RLC and LLC methods. However, LLC method is less sensitive to the scale factor s, and the numerical results based on all the three methods are in good agreement with those obtained from generalized Laguerre polynomials. The obtained results clearly show that the present spectral collocation method is a powerful mathematical tool for finding the numerical solutions of integro-differential equations on the half line.

For contribution on the methods of numerical resolution of integro-differential equations on an unbounded interval, and to improve the results obtained, it is essential

- To master the various recent approximation techniques, also to master programming languages evolves.
- This problem can be subjected to other numerical methods like Galerkin and Tau where we expect good results.
- It is possible also to extend this method to the high-order integro-differential equations.
- Looking forward to find another mapping from $[0, \infty)$ to $[-1, 1]$

Bibliography

[1] A. C. L. Barnard, I. M. Duck, and M. S. Lynn, *The application of electromagnetic theory to electrocardiology.*Biophysical Journal, Texas, 1967.

[2] A. Guechi, A. Rahmoune, *Mapped Chebyshev spectral methods for solving second kind integral equations on the real line.* The Australian Journal of Mathematical Analysis and Applications, 2017.

[3] A. Mohsen, M. El-Gamel, *A sinc-collocation method for the linear fredholm integro-differential equations*Z. Angew. Math. Phys. 58 (2007) 380-390.

[4] A. Rahmoune, *Sur la Résolution Numérique des équations Intégrales en utilisant des Fonctions Spéciales.* Thèse de doctorat,2011.

[5] A. Rahmoune, *Spectral collocation method for solving Fredholm integral equations on the half-line*Appl. Math. Comput. 219 (17) (2013) 9254-9260.

[6] A.I.Garralda.Guillem, M.Ruiz.Galán, M.I.berenguer,M.V.Fernàndez.Muoz, *A sequential approch for solving the fredholm integro-differential equation.*Applied Numerical Mathematics (2011).

[7] A.Yildirim S. Yüzbasi, N.Sahin, *A collocation approach for solving high-ordre linear fredholm-volterra integro-differential equations.*Mathematical and Computer Modelling 55, 547-563 (2012).

[8] Ali Ebadian P. Darania, *A method for the numerical solution of the integro-differential equations.*Applied Mathematics and Computation 188, 657-668 (2007).

[9] B.Y. Guo, *Jacobi spectral approximation and its applications to differential equations on the half line*J. Comput. Math. 18 (2000) 95-112.

[10] B.Y. Guo, *Jacobi spectral method for differential equations with rough asymptotic behaviors at infinity, Comput*Math. Appl. 46 (2003) 95-104.

[11] B.Y. Guo, J. Shen, *Laguerre-Galerkin method for nonlinear partial differential equations on a semi-infinite interval*Numer. Math. 86 (2000) 635-654.

[12] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, *Spectral Methods.* Springer-Verlag, Berlin.2006.

[13] C.E. Grosch, S.A. Orszag, *Numerical solution in unbounded regions: Coordinate transforms*J. Comput. Phys. 25 (1977) 273-296.

[14] E.varley, J.D.A.walker, *A method for solving singular integro-differential equations.*IMA Journal of Applied Mathematics 43, 11-45 (1989).

[15] F. Baharifard, S. Kazem, K. Parand, *Rational and Exponential Legendre Tau Method on Steady Flow of a Third Grade Fluid in a Porous Half Space*Int. J. Appl. Comput. Math (2016) 2:679–698.

[16] Funaro, D, *Polynomial Approximation of Differential Equations.*Springer-Verlag, New York 1992.

[17] G. Alvarado Barrios, J. C. Retamal, E. Solano and M. Sanz. *Analog simulator of integro-differential equations with classical memristors.*Scientific Reports,2019.

[18] G. Szegö, *Orthogonal Polynomials* Amer. Math. Soc., New York, 1967.

[19] George B. Rybicki, *Radiative Transfer.*J. Astrophys. Astr,USA,1996.

[20] H.Jaradat, O.Alsayyed and S.Al-Shara, *Numerical Solution of Linear Integro-Differential Equations.*Journal of Mathematics and Statistics 4 (4): 250-254, 2008, Jordan.

[21] Han Danfu, Shang Xufeng, *Numerical solution of integro-differential equations by using CAS wavelet operational matrix of integration.*Applied Mathematics and Computation 194 (2007) 460-466, China.

[22] Ian H. Sloan, *Quadrature Methods for Integral Equations of the Second Kind Over Infinite Intervals.*Mathematics Of Computation 1981. in Applied Mathematics (1985) .

