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PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
وزارة التعليـم العالـي و البحـث العلمــي
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

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FACULTY OF TECHNOLOGY
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شهادة إدارية

بعد الإطلاع على التقارير الايجابية الواردة من السادة الخبراء أعضاء لجنة دراسة المطبوعة الجامعية والاتية أسماؤهم:

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صادق أعضاء اللجنــة العلميــة على قبول المطبوعة البيداغوجية مع إمكانية إتخاذها سندا في تدريس طلبة السنة الثانية ليسانس الكترونيك واتصالات، في ميدان علوم و تكنولوجيا و أن تعتمد في أي تقييم المسار العلمي للأستاذ المعني **غلاب تركية** (أستاذ محاضر قسم "أ"– جامعة محمد بوضياف - المسيلة) تحت عنوان :

Vibrations and Mechanical Waves

رئيس القسم رئيس اللجنة العلمية

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جامعة محمد بوضياف المسيلة

MOHAMED BOUDIAF UNIVERSITY -M'SILA-

Faculty of Technology

Department: Electronics

2nd year License

Handout:

Vibrations and Mechanical Waves

Presented by

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Summary

Foreword

This course handout, entitled: “Vibrations and Mechanical Waves” is developed and presented in accordance with the outline relating to the LMD-S3 License training in the field of “Material Science (SM) and Science and Technology (ST)".

This course is structured in two parts:

The first, divided into five chapters, deals with the problem of vibrations. The first chapter concerns the use of the Lagrange formalism which describes the oscillations of physical systems. The study of free linear (low amplitude) oscillations of systems with one degree of freedom is presented in the second chapter. The third chapter deals with the damped movement which takes into account the viscosity friction forces proportional to the speed of the mobile. The notion of resonance devoted to forced oscillations is presented in the fourth chapter. The fifth chapter presents vibrations with several degrees of freedom.

The second part, which constitutes the last two chapters, is devoted to the treatment of wave propagation phenomena.

The course presented with a logical sequence, each new concept defined is clarified by simple and useful examples, a series of problems enriching the course, everything was carried out with the spirit of allowing better assimilation by the student.

PART ONE : VIBRATIONS

Chapter 1 :

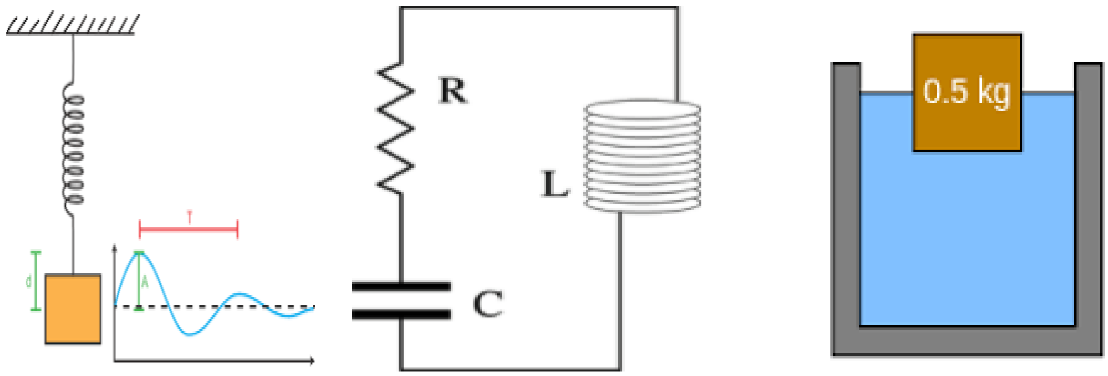
General information on oscillations

1.1 Definition of an oscillation (Vibration)

Vibration is an oscillatory physical phenomenon of a body moving around its equilibrium position.

Among the most varied mechanical movements, there are movements which are repeated: the beating of the heart, the movement of a swing, the alternating movement of the pistons of an internal combustion engine. All of these movements have one common trait: a repetition of the movement over a *cycle*.

1.1.1 Examples



a) Spring mass b) Oscillating electrical circuit c) cylinder floating in a liquid

A *cycle* is an uninterrupted series of movements or phenomena which are always renewed in the same order. Take as an example the four-stroke cycle of an internal combustion engine. A complete cycle includes four stages (intake, compression, explosion, exhaust) that repeat during an engine cycle.

1.2 Definition of a periodic movement

We call periodic movement a movement which repeats itself and each cycle of which reproduces itself identically. The duration of a cycle is called period measured by the second and is defined as follows: $T_0 = \frac{2\pi}{\omega_0}$.

Where ω_0 is called the pulsation which relates to the frequency of oscillations and is measured in rad,s^{-1} $\omega_0 = 2\pi f_0$

Frequency is defined as the number of oscillations that take place per unit of time *t*, and is measured in Hertz, $f_0 = \frac{1}{T_0}$

Dr.Ghellab Torkia

A particularly interesting periodic movement in the field of mechanics is that of an object which moves from its equilibrium position and returns to it by performing a back and forth movement relative to this position.

This type of periodic movement is called oscillation or oscillatory movement. The oscillations of a mass connected to a spring, the movement of a pendulum or the vibrations of a stringed instrument are examples of oscillatory movements.

Any mechanical system, including the most complex industrial machines, can be represented by models consisting of a spring, a shock absorber and a mass. The human body, often described as "beautiful mechanics", is broken down in Figure 1.1 into several "spring-damper mass" subsystems representing the head, shoulders, rib cage and legs or feet.

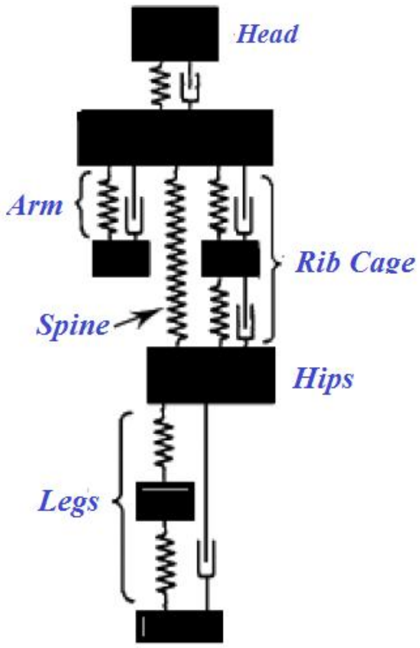


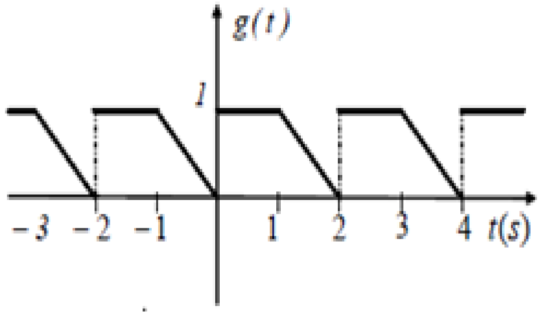
Figure 1.1: Mass-spring-damper modeling of man.

Mathematically, periodicity is expressed by $g(t+T)=g(T)$. A periodic quantity is called *Sinusoidal* when it is of the form $g(t)=A\sin(\omega t+\varphi)$.$A$ is called amplitude, ω : the *pulsation*, φ: the *initial phase*. Among the physical quantities studied of oscillating systems, we find:

- ✓ The movement x.
- ✓ The angle θ.
- ✓ The load q.
- ✓ The current i.
- ✓ Voltage U.
- ✓ A field E.

1.2.1 Example

a) Let the periodic quantity g (t) shown opposite.



$T = 2s.$

$f = \frac{1}{T} = 0.5\,Hz.$

$\omega = 2\pi f = \pi rads^{-1}$

1.3 Complex representation

To facilitate calculations, we transform the sinusoidal quantities into exponentials which are simpler to handle. This is possible thanks to Euler's formula (1748).

$$\cos\theta + j\sin\theta = e^{j\theta} \text{ with } j^2 = -1$$

1.3.1 Examples

b) Consider a capacitor and a current $i(t) = I_0 \cos\omega t$.

Find the complex impedance $\tilde{z}_c = \frac{\tilde{u}_c}{\tilde{i}}$. Reminder: $V_c = \frac{q}{c} = \frac{\int idt}{c}$ *because* $i = \frac{dq}{dt}$

$$\begin{cases} i(t) = I_0\cos\omega t \rightarrow \tilde{i}(t) = I_0 e^{j\omega t} \\ V_c(t) = \frac{\int i(t)dt}{c} \rightarrow \tilde{V}_c(t) = \frac{\int I_0 e^{j\omega t}dt}{c} = \frac{I_0 e^{j\omega t}}{jc\omega} = \frac{\tilde{i}(t)}{jc\omega} \Rightarrow \tilde{z}_c = \frac{\tilde{V}_c}{\tilde{i}} = \frac{\tilde{i}(t)}{jc\omega\tilde{i}} = \frac{1}{jc\omega} \end{cases}$$

c) Consider a coil L and a current $i(t) = I_0 \cos\omega t$.

Find the complex impedance $\tilde{z}_L = \frac{\tilde{V}_L}{\tilde{i}}$. Reminder: $\tilde{V}_L = L\frac{di}{dt}$.

$$\begin{cases} i(t) = I_0\cos\omega t \rightarrow \tilde{i}(t) = I_0 e^{j\omega t} \\ V(t) = \frac{di}{dt} \rightarrow \tilde{V}_L(t) = L\frac{d\tilde{i}}{dt} = L\frac{d\left(I_0 e^{j\omega t}\right)}{dt} = jL\omega I_0 e^{j\omega t} = jL\omega\tilde{i} \end{cases}$$

$$\Rightarrow \tilde{z}_L = \frac{\tilde{V}_L}{\tilde{i}} = \frac{jL\omega\tilde{i}}{\tilde{i}} = jL\omega$$

1.4 Superposition of periodic quantities

The addition of two or more quantities of the same nature is called superposition.

1.4.1 Sinusoidal quantities of the same pulsation

The superposition of two sinusoidal quantities of the same pulsation ω is a sinusoidal pulsation quantity ω.

1.4.2 Example

a) Let the two sinusoidal quantities be:

$$g_1(t)=\sqrt{2}\cos\left(3t-\frac{\pi}{4}\right)\ and\ g_2(t)=\sqrt{2}\sin(3t+\pi)$$

$$g_1(t)+g_2(t)=\sqrt{2}\cos\left(3t-\frac{\pi}{4}\right)+\sqrt{2}\sin(3t+\pi)\rightarrow\sqrt{2}e^{j\left(\omega t-\frac{\pi}{4}\right)}+e^{j\left(\omega t+\pi-\frac{\pi}{2}\right)}=e^{j\omega t}\left(\sqrt{2}e^{j\left(-\frac{\pi}{4}\right)}+e^{j\left(\frac{\pi}{2}\right)}\right)$$

$$=1\times e^{j\omega t}=\cos(\omega t)$$

So : A=1. φ=0.

1.4.3 Sinusoidal quantities of the same amplitudes

The superposition of two sinusoidal quantities of the same *amplitude* is a sinusoidal quantity with modulated amplitude if the two pulsations are *different*.

1.4.4 Example

Let the two sinusoidal quantities be:

$$g_1(t)=a\cos(\omega_1 t)\ and\ g_2(t)=a\sin(\omega_2 t)$$

$$g_1(t)=(a\cos(\omega_1 t))+g_2(t)=(a\sin(\omega_2 t))$$

The superposition is: $$g(t)=g_1(t)+g_2(t)=2a\cos\left(\frac{\omega_1-\omega_2}{2}t\right)\cos\left(\frac{\omega_1+\omega_2}{2}t\right)$$

1.4.5 Any sinusoidal quantities

The superposition of two sinusoidal quantities with different pulsations ω_1 and ω_2 will only be a periodic quantity *if* the ratio between their periods is a *rational* number: $\frac{T_1}{T_2}=\frac{n}{m}$. The resulting period is the smallest common multiple: $T=mT_1+nT_2$.

1.4.6 Example

a) Let the two sinusoidal quantities be: $g_1(t)=5\cos(5t+2)\ \ and\ g_2(t)=2\cos(7t+3)$

Their superposition is: $5\cos(5t+2)+2\cos(7t+3)$.

As $\frac{T_1}{T_2} = \frac{2\pi/5}{2\pi/7} = \frac{n}{m}$ is a rational number (n=7, m=5), the superposition is a periodic quantity of $T = m\left(2\pi/5\right) = n\left(2\pi/7\right) = 2\pi s$.

1.5 Definition of Fourier series

It is possible to express a periodic quantity by a sum of sines and cosines which are simpler to manipulate physically and mathematically. This sum is called the Fourier series (1807).

The Fourier series of a periodic function *f (t)* of period *T* is defined by:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

- ✓ The a_0, the a_n, and the b_n are called the Fourier *coefficients*.
- ✓ The pulsation ω= 2π/*T* is called the *fundamental* pulsation.
- ✓ The higher pulsations $n\omega$ (multiple of ω) are called the *harmonics*.
- ✓ The Fourier coefficients are defined by:

$$a_0 = \frac{1}{T}\int_0^T f(t)dt, \quad a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)dt, \quad b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)dt$$

The graph of a_n and b_n (and sometimes $\sqrt{a_n^2 + b_n^2}$) in terms of $n\omega$ is called the spectrum of the function.

1.5.1 Case of even and odd functions

• Even functions: A function is said to be even if $f(-t) = f(t)$.

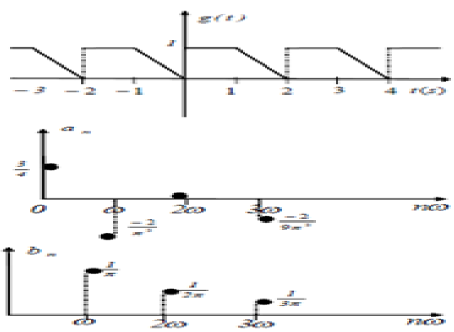
In the Fourier series of even functions, there are only the cosine terms and sometimes the constant a0 which is the average value of the function b_n =0.

• Odd functions: A function is said to be odd if $f(-t) = -f(t)$.

In the Fourier series of odd functions, there are only sine terms:

$a_0 = a_n = 0$.

1.5.2 Example



1. The period of the function is T=2s.

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\left[\int_0^1 (1)\,dt + \int_1^2 (-t+2)dt\right] = \frac{3}{4}.$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\frac{(2\pi n\,t)}{T}dt = \frac{2}{2}\left[\int_0^1 (1)\cos(n\pi\,t)\,dt + \int_1^2 (-t+2)\cos(n\pi\,t)\,dt\right] = \frac{\cos(n\pi)-1}{n^2\pi^2}$$

$$a_n = \frac{(-1)^n - 1}{n^2\pi^2} = \begin{cases} 0 & if\ n \text{ is an even number} \\ \dfrac{-2}{n^2\pi^2} & if\ n \text{ is an odd number} \end{cases}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\frac{(2\pi n\,t)}{T}dt = \frac{2}{2}\left[\int_0^1 (1)\sin(n\pi\,t)\,dt + \int_1^2 (-t+2)\sin(n\pi\,t)\,dt\right] = \frac{1}{\pi n}$$

So the Fourier series is $f(t) = \frac{3}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2\pi^2}\cos(n\omega t) + \sum_{n=1}^{\infty}\frac{1}{n\pi}\sin(n\omega t)$

The spectrum of f (t) will be taken to be the graph of a_n and b_n versus $n\omega$.

1.6 Physical modeling

To understand the vibrational phenomenon, we associate with all physical systems a "mass-spring" system which constitutes an excellent representative model for studying oscillations as follows, figure 1.2:

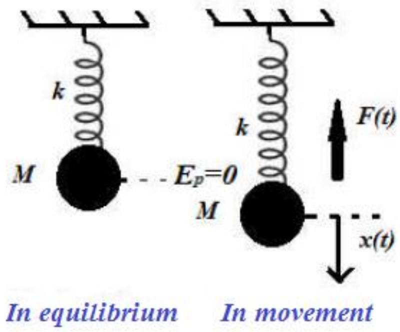


Figure 1.2: Mass-spring diagram.

F(t) is called the restoring force which is proportional to the elongation ***x(t)***. The constant ***k*** is called the stiffness constant.

1.6.1 The representation of several springs

There are two other configurations for the mass-spring system, figure 1.3:

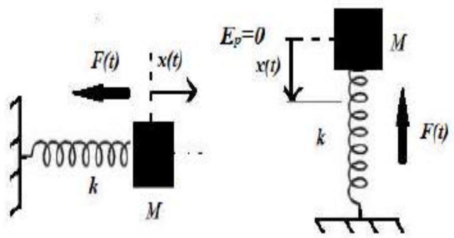


Figure 1.3: Configurations for the mass-spring system.

The representation of several springs is presented in two cases:

- ✓ In parallel, we have the figure 1.4:

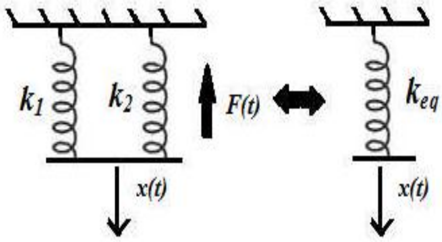


Figure 1.4: Parallel springs.

The equivalent stiffness is the sum of the stiffnesses k_1 and k_2 such that: $k_{eq} = k_1 + k_2$

- ✓ In series, we have figure 1.5

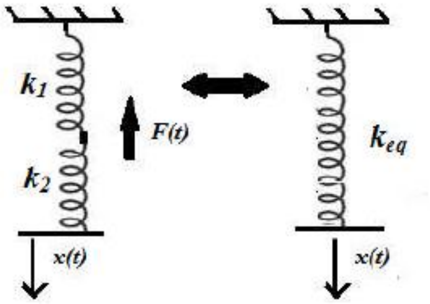


Figure 1.5: Springs in series.

The equivalent stiffness is the sum of the stiffnesses k_1 and k_2 such that: $\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$

An oscillating physical system is identified by the generalized coordinate q which is defined by the deviation from the stable equilibrium position.

We define n the number of degrees of freedom by the number of independent movements of a physical system which determines the number of differential equations of motion.

1.7 Total energy

The total energy of the system is defined by the sum of two types of energy

The kinetic energy of a mechanical system is written in the form:

$$T = E_C = \sum_{n>1} \frac{1}{2} m_i \dot{q}_1^2$$

The potential energy of a mechanical system is written using the limited Taylor expansion in the form:

$$E_p = U = U(0) + \left.\frac{\partial U}{\partial q}\right|_{q=0} q + \frac{1}{2}\left.\frac{\partial^2 U}{\partial q^2}\right|_{q=0} q^2 + \frac{1}{6}\left.\frac{\partial^3 U}{\partial q^3}\right|_{q=0} q^3 + ...+ \frac{1}{n!}\left.\frac{\partial^n U}{\partial q^n}\right|_{q=0} q^n$$

❖ The value q=0 corresponds to the equilibrium position of the system characterized by

$$\left.\frac{\partial U}{\partial q_i}\right|_{q=0} = 0$$

There are two types of equilibrium:

➢ Stable equilibrium, represented by figure 1.6:

❖ In this case, the necessary condition is that:

$$\left.\frac{1}{2}\frac{\partial^2 U}{\partial q^2}\right|_{q=0} > 0$$

.

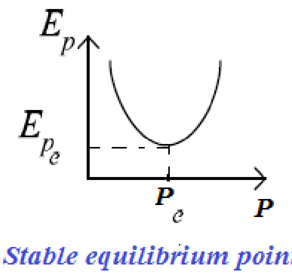


Figure 1.6: Stable equilibrium.

➢ Unstable equilibrium represented by figure 1.7

❖ In this case, the necessary condition is that

$$\left.\frac{1}{2}\frac{\partial^2 U}{\partial q^2}\right|_{q=0} < 0 .$$

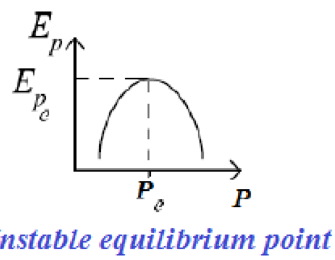


Figure 1.7: Unstable equilibrium.

➢ The oscillatory movement is said to be linear if this deviation is infinitesimal. For this purpose the potential energy takes the quadratic form as a function of the deviation from the equilibrium position represented as follows:

$$U = E_p = \left.\frac{1}{2}\frac{\partial^2 U}{\partial q^2}\right|_{q=0} q^2$$

The constant $\frac{\partial^2 U}{\partial q^2}$ is called the recall constant.

So; the restoring force takes the linear form as a function of the elongation and opposite to the movement such that:

$$\vec{F}(t) = -\left.\frac{\partial^2 U}{\partial q^2}\right|_{q=0} \vec{q}$$

1.8 Calculation methods

The equation of motion for a conservative system can be determined by three methods:

1. ***Principle of total energy conservation***

$$E_T = E_c + E_p = T + U = Constant \quad \Rightarrow \quad \frac{dE_T}{dt} = 0$$

Where E_T is called the total energy of the system.

2. ***Newton's dynamic law:***

$$\sum_{n>1} \vec{F}_i = m_i \vec{a}_i$$

Where $\vec{a}_i$ is called the acceleration of the components of the system.

3. ***Lagrange-Euler method:*** $L(q,\dot{q}) = T - U = E_c - E_p = Constant$

Where L: is the Lagrangian of the system.

In the case of a so-called conservative system, the forces derive from a potential.

We define the action of the system as the summation, between the time interval, t_0, t_1 along the path of the system, of the difference between the kinetic energy and the potential energy.

$$\Gamma = \int_{t_0}^{t_1} L(q,\dot{q})\,dt$$

The path is determined using a variational method. This method results in the Euler-Lagrange equations which give paths on which the action is minimal.

- By applying the principle of least action $\partial\Gamma = 0$, we obtain the Euler-Lagrange equation for a conservative system as follows:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = 0 \quad i = 1, n$$

The equation of motion for a dissipative (non-conservative) system can be determined as follows:

- ✓ ***System in translation:***

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\left(\frac{\partial L}{\partial q_i}\right)=\overline{\sum\left|\vec{F}_{ext}\right|} \quad i=1,n$$

Where $\vec{F}_{ext}$ are the external forces applied to the system.

- ✓ ***Rotating system:***

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\left(\frac{\partial L}{\partial q_i}\right)=\overline{\sum\left|\vec{M}_{ext}\right|} \quad i=1,n$$

Where $\vec{M}_{ext}$ are the external moments applied to the system. In this case the forces do not derive from a potential.

Chapter 2:

Free linear systems with one degree of freedom

2.1 Free oscillators

A system oscillating in the absence of any excitation force is called a free oscillator. The number of independent quantities involved in the movement is called the degree of freedom.

An isolated system oscillating with one degree of freedom is determined by the generalized coordinate q which is the deviation from the stable equilibrium.

2.2 Harmonic oscillator

In mechanics, we call a harmonic oscillator which, as soon as it is separated from its equilibrium position by a distance x (or angle θ), is subject to a restoring force opposite and proportional to the separation x (or θ):

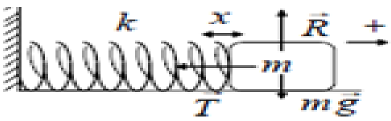
$$F = -Cx$$

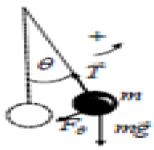
C: A *positive* constant

2.2.1 Examples

a) The mass-spring system opposite is a harmonic oscillator because the restoring force is $T = -kx$.

b) The restoring force of the simple pendulum is $F_\theta = -mg\sin\theta$. The pendulum becomes a harmonic oscillator when $\theta \ll 1$: $F_\theta = -mg\sin\theta \approx -mg\theta$.





