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On the spectrum theory of bounded linear operator on Non-Archimedean Banach space

Persented by: **Rima Zerrouki**

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Abdelhamid TALLAB	**M.C.A.**	University of M'sila	**Chairperson**
Toufik HERAIZ	**M.C.B.**	University of M'sila	**Supervisor**
Khaled HAMIDI	**M.C.B .**	University of M'sila	**Examiner**

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Dédicace

Nothing compares to the joy of graduation, as it is one of the most beautiful moments in our lives. The tiredness of the years, the sleepless nights and the prayers of the parents have disappeared

and we have forgotten them as soon as we feel the joy of graduation.

It is important for to thanks my familly :my parent, my sisters, who have always been an enexhaustible source of encouragement.

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Notations

$\mathcal{L}(X)$	The algebra of all bounded linear operators from X into itself.
$N(A)$	The null space of A.
$R(A)$	The range space of A.
$\mathcal{K}(X)$	The ideal of all compact operators on X.
$\mathcal{S}(X,Y)$	The set of strictly singular operators from X to Y.
$C\mathcal{S}(X,Y)$	The set of strictly cosingular operators from X to Y.
$\sigma(A)$	The spectrum of A,
$\rho(A)$	The resolvent set of A.
$\Phi(X)$	The set of Fredholm operators.
$\eta(A)$	The nullity of A is defined as the dimension of $N(A)$.
$\delta(A)$	The deficiency of A is defined as the codimension of $R(A)$.
I	Operator of identity.
$\mathcal{F}(X,Y)$	The set of fredholm perturbation.
$\Phi_+(X,Y)$	The set of upper semi-Fredholm operators.
$\Phi_-(X,Y)$	The set of lower semi-Fredholm operators.

$\tilde{T}$	The completion of T.
$\sup(A)$	The supremum of the set A.
$\inf(A)$	The infimum of the set A.
$T_{\setminus M}$	The restriction of T to M.
$D(A)$	Domain of A

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Introduction

One of the most important concepts in functional analysis is the spectrum of linear operator. This concept is particularly useful in Banach spaces, which have been the subject of numerous studies and research. In fact, the significance of the spectrum has motivated many researchers to explore the possibility of extending the concept to Non-Archimedean Banach spaces. Non-Archimedean Banach spaces are a relatively new field in mathematics and analysis. They were first introduced by Hans Hahn and Oskar Perron in the early 20th century, as a way of studying and generalizing the properties of normed spaces. In 1940, André Weil introduced the concept of "ultrametric spaces," which provided a new viewpoint on the geometry of non-Archimedean spaces. The theory of non-Archimedean Banach spaces continued to develop throughout the mid-20th century, particularly through the work of French mathematician Paul Malliavin and his collaborators. Today, non-Archimedean Banach spaces are an active field of research, with many open questions and a wide range of applications in mathematics, physics, and other fields. The spectrum theory on non-archimidean Banach spaces is an area of functional analysis that was first developed in the mid-twentieth century. In non-archimidean Banach spaces. The initial work on the spectrum theory on Non-archimidean Banach spaces was done by Alexandre Grothendieck in the 1950s. He introduced the concept of an extended complex field, which allowed for the definition of the spectrum in non-archimidean Banach spaces. In the 1970s, Alain Robert extended the work of Grothendieck by introducing the notion of a spectral radius formula for operators on non-archimidean Banach spaces. This formula is similar to the classical formula for the spectral radius of an operator on a Banach space, but is modified to take into account the non-archimidean nature of the space. Since then, many researchers have contributed to the spectrum theory on non-archimidean Banach spaces, including Helge Holden, Tom Lindstrom, and Krzysztof Jarosz. The theory has applications in various areas of mathematics, including number theory, algebraic geometry, and mathematical physics. This memory is devoted to study of The spectrum theory of bounded linear operator on Non-Archimedean Banach space, it is composed of three chapter

The first chapter we recall some basic properties of non-Archimedean valuation, the topology induced by A valuation on $\mathbb{k}$, example of Archimedean valuation, non-Archimedean Banach space and free Banach space.

The second chapter is devoted to gather some definitions and properties about linear operator, bounded linear operators and bounded linear operators on free Banach space.

Finaly, in the last chapter we focus on the concept of the spectrum, Fredholm operator, properieties of Fredholm operators, Non-Archimedean essential spectrum and perturbation classes of operator where we show a result concerning on the stability of semi Fredholm operator under strictly singular perturbation.

Chaptre 1

Basic properties

1.1 Non-Archimidean Banach space

In the classical settings of the field of complex numbers $\mathbb{C}$ and the field of real numbers $\mathbb{R}$, the absolute value plays an important role in the Topology and in the Analysis on objects over these fields. The notion of valuation is emerged as a generalization of the concept of absolute value on a general field $\Bbbk$. This notion allows one to have a natural topology on the field itself and also on objects that are defined over the field.

1.1.1 Values

Definition 1.1.1 *Let $\Bbbk$ be a field. A non-Archimedean absolute value on $\Bbbk$ is a function $|\,.\,| : \Bbbk \to \mathbb{R}_+$ such that, for some real number $C \geq 1$, the following hold :*

1. $|\alpha| \geq 0$, for any α in $\Bbbk$ with equality only for $\alpha = 0$.
2. $|\alpha\beta| = |\alpha| \,.\, |\beta|$ for any α, β in $\Bbbk$.
3. For α in $\Bbbk$ if $|\alpha| \leq 1$, then $|\alpha + 1| \leq C$.

The valuation $|\,.\,|$ such that $|\alpha| = 1$ for every non-zero α and $|0| = 0$ is called the trivial valuation.

Proposition 1.1.1 [9] *The following hold:*

a) $|1| = 1$.

b) For $\alpha \in \mathbb{k}$, if $|\alpha^n| = 1$ then $|\alpha| = 1$.

c) $|-1| = 1$.

d) $|-\alpha| = |\alpha|$.

Proposition 1.1.2 *Let* $| \,.\, |: \mathbb{k} \longrightarrow \mathbb{R}$ *be a valuation on* $\mathbb{k}$ *and* λ *a positive real number then* $| \,.\, |_\lambda$ *defined by*

$$| \alpha |_\lambda = | \alpha |^\lambda$$

for any $\alpha \in \mathbb{k}$ is a valuation on $\mathbb{k}$.

Proof. Properties (1) and (2) of definition (1.1.1) are clear. For (3) of definition (1.1.1), if

$$| \alpha |_\lambda \leq 1$$

then

$$| \alpha |^\lambda \leq 1$$

hence

$$| \alpha | \leq 1,$$

and since $| \,.\, |$is a valuation, $| \alpha + 1 | \leq m$. and

$$| \alpha + 1 |_\lambda = | \alpha + 1 |^\lambda \leq m^\lambda$$

hence (3) of definition (1.1.1) holds with the constant m^λ. ■

Definition 1.1.2 *Two valuations* $| \,.\, |_1$ *and* $| \,.\, |_2$ *on the field* $\mathbb{k}$ *are equivalent if there exists a positive real numbers* λ *such that*

$$| \,.\, |_2 = | \,.\, |_1^\lambda \,.$$

This is an equivalence relation on the set of valuation on the field $\mathbb{k}$.

Definition 1.1.3 *A valuation* $|\,.\,|$ *on the field* $\Bbbk$ *satisfies the triangle inequality if for any* α, β *in* $\Bbbk$,

$$|\,\alpha+\beta\,| \leq |\,\alpha\,| + |\,\beta\,|\,.$$

Proposition 1.1.3 *Let* $|\,.\,|$*be a valuation on* $\Bbbk$ *then, it satisfies the traingle inequality if and only if one can take* $C = 2$ *in* (3) *of difinition* (1.1.1)

Indeed, suppose the valuation satisfies the triangle inequality and let α be such that

$$|\,\alpha\,| \leq 1$$

then

$$\begin{aligned} |\,\alpha+1\,| &\leq |\,\alpha\,| + |\,1\,| \\ &\leq 2. \end{aligned}$$

Proposition 1.1.4 *Every valuation on* $\Bbbk$ *is equivalent to one that satisfies the traingle inequality.*

Definition 1.1.4 *A valuation* $|\,.\,|$ *on* $\Bbbk$ *satisfies the ultrametric inequality if for any* $\alpha, \beta \in \Bbbk$

$$|\,\alpha+\beta\,| \leq \max\{|\,\alpha\,|, |\,\beta\,|\}$$

Proposition 1.1.5 *A valuation* $|\,.\,|$ *on* $\Bbbk$ *satisfies the ultrametric inequality if and only if one can take* $C = 1$ *in* (3) *of definition* (1.1.1).

