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Properties and classes of ternary fuzzy relations

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Introduction

The study of relations is a fundamental tool in mathematics with far-reaching implications across numerous fields, including logic, computer science, algebra, and artificial intelligence. However, the formalization of relational theory did not gain significant momentum until the last century.

The historical development of the theory of relations can be traced back to the mid-19th century. In 1860, Augustus De Morgan [14] first introduced the concept of relations as a formal mathematical object, laying the groundwork for what would become a central area of study in logic and set theory. His work was soon expanded upon by Charles Sanders Peirce [31], who developed the calculus of binary relations. Peirce's work was further advanced by Schröder in 1890 [36]. Despite these early advances, the historical trajectory of the field did not allow for an extensive study of relations beyond the binary case.

The first type of higher-order relation studied was the ternary relation of betweenness [22, 28, 43]. Novák and Novotný laid the groundwork for the study of ternary relations in numerous studies published in the late 20th century [28, 29, 30]. Ternary relations have since been found useful in a variety of scientific fields, including theoretical physics (e.g., computational physics by [41]), mathematics (e.g., group theory by [10]), artificial intelligence (e.g., qualitative spatial reasoning by [9, 23]), computer science (e.g., knowledge graphs by [15, 44], data structures by [1], string matching by [24], video recognition by [37], information modeling by [32, 33], and social networks by [17]), and biology (e.g., phylogenetic modeling by [38]).

Composition of relations, also referred to as calculus of relations is by far the most importent aspect in the study of relations. I has been the focus of numerous textbooks [6, 7, 11, 20, 35]. In the latter half of the 20th century, Bandler and Kohout [4, 39] introduced a novel kind of composition of binary relations by replacing existential quantifiers and logical conjunctions with logical implications, providing a new perspective on relational operations [4]. BK-compositions have since been recognized for their utility in a wide range of applications, from medical diagnosis [5, 40] to decision-making [19, 25].

However, the study of ternary and higher-order relations has remained comparatively underdeveloped, particularly concerning their compositions. This thesis aims to address this gap by providing a systematic approach to the compositions of ternary relations, whose definitions have been the subject of ongoing discussions among researchers [28, 27, 43]. Various alternative definitions, each with its motivation, have been proposed. In this thesis, we propose a general definition that encompasses all four-point and five-point compositions of ternary relations. Moreover, we examine the interaction between these compositions and establish links between them. We also explore their properties, such as associativity, mixed associativity, mixed commutativity, and the existence of a neutral element.

Additionally, we utilize the valuable tools of projections of ternary relations and cylindrical extensions of binary relations to express these newly defined compositions of ternary relations in terms of the well-known and well-established compositions of binary relations. We observed a close connection between the traces of binary relations and their BK-compositions. Leveraging this insight, we introduce the traces of ternary relations based on the BK-compositions of ternary relations.

Lastly, we have shown generalized the above results to the setting of fuzzy L-relation where L is a complete residuated lattice

The thesis is divided into five chapters, each reviewed as follows:

- **Chapter 1** reviews the necessary basic concepts and properties of binary and ternary relations.
- **Chapters 2 and 3** present a systematic approach to studying the composition of ternary relations, focusing on the degrees of freedom available when linking a 3-tuple to two given 3-tuples. It outlines a method for enumerating all possible 4-point compositions (one degree of freedom) and 5-point compositions (two degrees of freedom) of ternary relations and establishes a correspondence between them. Additionally, it identifies associative compositions and explores intriguing mixed-associativity cases. Using projection and cylindrical extension tools, it relates the 4-point and 5-point compositions of ternary relations to the 3-point compositions of binary relations.
- **Chapter 4** introduces the BK-compositions of ternary relations and examines their interaction with the usual compositions. It also revisits the notions of left and right traces of a binary relation and introduces the left, middle, and right traces of a ternary relation, characterizing the main properties of a ternary relation in terms of its traces.
- **Chapter 5** extends the previous notions and results to the fuzzy setting. The thesis concludes with a discussion of findings and future research directions.

1 Preliminaries

In this chapter, we recall the basic definitions and properties of binary relations as well as of ternary relations that will be needed throughout this thesis.

An n-ary relation over a sequence of sets $X_1, \ldots, X_n$ is a subset of the cartesian product $X_1 \times \ldots \times X_n$. For the purposes of simplicity, through this thesis, we assume $X_1 = X_2 = \ldots = X_n$, unless otherwise mentioned.

1.1. Binary relations

In this section, we recall the basic definitions and properties of binary relations. as well as some advanced notions such as the Bandler–Kohout compositions, the traces, and other concepts that will be needed throughout this thesis.

1.1.1. Definitions and properties

A binary relation R on a set X is a subset of X^2, i.e., it is a set of couples $(x, y) \in X^2$. Given a binary relation R on X, and two elements x and y of X if $(x, y) \in R$ or $(y, x) \in R$, we say that x and y are comparable, otherwise, we say that they are incomparable.

Every binary relation on a countable set $X = \{x_1, x_2, \ldots\}$ can be represented by the boolean matrix $\mathcal{M}_R$ given as follows

$$\mathcal{M}_R = (m_{ij}), \qquad m_{ij} = \begin{cases} 1 & \text{, if } (x_i, x_j) \in R \\ 0 & \text{, if } (x_i, x_j) \notin R \end{cases} .$$

Being subsets, the inclusion, union, intersection, and complementation of two binary relations are defined accordingly. The *transpose* of a binary relation R is binary relation R^t defined as $(x, y) \in X^2 \mid (y, x) \in R$. A binary relation R on a set X said to be :

(i) *reflexive*, if, for any $x \in X$, it holds that $(x, x) \in R$;

(ii) *irreflexive*, if, for any $x \in X$, it holds that $(x, x) \in R^c$;

(iii) *symmetric*, if, for any $x, y \in X$, it holds that $(x, y) \in R$ implies $(y, x) \in R$;

(iv) *asymmetric*, if, for any $x, y \in X$, it holds that $(x, y) \in R$ implies $(y, x) \in R^c$;

(v) *antisymmetric*, if, for any $x, y \in X$, it holds that $(x, y) \in R \wedge (y, x) \in R$ implies $x = y$;

(vi) *transitive*, if, for any $x, y, z \in X$, it holds that $(x, y) \in R \wedge (y, z) \in R$ implies $(x, z) \in R$;

(vii) *complete*, if, for any $x, y \in X$, it holds that $(x, y) \in R \vee (y, x) \in R$.

1.1.2. Compositions of binary relations

The composition of two binary relations R and S on X is defined as follows [31]:

$$R \circ S = \{(x, z) \in X^2 \mid (\exists t \in X)((x, t) \in R \wedge (t, z) \in S)\}.$$

One could imagine other definitions of the composition of binary relations:

$$\begin{aligned}
R \circ_1 S &= \{(x, z) \in X^2 \mid (\exists t \in X)((t, x) \in R \wedge (t, z) \in S)\};\\
R \circ_2 S &= \{(x, z) \in X^2 \mid (\exists t \in X)((x, t) \in R \wedge (z, t) \in S)\};\\
R \circ_3 S &= \{(x, z) \in X^2 \mid (\exists t \in X)((t, x) \in R \wedge (z, t) \in S)\};\\
R \circ_4 S &= \{(x, z) \in X^2 \mid (\exists t \in X)((z, t) \in R \wedge (t, x) \in S)\};\\
R \circ_5 S &= \{(x, z) \in X^2 \mid (\exists t \in X)((t, z) \in R \wedge (t, x) \in S)\};\\
R \circ_6 S &= \{(x, z) \in X^2 \mid (\exists t \in X)((z, t) \in R \wedge (x, t) \in S)\};\\
R \circ_7 S &= \{(x, z) \in X^2 \mid (\exists t \in X)((t, z) \in R \wedge (x, t) \in S)\}.
\end{aligned}$$

As all of these compositions link two 2-tuples while allowing for a degree of freedom, we refer to them as 3-*point compositions*.

Any of the above 3-point compositions is determined by three 2-permutations ρ_i, ρ_j, ρ_k, $i, j, k \in \{0, 1\}$, as explained next. First of all, let us fix two (possibly identical) 2-permutations ρ_i and ρ_j. If we say that a 2-tuple $(x, z) \in X^2$ belongs to some composition of two binary relations R and S on X if there exists an element $t \in X$ such that $\rho_i(x, t) \in R$ and $\rho_j(z, t) \in S$, then this allows to retrieve the first four compositions above. If, additionally, we allow to permute the 2-tuple (x, z), then we can also retrieve the last four compositions (after a proper renaming of variables). This view allows to develop the following enumeration scheme. For any $q \in \{0, \dots, 7\}$, with $\circ_0 := \circ$ the basic composition, the composition $R \circ_q S$ can be written as:

$$R \circ_q S = \{\rho_k(x, z) \in X^2 \mid (\exists t \in X)(\rho_i(x, t) \in R \wedge \rho_j(t, z) \in S)\}, \qquad (1.1)$$

with $q = (kji)_2 = 4k + 2j + i$ and $i, j, k \in \{0, 1\}$. Hence, the compositions $\circ_q$, $q = 1, \dots, 7$, can be expressed in terms of the basic composition $\circ_0$ as follows:

$$R \circ_q S = (R^{\rho_i} \circ_0 S^{\rho_j})^{\rho_k}.$$

Note that for a given 3-point composition $\circ_q$, with $q = (kji)_2$, there exists a 3-

point composition $\circ_{q'}$ such that for any binary relations R and S on X, it holds that

$$R \circ_q S = S \circ_{q'} R\,,$$

with $q' = (\bar{k}\bar{i}\bar{j})_2$, $\bar{0} = 1$ and $\bar{1} = 0$.

1.1.3. BK-compositions of binary relations

We first recall the BK-compositions of binary relations.

Definition 1.1. [3] *The sub-composition $R \lhd S$, super-composition $R \rhd S$ and square-composition $R \diamond S$ of two binary relations R and S on X are defined as:*

$$\begin{aligned}
R \lhd S &= \{(x,z) \in X^2 \mid (\forall y \in X)((x,y) \in R \Rightarrow (y,z) \in S)\}\,;\\
R \rhd S &= \{(x,z) \in X^2 \mid (\forall y \in X)((x,y) \in R \Leftarrow (y,z) \in S)\}\,;\\
R \diamond S &= \{(x,z) \in X^2 \mid (\forall y \in X)((x,y) \in R \Leftrightarrow (y,z) \in S)\}\,.
\end{aligned}$$

Note that slight modifications of these notions (essentially differing in the way empty sets are handled) were studied in [13]. Similarly to Eq. (1.1.2), for any $q = (kji)_2 \in \{0, \ldots, 7\}$, we define the compositions $R \lhd_q S$, $R \rhd_q S$ and $R \diamond_q S$ as:

$$\begin{aligned}
R \lhd_q S &= (R^{\rho_i} \lhd S^{\rho_j})^{\rho_k}\,;\\
R \rhd_q S &= (R^{\rho_i} \rhd S^{\rho_j})^{\rho_k}\,;\\
R \diamond_q S &= (R^{\rho_i} \diamond S^{\rho_j})^{\rho_k}\,.
\end{aligned}$$

Moreover, the above-mentioned compositions verify the following commutativity-like properties, with $q' = 7 - q$:

$$R \lhd_q S = S \rhd_{q'} R\,; \tag{1.2}$$

$$R \diamond_q S = S \diamond_{q'} R\,. \tag{1.3}$$

Some of the key characteristics of the BK-compositions of binary relations shown by are the *Convertibility* i.e.,

$$\begin{aligned}
(R \lhd S)^t &= S^t \rhd R^t;\\
(R \rhd S)^t &= S^t \lhd R^t;\\
(R \diamond S)^t &= S^t \diamond R^t\,.
\end{aligned}$$

and the *Associativity* i.e.,

$$(R \lhd S) \rhd T = R \lhd (S \rhd T)$$

1.1.4. Traces of binary relation

In this subsection, we recall the notions of left and right trace of a binary relation introduced by Doignon et al. [16]. We then express these traces in terms of the BK-compositions and recall some of their properties.

Definition 1.2. [18] *Let R be a binary relation on X.*

(i) *The left trace of R is the binary relation R^ℓ on X defined as*

$$R^\ell = \{(x,y) \in X^2 \mid (\forall z \in X)((x,z) \in R \Rightarrow (y,z) \in R)\}.$$

(ii) *The right trace of R is the binary relation R^r on X defined as*

$$R^r = \{(x,y) \in X^2 \mid (\forall z \in X)((z,x) \in R \Rightarrow (z,y) \in R)\}.$$

Theorem 1.1. [18] *Let R be a binary relation on X. The following statements are equivalent:*

(i) *R is reflexive;*

(ii) *$(R^\ell)^t \subseteq R$;*

(iii) *$R^r \subseteq R$.*

Theorem 1.2. [18] *Let R be a binary relation on X. The following statements are equivalent:*

(i) *R is transitive;*

(ii) *$R \subseteq (R^\ell)^t$;*

(iii) *$R \subseteq R^r$.*

Theorem 1.3. [18] *For any binary relation R on X, it holds that*

$$(R^\ell)^t \circ R = R \circ R^r = R.$$

Remark 1.1. *It is interesting to note that these traces can be expressed in terms of the BK-compositions:*

$$\begin{aligned} R^r &= R \lhd_1 R = R \rhd_6 R; \\ R^\ell &= R \lhd_2 R = R \rhd_5 R. \end{aligned}$$

Following this line of reasoning, we can define other traces as follows. For any binary relation R on X and any $n \in \{0, \dots, 7\}$, the n-th trace of

R, denoted by $R^{\triangleleft_n}$, is defined as:

$$R^{\triangleleft_n} := R \triangleleft_n R = R \triangleright_{7-n} R \,.$$

1.2. Ternary relations

In this section, we recall the basic definitions and properties of ternary relations.

1.2.1. Definitions and examples

A ternary relation T on a set X is a subset of X^3, i.e., it is a set of triplets $(x, y, z) \in X^3$. Three special ternary relations on X are the null relation $O_{X^3} = \emptyset$, the ternary identity relation $I_{X^3} = \{(x, x, x) \mid x \in X\}$ and the universal ternary relation X^3.

Example 1.1. *Let T be the ternary relation on $\mathbb{N}$ given as follows: $(a, b, c) \in T$ if $a \bmod b \equiv c$. Then, it is clear that $(42, 13, 3) \in T$, whereas $(21, 10, 2) \notin T$.*

Example 1.2. *Given any set X whose elements are arranged on a circle, one can define a ternary relation T on X, i.e., a subset of X^3, by stipulating that $(x, y, z) \in T$ holds if and only if the elements x, y and z are pairwise different and when going from x to z in a clockwise direction one passes through y. For example, if $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ represents the hours on a clock face, then $(8, 12, 4) \in T$ holds and $(12, 8, 4) \in T$ does not hold.*

A ternary relation T_1 on a set X is said to be included in a ternary relation T_2 on the same set X, denoted by $T_1 \subseteq T_2$, if, for any $x, y, z \in X$, $(x, y, z) \in T_1$ implies that $(x, y, z) \in T_2$. The intersection of two ternary relations T_1 and T_2 on X is the ternary relation $T_1 \cap T_2$ on X defined as $T_1 \cap T_2 = \{(x, y, z) \in X^3 \mid (x, y, z) \in T_1 \wedge (x, y, z) \in T_2\}$. If $T_1 \cap T_2 = \emptyset$, then T_1 and T_2 are called disjoint ternary relations. Also, the union of two ternary relations T_1 and T_2 on X is the ternary relation $T_1 \cup T_2$ on X defined as $T_1 \cup T_2 = \{(x, y, z) \in X^3 \mid (x, y, z) \in T_1 \vee (x, y, z) \in T_2\}$.

Example 1.3.

(i) Let T_1 and T_2 be two ternary relations on $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ given by

$$T_1 = \{(x_1, x_2, x_4), (x_1, x_3, x_2), (x_5, x_1, x_6)\} \,,$$
$$T_2 = \{(x_1, x_2, x_4), (x_1, x_3, x_2), (x_5, x_1, x_6), (x_6, x_6, x_3)\} \,.$$

It is clear that $T_1 \subseteq T_2$.

(ii) Let T_1 and T_2 be two ternary relations on $X = \{x_1, x_2, x_3, x_4, x_5\}$ given by

$$T_1 = \{(x_1, x_2, x_4), (x_1, x_3, x_2), (x_5, x_1, x_4)\}\,,$$
$$T_2 = \{(x_1, x_1, x_2), (x_1, x_2, x_4)\}\,.$$

One easily verifies that

$$T_1 \cap T_2 = \{(x_1, x_2, x_4)\}\,;$$
$$T_1 \cup T_2 = \{(x_1, x_1, x_2), (x_1, x_2, x_4), (x_1, x_3, x_2), (x_5, x_1, x_4)\}\,.$$

For a given ternary relation T on a set X, we denote the *transpose* of T by T^t, i.e., for any $x, y, z \in X$, $(x, y, z) \in T^t$ means that $(z, y, x) \in T$. Also, we denote the *complement* of T by T^c, i.e., for any $x, y, z \in X$, $(x, y, z) \in T^c$ means that $(x, y, z) \notin T$. We denote the *dual* of T by T^d, i.e., for any $x, y, z \in X$, $(x, y, z) \in T^d$ means that $(z, y, x) \notin T$.