2.3 Proper pulsation of a harmonic oscillator

We define the harmonic oscillation by the following differential equation:

$$\ddot{q}(t) + \omega_0^2 q(t) = 0$$

(In mechanics $q = x, y, z, \theta, \varphi$ .... In electricity $q = i, u, q$). where ω_0 is called the system's proper pulsation. We define the proper period T_0 as follows: $T_0 = \dfrac{2\pi}{\omega_0}$

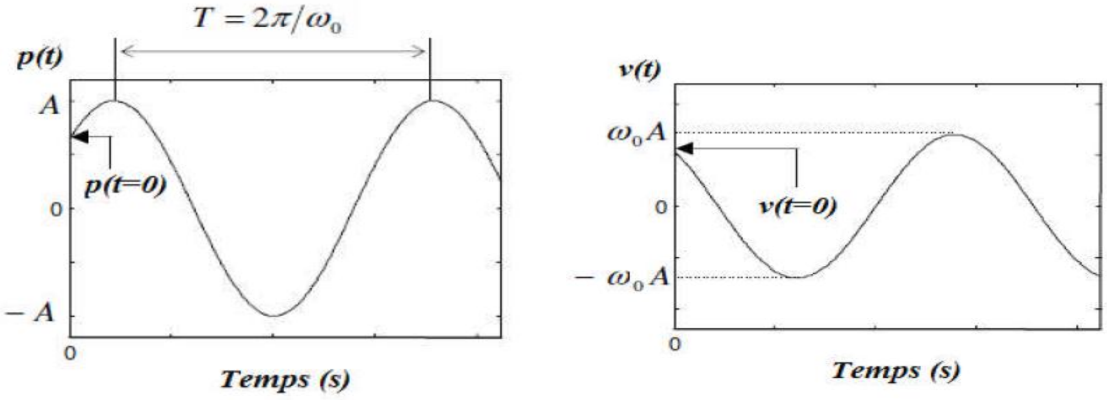
The solution to this differential equation is in sinusoidal form such that:

$$q(t) = A\cos(\omega_0 t + \phi)$$

Where ω_0 is called proper pulsation because it only depends on the quantities specific to the oscillator.

A represents the amplitude of the oscillations and ϕ is the phase shift. The constants A and ϕ are determined by the following initial conditions:

$$\begin{cases} q(t=0) = q_0 \\ \dot{q}(t=0) = \dot{q}_0 \end{cases}$$



A- Position response *B- Speed Response*

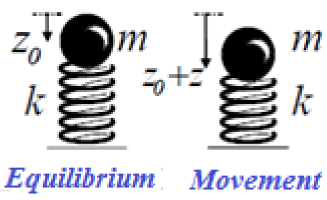
Figure 2.1: Free oscillatory movement.

It should be noted that all low amplitude oscillations around the equilibrium position can be assimilated to linear movements and the potential energy can be expressed in quadratic form of the generalized coordinate q.

On the other hand, beyond a certain amplitude the oscillation becomes non-linear. Some examples of applications:

2.3.1 Examples

a) Using the PFD, find the equation of motion of the system opposite.



- ✓ Calculate its own pulsation for m=1kg et k=3N/m.
- ✓ Find the amplitude A and the phase $\emptyset$ knowing that initially the mass is pushed 2cm downwards then launched upwards at a speed of 2cm/s.

Solution

- ✓ PFD in equilibrium: $\sum \vec{F} = \vec{0} \Rightarrow m\vec{g} + \vec{T} = \vec{0} \Rightarrow mg - kz_0 = 0$.
- ✓ PFD in movement: $\sum \vec{F} = m\,\vec{a} \Rightarrow m\vec{g} + \vec{T} = m\,\vec{a} \Rightarrow mg - k\,(z + z_0) = m\ddot{z}$.

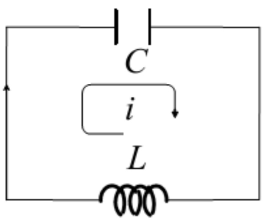
Thanks to the equilibrium equation $mg - kz_0 = 0$, the equation of motion simplifies :

$$\ddot{z} + \frac{k}{m} z = 0$$

- ✓ The proper pulsation is $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{3}$ rad/s.
- ✓ The time equation is: $z(t) = A\cos(\omega_0 t + \phi) = A\cos\left(\sqrt{3}t + \phi\right)$. Let's use the initial conditions to find A and $\emptyset$:

$$\begin{cases} z(0) = A\cos\phi = 2cm \\ \dot{z}(0) = -A\sqrt{3}\sin\phi = -2cm/s \end{cases} \Rightarrow \begin{cases} \tan\phi = \frac{1}{\sqrt{3}} \Rightarrow \phi = \frac{\pi}{6} \\ A = \frac{2}{\cos\phi} = \frac{2}{\cos\frac{\pi}{6}} \approx \frac{2}{0.866} \simeq 1.155cm/s \end{cases}$$

b) Using the mesh law, find the equation of motion of the charge q in the circuit opposite, then deduce the natural pulsation ω_0.



Solution

The law of meshes is written:

$$\sum V_i = 0 \Rightarrow V_L + V_C = 0$$

$$V_C = \frac{q}{C} \; et \; V_L = L\frac{di}{dt}, \mathrm{i} = \frac{dq}{dt} = \dot{q}$$

$$V_L = L\frac{d}{dt}\dot{q} = L\ddot{q}$$

$$L\ddot{q} + \frac{q}{C} = 0 \Rightarrow L\ddot{q} + \frac{1}{C}q = 0 \Rightarrow \begin{cases} \ddot{q} + \frac{1}{LC}q = 0 \\ \ddot{q} + \omega_0^2 q = 0 \end{cases}$$

$$\omega_0^2 = \frac{1}{LC} \Rightarrow \omega_0 = \sqrt{\frac{1}{LC}}$$

The proper pulsation is therefore $\omega_0 = \frac{1}{\sqrt{LC}}$

2.4 The energy of a harmonic oscillator

The energy of a harmonic oscillator is the sum of its kinetic and potential energies:

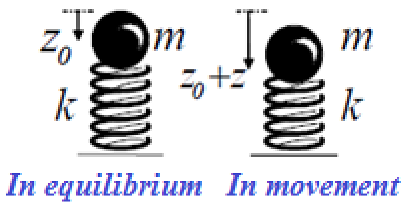
$$E = T + U$$

- ✓ The kinetic energy of translation of a body of mass m and speed v is $T_{translation} = \frac{1}{2} m v^2$. For a spool $T = \frac{1}{2} L i^2$.
- ✓ The kinetic energy of rotation of a Body with moment of inertia I_Δ around an axis Δ and angular velocity $\dot{\theta}$ is $T_{translation} = \frac{1}{2} I_\Delta \dot{\theta}^2$
- ✓ The potential energy of a mass m in a constant gravitational field g is: $U_{mass} = + mgh$ during an ascent of a height h.
- ✓ The potential energy of a coil spring of stiffness k lors of a deformation d is $U_{\text{spring}} = \frac{1}{2} k d^2$. For a capacitor $U = \frac{1}{2} \frac{1}{C} q^2$.
- ✓ The potential energy of a torsion spring of stiffness k during deformation θ is: $U_{\text{spring}} = \left(\frac{1}{2}\right) k \theta^2$.

Noticed : Total energy $E = T + U$ is preserved (constant) during the movement: $\frac{dE}{dt} = 0$

This *conservation equation* gives the equation of motion of conserved systems.

2.4.1 Example



Solution

$$T = \frac{1}{2} m \dot{z}^2$$

$$U = \frac{1}{2} k (z + z_0)^2 - mg (z + z_0)$$

$$U=\frac{1}{2}kz^2+kzz_0+\frac{1}{2}kz_0^2-mgz-mgz_0=\frac{1}{2}kz^2+kzz_0-mgz+\frac{1}{2}kz_0^2-mgz_0$$

$$U=\frac{1}{2}kz^2+\left(kz_0-mg\right)z+\frac{1}{2}kz_0^2-mgz_0$$

Thanks to the equilibrium condition

$$\left.\frac{\partial U}{\partial z}\right|_{z=0}=\left.\left(\frac{1}{2}2kz+kz_0-mg=0\right)\right|_{z=0}$$

$$z=0:kz_0-mg=0$$

Then U simplifies

$$U=\frac{1}{2}kz^2+\frac{1}{2}kz_0^2-mgz_0\Rightarrow U=\frac{1}{2}kz^2+Cte$$

$$E=T+U=\frac{1}{2}m\dot{z}^2+\frac{1}{2}kz^2+Cte\,.$$

$$\frac{dE}{dt}=0=\frac{d}{dt}\left(\frac{1}{2}m\dot{z}^2+\frac{1}{2}kz^2+Cte\right)$$

$$\frac{dE}{dt}=\frac{d}{dt}\left(\frac{1}{2}m\dot{z}^2\right)+\frac{d}{dt}\left(\frac{1}{2}kz^2\right)=0$$

$$\frac{dE}{dt}=\left(\frac{1}{2}m\,2\dot{z}\,\ddot{z}\right)+\left(\frac{1}{2}k\,2z\,\dot{z}\right)=0$$

$$m\dot{z}\,\ddot{z}+kz\,\dot{z}=0\Rightarrow m\,\ddot{z}+kz=0\Rightarrow\ddot{z}+\frac{k}{m}z=0$$

Which is the equation of motion found using the PFD.

2.5 Equilibrium condition

The equilibrium condition is $F=0$ If the equilibrium is at $x = x_0$, we write $\left.F\right|_{x=x_0}=0$. For a force deriving from a potential $\left(-\frac{\partial U}{\partial x}\right)$, the equilibrium condition is written:

$$\left.\frac{\partial U}{\partial x}\right|_{x=x_0}>0\,.$$

The equilibrium of a system is stable if, once removed from its equilibrium position, it returns. The system returns to its equilibrium if F is a restoring force. Since $F=-Cx$ we will have a restoring force if $C>0$.

As $\left(-\frac{\partial F}{\partial x}\right)=\left(\frac{\partial^2 U}{\partial x^2}\right)$, the *stable* equilibrium condition is written:

$$\left(\frac{\partial^2 U}{\partial x^2}\right)\Bigg|_{x=x_0} > 0.$$

This condition is also an oscillation condition.

The equilibrium of a system is *unstable* if the system does not regain its equilibrium during a deviation, i.e. if C <0 the unstable equilibrium condition is therefore written:

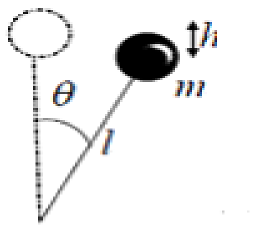
$$\left(\frac{\partial^2 U}{\partial x^2}\right)\Bigg|_{x=x_0} < 0$$

For the rotations the previous equations become:

$$\left(\frac{\partial U}{\partial \theta}\right)\Bigg|_{\theta=\theta_0} > 0,\ \left(\frac{\partial^2 U}{\partial \theta^2}\right)\Bigg|_{\theta=\theta_0} > 0,\ \left(\frac{\partial^2 U}{\partial \theta^2}\right)\Bigg|_{\theta=\theta_0} < 0.$$

2.5.1 Example

Find the equilibrium positions and their nature for the system opposite.



Solution

The potential energy during a separation θ from the vertical is:

$U = -mgh = -mgl\,(1-\cos\theta)$. The equilibrium positions are given by

$$\frac{\partial U}{\partial \theta} = 0 \Rightarrow \frac{\partial}{\partial \theta}\left(-mgl\,(1-\cos\theta)\right) = -mgl\,\frac{\partial}{\partial \theta}(1-\cos\theta) = -mgl\,\sin\theta = 0 \Rightarrow \sin\theta = 0$$

The equilibrium positions are therefore: $\theta=0$ or $\theta=\pi$.

$$U - mgl\,(1-\cos\theta)$$

$$\frac{\partial U}{\partial \theta} = -mgl\sin\theta$$

$$\left(\frac{\partial^2 U}{\partial \theta^2}\right) = \frac{\partial}{\partial \theta}(-mgl\sin\theta) = -mgl\,\frac{\partial}{\partial \theta}\sin\theta = -mgl\cos\theta$$

$$So\left(\frac{\partial^2 U}{\partial \theta^2}\right) = -mgl\cos\theta$$

$\left(\frac{\partial^2 U}{\partial \theta^2}\right)\Bigg|_{\theta=0} = -mgl\,\cos 0 = -mgl < 0$ So $\theta=0$ is an unstable equilibrium position

$$\left(\frac{\partial^2 U}{\partial\theta^2}\right)\Bigg|_{\theta=\pi} = -mgl\cos\pi = +mgl > 0$$ Which implies $\theta=\pi$ is a stable equilibrium position.

2.6 Lagrange equation (1788)

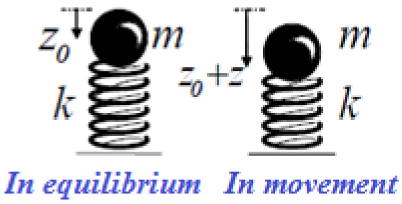
The Lagrange equation (also called the Euler-Lagrange equation) is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) = 0$$

L=T-U is called the *Lagrangian*

The Lagrange equation also directly gives the equation of motion (For translations $q=x$, y, z. For rotations $q=\theta$, φ,… In electricity $q=q$).

2.6.1 Example



$$T = \frac{1}{2}m\dot{z}^2$$

$$U = \frac{1}{2}kz^2 + Cte$$

The Lagrangian is: $L = T - U = \frac{1}{2}m\dot{z}^2 - \frac{1}{2}kz^2 + Cte$

The equation of motion is therefore:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \left(\frac{\partial L}{\partial z}\right) = 0$$

With $L = T - U = \frac{1}{2}m\dot{z}^2 - \frac{1}{2}kz^2 + cte$

$$\frac{\partial L}{\partial \dot{z}} = \frac{\partial}{\partial \dot{z}}\left(\frac{1}{2}m\dot{z}^2\right) = \frac{1}{2}m\frac{\partial}{\partial \dot{z}}\left(\dot{z}^2\right) = \frac{1}{2}m(2\dot{z}) = m\dot{z} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = \frac{d}{dt}(m\dot{z}) = m\frac{d}{dt}(\dot{z}) = m\ddot{z}$$

$$\frac{\partial L}{\partial z} = \frac{\partial}{\partial z}\left(-\frac{1}{2}kz^2\right) = -\frac{1}{2}k\frac{\partial}{\partial z}\left(z^2\right) = -\frac{1}{2}k(2z) = -kz$$

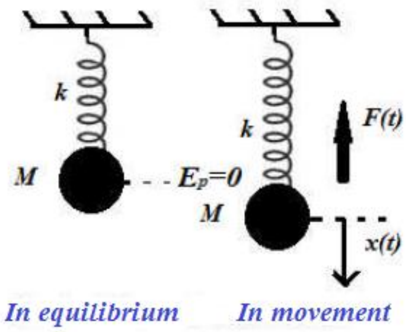
So $m\ddot{z} + kz = 0$

$$\begin{cases} \ddot{z} + \dfrac{k}{m}z = 0 \\ \ddot{z} + \omega_0^2 z = 0 \end{cases} \Rightarrow \omega_0^2 = \frac{k}{m} \Rightarrow \omega_0 = \sqrt{\frac{k}{m}}$$

Which is indeed the equation obtained using the PFD then using the conservation equation.

The solution to the differential equation is then written: $z(t) = A\cos(\omega_0 t + \varphi)$

2.6.1.1 Springs:



The kinetic energy is written:

$$T = E_c = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$$

The potential energy for small oscillations is written in the form:

$$E_p = U = U_k + U_m$$

$$U_k = \frac{k}{2}x^2 + kxx_0 + cte$$

$$U_m = -mg(x + x_0) = -mgx - mgx_0$$

$$U = U_k + U_m = \frac{k}{2}x^2 + kxx_0 - mgx - mgx_0 + cte$$

$$U = \frac{k}{2}x^2 + (kx_0 - mg)x - mgx_0 + cte \text{ with } -mgx_0 = cte$$

$$U = \frac{k}{2}x^2 + (kx_0 - mg)x + cte$$

In equilibrium

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = 0$$

$$\frac{\partial U}{\partial x} = \frac{k}{2}\times 2x + (kx_0 - mg)$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} : x = 0 \Rightarrow \frac{k}{2}\times 2(x=0) + (kx_0 - mg)$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = 0 \Rightarrow (kx_0 - mg) = 0 \Rightarrow x_0 = \frac{mg}{kx_0}$$

$$U = \frac{k}{2}x^2 + cte$$

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + cte$$

The equation of motion is of the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\left(\frac{\partial L}{\partial x}\right)=0$$

$$With\ L=T-U=\frac{1}{2}m\dot{x}^2-\frac{1}{2}kx^2+cte$$

$$\frac{\partial L}{\partial \dot{x}}=\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right)=\frac{1}{2}m\frac{\partial}{\partial \dot{x}}\left(\dot{x}^2\right)=\frac{1}{2}m\left(2\dot{x}\right)=m\dot{x}\Rightarrow\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=\frac{d}{dt}\left(m\dot{x}\right)=m\frac{d}{dt}\left(\dot{x}\right)=m\ddot{x}$$

$$\frac{\partial L}{\partial x}=\frac{\partial}{\partial x}\left(-\frac{1}{2}kx^2\right)=-\frac{1}{2}k\frac{\partial}{\partial x}\left(x^2\right)=-\frac{1}{2}k\left(2x\right)=-kx$$

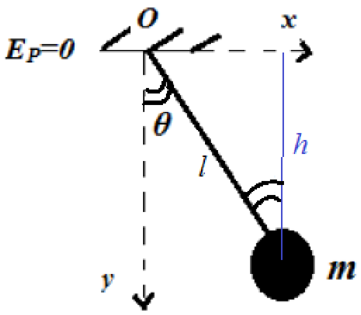
So $m\ddot{x}+kx=0$

$$\begin{cases}\ddot{x}+\dfrac{k}{m}x=0\\ \ddot{x}+\omega_0^2x=0\end{cases}\Rightarrow\omega_0^2=\frac{k}{m}\Rightarrow\omega_0=\sqrt{\frac{k}{m}}$$

The solution to the differential equation is then written:

$$x\left(t\right)=A\cos\left(\omega_0t+\varphi\right)$$

2.6.1.2 Simple pendulum



$$o\vec{m}=\begin{cases}x=l\sin\theta\\ y=l\cos\theta\end{cases}\Rightarrow\vec{v}=\begin{cases}\dot{x}=l\dot{\theta}\cos\theta\\ \dot{y}=-l\dot{\theta}\sin\theta\end{cases}$$

$$\left|v\right|^2=\dot{x}^2+\dot{y}^2=\left(l\dot{\theta}\right)^2$$

The kinetic energy is written:

$$T=E_c=\frac{1}{2}mv^2=\frac{1}{2}m\dot{x}^2$$

$$\dot{x}=l\dot{\theta}$$

$$T=\frac{1}{2}m\left(l\dot{\theta}\right)^2=\frac{1}{2}ml^2\dot{\theta}^2$$

For the potential energy we have:

$$U_m=-mgh$$

$$\cos\theta=\frac{h}{l}\Rightarrow h=l\cos\theta\Rightarrow U_m=-mgl\cos\theta$$

So $U = E_p = -mgl\cos\theta + cte$

Then, the Lagrangian of the system is written:

$$L = E_c - E_p = T - U = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta + cte$$

The equation of motion for small oscillations is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) = 0$$

With $L = E_c - E_p = T - U = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta + cte$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}ml^2\dot{\theta}^2\right) = \frac{1}{2}ml^2\frac{\partial}{\partial \dot{\theta}}\left(\dot{\theta}^2\right) = \frac{1}{2}ml^2\left(2\dot{\theta}\right) = ml^2\dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(ml^2\dot{\theta}\right) = ml^2\frac{d}{dt}\left(\dot{\theta}\right) = ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = \frac{\partial}{\partial \theta}\left(mgl\cos\theta\right) = mgl\frac{\partial}{\partial \theta}\left(\cos\theta\right) = -mgl\sin\theta$$

So $ml^2\ddot{\theta} + mgl\sin\theta = 0$

At low amplitude $\sin\theta \approx \theta$

So : $ml^2\ddot{\theta} + mgl\theta = 0$

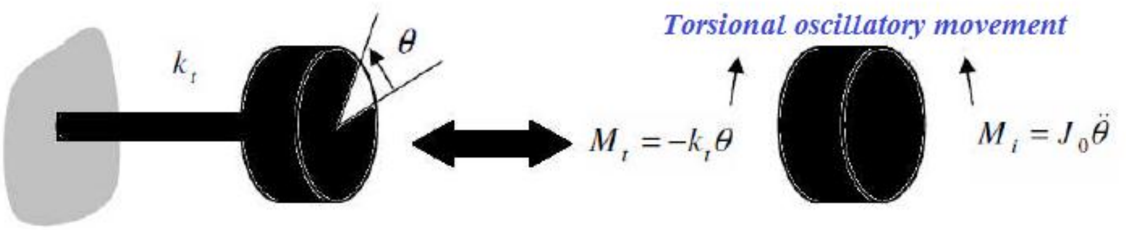
$$\begin{cases} \ddot{\theta} + \frac{g}{l}\theta = 0 \\ \ddot{\theta} + \omega_0^2\theta = 0 \end{cases}$$

$$\omega_0^2 = \frac{g}{l} \Rightarrow \omega_0 = \sqrt{\frac{g}{l}}$$

The solution to the differential equation is then written: $\theta(t) = A\cos(\omega_0 t + \varphi)$

2.6.1.3 Torsion system:

A rigid body with moment of inertia J_0 oscillates around an axis with a torsion constant k_t.



The kinetic energy is written: $E_c = T = \frac{1}{2}J_0\dot{\theta}^2$

For the potential energy we have: $E_p = U = \frac{1}{2}k_t\theta^2$

The Lagrangian of the system is then written: $L = E_c - E_p = T - U = \frac{1}{2}J_0\dot{\theta}^2 - \frac{1}{2}k_t\theta^2 + cte$

The differential equation is written:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) = 0 \quad with: L = E_c - E_p = \frac{1}{2}J_0\dot{\theta}^2 - \frac{1}{2}k_t\theta^2 + cte$$

$$\frac{\partial L}{\partial \dot{\theta}} = J_0\dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = J_0\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -k_t\theta$$

$$J_0\ddot{\theta} + k_t\theta = 0$$

$$\begin{cases} \ddot{\theta} + \frac{k_t}{J_0}\theta = 0 \\ \ddot{\theta} + \omega_0^2\theta = 0 \end{cases} \Rightarrow \omega_0^2 = \frac{k_t}{J_0} \Rightarrow \omega_0 = \sqrt{\frac{k_t}{J_0}}$$

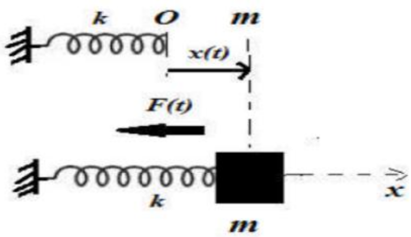
The solution to the differential equation is then written: $\theta(t) = \theta_0 \cos(\omega_0 t + \varphi)$

2.6.2 Exercises and problems

Exercise No. 1

Compare between the own pulsation of a *k+m* mechanical system and an *LC* electrical system

We give: k=100 *N/m*, *m*=250*g*, *L*=0.1 *H*, *C*=100 *pF*.