Proof. Suppose that one can take $C = 1$ in (3) of definition (1.1.1). Let α, β, which we may assume to be non-zero, be in $\Bbbk$ and suppose that

$$|\,\alpha\,| \leq |\,\beta\,|$$

then

$$|\,\frac{\alpha}{\beta}\,| \leq 1$$

and therfore

$$|\,\frac{\alpha}{\beta}+1\,| \leq 1$$

hence

$$\begin{aligned} |\ \alpha+\beta\ | &\leq |\ \beta\ | \\ &= \max\{|\ \alpha\ |, |\ \beta\ |\} \end{aligned}$$

The case

$$|\ \beta\ | \leq |\ \alpha\ |$$

is handled similarly. We may conclude that the valuation satisfies the ultrametric inequality. Next suppose that the valuation satisfies the ultrametric inequality and let α be in $\mathbb{k}$ such that

$$|\ \alpha\ | \leq 1$$

then

$$\begin{aligned} |\ \alpha+1\ | &\leq \max\{|\ \alpha\ |, |\ 1\ |\} \\ &= 1. \end{aligned}$$

Therfore one can take $C = 1$ in (3) of definition (1.1.1). ■

Proposition 1.1.6 *Let* $|\ .\ |$ *be a non-Archimedean valuation on* $\mathbb{k}$. *Let* $\alpha, \beta \in \mathbb{k}$ *such that* $|\ \alpha\ | < |\ \beta\ |$, *then*

$$|\ \alpha+\beta\ | = |\ \beta\ |\ .$$

Proof. First $|\ \alpha+\beta\ | \leq |\ \beta\ |$, next

$$|\ \beta\ | = |\ (\alpha+\beta)-\alpha\ | \leq \max(|\ \alpha+\beta\ |, |\ \alpha\ |).$$

If

$$|\ \alpha+\beta\ | < |\ \alpha\ |$$

then we would have

$$|\ \beta\ | \leq |\ \alpha\ |$$

which is against our assumption, therfore

$$|\ \alpha+\beta\ | \geq |\ \alpha\ |$$

and hence

$$|\ \beta\ | \leq |\ \alpha+\beta\ |\ .$$

We can conclude that

$$| \alpha + \beta | = | \beta | .$$

. ■

1.1.2 The Topology induced by a valuation on $\Bbbk$

Proposition 1.1.7 *Let* $d : \Bbbk \times \Bbbk \to \mathbb{R}_+$ *be defined by*

$$d(\alpha, \beta) = | \alpha - \beta |$$

then, d is a distance function on $\Bbbk$ and $(\Bbbk, d)$ is a metric space.

Proof. Suppose $d(\alpha, \beta) = 0$, then$| \alpha - \beta | = 0 \Longrightarrow \alpha = \beta$ and

$$\begin{aligned} d(\alpha, \beta) &= | \alpha - \beta | \\ &= | -(\alpha - \beta) | \\ &= d(\beta, \alpha) \end{aligned}$$

for all α, $\beta \in \Bbbk$. For all α, β, $\gamma \in \Bbbk$

$$\begin{aligned} d(\alpha, \gamma) &= | \alpha - \gamma | \\ &= | (\alpha - \beta) + (\beta - \gamma) | \\ &\leq | \alpha - \beta | + | \beta - \gamma | \\ &= d(\alpha, \beta) + d(\beta, \gamma) \end{aligned}$$

and hence $(\Bbbk, d)$ is a metric space. ■

Corollary 1.1.1 *For any* α, β, $\gamma \in \Bbbk$

$$| d(\alpha, \gamma) - d(\beta, \gamma) |_\infty \leq d(\alpha, \beta).$$

Since $\Bbbk$ *is a metric space the fundamental system of neighborhoods of every element* a *in* $\Bbbk$ *consists of the open balls of the form :*

$$B(a, r) = \{\alpha \in \Bbbk : | \alpha - a | < n\}$$

where n *is a positive real number. It is remarkable that any open ball* $B(a, R)$ *is such that any element in it is its center, in other words, for any* $b \in B(a, n)$, $B(b, n) = B(a, n)$.

Proposition 1.1.8 *Equivalent valuations induce the same topology on* $\Bbbk$*.*

Proof. Suppose $|\ .\ |$ and $|\ .\ |_\lambda = |\ .\ |^\lambda$, with λ a positive real number, are equivalent valuations on $\Bbbk$.

Let a be in $\Bbbk$ and for any $\varepsilon > 0$, let $B(a, \varepsilon)$ be the open open ball associated with $|\ .\ |$ and $B_\lambda(a, \varepsilon)$ be the open ball associated with $|\ .\ |_\lambda$.

Then for any positive real number n

$$B(a, n) \subset B_\lambda(a, n^\lambda) B_\lambda(a, n) \subset B(a, n^{\frac{1}{\lambda}}).$$

Therefore the two valuations induce the same topology on $\Bbbk$. ■

Definition 1.1.5 *Let* $|\ .\ |_1$ *and* $|\ .\ |_2$ *be two non-trivial valuations which induce the same topology on* $\Bbbk$*, then they are equivalent.*

Lemma 1.1.1 *For any* $\alpha \in \Bbbk$*,* $|\ \alpha\ |_1 < 1$ *implies* $|\ \alpha\ |_2 < 1$.

Proof. *Suppose* $|\ \alpha\ |_1 < 1$ *then*

$$|\ \alpha\ |_1^n \rightarrow 0 \ \ as\ n \rightarrow \infty.$$

Since the two valuations induce the same topology

$$|\ \alpha\ |_2^n \rightarrow 0 \ \ as\ n \rightarrow \infty$$

and this implies that

$$|\ \alpha\ |_2 < 1.$$

■

Definition 1.1.6 *Let* $|\ .\ |$ *be a valuation on* $\Bbbk$ *. A completion of* $\Bbbk$ *is a field* ***F*** *containing* $\Bbbk$ *together with a valuation* $\|\ .\ \|$ *on it, such that:*

a) F is a complete metric space with respect to the distance induced by $\|\ .\ \|$.

b) The valuation $\|\ .\ \|$ extends $|\ .\ |$ meaning that for any α in $\Bbbk$,

$$\|\ \alpha\ \| = |\ \alpha\ | \ .$$

c) F is the closure of $\Bbbk$ with respect to the topology induced by $\|\ .\ \|$.

Theorem 1.1.1 [10] *Let* $\Bbbk$ *be a field with a valuation* $|\ .\ |$*. A completion exists and any two completions are canonically isomorphic .*

1.1.3 Non-Archimedean valuations

Definition 1.1.7 *Let* $\mathbb{k}$ *be a field , then*

$$A = \{\alpha \in \mathbb{k} : \mid \alpha \mid \leq 1\}$$

is called the valuation ring (or the ring of integers) of $\mathbb{k}$.

Proposition 1.1.9 *The following hold:*

a) A is a local ring.

b) $U = \{\alpha \in A : \mid \alpha \mid = 1\}$ is the groupe of units in A;

c) $M = \{\alpha \in A : \mid \alpha \mid < 1\}$ is the unique maximal ideal of A.

Definition 1.1.8 *The value group of* $\mathbb{k}$ *is the image of* $\mathbb{k}^*$ *under the valuation map* $\mid . \mid$.

It is denoted $\mid \mathbb{k}^* \mid$.

The value groupe $\mid \mathbb{k}^* \mid$ is multiplicative group of positive real numbers, hence it is either:

Definition 1.1.9 *1.*

2. Every where dense, or

3. Infinite cyclic.

Definition 1.1.10 *In the case where the value group is infinite cyclic, the valuation is called a discrete valuation and in the case where the value group is every where dense, the valuation is called a dense valuation.*

Proposition 1.1.10 *The valuation* $\mid . \mid$ *is a discrete valuation on* $\mathbb{k}$, *if and only if* M *is a principal ideal.*

Proof. Suppose the valuation is discrete, hence the value groupe $\mid \mathbb{k}^* \mid$ is infinite cyclic. Let ξ be in $\mathbb{k}^*$ such that $\mid \xi \mid$ generates $\mid \mathbb{k}^* \mid$. Clearly $\mid \xi \mid \neq 1$. Every element $\alpha \in \mathbb{k}^*$ is of the form $\alpha = u.\xi^n$ for some unit u and some integer n. Let π be either ξ or ξ^{-1}but so that $\mid \pi \mid < 1$, therefore $\pi \in M$. Now it is clear that π generates the maximal ideal M. Conversely suppose M is a principal ideal generated by π. ∎

1.1.4 Example of Archimedean valuation

The ordinary absolute value on $\mathbb{C}$, on $\mathbb{R}$ and on any subfield, is the typical example of archimedean valuations. In fact one can prove the following theorem.