A ternary relation T on a set X is called:

(i) *reflexive*, if, for any $x \in X$, it holds that $(x, x, x) \in T$;

(ii) *strongly reflexive*, if, for any $x, y, z \in X$ with $\text{card}\{x, y, z\} \leq 2$, it holds that $(x, y, z) \in T$;

(iii) *irreflexive* , if, for any $x \in X$, it holds that $(x, x, x) \notin T$;

(iv) *strongly irreflexive*, if, for any $x, y, z \in X$ with $\text{card}\{x, y, z\} \leq 2$, it holds that $(x, y, z) \notin T$;

(v) *symmetric* , if, for any $x, y, z \in X$, it holds that $(x, y, z) \in T$ implies $(z, y, x) \in T$, i.e., $T = T^t$;

(vi) *strongly symmetric* , if $T = T^{\sigma_i}$, for any $i \in \{1, \ldots, 5\}$;

(vii) *asymmetric* , if, for any $x, y, z \in X$ such that $\text{card}\{x, y, z\} \geq 2$, it holds that $(x, y, z) \in T$ implies $(z, y, x) \notin T$;

(viii) *strongly asymmetric*, if, for any $x, y, z \in X$ such that $\text{card}\{x, y, z\} \geq 2$, it holds that $(x, y, z) \in T$ implies $\sigma_i(x, y, z) \notin T$, for any $i \in \{1, \ldots, 5\}$;

(ix) *cyclic* , if, for any $x, y, z \in X$, it holds that $(x, y, z) \in T$ implies $(y, z, x) \in T$;

(x) *complete* , if, for any $x, y, z \in X$ such that $\text{card}\{x, y, z\} \geq 2$, it holds that $(x, y, z) \in T \vee (z, y, x) \in T$;

(xi) *strongly complete*, if, for any $x, y, z \in X$ such that $\text{card}\{x, y, z\} \geq 2$, it holds that $\sigma_i(x, y, z) \in T$, for some $i \in \{0, ..., 5\}$.

It is clear that a ternary relation T on X is called:

(a) *left reflexive*, if, for any $x, y \in X$, it holds that $(x, y, y) \in T$;

(b) *middle reflexive*, if, for any $x, y \in X$, it holds that $(x, y, x) \in T$;

(c) *right reflexive*, if, for any $x, y \in X$, it holds that $(x, x, y) \in T$.

A cyclic order is a ternary relation that is asymmetric, transitive and cyclic. A ternary relation T on a set X is a betweenness relation if it satisfies the following conditions:

(i) for any $x, y, z \in X$, $(x, y, z) \in T$ if and only if $(z, y, x) \in T$;

(ii) for any $x, y, z \in X$, $(x, y, z) \in T$ and $(x, z, y) \in T$ if and only if $y = z$;

(iii) for any $x, y, z, u \in X$, $(x, y, u) \in T$ and $(x, u, z) \in T$ implies $(x, y, z) \in T$.

Also, a ternary relation T is said to be a strict order-betweenness relation if, for any $x, y, z \in X$, it holds that $(x, y, z) \in T$ if and only if $x < y < z$ or $z < y < x$. An order-betweenness relation is a ternary relation T such that for any $x, y, z \in X$, it holds that $(x, y, z) \in T$ if and only if $x \leq y \leq z$ or $z \leq y \leq x$.

For more details on ternary relations, we refer to [2, 28, 8, 27].

1.2.2. Traces of ternary relations

In this section, we recall the definitions and properties of traces of a ternary relation introduced in [27]. As in the binary case, these traces facilitate the study and characterization of properties of a ternary relation. Interestingly, the traces themselves turn out to be the greatest solutions of relational inequalities associated with newly introduced compositions of ternary relations.

Definition 1.3. [27] *Let T be a ternary relation on a set X.*

(i) The left trace of T is the binary relation T^ℓ on X defined as

$$T^\ell = \{(x, y) \in X^2 \mid (\forall (a, b) \in X^2)((x, a, b) \in T \Rightarrow (y, a, b) \in T)\} ;$$

(ii) The middle trace of T is the binary relation T^m on X defined as

$$T^m = \{(x, y) \in X^2 \mid (\forall (a, b) \in X^2)((a, x, b) \in T \Rightarrow (a, y, b) \in T)\} ;$$

(iii) The right trace of T is the binary relation T^r on X defined as

$$T^r = \{(x, y) \in X^2 \mid (\forall (a, b) \in X^2)((a, b, x) \in T \Rightarrow (a, b, y) \in T)\} .$$

The following result discusses the traces of some particular ternary relations.

Proposition 1.1. [27] *Let T be a ternary relation on a set X. The following statements hold:*

(i) If $T = X^3$ or $T = \emptyset$, then $T^\ell = T^m = T^r = X^2$;

(ii) If $T = I_{X^3}$, then $T^\ell = T^m = T^r = I_{X^2} = \{(x, x) \mid x \in X\}$;

(iii) If $T^\ell = T^m = T^r = X^2$, then $T = X^3$ or $T = \emptyset$.

The following proposition discusses the traces of ternary relations obtained by permutation.

Proposition 1.2. [27] *Let T be a ternary relation on a set X. The left, middle and right traces of the corresponding ternary relations obtained by permutation are listed in the following table:*

	$(.)^\ell$	$(.)^m$	$(.)^r$
$T^\dashv$	T^ℓ	T^r	T^m
$T^\vdash$	T^m	T^ℓ	T^r
T^+	T^r	T^ℓ	T^m
T^-	T^m	T^r	T^ℓ
T^t	T^r	T^m	T^ℓ

2 Four-point compositions of ternary relations

In the literature, various compositions of two ternary relations S and T have been proposed, each with its own motivation. A first example is the composition $\circ_B$ (used in the definition of the transitivity of betweenness relations [22]) defined as:

$$S \circ_B T = \{(x, y, z) \in X^3 \mid (\exists t \in X)((x, y, t) \in S \wedge (x, t, z) \in T)\} . \quad (2.1)$$

A 3-tuple belongs to this composition if there exist two 3-tuples that each share two components with the given tuple, of which exactly one in common, and one common degree of freedom t. We will refer to such a composition as a 4-point composition. A second example is the composition $\circ_{c_1}$ (derived from the composition of a ternary relation with a binary relation in [27]) defined as:

$$S \circ_{c_1} T = \{(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)((x, y, t) \in S \wedge (s, t, z) \in T)\} . \quad (2.2)$$

A 3-tuple belongs to this composition if there exist two 3-tuples of which one shares two components with the given tuple, while the other one contains the third component, complemented by two degrees of freedom s and t, of which one in common. We will refer to such a composition as a 5-point composition. In the Subsection 2.1, we will identify all 4-point compositions of ternary relations and study their properties, while in Subsection 3.1, we will do the same for the 5-point compositions.

2.1. Definition and basic properties

The reasoning underlying the 3-point compositions of binary relations can be naturally extended to enumerate the 4-point compositions of ternary relations. We start from the composition $\circ_B$ in Eq. (2.1) and use the 3-permutations to allow for a repositioning of the variables x, y, z and t, resulting in as many as 216 different 4-point compositions.

Definition 2.1. *Let $p \in \{0, \ldots, 215\}$ with $p = (kji)_6 = 36k + 6j + i$ and $i, j, k \in \{0, \ldots, 5\}$. For any ternary relations S and T on X, we define the composition $S \,\square_p^0\, T$ as follows:*

$$S \,\square_p^0\, T = \{\sigma_k(x, y, z) \in X^3 \mid (\exists t \in X)(\sigma_i(x, y, t) \in S \wedge \sigma_j(x, t, z) \in T)\} .$$

It is indeed easy to see that the compositions $\Box^0_p$, $p \in \{0, \ldots, 215\}$, are all different.

However, not all 4-point compositions of ternary relations can be obtained in this way. Indeed, one can also envisage compositions that match the following description. A 3-tuple belongs to such a composition if there exist two 3-tuples of which one shares two components with the given tuple, while the other one contains the third component and a second occurrence of the corresponding element, again complemented with one common degree of freedom t.

Definition 2.2. *Let $p \in \{0, \ldots, 215\}$ with $p = (kji)_6 = 36k + 6j + i$ and $i, j, k \in \{0, \ldots, 5\}$. For any ternary relations S and T on X, we define the compositions $S \Box^1_p T$ and $S \Box^2_p T$ as follows:*

$$S \Box^1_p T = \{\sigma_k(x, y, z) \in X^3 \mid (\exists t \in X)(\sigma_i(x, y, t) \in S \wedge \sigma_j(z, t, z) \in T)\} ;$$
$$S \Box^2_p T = \{\sigma_k(x, y, z) \in X^3 \mid (\exists t \in X)(\sigma_i(y, y, t) \in S \wedge \sigma_j(x, t, z) \in T)\} .$$

As the following proposition shows, we have only 54 different compositions of type $\Box^1$ and as many of type $\Box^2$.

Proposition 2.1. *Let $i, j, k \in \{0, \ldots, 5\}$. For any ternary relations S and T on X, it holds that*

$$S \Box^1_{(kji)_6} T = S \Box^1_{(\pi_1(k) j \pi_1(i))_6} T = S \Box^1_{(k \pi_2(j) i)_6} T = S \Box^1_{(\pi_1(k) \pi_2(j) \pi_1(i))_6} T$$

and

$$S \Box^2_{(kji)_6} T = S \Box^2_{(\pi_2(k) \pi_2(j) i)_6} T = S \Box^2_{(kj \pi_1(i))_6} T = S \Box^2_{(\pi_2(k) \pi_2(j) \pi_1(i))_6} T,$$

with π_1 the permutation of $\{0, \ldots, 5\}$ given in Table 2.1:

Table 2.1: The permutation π_1.

u	0	1	2	3	4	5
$\pi_1(u)$	2	3	0	1	5	4

and π_2 the permutation of $\{0, \ldots, 5\}$ given in Table 2.2:

Table 2.2: The permutation π_2.

u	0	1	2	3	4	5
$\pi_2(u)$	5	4	3	2	1	0

Note that if $u \in \{0,3,4\}$, then $\pi_1(u), \pi_2(u) \in \{1,2,5\}$. This implies that we can select the unique 54 compositions of type $\square^1$ and the 54 unique compositions of type $\square^2$ by restricting i, j to belong to $\{0,3,4\}$ (the same could be achieved by restricting i, j to belong to $\{1,2,5\}$). This corresponds to the compositions $\square_p^r$ ($r \in \{1,2\}$) with $p \bmod 36$ belonging to $\{0,3,4,18,21,22,24,27,28\}$.

The following examples illustrate the above definition of 4-point compositions of ternary relations.

Example 2.1. *Consider $p = 119$ and two ternary relations S and T on X. In senary notation, p can be written as $119 = 3*36+1*6+5 = (315)_6$. Hence,*

$$\begin{aligned} T \square_{119}^0 S &= \{\sigma_3(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_5(x,y,t) \in T \wedge \sigma_1(x,t,z) \in S)\} \\ &= \{(y,z,x) \in X^3 \mid (\exists t \in X)((t,y,x) \in T \wedge (x,z,t) \in S)\} \\ &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,z) \in T \wedge (z,y,t) \in S)\}, \end{aligned}$$

$$\begin{aligned} T \square_{119}^1 S &= \{\sigma_3(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_5(x,y,t) \in T \wedge \sigma_1(z,t,z) \in S)\} \\ &= \{(y,z,x) \in X^3 \mid (\exists t \in X)((t,y,x) \in T \wedge (z,z,t) \in S)\} \\ &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,z) \in T \wedge (y,y,t) \in S)\}, \end{aligned}$$

$$\begin{aligned} T \square_{119}^2 S &= \{\sigma_3(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_5(y,y,t) \in T \wedge \sigma_1(x,t,z) \in S)\} \\ &= \{(y,z,x) \in X^3 \mid (\exists t \in X)((t,y,y) \in T \wedge (x,z,t) \in S)\} \\ &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,x) \in T \wedge (z,y,t) \in S)\}. \end{aligned}$$

Conversely, the following example shows how to find $r \in \{0,1,2\}$ and $p = (kji)_6 \in \{0,\ldots,215\}$ for a given composition.

Example 2.2. *Consider two ternary relations S and T on X and the binary operation $*$ defined as follows:*

$$S * T = \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,x) \in T \wedge (y,z,t) \in S)\}.$$

The operation $*$ corresponds to a composition of the type $\square_p^0$, with $p \in \{0, \ldots, 215\}$.

We express this composition as in Definition 2.1 by identifying the proper 3-permutations:

$$\begin{aligned} S * T &= \{(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_4(z,x,t) \in T \wedge \sigma_4(z,t,y) \in S)\} \\ &= \{(y,z,x) \in X^3 \mid (\exists t \in X)(\sigma_4(x,y,t) \in T \wedge \sigma_4(x,t,z) \in S)\} \\ &= \{\sigma_3(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_4(x,y,t) \in T \wedge \sigma_4(x,t,z) \in S)\}\,. \end{aligned}$$

Hence, $i = 4$*,* $j = 4$ *and* $k = 3$*. Thus,* $p = (344)_6 = 136$ *and* $* = \square^0_{136}$.

Example 2.3. *Consider two ternary relations* S *and* T *on* X *and the binary operation* $\star$ *defined as follows:*

$$S \star T = \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,y) \in T \wedge (t,x,x) \in S)\}\,.$$

The operation $\star$ *corresponds to a composition of the type* $\square^1_p$*, with* $p \in \{0, \ldots, 215\}$*. We express this composition as in Definition 2.2 by identifying the proper 3-permutations:*

$$\begin{aligned} S \star T &= \{(z,y,x) \in X^3 \mid (\exists t \in X)((t,x,y) \in T \wedge (t,z,z) \in S)\} \\ &= \{\sigma_5(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_4(x,y,t) \in T \wedge \sigma_3(z,t,z) \in S)\}\,. \end{aligned}$$

Hence, $i = 4$*,* $j = 3$ *and* $k = 5$*. Thus,* $p = (534)_6 = 202$ *and* $\star = \square^1_{202}$*. According to Proposition 2.1,* $*$ *can also be written as* $\square^1_{161}$*,* $\square^1_{167}$ *and* $\square^1_{196}$.

The following proposition expresses all the 4-point compositions in terms of the compositions $\square^r_0$, $r \in \{0, 1, 2\}$, which we will refer to as the basic 4-point compositions.

Proposition 2.2. *Let* $p = (kji)_6 \in \{0, \ldots, 215\}$ *and* $r \in \{0, 1, 2\}$*. For any ternary relations* S *and* T *on* X*, the 4-point composition* $S \,\square^r_p\, T$ *can be written in terms the composition* $\square^r_0$ *as follows:*

$$S \,\square^r_p\, T = (S^{\sigma_i^{-1}} \,\square^r_0\, T^{\sigma_j^{-1}})^{\sigma_k}\,.$$

Proof. We give the proof for $r = 0$.

$$\begin{aligned} S \Box_p^0 T &= \{\sigma_k(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_i(x,y,t) \in S \wedge \sigma_j(x,t,z) \in T)\} \\ &= \{\sigma_k(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in S^{\sigma_i^{-1}} \wedge (x,t,z) \in T^{\sigma_j^{-1}})\} \\ &= \{\sigma_k(x,y,z) \in X^3 \mid (x,y,z) \in S^{\sigma_i^{-1}} \Box_0^0 T^{\sigma_j^{-1}}\} \\ &= (S^{\sigma_i^{-1}} \Box_0^0 T^{\sigma_j^{-1}})^{\sigma_k} . \end{aligned}$$

□

Next, we study some basic properties of the 4-point compositions of ternary relations. The following proposition identifies the right and left neutral elements of the basic 4-point compositions. To that end, we introduce the following definition and lemma.

Definition 2.3. *We define the special ternary relations E, E_ℓ, E_m and E_r on X as follows:*

(i) $E = \{(x,x,x) \in X^3 \mid x \in X\}$;

(ii) $E_\ell = \{(x,x,y) \in X^3 \mid x,y \in X\}$;

(iii) $E_m = \{(x,y,x) \in X^3 \mid x,y \in X\}$;

(iv) $E_r = \{(x,y,y) \in X^3 \mid x,y \in X\}$.

Lemma 2.1. *For any ternary relation T on X, it holds that*

(i) $T \Box_0^0 E_r = T$;

(ii) $E_r \Box_0^0 T = T$.

Proof. We prove (i). Let $(x,y,z) \in T$, then with $(x,z,z) \in E_r$, it follows that $(x,y,z) \in T \Box_0^0 E_r$. Conversely, let $(x,y,z) \in T \Box_0^0 E_r$, then there exists $t \in X$ such that $(x,y,t) \in T$ and $(x,t,z) \in E_r$. The fact that $(x,t,z) \in E_r$ implies that $t = z$. Hence $(x,y,z) \in T$, and thus $T \Box_0^0 E_r \subseteq T$. □

Proposition 2.3. *Let $i,j,k \in \{0,\ldots,5\}$. For any ternary relation T on X, it holds that*

(i) $T \Box^0_{(kjk)_6} E_{\zeta(j)} = T$;

(ii) $E_{\zeta(i)} \Box^0_{(kki)_6} T = T$,

with $\zeta : \{0, \ldots, 5\} \to \{\ell, m, r\}$ the mapping given in Table 2.3:

Table 2.3: The mapping ζ.

u	0	1	2	3	4	5
$\zeta(u)$	r	r	m	ℓ	m	ℓ

Proof. We prove (i). Due to Lemma 2.1, it holds for any $k \in \{0, \ldots, 5\}$ that

$$T^{\sigma_k^{-1}} \square_0^0 E_r = T^{\sigma_k^{-1}} .$$

Moreover, one can verify that $E_r = (E_{\zeta(j)})^{\sigma_j^{-1}}$ for any $j \in \{0, \ldots, 5\}$. Hence,

$$T^{\sigma_k^{-1}} \square_0^0 E_{\zeta(j)}{}^{\sigma_k^{-1}} = T^{\sigma_k^{-1}} ,$$

and thus

$$\left(T^{\sigma_k^{-1}} \square_0^0 E_{\zeta(j)}{}^{\sigma_j^{-1}}\right)^{\sigma_k} = T .$$

Therefore, Proposition 2.2 ensures that

$$T \square_{(kjk)_6}^0 E_{\zeta(j)} = T .$$

□

Note that $E_{\zeta(k)}$ is simultaneously the left and right neutral element of $\square_{(kkk)_6}^0$, and can hence be called the neutral element of $\square_{(kkk)_6}^0$.

Proposition 2.4. *Let $p \in \{0, \ldots, 215\}$. For any ternary relation T on X, it holds that*

(i) $T \square_p^1 E = T$;

(ii) $E \square_p^2 T = T$.

Proof. The proof is similar to that of Proposition 2.3. □

Note that the compositions $\square_p^1$ and $\square_p^2$ do not have a neutral element.

In the following definition, we introduce the notion of mixed-commutativity of compositions of ternary relations.