Solution :

$$T = E_c = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$$

$$E_p = U = U_k + U_m$$

$U_m=0$ because we have: ($U_m=mgh$ and $h=0$).

$$U = \frac{k}{2}x^2 + kxx_0 + cte$$

In equilibrium

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = 0 \Rightarrow \frac{\partial U}{\partial x} = \frac{k}{2}\times 2x + kx_0$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} : x = 0 \Rightarrow \frac{k}{2}\times 2(x=0) + kx_0 \Rightarrow \left.\frac{\partial U}{\partial x}\right|_{x=0} = 0 \Rightarrow kx_0 = 0 \Rightarrow x_0 = 0$$

$$U = \frac{k}{2}x^2 + cte$$

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + cte$$

The equation of motion is of the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0$$

$$With\ L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + cte$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right) = \frac{1}{2}m\frac{\partial}{\partial \dot{x}}\left(\dot{x}^2\right) = \frac{1}{2}m\left(2\dot{x}\right) = m\dot{x} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}\left(m\dot{x}\right) = m\frac{d}{dt}\left(\dot{x}\right) = m\ddot{x}$$

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{2}kx^2\right) = -\frac{1}{2}k\frac{\partial}{\partial x}\left(x^2\right) = -\frac{1}{2}k\left(2x\right) = -kx$$

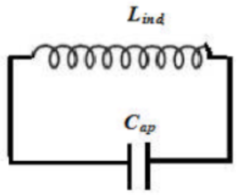
So $m\ddot{x} + kx = 0$

$$\begin{cases} \ddot{x} + \frac{k}{m}x = 0 \\ \ddot{x} + \omega_0^2 x = 0 \end{cases} \Rightarrow \omega_0^2 = \frac{k}{m} \Rightarrow \omega_0 = \sqrt{\frac{k}{m}}$$

k = 100 N/m, m=250 g=0.250 kg.

$$\omega_0\left(\text{mechanical}\right) = \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{0.250}} = 20\ rad.s^{-1}$$

2/LC electrical circuit



$$\sum V_i = 0 \Rightarrow V_L + V_C = 0$$

$$V_C = \frac{q}{C}, V_L = L\frac{di}{dt}, \text{i} = \frac{dq}{dt} = \dot{q} \Rightarrow V_L = L\frac{d}{dt}\dot{q} = L\ddot{q}$$

$$L\ddot{q} + \frac{q}{C} = 0 \Rightarrow L\ddot{q} + \frac{1}{C}q = 0 \Rightarrow \ddot{q} + \frac{1}{LC}q = 0$$

$$\ddot{q} + \omega_0^2 q = 0 / \omega_0^2 = \frac{1}{LC} \Rightarrow \omega_0 = \sqrt{\frac{1}{LC}}$$

L=0.1 H, C=100 pF=$100\ 10^{-12}$ F.

$$\omega_0\left(\text{electric}\right) = \omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{0.1\times100\times10^{-12}}} = 3.16\times10^{16}\ rad.s^{-1}$$

$$\omega_0\left(\text{electric}\right) >>>> \omega_0\left(\text{mechanical}\right)$$

Electric circuits have a fairly broad frequency spectrum compared to that of electrical spectra.

Exercise No. 2

A vibratory system (k+m) (arranged horizontally), consisting of a mass m=0.01 kg and a spring k=36Nm^{-1}. At time t=0 we observe that the mass is 50 mm to the right of its equilibrium position and is still moving to the right with a speed of 1.7ms^{-1}. Calculate the frequency, amplitude, initial phase and energy of the system. A second system identical to the first is vibrated with the same amplitude but with a phase advance of $\frac{\pi}{2}$ calculate the displacement and speed at $t=0$. At what moment will it then pass through the equilibrium position?

Solution

The equation of motion of a vibrational system k+m (arranged horizontally) is

$$\begin{cases} \ddot{x}+\dfrac{k}{m}x=0 \\ \ddot{x}+\omega_0^2 x=0 \end{cases}$$

$$x(t)=A\cos(\omega_0 t+\varphi)$$

$$\omega_0^2=\frac{k}{m}\Rightarrow\omega_0=\sqrt{\frac{k}{m}}=\sqrt{\frac{36}{0.01}}=60\ rad.s^{-1}$$

Frequency calculation f

$$\omega_0=2\pi f\Rightarrow f=\frac{\omega_0}{2\pi}=\frac{60}{2\pi}=9.55\ Hertz$$

Calculation of A and φ

$$\begin{cases} x(t)=A\cos(\omega_0 t+\varphi) \\ v(t)=\dfrac{dx(t)}{dt}=\dot{x}(t)=\dfrac{d}{dt}\left[A\cos(\omega_0 t+\varphi)\right] \end{cases}\Rightarrow\begin{cases} x(t)=A\cos(\omega_0 t+\varphi) \\ \dot{x}(t)=-A\omega_0\sin(\omega_0 t+\varphi) \end{cases}$$

In the initial conditions $\begin{cases} x(t=0)=A\cos(\omega_0\times 0+\varphi) \\ \dot{x}(t=0)=-A\omega_0\sin(\omega_0\times 0+\varphi) \end{cases}\Rightarrow\begin{cases} x(t=0)=A\cos(\varphi) \\ \dot{x}(t=0)=-A\omega_0\sin(\varphi) \end{cases}$

Digital Application $\begin{cases} x(t=0)=A\cos(\varphi)=50mm=50.10^{-3}m \quad (1) \\ \dot{x}(t=0)=-A\omega_0\sin(\varphi)=1.7m/s \quad (2) \end{cases}$

$$\begin{cases} x(t=0)=A\cos(\varphi)=50mm=50.10^{-3}m \quad (1) \\ \dot{x}(t=0)=-A\omega_0\sin(\varphi)=1.7m/s \quad (2) \end{cases}$$

$$\frac{(2)}{(1)}=\frac{-A\omega_0\sin(\varphi)}{A\cos(\varphi)}=\frac{1.7}{50.10^{-3}}=-\omega_0\tan(\varphi)$$

$$\omega_0\tan(\varphi)=\frac{-1.7}{50.10^{-3}}\Rightarrow\tan(\varphi)=\frac{-1.7}{50.10^{-3}}\frac{1}{\omega_0}=\frac{-1.7}{50.10^{-3}}\frac{1}{60}\Rightarrow\varphi=-\frac{\pi}{6}rad$$

According to (1) : $A\cos(\varphi) = A\cos\left(-\frac{\pi}{6}\right) = 50.10^{-3} \Rightarrow A = \frac{50.10^{-3}}{\cos\left(-\frac{\pi}{6}\right)} = 0.057m$

Calculation of maximum system energy

$$E_T = T + U$$
$$E_T(Max) = U \ and \ (T = 0)$$

Since the energy is maximum when the elongation of the spring is maximum and in this instant the spring changes direction i.e. v=0 this implies that the kinetic energy becomes zero.

$$E_T(Max) = U = \frac{1}{2}kx^2$$
$$x = 0.057m, \ k = 36$$
$$E_T(Max) = \frac{1}{2}36(0.057)^2 = 0.06 \ J$$

Calculation of the position and speed of the second system t=0:

For the second system: It has the same amplitude as the first system, that is to say A does not change but with a phase advance of $\frac{\pi}{2}$ that's to say φ becomes $\varphi + \frac{\pi}{2}$

So $\begin{cases} x(t) = A\cos\left(\omega_0 t + \left(\varphi + \frac{\pi}{2}\right)\right) \\ \dot{x}(t) = -A\omega_0 \sin\left(\omega_0 t + \left(\varphi + \frac{\pi}{2}\right)\right) \end{cases}$

In the initial conditions

$$\begin{cases} x(t=0) = A\cos\left(\omega_0 \times (t=0) + \left(\varphi + \frac{\pi}{2}\right)\right) \\ \dot{x}(t=0) = -A\omega_0 \sin\left(\omega_0 \times (t=0) + \left(\varphi + \frac{\pi}{2}\right)\right) \end{cases} \Rightarrow \begin{cases} x(t=0) = A\cos\left(\left(\varphi + \frac{\pi}{2}\right)\right) \\ \dot{x}(t=0) = -A\omega_0 \sin\left(\left(\varphi + \frac{\pi}{2}\right)\right) \end{cases}$$

Digital Application

$$\begin{cases} x(t=0) = A\cos\left(\left(\varphi + \frac{\pi}{2}\right)\right) = A\cos\left(-\frac{\pi}{6} + \frac{\pi}{2}\right) = A\cos\left(\frac{\pi}{3}\right) = 0.057\cos\left(\frac{\pi}{3}\right) = 0.0288m \\ \dot{x}(t=0) = -A\omega_0 \sin\left(\left(\varphi + \frac{\pi}{2}\right)\right) = -A\omega_0 \sin\left(-\frac{\pi}{6} + \frac{\pi}{2}\right) = -A\omega_0 \sin\left(\frac{\pi}{3}\right) = -0.057 \times 60 \times \sin\left(\frac{\pi}{3}\right) = -2.9 \ m/s \end{cases}$$

So the second system at $t=0$ is 28.8 mm from the equilibrium position and moves with a speed of 3m/s.

At what moment will it then pass through the equilibrium position?

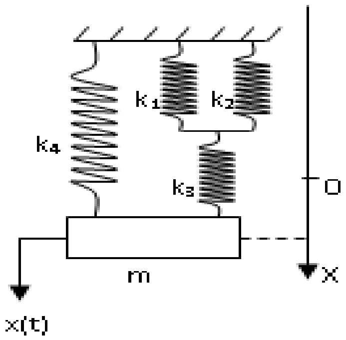
$$x(t) = A\cos\left(\omega_0 t + \varphi + \frac{\pi}{2}\right) = A\cos\left(\omega_0 t + \frac{\pi}{3}\right) = 0 \Rightarrow \cos\left(\omega_0 t + \frac{\pi}{3}\right) = 0$$

$$\cos(\alpha) = 0 \Rightarrow \alpha = \frac{\pi}{2}$$

Likewise
$$\cos\left(\omega_0 t + \frac{\pi}{3}\right) = 0 \Rightarrow \omega_0 t + \frac{\pi}{3} = \frac{\pi}{2}$$
$$\omega_0 t = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6} \Rightarrow t = \frac{\pi}{6} \times \frac{1}{\omega_0} = \frac{\pi}{6} \times \frac{1}{60} = 0.01\,s$$

Exercise No. 3

Simplify the system in the following figure by replacing the springs with an equivalent spring k_e with $k_1=k_2=k_3=k$ and $k_4=2k$. Deduce the nature of the movement and its own pulsation ω_0 which we ask to calculate knowing that $k = 150$ N/m, m=1kg.



Solution

k_1 is in parallel with k_2 so $k_{1(equivalent)} = k_1 + k_2 = 2k$

$k_{1(equivalent)}$ is in series with k_3

$$\frac{1}{k_{2(equivalent)}} = \frac{1}{k_{1(equivalent)}} + \frac{1}{k_3} = \frac{1}{2k} + \frac{1}{k} = \frac{3}{2k}$$

$$k_{2(equivalent)} = \frac{2k}{3}$$

$k_{2(equivalent)}$ is in parallel with k_4

$$k_{3(equivalent)} = k_{2(equivalent)} + k_4 = \frac{2k}{3} + 2k = \frac{8k}{3}$$

$$k_{3(equivalent)} = k_{(equivalent)} = k_e$$

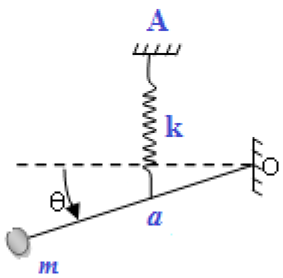
$$\begin{cases} \ddot{x} + \dfrac{k}{m}x = 0 \\ \ddot{x} + \omega_0^2 x = 0 \end{cases}$$

$$\omega_0^2 = \frac{k}{m} \Rightarrow \omega_0 = \sqrt{\frac{k_e}{m}} = \sqrt{\frac{8k}{3m}} = 20\,rad.s^{-1}$$

Exercise No. 4

A rod of length *l* and negligible mass articulated at point *O* carrying at its free end a point mass *m*. At a distance *a* of *O* from the rod we attach vertically a spring of stiffness k, the other end being fixed to a fixed frame at point *A*. At static equilibrium the rod takes a horizontal position ($\theta=0$).

1- Say if at this position is the spring extended or not? deduce the equilibrium condition.
2- Establish the differential equation of weak oscillations and deduce their period.

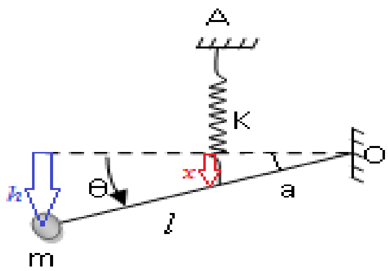


Solution

$$U = U_k + U_m$$

The potential energy of the rod equals zero because the mass of the rod is neglected.

$$U_k = \frac{k}{2}x^2 + kxx_0 + cte$$



With $\sin\theta = \frac{x}{a} \Rightarrow x = a\sin\theta$

$$U_k = \frac{k}{2}x^2 + kxx_0 + cte = \frac{k}{2}(a\sin\theta)^2 + k(a\sin\theta)x_0 + cte$$

$$U_m = -mgh \text{ with } \sin\theta = \frac{h}{l} \Rightarrow h = l\sin\theta \Rightarrow U_m = -mgl\sin\theta$$

$$U = U_k + U_m = \frac{k}{2}(a\sin\theta)^2 + k(a\sin\theta)x_0 - mgl\sin\theta + cte$$

$$U = \frac{k}{2}a^2(\sin\theta^2) + kax_0(\sin\theta) - mgl(\sin\theta) + cte \Rightarrow U = \frac{k}{2}a^2(\sin\theta^2) + (kax_0 - mgl)(\sin\theta) + cte$$

At low amplitude $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{\theta^2}{2}$

$$U = \frac{k}{2}a^2\left(\theta^2\right) + \left(kax_0 - mgl\right)\left(\theta\right) + cte$$

In equilibrium

$$\left.\frac{\partial U}{\partial \theta}\right|_{\theta=0} = 0 \Rightarrow \frac{\partial U}{\partial \theta} = \frac{k}{2} \times a^2 \times 2\theta + \left(kax_0 - mgl\right)$$

$$\left.\frac{\partial U}{\partial \theta}\right|_{\theta=0} : \theta = 0 \Rightarrow \frac{k}{2} \times a^2 \times 2\left(\theta = 0\right) + \left(kax_0 - mgl\right) = \left(kax_0 - mgl\right)$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = 0 \Rightarrow \left(kax_0 - mgl\right) = 0 \Rightarrow x_0 = +\frac{mgl}{ka}$$

That is to say, the spring is elongated by $x_0 = +\frac{mgl}{ka}$

$$\Rightarrow U = \frac{k}{2}a^2\left(\theta^2\right) + cte \cdot$$

$$T = E_c = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$$

$$\dot{x} = l\dot{\theta}$$

$$T = \frac{1}{2}m\left(l\dot{\theta}\right)^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

Then, the Lagrangian of the system is written:

$$L = E_c - E_p = T - U = \frac{1}{2}ml^2\dot{\theta}^2 - \frac{k}{2}a^2\left(\theta^2\right) + cte$$

The equation of motion for small oscillations is : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) = 0$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}ml^2 \times 2\dot{\theta} = ml^2\dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -\frac{k}{2}a^2 \times 2\theta = -ka^2\theta$$

$$ml^2\ddot{\theta} + ka^2\theta = 0$$

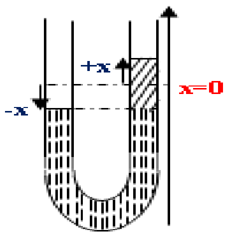
$$\begin{cases} \ddot{\theta} + \frac{ka^2}{ml^2}\theta = 0 \\ \ddot{\theta} + \omega_0^2\theta = 0 \end{cases}$$

$$\omega_0^2 = \frac{ka^2}{ml^2} \Rightarrow \omega_0 = \sqrt{\frac{ka^2}{ml^2}} = \frac{a}{l}\sqrt{\frac{k}{m}}$$

Exercise No. 5

A *U*-shaped tube of section *S* contains a liquid of density ρ and length l in the tube.

- ✓ Establish the differential equation for free vibrations of low amplitudes.
- ✓ Deduce their own pulsation.



Solution :

1. *Kinetic energy*

$$T = E_c = \frac{1}{2}Mv^2 = \frac{1}{2}M\dot{x}^2$$

M : The mass of liquid

$$\rho = \frac{M}{V} \Rightarrow M = \rho V = \rho Sl$$

$$\frac{M}{l} = S\rho \text{ ....(1)}$$

l : Liquid length.

$$T = \frac{1}{2}M\dot{x}^2$$

2. *Potential energy*

$$dU_m = -\vec{F}_m d\vec{l} = -F_m dl\cos\left(\vec{F}_m(\downarrow), d\vec{l}(\uparrow)\right)$$

$$dU_m = -F_m dl\cos(180°) \Rightarrow dU_m = +F_m dl = +mgdl \ with \ F_m = mg$$

m: The mass of the tube 2x

$$\rho = \frac{m}{V_m} \Rightarrow m = \rho V_m = \rho S2x \Rightarrow F_m = mg = \rho S2xg = 2\rho gSx$$

$$U_m = \int dU_m = \int_0^x +mgdl = +\int_0^x 2\rho gSxdx = +\int_0^x 2\rho gSxdx = +2\rho gS\int_0^x xdx = +2\rho gS\left[\frac{x^2}{2}\right]_0^x$$

$$U_m = 2\rho gS\left[\frac{x^2}{2} - 0\right] \Rightarrow U_m = 2\rho gS\left[\frac{x^2}{2}\right]$$

$$U_m = \rho gSx^2 + cte$$

$$L = T - U$$

$$L = \frac{1}{2}M\dot{x}^2 - \rho gSx^2 + cte$$

According to (1) : $\frac{M}{l} = S\rho$

$$L = T - U$$

$$L = \frac{1}{2}M\dot{x}^2 - \rho g S x^2 + cte \,/\, \frac{M}{l} = S\rho$$

$$L = \frac{1}{2}M\dot{x}^2 - \frac{Mg}{l}x^2 + cte$$

The equation of motion is of the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{1}{2}M\,2\dot{x} = M\dot{x} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = M\ddot{x}$$

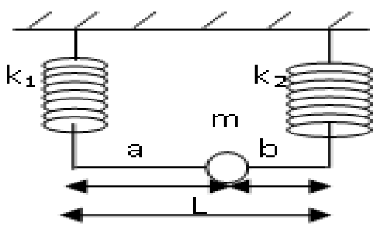
$$\frac{\partial L}{\partial x} = -\frac{Mg}{l}2x = -\frac{2Mg}{l}x$$

$$\Rightarrow M\ddot{x} + \frac{2Mg}{l}x = 0$$

$$\begin{cases} \ddot{x} + \dfrac{2g}{l}x = 0 \\ \ddot{x} + \omega_0^2 x = 0 \end{cases} \Rightarrow \omega_0^2 = \frac{2g}{l} \Rightarrow \omega_0 = \sqrt{\frac{2g}{l}}$$

Exercise No. 6

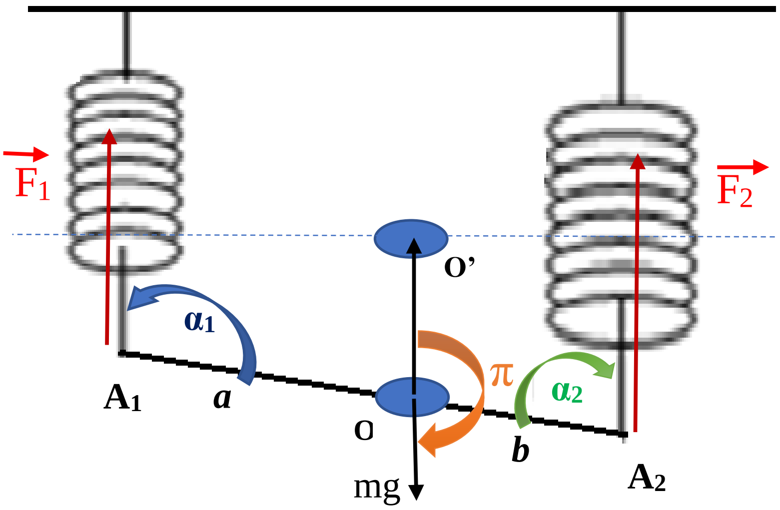
Find the simple system equivalent of the system shown in the figure, then calculate the proper pulsation. We admit that the displacement of the mass is only *v*.