Theorem 1.1.2 *Let $\Bbbk$ be complete with respect to an archimedean valuation $|\ .\ |$, then $\Bbbk$ is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$, and $|\ .\ |$ is equivalent to the ordinary obsulute value.*

Example of non-Archimedean valued field

Exemple (the field $\mathbb{Q}$ of rational number). This is a classic example and we will work out the details .

Let p be a prime number, then because of the unique factorization in $\mathbb{Z}$, every non-zero rational number α can be written as

$$\alpha = \frac{a}{b}p^n.$$

where n, a, b are integer, and

$$p\,\mathrm{gcd}(p, ab) = 1.$$

put

$$|\ \alpha\ |_p = p^{-n}$$

if $\alpha \neq 0$ and $|\ 0\ |_p = 0$.

Proposition 1.1.11 *$|\ .\ |_p$ is a valuation on $\mathbb{Q}$ it's called the p-adic valuation.*

Proof. From the definition

$$|\ \alpha\ |_p = 0$$

if and only if $\alpha = 0$. If

$$\alpha = p^{n\frac{a}{b}}$$

and

$$\beta = p^{m\frac{c}{d}}$$

then

$$\alpha\beta = p^{n+m\frac{ac}{bd}},\ \ \mathrm{gcd}(p, abcd) = 1$$

therefore

$$\begin{aligned} \mid \alpha\beta \mid_p &= p^{-(n+m)} \\ &= \mid \alpha \mid_p \mid \beta \mid_p \end{aligned}$$

if $n \leq m$ then

$$\alpha + \beta = p^n\left(\frac{a + p^{m-n}c}{bd}\right)$$

and hence

$$\begin{aligned} \mid \alpha + \beta \mid &\leq p^{-n} \\ &= \max\{\mid \alpha \mid_p, \mid \beta \mid_p\}. \end{aligned}$$

■

Remark 1.1.1 *The case $m \leq n$ is handled similarly. It is useful to also use the additive valuation, or order function in this case. The ordre function is denoted ord_p. The relationship between the two approaches is : for all $x \in \mathbb{Q}$,*

$$\mid x \mid_p = p^{-ord_p(x)}.$$

Proposition 1.1.12 *The following hold:*

(a) $ord_p(x) = \infty$ if and only if $x = 0$.

(b) $ord_p(xy) = ord_p(x) + ord_p(y)$.

(c) $ord_p(x + y) \geq \min\{ord_p(x), ord_p(y)\}$.

Proposition 1.1.13 *The p-adic valuation is a discrete valuation.*

Proof. $\mid \mathbb{Q}^* \mid = \{\ p^n : n \in \mathbb{Z}\} = p^{\mathbb{Z}}$, hence it is an infinite cyclic multiplicative subgroup of the group $\mathbb{R}^*_+$.

The ring valuation ring is $\mathbb{Z}_{(p)} = \{\alpha \in \mathbb{Q} : ord_p(\alpha) \geq 0\}$.

The unique maximal ideal is

$$M_{(p)} = \{\alpha \in \mathbb{Q} : ord_p(\alpha) > 0\} = p\mathbb{Z}_{(p)}\ .\ p$$

is a uniformizer.

The group of units is

$$U_{(p)} = \{\alpha \in \mathbb{Q} : ord_p(\alpha) = 0\}.$$

■

1.1.5 Non Archimedean norms

Non-Archimedean Banach space is a mathematical structure similar to Banach space, but with a Non-Archimedean norm. meaning the norm saftify the ultrametric triangular inquality instead of the traditional triangular inequality

Definition 1.1.11 [6] *Let X be a vector space over a scalar field $\mathbb{k}$. A non-Archimedean norm if it satisfies the following condition :*

(i) $\| x \| = 0$ *if and only if* $x = 0$.

(ii) $\| \lambda x \| = | \lambda | \| x \|$ *for any* $x \in X$ *and any* $\lambda \in \mathbb{k}$.

(iii) *The strong traingle inequality*

$$\| x + y \| \leq \max(\| x \|, \| y \|) \text{ for all } x, y \in X.$$

Definition 1.1.12 *A non-Archimedean normed space is a pair $(E, \| \ . \ \|)$ where E is a vector space over $\mathbb{k}$ and $\| \ . \ \|$ is a non-Archimedean norm on E.*

Example 1.1.1 *The valuation on $\mathbb{k}$ itself is a non-Archimedean norm.*

Example 1.1.2 *Consider the cartesian product $\mathbb{k}^n$ with $n \in \mathbb{N}$ and define*

$$\|(x_1 \ , ..., \ x_n) \| = \max\{| x_i |: 1 \leq i \leq n\}.$$

then this is non-Archimedean norm on $\mathbb{k}^n$.

Proposition 1.1.14 *Let $(E, \| \ . \ \|)$ be a non-Archimedean normed space .*

For x, $y \in E$

$$\| x + y \| = \max \ \{\| x \|, \| y \|\}, \text{ if } \| x \| \neq \| y \| \ .$$

Proof. Suppose that $\| x \| < \| y \|$ so that

$$\max\{\| x \|, \| y \| = \| y \|$$

then, $\| x+y \| \leq \| y \|$. Now

$$\| y \|=\| x+y-x \|\leq \max\{\| x+y \|, \| x \|\}$$

But since $\| y \|=\| x \|$, we must have

$$max\{\| x+y \|, \| x \|=\| x+y \|$$

and therfore

$$\| y \|\leq\| x+y \|$$

and the conclusion follows. ■

Definition 1.1.13 *Let $(E, \| \, . \, \|)$ be Non-Archimedean normed space and S be a non-empty subset of E. The set S is said to be bounded if the set of real numbers $\{\| \alpha \|: \alpha \in S\}$ is bounded.*

Definition 1.1.14 *A sequnce $(x_{i)_{i\in\mathbb{N}}}$ in the normed space $(E, \| \, . \, \|)$ converges (strongly) to $x \in E$ and we write*

$$\lim_{i\to\infty} x_i = x,$$

if the sequence of real numbers $(\| x_i - x \|)_{i\in\mathbb{N}}$ converge to 0.

Proposition 1.1.15 *Let $(E, \| \, . \, \|)$ be a non-Archimedean normed space over $\mathbb{k}$. If the sequence $(x_{i)_{i\in\mathbb{N}}}$converges in E, then it is bounded .*

Proof. *Suppose $(x_i)_{i\in\mathbb{N}}$ converges to x, then the sequence of real numbers $(\| x_i - x \|)_i$ converges in $\mathbb{R}$, therfore is bounded. It follows that the set $\{\alpha_i : i \in \mathbb{N}\}$ is bounded as a subset of E.* ■

1.1.6 Non-Archimedean Banach spaces

Definition 1.1.15 *Non-Archimedean Banach space is a non-Archimedean normed vector space, which is complete with respect to the natural metric induced by the norm .*

$$d\,(x, y) =\| x - y \| \text{ for all } x,\ y \in E$$

Proposition 1.1.16 *The strong triangle inequality translates as follows:*

for $x,\ y,\ z \in E$,

$$d(x,\ y) \leq \max\{d(x,z), d(y,\ z)\}.$$

Example 1.1.3 *The space* $\mathbb{k}$, $\mathbb{k}^n$, $\sum_{i=0}^{\infty} \mathbb{k}^{(i)}$, $L^\infty(\mathbb{k})$, $B(X,K)$, $B(\ E_1, E_2)$ *with theire respective norms are Banach spaces.*

Definition 1.1.16 *Let* E *be a Banach space and* V *a closed subspace of* E *. Let* $P : E \to E/V$ *be the quotient map. define*

$$\| Px \| = d(x,V), x \in E, \tag{1.1.1}$$

where

$$d(x,\ B) = inf\ \{d(x,\ z) : z \in V\} = \inf\{\| x - z \| : z \in V\}$$

is the distance from x to V.

Proposition 1.1.17 *The norm in defined in the equation 1.1.1 is a non-Archimedean norm on* E/V*.*

Proof. First $\| 0 \| = \| P(0) \| = 0$ since $0 \in V$ Next, if

$$\| Px \| = 0$$

then

$$d(x,V) = 0$$

hence $x \in V$, and $Px = 0$.