Definition 2.4. *Let $*$ and $\star$ be two binary operations on a set X. We say that the couple $(*, \star)$ is mixed-commutative if $x * y = y \star x$, for any $x, y \in X$.*

Obviously, if a couple $(*, \star)$ is mixed-commutative, then also the converse couple $(\star, *)$ is mixed-commutative. Four-point compositions of ternary relations are not commutative in general, but to any 4-point composition corresponds another one so that they make up a mixed-commutative couple.

Proposition 2.5. *Let* $p = (kji)_6 \in \{0, \ldots, 215\}$*. The couples* $(\square^0_p, \square^0_{p'})$ *and* $(\square^1_p, \square^2_{p'})$ *are mixed-commutative, with* $p' = (\pi_3(k)\pi_3(i)\pi_3(j))_6$ *and* π_3 *the permutation of* $\{0, \ldots, 5\}$ *given in Table 4.1:*

Table 2.4: The permutation π_3.

u	0	1	2	3	4	5
$\pi_3(u)$	1	0	4	5	2	3

Proof. We give the proof for $\left(\square^0_p, \square^0_{p'}\right)$. Let S and T be two ternary relations on X, then it holds that

$$\begin{aligned} S\,\square^0_p\,T &= \{\sigma_k(x, y, z) \in X^3 \mid (\exists t \in X)(\sigma_i(x, y, t) \in S \wedge \sigma_j(x, t, z) \in T)\} \\ &= \{\sigma_k(x, y, z) \in X^3 \mid (\exists t \in X)(\sigma_j(x, t, z) \in T \wedge \sigma_i(x, y, t) \in S)\} \\ &= \{\sigma_k(\sigma_1(x, z, y)) \in X^3 \mid (\exists t \in X)(\sigma_j(\sigma_1(x, z, t)) \in T \wedge \sigma_i(\sigma_1(x, t, y)) \in S)\} \\ &= \{\sigma_k(\sigma_1(x, y, z)) \in X^3 \mid (\exists t \in X)(\sigma_j(\sigma_1(x, y, t)) \in T \wedge \sigma_i(\sigma_1(x, t, z)) \in S)\}\,. \end{aligned}$$

Since $\sigma_u(\sigma_1(x, y, z)) = \sigma_{\pi_3(u)}(x, y, z)$, it follows that

$$S\square^0_p T = \{\sigma_{\pi_3(k)}(x, y, z) \in X^3 \mid (\exists t \in X)(\sigma_{\pi_3(j)}(x, y, t) \in T \wedge \sigma_{\pi_3(i)}(x, t, z) \in S)\} = T\square^0_{p'} S\,.$$

Hence, the couple $\left(\square^0_p, \square^0_{p'}\right)$ is mixed-commutative. $\square$

In view of Proposition 2.5, indeed no 4-point composition is commutative.

2.2. Associativity of the four-point compositions of ternary relations

The following proposition identifies the associative 4-point compositions.

Proposition 2.6. *The* 4*-point composition* $\square^0_p$ *is associative if and only if* $p = (iii)_6 = 43i$*, with* $i \in \{0, \ldots, 5\}$*.*

Proof. Let $p = (kji)_6$. For any three ternary relations R, S, T on X, we have

$$(R \,\square_p^0\, S) \,\square_p^0\, T = \{\sigma_k(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_i(x,y,t) \in R \,\square_p^0\, S \wedge \sigma_j(x,t,z) \in T)\}\,.$$

Since the set of 3-permutations is a group, there exists $m \in \{0,\dots,5\}$ such that $\sigma_i = \sigma_k\sigma_m$. Hence, we can write

$$\begin{aligned}
&(R \,\square_p^0\, S) \,\square_p^0\, T \\
&= \{\sigma_k(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_k(\sigma_m(x,y,t)) \in R \,\square_p^0\, S \wedge \sigma_j(x,t,z) \in T)\} \\
&= \{\sigma_k(x,y,z) \in X^3 \mid (\exists (s,t) \in X^2)(\sigma_i(\sigma_m(x,y,s)) \in R \wedge \sigma_j(\sigma_m(x,s,t)) \in S \wedge \sigma_j(x,t,z) \in T)\}.
\end{aligned}$$

Using the same argument, there exists $n \in \{0,\dots,5\}$ such that $\sigma_j = \sigma_k\sigma_n$, and we can write

$$\begin{aligned}
&R \,\square_p^0\, (S \,\square_p^0\, T) \\
&= \{\sigma_k(x,y,z) \in X^3 \mid (\exists u \in X)(\sigma_i(x,y,u) \in R \wedge \sigma_j(x,u,z) \in S \,\square_p^0\, T)\} \\
&= \{\sigma_k(x,y,z) \in X^3 \mid (\exists u \in X)(\sigma_i(x,y,u) \in R \wedge \sigma_k(\sigma_n(x,u,z)) \in S \,\square_p^0\, T)\} \\
&= \{\sigma_k(x,y,z) \in X^3 \mid (\exists (u,v) \in X^2)(\sigma_i(x,y,u) \in R \wedge \sigma_i(\sigma_n(x,u,v)) \in S \wedge \sigma_j(\sigma_n(x,v,z)) \in T)\}.
\end{aligned}$$

It clearly holds that $(R \,\square_p^0\, S) \,\square_p^0\, T = R \,\square_p^0\, (S \,\square_p^0\, T)$ if and only if $m = n = 0$, *i.e.*, if and only if $i = j = k$. Therefore, $\square_p^0$ is associative if and only if $i = j = k$. □

Remark 2.1. *For any $p \in \{0, \dots, 215\}$, the compositions $\square_p^1$ and $\square_p^2$ are not associative. Indeed, consider the three ternary relations R, S, T on the set $X = \{a,b,c,d,e\}$ given as:*

$$\begin{aligned}
R &= \{(a,b,c),(a,c,b),(b,a,c),(b,c,a),(c,a,b),(c,b,a)\}; \\
S &= \{(d,c,d),(d,d,c),(c,d,d)\}; \\
T &= \{(e,d,e),(e,e,d),(d,e,e)\}.
\end{aligned}$$

One can verify that for any $p \in \{0, \dots, 215\}$, there exists $i \in \{0, \dots 5\}$ such that

$$\sigma_i(a,b,e) \in (R \,\square_p^1\, S) \,\square_p^1\, T\,,$$

while $(R \,\square_p^1\, S) \,\square_p^1\, T = \emptyset$. The same holds for the compositions $\square_p^2$ by considering the relations

$$\begin{aligned}
R &= \{(a,a,b),(a,b,a),(b,a,a)\}; \\
S &= \{(b,b,c),(b,c,b),(c,c,b)\}; \\
T &= \{(c,d,e),(c,e,d),(d,c,e),(d,e,c),(e,c,d),(e,d,c)\}.
\end{aligned}$$

In Table 2.5, we present the six associative 4-point compositions denoted by the new notation $\diamond_i$, $i \in \{1, \dots, 6\}$.

Table 2.5: The associative 4-point compositions.

$\diamond_i$	$S \diamond_i T$	S	T
$\diamond_1 = \square_0^0$	(x, y, z)	(x, y, t)	(x, t, z)
$\diamond_2 = \square_{43}^0$	(x, y, z)	(x, t, z)	(x, y, t)
$\diamond_3 = \square_{86}^0$	(x, y, z)	(x, y, t)	(t, y, z)
$\diamond_4 = \square_{129}^0$	(x, y, z)	(x, t, z)	(t, y, z)
$\diamond_5 = \square_{172}^0$	(x, y, z)	(t, y, z)	(x, y, t)
$\diamond_6 = \square_{215}^0$	(x, y, z)	(t, y, z)	(x, t, z)

The 3-tuple (x, y, z) belongs to $S \diamond_i T$ if there exists an element $t \in X$ such that the other listed 3-tuples belong to S and T, respectively.

Proposition 2.5 leads to the following corollary. It expresses that the associative 4-point compositions make up three couples of compositions that are mixed-commutative.

Corollary 2.1. *The couples* $(\diamond_1, \diamond_2)$*,* $(\diamond_3, \diamond_5)$ *and* $(\diamond_4, \diamond_6)$ *are mixed-commutative.*

The following result is a particular case of Proposition 2.3. It identifies the neutral element of the associative 4-point compositions.

Corollary 2.2.

(i) E_ℓ *is the neutral element of* $\diamond_1$ *and* $\diamond_2$*;*

(ii) E_m *is the neutral element of* $\diamond_3$ *and* $\diamond_5$*;*

(iii) E_r *is the neutral element of* $\diamond_4$ *and* $\diamond_6$*.*

2.3. Mixed-associativity of the four-point composition ternary relations

As a sequel to the study of the associativity property of the compositions of ternary relations, we next investigate the mixed-associativity property for the 4-point compositions. First, we present a proposition expressing a useful identity for further purposes.

Proposition 2.7. *Let $\alpha, \beta, \gamma, \delta \in \{0, \dots, 5\}$. For any ternary relations R, S, T on X, the following equalities hold:*

(i) $(R \ \square^0_{(\alpha\beta\gamma)_6} \ S) \ \square^0_{(\alpha\delta\alpha)_6} \ T = R \ \square^0_{(\alpha\alpha\gamma)_6} \ (S \ \square^0_{(\alpha\delta\beta)_6} \ T)$;

(ii) $(R \ \square^0_{(\alpha\beta\gamma)_6} \ S) \ \square^1_{(\alpha\delta\alpha)_6} \ T = R \ \square^0_{(\alpha\alpha\gamma)_6} \ (S \ \square^1_{(\alpha\delta\beta)_6} \ T)$;

(iii) $(R \ \square^2_{(\alpha\beta\gamma)_6} \ S) \ \square^0_{(\alpha\delta\alpha)_6} \ T = R \ \square^2_{(\alpha\alpha\gamma)_6} \ (S \ \square^0_{(\alpha\delta\beta)_6} \ T)$;

(iv) $(R \ \square^2_{(\alpha\beta\gamma)_6} \ S) \ \square^1_{(\alpha\delta\alpha)_6} \ T = R \ \square^2_{(\alpha\alpha\gamma)_6} \ (S \ \square^1_{(\alpha\delta\beta)_6} \ T)$.

Proof. We prove case (i).

$$
\begin{aligned}
(R \,\square^0_{(\alpha\beta\gamma)_6}\, S) \,\square^0_{(\alpha\delta\alpha)_6}\, T &= \left(\left(R \,\square^0_{(\alpha\beta\gamma)_6}\, S\right)^{\sigma_\alpha^{-1}} \square^0_0\, T^{\sigma_\delta^{-1}}\right)^{\sigma_\alpha} \\
&= \left(\left(\left(R^{\sigma_\gamma^{-1}} \,\square^0_0\, S^{\sigma_\beta^{-1}}\right)^{\sigma_\alpha}\right)^{\sigma_\alpha^{-1}} \square^0_0\, T^{\sigma_\delta^{-1}}\right)^{\sigma_\alpha} \\
&= \left(\left(R^{\sigma_\gamma^{-1}} \,\square^0_0\, S^{\sigma_\beta^{-1}}\right) \square^0_0\, T^{\sigma_\delta^{-1}}\right)^{\sigma_\alpha} \\
&= \left(R^{\sigma_\gamma^{-1}} \,\square^0_0\, \left(S^{\sigma_\beta^{-1}} \,\square^0_0\, T^{\sigma_\delta^{-1}}\right)\right)^{\sigma_\alpha} \\
&= \left(R^{\sigma_\gamma^{-1}} \,\square^0_0\, \left(\left(S^{\sigma_\beta^{-1}} \,\square^0_0\, T^{\sigma_\delta^{-1}}\right)^{\sigma_\alpha}\right)^{\sigma_\alpha^{-1}}\right)^{\sigma_\alpha} \\
&= R \,\square^0_{(\alpha\alpha\gamma)_6} \left(\left(S^{\sigma_\beta^{-1}} \,\square^0_0\, T^{\sigma_\delta^{-1}}\right)^{\sigma_\alpha}\right) \\
&= R \,\square^0_{(\alpha\alpha\gamma)_6} \left(S \,\square^0_{(\alpha\delta\beta)_6}\, T\right) .
\end{aligned}
$$

□

One can verify that for any $r \in \{0, 1, 2\}$ and $p, q \in \{0, \dots, 215\}$, the equalities

$$(R \,\square^1_p\, S) \,\square^r_q\, T = R \,\square^1_p\, (S \,\square^r_q\, T) \text{ and } (R \,\square^r_p\, S) \,\square^2_q\, T = R \,\square^r_p\, (S \,\square^2_q\, T)$$

do not hold in general.

Next, we recall the property of mixed-associativity of binary operations, also called linear distributivity. Note that remarkable instances of this property have been identified for the relational compositions (in the crisp as well as in the fuzzy case) in the groundbreaking studies of Bandler and Kohout [3, 13, 12].

Definition 2.5. *Let $*$ and $\star$ be two binary operations on a set X. We say that the couple $(*, \star)$ is mixed-associative if $(x * y) \star z = x * (y \star z)$, for any $x, y, z \in X$.*

Obviously, if $*$ is an associative operation, then $(*, *)$ is a trivial mixed-associative couple.

The following proposition presents the couples of 4-point compositions that satisfy the mixed-associativity property. This result follows from Proposition 2.7.

Corollary 2.3. *Let $\alpha, \beta, \gamma \in \{0, \ldots, 5\}$. The following couples of 4-point compositions are mixed-associative:*

(i) $(\square^0_{(\alpha\alpha\beta)_6}, \square^0_{(\alpha\gamma\alpha)_6})$;

(ii) $(\square^0_{(\alpha\alpha\beta)_6}, \square^1_{(\alpha\gamma\alpha)_6})$;

(iii) $(\square^2_{(\alpha\alpha\beta)_6}, \square^0_{(\alpha\gamma\alpha)_6})$;

(iv) $(\square^2_{(\alpha\alpha\beta)_6}, \square^1_{(\alpha\gamma\alpha)_6})$.

Remark 2.2. *Note that $(\square^0_p, \square^0_{p'})$ and $(\square^0_{p'}, \square^0_p)$ are simultaneously mixed-associative if and only if $p = p'$ and $\square^0_p$ is an associative 4-point composition.*

2.4. Link between compositions of binary relations and four-point compositions of ternary relations

In this section, we show that any cylindrical extension of a 3-point composition of binary relations contains a 4-point composition of their cylindrical extensions. First, we recall the definition of the cylindrical extensions of a binary relation.

Definition 2.6. [27] *Let R be a binary relation on X.*

(i) The left cylindrical extension of R is the ternary relation $C_\ell(R)$ on X defined as:

$$C_\ell(R) = \{(x, y, z) \in X^3 \mid (y, z) \in R\};$$

(ii) The middle cylindrical extension of R is the ternary relation $C_m(R)$ on X defined as:

$$C_m(R) = \{(x, y, z) \in X^3 \mid (x, z) \in R\};$$

(iii) The right cylindrical extension of R is the ternary relation $C_r(R)$ on X defined as:

$$C_r(R) = \{(x, y, z) \in X^3 \mid (x, y) \in R\}.$$

For further use, we introduce the following notation and lemma. Consider the mapping $\Gamma : \{0, 1\} \times \{\ell, m, r\} \to \{0, \ldots, 5\}$ given in Table 2.6:

Table 2.6: The mapping Γ.

i \ λ	ℓ	m	r
0	0	2	3
1	1	4	5

Lemma 2.2. *Let $i \in \{0, 1\}$ and $\lambda \in \{\ell, m, r\}$. For any binary relation R on X, it holds that*

$$\sigma_{\Gamma(\lambda,i)}(x, y, z) \in C_\lambda(R) \iff \rho_i(y, z) \in R\,.$$

Proof. We give the proof for $\lambda = \ell$. From Table 2.6, it follows that

$$\begin{aligned} \sigma_{\Gamma(\ell,i)}(x, y, z) \in C_\ell(R) &\iff \sigma_i(x, y, z) \in C_\ell(R) \\ &\iff \rho_i(y, z) \in R\,. \end{aligned}$$

□

Proposition 2.8. *Let $q = (kji)_2 \in \{0, \ldots, 7\}$, $r \in \{0, 1, 2\}$ and $\alpha, \beta, \gamma \in \{\ell, m, r\}$. For any binary relations R and P on X, the following inclusion holds:*

$$C_\alpha(R) \,\square_p^r\, C_\beta(P) \subseteq C_\gamma(R \circ_q P)\,,$$

with $p = (\Gamma(\gamma, k)\Gamma(\beta, j)\Gamma(\alpha, i))_6$.

Proof. We give the proof for $r = 0$.

$$\begin{aligned}
&C_\alpha(R) \,\square_p^0\, C_\beta(P) \\
&= \left\{\sigma_{\Gamma(\gamma,k)}(x,y,z) \mid (\exists t \in X)(\sigma_{\Gamma(\alpha,i)}(x,y,t) \in C_\alpha(R) \wedge \sigma_{\Gamma(\beta,j)}(x,t,z) \in C_\beta(P))\right\} \\
&\subseteq \left\{\sigma_{\Gamma(\gamma,k)}(x,y,z) \mid (\exists t \in X)(\rho_i(y,t) \in R \wedge \rho_j(t,z) \in P)\right\} \\
&= \left\{\sigma_{\Gamma(\gamma,k)}(x,y,z) \mid \rho_k(y,z) \in R \circ_q P\right\} \\
&= \left\{\sigma_{\Gamma(\gamma,k)}(x,y,z) \mid \sigma_{\Gamma(\gamma,k)}(x,y,z) \in C_\gamma\left(R \circ_q P\right)\right\} \\
&= C_\gamma\left(R \circ_q P\right)\ .
\end{aligned}$$

□

The following example shows that the inclusions in Proposition 2.8 are proper inclusions, *i.e.*, the cylindrical extension of the 3-point composition of two binary relations is in general not equal to the corresponding 4-point composition of the cylindrical extensions of these binary relations.