[23] Ishak Hashim, *Adomian decomposition method for solving BVPs for fourth-order integro-differential equations* .Journal of Computational and Applied Mathematics 193 (2006) 658-664, Malaysia.

[24] J. Biazar and F. Salehi, *Chebyshev Galerkin method for integro-differential equations of the second kind.*Iranian Journal of Numerical Analysis and Optimization Vol. 6, No. 1, (2016), pp 31-42.

[25] J. Rashidinia, M. Zarebnia, *The numerical solution of integro-differential equation by means of the Sinc method.*Applied Mathematics and Computation 188 (2007) 1124-1130, Iran.

[26] J. Shen, *Stable and efficient spectral methods in unbounded domains using Laguerre functions*SIAM J. Numer. Anal. 38 (4) (2001) 1113-1133.

[27] J. Shen, L.L. Wang, *Some recent advances on spectral methods for unbounded domains*Commun. Comput. Phys. 5 (2-4) (2009) 195-241.

[28] J.P. Boyd, *Chebyshev and Fourier Spectral Methods.*second ed., Dover Publications INC, Mineola, NY, USA, 2001.

[29] J.P. Boyd, *Orthogonal rational functions on a semi-infinite interval.*J. Comput. Phys. 70 (1987) 63-88.

[30] Jafar Biazar. and Mohammad Ali Asadi, *Indirect RBF for High-Order Integro-Differential Equations.*British Journal of Mathematics and Computer Science 11(1): 1-16, 2015, Article no.BJMCS.19199.

[31] Jan Medlock, Mark Kot, *Spreading disease: integro-differential equations old and new.* Mathematical Biosciences 184 (2003) 201–222

[32] Jie Shen .Li-Lian Wang .Haijun Yu, *Approximations by orthonormal mapped Chebyshev functions for higher-dimensional problem in unbounded domains* . Journal of Computational and applied mathematics. 2014

[33] Jie Shen.Tao Tang .Li-Lian Wang, *Spectral Methods.* Algorithms, Analysis and Applications. 2010

[34] K. Maleknejad, B. Basirat, E. Hashemizadeh, *Hybrid Legendre polynomials and Block-Pulse functions approach for nonlinear Volterra-Fredholm integro-differential equations.*Computers and Mathematics with Applications 61 (2011) 2821-2828, Iran.

[35] K. Maleknejad, F. Mirzaee, *Using rationalized Haar wavelet for solving linear integral equations.*Appl. Math. Comput. 160 (2005) 579-587.

[36] M. Matinfar, A. Riahifar, *Numerical solution of Fredholm integral-differential equations on unbounded domain.* Journal of Linear and Topological Algebra,2015.

[37] M. Rahman, *Integral Equations and their Applications.* Dalhousie University, Canada,2007.

[38] M.Sezer. S.Yalçinbaçs, *A Taylor collocation method for the approximate solution of general linear fredholm-volterra integro-difference equations with mixed argument.*Mathematics and Computation 175, 675-690 (2006).

[39] M.T. Rashed, *Lagrange interpolation to compute the numerical solutions of differential, integral and integro-differential equations..*Applied Mathematics and Computation 151, 869-878 (2004).

[40] M.T. Rashed, *Numerical solution of a special type of integro-differential equations.*Applied Mathematics and Computation 143, 73-88 (2003).

[41] Matt Young, *The Stone-Weierstrass Theorem.* MATH 328 Notes Queen's University at Kingston Winter Term,2006.

[42] Mohamed Fathy, Mohamed El-Gamel, M.S. El-Azab, *Legendre-Galerkin method for the linear Fredholm integro-differential equations* .Applied Mathematics and Computation, Egypt,2014.

[43] M. Nadir, A. Rahmoune, *Modified Method for Solving Linear Volterra Integral equations of the Second Kind Using Simpson's Rule*.in International Journal Mathematical Manuscripts. (IJMM) 1(2) (2007), 133-140.

[44] M. Nadir, A. Rahmoune, *Solving linear Fredholm integral equations of the second kind using Newton divided difference interpolation polynomial,*.in International journal of Mathematics and Computation. (IJMC) 7(10) (2010), 1-6.

[45] M. Nadir, B. Gagui, *Two Points for the Adaptive Method for the Numerical Solution of Volterra Integral Equations* .in International Journal Mathematical Manuscripts. (IJMM) 1(2) (2007), 133-140.

[46] M. Nadir, B. Gagui, *Numerical Approximation for Solutions of Ham-merstein Integral Equations in Lp Spaces* .in Sao Paulo Journal of Mathematical Sciences. 8(1) (2014), 23-31.