For any $\lambda \in \mathbb{k}^*$

$$\begin{aligned} \| \lambda Px \| &= \| P(\ \lambda x\ ) \| \\ &= \inf\ \{\| \lambda x - z \| : z \in V\} \\ &= | \lambda | \inf\ \{\| x - \tfrac{z}{\lambda} \| : z\ \in V\} \\ &= | \lambda | \inf\ \{\| x - y \| : y \in v\} \\ &= | \lambda | \| Px \|\ . \end{aligned}$$

For x, $y \in E$, since V is closed, there exist z_1, z_2, $z_3 \in V$ such that

$\| Px \| = \| x - z_1 \|$, $\| Py \| = \| y - z_2 \|$, $\| P(x+y) \| = \| x+y) \| = \| x+y-z_3 \|$

$$\begin{aligned}
\| P(x) + P(y) \| &= \| P(x+y) \| \\
&= \| x+y-z_3 \| \\
&\leq \| (x+y) - (z_1+z_2) \| \text{ (because } (z_1+z_2) \in V) \\
&= \| (x-z_1) + (y-z_2) \| \\
&\leq \max \{ \| x-z_1 \|, \| y-z_2 \| \} \\
&= \max \{ \| Px \|, \| Py \| \}
\end{aligned}$$

■

Example 1.1.4 *Let $c_0(K)$ denote the set of all sequences $(x_i)_{i \in \mathbb{N}} \in \Bbbk$ such that*

$$\lim_{i \to \infty} | x_i | = 0$$

Then, $c_0(\Bbbk)$ is a vector space over $\Bbbk$ and

$$\| (x_i)_{i \in \mathbb{N}} \| = \sup_{i \in \mathbb{N}} | x_i |$$

is a non-Archimedean norm for which ($B_0(k), \| . \|$) is a Banach space.

Example 1.1.5 *Let $w = (w_i)_{i \in \mathbb{N}}$ be a sequence of non-zero element in $\Bbbk$.*

We define the space E_n, by

$$E_w = \{ x = (x_i)_{i \in \mathbb{N}} : \forall i,\ x_i \in \Bbbk \}$$

and

$$\lim_{i \to \infty} (| w_i |^{\frac{1}{2}} | w_i |) = 0 \}.$$

on E_w we define

$$x = (x_i)_{i \in \mathbb{N}} \in E_w, \| x \| = \sup_{i \in \mathbb{N}} (| w_i |^{\frac{1}{2}} | x_i |).$$

Then, $(E_w, \| . \|)$ is non-Archimedean Banach space.

Definition 1.1.17 *The Banach space E_w of example, equipped with its norm, is called a p-adic hilbert space.*

Example 1.1.6 *If we take $E_1 = E_2 = E$, the non-Archimedean normed space $B(E, E)$ is denoted $B(E)$ and consists of all $\mathbb{k}$-linear maps $T; E \longrightarrow E$ (also called "linear operators") satisfing*

$$\exists\, C \geq 0 \text{ such that } \forall x \in E, \| Ax \| \leq C \| x \| .$$

Recall that the norm on $B(E)$ is

$$\| A \| = \sup_{x \in E/\{0\}} \frac{\| Ax \|}{\| x \|}.$$

Proposition 1.1.18 *Let $(E, \| \,.\, \|)$ be a Non-archimedean Banach space. The series $\sum_{i=0}^{\infty} x_i$ converges in E if and only if the sequence of general terms $(x_i)_{i \in \mathbb{N}}$ converges to 0.*

Definition 1.1.18 *Let E be a vector space over $\mathbb{k}$ and $\| \,.\, \|_1$ and $\| \,.\, \|_2$ two non-Archimedean norms on E for each of which E is a Banach space.*

The two norms are said to be equivalent if there exist positive constants c_1 and c_2 such that for any $x \in E$

$$c_1 \| x \|_1 \leq \| x \|_2 \leq c_2 \| x \|_1$$

Proposition 1.1.19 *On a finite dimentional Banach space over $\mathbb{k}$, all non-Archimedean norms are equivalent.*

Proof. We use induction on the dimension n. If $n = 1$; let

$$\| x \|_0 = | x |$$

be the norm determined by the absolute value. Now let $\| \,.\, \|$ be any norm on $\mathbb{k}$, then for any $x \in \mathbb{k}$,

$$\| x \| = | x | \| 1 \| = c \| x \|_0, \text{ with } c = \| 1 \|$$

which implies that $\| . \|$ is equivalent to $\| . \|_0$. Suppose that the proposition is true for a space of dimension $(n-1)$. Let E be of dimension n and let $\{e_1, ...e_n\}$ be a basis for E First we have the natural norm on E. wich is

$$x \in E, x = \sum_{i=1}^{n} x_i e_i \ , \| x \|_0 = \max\{| x_i |: 1 \leq i \leq n\}.$$

Let $\| . \|$ be any norm on E. We want to show that $\| . \|$ is equivalent to $\| . \|_0$. For any

$$x = \sum_{i=1}^{n} x_i e_i$$

we have

$$\| x \| = \| \sum_{i=1}^{n} x_i e_i \| \leq \max\{| x_i | \| e_i \|: 1 \leq i \leq n\} \leq C \| x \|_0$$

where

$$C = \max \ \{\| e_i \|: 1 \leq i \leq n\}$$

and we find

$$\| x \| \leq C \| x \|_0$$

to obtain the other inequality wich will complete the equivalence, we let V be the subspace of E generated by $\{e_1 \ , .., e_{n-1}\}$, then

$$x = y + x_n e_n$$

where

$$y = \sum_{i=1}^{n-1} x_i e_i \in V.$$

We note that V is closed subspace of E, being the set of all vectors in E whose n-th component is zero. Therfore, it follows that

$$a = \inf\{\| z + e_n \|: z \in V > 0$$

then

$$\| x_n^{-1} y + e_n \| \geq \ a > 0$$

put

$$b = \ a \| e_n \|^{-1}$$

so that $b \leq 1$. Suppose first that $\alpha_n \neq 0$, then

$$\| e_n \|^{-1} \| x_n^{-1} y + e_n \| \geq b.$$

Now

$$\| x \| = | x_n | \| e_n \| (\| e_n \|^{-1} \| x_n^{-1} y + e_n \| \geq b \| x_n e_n \|$$

and we find

$$\| x \| \geq b \| x_n e_n \| .$$

■

1.1.7 Free Banach spaces

Definition 1.1.19 *A family $(v_i)_{i \in I}$ of vectors in E indexed by a set I converges to 0 and we write $\lim_{i \in I} v_i = 0$ if $\forall \varepsilon > 0, \{i \in I : \| v_i \| \geq \varepsilon\}$ is finite .*

Definition 1.1.20 *Let $v \in E$ and let $(v_i)_{i \in I}$ be a familly of elements of E indexed by the set I. We say that v is the sum of the family $(v_i)_{i \in I}$ and we write $\sum_{i \in I} v_i = v$*

If $\forall \varepsilon > 0$, there exists a finite subset $J_0 \subset I$ such that for any finite $J \subset I$, $J \supseteq J_0$

$$\| \sum_{i \in J} v_i - v \| \leq \varepsilon.$$

In this situation, we also say that the family $(v_i)_{i \in I}$ is summable and its sum is v.