Example 2.4. *Let R and S be the binary relations on the set $X = \{x_1, x_2, x_3, x_4\}$ given by $R = \{(x_1, x_2), (x_4, x_1)\}$ and $S = \{(x_1, x_3), (x_3, x_2)\}$. On the one hand, it holds that $R \circ_0 S = \{(x_4, x_3)\}$, and thus*

$$C_\ell(R \circ_0 S) = \{(x_1, x_4, x_3), (x_2, x_4, x_3), (x_3, x_4, x_3), (x_4, x_4, x_3)\}.$$

On the other hand, we have $C_\ell(R) \ \square_0^0\ C_\ell(S) = \{(x_1, x_4, x_3)\}$. It is clear that $C_\ell(R) \ \square_0^0\ C_\ell(S) \subsetneq C_\ell(R \circ_0 S)$.

Next, we show that any projection of a 4-point composition of two ternary relations is included in a corresponding 3-point composition of their binary projections. First, we recall the definition of the binary projections of a ternary relation.

Definition 2.7. [27] *Let T be a ternary relation on X.*

(i) The left projection of T is the binary relation $P_\ell(T)$ on X dfined as:

$$P_\ell(T) = \{(x, y) \in X^2 \mid (\exists z \in X)((z, x, y) \in T)\};$$

(ii) The middle projection of T is the binary relation $P_m(T)$ on X defined as:

$$P_m(T) = \{(x, y) \in X^2 \mid (\exists z \in X)((x, z, y) \in T)\};$$

(iii) The right projection of T is the binary relation $P_r(T)$ on X defined as:

$$P_r(T) = \{(x, y) \in X^2 \mid (\exists z \in X)((x, y, z) \in T)\}.$$

Consider the mapping $\Pi : \{0, \ldots, 5\} \times \{\ell, m, r\} \rightarrow \{0,1\}$ defined as:

$$\Pi(k,\lambda) = \begin{cases} 1 \text{ , if } \Gamma(1,\lambda) = k \\ 0 \text{ , otherwise.} \end{cases} \tag{2.3}$$

Explicitly, $\Pi(k,\lambda)$ is given in Table 2.7. The following lemma is straightforward.

Table 2.7: The mapping Π.

k \ λ	ℓ	m	r
0	0	0	0
1	1	0	0
2	0	0	0
3	0	0	0
4	0	1	0
5	0	0	1

Lemma 2.3. *Let T be a ternary relation on X. For any $k \in \{0, \ldots, 5\}$, it holds that*

$$P_{\omega(k)}\left(T^{\sigma_k}\right) = \left(P_\ell(T)\right)^{\rho_{\Pi(k,\omega(k))}} ,$$

with $\omega : \{0, \ldots, 5\} \rightarrow \{\ell, m, r\}$ the mapping given in Table 2.8:

Table 2.8: The mapping ω.

k	0	1	2	3	4	5
$\omega(k)$	ℓ	ℓ	m	r	m	r

Proposition 2.9. *Let $p = (kji)_6 \in \{0, \ldots, 215\}$ and $r \in \{0,1,2\}$. For any ternary relations S and T on X, the following inclusion holds:*

$$P_{\omega(k)}(S \,\square_p^r\, T) \subseteq P_{\omega(i)}(S) \circ_q P_{\omega(j)}(T) , \tag{2.4}$$

with $q = (\Pi(k,\omega(k))\Pi(j,\omega(j))\Pi(i,\omega(i)))_2$.

Proof. We give the proof for the case $r = 0$. From Proposition 2.2 and Lemma 2.3, it follows that

$$\begin{aligned} P_{\omega(k)}(S \,\Box_p^0\, T) &= P_{\omega(k)}\left((S^{\sigma_i^{-1}} \,\Box_0^0\, T^{\sigma_j^{-1}})^{\sigma_k}\right) \\ &= \left(P_\ell(S^{\sigma_i^{-1}} \,\Box_0^0\, T^{\sigma_j^{-1}})\right)^{\rho_{\Pi(k,\omega(k))}} \\ &\subseteq \left(P_\ell(S^{\sigma_i^{-1}}) \circ P_\ell(T^{\sigma_j^{-1}})\right)^{\rho_{\Pi(k,\omega(k))}} . \end{aligned}$$

Lemma 2.3 also implies that $P_\ell(T^{\sigma_i^{-1}}) = (P_{\omega(i)}(T))^{\rho_{\Pi(i,\omega(i))}}$, and hence,

$$P_{\omega(k)}(S \,\Box_p^0\, T) \subseteq \left(P_{\omega(i)}\,(S)^{\rho_{\Pi(i,\omega(i))}} \circ P_{\omega(j)}\,(T)^{\rho_{\Pi(j,\omega(j))}}\right)^{\rho_{\Pi(k,\omega(k))}} = P_{\omega(i)}(S) \circ_q P_{\omega(j)}(T)\,.$$

□

The following example shows that the inclusions in Proposition 2.9 are proper inclusions.

Example 2.5. *Let S and T be the ternary relations on the set $X = \{x_1, x_2, x_3, x_4\}$ given by*

$$\begin{aligned} S &= \{(x_1, x_1, x_2), (x_1, x_2, x_3)\}, \\ T &= \{(x_1, x_2, x_4), (x_2, x_4, x_1), (x_3, x_2, x_2)\}. \end{aligned}$$

We have $S \,\Box_0^0\, T = \{(x_1, x_1, x_4)\}$ and $P_\ell(S \,\Box_0^0\, T) = \{(x_1, x_4)\}$. Since $P_\ell(S) \circ P_\ell(T) = \{(x_1, x_2), (x_1, x_4)\}$, it holds that $P_\ell(S \,\Box_0^0\, T) \neq P_\ell(S) \circ P_\ell(T)$.

3 Five-point compositions of ternary relations

3.1. Defenitions and basic properties

In this section, we study the 5-point compositions of ternary relations. In view of the notation used for 4-point compositions, we use the mnemonic notation $\pentagon$ for 5-point compositions. A 4-point composition can be modified to obtain a 5-point composition by introducing an additional degree of freedom, substituting one of the occurrences of the element appearing twice. More precisely, the 4-point composition $\square_p^0$ in Definition 2.1 can be modified in two ways to obtain a 5-point composition. The first one is obtained by replacing the element x in the first 3-tuple (x, y, t) by a new degree of freedom s as follows:

$$S \pentagon'_p T = \{\sigma_k(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)(\sigma_i(s, y, t) \in S \wedge \sigma_j(x, t, z) \in T)\}\,, \quad (3.1)$$

while the second one is obtained by replacing the element x in the second 3-tuple (x, t, z) as follows:

$$S \pentagon''_p T = \{\sigma_k(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)(\sigma_i(x, y, t) \in S \wedge \sigma_j(s, t, z) \in T)\}\,. \quad (3.2)$$

This reasoning could give the impression that there are twice as many 5-point compositions as there are 4-point compositions. However, there are only 108 different compositions of type $\pentagon'_p$ and 108 different compositions of type $\pentagon''_p$. Indeed, each 5-point composition is obtained in two different ways. For $p = (kji)_6 \in \{0, \ldots, 215\}$, it holds that $S \pentagon'_p T = S \pentagon'_q T$, with $q = (\pi_1(k) j \pi_1(i))_6$ and π_1 the permutation of $\{0, \ldots, 5\}$ given in Table 2.1. Similarly, for $p = (kji)_6 \in \{0, \ldots, 215\}$, it holds that $S \pentagon''_p T = S \pentagon''_q T$, with $q = (\pi_2(k) \pi_2(j) i)_6$ and π_2 the permutation of $\{0, \ldots, 5\}$ given in Table 2.2.

In the following definition, we identify the 216 different 5-point compositions of ternary relations by using an elegant enumeration system.

Definition 3.1. *Let $p \in \{0, \ldots, 215\}$ with $p = (kji)_6$. For any ternary relations S and T on X, we define the composition $S \pentagon_p T$ as follows:*

$$S \pentagon_p T = \{\sigma_k(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)(\sigma_i(\hat{x}, y, t) \in S \wedge \sigma_j(\check{x}, t, z) \in T)\}\,, \quad (3.3)$$

with

$$\hat{x} = \begin{cases} x \ , \ \textit{if } k \in \{0, 3, 4\} \\ s \ , \ \textit{if } k \in \{1, 2, 5\} \end{cases}, \quad \check{x} = \begin{cases} s \ , \ \textit{if } k \in \{0, 3, 4\} \\ x \ , \ \textit{if } k \in \{1, 2, 5\} \end{cases}.$$

The same 5-point compositions would have been obtained when starting from the 4-point compositions $\square_p^1$ and $\square_p^2$. Indeed, by substituting one of the occurrences of the element z by s in the 54 compositions of the type $\square_p^1$, we obtain the 108 five-point compositions $\pentagon_p$, with $k \in \{0, 3, 4\}$. Similarly, starting from the compositions of the type $\square_p^2$, we obtain the other 108 five-point compositions $\pentagon_p$, with $k \in \{1, 2, 5\}$. Note that the 5-point compositions $\pentagon_0$, $\pentagon_{12}$, $\pentagon_{18}$, $\pentagon_{84}$, $\pentagon_{86}$ and $\pentagon_{87}$ were already introduced in [2].

The following examples illustrate Definition 3.1.

Example 3.1. *Consider $p = 175$ and two ternary relations S and T on X. In senary notation, it holds that $p = (451)_6$. Since $k = 4$, we have*

$$\begin{aligned}
S \pentagon_{175} T &= \{\sigma_4(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)(\sigma_1(\hat{x}, y, t) \in S \wedge \sigma_5(\check{x}, t, z) \in T)\} \\
&= \{\sigma_4(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)(\sigma_1(x, y, t) \in S \wedge \sigma_5(s, t, z) \in T)\} \\
&= \{(z, x, y) \in X^3 \mid (\exists (s, t) \in X^2)((x, t, y) \in S \wedge (z, t, s) \in T)\} \\
&= \{(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)((y, t, z) \in S \wedge (x, t, s) \in T)\}\,.
\end{aligned}$$

Example 3.2. *Consider two ternary relations S and T on X and the binary operation $*$ defined as follows:*

$$S * T = \{(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)((s, t, z) \in S \wedge (y, t, x) \in T)\}\,.$$

The operation $*$ corresponds to a composition of the type $\pentagon_p$, with $p \in \{0, \ldots, 215\}$. We express this composition as in Definition 3.1 by identifying the proper 3-permutations. We distinguish two possibilities:

$$S * T = \{\sigma_1(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)(\sigma_1(s, y, t) \in S \wedge \sigma_5(x, t, z) \in T)\} \tag{3.4}$$

or

$$S * T = \{\sigma_3(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)(\sigma_1(s, y, t) \in S \wedge \sigma_0(x, t, z) \in T)\}\,. \tag{3.5}$$

According to our enumeration scheme, it holds that $k \in \{1, 2, 5\}$. Therefore, Eq. (3.4) is the proper one, i.e., *$i = 1$, $j = 5$ and $k = 1$. Hence $* = \pentagon_{67}$.*

The following proposition expresses all the 5-point compositions in terms of the compositions $\pentagon_0$ or $\pentagon_{210}$, which we will refer to as the basic 5-point compositions.

Proposition 3.1. *Let $p=(kji)_6 \in \{0,\ldots,215\}$. For any ternary relations S and T on X, the 5-point composition $S \pentagon_p T$ can be written in terms of the compositions $\pentagon_0$ or $\pentagon_{210}$ as follows:*

$$S \pentagon_p T = \begin{cases} \left(S^{\sigma_i^{-1}} \pentagon_0 T^{\sigma_j^{-1}}\right)^{\sigma_k} & , \text{ if } k \in \{0,3,4\}\,, \\ \left(S^{\sigma_i^{-1}} \pentagon_{210} T^{\sigma_j^{-1}}\right)^{\sigma_k} & , \text{ if } k \in \{1,2,5\}\,. \end{cases}$$

The following proposition shows some inclusion relationships between the 4-point and 5-point compositions.

Proposition 3.2. *Let $p=(kji)_6 \in \{0,\ldots,215\}$. For any ternary relations S and T on X, it holds that*

(i) $S \square_p^0 T \subseteq S \pentagon_p T$;

(ii) $S \square_p^1 T \subseteq S \pentagon_p T$, *if* $k \in \{0,3,4\}$;

(iii) $S \square_p^2 T \subseteq S \pentagon_p T$, *if* $k \in \{1,2,5\}$.

Proof. We give the proof of (i). If $k \in \{0,3,4\}$, then

$$\begin{aligned} S \square_p^0 T &= \{\sigma_k(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_i(x,y,t) \in S \wedge \sigma_j(x,t,z) \in T)\} \\ &\subseteq \{\sigma_k(x,y,z) \in X^3 \mid (\exists (s,t) \in X^2)(\sigma_i(x,y,t) \in S \wedge \sigma_j(s,t,z) \in T)\} \\ &= S \pentagon_p T\,. \end{aligned}$$

Similarly, if $k \in \{1,2,5\}$, then

$$\begin{aligned} S \square_p^0 T &= \{\sigma_k(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_i(x,y,t) \in S \wedge \sigma_j(x,t,z) \in T)\} \\ &\subseteq \{\sigma_k(x,y,z) \in X^3 \mid (\exists (s,t) \in X^2)(\sigma_i(s,y,t) \in S \wedge \sigma_j(x,t,z) \in T)\} \\ &= S \pentagon_p T\,. \end{aligned}$$

□

Next, we study some basic properties of the 5-point compositions of ternary relations, some of them being similar to those of the 4-point compositions, while others being different.

The following proposition identifies the right and left neutral elements of the 5-point compositions of ternary relations.

Proposition 3.3. *Let $i,j,k \in \{0,\ldots,5\}$. For any ternary relation T on X, it holds that*

(i) $T \pentagon_{(kjk)_6} E_{\zeta(j)} = T$, *if* $k \in \{0,3,4\}$;

(ii) $E_{\zeta(i)} \pentagon_{(kki)_6} T = T$, *if* $k \in \{1,2,5\}$,

with ζ *as in Proposition 2.3.*

It is worth noting that in contrast to the 4-point case, no 5-point composition has a neutral element.

The following proposition discusses the mixed-commutativity of the 5-point compositions. This result is analogous to that for the 4-point compositions.

Proposition 3.4. *Let* $p = (kji)_6 \in \{0,\ldots,215\}$. *The couple* $(\pentagon_p, \pentagon_{p'})$ *is mixed-commutative, with* $p' = (\pi_3(k)\pi_3(i)\pi_3(j))_6$ *and* π_3 *as in Proposition 2.5.*

3.2. Associativity of five-point compositions of ternary relations

Although there are more 4-point compositions than 5-point compositions, it turns out that there are more associative 5-point compositions than associative 4-point compositions.

The following proposition identifies the associative 5-point compositions.

Proposition 3.5. *Let* $p = (kji)_6 \in \{0,\ldots,215\}$. *The composition* $\pentagon_p$ *is associative if and only if one of the following conditions holds:*

(i) $k = j = i$;

(ii) $k \in \{0,3,4\}$, $i = k$ *and* $j = \pi_4(k)$;

(iii) $k \in \{1,2,5\}$, $i = \pi_4(k)$ *and* $j = k$,

with π_4 *is the permutation of* $\{0,\ldots,5\}$ *given in Table 3.1:*

Table 3.1: The permutation π_4.

u	0	1	2	3	4	5
$\pi_4(u)$	2	4	3	1	5	0

Proof. Let R, S, T be three ternary relations on X. We consider the case that $k \in \{0,3,4\}$, the other case being similar. For m as in Proposition 2.6, we have

$$\begin{aligned}(R \pentagon_p S) \pentagon_p T &= \{\sigma_k(x,y,z) \in X^3 \mid (\exists(s,t) \in X^2)(\sigma_i(x,y,t) \in R \pentagon_p S \wedge \sigma_j(s,t,z) \in T)\}\\ &= \{\sigma_k(x,y,z) \in X^3 \mid (\exists(s,t) \in X^2)(\sigma_k(\sigma_m(x,y,t)) \in R \pentagon_p S \wedge \sigma_j(s,t,z) \in T)\}\\ &= \{\sigma_k(x,y,z) \in X^3 \mid (\exists(s,t,s',t') \in X^4)(\sigma_i(\sigma_m(x,y,t')) \in R\\ &\quad \wedge\sigma_j(\sigma_m(s',t',t)) \in S \wedge \sigma_j(s,t,z) \in T)\}\,.\end{aligned}$$

For n as in Proposition 2.6, we have

$$
\begin{aligned}
R \pentagon_p (S \pentagon_p T) &= \{\sigma_k(x,y,z) \in X^3 \mid (\exists (u,v) \in X^2)(\sigma_i(x,y,u) \in R \wedge \sigma_j(v,u,z) \in S \pentagon_p T)\} \\
&= \{\sigma_k(x,y,z) \in X^3 \mid (\exists (u,v) \in X^2)(\sigma_i(x,y,u) \in R \wedge \sigma_k(\sigma_n(v,u,z)) \in S \pentagon_p T)\} \\
&= \{\sigma_k(x,y,z) \in X^3 \mid (\exists (u,u',v,v') \in X^4)(\sigma_i(x,y,u) \in R \\
&\quad \wedge \sigma_i(\sigma_n(v,u,u')) \in S \wedge \sigma_j(\sigma_n(v',u',z)) \in T)\}\,.
\end{aligned}
$$

By identification,

$$k = j = i \qquad \text{or} \qquad i = k \text{ and } j = \pi_4(k).$$

□

In Table 3.2, we present the associative 5-point compositions denoted by the new notation ${\scriptstyle\pentagon}_i$, $i \in \{1, \ldots, 12\}$.

Table 3.2: The associative 5-point compositions.