In Equilibrium

$$\sum \vec{F} = 0$$

$$\vec{F}_1 + \vec{F}_2 + m\vec{g} = 0$$



Following the movement of the system we find

$F_1 + F_2 = mg$

$F_1 = k_1 x_1, F_2 = k_2 x_2$

$\boldsymbol{k_1 x_1 + k_2 x_2 = mg}$ ....(1)

$\sum \vec{\mu} = 0$

$\overrightarrow{OA_1} \wedge \vec{F_1} + \overrightarrow{OA_2} \wedge \vec{F_2} + \overrightarrow{OO'} \wedge m\vec{g} = 0$

$\overrightarrow{OA_1} \wedge \vec{F_1} - \overrightarrow{A_2O} \wedge \vec{F_2} + \overrightarrow{OO'} \wedge m\vec{g} = 0$

We have $\vec{A} \wedge \vec{B} = \|\vec{A}\| \times \|\vec{B}\| \sin(\vec{A}, \vec{B})$

So $\|\overrightarrow{OA_1}\| \times \|\vec{F_1}\| \times \sin(\overrightarrow{OA_1}, \vec{F_1}) - \|\overrightarrow{A_2O}\| \times \|\vec{F_2}\| \times \sin(\overrightarrow{A_2O}, \vec{F_2}) + \|\overrightarrow{OO'}\| \times \|m\vec{g}\| \times \sin(\overrightarrow{OO'}, m\vec{g}) = 0$

We have

$\|\overrightarrow{OA_1}\| = a, \|\overrightarrow{A_2O}\| = b, \|\vec{F_1}\| = F_1, \|\vec{F_2}\| = F_2$

So

$aF_1 \sin(\overrightarrow{OA_1}, \vec{F_1}) - bF_2 \sin(\overrightarrow{A_2O}, \vec{F_2}) + OO' \times mg \times \sin(\overrightarrow{OO'}, m\vec{g}) = 0$

$\sin(\overrightarrow{OO'}, m\vec{g}) = \sin 180° = 0$

$aF_1 \sin(\overrightarrow{OA_1}, \vec{F_1}) - bF_2 \sin(\overrightarrow{A_2O}, \vec{F_2}) = 0$

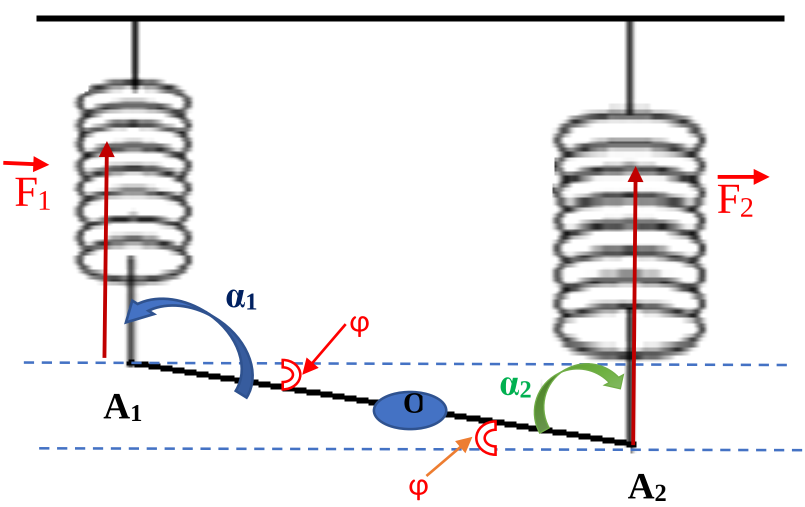
We pose

$\sin(\overrightarrow{OA_1}, \vec{F_1}) = \alpha_1, \sin(\overrightarrow{A_2O}, \vec{F_2}) = \alpha_2$

$aF_1 \sin(\alpha_1) - bF_2 \sin(\alpha_2) = 0$

$\alpha_1 = \varphi + \frac{\pi}{2}, \alpha_2 = \frac{\pi}{2} - \varphi$

 Dr.Ghellab Torkia



We know that

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

$$\sin(\alpha_1)=\sin\left(\varphi+\frac{\pi}{2}\right)=\sin\varphi\cos\frac{\pi}{2}+\cos\varphi\sin\frac{\pi}{2}$$

$$\cos\frac{\pi}{2}=0,\ \sin\frac{\pi}{2}=1$$

$$\sin(\alpha_1)=\sin\left(\varphi+\frac{\pi}{2}\right)=\cos\varphi$$

$$\sin(A-B)=\sin A\cos B-\cos A\sin B$$

$$\sin(\alpha_2)=\sin\left(\frac{\pi}{2}-\varphi\right)=\sin\frac{\pi}{2}\cos\varphi-\cos\frac{\pi}{2}\sin\varphi$$

$$\cos\frac{\pi}{2}=0,\ \sin\frac{\pi}{2}=1$$

$$\sin(\alpha_2)=\sin\left(\frac{\pi}{2}-\varphi\right)=\cos\varphi$$

So $aF_1\sin(\alpha_1)-bF_2\sin(\alpha_2)=0$

Becomes

$$aF_1\cos(\varphi)-bF_2\cos(\varphi)=0$$
$$aF_1=bF_2$$

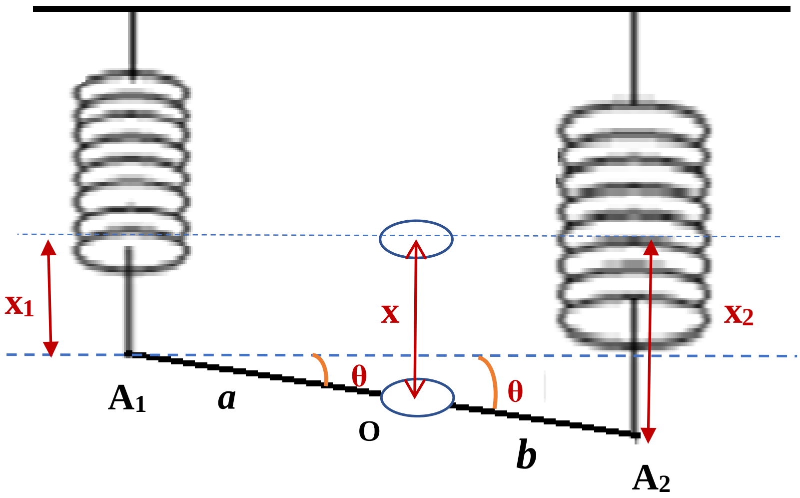
So $ak_1x_1=bk_2x_2$ …… ……..(2)

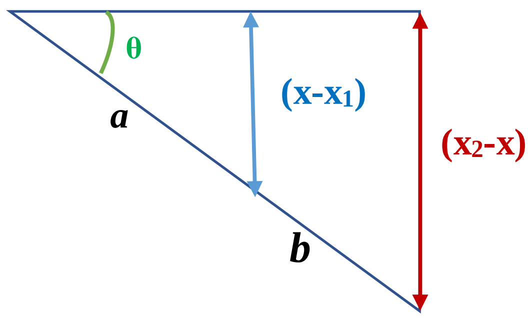
In equilibrium $k_{(equivalent)}x=mg$

According to (1): $k_{(equivalent)}x=mg=k_1x_1+k_2x_2$

$k_{(equivalent)}x=k_1x_1+k_2x_2$ ………(3)

Triangle: $A_1A_2A_3$





$$\sin\theta = \frac{x - x_1}{a} = \frac{x_2 - x_1}{a+b} \quad \ldots\ldots\ldots(4)$$

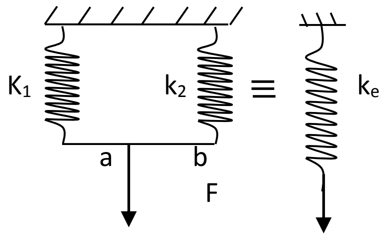
By replacing (2) and (4) in (3) we find

$$k_{(equivalent)} = \frac{(a+b)^2}{\dfrac{a^2}{k_1} + \dfrac{b^2}{k_2}}$$

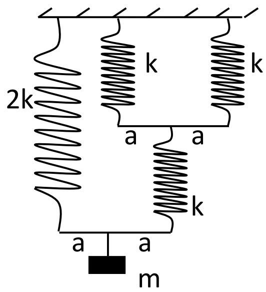
$$\begin{cases} \ddot{x} + \dfrac{k_{(equivalent)}}{m} x = 0 \\ \ddot{x} + \omega_0^2 x = 0 \end{cases} \Rightarrow \omega_0^2 = \frac{k_{(equivalent)}}{m} \Rightarrow \omega_0 = \sqrt{\frac{k_{(equivalent)}}{m}} = \sqrt{\frac{(a+b)^2}{\left(\dfrac{a^2}{k_1} + \dfrac{b^2}{k_2}\right) m}}$$

Exercise No. 7

We demonstrate that the simple equivalent system of the system of figure **(A)** is the system of figure **(B)** such that $k_{(equivalent)} = \dfrac{(a+b)^2}{\dfrac{a^2}{k_1} + \dfrac{b^2}{k_2}}$



(A) **(B)**



(C)

1- Taking into account the previous result finds the simple equivalent system of the system shown in Figure **(C)** and calculate the equivalent elements K_e and M_e. What happens to this result if we neglect a.

2- Then calculate in each of the two cases the proper pulsation ω_0. We give: k=150 N/m and m=1Kg (all springs have negligible mass).

Solution

By applying the result of figure (1) to the system of figure (2) $a=a$, $b=a$, $k_1=k_2=k$ we will have:

$$k_{1(equivalent)}=\frac{(a+a)^2}{\frac{a^2}{k}+\frac{a^2}{k}}=\frac{(2a)^2}{\frac{a^2+a^2}{k}}=\frac{4a^2}{\frac{2a^2}{k}}=\frac{2}{\frac{1}{k}}=2k$$

$k_{1(equivalent)}$ and k_3 are in series:

$$\frac{1}{k_{2(equivalent)}}=\frac{1}{k_{1(equivalent)}}+\frac{1}{k}=\frac{1}{2k}+\frac{1}{k}=\frac{1+2}{2k}=\frac{3}{2k}\Rightarrow k_{2(equivalent)}=\frac{2k}{3}$$

For ($k_{2(equivalent)}$) and ($2k$) by applying the result of the figure (1) $a=a$, $b=a$, $k_1=2k$, $k_2=\frac{2k}{3}$

$k_{2(equivalent)}$ we will have:

$$k_{3(equivalent)}=k_{(equivalent)}=\frac{(a+a)^2}{\frac{a^2}{2k}+\frac{a^2}{\frac{2k}{3}}}=\frac{(2a)^2}{\frac{a^2}{2k}+\frac{3a^2}{2k}}=\frac{4a^2}{\frac{4a^2}{2k}}=\frac{1}{\frac{1}{2k}}=2k\Rightarrow k_{(equivalent)}=2k$$

$$\begin{cases}\ddot{x}+\frac{k_{(equivalent)}}{m}x=0\\ \ddot{x}+\omega_0^2x=0\end{cases}$$

$$\omega_0^2=\frac{k_{(equivalent)}}{m}\Rightarrow\omega_0=\sqrt{\frac{k_{(equivalent)}}{m}}=\sqrt{\frac{2k}{m}}=\sqrt{\frac{2\times150}{1}}=\sqrt{300}=17.3\,rads^{-1}$$

If a<<< (a negligible)

k_1 and k_2 are in parallel:

$$k_{1(equivalent)}=k_1+k_2=k+k=2k$$

$k_{1(equivalent)}$ and k are in series:

$$\frac{1}{k_{2(equivalent)}}=\frac{1}{k_{1(equivalent)}}+\frac{1}{k}=\frac{1}{2k}+\frac{1}{k}=\frac{1+2}{2k}=\frac{3}{2k}\Rightarrow k_{2(equivalent)}=\frac{2k}{3}$$

$k_{2(equivalent)}$ and $2k$ are in parallel:

$$k_{3(equivalent)} = k_{(equivalent)} = k_{2(equivalent)} + 2k = \frac{2k}{3} + 2k = \frac{2k+6k}{3} = \frac{8k}{3}$$

$$k_{(equivalent)} = \frac{8k}{3}$$

$$\begin{cases} \ddot{x} + \dfrac{k_{(equivalent)}}{m} x = 0 \\ \ddot{x} + \omega_0^2 x = 0 \end{cases}$$

$$\omega_0^2 = \frac{k_{(equivalent)}}{m} \Rightarrow \omega_0 = \sqrt{\frac{k_{(equivalent)}}{m}} = \sqrt{\frac{8k}{3m}} = \sqrt{\frac{8\times 150}{3\times 1}} = \sqrt{400} = 20\ rads^{-1}$$

Problem 1

1) kinetic energy and potential energy

1.1 The diagrams below represent systems in a state of movement. The initial positions are shown in dotted lines. A bold line stem is massive and homogeneous while a thin line stem is negligible. The black balls are punctual. The wires are inextensible and do not slip on the discs. It will be assumed that the springs keep their vertical or horizontal directions during spacing. Find the kinetic energy T and the potential energy U as a function of θ for each of these systems.

2) Equilibrium condition, stable equilibrium, and unstable equilibrium

2.1

a) Find the equilibrium positions for each system.

b) Study the nature of the equilibrium at $\theta = \pi/2$ of system (*i*).

c) Find the oscillation condition of systems (*ii*) and (*iii*) at $\theta=0$.

3) Lagrangian equation of motion

3.1 Find in each case the Lagrangian for $\theta \ll 1$ then deduce the equation of motion and the proper pulsation.

Reminders

The moment of inertia of a rod of mass M and length l around its center of gravity G is:

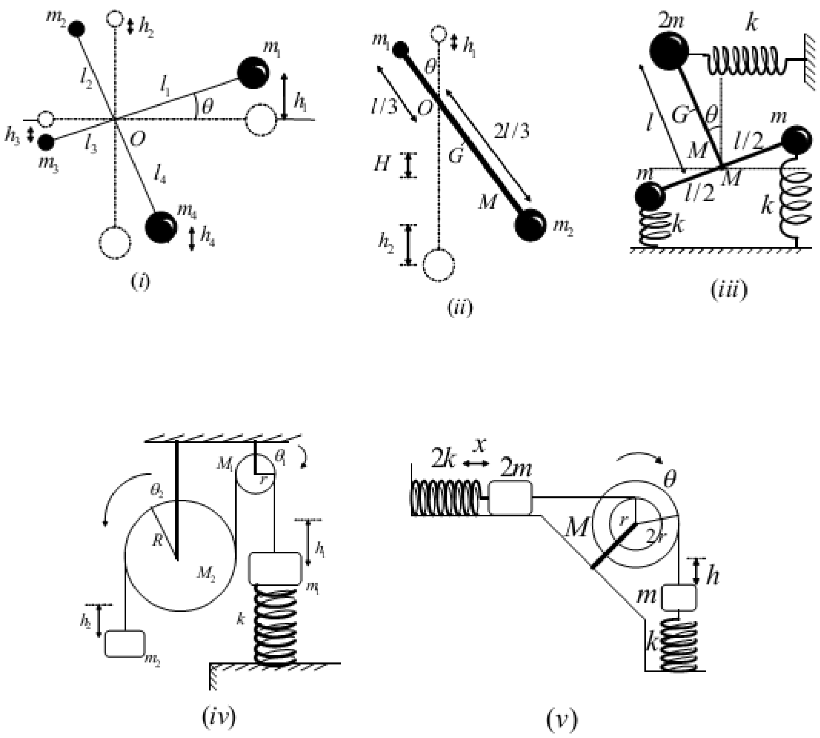
$$I_{/G} = \frac{1}{12} M l^2$$

The moment of inertia of a disk of mass M and radius R around its center of gravity G is:

$$I_{/G} = \frac{1}{2} M R^2$$

The moment of inertia of a rod of mass M and length l around a point O far from its center of gravity by a distance D is, according to the Huygens-Steiner theorem:

$$I_{/O} = I_{/G} + M\left(OG\right)^2 = \frac{1}{12} M l^2 + M D^2$$



Solution

1) kinetic energy and potential energy

1.1 The kinetic energy T and the potential energy U as a function of θ

i)

$$T = T_{m1} + T_{m2} + T_{m3} + T_{m4} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2\frac{1}{2}m_3v_3^2\frac{1}{2}m_4v_4^2 = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 + \frac{1}{2}m_4\dot{x}_4^2$$

$$\dot{x}_1 = l_1\dot{\theta},\ \dot{x}_2 = l_2\dot{\theta},\ \dot{x}_3 = l_3\dot{\theta},\ \dot{x}_4 = l_4\dot{\theta}$$

$$T = \frac{1}{2}m_1\left(l_1\dot{\theta}\right)^2 + \frac{1}{2}m_2\left(l_2\dot{\theta}\right)^2 + \frac{1}{2}m_3\left(l_3\dot{\theta}\right)^2 + \frac{1}{2}m_4\left(l_4\dot{\theta}\right)^2 \Rightarrow T = \frac{1}{2}\left(m_1l_1^2 + m_2l_2^2 + m_3l_3^2 + m_4l_4^2\right)\dot{\theta}^2$$

$$U = U_{m1} + U_{m2} + U_{m3} + U_{m4}$$

$$U_{m1} = +m_1gh_1$$

$$\sin\theta = \frac{h_1}{l_1} \Rightarrow h_1 = l_1\sin\theta \Rightarrow U_{m1} = +m_1gl_1\sin\theta$$

$$U_{m2} = -m_2gh_2$$

$$l_2 = h_2 + x \Rightarrow h_2 = l_2 - x$$

$$\cos\theta = \frac{x}{l_2} \Rightarrow x = l_2\cos\theta \Rightarrow h = l_2 - l_2\cos\theta = l_2\left(1-\cos\theta\right)$$

$$\Rightarrow U_{m2} = -m_2gl_2\left(1-\cos\theta\right)$$

$$U_{m3} = -m_3 g h_3$$

$$\sin\theta = \frac{h_3}{l_3} \Rightarrow h_3 = l_3 \sin\theta \Rightarrow U_{m3} = -m_3 g l_3 \sin\theta$$

$$U_{m4} = +m_4 g h_4$$

$$l_4 = h_4 + x \Rightarrow h_4 = l_4 - x$$

$$\cos\theta = \frac{x}{l_4} \Rightarrow x = l_4 \cos\theta \Rightarrow h = l_4 - l_4\cos\theta = l_4\left(1-\cos\theta\right)$$

$$\Rightarrow U_{m4} = +m_4 g l_4\left(1-\cos\theta\right)$$

$$U = +m_1 g l_1 \sin\theta - m_2 g l_2\left(1-\cos\theta\right) - m_3 g l_3 \sin\theta + m_4 g l_4\left(1-\cos\theta\right) + cte$$

ii)

$$T = T_{m1} + T_{m2} + T_{\text{Stem}} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}I_{/O}\dot{\theta}^2$$

The moment of inertia of the rod at the turn of O (according to Huygens' theorem):

$$I_{/O} = I_{/G} + M\left(OG\right)^2 \Rightarrow I_{/O} = \frac{1}{12}Ml^2 + M\left(\frac{l}{2}-\frac{l}{3}\right)^2$$

$$I_{/O} = \frac{1}{12}Ml^2 + M\left(\frac{l}{6}\right)^2 = \frac{1}{12}Ml^2 + \frac{1}{36}Ml^2 = \frac{4}{36}Ml^2 = \frac{1}{9}Ml^2$$

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}I_{/O}\dot{\theta}^2 = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}\left(\frac{1}{9}Ml^2\right)\dot{\theta}^2$$

$$\dot{x}_1 = \frac{l}{3}\dot{\theta},\ \dot{x}_2 = \frac{2l}{3}\dot{\theta}, \Rightarrow T = \frac{1}{2}m_1\left(\frac{l}{3}\dot{\theta}\right)^2 + \frac{1}{2}m_2\left(\frac{2l}{3}\dot{\theta}\right)^2 + \frac{1}{2}\left(\frac{1}{9}Ml^2\right)\dot{\theta}^2$$

$$T = \frac{1}{2}\left(\frac{l^2}{9}m_1 + \frac{4l^2}{9}m_2 + \frac{1}{9}Ml^2\right)\dot{\theta}^2$$

$$U = U_{m1} + U_{m2} + U_{\text{Stem}}$$

$$U_{m1} = -m_1 g h_1$$

$$\frac{l}{3} = h_1 + x \Rightarrow h_1 = \frac{l}{3} - x$$

$$\cos\theta = \frac{x}{\frac{l}{3}} \Rightarrow x = \frac{l}{3}\cos\theta \Rightarrow h = \frac{l}{3} - \frac{l}{3}\cos\theta = \frac{l}{3}\left(1-\cos\theta\right)$$

$$\Rightarrow U_{m1} = -m_1 g\frac{l}{3}\left(1-\cos\theta\right)$$

$$U_{m2}=+m_2gh_2$$
$$\frac{2l}{3}=h_2+x\Rightarrow h_2=\frac{2l}{3}-x$$
$$\cos\theta=\frac{x}{\frac{2l}{3}}\Rightarrow x=\frac{2l}{3}\cos\theta\Rightarrow h=\frac{2l}{3}-\frac{2l}{3}\cos\theta=\frac{2l}{3}(1-\cos\theta)$$
$$\Rightarrow U_{m2}=+m_2g\frac{2l}{3}(1-\cos\theta)$$
$$U_{m2}=+MgH$$
$$\frac{l}{6}=H+x\Rightarrow H=\frac{l}{6}-x$$
$$\cos\theta=\frac{x}{\frac{l}{6}}\Rightarrow x=\frac{l}{6}\cos\theta\Rightarrow H=\frac{l}{6}-\frac{l}{6}\cos\theta=\frac{l}{6}(1-\cos\theta)\Rightarrow U_{\text{Stem}}=+Mg\frac{l}{6}(1-\cos\theta)$$
$$U=-m_1g\frac{l}{3}(1-\cos\theta)+m_2g\frac{2l}{3}(1-\cos\theta)+Mg\frac{l}{6}(1-\cos\theta)+cte$$

iii)

$$T=T_m+T_{2m}+T_m+T_{\text{Stem1}}+T_{\text{Stem2}}=\frac{1}{2}mv_1^2+\frac{1}{2}2mv_2^2+\frac{1}{2}mv_3^2+\frac{1}{2}I_1\dot{\theta}^2+\frac{1}{2}I_2\dot{\theta}^2$$
$$I_1=\frac{1}{12}Ml^2$$
$$I_2=I_{/G}+M(OG)^2=\frac{1}{12}Ml^2+M\left(\frac{l}{2}\right)^2=\frac{1}{12}Ml^2+\frac{1}{4}Ml^2=\frac{4}{12}Ml^2=\frac{1}{3}Ml^2$$
$$T=\frac{1}{2}m\dot{x}_1^2+\frac{1}{2}2m\dot{x}_2^2+\frac{1}{2}m\dot{x}_3^2+\frac{1}{2}I_1\dot{\theta}^2+\frac{1}{2}I_2\dot{\theta}^2$$
$$T=\frac{1}{2}m\dot{x}_1^2+\frac{1}{2}2m\dot{x}_2^2+\frac{1}{2}m\dot{x}_3^2+\frac{1}{2}\left(\frac{1}{12}Ml^2\right)\dot{\theta}^2+\frac{1}{2}\left(\frac{1}{3}Ml^2\right)\dot{\theta}^2$$
$$\dot{x}_1=\frac{l}{2}\dot{\theta},\ \dot{x}_2=l\dot{\theta},\ \dot{x}_3=\frac{l}{2}\dot{\theta}$$
$$\Rightarrow T=\frac{1}{2}m\left(\frac{l}{2}\dot{\theta}\right)^2+\frac{1}{2}2m\left(l\dot{\theta}\right)^2+\frac{1}{2}m\left(\frac{l}{2}\dot{\theta}\right)^2+\frac{1}{2}\left(\frac{1}{12}Ml^2\right)\dot{\theta}^2+\frac{1}{2}\left(\frac{1}{3}Ml^2\right)\dot{\theta}^2$$
$$T=\frac{1}{2}\left(\frac{l^2}{4}ml^2+2ml^2+\frac{l^2}{4}m+\frac{4}{12}Ml^2\right)\dot{\theta}^2=\frac{1}{2}\left(\frac{5l^2}{2}m+\frac{1}{3}Ml^2\right)\dot{\theta}^2$$
$$U=U_m+U_{2m}+U_m+U_{\text{Stem2}}+U_{\text{Spring1}}+U_{\text{Spring2}}+U_{\text{Spring3}}$$
$$U_m=-mgh_1$$
$$\sin\theta=\frac{h_1}{\frac{l}{2}}\Rightarrow h_1=\frac{l}{2}\sin\theta\Rightarrow U_m=-mg\frac{l}{2}\sin\theta$$

$$U_{2m} = -2mgh_2$$

$$l = h_2 + x \Rightarrow h_2 = l - x$$

$$\cos\theta = \frac{x}{l} \Rightarrow x = l\cos\theta \Rightarrow h = l - l\cos\theta = l(1-\cos\theta)$$

$$\Rightarrow U_{2m} = -2mgl(1-\cos\theta)$$

$$U_m = +mgh_3$$

$$\sin\theta = \frac{h_3}{\frac{l}{2}} \Rightarrow h_1 = \frac{l}{2}\sin\theta \Rightarrow U_m = +mg\frac{l}{2}\sin\theta$$

$$U_{\text{Stem}} = -MgH$$

$$\frac{l}{2} = H + x \Rightarrow H = \frac{l}{2} - x$$

$$\cos\theta = \frac{x}{\frac{l}{2}} \Rightarrow x = \frac{l}{2}\cos\theta \Rightarrow H = \frac{l}{2} - \frac{l}{2}\cos\theta = \frac{l}{2}(1-\cos\theta)$$

$$\Rightarrow U_{\text{Stem}} = -Mg\frac{l}{2}(1-\cos\theta)$$

$$U_{\text{Spring1}} = \frac{k}{2}x^2 + kxx_0 + cte$$

$$We\ have\ \sin\theta = \frac{x}{\frac{l}{2}} \Rightarrow x = \frac{l}{2}\sin\theta$$

$$U_{\text{Spring1}} = \frac{k}{2}\left(\frac{l}{2}\sin\theta\right)^2 + k\left(\frac{l}{2}\sin\theta\right)x_0 + cte$$

$$U_{\text{Spring2}} = \frac{k}{2}x^2 + kxx_0 + cte$$

$$We\ have\ \sin\theta = \frac{x}{l} \Rightarrow x = l\sin\theta$$

$$U_{\text{Spring2}} = \frac{k}{2}(l\sin\theta)^2 + k(l\sin\theta)x_0 + cte$$

$$U_{\text{Spring3}} = \frac{k}{2}x^2 + kxx_0 + cte$$

$$We\ have\ \sin\theta = \frac{x}{\frac{l}{2}} \Rightarrow x = \frac{l}{2}\sin\theta$$

$$U_{\text{Spring3}} = \frac{k}{2}\left(\frac{l}{2}\sin\theta\right)^2 + k\left(\frac{l}{2}\sin\theta\right)x_0 + cte$$

$$U = U_m + U_{2m} + U_m + U_{\text{Stem2}} + U_{\text{Spring1}} + U_{\text{Spring2}} + U_{\text{Spring3}}$$

$$U = -mg\frac{l}{2}\sin\theta - 2mgl\left(1-\cos\theta\right) + mg\frac{l}{2}\sin\theta - Mg\frac{l}{2}\left(1-\cos\theta\right)$$

$$+\frac{k}{2}\left(\frac{l}{2}\sin\theta\right)^2 + k\left(\frac{l}{2}\sin\theta\right)x_0 + \frac{k}{2}\left(l\sin\theta\right)^2 + k\left(l\sin\theta\right)x_0 + \frac{k}{2}\left(\frac{l}{2}\sin\theta\right)^2 + k\left(\frac{l}{2}\sin\theta\right)x_0 + cte$$

$$U = -\left(2m + \frac{1}{2}M\right)gl\left(1-\cos\theta\right) + \frac{3l^2}{4}k\sin\theta^2 + 2kl\sin\theta x_0 + cte$$

iv)

$$T = T_{m1} + T_{m2} + T_{M1} + T_{M2} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\dot{\theta}_2^2$$