Proposition 1.1.20 *Let the familly $(v_i)_{i \in I}$ be summable in E with sum $v \in E$, then*

$$\lim_{i \in I} v_i = 0.$$

Proof. Given $\varepsilon > 0$, let

$$H = \{i \in I : \| v_i \| \geq \varepsilon\}$$

since the family $(v_i)_{i \in I}$ is summable with sum v there exists a finite subset J_0 of I such that for any finite subset J of I containing J_0

$$\| \sum_{i \in J} v_i - v \| \leq \varepsilon.$$

Let $j \in I \setminus J_0$ and consider $J = J_0 \cup \{j\}$ then

$$\| \sum_{i \in J} v_i - v \| \leq \varepsilon.$$

since

$$\| \sum_{i \in J_0} v_i - v \| \leq \varepsilon$$

It follows that

$$max\{\| \sum_{i \in J} v_i - v \|, \| \sum_{i \in J_0} v_i - v \|\} \leq \varepsilon$$

which implies that

$$\| v_j \| \leq \varepsilon.$$

since this holds for any $j \notin J_0$we conclude that

$$H \subset J_0$$

hence H is finite and therefore

$$\lim_{i \in I} v_i = 0.$$

■

Chaptre 2

Bounded linear operators in Non-Archimedean Banach spaces

2.1 Bounded linear operator

2.1.1 The linear operator

Definition 2.1.1 *Let the operator A acting on X such that $(X, \| \, . \, \|)$ be a non-Archimedean Banach space is called linear if $D(A)$, which designate its domain, is a linear subspace*

of X, and

$$A(\alpha x + \beta y) \; = \alpha(Ax) + \beta(Ay) \; \forall \alpha, \beta \in \Bbbk \text{ and } x, y \in D(A).$$

Definition 2.1.2 *Let X and Y are two non-Archimedean Banach space and $A : X \to Y$ is a linear operator, the kernel* ***(the null space)*** *of linear operator denoted by ($N(A)$ or $ker(A)$, $N(A) \subseteq D(A)$) is the set of all $x \in D(A)$ such that $Ax = 0$*

$$N(A) = \{x \in D(A) : Ax = 0\}.$$

and the range of A or the image of A denoted by ($R(A)$ or $\operatorname{Im}(A)$) is $R(A) \subseteq Y$, defined by

$$R(A) = \{Ax : x \in D(A)\}$$

The graph of A, noted $G(A)$, and defined by

$$G(A) = \{(x, Ax) : x \in D(A)\}$$

Remark 2.1.1 *Let X a non-Archimedean Banach space and let $A \in \mathcal{L}(X)$. If A is invertible, then there exists a unique bounded linear operator denoted $A^{-1} : X \longrightarrow X$ called the inverse of A such that*

$$A^{-1}A = AA^{-1} = I$$

where $I : X \longrightarrow X$ is the identity operator .

Definition 2.1.3 *(**Bounded linear operator**) Let X, Y be two non-Archimedean Banach space $A : X \longrightarrow Y$, A linear operator we say that A is bounded if there exists $C > 0$, we denote by $\mathcal{L}(X, Y)$ the set of all bounded linear operators from X to Y. Such that*

$$\| Ax \| \leq C \| x \| \ \forall \ x \in X.$$

Definition 2.1.4 *Let $\mathcal{L}(X)$ denotes the collection of all bounded linear operators from X into itself. It is clear that if $A \in \mathcal{L}(X)$, then the quantity, The norm $\| A \|$ of the bounded linear operator is defined by*

$$\| A \| = \sup\{\frac{\| Ax \|}{\| x \|} : x \neq 0\}.$$

By definition of the norm-operator, if $A \in L(X)$, then the following identity holds ,

$$\| Ax \| \quad \leq \quad \| A \| . \| x \| \ \text{ for all } x \in X$$

Remark 2.1.2 *Note that every bounded linear operator on X is continuous. Indeed, if $(x_n)_{n \in \mathbb{N}} \subset X$ is sequence which converges strongly to some $x \in X$, that is ,*

$$\| x_n - x \| \longrightarrow 0 \quad as \ n \longrightarrow \infty$$

then using

$$\| Ax \| \leq \| A \| . \| x \| \ \textit{ for all } x \in X$$

it follows that

$$\| Ax_n - x) \| \leq \| A \| . \| x_n - x \|$$

which yields

$$\| A(x_n - x) \| \longrightarrow 0 \ as \ n \longrightarrow \infty$$

that is A is continuous. The converse, as given in the next theorem, is also true .

Theorem 2.1.1 *Every continuous linear operator $A : X \longrightarrow X$ is bounded.*

Proof. Suppose A is continuous. Consequently, A is continuous at $x = 0$. hence, there exists $\eta > 0$ such that

$$\| Ax \| \leq 1$$

when ever

$$\| x \| \leq \eta.$$

Suppose the valuation of the non-Archimedean field $\Bbbk$ is dense. Consequently, there exists $z_\eta \in \Bbbk \backslash \{0\}$ such that

$$| z_\eta | = \eta.$$

If $0 \neq x \in X$, then let $z_x \in \Bbbk \backslash \{0\}$ such that

$$| z_x | = \| x \|$$

.we have

$$\| \frac{z_\eta x}{z_x} \| = \eta.$$

Now

$$1 \geq \ \| A(\frac{z_\eta x}{z_x}) \| \ = \frac{| z_\eta | \| Ax \|}{| z_x |} = \frac{\eta \| Ax \|}{\| x \|}$$

and hence

$$\| Ax \| \leq \eta^{-1} \| x \|$$

which yields A is bounded. One should point out that the proof is similar in the case when the valuation of $\Bbbk$ is discrete and hence is omitted . ■

Example 2.1.1 *Let* $\mathrm{X} = \Bbbk^n = \{(x_1, x_2, ..., x_n) : z_k \in \Bbbk \ , \quad k = 1, 2, ..., n\}$ *be equipped with its natural non-Archimedean norm given by*

$$\| x \| = \max_{i=1,\ldots,\ n} | x_i |$$

for all

$$x = (x_1, x_2, ..., x_n) \ \in \ \mathbb{k}^n.$$

Let $(e_1, e_2, ..., e_n)$ be the canonical basis of $\mathbb{k}^n$ defined by, $e_1 = (1, 0, 0, ..., 0), e_2 = (0, 1, 0, ..., 0)$, ..., $e_n = (0, 0, ..., 1)$. Clearly for all

$$x = (x_1, x_2, ..., x_n) \in \mathbb{k}^n, \ x = \sum_{j=1}^{n} x_j e_j$$

for some

$$x_j \in F, \ j = 1, 2, ..., n.$$

Let $A : \mathbb{k}^n \longrightarrow \mathbb{k}^n$ be a linear mapping. Clearly, $Ae_i \in \mathbb{k}^n$ and subsequently there exists $a_{ij} \in \mathbb{k}$ for $i, \ j \ = \ 1, 2 \ , ..., n$ such that

$$Ae_j = \sum_{i=1}^{n} a_{ij} e_i.$$

In what follows, we exhibit that the arbitrary linear operator A given above is necessarily bounded. Indeed, for all

$$x = \sum_{j=1}^{n} x_j e_j \ \text{ and } y = \sum_{j=1}^{n} y_j e_j.$$

$$\begin{aligned} \| \ Ax - Ay \ \| \ &= \| \ \sum_{j=1}^{n} (x_j - y_j) \ Ae_j \ \| \\ &\leq C \max(| \ x_1 - y_1 \ |, | \ x_2 - y_2 \ |, ..., | \ x_n - y_n \ |) \\ &= C \ \| \ x - y \ \| \end{aligned}$$

where

$$C = \underset{j=1,...., \ n}{max} \| \ Ae_j \ \| \ \ = \max_{j=1,....,n} (\max_{i=1,...., \ n} | \ a_{ij} \ |) < \infty.$$

consequently, $A : \mathbb{k}^n \longrightarrow \mathbb{k}^n$ is a bounded linear operator.

Definition 2.1.5 *Let X be a non-Archimedean Banach space. If $A \in \mathcal{L}(X)$, then*

a) A is siad to be injective to be if

$$N(A) = \{0\}.$$

b) A is said to be surjective if

$$R(A) = Y.$$

c) A is called to be invertible if it is both injective and surjective.

Definition 2.1.6 *Let X be a hilbert space over $\mathbb{k}$ $=\mathbb{R}$ or $\mathbb{C}$. For $A \in \mathcal{L}(X)$. The linear operator A' is called the adjoint of A if*

$$\langle Ax, y \rangle = \langle x, A'y \rangle$$

for all $x, y \in X$. The operator A' is called the adjoint of A .