${\scriptstyle\pentagon}_i$	$S \ {\scriptstyle\pentagon}_i T$	S	T
${\scriptstyle\pentagon}_1 = \pentagon_0$	(x,y,z)	(x,y,t)	(s,t,z)
${\scriptstyle\pentagon}_2 = \pentagon_{12}$	(x,y,z)	(x,y,t)	(t,s,z)
${\scriptstyle\pentagon}_3 = \pentagon_{43}$	(x,y,z)	(s,t,z)	(x,y,t)
${\scriptstyle\pentagon}_4 = \pentagon_{46}$	(x,y,z)	(t,s,z)	(x,y,t)
${\scriptstyle\pentagon}_5 = \pentagon_{86}$	(x,y,z)	(x,s,t)	(t,y,z)
${\scriptstyle\pentagon}_6 = \pentagon_{87}$	(x,y,z)	(x,t,s)	(t,y,z)
${\scriptstyle\pentagon}_7 = \pentagon_{117}$	(x,y,z)	(x,t,z)	(s,y,t)
${\scriptstyle\pentagon}_8 = \pentagon_{129}$	(x,y,z)	(x,t,z)	(t,y,s)
${\scriptstyle\pentagon}_9 = \pentagon_{172}$	(x,y,z)	(t,y,z)	(x,s,t)
${\scriptstyle\pentagon}_{10} = \pentagon_{178}$	(x,y,z)	(t,y,z)	(x,t,s)
${\scriptstyle\pentagon}_{11} = \pentagon_{210}$	(x,y,z)	(s,y,t)	(x,t,z)
${\scriptstyle\pentagon}_{12} = \pentagon_{215}$	(x,y,z)	(t,y,s)	(x,t,z)

The 3-tuple (x,y,z) belongs to $S {\scriptstyle\pentagon}_i T$ if there exists two elements $s,t \in X$ such that the other listed 3-tuples belong to S and T, respectively.

Proposition 3.4 leads to the following corollary. It expresses that the associative 5-point compositions make up six couples of compositions that are mixed-commutative.

Corollary 3.1. *The couples* $({\scriptstyle\pentagon}_1, {\scriptstyle\pentagon}_3)$, $({\scriptstyle\pentagon}_2, {\scriptstyle\pentagon}_4)$, $({\scriptstyle\pentagon}_5, {\scriptstyle\pentagon}_9)$, $({\scriptstyle\pentagon}_6, {\scriptstyle\pentagon}_{10})$, $({\scriptstyle\pentagon}_7, {\scriptstyle\pentagon}_{11})$ *and* $({\scriptstyle\pentagon}_8, {\scriptstyle\pentagon}_{12})$ *are mixed-commutative.*

3.3. Mixed associativity of five-point compositions of ternary relations

Next, we study the mixed-associativity property for the 5-point compositions. To that end, we first present some more general compositional identities. The following lemma is a matter of simple verification.

Lemma 3.1. *The permutations π_3 in Proposition 2.5 and π_4 in Proposition 5.9 commute,* i.e., *for any $u \in \{0, \ldots, 5\}$, it holds that*

$$\pi_3(\pi_4(u)) = \pi_4(\pi_3(u)) \,.$$

Proposition 3.6. *Let $\alpha, \beta, \gamma, \delta \in \{0, \ldots, 5\}$. For any ternary relations R, S, T on X, the following equalities hold:*

(i) $(R \pentagon_{(\alpha\beta\gamma)_6} S) \pentagon_{(\alpha\delta\alpha)_6} T = R \pentagon_{(\alpha\alpha\gamma)_6} (S \pentagon_{(\alpha\delta\beta)_6} T)$;

(ii) $(R \pentagon_{(\alpha\,\pi_4(\beta)\gamma)_6} S) \pentagon_{(\alpha\delta\alpha)_6} T = R \pentagon_{(\alpha\,\pi_4(\alpha)\gamma)_6} (S \pentagon_{(\alpha\delta\beta)_6} T)$, *if* $\alpha, \beta \in \{0, 3, 4\}$;

(iii) $(R \pentagon_{(\pi_4(\alpha)\pi_4(\beta)\gamma)_6} S) \pentagon_{(\alpha\delta\alpha)_6} T = R \pentagon_{(\alpha\,\pi_4(\alpha)\pi_4(\gamma))_6} (S \pentagon_{(\alpha\delta\beta)_6} T)$, *if* $\alpha, \beta \in \{0, 3, 4\}$;

(iv) $(R \pentagon_{(\alpha\beta\gamma)_6} S) \pentagon_{(\alpha\delta\,\pi_4(\alpha))_6} T = R \pentagon_{(\alpha\alpha\gamma)_6} (S \pentagon_{(\alpha\delta\,\pi_4(\beta))_6} T)$, *if* $\alpha, \beta \in \{1, 2, 5\}$;

(v) $(R \pentagon_{(\alpha\beta\gamma)_6} S) \pentagon_{(\pi_4(\alpha)\delta\,\pi_4(\alpha))_6} T = R \pentagon_{(\alpha\alpha\gamma)_6} (S \pentagon_{(\pi_4(\alpha)\delta\,\pi_4(\beta))_6} T)$, *if* $\alpha, \beta \in \{1, 2, 5\}$.

Proof. The proof of (i) is similar to that of Proposition 2.7. It follows by expressing $(R \pentagon_{(\alpha\beta\gamma)_6} S) \pentagon_{(\alpha\delta\alpha)_6} T$ in terms of $\pentagon_0$ in the case that $\alpha \in \{0, 3, 4\}$, and in terms of $\pentagon_{210}$ in the case that $\alpha \in \{1, 2, 5\}$. Next, we prove items (ii) and (iv), items (iii) and (v) being similar. Let α, $\beta \in \{0, 3, 4\}$. Using $\alpha \in \{0, 3, 4\}$, it follows that

$$\begin{aligned}
&(R \pentagon_{(\alpha\pi_4(\beta)\gamma)_6} S) \pentagon_{(\alpha\delta\alpha)_6} T \\
&= \{\sigma_\alpha(x,y,z) \in X^3 \mid (\exists (s,t) \in X^2)(\sigma_\alpha(x,y,t) \in R \pentagon_{(\alpha\pi_4(\beta)\gamma)_6} S \wedge \sigma_\delta(s,t,z) \in T)\} \\
&= \{\sigma_\alpha(x,y,z) \in X^3 \mid (\exists (s,t,s',t') \in X^4)(\sigma_\gamma(x,y,t') \in R \wedge \sigma_{\pi_4(\beta)}(s',t',t) \in S \wedge \sigma_\delta(s,t,z) \in T)\} \,.
\end{aligned}$$

Moreover, since $\beta \in \{0, 3, 4\}$, it holds that $\sigma_{\pi_4(\beta)} = \sigma_\beta \sigma_2$, and the proof continues

$$\begin{aligned}
&= \{\sigma_\alpha(x,y,z) \in X^3 \mid (\exists (s,t,s',t') \in X^4)(\sigma_\gamma(x,y,t') \in R \wedge \sigma_{\pi_4(\beta)}(s',t',t) \in S \wedge \sigma_\delta(s,t,z) \in T)\} \\
&= \{\sigma_\alpha(x,y,z) \in X^3 \mid (\exists (s,t,s',t') \in X^4)(\sigma_\gamma(x,y,t') \in R \wedge \sigma_\beta(t',s',t) \in S \wedge \sigma_\delta(s,t,z) \in T)\} \\
&= \{\sigma_\alpha(x,y,z) \in X^3 \mid (\exists (s,t,s',t') \in X^4)(\sigma_\gamma(x,y,t') \in R \wedge \sigma_\alpha(t',s',z) \in S \pentagon_{(\alpha\delta\beta)_6} T)\} \\
&= \{\sigma_\alpha(x,y,z) \in X^3 \mid (\exists (s,t,s',t') \in X^4)(\sigma_\gamma(x,y,t') \in R \wedge \sigma_{\pi_4(\alpha)}(s',t',z) \in S \pentagon_{(\alpha\delta\beta)_6} T)\} \\
&= R \pentagon_{(\alpha\pi_4(\alpha)\gamma)_6} (S \pentagon_{(\alpha\delta\beta)_6} T) \,.
\end{aligned}$$

This concludes the proof of (ii). Next, starting from (ii), we obtain using Proposition 3.4 that

$$T\pentagon_{(\pi_3(\alpha)\pi_3(\alpha)\pi_3(\delta))_6}(S\pentagon_{(\pi_3(\alpha)\pi_3(\gamma)\pi_3(\pi_4(\beta)))_6}R) = (T\pentagon_{(\pi_3(\alpha)\pi_3(\beta)\pi_3(\delta))_6}S)\pentagon_{(\pi_3(\alpha)\pi_3(\gamma)\pi_3(\pi_4(\alpha)))_6}R\,.$$

Using Lemma 3.1, we can rewrite this as:

$$T\pentagon_{(\pi_3(\alpha)\pi_3(\alpha)\pi_3(\delta))_6}(S\pentagon_{(\pi_3(\alpha)\pi_3(\gamma)\pi_4(\pi_3(\beta)))_6}R) = (T\pentagon_{(\pi_3(\alpha)\pi_3(\beta)\pi_3(\delta))_6}S)\pentagon_{(\pi_3(\alpha)\pi_3(\gamma)\pi_4(\pi_3(\alpha)))_6}R\,.$$

Now, a simple renaming $(\alpha',\beta',\delta',\gamma') := (\pi_3(\alpha),\pi_3(\beta),\pi_3(\delta),\pi_3(\gamma))$ and $(R',S',T') := (T,S,R)$ yields

$$R'\pentagon_{(\alpha'\alpha'\gamma')_6}(S'\pentagon_{(\alpha'\delta'\pi_4(\beta'))_6}T') = (R'\pentagon_{(\alpha'\beta'\gamma'))_6}S')\pentagon_{(\alpha'\delta'\pi_4(\alpha'))_6}T'\,.$$

Realizing that now $\alpha',\beta' \in \{1,2,5\}$, the proof of (iv) is complete. □

By taking $\alpha = \beta$ in Proposition 3.6, we obtain the following proposition about the mixed-associativity of 5-point compositions.

Corollary 3.2. *Let $\alpha,\beta,\gamma \in \{0,\dots,5\}$. The following couples of 5-point compositions are mixed-associative:*

(i) $\big(\pentagon_{(\alpha\alpha\beta)_6},\pentagon_{(\alpha\gamma\alpha)_6}\big)$;

(ii) $\big(\pentagon_{(\alpha\,\pi_4(\alpha)\beta)_6},\pentagon_{(\alpha\gamma\alpha)_6}\big)$, *if* $\alpha \in \{0,3,4\}$;

(iii) $\big(\pentagon_{(\pi_4(\alpha)\pi_4(\alpha)\beta)_6},\pentagon_{(\alpha\gamma\alpha)_6}\big)$, *if* $\alpha \in \{0,3,4\}$;

(iv) $\big(\pentagon_{(\alpha\alpha\beta)_6},\pentagon_{(\alpha\gamma\,\pi_4(\alpha))_6}\big)$, *if* $\alpha \in \{1,2,5\}$;

(v) $\big(\pentagon_{(\alpha\alpha\beta)_6},\pentagon_{(\pi_4(\alpha)\gamma\,\pi_4(\alpha))_6}\big)$, *if* $\alpha \in \{1,2,5\}$.

Remark 3.1. *Although there exists no mixed-associative couple consisting of different 4-point compositions for which also the converse couple is mixed-associative, such couples can be found in the case of 5-point compositions:*

(i) $(\pentagon_{(\alpha\alpha\alpha)_6},\pentagon_{(\alpha\pi_4(\alpha)\alpha)_6})$, *with* $\alpha \in \{0,3,4\}$;

(ii) $(\pentagon_{(\alpha\alpha\alpha)_6},\pentagon_{(\alpha\alpha\pi_4(\alpha))_6})$, *with* $\alpha \in \{1,2,5\}$.

Interestingly, according to Proposition 5.9, these couples consist of two different associative compositions.

The following proposition allows to identify mixed-associative couples consisting of a 4-point and a 5-point composition.

Proposition 3.7. *Let* $\alpha,\beta,\gamma,\delta \in \{0,\dots,5\}$*. For any ternary relations* R,S,T *on* X*, it holds that*

(i) $(R \;\square^0_{(\alpha\beta\gamma)_6}\; S) \;\pentagon_{(\alpha\delta\alpha)_6}\; T = R \;\square^0_{(\alpha\alpha\gamma)_6}\; (S \;\pentagon_{(\alpha\delta\beta)_6}\; T)$, *if* $\alpha \in \{0,3,4\}$*;*

(ii) $(R \;\pentagon_{(\alpha\beta\gamma)_6}\; S) \;\square^0_{(\alpha\delta\alpha)_6}\; T = R \;\pentagon_{(\alpha\alpha\gamma)_6}\; (S \;\square^0_{(\alpha\delta\beta)_6}\; T)$, *if* $\alpha \in \{1,2,5\}$*;*

(iii) $(R \;\pentagon_{(\alpha\beta\gamma)_6}\; S) \;\square^1_{(\alpha\delta\alpha)_6}\; T = R \;\pentagon_{(\alpha\alpha\gamma)_6}\; (S \;\square^1_{(\alpha\delta\beta)_6}\; T)$, *if* $\alpha \in \{1,2,5\}$*;*

(iv) $(R \;\square^2_{(\alpha\beta\gamma)_6}\; S) \;\pentagon_{(\alpha\delta\alpha)_6}\; T = R \;\square^2_{(\alpha\alpha\gamma)_6}\; (S \;\pentagon_{(\alpha\delta\beta)_6}\; T)$, *if* $\alpha \in \{0,3,4\}$.

Proof. We give the proof of (i).

$$
\begin{aligned}
&(R\,\square^0_{(\alpha\beta\gamma)_6}\,S)\,\pentagon_{(\alpha\delta\alpha)_6}\,T\\
&= \{\sigma_\alpha(x,y,z)\in X^3 \mid (\exists (s,t)\in X^2)(\sigma_\alpha(x,y,t)\in R\,\square^0_{(\alpha\beta\gamma)_6}\,S \wedge \sigma_\delta(s,t,z)\in T)\}\\
&= \{\sigma_\alpha(x,y,z)\in X^3 \mid (\exists (s,t,u)\in X^3)(\sigma_\gamma(x,y,u)\in R \wedge \sigma_\beta(x,u,t)\in S \wedge \sigma_\delta(s,t,z)\in T)\}\\
&= \{\sigma_\alpha(x,y,z)\in X^3 \mid (\exists u\in X)(\sigma_\gamma(x,y,u)\in R \wedge \sigma_\alpha(x,u,z)\in S\,\pentagon_{(\alpha\delta\beta)_6}\,T)\}\\
&= R\,\square^0_{(\alpha\alpha\gamma)_6}\,(S\,\pentagon_{(\alpha\delta\beta)_6}\,T)\,.
\end{aligned}
$$

□

The following proposition presents the couples of 4-point and 5-point compositions that satisfy the mixed-associativity property. This result follows from Proposition 3.7.

Corollary 3.3. *Let* $\alpha,\beta,\gamma \in \{0,\dots,5\}$*. The following couples of 4-point and 5-point compositions are mixed-associative:*

(i) $\left(\square^0_{(\alpha\alpha\beta)_6}, \pentagon_{(\alpha\gamma\alpha)_6}\right)$, *if* $\alpha \in \{0,3,4\}$*;*

(ii) $\left(\pentagon_{(\alpha\alpha\beta)_6}, \square^0_{(\alpha\gamma\alpha)_6}\right)$, *if* $\alpha \in \{1,2,5\}$*;*

(iii) $\left(\pentagon_{(\alpha\alpha\beta)_6}, \square^1_{(\alpha\gamma\alpha)_6}\right)$, *if* $\alpha \in \{1,2,5\}$*;*

(iv) $\left(\square^2_{(\alpha\alpha\beta)_6}, \pentagon_{(\alpha\gamma\alpha)_6}\right)$, *if* $\alpha \in \{0,3,4\}$.

3.4. Link between compositions of binary relations and five-point compositions of ternary relations

The following result shows that any cylindrical extension of a 3-point composition of binary relations is equal to a corresponding 5-point composition of their cylindrical extensions.

Proposition 3.8. *Let $q = (kji)_2 \in \{0, \dots, 7\}$ and $\alpha, \beta, \gamma \in \{\ell, m, r\}$. For any binary relations R and P on X, the following equality holds:*

$$C_\gamma(R \circ_q P) = C_\alpha(R) \pentagon_p C_\beta(P)\,,$$

with p as in Proposition 2.8.

Proof. For $p = (\Gamma(\gamma,k)\Gamma(\beta,j)\Gamma(\alpha,i))_6$, it holds that

$$\begin{aligned} C_\alpha(R) \pentagon_p C_\beta(P) &= \Big\{\sigma_{\Gamma(\gamma,k)}(x,y,z) \mid (\exists t \in X)(\sigma_{\Gamma(\alpha,i)}(x,y,t) \in C_\alpha(R) \wedge \sigma_{\Gamma(\beta,j)}(x,t,z) \in C_\beta(P))\Big\} \\ &= \Big\{\sigma_{\Gamma(\gamma,k)}(x,y,z) \mid (\exists t \in X)(\rho_i(y,t) \in R \wedge \rho_j(t,z) \in P)\Big\} \\ &= \Big\{\sigma_{\Gamma(\gamma,k)}(x,y,z) \mid \rho_k(y,z) \in R \circ_q P\Big\} \\ &= \Big\{\sigma_{\Gamma(\gamma,k)}(x,y,z) \mid \sigma_{\Gamma(\gamma,k)}(x,y,z) \in C_\gamma\left(R \circ_q P\right)\Big\} \\ &= C_\gamma\left(R \circ_q P\right)\,. \end{aligned}$$

□

Conversely, we have the following result.

Proposition 3.9. *Let $p = (kji)_6 \in \{0, \dots, 215\}$. For any ternary relations S and T on X, the following equality holds:*

$$P_{\omega(k)}(S \pentagon_p T) = P_{\omega(i)}(S) \circ_q P_{\omega(j)}(T), \tag{3.6}$$

with q as in Proposition 2.9.

Proof. We set $\eta = 0$ if $k \in \{0, 3, 4\}$, and $\eta = 210$ if $k \in \{1, 2, 5\}$. From Proposition 3.1 and Lemma 2.3, it follows that

$$\begin{aligned} P_{\omega(k)}(S \pentagon_p T) &= P_{\omega(k)}\left((S^{\sigma_i^{-1}} \pentagon_\eta T^{\sigma_j^{-1}})^{\sigma_k}\right) \\ &= \left(P_\ell(S^{\sigma_i^{-1}} \pentagon_\eta T^{\sigma_j^{-1}})\right)^{\rho_{\Pi(k,\omega(k))}} \\ &= \left(P_\ell(S^{\sigma_i^{-1}}) \circ P_\ell(T^{\sigma_j^{-1}})\right)^{\rho_{\Pi(k,\omega(k))}} . \end{aligned}$$

From Lemma 2.3, it follows that

$$P_{\omega(k)}(S \pentagon_p T) = \left(P_{\omega(i)}(S)^{\rho_{\Pi(i,\omega(i))}} \circ P_{\omega(j)}(T)^{\rho_{\Pi(j,\omega(j))}}\right)^{\rho_{\Pi(k,\omega(k))}} = P_{\omega(i)}(S) \circ_q P_{\omega(j)}(T) .$$

□

4 Bandler–Kohout compositions and traces of ternary relations

In this chapter, we introduce Bandler–Kohout-like compositions of ternary relation, we show theire properties and use these compositions to introduce the notion of the traces of ternary relations.