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\dot{\theta}_2^2$$

$$\dot{x}_1 = r\dot{\theta}_1,\ \dot{x}_2 = R\dot{\theta}_2,\ J_1 = \frac{1}{2}M_1r^2,\ J_2 = \frac{1}{2}M_2R^2$$

$$T = \frac{1}{2}m_1\left(r\dot{\theta}_1\right)^2 + \frac{1}{2}m_2\left(R\dot{\theta}_2\right)^2 + \frac{1}{2}\left(\frac{1}{2}M_1r^2\right)\dot{\theta}_1^2 + \frac{1}{2}\left(\frac{1}{2}M_2R^2\right)\dot{\theta}_2^2$$

Since the Wire is inextensible and does not slip on the discs, we have $r\theta_1=R\theta_2$ So

$$T = \frac{1}{2}\left(m_1 + m_2 + \frac{1}{2}M_1 + \frac{1}{2}M_2\right)r^2\dot{\theta}_1^2$$

$$U = U_{m1} + U_{m2} + U_{\text{Spring}}$$

$$U_{m1} = +m_1gh_1$$

$$\sin\theta_1 = \frac{h_1}{r} \Rightarrow h_1 = r\sin\theta_1 \Rightarrow U_{m1} = +m_1gr\sin\theta_1$$

$$U_{m1} = -m_2gh_2$$

$$\sin\theta_2 = \frac{h_2}{R} \Rightarrow h_2 = R\sin\theta_2 \Rightarrow U_{m2} = -m_2gr\sin\theta_2$$

$$U_{\text{Spring}} = \frac{k}{2}x^2 + kxx_0 + cte$$

We have $\sin\theta_1 = \frac{x}{r} \Rightarrow x = r\sin\theta_1$

$$U_{\text{Spring}} = \frac{k}{2}\left(r\sin\theta_1\right)^2 + k\left(r\sin\theta_1\right)x_0 + cte$$

$$U = \frac{k}{2}\left(r\sin\theta_1\right)^2 + k\left(r\sin\theta_1\right)x_0 + m_1gr\sin\theta_1 - m_2gR\sin\theta_2 + cte$$

$$U = \frac{k}{2}r^2\sin\theta_1^2 + kr\sin\theta_1x_0 + \left(m_1 - m_2\right)r\sin\theta_1 + cte$$

v)

$$T = T_m + T_{2m} + T_M = \frac{1}{2}mv_1^2 + \frac{1}{2}2mv_2^2 + \frac{1}{2}I\dot{\theta}^2$$

$$I = \frac{1}{2}M\left(2r\right)^2$$

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}2m\dot{x}_2^2 + \frac{1}{2}\left(\frac{1}{2}M\left(2r\right)^2\right)\dot{\theta}^2$$

$$\dot{x}_1 = 2r\dot{\theta},\ \dot{x}_2 = r\dot{\theta}$$

$$T = \frac{1}{2}m\left(2r\dot{\theta}\right)^2 + \frac{1}{2}2m\left(r\dot{\theta}\right)^2 + \frac{1}{2}\left(\frac{1}{2}M\left(2r\right)^2\right)\dot{\theta}^2$$

$$T = \frac{1}{2}\left(6m + 2M\right)r^2\dot{\theta}^2$$

$$U = U_m + U_{\text{Spring1}} + U_{\text{Spring2}}$$

$$U_m = -mgh$$

$$\sin\theta = \frac{h}{2r} \Rightarrow h_1 = 2r\sin\theta \Rightarrow U_m = -2mgr\sin\theta$$

$$U_{\text{Spring1}} = \frac{k}{2}x^2 + kxx_0 + cte$$

$$We\ have\ \sin\theta = \frac{x}{2r} \Rightarrow x = 2r\sin\theta$$

$$U_{\text{Spring1}} = \frac{k}{2}\left(2r\sin\theta\right)^2 + k\left(2r\sin\theta\right)x_0 + cte$$

$$U_{\text{Spring2}} = \frac{2k}{2}x^2 + 2kxx_0 + cte$$

$$We\ have\ \sin\theta = \frac{x}{r} \Rightarrow x = r\sin\theta$$

$$U_{\text{Spring2}} = k\left(r\sin\theta\right)^2 + 2k\left(r\sin\theta\right)x_0 + cte$$

$$U = -2mgr\sin\theta + \frac{k}{2}\left(2r\sin\theta\right)^2 + k\left(2r\sin\theta\right)x_0 + k\left(r\sin\theta\right)^2 + 2k\left(r\sin\theta\right)x_0 + cte$$

$$U = 3kr^2\sin\theta^2 + 4kr\sin\theta x_0 - 2mgr\sin\theta + cte$$

$$U = 3kr^2\sin\theta^2 + \left(4krx_0 - 2mgr\right)\sin\theta$$

2) Equilibrium condition, stable equilibrium, and unstable equilibrium

i) $U = +m_1gl_1\sin\theta - m_2gl_2\left(1-\cos\theta\right) - m_3gl_3\sin\theta + m_4gl_4\left(1-\cos\theta\right) + cte$

ii) $U = -m_1g\frac{l}{3}\left(1-\cos\theta\right) + m_2g\frac{2l}{3}\left(1-\cos\theta\right) + Mg\frac{l}{6}\left(1-\cos\theta\right) + cte$

iii) $U = -\left(2m + \frac{1}{2}M\right)gl\left(1-\cos\theta\right) + \frac{3l^2}{4}k\sin\theta^2 + 2kl\sin\theta x_0 + cte$

iv) $U = \frac{k}{2}r^2\sin\theta_1^2 + \sin\theta r\sin\theta_1 x_0 + \left(m_1 - m_2\right)r\sin\theta_1 + +cte$

v) $U = U = 3kr^2\sin\theta^2 + \left(4krx_0 - 2mgr\right)\sin\theta$

a) *The variable being θ the equilibrium condition is* $\frac{\partial U}{\partial\theta} = 0$

i)

$$\frac{\partial U}{\partial\theta} = 0 \Rightarrow +m_1gl_1\cos\theta - m_2gl_2\sin\theta - m_3gl_3\cos\theta + m_4gl_4\sin\theta = 0$$
$$\frac{\partial U}{\partial\theta} = 0 \Rightarrow \left(+m_1gl_1 - m_3gl_3\right)\cos\theta + \left(-m_2gl_2 + m_4gl_4\right)\sin\theta = 0$$
$$\frac{\partial U}{\partial\theta} = 0 \Rightarrow \cos\theta\left[\left(+m_1gl_1 - m_3gl_3\right) + \left(-m_2gl_2 + m_4gl_4\right)\frac{\sin\theta}{\cos\theta}\right] = 0$$
$$\frac{\partial U}{\partial\theta} = 0 \Rightarrow \cos\theta\left[\left(+m_1gl_1 - m_3gl_3\right) + \left(-m_2gl_2 + m_4gl_4\right)\tan g\theta\right] = 0$$
$$\cos\theta = 0\left(\theta = \frac{\pi}{2}\right) Or\ \tan g\theta = \frac{-\left(+m_1gl_1 - m_3gl_3\right)}{\left(-m_2gl_2 + m_4gl_4\right)}$$

ii)

$$\frac{\partial U}{\partial\theta} = 0 \Rightarrow -m_1g\frac{l}{3}\sin\theta + m_2g\frac{2l}{3}\sin\theta + Mg\frac{l}{6}\sin\theta = 0$$
$$\frac{\partial U}{\partial\theta} = 0 \Rightarrow \left(-m_1g\frac{l}{3} + m_2g\frac{2l}{3} + Mg\frac{l}{6}\right)\sin\theta = 0$$
$$\frac{\partial U}{\partial\theta} = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$$

iii)

$$\frac{\partial U}{\partial\theta} = 0 \Rightarrow -\left(2m + \frac{1}{2}M\right)gl\sin\theta + \frac{3l^2}{4}k2\cos\theta\sin\theta = 0$$
$$\frac{\partial U}{\partial\theta} = 0 \Rightarrow \sin\theta\left[-\left(2m + \frac{1}{2}M\right)gl + \frac{3l^2}{2}k\cos\theta\right] = 0$$
$$\frac{\partial U}{\partial\theta} = 0 \Rightarrow \sin\theta = 0\left(\Rightarrow\theta = 0\right) Or \left(\begin{array}{c}\left[-\left(2m + \frac{1}{2}M\right)gl + \frac{3l^2}{2}k\cos\theta\right] = 0 \\ \left(\Rightarrow \cos\theta = \frac{\left(2m + \frac{1}{2}M\right)gl}{\frac{3l^2}{2}k} = \frac{\left(4m + M\right)g}{3lk}\right)\end{array}\right)$$

iv)

$$\frac{\partial U}{\partial \theta}=0 \Rightarrow \frac{k}{2}r^2 2\cos\theta_1 \sin\theta_1+(m_1-m_2)r\cos\theta_1=0$$

$$\frac{\partial U}{\partial \theta}=0 \Rightarrow \cos\theta_1\left(kr^2\sin\theta_1+(m_1-m_2)r\right)=0$$

$$\frac{\partial U}{\partial \theta}=0 \Rightarrow \cos\theta_1=0 \Rightarrow \left(\theta_1=\frac{\pi}{2}\right) Or \left(\sin\theta_1=\frac{-(m_1-m_2)}{kr}\right)=0$$

v)

$$\frac{\partial U}{\partial \theta}=0 \Rightarrow 3kr^2 2\sin\theta\cos\theta+(4krx_0-2mgr)\cos\theta=0$$

$$\frac{\partial U}{\partial \theta}=0 \Rightarrow \cos\theta\left[6kr^2\sin\theta+(4krx_0-2mgr)\right]=0$$

$$\frac{\partial U}{\partial \theta}=0 \Rightarrow \cos\theta=0 \Rightarrow \left(\theta=\frac{\pi}{2}\right) Or \begin{cases}\left[6kr^2\sin\theta+(4krx_0-2mgr)\right]=0\\ \left[\sin\theta=\frac{-(4krx_0-2mgr)}{6kr^2}\right]=0\end{cases}$$

b) Let's calculate $\frac{\partial^2 U}{\partial \theta^2}$ *and check its sign at* $\theta=\frac{\pi}{2}$ of system (i)

We have $\frac{\partial U}{\partial \theta}=(+m_1gl_1-m_3gl_3)\cos\theta+(-m_2gl_2+m_4gl_4)\sin\theta$

So $\left.\frac{\partial^2 U}{\partial \theta^2}\right|_{\theta=\frac{\pi}{2}}=\left[-(+m_1gl_1-m_3gl_3)\sin\theta+(-m_2gl_2+m_4gl_4)\cos\theta\right]_{\theta=\frac{\pi}{2}}$

$$\left.\frac{\partial^2 U}{\partial \theta^2}\right|_{\theta=\frac{\pi}{2}}=\left[-(+m_1gl_1-m_3gl_3)\sin\frac{\pi}{2}+(-m_2gl_2+m_4gl_4)\cos\frac{\pi}{2}\right]$$

$$\left.\frac{\partial^2 U}{\partial \theta^2}\right|_{\theta=\frac{\pi}{2}}=g(+m_3l_3-m_1l_1)$$

If $m_3l_3>m_1l_1$ *the* system is in stable equilibrium

If $m_3l_3<m_1l_1$ *then the* system is in instable equilibrium

b) *For a system to oscillate it must regain its equilibrium position after each spacing, therefore the condition for oscillation of the systems (ii) and (iii) is that the equilibrium is stable:* $\frac{\partial^2 U}{\partial \theta^2}>0$

ii) :

$$\frac{\partial U}{\partial \theta}=\left(-m_1 g\frac{l}{3}+m_2 g\frac{2l}{3}+Mg\frac{l}{6}\right)\sin\theta \Rightarrow \frac{\partial^2 U}{\partial \theta^2}=\frac{1}{3}\left(-m_1+2m_2+\frac{1}{2}M\right)gl\cos\theta$$

$$\left.\frac{\partial^2 U}{\partial \theta^2}\right|_{\theta=0}=\frac{1}{3}\left(-m_1+2m_2+\frac{1}{2}M\right)gl$$

$$\left.\frac{\partial^2 U}{\partial \theta^2}\right|_{\theta=0}>0 \Rightarrow \left(-m_1+2m_2+\frac{1}{2}M\right)>0 \Rightarrow \left(+2m_2+\frac{1}{2}M\right)>m_1$$

iii):

$$\frac{\partial U}{\partial \theta}=\sin\theta\left[-\left(2m-\frac{1}{2}M\right)gl+\frac{3l^2}{2}k\cos\theta\right]$$

$$\Rightarrow \frac{\partial^2 U}{\partial \theta^2}=\cos\theta\left[-\left(2m+\frac{1}{2}M\right)gl+\frac{3l^2}{2}k\cos\theta\right]-\frac{3l^2}{2}k\sin\theta^2$$

$$\Rightarrow \frac{\partial^2 U}{\partial \theta^2}=+\frac{3l^2}{2}k\cos\theta^2-\left(2m+\frac{1}{2}M\right)gl\cos\theta-\frac{3l^2}{2}k\sin\theta^2$$

$$\Rightarrow \left.\frac{\partial^2 U}{\partial \theta^2}\right|_{\theta=0}=+\frac{3l^2}{2}k-\left(2m+\frac{1}{2}M\right)gl$$

$$\Rightarrow \left.\frac{\partial^2 U}{\partial \theta^2}\right|_{\theta=0}>0 \Rightarrow \frac{3l^2}{2}k-\left(2m+\frac{1}{2}M\right)gl>0 \Rightarrow \frac{3l^2}{2}k>\left(2m+\frac{1}{2}M\right)gl \Rightarrow k>\frac{(4m+M)g}{3l}$$

3) Lagrangian equation of motion

3.1 The Lagrangian for $\theta \ll 1$, equation of motion and the proper pulsation for each case.

i)

$$T=\frac{1}{2}\left(m_1 l_1^2+m_2 l_2^2+m_3 l_3^2+m_4 l_4^2\right)\dot{\theta}^2$$

$$U=+m_1 g l_1\sin\theta-m_2 g l_2(1-\cos\theta)-m_3 g l_3\sin\theta+m_4 g l_4(1-\cos\theta)+cte$$

For $\theta \ll 1$ $\sin\theta \approx \theta$ *and* $\cos\theta = 1-\frac{\theta^2}{2}$

$$U=+m_1 g l_1\theta-m_2 g l_2\left(1-(1-\frac{\theta^2}{2})\right)-m_3 g l_3\sin\theta+m_4 g l_4\left(1-(1-\frac{\theta^2}{2})\right)+cte$$

$$U=+m_1 g l_1\theta-m_2 g l_2\frac{\theta^2}{2}-m_3 g l_3\theta+m_4 g l_4\frac{\theta^2}{2}+cte$$

$$U=\frac{1}{2}\left(m_4 g l_4-m_2 g l_2\right)\theta^2+\left(m_1 g l_1-m_3 g l_3\right)\theta+cte$$

In equilibrium

$$\left.\frac{\partial U}{\partial \theta}\right|_{\theta=0}=0$$

$$\frac{\partial U}{\partial \theta}=\left(m_4gl_4-m_2gl_2\right)\theta+\left(m_1gl_1-m_3gl_3\right)$$

$$\left.\frac{\partial U}{\partial \theta}\right|_{\theta=0}:\theta=0\Rightarrow\left(m_1gl_1-m_3gl_3\right)=0$$

So $U=\frac{1}{2}\left(m_4gl_4-m_2gl_2\right)\theta^2+cte$.

Then, the Lagrangian of the system is written:

$$L=T-U=\frac{1}{2}\left(m_1l_1^2+m_2l_2^2+m_3l_3^2+m_4l_4^2\right)\dot{\theta}^2-\frac{1}{2}\left(m_4gl_4-m_2gl_2\right)\theta^2+cte$$

The equation of motion for small oscillations is: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\left(\frac{\partial L}{\partial \theta}\right)=0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)=\left(m_1l_1^2+m_2l_2^2+m_3l_3^2+m_4l_4^2\right)\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta}=\frac{\partial}{\partial \theta}\left(-\frac{1}{2}\left(m_4gl_4-m_2gl_2\right)\theta^2\right)=-\left(m_4gl_4-m_2gl_2\right)\theta$$

$$\left(m_1l_1^2+m_2l_2^2+m_3l_3^2+m_4l_4^2\right)\ddot{\theta}+\left(m_4gl_4-m_2gl_2\right)\theta=0$$

$$\begin{cases}\ddot{\theta}+\dfrac{\left(m_4gl_4-m_2gl_2\right)}{\left(m_1l_1^2+m_2l_2^2+m_3l_3^2+m_4l_4^2\right)}\theta=0\\ \ddot{\theta}+\omega_0^2\theta=0\end{cases}$$

The proper pulsation

$$\omega_0^2=\frac{\left(m_4gl_4-m_2gl_2\right)}{\left(m_1l_1^2+m_2l_2^2+m_3l_3^2+m_4l_4^2\right)}\Rightarrow\omega_0=\sqrt{\frac{\left(m_4gl_4-m_2gl_2\right)}{\left(m_1l_1^2+m_2l_2^2+m_3l_3^2+m_4l_4^2\right)}}$$

ii)

$$T=\frac{1}{2}\left(\frac{l^2}{9}m_1+\frac{4l^2}{9}m_2+\frac{1}{9}Ml^2\right)\dot{\theta}^2$$

$$U=-m_1g\frac{l}{3}(1-\cos\theta)+m_2g\frac{2l}{3}(1-\cos\theta)+Mg\frac{l}{6}(1-\cos\theta)+cte$$

For $\theta\ll 1$ $\sin\theta\approx\theta$ and $\cos\theta=1-\frac{\theta^2}{2}$

$$U=-m_1 g\frac{l}{3}\left(1-(1-\frac{\theta^2}{2})\right)+m_2 g\frac{2l}{3}\left(1-(1-\frac{\theta^2}{2})\right)+Mg\frac{l}{6}\left(1-(1-\frac{\theta^2}{2})\right)+cte$$

$$U=-m_1 g\frac{l}{3}\frac{\theta^2}{2}+m_2 g\frac{2l}{3}\frac{\theta^2}{2}+Mg\frac{l}{6}\frac{\theta^2}{2}+cte$$

$$U=\frac{1}{2}\left(2m_2-m_1+\frac{1}{2}M\right)\frac{l}{3}g\theta^2+cte$$

Then, the Lagrangian of the system is written:

$$L=T-U=\frac{1}{2}\left(\frac{l^2}{9}m_1+\frac{4l^2}{9}m_2+\frac{1}{9}Ml^2\right)\dot{\theta}^2-\frac{1}{2}\left(2m_2-m_1+\frac{1}{2}M\right)\frac{l}{3}g\theta^2+cte$$

The equation of motion for small oscillations is: $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)-\left(\frac{\partial L}{\partial\theta}\right)=0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)=\left(\frac{l^2}{9}m_1+\frac{4l^2}{9}m_2+\frac{1}{9}Ml^2\right)\ddot{\theta}$$

$$\frac{\partial L}{\partial\theta}=\frac{\partial}{\partial\theta}\left(-\frac{1}{2}\left(2m_2-m_1+\frac{1}{2}M\right)\frac{l}{3}g\theta^2\right)=-\left(2m_2-m_1+\frac{1}{2}M\right)\frac{l}{3}g\theta$$

$$\left(\frac{l^2}{9}m_1+\frac{4l^2}{9}m_2+\frac{1}{9}Ml^2\right)\ddot{\theta}+\left(2m_2-m_1+\frac{1}{2}M\right)\frac{l}{3}g\theta=0$$

$$\begin{cases}\ddot{\theta}+\dfrac{\left(2m_2-m_1+\frac{1}{2}M\right)\frac{l}{3}g}{\left(\frac{l^2}{9}m_1+\frac{4l^2}{9}m_2+\frac{1}{9}Ml^2\right)}\theta=0\\ \ddot{\theta}+\omega_0^2\theta=0\end{cases}$$

$$\begin{cases}\ddot{\theta}+\dfrac{3\left(2m_2-m_1+\frac{1}{2}M\right)g}{\left(m_1+4m_2+M\right)l}\theta=0\\ \ddot{\theta}+\omega_0^2\theta=0\end{cases}$$

The proper pulsation

$$\omega_0^2=\frac{3\left(2m_2-m_1+\frac{1}{2}M\right)g}{\left(m_1+4m_2+M\right)l}\Rightarrow\omega_0=\sqrt{\frac{3\left(2m_2-m_1+\frac{1}{2}M\right)g}{\left(m_1+4m_2+M\right)l}}$$

iii)

$$T=\frac{1}{2}\left(\frac{5l^2}{2}m+\frac{1}{3}Ml^2\right)\dot{\theta}^2$$

$$U=-\left(2m+\frac{1}{2}M\right)gl\left(1-\cos\theta\right)+\frac{3l^2}{4}k\sin\theta^2+2kl\sin\theta x_0+cte$$