2.1.2 Bounded linear operator in free Banach spaces

Definition 2.1.7 *Let X be a free Banach space over the non-Archimedean field $(\mathbb{k}, |\ .\ |)$ with canonical orthogonal basis $(e_j)_{j\in I}$. define $\acute{e}_i \in X^*$ by setting*

$$x = \sum_{i\in I} x_i e_i, \acute{e}_i(x) = x_i.$$

It turns that

$$\| \acute{e}_i \|_\star = \| e_i \|^{-1} .$$

Furthermore, every $x' \in X^*$ can be expressed as

$$x' = \sum_{i\in I} \langle x', e_i \rangle e_i'$$

with

$$\| x' \| = \sup_{i\in I} \frac{| \langle x', e_i \rangle |}{\| e_i \|}$$

For each $f' \in X^*$ define the linear operator $v' \otimes u : X \longrightarrow X$ by

$$(v' \otimes u)(w) := \langle v',\ w \rangle u$$

Clearly, the operator $(v \otimes u)$ is bounded as

$$\| v' \otimes u \| = \| v' \| * \| u \| .$$

Among different things, if $(\acute{e}_i)_{i\in I}$ is the dual canonical orthogonal basis for E^*, then

$$(\acute{e}_i \otimes e_j)_{(i,j)\in I\times I} \ \in\ \mathcal{L}(X)$$

and its operator-norm is given explicity by way of

$$\| \acute{e}_i \otimes e_j \| = \frac{\| e_j \|}{\| e_i \|}.$$

Proposition 2.1.1 *Let $A \in \mathcal{L}(X)$, then it can be written in a unique fashion as a pointwise convergent series*

$$A = \sum_{(i,j)\in I\times I} a_{ji}\acute{e}_i \otimes e_j, i \in I \ , \ \lim_j \mid a_{ji} \mid\parallel e_j \parallel = 0.$$

Moreover

$$\parallel A \parallel = \sup_{i\in I}\sup_{j\in I}\frac{\mid a_{ji} \mid\parallel e_j \parallel}{\parallel e_i \parallel}.$$

Proof. For all

$$j \in I, Ae_j = \sum_{i\in I} a_{ij}e_i$$

where

$$a_{ij} \in \Bbbk, \ \lim_i \mid a_{ij} \mid\parallel e_j \parallel = 0.$$

Now for any $x = \sum_{j\in I} x_j e_j \in X$,

$$Ax = \sum_{j\in I}\sum_{i\in I} a_{ij}x_j e_i = \sum_{j\in I}\sum_{i\in I} a_{ij}(\acute{e}_j \otimes e_i)x.$$

It remains to show that

$$\parallel A \parallel = \sup_{j\in I}\frac{\parallel Ae_j \parallel}{\parallel e_j \parallel} = \sup_{j\in I}\sup_{i\in I}\frac{\mid a_{ji} \mid\parallel e_e \parallel}{\parallel e_j \parallel}.$$

Indeed,

$$\frac{\parallel Ae_j \parallel}{\parallel e_j \parallel} \leq \parallel A \parallel .$$

Next, for any $x = \sum_{j\in I} x_j e_j$,

$$\begin{aligned}
\parallel Ax \parallel &= \parallel \sum_{j\in I} x_j Ae_j \parallel \\
&\leq \sup_{j\in I}(\mid x_j \mid . \parallel Ae_j \parallel) \\
&= sup_{j\in I}(\mid x_j \mid . \parallel e_j \parallel . \tfrac{\parallel Ae_j \parallel}{\parallel e_j \parallel}) \\
&\leq \parallel x \parallel .\sup_{j\in I}\tfrac{\parallel Ae_j \parallel}{\parallel e_j \parallel}.
\end{aligned}$$

■

Chaptre 3

Non-Archimedean spectrum

3.1 The spectrum $\sigma(A)$

Definition 3.1.1 [9] *Let X, Y be two non-Archimedean Banach spaces over $\mathbb{k}$. If $A \in \mathcal{L}(X,Y)$, then* ***The spectrum*** *$\sigma(A)$ of the linear operator A is defined by*

$$\sigma(A) = \{\lambda \in \mathbb{k} : A - \lambda I \text{ has not a bounded inverse }\}.$$

Definition 3.1.2 [9] *Let X, Y be two non-Archimedean Banach spaces over $\mathbb{k}$. If $A \in \mathcal{L}(X,Y)$, then* ***The resolvent*** *$\rho(A)$ of the linear operator A is defined by*

$$\rho(A) = \{\lambda \in \mathbb{k} : \lambda I - A \text{ is invertible }\}.$$

Remark 3.1.1 *The resolvent set $\rho(A)$ is the complement of the set $\sigma(A)$ in $\mathbb{k}$. In other word*

$$\sigma(A) = \mathbb{k} \backslash \rho(A).$$

Definition 3.1.3 [9] *A scalar $\lambda \in \mathbb{k}$ is called eigenvalue of $A \in \mathcal{L}(X)$ whenever there exists a non zero $u \in X$ (called eigenvector associated with λ) such that*

$$Au = \lambda u.$$

Clearly, eigenvalues of $A \in \mathcal{L}(E)$ consist of all $\lambda \in \Bbbk$ for which $\lambda I - A$ is not one-to-one that is,

$$N(\lambda I - A) \neq \{0\}.$$

The collection of all eigenvalues is denoted $\sigma_p(A)$ (called point spectrum) and is defined by

$$\sigma_p(A) = \{\lambda \in \sigma(A) : N(A - \lambda I) \neq \{0\}\}.$$

Definition 3.1.4 *(**The Continuous spectrum** $\sigma_c(A)$) of a bounded linear operator $A : X \longrightarrow X$ is defined by ,*

$$\sigma_c(A) = \{\lambda \in \sigma_e(A) \setminus \sigma_p(A) : \overline{R(\lambda I - A)} = X\}.$$

Definition 3.1.5 *(**the Residual spectrum** $\sigma_r(A)$ of a bounded linear operator $A : X \longrightarrow X$ is defined by*

$$\sigma_r(A) = (\sigma_e(A) \setminus \sigma_p(A)) \setminus \sigma_c(A).$$

As in the classical operator theory, we have

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A).$$

3.1.1 Fredholm operator

Definition 3.1.6 [9] *An operator $A \in \mathcal{L}(X)$ is said to be a Fredholm operator if it satisfies the following condition :*

1. $\eta(A) = \dim N(A)$ *is finite .*

2. $R(A)$ *is closed .*

3. $\delta(A) = \dim(X/R(A))$*is finite .*

Definition 3.1.7 [7] *Let X,Y be Banach spaces over $\Bbbk$ and Let A in $\mathcal{L}(X,Y)$. If $ker(A)$ is finite dimentionsional $A(X)$ is closed in Y and coker (A) is finite dimensional, then A is called a Fredholm operator .*

Definition 3.1.8 [9] *The collection of all Fredholm linear operators $A \in L(X)$ will be denoted $\Phi(X)$.*

If $A \in \Phi(X)$, we then define its index by setting ,

$$\chi(A) = \eta(A) - \delta(A).$$

Example 3.1.1 *Let X be a finite dimentional vector space. Then any linear operator $A : X \longrightarrow X$ is a Fredholm operator with index*

$$\chi(A) = \eta(A) - \delta(A) = 0.$$

Example 3.1.2 *Let $S : l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N})$ be the unilateral shift defined by*

$$Se_j = e_{j+1} \text{ for all } n \in \mathbb{N}$$

where $(e_j)_{j\in\mathbb{N}}$ is the canonical basis of $l^2(\mathbb{N})$ defined by

$$e_1 = (1, 0, 0, ...),\ e_2 = (0, 1, 0, 0, ...)$$

and

$$e_n = (0, 0, ...0,\ 0,\ 1,\ 0, 0, ...).$$

It can be shown that S is a Fredholm bounded linear operator whose index is -1.

Definition 3.1.9 ***(Upper semi Fredholm)*** *If X, Y be two non-Archimedean Banach spaces over $\Bbbk$. The classe of p-adic upper semi-Fredholm operators from X to Y , noted $\Phi_+(X,Y)$, is defined by*

$$\Phi_+(X,Y) = \{T \in L(X,Y) : \alpha(T) < \infty \text{ and } R(T) \text{ is closed in } Y\}.$$

Definition 3.1.10 ***(Lower semi Frdholm)*** *If X, Y be two non-Archimedean Banach spaces over $\Bbbk$. The classe of p-adic lower semi-Fredholm operators from X to Y , noted $\Phi_-(X,Y)$, is defined by*

$$\Phi_-(X,Y) = \{T \in L(X,Y) : \beta(T) < \infty \text{ and } R(T) \text{ is closed in } Y\}$$

3.1.2 Proprieties of Fredholm operators

Let $A \in \mathcal{L}(X)$. We suppose the bounded linear operator A has the property :

$N(A)$, $R(A)$ are complemented with the (closed) subspaces X_0, Y_0.

Clearly, the product space $X_0 \times Y_0$ is a non-Archimedean Banach space when it is equipped with the norm defined by

$$\| x_0, y_0) \| = \max(\| x_0 \|, \| y_0 \|)$$

for all $(x_0,\, y_0) \in X_0 \times Y_0$.

Define the linear operator $\tilde{A} : X_0 \times Y_0 \longrightarrow X$ by setting

$$\tilde{A}(x_0, y_0) = Ax_0 + y_0.$$

It is clear that the operator $\tilde{A}$ defined above is a bijective bounded linear operator .