4.1. Definition and Properties of BK-compositions of ternary relations

In this section, we introduce the BK-compositions of ternary relations and show thier basic properties.

Definition 4.1. *Let* $p \in \{0, \dots, 215\}$ *with* $p = (kji)_6$. *For any two ternary relations* S *and* T *on* X*, the sub-composition* $S \boxed{\lhd} T$*, super-composition* $S \boxed{\rhd} T$ *and square-composition* $S \boxed{\diamond} T$ *are defined as:*

$$
\begin{aligned}
S \boxed{\lhd}_p^0 T &= \{\sigma_k(x,y,z) \in X^3 \mid (\forall t \in X)(\sigma_i(x,y,t) \in S \Rightarrow \sigma_j(x,t,z) \in T)\}\,;\\
S \boxed{\rhd}_p^0 T &= \{\sigma_k(x,y,z) \in X^3 \mid (\forall t \in X)(\sigma_i(x,y,t) \in S \Leftarrow \sigma_j(x,t,z) \in T)\}\,;\\
S \boxed{\diamond}_p^0 T &= \{\sigma_k(x,y,z) \in X^3 \mid (\forall t \in X)(\sigma_i(x,y,t) \in S \Leftrightarrow \sigma_j(x,t,z) \in T)\}\,;\\
S \boxed{\lhd}_p^1 T &= \{\sigma_k(x,y,z) \in X^3 \mid (\forall t \in X)(\sigma_i(x,y,t) \in S \Rightarrow \sigma_j(z,t,z) \in T)\}\,;\\
S \boxed{\rhd}_p^1 T &= \{\sigma_k(x,y,z) \in X^3 \mid (\forall t \in X)(\sigma_i(x,y,t) \in S \Leftarrow \sigma_j(z,t,z) \in T)\}\,;\\
S \boxed{\diamond}_p^1 T &= \{\sigma_k(x,y,z) \in X^3 \mid (\forall t \in X)(\sigma_i(x,y,t) \in S \Leftrightarrow \sigma_j(z,t,z) \in T)\}\,;\\
S \boxed{\lhd}_p^2 T &= \{\sigma_k(x,y,z) \in X^3 \mid (\forall t \in X)(\sigma_i(y,y,t) \in S \Rightarrow \sigma_j(x,t,z) \in T)\}\,;\\
S \boxed{\rhd}_p^2 T &= \{\sigma_k(x,y,z) \in X^3 \mid (\forall t \in X)(\sigma_i(y,y,t) \in S \Leftarrow \sigma_j(x,t,z) \in T)\}\,;\\
S \boxed{\diamond}_p^2 T &= \{\sigma_k(x,y,z) \in X^3 \mid (\forall t \in X)(\sigma_i(y,y,t) \in S \Leftrightarrow \sigma_j(x,t,z) \in T)\}\,.
\end{aligned}
$$

The following example illustrate the composition $S \boxed{\lhd}_0^0 T$ for two simple ternary relations S and T.

Example 4.1. *Let* S *and* T *be the ternary relations on the set* $X = \{x_1, x_2, x_3, x_4\}$ *given by*

$$
\begin{aligned}
S &= \{(x_3, x_3, x_1), (x_3, x_2, x_3)\},\\
T &= \{(x_3, x_1, x_3), (x_3, x_1, x_2), (x_1, x_1, x_1)\}.
\end{aligned}
$$

We have $S \boxed{\lhd}_0^0 T = \{(x_1,x_1,x_3),(x_2,x_2,x_1),(x_1,x_3,x_1),(x_2,x_1,x_1),(x_1,x_2,x_2),(x_2,x_3,x_2),$ $(x_1,x_3,x_2),(x_1,x_1,x_2),(x_2,x_1,x_3),(x_1,x_2,x_1),(x_2,x_2,x_2),(x_1,x_1,x_1),(x_2,x_1,x_1),(x_2,x_2,x_3)\}$

These compositions can be expressed in terms of three representative ones, just as in the case of the composition of binary relations.

Proposition 4.1. *Let* $p = (kji)_6 \in \{0, \ldots, 215\}$ *and* $r \in \{0, 1, 2\}$*. For any two ternary relations* S *and* T *on* X*, the compositions* $S \boxed{\lhd}_p^r T$*,* $S \boxed{\rhd}_p^r T$ *and* $S \boxed{\diamond}_p^r T$ *can be written in terms of the compositions* $S \boxed{\lhd}_0^r T$*,* $S \boxed{\rhd}_0^r T$ *and* $S \boxed{\diamond}_0^r T$ *as:*

(i) $S \boxed{\lhd}_p^r \ T = (S^{\sigma_i^{-1}} \boxed{\lhd}_0^r T^{\sigma_j^{-1}})^{\sigma_k}$;

(ii) $S \boxed{\rhd}_p^r \ T = (S^{\sigma_i^{-1}} \boxed{\rhd}_0^r T^{\sigma_j^{-1}})^{\sigma_k}$;

(iii) $S \boxed{\diamond}_p^r \ T = (S^{\sigma_i^{-1}} \boxed{\diamond}_0^r T^{\sigma_j^{-1}})^{\sigma_k}$.

Proof. We give the proof for $\boxed{\lhd}_p^0$:

$$
\begin{aligned}
S \boxed{\lhd}_p^0 T &= \{\sigma_k(x, y, z) \in X^3 \mid (\exists t \in X)(\sigma_i(x, y, t) \in S \Rightarrow \sigma_j(x, t, z) \in T)\} \\
&= \{\sigma_k(x, y, z) \in X^3 \mid (\exists t \in X)((x, y, t) \in S^{\sigma_i^{-1}} \Rightarrow (x, t, z) \in T^{\sigma_j^{-1}})\} \\
&= \{\sigma_k(x, y, z) \in X^3 \mid (x, y, z) \in S^{\sigma_i^{-1}} \boxed{\lhd}_0^0 T^{\sigma_j^{-1}}\} \\
&= (S^{\sigma_i^{-1}} \boxed{\lhd}_0^0 T^{\sigma_j^{-1}})^{\sigma_k}.
\end{aligned}
$$

□

In this section, we study the properties of the newly introduced compositions.

The following proposition concerns the refinement of the BK-compositions of ternary relations.

Proposition 4.2. *Let* $p \in \{0, \ldots, 215\}$ *and* $r \in \{0, 1, 2\}$*. For any two ternary relations* S *and* T *on* X*, the following inclusions hold:*

(i) $S \boxed{\diamond}_p^r \ T \subseteq S \boxed{\lhd}_p^r \ T \subseteq S \square_p^r \ T$;

(ii) $S \boxed{\diamond}_p^r \ T \subseteq S \boxed{\rhd}_p^r \ T \subseteq S \square_p^r \ T$.

Although the compositions $\boxed{\lhd}$, $\boxed{\rhd}$ and $\boxed{\diamond}$ are not commutative in general, interesting mixed-commutativity properties can be established.

Proposition 4.3. *Let* $p = (kji)_6 \in \{0, \ldots, 215\}$*. For any two ternary relations* S *and* T *on* X*, the following equalities hold:*

(i) $S \boxed{\lhd}_p^0 \ T = T \boxed{\rhd}_{p'}^0 \ S$;

(ii) $S \boxed{\lhd}_p^1 \ T = T \boxed{\rhd}_{p'}^2 \ S$;

(iii) $S \boxed{\diamond}_p^0 \ T = T \boxed{\diamond}_{p'}^0 \ S$;

(iv) $S \boxed{\diamond}_p^1 \ T = T \boxed{\diamond}_{p'}^2 \ S$;

with $p' = (\pi(k)\pi(i)\pi(j))_6$ *and* π *the permutation of* $\{0,\ldots,5\}$ *given by Table 4.1:*

Table 4.1: The permutation π.

u	0	1	2	3	4	5
$\pi(u)$	1	0	4	5	2	3

Proof. We only give the proof for (i):

$$\begin{aligned} S \boxdot_p^0 T &= \{\sigma_k(x,y,z) \in X^3 \mid (\forall t \in X)(\sigma_i(x,y,t) \in S \Rightarrow \sigma_j(x,t,z) \in T)\} \\ &= \{\sigma_k(x,y,z) \in X^3 \mid (\forall t \in X)(\sigma_j(x,t,z) \in T \Leftarrow \sigma_i(x,y,t) \in S)\} \\ &= \{\sigma_k(\sigma_1(x,z,y)) \in X^3 \mid (\forall t \in X)(\sigma_j(\sigma_1(x,z,t)) \in T \Leftarrow \sigma_i(\sigma_1(x,t,y)) \in S)\} \\ &= \{\sigma_k(\sigma_1(x,y,z)) \in X^3 \mid (\forall t \in X)(\sigma_j(\sigma_1(x,y,t)) \in T \Leftarrow \sigma_i(\sigma_1(x,t,z)) \in S)\}\,. \end{aligned}$$

Since $\sigma_u(\sigma_1(x,y,z)) = \sigma_{\pi(u)}(x,y,z)$, it follows that

$$S \boxdot_p^0 T = \{\sigma_{\pi(k)}(x,y,z) \in X^3 \mid (\forall t \in X)(\sigma_{\pi(j)}(x,y,t) \in T \Leftarrow \sigma_{\pi(i)}(x,t,z) \in S)\} = T \boxdot_{p'}^0 S\,.$$

□

4.2. Interactions between the compositions of ternary relations and their BK-compositions

In this section, we show some interactions between the 4-point compositions and the BK-compositions of ternary relations.

The following proposition shows a mixed-associativity property of BK-compositions.

Proposition 4.4. *Let* $p \in \{0, 43, 86, 129, 172, 215\}$. *For any three ternary relations* R, S, *and* T *on* X, *the following equality holds:*

$$R \boxdot_p^0 (S \boxdot_p^0 T) = (R \boxdot_p^0 S) \boxdot_p^0 T\,.$$

Proof. We only give the proof for the case $p = 0$. Using the following tautology which holds for any three Boolean propositions P, Q, R,

$$P \implies (Q \implies R) \iff Q \implies (P \implies R)\,,$$

we get

$$
\begin{aligned}
& (x,y,z) \in R \boxleft^0_0 (S \boxright^0_0 T) \\
\Longleftrightarrow\ & (\forall t \in X)((x,y,t) \in R \Rightarrow (x,t,z) \in (S \boxright^0_0 T)) \\
\Longleftrightarrow\ & (\forall t \in X)((x,y,t) \in R \Rightarrow ((\forall s \in X)(x,t,s) \in S \Leftarrow (x,s,z) \in T)) \\
\Longleftrightarrow\ & (\forall (s,t) \in X^2)\,((x,y,t) \in R \Rightarrow ((x,s,z) \in T \Rightarrow (x,t,s) \in S)) \\
\Longleftrightarrow\ & (\forall (s,t) \in X^2)\,((x,s,z) \in T \Rightarrow ((x,y,t) \in R \Rightarrow (x,t,s) \in S)) \\
\Longleftrightarrow\ & (\forall s \in X)\,((\forall t \in X)\ ((x,y,t) \in R \Rightarrow (x,t,s) \in S) \Leftarrow (x,s,z) \in T) \\
\Longleftrightarrow\ & (\forall s \in X)\Big((x,y,s) \in R \boxleft^0_0 S\Big) \Leftarrow (x,s,z) \in T) \\
\Longleftrightarrow\ & (x,y,z) \in (R \boxleft^0_0 S) \boxright^0_0 T.
\end{aligned}
$$

$\square$

The following example illustrates that the above property does not hold for the compositions $\boxleft^r_p$ and $\boxright^r_p$ with $r \in \{1,2\}$.

Example 4.2. *Consider the ternary relations R, S and T on the set $X = \{x_1, x_2, x_3, x_4, x_5\}$ given as:*

$$
\begin{aligned}
R &= \{(x_1, x_2, x_3)\}, \\
S &= \{(x_4, x_3, x_5)\}, \\
T &= \{(x_4, x_5, x_4)\}.
\end{aligned}
$$

It is clear that $R \boxleft^1_0 (S \boxright^1_0 T) = \{(x_1, x_2, x_4)\} \neq (R \boxleft^1_0 S) \boxright^1_0 T = \emptyset$.

Proposition 4.5. *Let $p \in \{0, 43, 86, 129, 172, 215\}$. For any three ternary relations R, S, and T on X, the following equalities hold:*

(i) $R \boxleft^0_p (S \boxleft^0_p T) = (R \square^0_p S) \boxleft^0_p T$;

(ii) $R \boxright^0_p (S \square^0_p T) = (R \boxright^0_p S) \boxright^0_p T$.

Proof. We only give the proof for (i) with $p = 0$. Again using the Boolean tautology from the proof of Proposition 4.4, we get

$$
\begin{aligned}
& (x,y,z) \in R \mathbin{\boxed{\lhd}}_0^0 (S \mathbin{\boxed{\lhd}}_0^0 T) \\
\iff\ & (\forall t \in X) \big((x,y,t) \in R \Rightarrow (x,t,z) \in (S \mathbin{\boxed{\lhd}}_0^0 T)\big) \\
\iff\ & (\forall t \in X) ((x,y,t) \in R \Rightarrow ((\forall s \in X)((x,t,s) \in S \Rightarrow (x,s,z) \in T)) \\
\iff\ & (\forall (s,t) \in X^2) ((x,y,t) \in R \Rightarrow ((x,t,s) \in S \Rightarrow (x,s,z) \in T)) \\
\iff\ & (\forall (s,t) \in X^2) (((x,y,t) \in R \wedge (x,t,s) \in S) \Rightarrow (x,s,z) \in T) \\
\iff\ & (\forall s \in X) ((\exists t \in X) ((x,y,t) \in R \wedge (x,t,s)) \in S \Rightarrow (x,s,z) \in T) \\
\iff\ & (\forall s \in X) \big((x,y,s) \in R \mathbin{\square}_0^0 S \Rightarrow (x,s,z) \in T\big) \\
\iff\ & (x,y,z) \in (R \mathbin{\square}_0^0 S) \mathbin{\boxed{\lhd}}_0^0 T \,.
\end{aligned}
$$

□

Now, we show that the BK-compositions of ternary relations are a valuable tool to express the greatest solutions of relational equations. The following proposition expresses an adjointness-like relationship [21] between the different types of compositions of t ernary relations.

Proposition 4.6. *For any three ternary relations R, S, and T on X, the following equivalences hold:*

$$R \mathbin{\square}_0^0 S \subseteq T \iff R \subseteq T \mathbin{\boxed{\rhd}}_0^0 S^{\sigma_1} \iff S \subseteq R^{\sigma_1} \mathbin{\boxed{\lhd}}_0^0 T \,.$$

Proof. We only give the proof for the first equivalence. For the direct implication, suppose that $R \mathbin{\square}_0^0 S \subseteq T$. Let $(x,y,z) \in R$, then we need to show that $(x,y,z) \in T \mathbin{\boxed{\rhd}}_0^0 S^{\sigma_1}$, *i.e.*, for any $t \in X$, it holds that $(x,y,t) \in T \Leftarrow (x,z,t) \in S$. Suppose that $(x,z,t) \in S$, then the fact that $(x,y,z) \in R$ implies that $(x,y,t) \in R \mathbin{\square}_0^0 S$. The hypothesis $R \mathbin{\square}_0^0 S \subseteq T$ guarantees that $(x,y,t) \in T$. Thus, $(x,y,t) \in T \Leftarrow (x,z,t) \in S$. For the converse implication, let $(x,y,z) \in R \mathbin{\square}_0^0 S$, then there exists $t \in X$ such that $(x,y,t) \in R$ and $(x,t,z) \in S$. Since $R \subseteq T \mathbin{\boxed{\rhd}}_0^0 S^{\sigma_1}$, it follows that $(x,y,t) \in T \mathbin{\boxed{\rhd}}_0^0 S^{\sigma_1}$. Thus, for any $s \in X$, it holds that $(x,y,s) \in T \Leftarrow (x,t,s) \in S$. Since $(x,t,z) \in S$, it holds that $(x,y,z) \in T$.

□

The following proposition extends the previous result to account for all $p \in \{0, \ldots, 215\}$.

Proposition 4.7. *Let $p = (kji)_6 \in \{0, \ldots, 215\}$. For any three ternary relations R, S, and T on X, the following equivalences hold:*

$$R \mathbin{\square}_p^0 S \subseteq T \iff R \subseteq T \mathbin{\boxed{\rhd}}_{p'}^0 S \iff S \subseteq R \mathbin{\boxed{\lhd}}_{p''}^0 T \,,$$

with $p' = (i\pi(j)k)$ *and* $p'' = (jk\pi(i))$.

Proof. We only show the first equivalence. Suppose that $R \ \square^0_p \ S \subseteq T$. From Proposition 2.2, it follows that $R^{\sigma_i^{-1}} \square^0_0 S^{\sigma_j^{-1}} \subseteq T^{\sigma_k^{-1}}$. Proposition 4.6 implies that $R^{\sigma_i^{-1}} \subseteq T^{\sigma_k^{-1}} \boxright^0_0 (S^{\sigma_j^{-1}})^{\sigma_1}$. Hence, $R \subseteq (T^{\sigma_k^{-1}} \boxright^0_0 S^{\sigma_{\pi(j)}^{-1}})^{\sigma_i}$. Proposition 5.2 guarantees that $R \subseteq T \boxright^0_{p'} S$, with $p' = (i\pi(j)k)$.

□

The following proposition concerns the greatest solution of the most basic ternary relational equation.