For $\theta \ll 1$ $\sin\theta \approx \theta$ *and* $\cos\theta = 1-\dfrac{\theta^2}{2}$

$$U = -\left(2m+\frac{1}{2}M\right)gl\left(1-(1-\frac{\theta^2}{2})\right)+\frac{3l^2}{4}k\theta^2+2kl\theta x_0+cte$$

$$U = -\left(2m+\frac{1}{2}M\right)gl\frac{\theta^2}{2}+\frac{3l^2}{4}k\theta^2+2kl\theta x_0+cte$$

$$U = \frac{1}{2}\left[\left(2m+\frac{1}{2}M\right)gl+\frac{3l^2}{2}k\right]\theta^2+2kl\theta x_0+cte$$

In equilibrium

$$U = \frac{1}{2}\left[\left(2m+\frac{1}{2}M\right)gl+\frac{3l^2}{2}k\right]\theta^2+2kl\theta x_0+cte \Rightarrow \left.\frac{\partial U}{\partial\theta}\right|_{\theta=0}=0$$

$$\frac{\partial U}{\partial\theta} = \left[\left(2m+\frac{1}{2}M\right)gl+\frac{3l^2}{2}k\right]\theta+2klx_0 \Rightarrow \left.\frac{\partial U}{\partial\theta}\right|_{\theta=0} : \theta=0 \Rightarrow +2klx_0=0$$

$$U = \frac{1}{2}\left[\left(2m+\frac{1}{2}M\right)gl+\frac{3l^2}{2}k\right]\theta^2+cte.$$

Then, the Lagrangian of the system is written:

$$L = T-U = \frac{1}{2}\left(\frac{5l^2}{2}m+\frac{1}{3}Ml^2\right)\dot\theta^2-\frac{1}{2}\left[\left(2m+\frac{1}{2}M\right)gl+\frac{3l^2}{2}k\right]\theta^2+cte$$

The equation of motion for small oscillations is: $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot\theta}\right)-\left(\dfrac{\partial L}{\partial\theta}\right)=0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right)=\left(\frac{5l^2}{2}m+\frac{1}{3}Ml^2\right)\ddot\theta$$

$$\frac{\partial L}{\partial\theta}=\frac{\partial}{\partial\theta}\left(-\frac{1}{2}(m_4gl_4-m_2gl_2)\theta^2\right)=-\left[\left(2m+\frac{1}{2}M\right)gl+\frac{3l^2}{2}k\right]\theta$$

$$\left(\frac{5l^2}{2}m+\frac{1}{3}Ml^2\right)\ddot\theta+\left[\left(2m+\frac{1}{2}M\right)gl+\frac{3l^2}{2}k\right]\theta=0$$

$$\begin{cases}\ddot\theta+\dfrac{\left[\left(2m+\frac{1}{2}M\right)gl+\frac{3l^2}{2}k\right]}{\left(\frac{5l^2}{2}m+\frac{1}{3}Ml^2\right)}\theta=0\\ \ddot\theta+\omega_0^2\theta=0\end{cases} \Rightarrow \begin{cases}\ddot\theta+\dfrac{\left[(4m+M)g+3lk\right]}{\left(5m+\frac{2}{3}M\right)l}\theta=0\\ \ddot\theta+\omega_0^2\theta=0\end{cases}$$

The proper pulsation

$$\omega_0^2=\frac{\left[(4m+M)g+3lk\right]}{\left(5m+\frac{2}{3}M\right)l} \Rightarrow \omega_0=\sqrt{\frac{\left[(4m+M)g+3lk\right]}{\left(5m+\frac{2}{3}M\right)l}}$$

iv)

$$T=\frac{1}{2}\left(m_1+m_2+\frac{1}{2}M_1+\frac{1}{2}M_2\right)r^2\dot{\theta}_1^2$$

$$U=\frac{k}{2}r^2\sin\theta_1^2+kr\sin\theta_1 x_0+\left(m_1-m_2\right)r\sin\theta_1++cte$$

For $\theta \ll 1 \Rightarrow \sin\theta\approx\theta$ *and* $\cos\theta=1-\frac{\theta^2}{2}$

$$U=\frac{k}{2}r^2\theta_1^2+krx_0\theta_1+\left(m_1-m_2\right)r\theta_1+cte$$

$$U=\frac{k}{2}r^2\theta_1^2+\left[krx_0+\left(m_1-m_2\right)r\right]\theta_1+cte$$

In equilibrium

$$\left.\frac{\partial U}{\partial\theta}\right|_{\theta_1=0}=0$$

$$\frac{\partial U}{\partial\theta}=\left[kr^2\right]\theta_1+\left[krx_0+\left(m_1-m_2\right)r\right]$$

$$\left.\frac{\partial U}{\partial\theta}\right|_{\theta_1=0}:\theta_1=0\Rightarrow+\left[krx_0+\left(m_1-m_2\right)r\right]=0$$

$$\Rightarrow U=\frac{k}{2}r^2\theta_1^2+cte\cdot$$

Then, the Lagrangian of the system is written:

$$L=T-U=\frac{1}{2}\left(m_1+m_2+\frac{1}{2}M_1+\frac{1}{2}M_2\right)r^2\dot{\theta}_1^2-\frac{k}{2}r^2\theta_1^2+cte$$

The equation of motion for small oscillations is: $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right)-\left(\frac{\partial L}{\partial\theta_1}\right)=0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right)=\left(m_1+m_2+\frac{1}{2}M_1+\frac{1}{2}M_2\right)r^2\ddot{\theta}_1$$

$$\frac{\partial L}{\partial \theta_1} = \frac{\partial}{\partial \theta_1}\left(-\frac{k}{2}r^2\theta_1^2\right) = -kr^2\theta_1$$

$$\left(m_1+m_2+\frac{1}{2}M_1+\frac{1}{2}M_2\right)r^2\ddot{\theta}_1 + kr^2\theta_1 = 0 \Rightarrow \begin{cases} \ddot{\theta}_1 + \dfrac{kr^2}{\left(m_1+m_2+\frac{1}{2}M_1+\frac{1}{2}M_2\right)r^2}\theta_1 = 0 \\ \ddot{\theta}_1 + \omega_0^2\theta_1 = 0 \end{cases}$$

$$\begin{cases} \ddot{\theta}_1 + \dfrac{k}{\left(m_1+m_2+\frac{1}{2}M_1+\frac{1}{2}M_2\right)}\theta_1 = 0 \\ \ddot{\theta} + \omega_0^2\theta_1 = 0 \end{cases}$$

The proper pulsation

$$\omega_0^2 = \frac{k}{\left(m_1+m_2+\frac{1}{2}M_1+\frac{1}{2}M_2\right)} \Rightarrow \omega_0 = \sqrt{\frac{k}{\left(m_1+m_2+\frac{1}{2}M_1+\frac{1}{2}M_2\right)}}$$

v)

$$T = \frac{1}{2}(6m+2M)r^2\dot{\theta}^2$$

$$U = 3kr^2\sin\theta^2 + (4krx_0 - 2mgr)\sin\theta$$

For $\theta \ll 1 \Rightarrow \sin\theta \approx \theta \text{ and } \cos\theta = 1 - \frac{\theta^2}{2}$

$$U = 3kr^2\theta^2 + (4krx_0 - 2mgr)\theta + cte$$

In equilibrium

$$\left.\frac{\partial U}{\partial \theta}\right|_{\theta_1=0} = 0$$

$$\frac{\partial U}{\partial \theta} = 6kr^2\theta + (4krx_0 - 2mgr)$$

$$\left.\frac{\partial U}{\partial \theta}\right|_{\theta_1=0} : \theta_1 = 0 \Rightarrow +(4krx_0 - 2mgr) = 0$$

$U = 3kr^2\theta^2 + cte$.

Then, the Lagrangian of the system is written:

$$L = T - U = \frac{1}{2}(6m+2M)r^2\dot{\theta}^2 - 3kr^2\theta^2 + cte$$

The equation of motion for small oscillations is: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)=\left(6m+2M\right)r^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta}=\frac{\partial}{\partial \theta}\left(-3kr^2\theta^2\right)=-6kr^2\theta$$

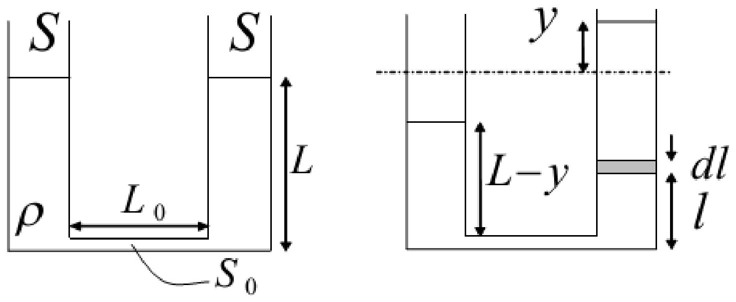
$$\left(6m+2M\right)r^2\ddot{\theta}+6kr^2\theta=0\Rightarrow\begin{cases}\ddot{\theta}+\dfrac{6kr^2}{\left(6m+2M\right)r^2}\theta=0\\ \ddot{\theta}+\omega_0^2\theta=0\end{cases}\Rightarrow\begin{cases}\ddot{\theta}+\dfrac{k}{\left(3m+M\right)}\theta=0\\ \ddot{\theta}+\omega_0^2\theta=0\end{cases}$$

The proper pulsation

$$\omega_0^2=\frac{k}{\left(3m+M\right)}\Rightarrow\omega_0=\sqrt{\frac{k}{\left(3m+M\right)}}$$

Problem 2

1. Find, based on the height, the kinetic energy and the potential energy of the liquid in the *U-shaped* tube below. The volume density of the liquid is ρ: The initial length of the liquid columns as well as the section of each part of the tube are indicated on the diagram.
2. Find the total energy then deduce the equation of motion and the proper pulsation.
3. Find the Lagrangian then deduce the equation of motion.



At rest ***In motion***

Solution

1. Let T be the kinetic energy and U the potential energy.

To find the kinetic energy of the liquid we need the kinetic energy of each of the liquid columns. Since the left and right vertical parts of the tube have the same section S, the liquid columns in these parts will have the same speed $v=\dot{y}$: Since the horizontal part of the tube has a different (smaller) section, the liquid column in this part will have a different (higher) speed $v_0=\dot{y}_0$. When the liquid on the right rises a height *dy*, it draws with it a horizontal liquid column dy_0 such that

$$S_0 dy_0 = Sdy \Rightarrow dy_0 = \frac{S}{S_0}dy \Rightarrow y_0 = \frac{S}{S_0}y$$

$$T_{\text{Left}} = \frac{1}{2}m_{\text{Left}}\dot{y}^2 = \frac{1}{2}\rho S\left(L-y\right)\dot{y}^2$$

$$T_{Right} = \frac{1}{2}m_{Right}\dot{y}^2 = \frac{1}{2}\rho S\left(L+y\right)\dot{y}^2$$

$$T_{\text{Horizontal}} = \frac{1}{2}m_{\text{Horizontal}}\dot{y}_0^2 = \frac{1}{2}\rho S_0\left(L_0\frac{S^2}{S_0{}^2}\right)\dot{y}^2 = \frac{1}{2}\rho\left(L_0\frac{S^2}{S_0}\right)\dot{y}^2$$

$$T = T_{\text{Left}} + T_{Right} + T_{\text{Horizontal}} = \frac{1}{2}\rho S\left(L-y\right)\dot{y}^2 + \frac{1}{2}\rho S\left(L+y\right)\dot{y}^2 + \frac{1}{2}\rho\left(L_0\frac{S^2}{S_0}\right)\dot{y}^2$$

$$T = \rho SL\dot{y}^2 + \frac{1}{2}\rho\left(L_0\frac{S^2}{S_0}\right)\dot{y}^2 = \frac{1}{2}\rho\left(2SL + L_0\frac{S^2}{S_0}\right)\dot{y}^2$$

The potential energy of the liquid column is the sum of the potential energies $dU = dmgl = \rho Sdl\,g\,l$ of the infinitesimal elements dm with height *l*.

$$U_{\text{Left}} = \int dU = \int_0^{L-y}\rho Sd\,\text{l}\,\text{g}\,l = \frac{1}{2}\rho Sg\left(L-y\right)^2$$

$$U_{Right} = U_{\text{Left}} = \int dU = \int_0^{L+y}\rho Sd\,\text{l}\,\text{g}\,l = \frac{1}{2}\rho Sg\left(L+y\right)^2$$

$T_{\text{Horizontal}} = Cte.$ (The horizontal column does not change height)

$$U = U_{\text{Left}} + U_{Right} + U_{\text{Horizontal}} = \frac{1}{2}\rho Sg\left(L-y\right)^2 + \frac{1}{2}\rho Sg\left(L+y\right)^2 + Cte$$

$$U = \frac{1}{2}\rho SgL^2 + \frac{1}{2}\rho Sgy^2 - \rho Sg\left(Ly\right) + \frac{1}{2}\rho SgL^2 + \frac{1}{2}\rho Sgy^2 + \rho Sg\left(Ly\right) + Cte$$

$U = \rho Sgy^2 + \rho SgL^2 + Cte$ *We have* $\rho SgL^2 = Cte$

So $U = \rho Sgy^2 + Cte$

2. The total energy and then deduce the equation of motion and the proper pulsation of the system

The total energy of a system is *E*=*T*+*U*. To find the equation of motion, it suffices to write the equation of conservation of total energy: $\frac{dE}{dt} = 0$

$$E = T + U = \frac{1}{2}\rho\left(2SL + L_0\frac{S^2}{S_0}\right)\dot{y}^2 + \rho Sgy^2 + Cte$$

The equation of motion is:

$$\frac{dE}{dt}=0\Rightarrow\rho\left(2SL+L_0\frac{S^2}{S_0}\right)\ddot{y}+2\rho Sgy=0\Rightarrow\ddot{y}+\frac{2\rho Sg}{\rho\left(2SL+L_0\frac{S^2}{S_0}\right)}y=0$$

$$\frac{dE}{dt}=0\Rightarrow\ddot{y}+\frac{2SS_0g}{\left(2SS_0L+L_0S^2\right)}y=0$$

$$\omega_0=\sqrt{\frac{2SS_0g}{\left(2SS_0L+L_0S^2\right)}}$$

3. The Lagrangian and the equation of motion of the system

The Lagrangian of a system is $L=T-U$. To find the equation of motion, simply write the Lagrange equation $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{y}}\right)-\left(\frac{\partial L}{\partial y}\right)=0$

$$L=T-U=\frac{1}{2}\rho\left(2SL+L_0\frac{S^2}{S_0}\right)\dot{y}^2-\rho Sgy^2+Cte$$

The equation of motion is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{y}}\right)-\left(\frac{\partial L}{\partial y}\right)=0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{y}}\right)=\rho\left(2SL+L_0\frac{S^2}{S_0}\right)\ddot{y}$$

$$\left(\frac{\partial L}{\partial y}\right)=-\rho Sgy$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{y}}\right)-\left(\frac{\partial L}{\partial y}\right)=0\Rightarrow\rho\left(2SL+L_0\frac{S^2}{S_0}\right)\ddot{y}+\rho Sgy=0$$

$$\begin{cases}\ddot{y}+\frac{2SS_0g}{\left(2SS_0L+L_0S^2\right)}y=0\\ \ddot{y}+\omega_0^2y=0\end{cases}\Rightarrow\omega_0^2=\frac{2SS_0g}{\left(2SS_0L+L_0S^2\right)}\Rightarrow\omega_0=\sqrt{\frac{2SS_0g}{\left(2SS_0L+L_0S^2\right)}}$$

Chapter 3:

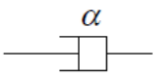
Free linear system damped to one degree of freedom

3.1 Damping force

A system subjected to friction is said to be damped. The simplest friction is *viscous* friction. Viscous frictions are of the form

$$f = -\alpha v$$

α: is a positive constant called coefficient of friction and v is the speed of the moving body. In mechanics, damping is schematized by:



The speed v is in this case the relative speed of the two damping arms.

3.2 Lagrange equation of damped systems

If there is friction $f = -\alpha\dot{q}$, the Lagrange equation becomes:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) = -\alpha\dot{q} \quad .$$

By introducing the dissipation function $D = \frac{1}{2}\alpha\dot{q}^2$, we can write:

$f = -\alpha\dot{q} = -\frac{\partial D}{\partial \dot{q}}$. (In translation $D = \frac{1}{2}\alpha v^2 = \frac{1}{2}\alpha\dot{x}^2$. In electricity $D = \frac{1}{2}Ri^2 = \frac{1}{2}R\dot{q}^2$). The Lagrange equation for damped systems is then written (where q=x, y, z, q, θ…)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) = -\frac{\partial D}{\partial \dot{q}}$$

3.3 Equation of motion of damped systems

The equation of motion of linear systems damped by $f = -\alpha\dot{q}$ is of the following form

$$\ddot{q}(t) + 2\delta\dot{q}(t) + \omega_0^2 q(t) = 0$$

Where δ is a positive coefficient and is called damping factor. ω_0 is the proper pulsation. $\frac{\omega_0}{2\delta} = Q$ is called *quality factor.*

3.4 Solving the equation of motion

$$\ddot{q}(t) + 2\delta\dot{q}(t) + \omega_0^2 q(t) = 0$$

The solution to this equation is done by changing the variable $q(t) = Ae^{rt} \Rightarrow \dot{q}(t) = Are^{rt} \Rightarrow \ddot{q}(t) = Ar^2e^{rt}$, the equation then becomes:

$$\ddot{q}(t) + 2\delta\dot{q}(t) + \omega_0^2 q(t) = 0$$
$$Ar^2e^{rt} + 2\delta Are^{rt} + \omega_0^2 Ae^{rt} = 0$$
$$Ae^{rt}\left(r^2 + 2\delta r + \omega_0^2\right) = 0$$

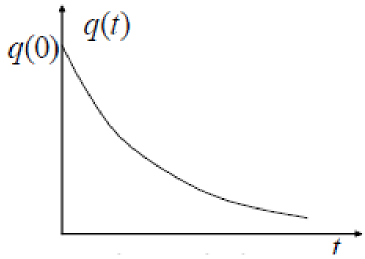
$$r^2 + 2\delta r + \omega_0^2 = 0$$

We calculate the discriminant Δ and then obtain:

$$\Delta = (2\delta)^2 - 4\omega_0^2$$
$$\Delta = 4\delta^2 - 4\omega_0^2$$
$$\Delta = 4\left(\delta^2 - \omega_0^2\right)$$
$$\sqrt{\Delta} = \sqrt{4\left(\delta^2 - \omega_0^2\right)} = 2\sqrt{\left(\delta^2 - \omega_0^2\right)}$$

There are three types of solutions:

- ***Case where the system is strongly damped:*** $\Delta > 0 \Rightarrow \delta > \omega_0$ et Q < 0.5



The solution to the differential equation is written as follows:

$$q(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t} \ avec \ r_{1,2} = \frac{-2\delta \pm 2\sqrt{\delta^2 - \omega_0^2}}{2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2}$$

$$q(t) = A_1 e^{\left(-\delta - \sqrt{\delta^2 - \omega_0^2}\right)t} + A_2 e^{\left(-\delta + \sqrt{\delta^2 - \omega_0^2}\right)t} = e^{-\delta t}\left(A_1 e^{\left(-\sqrt{\delta^2 - \omega_0^2}\right)t} + A_2 e^{\left(\sqrt{\delta^2 - \omega_0^2}\right)t}\right)$$

Where A_1 and A_2 are coefficients to be determined by the initial conditions

$$\begin{cases} q(t=0) \\ \dot{q}(t=0) \end{cases}$$

The system is said to have aperiodic motion.

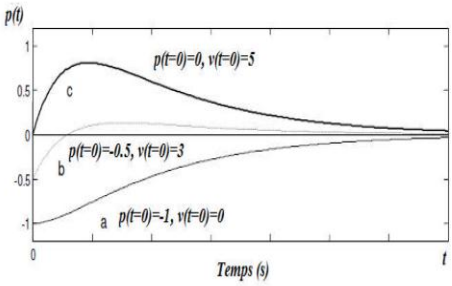
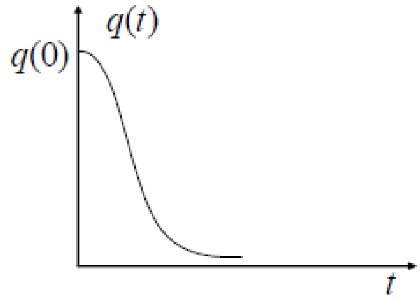


Figure 3.1: Aperiodic damped movement.

- **Case where depreciation is critical:** $\Delta = 0 \Rightarrow \delta = \omega_0$ et Q = 0.5



The solution to the equation is of the form:

$$q(t) = (A_1 t + A_2) e^{rt}$$

$$r_1 = r_2 = r = \frac{-2\delta}{2} = -\delta$$

$$q(t) = (A_1 t + A_2) e^{-\delta t}$$

Where A_1 and A_2 are coefficients to be determined by the initial conditions

$$\begin{cases} q(t=0) \\ \dot{q}(t=0) \end{cases}$$

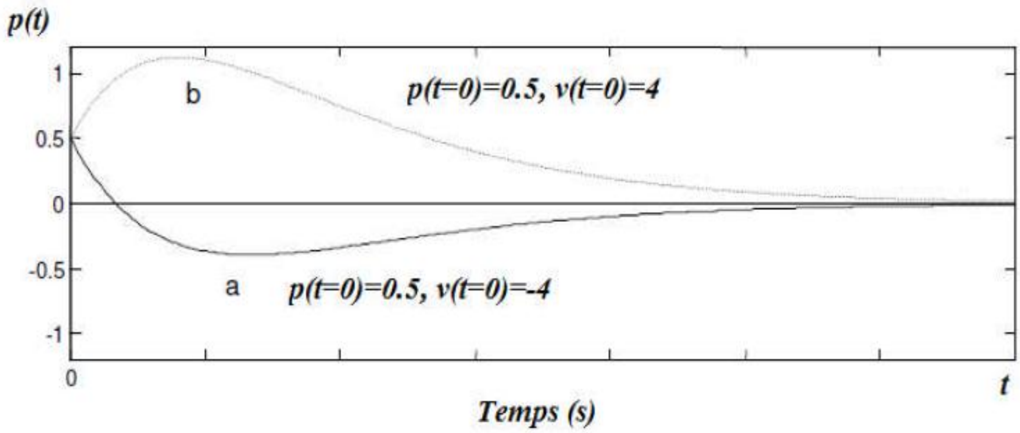
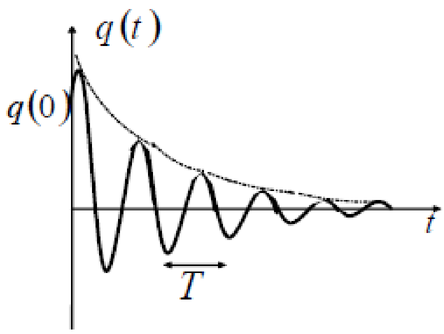


Figure 3.2: Critical damped movement.