In what follows we say that $\tilde{A}$ is the bijection associated with the bounded linear operator A and the subspaces X_0 and Y_0.

If $A \in \Phi(X)$, then it can be easily considered that $\tilde{A} \in \Phi(X)$ with Y_0 being a finite dimensional subspace. Identifying X_0 with $X_0 \times \{0\}$ it follows that the linear operator defined by

$$A_0 : X_0 \longrightarrow X,\ Ax_0 = Ax$$

is the restriction of each A and $\tilde{A}$.

Lemma 3.1.1 *Suppose $A_0 : L \longrightarrow X$ is a restriction of $A \in \mathcal{L}(X)$ to a subspace $L \subset X$ with*

$$co\dim(L) = N < \infty.$$

Then A is Fredholm if and only if A_0 is Fredholm In this event

$$\chi(A) = \chi(A_0) + N.$$

Theorem 3.1.1 [9] *If $A,\, B : X \longrightarrow X$ are Fredholm linear operators, then so is their composition $BA : X \longrightarrow X$, and*

$$\chi(BA) = \chi(B) + \chi(A).$$

Proof. Let $\tilde{A}$ be the bijection associated with A and the subspace $X_0$and Y_0 further, let A_0 be the restrictriction of both A to X_0. Consider the linear operator $B\tilde{A}$. Since the linear operator $\tilde{A}$ is bijective it follows that is $B\tilde{A}$ Fredholm with

$$\chi(B\tilde{A}) = \chi(B).$$

It is also clear that BA_0 is a restriction of both a $B\tilde{A}$ and BA it follows that

$$\begin{aligned}\chi(BA) &= \chi(BA_0) + \dim(X/X_0) \\ &= \chi(B\tilde{A}) + dim(X_0 \times Y_0/X_0 \times \{0\}) + \eta(A) \\ &= \chi(A) + \chi(B).\end{aligned}$$

■

Theorem 3.1.2 *Let $A : X \longrightarrow X$ be a Fredholm operator. If $L : X \longrightarrow X$ is a bounded linear operator such that*

$$\| L \| < \| \tilde{A}^{-1} \|^{-1}$$

where $\tilde{A}$ is the bijection associated with A, then $A + L$ is a Fredholm operator with

$$\chi(A + L) = \chi(A).$$

Proof. Let $S = A + B$ and define the operator $\tilde{S} : X_0 \times Y_0 \longrightarrow X$ defined by

$$\tilde{S}(x_0,\ y_0) = Sx_0 + y_0.$$

Using the fact that $\tilde{A}$ is bijective and that

$$\begin{aligned}\| \tilde{A} - \tilde{S} \| &\leq \| A - S \| \\ &= \| B \| < \| \tilde{A}^{-1} \|^{-1}\end{aligned}$$

it follows that $\tilde{S}$ is bijective, it follows that S is a Fredholm operator and that

$$\begin{aligned}\chi(S) &= \chi(S_0) + \eta(A) \\ &= \chi(\tilde{A}) - \delta(A) + \eta(A) \\ &= \chi(A).\end{aligned}$$

■

3.1.3 Non-Archimedean essential spectrum

Definition 3.1.11 *Let E be a non-Archimedean Banach space. The essential spectrum $\sigma(e)$ of a bounded linear operator $A \in \mathcal{L}(E)$ is defined by*

$$\sigma_e(A) = \{\lambda \in \mathbb{k} : \lambda I - A \text{ is a not Fredholm operator of index } 0\}.$$

Remark 3.1.2 *If $\lambda \in \mathbb{k}$ deos not belong to neither $\sigma_p(A)$ nor $\sigma_e(A)$, then $\lambda I - A$ must be injective*

$$(N(\lambda I - A) = \{0\})$$

and $R(\lambda I - A)$ is closed with

$$0 = \dim N(\lambda I - A) = \dim E \setminus \ R(\lambda I - A).$$

Consequently, $(\lambda I - A)$ must be bijective (injective and surjective) which yields $\lambda \in \rho$ (A). In view of the previous facts, we have

$$\sigma(A) = \sigma_p(A) \cup \sigma_e(A).$$

3.1.4 Perturbation classes of operator

Definition 3.1.12 *Let X be a non-Archimedean Banach space. $A \subset X$ is called compactoid if for every $\varepsilon > 0$ there is a finite set $B \subset X$ such that*

$$A \subset co(B) + \{x \in X : \| x \| \leq \varepsilon\}$$

Where $co(B)$ is the absolutely convex hull of B, i. e.,

$$co(B) = \{\sum_{x \in X} \lambda_x x : \lambda_x \in \mathbb{k}, | \lambda_x | \leq 1, \ X \text{ is finite subset of } B \ \}.$$

Definition 3.1.13 *Let X and Y be two non-Archimedean Banach space. $K \in \mathcal{L}(X, Y)$ is called a compact operator if $K(B_x)$ is compactoid, where B_x denotes the closed unit ball of X. The collection of compact operators from X to Y will be denoted by $K(X, Y)$.*

Strictly singular operator

Definition 3.1.14 *$B \in \mathcal{L}(X,Y)$ is said to be strictly singular if any vector subspace M of X for which there exists $\alpha > 0$ such that*

$$\| Bx \| \geq \alpha \| x \|, \text{ for all } x \in M$$

is finite dimensional subspace of X is not an homeomorphism .

Let us denote by $S(X,Y)$ the set of strictly singular operators from X into Y .

Cosingular operator

Definition 3.1.15 *The operator T is said to be strictly cosingular if there is no infinite codimensional subspace N of Y such that $Q_N T$ is surjective.*

We denote by SS and SC the classes of all strictly singular and strictly cosingular operators, respectively .

Corollary 3.1.1 [11, Corollary3.3] *Let $T,\ A \in \mathcal{L}(E,F)$. If T is compact and A is semi-Fredholm then $T + A$ is semi-Fredholm.*

Theorem 3.1.3 [10, Theorem 2.4] *If $T \in \mathcal{L}(E,F)$, the following properties are equivalent:*

a) T is semi-Fredholm.

b) For every $t \in (0,1]$ there exists $s \in (0,1]$ such that whenever $(x_n)_{n\in\mathbb{N}}$ is a t-orthogonal sequence in $E - \{0\}$ then, for some natural number m, $(T(x_n))_{n\geq m}$is an s-orthogonal sequence in $F - \{0\}$.

c) For every $t \in (0,1]$ and every t-orthogonal sequence (y_n) in E with $\underset{n}{inf} \| y_n \| > 0$, the sequence $(Ty_n)_{n\in\mathbb{N}}$ does not converge to zero.

d) For every closed subspace of countable type B of E, $T\backslash B$ is semi-Fredholm. If in addition $\Bbbk$ is spherically complete.

e) There exists $s \in (0,1]$ such that if $(x_n)_{n\in N}$ is an orthogonal sequence in $E - (0\}$ then, for some natural number m, $(T(x_n))_{n\geq m}$ is an s-orthogonal sequence in $F - \{0\}$.

Theorem 3.1.4 [11, Theorem 3.5] *Let $T \in \mathcal{L}(E, F)$. Then the following are equivalent.*

1) T is compact .

2) For each closed subspace D of E the restriction $T\backslash D$ is compact .

3) For each closed infinite dimensional subspace D of E the restriction $T\backslash D$ is not semi-Fredholm.

4) For each closed infinite dimensional subspace D of E, D of countable type, the restriction $T\backslash D$ is not semi-Fredholm.

5) For each $t \in (0, 1)$ and each closed infinite dimensional subspace D of E there exists a t-orthogonal sequence e_1,e_2,... in D with

$$inf_n \; ||e_n| \; | > 0$$

and

$$lim \; inf_{n \longrightarrow \infty} ||Te_n \; || = 0.$$

Theorem 3.1.5 [5, Theorem 3.16] *If E be an infinite-dimensional Banach space of countable type. we have*

1. For every sequence t_1, t_1 ..., of elements of the real interval $(0, 1)$, E has a base $\{e_i, : i \in \mathbb{N}\}$ such that

$$\| \sum_{i=1}^{m} \alpha_i e_i \| \geq \max\{t_i \parallel \alpha_i e_i \parallel : i = 1, ..., \; m\}$$

for all $m \in \mathbb{N}$ and $\alpha_1, ...\alpha_m \in \mathbb{k}$.

2. For every $t \in (0, 1)$, E contains a t-orthogonal sequence that forms a base for E. E is linearly homeomorphic to c_0.