Proposition 4.8. *Let* S *and* T *be two ternary relations on* X. *The following statements hold:*

(i) *The relational equation* $S \ \square^0_0 U = T$ *in the unknown relation* U *is solvable if and only if* $S^{\sigma_1} \boxleft^0_0 T$ *is its greatest solution.*

(ii) *The relational equation* $U \ \square^0_0 S = T$ *in the unknown relation* U *is solvable if and only if* $T \boxright^0_0 S^{\sigma_1}$ *is its greatest solution.*

Proof. Suppose that the relational equation $S \ \square^0_0 U = T$ has a solution U, then Proposition 4.6 guarantees that $U \subseteq S^{\sigma_1} \boxleft^0_0 T$. Next, it suffices to show that $S^{\sigma_1} \boxleft^0_0 T$ is a solution, *i.e.*, $T = S \ \square^0_0 \ (S^{\sigma_1} \boxleft^0_0 T)$. The fact that $U \subseteq S^{\sigma_1} \boxleft^0_0 T$ implies that $T = S \ \square^0_0 U \subseteq S \ \square^0_0 \ (S^{\sigma_1} \boxleft^0_0 T)$. Conversely, let $(x, y, z) \in S \ \square^0_0 \ (S^{\sigma_1} \boxleft^0_0 T)$, then there exists $t \in X$ such that $(x, y, t) \in S$ and $(x, t, z) \in S^{\sigma_1} \boxleft^0_0 T$. Hence, for any $t' \in X$, we have that $(x, t', t) \in S$ implies $(x, t', z) \in T$. Since $(x, y, t) \in S$, it follows that $(x, y, z) \in T$.

The converse implication is trivial.

□

Proposition 4.9. *Let* $p \in \{0, \ldots, 215\}$. *For any two ternary relations* S *and* T *on* X, *the following statements hold, where* p' *and* p'' *are as in Proposition 4.7:*

(i) *The relational equation* $S \ \square^0_p U = T$ *in the unknown relation* U *is solvable if and only if* $T \boxleft^0_{p'} S^{\sigma_1}$ *is its greatest solution.*

(ii) *The relational equation* $U \ \square^0_p S = T$ *in the unknown relation* U *is solvable if and only if* $S^{\sigma_1} \boxright^0_{p''} T$ *is its greatest solution.*

Proof. The proof is similar to that of Proposition 4.8.

□

4.3. Traces of ternary relations based of BK-compositions

In this section, we introduce the traces of ternary relation based on the BK-compositions of ternary relations

4.3.1. Traces of ternary relations

Similarly as for binary relations, we can introduce the traces of a ternary relation T as the BK-compositions of T with itself.

Definition 4.2. *Let $p = (kji)_6 \in \{0, \ldots, 215\}$ and $r \in \{0, 1, 2\}$. The traces of a ternary relation T on X are the ternary relations defined as:*

$$T^{\lhd_p^r} := T \lhd_p^r T = T \rhd_{p'}^{r'} T \,,$$

where r' and p' are as in Proposition 4.3.

Example 4.3. *The following example gives the relation $T^{\lhd_p^r}$ where T is given as in Example 4.1.*

We have $T^{\lhd_p^r} = \{(x_3, x_2, x_3), (x_3, x_3, x_3), (x_1, x_1, x_2), (x_1, x_2, x_1), (x_1, x_3, x_1), (x_3, x_1, x_2), (x_1, x_1, x_3), (x_3, x_2, x_2), (x_3, x_1, x_1), (x_3, x_3, x_2), (x_3, x_1, x_3), (x_1, x_1, x_1)\}$

Obviously, these traces are not all distinct. Among them, we identify those that preserve the properties in Theorems 1.1–1.3.

Proposition 4.10. *Let T be a ternary relation on X.*

(i) *The following traces are left-reflexive and left-transitive:*

(a) $T^{\lhd_1^0} = T^{\lhd_6^0} = T^{\lhd_{16}^0} = T^{\lhd_{23}^0} = T^{\lhd_{26}^0} = T^{\lhd_{33}^0}$;

(b) $T^{\lhd_{37}^0} = T^{\lhd_{42}^0} = T^{\lhd_{52}^0} = T^{\lhd_{59}^0} = T^{\lhd_{62}^0} = T^{\lhd_{69}^0}$.

(ii) *The following traces are middle-reflexive and middle-transitive:*

(a) $T^{\lhd_{73}^0} = T^{\lhd_{78}^0} = T^{\lhd_{88}^0} = T^{\lhd_{95}^0} = T^{\lhd_{98}^0} = T^{\lhd_{105}^0}$;

(b) $T^{\lhd_{109}^0} = T^{\lhd_{114}^0} = T^{\lhd_{124}^0} = T^{\lhd_{131}^0} = T^{\lhd_{134}^0} = T^{\lhd_{141}^0}$.

(iii) *The following traces are right-reflexive and right-transitive:*

(a) $T^{\lhd_{145}^0} = T^{\lhd_{150}^0} = T^{\lhd_{160}^0} = T^{\lhd_{167}^0} = T^{\lhd_{170}^0} = T^{\lhd_{177}^0}$;

(b) $T^{\lhd_{181}^0} = T^{\lhd_{186}^0} = T^{\lhd_{196}^0} = T^{\lhd_{203}^0} = T^{\lhd_{206}^0} = T^{\lhd_{213}^0}$.

The following proposition expresses the link between the traces in statements (a) and (b) of Proposition 4.10.

Proposition 4.11. *Let T be a ternary relation on X.*

(i) $T^{\boxed{\lhd}^0_{37}} = (T^{\boxed{\lhd}^0_1})^{\sigma_1}$;

(ii) $T^{\boxed{\lhd}^0_{109}} = (T^{\boxed{\lhd}^0_{73}})^{\sigma_1}$;

(iii) $T^{\boxed{\lhd}^0_{181}} = (T^{\boxed{\lhd}^0_{145}})^{\sigma_1}$.

Proof. We give the proof for (i). From Proposition 4.1, it follows that

$$T^{\boxed{\lhd}^0_{37}} = (T^{\sigma_1} \boxed{\lhd}^0_0 T)^{\sigma_1} = (T \boxed{\lhd}^0_1 T)^{\sigma_1} = (T^{\boxed{\lhd}^0_1})^{\sigma_1} .$$

□

Similarly as in the binary setting, we select three of the traces that will play a specific role in what follows, namely the left trace T^ℓ, the middle trace T^m and the right trace T^r:

(i) $T^\ell := T^{\boxed{\lhd}^0_1} = \{(x,y,z) \in X^3 \mid (\forall a \in X)((x,a,y) \in T \Rightarrow (x,a,z) \in T)\}$;

(ii) $T^m := T^{\boxed{\lhd}^0_{73}} = \{(x,y,z) \in X^3 \mid (\forall a \in X)((x,y,a) \in T \Rightarrow (z,y,a) \in T)\}$;

(iii) $T^r := T^{\boxed{\lhd}^0_{145}} = \{(x,y,z) \in X^3 \mid (\forall a \in X)((y,a,z) \in T \Rightarrow (x,a,z) \in T)\}$.

The following proposition expresses that some of these traces coincide in the face of symmetry.

Proposition 4.12. *For any ternary relation T on X, the following statements hold:*

(i) *if T is left-symmetric, then $T^\ell = T^m$;*

(ii) *if T is middle-symmetric, then $T^r = T^\ell$;*

(iii) *if T is left- symmetric, then $T^m = T^r$.*

The following theorem shows that the above-defined traces are solutions to relational equations. It extends Theorem 1.3 to the ternary setting.

Theorem 4.1. *For any ternary relation T on X, the following equalities hold:*

(i) $T = T \square_0 T^\ell = T^\ell \square_{43} T$;

(ii) $T = T \square_{86} T^m = T^m \square_{129} T$;

(iii) $T = T \square_{172} T^r = T^r \square_{215} T$.

Proof. We give the proof for the first equality of (i). Let $(x,y,z) \in T \square_0 T^\ell$, then there exists an element t of X such that $(x,y,t) \in T$ and $(x,t,z) \in T^\ell$, and thus it follows for any $a \in X$ that $(x,a,t) \in T$ implies $(x,a,z) \in T$, in particular $(x,y,t) \in T$ implies $(x,y,z) \in T$, thus $T \square_0 T^\ell \subseteq T$. Conversely, let $(x, y, z) \in T$, since T^ℓ is left-reflexive it holds that $(x, z, z) \in T^\ell$, thus $(x, y, z) \in T \square_0 T^\ell$ and hence $T \subseteq T \square_0 T^\ell$.

□

4.3.2. Properties of ternary relations in terms of traces

In this section, we characterize some properties of a ternary relation in terms of its traces. The first proposition characterizes different reflexivity properties and extends Theorem 1.1 to the ternary setting.

Proposition 4.13. *For any ternary relation T on X, the following equivalences hold:*

(i) *T is left-reflexive $\iff T^\ell \subseteq T$;*

(ii) *T is middle-reflexive $\iff T^m \subseteq T$;*

(iii) *T is right-reflexive $\iff T^r \subseteq T$.*

Proof. We give the proof for (i). Since T^ℓ is left-reflexive, the inclusion $T^\ell \subseteq T$ implies that T is also left-reflexive. Conversely, suppose that T is left-reflexive. Let $(x, y, z) \in T^\ell$, then for any $a \in X$, it holds that $(x, a, y) \in T \implies (x, a, z) \in T$. In particular, $(x, y, y) \in T \implies (x, y, z) \in T$, thus $T^\ell \subseteq T$.

□

The following proposition characterizes different transitivity properties of a ternary relation in terms of its traces. This result extends Theorem 1.2 to the ternary setting.

Proposition 4.14. *For any ternary relation T on X, the following equivalences hold:*

(i) *T is left-transitive $\iff T \subseteq T^\ell$;*

(ii) *T is middle-transitive $\iff T \subseteq T^m$;*

(iii) *T is right-transitive $\iff T \subseteq T^r$.*

Proof. We give the proof for (i). Theorem 4.1 states that $T \square_0 T^\ell \subseteq T$. If $T \subseteq T^\ell$, then $T \square_0 T \subseteq T \square_0 T^\ell \subseteq T$. Thus, T is left-transitive. Conversely, let $(x, y, z) \in T$, then for any $a \in X$, the left-transitivity of T implies $(x, a, y) \in T \implies (x, a, z) \in T$. Hence, $(x, y, z) \in T^\ell$.

□

The following corollary follows immediately from Propositions 4.13 and 4.14.

Corollary 4.1. *For any ternary relation T on X, the following equivalences hold:*

(i) T *is left-reflexive and left-transitive* $\iff T^\ell = T$;

(ii) T *is middle-reflexive and middle-transitive* $\iff T^m = T$;

(iii) T *is right-reflexive and right-transitive* $\iff T^r = T$.

4.3.3. Ternary equivalence relations

In this subsection, we consider formal ternary counterparts of the concept of a binary equivalence relation, *i.e.*, a binary relation that is reflexive, symmetric, and transitive. As each of these properties comes in many flavors, we aim to avoid a combinatorial explosion and select relevant combinations only.

Definition 4.3. *A ternary relation T on X is said to be:*

(i) *a left ternary equivalence relation if it is left-reflexive, left-symmetric and left-transitive;*

(ii) *a middle ternary equivalence relation if it is middle-reflexive, middle-symmetric and middle-transitive;*

(iii) *a right ternary equivalence relation if it is right-reflexive, right-symmetric and right-transitive.*

The following definition and proposition provide a setting in which ternary equivalence relations naturally arise.

Definition 4.4. *With any partition $\mathcal{P}$ of X^2, we associate the ternary relations T_ℓ, T_m and T_r on X defined as:*

(i) $T_\ell = \{(x,y,z) \in X^3 \mid (\exists\, \mathcal{C} \in \mathcal{P})(\,(x,y) \in \mathcal{C} \land (x,z) \in \mathcal{C}\,)\,\}$;

(ii) $T_m = \{(x,y,z) \in X^3 \mid (\exists\, \mathcal{C} \in \mathcal{P})(\,(x,y) \in \mathcal{C} \land (y,z) \in \mathcal{C}\,)\,\}$;

(iii) $T_r = \{(x,y,z) \in X^3 \mid (\exists\, \mathcal{C} \in \mathcal{P})(\,(x,z) \in \mathcal{C} \land (y,z) \in \mathcal{C}\,)\,\}$.

Proposition 4.15. *For any partition $\mathcal{P}$ of X^2, the following statements hold:*

(i) T_ℓ *is a left ternary equivalence relation;*

(ii) T_m *is a middle ternary equivalence relation;*

(iii) T_r *is a right ternary equivalence relation.*

Proof. We give the proof for (i). The left-reflexivity and left-symmetry of T_ℓ are obvious. Next, we show that T_ℓ is left-transitive. Let $(x,y,t),(x,t,z) \in T_\ell$, then there exist $\mathcal{C},\mathcal{D} \in \mathcal{P}$ such that $(x,y),(x,t) \in \mathcal{C}$ and $(x,t),(x,z) \in \mathcal{D}$. Since $(x,t) \in \mathcal{C} \cap \mathcal{D}$, it holds that $\mathcal{C} = \mathcal{D}$. Hence, (x,y) and (x,z) belong to the same class, and, thus, $(x, y, z) \in T$. Therefore, T_ℓ is left-transitive.

□

5 Compositions of ternary fuzzy relations

The notion of an L-relation on a set X generalizes the classical notion of a relation by expressing degrees of relationship in some bounded lattice L. In this chapter, we extend the previous results to the framework of L-relations

5.1. Fuzzy relation

5.1.1. Lattice

An algebra $(L, \wedge, \vee)$ is called a lattice [11] if L is a nonempty set, $\wedge$ and $\vee$ are binary operations on L, both $\wedge$ and $\vee$ are idempotent(ie $x \wedge x = x$ and $x \vee x = x$), commutative(ie $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$), and associative(i.e. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$ and $x \vee x = x$), and they satisfy the absorption law(i.e $x \wedge (x \vee y) = x \vee (x \wedge y) = x$) . A bounded lattice L is a lattice that has a smallest element 0 and a greatest element 1. A triangular norm $*$ on a bounded lattice is a binary operation that is commutative, associative , order-preserving(i.e $x \leq y$ implies $x * a \leq y * a$) and 1 is neutral for $*$ [21]. Moreover if $a * (x \vee y) = (a * x) \vee (a * y)$, $*$ is said distributive over $\vee$. In the latter case, $(L, \wedge, \vee, *,)$ is called *commutative* l-monoid, if is bounded $(L, \wedge, \vee, *, 0, 1)$ *commutative integral* l-monoid. Furthermore if there exist a second binary operation $\rightarrow_*$ for which $x * y \leq z$ if and only it $x \leq y \rightarrow_* z$, then $(L, \wedge, \vee, *, \rightarrow_*, 0, 1)$ is called *Residuated Lattice* [21]. Each residuated lattice satisfies the following conditions:

(M.R.1) $x * (x \rightarrow_* y) \leq y$

(M.R.2) $x \leq y \iff x \rightarrow_* y = 1$

(M.R.3) $x \rightarrow_* x = x \rightarrow_* 1 = 0 \rightarrow_* x = 1$ and $1 \rightarrow_* x = x$

(M.R.4) $x * y \leq x$ and $x \leq y \rightarrow_* x$

(M.R.5) $(x * y) \rightarrow_* z = x \rightarrow_* (y \rightarrow_* z)$

(M.R.6) $(x \rightarrow_* y) * (y \rightarrow_* z) \leq (x \rightarrow_* z)$

A complete lattice is lattice in which all subsets have both a supremum and an infimum. The following are true in every complete residuated lattice for each index set I.

(S.I.1) $x * \bigvee_{i \in I} y_i = \bigvee_{i \in I}(x * y_i)$

(S.I.2) $x \rightarrow_* \bigwedge_{i \in I} y_i = \bigwedge_{i \in I}(x \rightarrow_* y_i)$

(S.I.3) $\bigvee_{i \in I} x_i \rightarrow_* y = \bigwedge_{i \in I}(x_i \rightarrow_* y)$

(S.I.4) $x * \bigwedge_{i\in I} y_i \leq \bigwedge_{i\in I}(x * y_i)$

(S.I.5) $\bigvee_{i\in I}(x \to_* y_i) \leq x \to_* \bigvee_{i\in I} y_i$

(S.I.6) $\bigvee_{i\in I}(x_i \to_* y) \leq \bigwedge_{i\in I} x_i \to_* y$

Throughout this chapter, unless otherwise stated, L denotes a residuated lattice.

Definition 5.1. *A ternary L-relation on a set X is an L-set on X^3, i.e. a mapping from X^3 to L.*

Three special ternary L-relation on X are the universal defined as $\mathcal{U}(x,y,z) = 1$; $\forall(x,y,z) \in X^3$, the empty: $\phi(x,y,z) = 0$; $\forall(x,y,z) \in X^3$ and the identity: $I(x,y,z) = 1$ if $x = y = z$; $I(x,y,z) = 0$ otherwise.

5.2. Four-point compositions of ternary fuzzy relations

This subsection discusses the four-point compositions of ternary fuzzy relations, first, we generalise Defintions 2.1 2.2 to the fuzzy setting.

Definition 5.2. *Let $p \in \{0,\ldots,215\}$ with $p = (kji)_6 = 36k + 6j + i$ and $i,j,k \in \{0,\ldots,5\}$. For any ternary L-relations S and T on X, we define the composition $S \,\Box_p^0\, T$ as follows:*

$$S \,\Box_p^0\, T(\sigma_k(x,y,z)) = \bigvee_{t\in X} (S(\sigma_i(x,y,t)) * T(\sigma_j(x,t,z)))\,.$$
$$S \,\Box_p^1\, T(\sigma_k(x,y,z)) = \bigvee_{t\in X} (S(\sigma_i(x,y,t)) * T(\sigma_j(z,t,z)))\,.$$
$$S \,\Box_p^2\, T(\sigma_k(x,y,z)) = \bigvee_{t\in X} (S(\sigma_i(y,y,t)) * T(\sigma_j(x,t,z)))\,.$$

Proposition 5.1. *Let $p = (kji)_6 \in \{0,\ldots,215\}$ and $r \in \{0,1,2\}$. For any ternary L-relations S and T on X, the 4-point composition $S \,\Box_p^r\, T$ can be written in terms of the composition $\Box_0^r$ as follows:*

$$S \,\Box_p^r\, T = (S^{\sigma_i^{-1}} \,\Box_0^r\, T^{\sigma_j^{-1}})^{\sigma_k}\,.$$

Proof. We give the proof for $r = 0$.

$$\begin{aligned} S \,\Box_p^0\, T(\sigma_k(x,y,z) &= \bigvee_{t\in X} (S(\sigma_i(x,y,t)) * T(\sigma_j(x,t,z)) \\ &= \bigvee_{t\in X} (S^{\sigma_i}(x,y,t)) * T^{\sigma_j}(x,t,z)) \end{aligned}$$

□

Next, we study some basic properties of the 4-point compositions of ternary L-relations. The following proposition identifies the right and left neutral elements of the basic 4-point compositions. To that end, we introduce the following definition and lemma.