[47] M. Nadir, B. Gagui, *Numerical solution of Solutions of Hammerstein Integral Equations in Lp Spaces* .in Fasciculi mathematici. 52(2) (2014), 82-91.

[48] M. Nadir, B. Gagui, *Quadratic numerical treatment for singular integral equations with logarithmic kernel*.in Int. J. Computing Science and Mathematics. 10(3) (2019), 288-296.

[49] M. Nadir, B. Lakehali, *Adapted Quadratic Approximation ForWeakly Singular Integrals* .in Applied Mathematics E-Notes. 15 (2015), 225-232.

[50] M. Nadir, B. Lakehali, *Adapted linear approximation for singular integral equations*.in Malaya J. Mat. 2(4) (2014), 497.501.

[51] M. Nadir, N. Djaidja, *Comparison between Taylor and perturbed method for Volterra integral equation of the first kind* .in Numerical Algebra, Control and Optimization. xx(x) (2020), xxx-xxx.

[52] Necdet Bildik, Ali Konuralp , Salih Yalçinbaş, *Comparison of Legendre polynomial approximation and variational iteration method for the solutions of general linear Fredholm integro-differential equations*.Computers and Mathematics with Applications Computers and Mathematics with Applications 59 (2010) 1909-1917, Turkiye

[53] Nurcan Kurt, Mehmet Sezer, *Polynomial solution of high-orderlinear Fredholm integro-differential equations with constant coefficients*.Journal of the Franklin Institute 345 (2008)839-850, Turkey.

[54] O. Coulaud, D. Funaro, O. Kavian, *Laguerre spectral approximation of elliptic problems in exterior domains*Comput. Methods Appl. Mech.Engrg. 80 (1990) 451-458.

[55] P. Darania, Ali Ebadian, *A method for the numerical solution of the integro-differential equations.*Applied Mathematics and Computation 188 (2007) 657-668, Iran.

[56] Peter Linz, *A general theory for the approximate solution of operator equations of the second kind.*SIAM Journal on Numerical Analysis 14, 543-554 (1977).

[57] Peter Linz, *Analytical and Numerical Methods for Volterra Equations.*Society for Industrial and Applied Mathematics (1985).

[58] S. Akhavan, *Numerical solution of singular Fredholm integro-differential equations of the second kind via Petrov-Galerkin method by using Legendre multiwavelet.*J. Math. Comput. Sci. 9 (2014) 321-331.

[59] S. Yalçinbas, M. Sezer , H. Hilmi Sorkun, *Legendre polynomial solutions of high-order linear Fredholm integro-differential equations.*Applied Mathematics and Computation 210 (2009) 334-349.

[60] S.Benyoussef, A.Rahmoune, *Efficient spectral-collocation methods for a class of linear Fredholm integro-differential equations on the half-line*Journal of Computational and Applied Mathematics 377 (2020) 112894.

[61] S.M. Hosseini, S. Shahmorad, *Tau numerical solution of Fredholm integro-differential equations with arbitrary polynomial basis.*Applied Mathematical Modelling 27 (2003) 145-154, Iran.

[62] Salih Yalçinbaş, Mehmet Sezer, *A Taylor collocation method for the approximate solution of general linear Fredholm-Volterra integro-difference equations with mixed argument.*Applied Mathematics and Computation 175 (2006) 675-690, Turkey.

[63] V. V. Vlasov and N. A. Rautian, *Properties of solutions of integro-differential equations arising in heat and mass transfer theory.*Trans. Moscow Math. Soc,Russia, 2014.

[64] Walter Ambrosini, *Lecture Notes for the Course on numerical methods for nuclear reactors* .University of Pisa, Italy.

[65] Y. Maday, B. Pernaud-Thomas, H. Vandeven, *Une rehabilitation des methodes de type Laguerre*Rech. Aerosp. 6 (1985) 353-375.

[66] Y.Chen, T.Tang, *Convergence analysis of the Jacobi spectral-collocation methods for Volterra integral equations with a weakly singular kernel.* Mathematics of computation, 2010.

[67] Ying-Jun jiang, *On spectral methods for Volterra-type integro-differential equations.* Journal of Computational and Applied Mathematics, 2008.

[68] Z. Avazzadeh, M. Heydari, G.B. Loghmani, *Numerical solution of Fredholm integral equations of the second kind by using integral mean value theorem.*Applied Mathematical Modelling 35 (2011) 2374-2383, Iran.

[69] Zakieh Avazzadeh, Mohammad Heydari, *Integral mean value method for solving Fredholm integral equations of the second kind on the half line.*Journal of Multidisciplinary Engineering Science Studies 2016.