- **Case where depreciation is low:** $\Delta < 0 \Rightarrow \delta < \omega_0$ et Q > 0.5



Two complex solutions for the characteristic equation

$$\begin{cases} r_1 = -\delta - j\sqrt{\omega_0^2 - \delta^2} \\ r_2 = -\delta + j\sqrt{\omega_0^2 - \delta^2} \end{cases}$$

The resulting movement is $q(t) = A_1 e^{-r_1 t} + A_2 e^{-r_2 t}$ either :

$$q(t) = Ae^{-\delta t}\cos(\omega t + \varphi)$$

Where A and φ are constants to be determined by the initial conditions: $\begin{cases} q(t=0) \\ \dot{q}(t=0) \end{cases}$

The movement is called pseudo-periodic, $\omega = \sqrt{\omega_0^2 - \delta^2}$ is called *pseudo-pulsation.*

$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0^2 - \delta^2}}$ is called *pseudo period.*

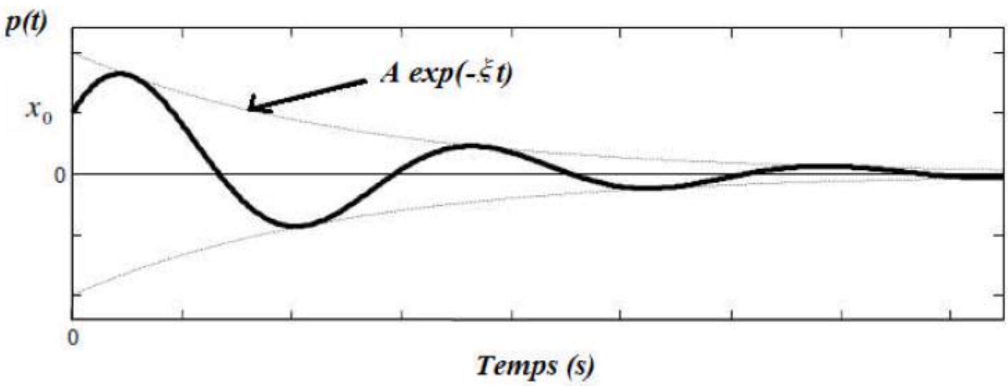


Figure 3.3: Damped oscillatory movement.

3.5 Logarithmic decrement

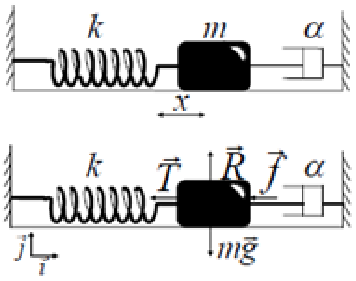
To evaluate the exponential decrease in the amplitude of the pseudo-periodic movement, we use the logarithm. The report is $D = Ln\frac{q(t)}{q(t+T)}$ Or $D = \frac{1}{n}Ln\frac{q(t)}{q(t+nT)}$ is called the logarithmic decrement. Using the equation $q(t) = Ae^{-\delta t}\cos(\omega t + \varphi)$, we find $\delta = Ln\frac{Ae^{-\delta t}}{Ae^{-\delta(t+T)}} \Rightarrow D = \delta T$

- ✓ It should be noted that the system undergoes a *total loss of energy* due to *the work of friction forces*.

$$dE_T(t) = -\alpha\dot{p}(t)^2 dt = -dW_{fr} \Rightarrow \Delta E_T + \Delta W_{fr} = 0$$

3.5.1 Examples

a) Consider the mass-spring system opposite. Find the equation of motion first with the Lagrangian then with PFD.



• Using the Lagrangian:

The Lagrangian is:

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2.$$

$$D = \frac{1}{2}\alpha v^2 = \frac{1}{2}\alpha\dot{x}^2$$

The Lagrange equation is then written:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = -\frac{\partial D}{\partial \dot{x}}$$

$$With\ L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + cte$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right) = \frac{1}{2}m\frac{\partial}{\partial \dot{x}}\left(\dot{x}^2\right) = \frac{1}{2}m\left(2\dot{x}\right) = m\dot{x} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}\left(m\dot{x}\right) = m\frac{d}{dt}\left(\dot{x}\right) = m\ddot{x}$$

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{2}kx^2\right) = -\frac{1}{2}k\frac{\partial}{\partial x}\left(x^2\right) = -\frac{1}{2}k\left(2x\right) = -kx$$

$$D = \frac{1}{2}\alpha v^2 = \frac{1}{2}\alpha\dot{x}^2 \Rightarrow \frac{\partial D}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}\alpha\dot{x}^2\right) = \frac{1}{2}\alpha\frac{\partial}{\partial \dot{x}}\left(\dot{x}^2\right) = \frac{1}{2}\alpha\left(2\dot{x}\right) = \alpha\dot{x}$$

$$m\ddot{x} + kx = -\alpha\dot{x} \Rightarrow \begin{cases} \ddot{x} + \dfrac{k}{m}x + \dfrac{\alpha}{m}\dot{x} = 0 \\ \ddot{x} + \omega_0^2 x + 2\delta\dot{x} = 0 \end{cases}$$

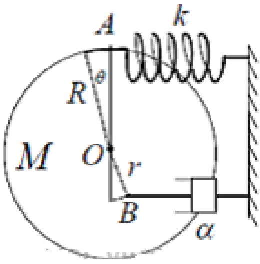
$$\omega_0^2 = \frac{k}{m} \Rightarrow \omega_0 = \sqrt{\frac{k}{m}} \text{ and } 2\delta = \frac{\alpha}{m} \Rightarrow \delta = \frac{\alpha}{2m}$$

Using the PFD: $\sum \vec{F} = m\,\vec{a} \Rightarrow m\vec{g} + \vec{T} + \vec{R} + \vec{f} = m\,\vec{a} \Rightarrow -kx\vec{i} - mg\vec{j} + R\vec{j} - \alpha\dot{x}\vec{i} = m\ddot{x}\vec{i}$

By projection on $x'Ox$:

$$-kx - \alpha\dot{x} = m\ddot{x} \Rightarrow m\ddot{x} + kx + \alpha\dot{x} = 0 \Rightarrow \ddot{x} + \frac{k}{m}x + \frac{\alpha}{m}\dot{x} = 0$$

b) Or the disk-spring system opposite $\theta \ll 1$. Find the equation of motion if M=1kg, k=2 N/m, R=10cm, r=5cm, α=8Ns/m.



Using the Lagrangian:

$$T = E_c = T\left[Disk\right] = \frac{1}{2}J\dot{\theta}^2$$

$$J = \frac{1}{2}MR^2$$

$$T = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot{\theta}^2$$

$$U = U_k = \frac{k}{2}x^2 + kxx_0 + cte$$

With $\sin\theta = \dfrac{x}{R} \Rightarrow x = R\sin\theta$

$$U = U_k = \frac{k}{2}x^2 + kxx_0 + cte = \frac{k}{2}\left(\mathrm{R}\sin\theta\right)^2 + k\left(\mathrm{R}\sin\theta\right)x_0 + cte$$

At low amplitude $\sin\theta \approx \theta$

$$U = \frac{k}{2}R^2\left(\theta^2\right) + \left(kRx_0\right)\left(\theta\right) + cte$$

In equilibrium

$$\left.\frac{\partial U}{\partial \theta}\right|_{\theta=0} = 0$$

$$\frac{\partial U}{\partial \theta} = \frac{k}{2} \times R^2 \times 2\theta - \left(kRx_0\right)$$

$$\left.\frac{\partial U}{\partial \theta}\right|_{\theta=0} : \theta = 0 \Rightarrow \frac{k}{2} \times R^2 \times 2\left(\theta = 0\right) - \left(kRx_0\right) = -\left(kRx_0\right)$$

$$\left.\frac{\partial U}{\partial x}\right|_{\theta=0} = 0 \Rightarrow -\left(kRx_0\right) = 0 \Rightarrow x_0 = 0$$

$$U = \frac{k}{2}R^2\theta^2 + cte\ .$$

$$D=\frac{1}{2}\alpha v^2=\frac{1}{2}\alpha\dot{x}^2$$

$$\dot{x}=r\dot{\theta}\Rightarrow D=\frac{1}{2}\alpha\left(r\dot{\theta}\right)^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)-\left(\frac{\partial L}{\partial\theta}\right)=-\frac{\partial D}{\partial\dot{\theta}}$$

$$With\ L=T-U=U=\frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot{\theta}^2-\frac{k}{2}R^2\theta^2+cte$$

$$\frac{\partial L}{\partial\dot{\theta}}=\frac{\partial}{\partial\dot{\theta}}\left(\frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot{\theta}^2\right)=\frac{1}{2}\left(\frac{1}{2}MR^2\right)\frac{\partial}{\partial\dot{\theta}}\left(\dot{\theta}^2\right)=\frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(2\dot{\theta}\right)=\left(\frac{1}{2}MR^2\right)\dot{\theta}$$

$$\Rightarrow\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)=\frac{d}{dt}\left(\left(\frac{1}{2}MR^2\right)\dot{\theta}\right)=\left(\frac{1}{2}MR^2\right)\frac{d}{dt}\left(\dot{\theta}\right)=\left(\frac{1}{2}MR^2\right)\ddot{\theta}$$

$$\frac{\partial L}{\partial\theta}=\frac{\partial}{\partial\theta}\left(-\frac{1}{2}kR^2\theta^2\right)=-\frac{1}{2}kR^2\frac{\partial}{\partial\theta}\left(\theta^2\right)=-\frac{1}{2}kR^2\left(2\theta\right)=-kR^2\theta$$

$$D=\frac{1}{2}\alpha v^2=\frac{1}{2}\alpha r^2\dot{\theta}^2\Rightarrow\frac{\partial D}{\partial\dot{\theta}}=\frac{\partial}{\partial\dot{\theta}}\left(\frac{1}{2}\alpha r^2\dot{\theta}^2\right)=\frac{1}{2}\alpha r^2\frac{\partial}{\partial\dot{\theta}}\left(\dot{\theta}^2\right)=\frac{1}{2}\alpha r^2\left(2\dot{\theta}\right)=\alpha r^2\dot{\theta}$$

$$\left(\frac{1}{2}MR^2\right)\ddot{\theta}+kR^2\theta=-\alpha r^2\dot{\theta}\Rightarrow\begin{cases}\ddot{\theta}+\dfrac{kR^2}{\left(\frac{1}{2}MR^2\right)}\theta+\dfrac{\alpha r^2}{\left(\frac{1}{2}MR^2\right)}\dot{\theta}=0\\ \ddot{\theta}+\omega_0^2\theta+2\delta\dot{\theta}=0\end{cases}\Rightarrow\begin{cases}\ddot{\theta}+\dfrac{2k}{M}\theta+\dfrac{2\alpha r^2}{MR^2}\dot{\theta}=0\\ \ddot{\theta}+\omega_0^2\theta+2\delta\dot{\theta}=0\end{cases}$$

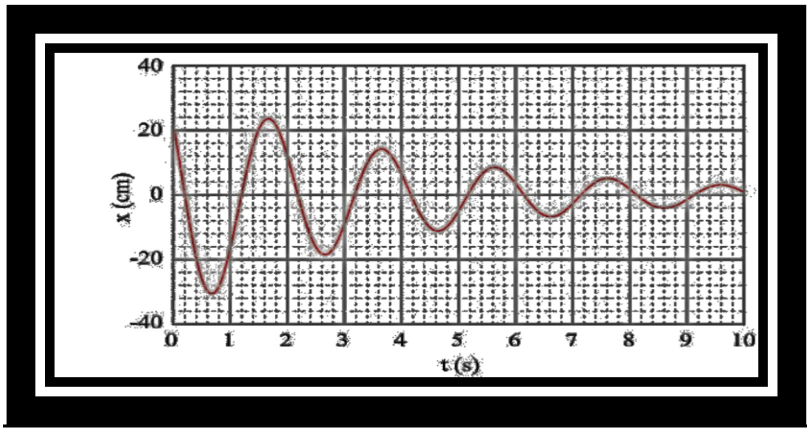
$$\omega_0^2=\frac{2k}{M}\Rightarrow\omega_0=\sqrt{\frac{2k}{M}}\ and\ 2\delta=\frac{2\alpha r^2}{MR^2}\Rightarrow\delta=\frac{\alpha r^2}{MR^2}$$

3.5.2 Exercises and problems

Exercise No. 1

We consider a mechanical oscillator $k+m+\alpha$. The instantaneous position *x(t)* of the mass *m* is represented by the graph in the following figure.

1. What is the evolution regime of the oscillator? Give the differential equation involving the damping coefficient and the natural pulsation. Give the expression of *D*(*t*).
2. Determine the pseudo-period graphically T_a.
3. Recall the definition of the logarithmic decrement *D*. Determine it graphically and deduce the damping coefficient and the proper period T_0.



Solution

1.a. The regime is pseudoperiodic.

1.b. Differential equation is:

$$\begin{cases} \ddot{x}+\dfrac{k}{m}x+\dfrac{\alpha}{m}\dot{x}=0 \\ \ddot{x}+\omega_0^2x+2\delta\dot{x}=0 \end{cases}$$

$$\omega_0^2=\frac{k}{m}\Rightarrow\omega_0=\sqrt{\frac{k}{m}}$$

$$2\delta=\frac{\alpha}{m}\Rightarrow\delta=\frac{\alpha}{2m}$$

ω_0 : Undamped system proper pulsation (that is to say when $\alpha=0$)

δ: Damping coefficient.

2. T_a value of the graph:

$$T_a=2s\Rightarrow\omega_a=2\pi f_a=\frac{2\pi}{T_a}=\frac{2\pi}{2}=\pi\ rad.s^{-1}.$$

3. The expression of D :

$$D=\ln\left(\frac{x_1}{x_2}\right)$$

x_1, x_2 : Two successive crests

$$D=\ln\left(\frac{24}{16}\right)=0.448$$

$$D=\delta T_a\Rightarrow\delta=\frac{D}{T_a}=\frac{0.448}{2}$$

$$\delta=0.224s^{-1}$$

On the other hand, we have

$$T_a = \frac{2\pi}{\omega_a} \Rightarrow \omega_a = \frac{2\pi}{T_a} = \frac{2\pi}{2} = \pi\ rad.s^{-1}$$

$$\omega_a = \sqrt{\omega_0^2 - \delta^2}$$

$$\omega_a^2 = \omega_0^2 - \delta^2 \Rightarrow \omega_0^2 = \omega_a^2 + \delta^2$$

$$T_0 = \frac{2\pi}{\omega_0} = 1.99\,s$$

Exercise No. 2

A body of mass m=0.5 kg resting on a horizontal plane is connected to a fixed frame by a spring of stiffness k=245 N/m.

Moved x_0 = 3cm from its equilibrium position then released without initial speed, it performs free oscillations damped by a dry friction coefficient $\mu = 0.1$.

1- Calculate the period of oscillations T_0.
2- How many half-periods does the body perform and at what distance does it stop?
3- What total distance will he have traveled? Deduce the resistive work of the friction force.
4- Compare the potential energies of the spring before and after these oscillations, conclusion?

Solution

1.1 : $$\begin{cases} \ddot{x} + \frac{k}{m}x + \frac{\alpha}{m}\dot{x} = 0 \\ \ddot{x} + \omega_0^2 x + 2\delta\dot{x} = 0 \end{cases}$$

$$2\delta = \frac{\alpha}{m} \Rightarrow \delta = \frac{\alpha}{2m}$$

$$\omega_0^2 = \frac{k}{m} \Rightarrow \omega_0 = \sqrt{\frac{k}{m}} = \frac{2\pi}{T_0} \Rightarrow T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}} = 0.28\,s$$

2. We know that the amplitudes decrease linearly according to:

$x_k = x_0 - n\left(\frac{2\mu mg}{k}\right)$ That is to say the amplitude decreases each half-period $\left(\frac{T_0}{2}\right)$ with a fixed pitch $\left(\frac{2\mu mg}{k}\right)$.

x_k : Absolute value of amplitudes.

2.1. The system stops when the restoring force becomes smaller than the friction force, that is to say:

$$kx_k \leq \mu mg$$
$$x_k = x_0 - n\left(\frac{2\mu mg}{k}\right)$$
$$kx_k = kx_0 - nk\left(\frac{2\mu mg}{k}\right)$$
$$kx_k = kx_0 - n(2\mu mg)$$
$$kx_k \leq \mu mg \Rightarrow kx_0 - n(2\mu mg) \leq \mu mg$$
$$-kx_0 + n(2\mu mg) \geq -\mu mg \Rightarrow n(2\mu mg) \geq -\mu mg + kx_0$$

$$\frac{n(2\mu mg)}{(2\mu mg)} \geq \frac{-\mu mg}{(2\mu mg)} + \frac{kx_0}{(2\mu mg)} \Rightarrow n \geq \frac{-\mu mg}{(2\mu mg)} + \frac{kx_0}{(2\mu mg)} \Rightarrow n \geq \frac{-1}{2} + \frac{kx_0}{(2\mu mg)}$$
$$n \geq \frac{kx_0}{(2\mu mg)} - \frac{1}{2}$$

Digital Application : $n \geq 6.85 \Rightarrow n = 7$

2.2. So the system stops after having completed 7 half-periods

$$x_7 = x_0 - 7\left(\frac{2\mu mg}{k}\right)$$
$$x_7 = 1.4 \times 10^{-3}\, m$$

3.1. Distance traveled

$$D = x_0 + 2(x_2 + x_3 + x_4 + x_5 + x_6) + x_7$$

$$x_1 = x_0 - 1\times\left(\frac{2\mu mg}{k}\right),\ x_2 = x_0 - 2\times\left(\frac{2\mu mg}{k}\right),\ x_3 = x_0 - 3\times\left(\frac{2\mu mg}{k}\right),$$
$$x_4 = x_0 - 4\times\left(\frac{2\mu mg}{k}\right),\ x_5 = x_0 - 5\times\left(\frac{2\mu mg}{k}\right),\ x_6 = x_0 - 6\times\left(\frac{2\mu mg}{k}\right)$$

$$D = 14x_0 - 49\frac{2\mu mg}{k} \Rightarrow D = 0.22\, m$$

3.2. The resistant work of friction force

$$\omega(f_s) = -f_s d = -\mu mgd$$
$$\omega(f_s) = -0.11\,J$$

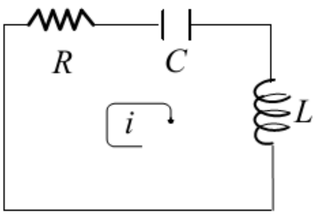
4. Variation of U:

$$\Delta U = U(x_7) - U(x_0)$$
$$\Delta U = \frac{1}{2}k\left(x_7^2 - x_0^2\right)$$
$$\Delta U = -0.11\,J$$

That's to say $\Delta U = \omega(f_s)$ the energy lost by the system is transformed by f_f of the heat.

Exercise No. 3

We consider a series *RLC* electrical circuit. We give $C = 10\mu F, L = 100mH$. The capacitor is initially charged. At time *t=0*, we close the *RLC* circuit and let it evolve freely. We set the resistance successively to the following three different values: $R = 100\,\Omega, 150\,\Omega, 250\,\Omega$.



1. Determine in each case the operating regime of this oscillator. In what cases are oscillations observed? Then deduce the pseudo-period.
2. What value should we give to R to be in critical mode?

Solution

$$\sum V_i = 0 \Rightarrow V_L + V_C + V_R = 0$$

$$V_C = \frac{q}{C}, V_L = L\frac{di}{dt}, \mathrm{i} = \frac{dq}{dt} = \dot{q} \Rightarrow V_L = L\frac{d}{dt}\dot{q} = L\ddot{q},\ V_R = Ri = R\dot{q}$$

$$L\ddot{q} + \frac{q}{C} + R\dot{q} = 0 \Rightarrow L\ddot{q} + \frac{1}{C}q + R\dot{q} = 0$$

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$$\begin{cases} \ddot{q} + \frac{1}{LC} q + \frac{R}{L} \dot{q} = 0 \\ \ddot{q} + \omega_0^2 q + 2\delta \dot{q} = 0 \end{cases}$$

$$\omega_0^2 = \frac{1}{LC} \Rightarrow \omega_0 = \sqrt{\frac{1}{LC}}$$

$$2\delta = \frac{R}{L} \Rightarrow \delta = \frac{R}{2L}$$

$C=10\mu F=10\times 10^{-6}F$, $L=100mH=100\times 10^{-3}H$.

Three Cases:

1) $\delta < \omega_0$: Low Damping $\Rightarrow$ Pseudoperiodic Regime.

2) $\delta = \omega_0$: Critical Damping $\Rightarrow$ Critical Regime.

3) $\delta > \omega_0$: High Damping.

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{0.1\times 10\times 10^{-6}}} = 10^{+3} rad.s^{-1}$$

Calculation of $\delta = \frac{R}{2L}$

$R = 100\,\Omega, 150\,\Omega, 250\,\Omega$

$R = 100\,\Omega \Rightarrow \delta = 500\ s^{-1} < \omega_0 = 10^3$

$R = 150\,\Omega \Rightarrow \delta = 750\ s^{-1} < \omega_0 = 10^3$

$R = 250\,\Omega \Rightarrow \delta = 1250\ s^{-1} > \omega_0 = 10^3$

So:

1. $R = 100\,\Omega$ *pseudoperiodic regime*

$$\omega_a = 2\pi f_a = \frac{2\pi}{T_a}$$

$$T_a = \frac{2\pi}{\omega_a}$$

$$\omega_a = \sqrt{\omega_0^2 - \delta^2} = 866\ rad.s^{-1} \Rightarrow T_a = 0.007\ s$$

2. $R = 150\,\Omega$ *pseudoperiodic regime*

$$\omega_a = 2\pi f_a = \frac{2\pi}{T_a}$$

$$T_a = \frac{2\pi}{\omega_a}$$

$$\omega_a = \sqrt{\omega_0^2 - \delta^2} = 661.4\, rad.s^{-1} \Rightarrow T_a = 0.009\, s$$

For the critical regime it is necessary that $\delta = \omega_0$:

$$\omega_0 = \sqrt{\frac{1}{LC}}$$

$$\delta = \frac{R}{2L}$$

$$\sqrt{\frac{1}{LC}} = \frac{R_{(critical)}}{2L} \Rightarrow R_{(critical)} = 2L\sqrt{\frac{1}{LC}} = \frac{2L}{\sqrt{L}}\sqrt{\frac{1}{C}} = 2\sqrt{\frac{L}{C}}$$

$$R_{(critical)} = 200\,\Omega.$$

Exercise No. 4

I-Undamped free regime

We consider a system with one degree of freedom in the following figure. The homogeneous disk of mass *M* and radius *R* can pivot around its fixed horizontal axis passing through its center. A rigid rod of length l and without mass is attached to the disk and carries at its free end a point mass *m*. A horizontally placed stiffness constant spring is connected to the disk as shown in the following figure, the other end being held fixed. The system is in static equilibrium when the rod is in its horizontal position. In movement the rod is identified in relation to this position by the nail $\theta(t)$. We place ourselves in the case of low amplitude vibrations and we admit that $sin\theta \approx \theta$ and $\cos\theta \approx 1$.

1- Calculate the total potential energy $U(\theta)$ of the system. Determine the deformation Δx of the spring at static equilibrium. Then simplify the expression of $U(\theta)$.

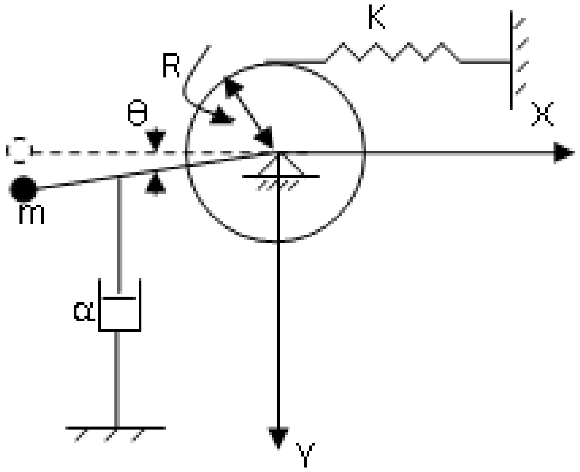
2- Calculate the kinetic energy $T(\theta)$ of the system. We give $J_{disk} = \frac{1}{2}MR^2$

3- Write the Lagrangian of the system $L(\dot{\theta}, \theta)$ and deduce the differential equation governing the movement of the system and its proper pulsation.