Definition 5.3. *We define the special ternary L-relations E, E_ℓ, E_m and E_r on X as follows:*

(i) $E(x,y,z)=1$ *if* $x=y=z$ *and* $E(x,y,z)=0$ *otherwise;*

(ii) $E_\ell(x,y,z)=1$ *if* $x=y$ *and* $E_\ell(x,y,z)=0$ *otherwise;*

(iii) $E_m(x,y,z)=1$ *if* $x=z$ *and* $E_m(x,y,z)=0$ *otherwise;*

(iv) $E_r(x,y,z)=1$ *if* $y=z$ *and* $E_r(x,y,z)=0$ *otherwise;*

Proposition 5.2. *For any ternary L-relation T on X, it holds that*

(i) $T\,\square^0_{(kjk)_6}\,E_{\zeta(j)}=T$;

(ii) $E_{\zeta(j)}\,\square^0_{(kki)_6}\,T=T$.

Proof. We only prove (i). Let $(x,y,z)\in X^3$

$$\begin{aligned}
T\,\square^0_{(kjk)_6}\,E_{\zeta(j)}(\sigma_k(x,y,z)) &= \bigvee_{t\in X} T(\sigma_k(x,y,t))*E_{\zeta(j)}(\sigma_j(x,t,z))\\
&= \bigvee_{t\in X\setminus\{z\}} \Big(T(\sigma_k(x,y,t))*E_{\zeta(j)}(\sigma_j(x,t,z))\Big)\\
&\quad\vee\quad T(\sigma_k(x,y,z))*E_r(\sigma_j(x,z,z))\\
&= \bigvee_{t\in X\setminus\{z\}} (T(\sigma_k(x,y,t))*0)\vee T(\sigma_k(x,y,z))*1\\
&= T(\sigma_k(x,y,z))
\end{aligned}$$

□

Proposition 5.3. *Let $p=(kji)_6\in\{0,\ldots,215\}$. The couples $(\square_p,\square'_p)$ is mixed-commutative, with $p'=(\pi_3(k),\pi_3(i),\pi_3(j))$*

Proof.

$$\begin{aligned}
S\,\square^0_p\,T(\sigma_k(x,y,z)) &= \bigvee_{t\in X}(S(\sigma_i(x,y,t))*T(\sigma_j(x,t,z)))\\
&= \bigvee_{t\in X}(T(\sigma_j(x,t,z))*S(\sigma_i(x,y,t)))\\
&= \bigvee_{t\in X}(T(\sigma_j\sigma_1(x,z,t))*S(\sigma_i\sigma_1(x,t,y)))\\
&= T\,\square^0_{p'}\,S(\sigma_k\sigma_1(x,z,y))
\end{aligned}$$

□

The following proposition identifies the associative 4-point compositions.

Proposition 5.4. *The* 4*-point composition* $\square_p^0$ *is associative if and only if* $p = (iii)_6 = 43i$*, with* $i \in \{0, \ldots, 5\}$.

Proof. The proof is analogous to the proof of Proposition 2.6 □

Next we show that any cylindrical extension of a 3-point composition of binary relations contains a 4-point composition of their cylindrical extensions. First, we recall the definition of the cylindrical extensions of a binary relation.

Definition 5.4. [27] *Let* R *be a binary relation on a set* X*.*

(i) The left cylindrical extension of R *is the ternary* L*-relation* $C_\ell(R)$ *on* X *defined as:*

$$C_\ell(R)(x, y, z) = R(y, z);$$

(ii) The middle cylindrical extension of R *is the ternary* L*-relation* $C_m(R)$ *on* X *defined as:*

$$C_\ell(R)(x, y, z) = R(x, z);$$

(iii) The right cylindrical extension of R *is the ternary* L*-relation* $C_r(R)$ *on* X *defined as:*

$$C_\ell(R)(x, y, z) = R(x, y);$$

Proposition 5.5. *Let* $q = (kji)_2 \in \{0, \ldots, 7\}$*,* $r \in \{0, 1, 2\}$ *and* $\alpha, \beta, \gamma \in \{\ell, m, r\}$*. For any binary relations* R *and* P *on* X*, the following inclusion holds:*

$$C_\alpha(R) \,\square_p^r\, C_\beta(P) \subseteq C_\gamma(R \circ_q P)\,,$$

with $p = (\Gamma(\gamma, k)\Gamma(\beta, j)\Gamma(\alpha, i))_6$.

Proof.

$$\begin{aligned}
C_\alpha(R) \,\square_p^r\, C_\beta(P)(\sigma_{\Gamma(\gamma,k)}(x, y, z)) &= \bigvee_{t \in X} C_\alpha(R)(\sigma_{\Gamma_{(\alpha,i)}}(x, y, t)) * C_\beta(R)(\sigma_{\Gamma_{(\beta,j)}}(x, t, z)) \\
&\leq \bigvee_{t \in X} R(\rho_i(y, t)) * P(\rho_j(t, z)) \\
&\leq R \circ P\ (\rho_k(y, z)) \\
&\leq C_\alpha(R \circ P)(\sigma_{\Gamma_{(\gamma,k)}}(x, y, z))
\end{aligned}$$

□

Next, we show that any projection of a 4-point composition of two ternary L-relations is included in a corresponding 3-point composition of their binary projections. First, we recall the definition of the binary projections of a ternary L-relation.

Definition 5.5. *[27] Let T be a ternary L-relation on a set X.*

(i) The left projection of T is the binary relation $P_\ell(T)$ on X defined as:

$$P_\ell(T)(x,y) = \bigvee_{t\in X} T(t,x,y)$$

(ii) The middle projection of T is the binary relation $P_m(T)$ on X defined as:

$$P_m(T)(x,y) = \bigvee_{t\in X} T(x,t,y)$$

(iii) The right projection of T is the binary relation $P_r(T)$ on X defined as:

$$P_r(T)(x,y) = \bigvee_{t\in X} T(x,y,t)$$

Proposition 5.6. *Let $p = (kji)_6 \in \{0,\ldots,215\}$ and $r \in \{0,1,2\}$. For any ternary L-relations S and T on X, the following inclusion holds:*

$$P_{\omega(k)}(S \,\square_p^r\, T) \subseteq P_{\omega(i)}(S) \circ_q P_{\omega(j)}(T)\,, \tag{5.1}$$

with $q = (\mathit{\Pi}(k,\omega(k))\mathit{\Pi}(j,\omega(j))\mathit{\Pi}(i,\omega(i)))_2$.

For the proof of this proposition we introduce the following lemma

Lemma 5.1.

$$P_{\omega(k)}(x,y,z) = P_\ell(\sigma_k(x,y,z))$$

Proof.

$$\begin{aligned}
P_{\omega(k)}(S \,\square_p^r\, T) &= P_\ell(S \,\square_p^r\, T)(\sigma_k(x,y,z)) \\
&= P_\ell(\bigvee_{t\in X} (S(\sigma_i(x,y,t)) * T(\sigma_j(x,t,z)))) \\
&= \bigvee_{x\in X} \bigvee_{t\in X} (S(\sigma_i(x,y,t)) * T(\sigma_j(x,t,z))) \\
&= \bigvee_{t\in X} \bigvee_{x\in X} (S(\sigma_i(x,y,t)) * T(\sigma_j(x,t,z))) \\
&\subseteq \bigvee_{t\in X} \bigvee_{x\in X} S(\sigma_i(x,y,t)) * \bigvee_{x\in X} T(\sigma_j(x,t,z)) \\
&\subseteq \bigvee_{t\in X} P_\ell(S\sigma_i(x,y,t)) * P_\ell(T\sigma_j(x,t,z)) \\
&\subseteq \bigvee_{t\in X} P_{\omega(i)}(S)(y,t) * P_{\omega(j)}(T)(t,z) \\
&\subseteq P_{\omega(i)}(S) \circ_q P_{\omega(j)}(T)
\end{aligned}$$

□

5.3. Five-point compositions of ternary fuzzy relations

Definition 5.6. *Let* $p \in \{0, \ldots, 215\}$ *with* $p = (kji)_6 = 36k + 6j + i$ *and* $i, j, k \in \{0, \ldots, 5\}$. *For any ternary* L*-relations* S *and* T *on* X, *we define the composition* $S \square_p^0 T$ *as follows:*

$$S \pentagon_p^0 T(\sigma_k(x,y,z)) = \bigvee_{t,s\in X} (S(\sigma_i(\hat{x},y,t)) * T(\sigma_j(\check{x},t,z)))\,.$$

with

$$\hat{x} = \begin{cases} x \ , \ \textit{if}\ k \in \{0,3,4\} \\ s \ , \ \textit{if}\ k \in \{1,2,5\} \end{cases}, \quad \check{x} = \begin{cases} s \ , \ \textit{if}\ k \in \{0,3,4\} \\ x \ , \ \textit{if}\ k \in \{1,2,5\} \end{cases}.$$

The following proposition shows some inclusion relationships between the 4-point and 5-point compositions.

Proposition 5.7. *Let* $p = (kji)_6 \in \{0, \ldots, 215\}$. *For any ternary* L*-relations* S *and* T *on* X, *it holds that*

(i) $S \,\square_p^0\, T \subseteq S \pentagon_p T$;

(ii) $S \,\square_p^1\, T \subseteq S \pentagon_p T$, *if* $k \in \{0,3,4\}$;

(iii) $S \,\square_p^2\, T \subseteq S \pentagon_p T$, *if* $k \in \{1,2,5\}$.

Proof. We give the proof for (i) in the case where $k \in \{0,3,4\}$, the other cases can be proven similarly.

$$
\begin{aligned}
S \,\square_p^0\, T &= \bigvee_{t\in X} (S(\sigma_i(x,y,t) * T(\sigma_j(x,t,z))) \\
&\leq \bigvee_{t,s\in X} (S(\sigma_i(x,y,t) * T(\sigma_j(s,t,z))) \\
&\leq S \pentagon_p T
\end{aligned}
$$

□

Proposition 5.8. *Let $i, j, k \in \{0, \ldots, 5\}$. For any ternary L-relation T on X, it holds that*

(i) $T \pentagon_{(kjk)_6} E_{\zeta(j)} = T$*, if* $k \in \{0, 3, 4\}$*;*

(ii) $E_{\zeta(i)} \pentagon_{(kki)_6} T = T$*, if* $k \in \{1, 2, 5\}$*,*

with ζ as in Proposition 2.3.

Proof. We only prove (i). Let $(x, y, z) \in X^3$

$$
\begin{aligned}
T \pentagon_{(kjk)_6} E_{\zeta(j)}(\sigma_k(x,y,z)) &= \bigvee_{s,t\in X} T(\sigma_k(x,y,t)) * E_{\zeta(j)}(\sigma_j(s,t,z)) \\
&= \bigvee_{s\in X} \bigvee_{t\in X\setminus\{z\}} (T(\sigma_k(x,y,t)) * E_{\zeta(j)}(\sigma_j(s,t,z)) \\
&\quad \vee \;\; T(\sigma_k(x,y,z)) * E_r(\sigma_j(s,z,z)) \\
&= \bigvee_{s\in X} (T(\sigma_k(x,y,t)) * 0) \vee T(\sigma_k(x,y,z)) * 1 \\
&= T(\sigma_k(x,y,z))
\end{aligned}
$$

□

Proposition 5.9. *Let $p = (kji)_6 \in \{0, \ldots, 215\}$. The composition $\pentagon_p$ is associative if and only if one of the following conditions holds:*

(i) $k = j = i$*;*

(ii) $k \in \{0, 3, 4\}$*,* $i = k$ *and* $j = \pi_4(k)$*;*

(iii) $k \in \{1, 2, 5\}$*,* $i = \pi_4(k)$ *and* $j = k$*,*

with π_4 is the permutation of $\{0, \ldots, 5\}$ given in Table 3.1:

Proposition 5.10. *Let $q = (kji)_2 \in \{0, \ldots, 7\}$ and $\alpha, \beta, \gamma \in \{\ell, m, r\}$. For any binary relations R and P on X, the following equality holds:*

$$
C_\gamma(R \circ_q P) = C_\alpha(R) \pentagon_p C_\beta(P)\,,
$$

with p as in Proposition 2.8.

Proof.

$$\begin{aligned}
C_\alpha(R) \pentagon_p C_\beta(P)(\sigma_{\Gamma(\gamma,k)}(x,y,z)) &= \bigvee_{s,t\in X} C_\alpha(R)(\sigma_{\Gamma_{(\alpha,i)}}(x,y,t)) * C_\beta(R)(\sigma_{\Gamma_{(\beta,j)}}(s,t,z)) \\
&= \bigvee_{t\in X} R(\rho_i(y,t)) * P(\rho_j(t,z)) \\
&= R \circ P\ (\rho_k(y,z)) \\
&= C_\alpha(R \circ P)(\sigma_{\Gamma_{(\gamma,k)}}(x,y,z))
\end{aligned}$$

□

Proposition 5.11. *Let $p = (kji)_6 \in \{0,\dots,215\}$. For any ternary L-relations S and T on X, the following equality holds:*

$$P_{\omega(k)}(S \pentagon_p T) \ = P_{\omega(i)}(S) \circ_q P_{\omega(j)}(T) \ , \qquad (5.2)$$

with q as in Proposition 2.9.

Proof. We only give the proof for the case where $k \in \{0,3,4\}$.

$$\begin{aligned}
P_{\omega(k)}(S \pentagon_p T) &= P_\ell(S \square_p^r T)(\sigma_k(x,y,z)) \\
&= P_\ell(\bigvee_{s,t\in X} (S(\sigma_i(x,y,t)) * T(\sigma_j(s,t,z)))) \\
&= \bigvee_{t\in X} \bigvee_{x\in X} S(\sigma_i(x,y,t)) * \bigvee_{s\in X} T(\sigma_j(s,t,z)) \\
&= \bigvee_{t\in X} P_\ell(S\sigma_i(x,y,t)) * P_\ell(T\sigma_j(x,t,z)) \\
&= \bigvee_{t\in X} P_{\omega(i)}(S)(y,t) * P_{\omega(j)}(T)(t,z) \\
&= P_{\omega(i)}(S) \circ_q P_{\omega(j)}(T)
\end{aligned}$$

□

6 General conclusions and future research

In this work, we have first expounded a general approach to the study of the composition of ternary relations, resulting in a first type of 324 four-point compositions and a second type of 216 five-point compositions. For each type, we have provided a convenient enumeration scheme based on senary numbers that allows to generate all compositions. Also, we have presented a way of expressing all compositions in terms of a limited number of representative ones. We have identified all associative 4-point and 5-point-compositions, as well as couples of compositions satisfying the mixed-associativity property. Although there exist plenty such couples, only in the case of 5-point compositions, there exist mixed-associative couples consisting of different compositions for which also the converse couples are mixed-associative. Furthermore, we have provided some links between the compositions of binary relations and the two types of compositions of ternary relations.

Second, we have explored three new types of compositions of ternary relations, inspired by the substitution of the underlying existential quantifier with the universal quantifier, an approach initially introduced by Bandler and Kohout in the binary case. In our study, we have examined these new compositions, showing their key properties, their relationship with the original Bandler–Kohout compositions of binary relations, their role in solving ternary relational equations, and, notably, their significance in defining the traces of ternary relations. Leveraging this insight, we have introduced the traces of ternary relations. Finally, we have provided characterizations of several types of ternary equivalence relations.

Given the importance of fuzzy relations, as amply illustrated in the introduction for binary relations, future efforts will be directed to the study of fuzzy ternary relations as well. We anticipate that it will be interesting to extend the different types of compositions of ternary relations to the fuzzy setting.

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خصائص وفئات العلاقات الثلاثية الضبابية

ملــخص: هذه الرسالة تستكشف بشكل منهجي تركيبات العلاقات الثلاثية، حيث تحدد جميع التركيبات الممكنة وتدرس خصائصها بما في ذلك خاصية التجميع والتجميع المختلط. كما تقدم تركيبات بندر-كوهوت للعلاقات الثلاثية، وتوسعتها للعلاقات الضبابية، وكذلك ترابط هذه التركيبات بالعلاقات الثنائية من خلال الإسقاط والتمديد الاسطواني.

الكلمات المفتاحية : العلاقات الثلاثية، تركيب العلاقات، تركيبات بندر-كوهوت، العلاقات الضبابية، خاصية التجميع.

Propriétés et classes de relations floues ternaires

Résumé: Cette thèse explore systématiquement les compositions des relations ternaires, en identifiant toutes les compositions possibles et étudie leurs propriétés d'associativité, y compris l'associativité mixte. Elle introduit les compositions BK pour les relations ternaires, étend le cadre aux contextes flous, et établit des liens entre ces compositions et
les relations binaires à travers l'projection et l'extension cylindrique.

Mots-clés: Relations ternaires, Composition des relations, Bandler-Kohout compositions, Relations floues, Associativité.

Properties and classes of ternary fuzzy relations

Abstract: This thesis systematically explores the compositions of ternary relations, identifying all possible compositions and examining their associativity properties, including mixed-associativity. It introduces Bandler-Kohout-compositions for ternary relations, extends the framework to fuzzy settings, and establishes connections between these compositions and binary relations through projection and cylindrical extension.

Keywords: Ternary relations, Composition of relations, Bandler-Kohout compositions, Fuzzy relations, Associativity.