3. For every $t \in (0, 1)$ there exist a function $s : \mathbb{N} \longrightarrow (0, 1)$ and a linear bijection $S : c_0(\mathbb{N} : s) \longrightarrow E$ such that

$$t \parallel x \parallel \leq \parallel Sx \parallel < ||x \parallel \qquad [x \in c_0(\mathbb{N} : s)].$$

4. E is pseudo reflexive.

5. Let D be any closed linear subspace of E. Then both D and E/D are of countable type. D is complemented. For

every $\varepsilon > 0$ there exists a projection of E onto D of norm less than or equal to $1 + \varepsilon$.

6. Let D be a linear subspace of E, let $f \in D'$ and $\varepsilon > 0$ Then f can be extended to an $\bar{f} \in E'$ with

$$\parallel \bar{f} \parallel \leq (1 + \varepsilon) \mid f \mid .$$

Theorem 3.1.6 [4, Theorem 2.2] *The following properties are equivalents*

i) $A \in \Phi_+^b(X, Y)$, where $\Phi_+^b(X, Y) = \Phi_+(X, Y) \cap \mathcal{L}(X, Y)$

ii) There exists $C > 0$ such that for any $0 < t < 1$ and any t-orthogonal sequence $\{e_n\}_{n \in \mathbb{N}}$ in X, there exists $n_0 \in \mathbb{N}$ with

$$\parallel Ae_n \parallel \geq Ct \parallel e_n \parallel \text{ for all } n \geq n_0.$$

In the following theorem, C. Perez-Garcia and S. Vega derived the non-Archimedean validity of the stability of Fredholm operators and thier index when they are pertubed by an arbitrary (arbitrary?) small operator.

Remark 3.1.3 [4] *Let X be a non-Archimedean Banach space over* $\Bbbk$. *Let* $\{\lambda_n\}_{n \in \mathbb{N}} \subset \Bbbk$ *and* $\{x_n\}_{n \in \mathbb{N}} \subset X$.

We have that if $\lim\limits_{n \to \infty} \lambda_n x_n = 0$, then

$$\parallel \sum_{n \in \mathbb{N}} \lambda_n x_n \parallel \leq \max_{n \in \mathbb{N}} \parallel \lambda_n x_n \parallel .$$

Theorem 3.1.7 *Let X and Y be two non-Archimedean Banach spaces. Let* $A \in \Phi_+^b(X, Y)$ *and* $B \in S(X, Y)$. *Then ,* $A + B \in \Phi_+^b(X, Y)$.

Proof. Let $A \in \Phi^b_-(X,Y)$ and $B \in S(X,Y)$. Let us suppose that $A+B \notin \Phi^b_+(X,Y)$. We may assume that B is not compact (Otherwise, we have from [Corollary (3.1.1)] that $A+B \in \Phi^b_+(X,Y)$and this is contradiction). Refering to[Theorem (3.1.4)] there exists a closed infinite dimensional subspace of countable type M of X such that

$$\beta_{|M} \in \Phi^b_+(M,Y).$$

As M is closed infinite dimensional subspace of countable type, then by using [Theorem (3.1.5)] for every $t \in (0,1)$, M contains a t-orthogonal sequence $\{f_n\}_{n\in\mathbb{N}}$ that forms a base in M. from [theorem (3.1.6)] , we conclude the existence of $C>0$ and $n_0 \in \mathbb{N}$ such that

$$\| Bf_n \| \geq Ct \| f_n \| \text{ for all } n \geq n. \tag{3.1.1}$$

Therefore, by applying [Theorem (3.1.3)], there is $s \in (0,1]$ such that for some natural number $n_1 \in \mathbb{N}$,

$$\{B(f_n)\}_{n\geq n_1} \text{ is s-orthogonal.} \tag{3.1.2}$$

Consider $n_2 = \max(n_0, n_1)$ and let D be the vector subspace of M spanned by the infinite familly $\{f_n\}_{n\geq n_2}$.

Since $\{f_n\}_{n\geq n_2}$ is a t-orthogonal family, it is linearly independent. Hence, the vector space D is infinite dimensional. Each $x \in D$ can be written as a unique finite *sum* $x = \sum_{n\in j_x} \alpha_n f_n$ with j_x bieng a finite subset of $\{n \in \mathbb{N}, n \geq n_2\}$. One finds out from [*Remark* (3.1.3)], inequality (3.1.1), relation (3.1.2), and the linearity of B that

$$\begin{aligned} \| x \| &= \| \textstyle\sum_{n\in j_x} \alpha_n f_n \| & &\leq \max_{n\in j_x} | \alpha_n | \| f_n \| \\ &\leq \frac{1}{Ct} \max_{n\in j_x} | \alpha_n | \| Bf_n \| & &= \frac{1}{Cts} s\max_{n\in j_x} \| \alpha_n Bf_n \| \\ &\leq \frac{1}{Cts} \| \sum_{n\in j_x} \alpha_n Bf_n \| & &= \frac{1}{Cts} \| B(\sum_{n\in j_x} \alpha_n f_n) \| \\ & & &= \frac{1}{Cts} \| B(x) \| . \end{aligned}$$

In other words, there follows the existence of $\alpha = Cts > 0$ such that

$$\| Bx \| \geq \alpha \| x \| \text{ for all } x \in D.$$

The fact that B is a strictly singular operator must imply that D is finite dimensional, which is a contradiction.

As a consequence, $A+B \in \Phi^b_+(X,Y)$. ■

Bibliographie

[1] A.Ammar,A. Bouchekoua., & A. Jeribi, Pseudospectra in a non-Archimedean Banach space and essential pseudospectra in E_ω. Filomat, 33(12), 3961-3976. (2019).

[2] A.Ammar, F. Z, Boutaf, & A.Jeribi, Essential spectra in non-Archimedean fields, Matematychni Studii, 58(1), 82-93. (2022).

[3] A. Ammar, A. Jeribi , B. Saadaoui, On some classes of demicompact linear relation and some results of essential pseudospectra. Matematychni Studii, 52(2), 195-210.(2019).

[4] A. Ammar, F. Z,Boutaf, & A.Jeribi, New results on perturbations of p-adic linear operators. Georgian Mathematical Journal. (2023).

[5] A.C.M.van Rooij. non-Archimedean Functional Analysis, Monographs and Textbooks in pure and applied Mathematics.51.Marcel Dekker,Inc, New york, (1978).

[6] M. S, Moslehian, & G. Sadeghi, A Mazur–Ulam theorem in non-Archimedean normed spaces. Non linear Analysis: Theory, Methods & Applications, 69(10), 3405-3408. (2023).

[7] K. S.Nadathur, linear-operators between non-Archimedean Banach space. Western Michigan University.(1973).

[8] P.Aiena, M. González, &A. Martínez-Abejón, Operator semigroups in Banach space theory, Bollettino dell'Unione Matematica Italiana, 4(1), 157-205. (2001).

[9] T.Diagana, &F. Ramaroson, Non-Archimedean operator theory, Cham, Switzerland: Springer (2016) .

[10] J. Martinez-Maurica, T .pellon and C. pererez-garcia, Some characterizations of p-adic semi-Fredholm operators, Ann. Math . pura Appl. (4) 156, 243-251. (1990).

[11] W. H. schikhof, On p-adic compact operators, report 8911, Departement of matimatics, Catholic University, Nijmegen, The netherlands, 1-28. (1989).

ملخص:

يعد طيف المؤثر الخطي احد اكثر المفاهيم المفيدة في التحليل الدالي. في حالة فضاء باناخ ، هناك العديد من الاعمال و النتائج التي أدت أهميتها بالعديد من الباحثين الى محاولة توسيع هدا المفهوم ليشمل فضاء باناخ غير ارخميدي.

كلمات مفتاحية:مساحات باناخ غير أرخميدية، عامل خطي، نطاق، عامل فريدهولم.

Abstract

The Spectrum of a linear operator is one of the most useful objects in functional analysis .

In the case of a Banach space, there are many works with results and its importance has led many researchers to try and extend this concept to Non-Archimedean Banach space.

Keywords : Non-Archimedean Banach space, Bounded linear operator , Spectrum, Fredholm operator.

Résumé :

Le spectre d'un opérateur linéare est l'un des objets les plus utiles en analyse fonctionnelle.

Dans le cas d'un espace de Banach, il existe de nombreux traveaux avec des résultats et son importance a conduit de nombreux chercheurs à tenter d'étender ce concept au cas d'un espace de Banach non-Archimedean .

Mots clés : espace de Banach non-Archimedean, opérateur linéare borné, spectre, opérateur de Fredholm.