II-Damped free regime

The system now experiences viscous friction represented by a damper with linear coefficient α placed as indicated in the following figure knowing that

$m = \frac{M}{8}, k = \frac{2mgl}{R^2}, R = \frac{l}{2}$, show that the differential equation of motion is written: $\ddot{\theta} + 2\delta\dot{\theta} + \omega_0^2\theta = 0$ we specify the expression of δ and ω_0.



Solution

I-Undamped free regime

$$U = U_k + U_m$$

$$U_k = \frac{k}{2}x^2 + kxx_0 + cte \text{ with } \sin\theta = \frac{-x}{R} \Rightarrow x = -R\sin\theta$$

$$U_k = \frac{k}{2}x^2 + kxx_0 + cte = \frac{k}{2}(-R\sin\theta)^2 + k(-R\sin\theta)x_0 + cte$$

$$U_m = -mgh$$

$$\sin\theta = \frac{h}{l} \Rightarrow h = l\sin\theta \Rightarrow U_m = -mgl\sin\theta$$

$$\Rightarrow U = U_k + U_m = \frac{k}{2}(-R\sin\theta)^2 + k(-R\sin\theta)x_0 - mgl\sin\theta + cte$$

$$U = \frac{k}{2}R^2(\sin\theta^2) - kRx_0(\sin\theta) - mgl(\sin\theta) + cte$$

$$U = \frac{k}{2}R^2(\sin\theta^2) - (kRx_0 + mgl)(\sin\theta) + cte$$

At low amplitude $\sin\theta \approx \theta$

$$U = \frac{k}{2}R^2(\theta^2) - (kax_0 + mgl)(\theta) + cte$$

In equilibrium

$$\left.\frac{\partial U}{\partial \theta}\right|_{\theta=0}=0$$

$$\frac{\partial U}{\partial \theta}=\frac{k}{2}\times R^2\times 2\theta-\left(kRx_0+mgl\right)$$

$$\left.\frac{\partial U}{\partial \theta}\right|_{\theta=0}:\theta=0\Rightarrow\frac{k}{2}\times R^2\times 2\left(\theta=0\right)-\left(kRx_0+mgl\right)=-\left(kRx_0+mgl\right)$$

$$\left.\frac{\partial U}{\partial \theta}\right|_{x=0}=0\Rightarrow-\left(kRx_0+mgl\right)=0\Rightarrow x_0=-\frac{mgl}{kR}$$

$$\Rightarrow U=\frac{k}{2}R^2\left(\theta^2\right)+cte.$$

$$T=E_c=T\left[mass\right]+T\left[Disk\right]=\frac{1}{2}mv^2\left[mass\right]+\frac{1}{2}J\dot{\theta}^2\left[Disk\right]=\frac{1}{2}m\dot{x}^2\left[mass\right]+\frac{1}{2}J\dot{\theta}^2\left[Disk\right]$$

$$T\left[mass\right]=\frac{1}{2}m\dot{x}^2\left[mass\right]$$

$$\dot{x}=l\dot{\theta}$$

$$T\left[mass\right]=\frac{1}{2}m\left(l\dot{\theta}\right)^2=\frac{1}{2}ml^2\dot{\theta}^2$$

$$T\left[Disk\right]=\frac{1}{2}J\dot{\theta}^2\left[Disk\right]$$

$$J=\frac{1}{2}MR^2$$

$$T\left[Disk\right]=\frac{1}{2}J\dot{\theta}^2\left[Disk\right]=\frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot{\theta}^2\left[Disk\right]$$

$$T=\frac{1}{2}ml^2\dot{\theta}^2+\frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot{\theta}^2$$

$$T=\frac{1}{2}\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)\dot{\theta}^2$$

Then, the Lagrangian of the system is written:

$$L=E_c-E_p=T-U=\frac{1}{2}\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)\dot{\theta}^2-\frac{k}{2}R^2\left(\theta^2\right)+cte$$

The equation of motion for small oscillations is: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\left(\frac{\partial L}{\partial \theta}\right)=0$

$$L=\frac{1}{2}\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)\dot{\theta}^2-\frac{k}{2}R^2\left(\theta^2\right)+cte$$

$$\frac{\partial L}{\partial\dot{\theta}}=\frac{\partial}{\partial\dot{\theta}}\left[\frac{1}{2}\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)\dot{\theta}^2\right]=\frac{1}{2}\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)\frac{\partial}{\partial\dot{\theta}}\left[\dot{\theta}^2\right]$$

$$\frac{\partial L}{\partial\dot{\theta}}=\frac{1}{2}\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)\times 2\dot{\theta}=\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)=\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)\ddot{\theta}$$

$$\frac{\partial L}{\partial\theta}=\frac{\partial}{\partial\theta}\left(-\frac{k}{2}R^2\left(\theta^2\right)\right)=-\frac{k}{2}R^2\frac{\partial}{\partial\theta}\left(\theta^2\right)=-\frac{k}{2}R^2\times 2\theta=-kR^2\theta$$

$$\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)\ddot{\theta}+kR^2\theta=0$$

$$\begin{cases}\ddot{\theta}+\dfrac{kR^2}{\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)}\theta=0\\ \ddot{\theta}+\omega_0^2\theta=0\end{cases}$$

$$\omega_0^2=\frac{kR^2}{\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)}\Rightarrow\omega_0=\sqrt{\frac{kR^2}{\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)}}$$

II-Damped free regime

The equation of motion for small oscillations is: $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)-\left(\frac{\partial L}{\partial\theta}\right)+\left(\frac{\partial D}{\partial\dot{\theta}}\right)=0$

$$L=\frac{1}{2}\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)\dot{\theta}^2-\frac{k}{2}R^2\left(\theta^2\right)+cte$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)=\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)\ddot{\theta}$$

$$\frac{\partial L}{\partial\theta}=-kR^2\theta$$

$$D=\frac{1}{2}\alpha\dot{x}^2$$

$$\dot{x}=b\dot{\theta}$$

$$D=\frac{1}{2}\alpha\left(b\dot{\theta}\right)^2=\frac{1}{2}\alpha\left(b^2\dot{\theta}^2\right)=\frac{1}{2}\alpha b^2\dot{\theta}^2$$

$$\frac{\partial D}{\partial\dot{\theta}}=\frac{\partial}{\partial\dot{\theta}}\left(\frac{1}{2}\alpha b^2\dot{\theta}^2\right)=\frac{1}{2}\alpha b^2\frac{\partial}{\partial\dot{\theta}}\left(\dot{\theta}^2\right)=\frac{1}{2}\alpha b^2\left(2\dot{\theta}\right)=\alpha b^2\dot{\theta}$$

$$\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)\ddot{\theta}+kR^2\theta+\alpha b^2\dot{\theta}=0$$

$$\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)\ddot{\theta}+kR^2\theta++\alpha b^2\dot{\theta}=0$$

$$\begin{cases}\ddot{\theta}+\dfrac{kR^2}{\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)}\theta+\dfrac{\alpha b^2}{\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)}\dot{\theta}=0\\ \ddot{\theta}+\omega_0^2\theta+2\delta\dot{\theta}=0\end{cases}$$

$$\omega_0^2=\frac{kR^2}{\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)}\Rightarrow\omega_0=\sqrt{\frac{kR^2}{\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)}}$$

$$2\delta=\frac{\alpha b^2}{\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)}\Rightarrow\delta=\frac{\alpha b^2}{2\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)}$$

When

$$m=\frac{M}{8},k=\frac{2mgl}{R^2},R=\frac{l}{2}$$

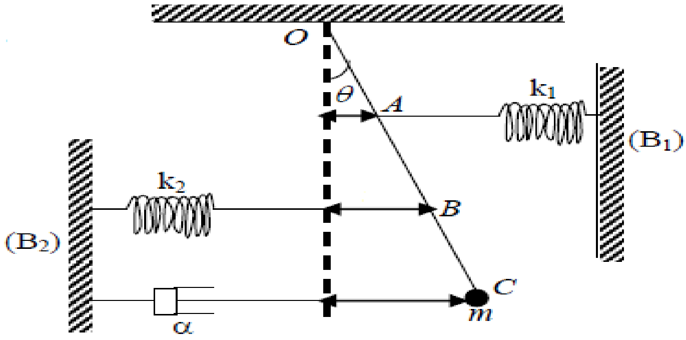
$$\omega_0=\sqrt{\frac{kR^2}{\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)}}=\sqrt{\frac{g}{2R}}$$

$$\delta=\frac{9\alpha}{8M}$$

Exercise No. 5

A mass m is welded to the end of a rod of length *l* and negligible mass. The other end of the wire is articulated at point O. The rod is linked at point A to a Frame (B_1) by a spring of stiffness k_1. At point B, the rod is connected to a Frame (B_2) by a spring of stiffness k_2. The mass m is linked to the Frame (B_2) by a damper with a friction coefficient α. *OA=l/3*, *OB=2l/3* and *OC=l*.

1. Calculate the kinetic and potential energy of the system.
2. Deduce the Lagrangian.
3. Find the differential equation of motion.
4. Determine the solution of the differential equation in the case of low damping, the damping coefficient δ, the natural pulsation ω_0 and the pseudo-pulsation ω_a.



Solution

1.1- Potential energy

$$U = U_{k_1} + U_{k_2} + U_m$$

$$U_{k_1} = \frac{k_1}{2}x^2 + k_1 x x_0 + cte$$

$$\sin\theta = \frac{x}{OA} = \frac{x}{\frac{l}{3}} \Rightarrow x = \frac{l}{3}\sin\theta$$

$$U_{k_1} = \frac{k_1}{2}x^2 + k_1 x x_0 + cte = \frac{k_1}{2}\left(\frac{l}{3}\sin\theta\right)^2 + k_1\left(\frac{l}{3}\sin\theta\right)x_0 + cte$$

$$U_{k_2} = \frac{k_2}{2}x^2 + k_2 x x_0 + cte$$

$$\sin\theta = \frac{x}{OB} = \frac{x}{\frac{2l}{3}} \Rightarrow x = \frac{2l}{3}\sin\theta$$

$$U_{k_2} = \frac{k_2}{2}x^2 + k_2 x x_0 + cte = \frac{k_2}{2}\left(\frac{2l}{3}\sin\theta\right)^2 + k_2\left(\frac{2l}{3}\sin\theta\right)x_0 + cte$$

$$U_m = mgh$$

$$l = h + x \Rightarrow h = l - x$$

$$\cos\theta = \frac{x}{l} \Rightarrow x = l\cos\theta \Rightarrow h = l - l\cos\theta = l(1-\cos\theta)$$

$$U_m = mgl(1-\cos\theta)$$

$$U = \frac{k_1}{2}\left(\frac{l}{3}\sin\theta\right)^2 + k_1\left(\frac{l}{3}\sin\theta\right)x_0 + \frac{k_2}{2}\left(\frac{2l}{3}\sin\theta\right)^2 + k_2\left(\frac{2l}{3}\sin\theta\right)x_0 + mgl(1-\cos\theta) + cte$$

At low amplitude $\sin\theta \approx \theta \; and \; co\theta \approx 1 - \frac{\theta^2}{2}$

$$U=\frac{k_1}{2}\left(\frac{l}{3}\theta\right)^2+k_1\left(\frac{l}{3}\theta\right)x_0+\frac{k_2}{2}\left(\frac{2l}{3}\theta\right)^2+k_2\left(\frac{2l}{3}\theta\right)x_0+mgl\,(1-\left(1-\frac{\theta^2}{2}\right))+cte$$

$$U=\frac{k_1}{2}\left(\frac{l}{3}\right)^2\theta^2+k_1\left(\frac{l}{3}\right)x_0\theta+\frac{k_2}{2}\left(\frac{2l}{3}\right)^2\theta^2+k_2\left(\frac{2l}{3}\right)x_0\theta+mg\,\frac{l}{2}\theta^2+cte$$

$$U=\left(\frac{k_1}{2}\left(\frac{l}{3}\right)^2+\frac{k_2}{2}\left(\frac{2l}{3}\right)^2+mg\,\frac{l}{2}\right)\theta^2+\left(k_1\left(\frac{l}{3}\right)+k_2\left(\frac{2l}{3}\right)\right)x_0\theta+cte$$

$$U=\frac{1}{2}\left(k_1\left(\frac{l}{3}\right)^2+k_2\left(\frac{2l}{3}\right)^2+mgl\right)\theta^2+\left(k_1\left(\frac{l}{3}\right)+k_2\left(\frac{2l}{3}\right)\right)x_0\theta+cte$$

In equilibrium

$$\left.\frac{\partial U}{\partial\theta}\right|_{\theta=0}=0$$

$$\frac{\partial U}{\partial\theta}=\frac{1}{2}\left(k_1\left(\frac{l}{3}\right)^2+k_2\left(\frac{2l}{3}\right)^2+mgl\right)\times 2\theta+\left(k_1\left(\frac{l}{3}\right)+k_2\left(\frac{2l}{3}\right)\right)x_0$$

$$\left.\frac{\partial U}{\partial\theta}\right|_{\theta=0}:\theta=0\Rightarrow\frac{1}{2}\left(kl^2-2mgl\right)\times 2\left(\theta=0\right)+\left(k_1\left(\frac{l}{3}\right)+k_2\left(\frac{2l}{3}\right)\right)x_0=+\left(k_1\left(\frac{l}{3}\right)+k_2\left(\frac{2l}{3}\right)\right)x_0$$

$$\left.\frac{\partial U}{\partial\theta}\right|_{\theta=0}=0\Rightarrow\left(k_1\left(\frac{l}{3}\right)+k_2\left(\frac{2l}{3}\right)\right)x_0=0$$

$$U=\frac{1}{2}\left(k_1\left(\frac{l}{3}\right)^2+k_2\left(\frac{2l}{3}\right)^2+mgl\right)\theta^2+cte$$

1.2- The kinetic energy

$$T=T_m=\frac{1}{2}m\dot{x}^2$$

$$\dot{x}=l\dot{\theta}$$

$$T=\frac{1}{2}m\left(l\dot{\theta}\right)^2=\frac{1}{2}ml^2\dot{\theta}^2$$

2- The Lagrangian of the system is written:

$$L=T-U=\frac{1}{2}ml^2\dot{\theta}^2-\frac{1}{2}\left(k_1\left(\frac{l}{3}\right)^2+k_2\left(\frac{2l}{3}\right)^2+mgl\right)\theta^2+cte$$

3- The Lagrange equation for a damped system is $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)-\left(\frac{\partial L}{\partial\theta}\right)+\left(\frac{\partial D}{\partial\dot{\theta}}\right)=0$

$$D=\frac{1}{2}\alpha\dot{x}^2$$

$$D=\frac{1}{2}\alpha\left(l\dot{\theta}\right)^2=\frac{1}{2}\alpha\left(l^2\dot{\theta}^2\right)=\frac{1}{2}\alpha l^2\dot{\theta}^2$$

$$\frac{\partial D}{\partial\dot{\theta}}=\frac{\partial}{\partial\dot{\theta}}\left(\frac{1}{2}\alpha l^2\dot{\theta}^2\right)=\frac{1}{2}\alpha l^2\frac{\partial}{\partial\dot{\theta}}\left(\dot{\theta}^2\right)=\frac{1}{2}\alpha l^2\left(2\dot{\theta}\right)=\alpha l^2\dot{\theta}$$

$$L=\frac{1}{2}ml^2\dot{\theta}^2-\frac{1}{2}\left(k_1\left(\frac{l}{3}\right)^2+k_2\left(\frac{2l}{3}\right)^2+mgl\right)\theta^2+cte$$

$$\frac{\partial L}{\partial\dot{\theta}}=\frac{\partial}{\partial\dot{\theta}}\left[\frac{1}{2}ml^2\dot{\theta}^2\right]=\left(\frac{1}{2}ml^2\right)\frac{\partial}{\partial\dot{\theta}}\left[\dot{\theta}^2\right]$$

$$\frac{\partial L}{\partial\dot{\theta}}=\left(\frac{1}{2}ml^2\right)\times2\dot{\theta}=ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)=ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial\theta}=\frac{\partial}{\partial\theta}\left(-\frac{1}{2}\left(k_1\left(\frac{l}{3}\right)^2+k_2\left(\frac{2l}{3}\right)^2+mgl\right)\left(\theta^2\right)\right)=-\frac{1}{2}\left(k_1\left(\frac{l}{3}\right)^2+k_2\left(\frac{2l}{3}\right)^2+mgl\right)\frac{\partial}{\partial\theta}\left(\theta^2\right)$$

$$\frac{\partial L}{\partial\theta}=-\frac{1}{2}\left(k_1\left(\frac{l}{3}\right)^2+k_2\left(\frac{2l}{3}\right)^2+mgl\right)\times2\theta=-\left(k_1\left(\frac{l}{3}\right)^2+k_2\left(\frac{2l}{3}\right)^2+mgl\right)\theta$$

$$ml^2\ddot{\theta}+\left(k_1\left(\frac{l}{3}\right)^2+k_2\left(\frac{2l}{3}\right)^2+mgl\right)\theta+\alpha l^2\dot{\theta}=0$$

$$\ddot{\theta}+\frac{\left(k_1\left(\frac{l}{3}\right)^2+k_2\left(\frac{2l}{3}\right)^2+mgl\right)}{ml^2}\theta+\frac{\alpha l^2}{ml^2}\dot{\theta}=0$$

4- For low dumping

4.1- $\theta(t)=Ae^{-\delta t}\cos(\omega_a t+\varphi)$

By analogy with:

$$\begin{cases}\ddot{\theta}+\frac{\left(k_1\left(\frac{l}{3}\right)^2+k_2\left(\frac{2l}{3}\right)^2+mgl\right)}{ml^2}\theta+\frac{\alpha}{m}\dot{\theta}=0\\ \ddot{\theta}+\omega_0^2\theta+2\delta\dot{\theta}=0\end{cases}$$

4.2- The proper pulsation:

$$\omega_0^2=\frac{\left(k_1\left(\frac{l}{3}\right)^2+k_2\left(\frac{2l}{3}\right)^2+mgl\right)}{ml^2}\Rightarrow\omega_0=\sqrt{\frac{\left(k_1\left(\frac{l}{3}\right)^2+k_2\left(\frac{2l}{3}\right)^2+mgl\right)}{ml^2}}$$

4.3- The damping coefficient:

$$2\delta=\frac{\alpha}{m}\Rightarrow\delta=\frac{\alpha}{2m}$$

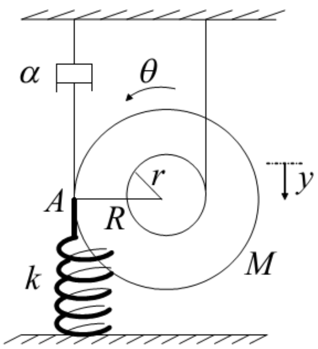
4.4- The pseudo pulsation:

$$\omega_a=\sqrt{\omega_0^2-\delta^2}$$

Problem 1

By turning the disc opposite can move up and down thanks to the non-slip and inextensible thread wrapped around the furrow circular of radius *r*. At equilibrium the spring *k* was compressed with a distance y_0. The α damper represents the friction

1. Find the kinetic energy of the system.
2. Find the potential energy according to *y*.
3. Simplify *U* using the equilibrium condition.
4. Find the Lagrangian and the dissipation function *D*.
5. Derive the equation of motion. ($\theta \ll 1$. The moment of inertia of the disk is $I = \frac{1}{2}MR^2$).
6. Knowing that *M*= 2kg, *R*= 50cm, *r*= 25cm; *k*=10N/m; find the maximum value that the coefficient α must not reach for the system to oscillate.
7. With a damper with coefficient α= 5N.m^{-1}.s, the system oscillates but its amplitude decreases over time. Find the time τ necessary for the amplitude to decrease to 1/2 of its value.
8. Calculate the logarithmic decrement *D*.
9. The previous damper is now replaced by another with coefficient α': We then notice that the amplitude decreases to 1/3 of its value after 22 complete oscillations. Deduce the value of the coefficient α'.



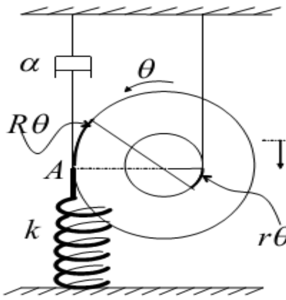
Solution

1. $$T = T\left[Disk\right]_{\text{(Translation)}} + T\left[Disk\right]_{\text{(Rotation)}} = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot{\theta}^2$$

Since the wire is inextensible and non-slippery, when the disk descends a distance *y* it rotates through an angle θ such that: $y=r\theta$, from where $\theta = \frac{y}{r} \Rightarrow R\theta = \frac{Ry}{r} \Rightarrow R\dot{\theta} = \frac{R\dot{y}}{r}$

$$\theta=\frac{y}{r}\Rightarrow R\theta=\frac{Ry}{r}\Rightarrow R\dot{\theta}=\frac{R\dot{y}}{r}$$

$$T=\frac{1}{2}M\dot{y}^2+\frac{1}{2}\left(\frac{1}{2}M\frac{R^2}{r^2}\right)\dot{y}^2=\frac{1}{2}M\left(1+\frac{R^2}{2r^2}\right)\dot{y}^2$$



2. Going down a distance y the disk also rotates through an angle θ, so the displacement of point A during the movement is $y+R\theta=\left(1+\frac{R}{r}\right)y$. As the spring is connected to point A and was already compressed initially, its total compression is $y_0+y=y_0+\left(1+\frac{R}{r}\right)y$. From where

$$U=U_{Disk}+U_{\text{Spring}}=-Mgy+\frac{1}{2}k\left(y_0+y+R\theta\right)^2=-Mgy+\frac{1}{2}k\left(y_0+\left(1+\frac{R}{r}\right)y\right)^2$$

$$U=-Mgy+\frac{1}{2}ky_0^{\ 2}+\frac{1}{2}k\left(1+\frac{R}{r}\right)^2y^2+k\left(\left(1+\frac{R}{r}\right)yy_0\right)$$

We have $\frac{1}{2}ky_0^{\ 2}=Cte$

$$U=\frac{1}{2}k\left(1+\frac{R}{r}\right)^2y^2+\left(\left(1+\frac{R}{r}\right)y_0-Mg\right)y+Cte$$

3. The condition of equilibrium is $\left.\frac{\partial U}{\partial y}\right|_{y=0}=0$, allows us to simplify U

$$U=\frac{1}{2}k\left(1+\frac{R}{r}\right)^2y^2+\left(\left(1+\frac{R}{r}\right)y_0-Mg\right)y+Cte$$

$$\frac{\partial U}{\partial y}=\frac{1}{2}k\left(1+\frac{R}{r}\right)^2\times2y+\left(\left(1+\frac{R}{r}\right)y_0-Mg\right)$$

$$\left.\frac{\partial U}{\partial y}\right|_{y=0}:y=0\Rightarrow\left(1+\frac{R}{r}\right)^2\times(y=0)+\left(\left(1+\frac{R}{r}\right)y_0-Mg\right)=\left(\left(1+\frac{R}{r}\right)y_0-Mg\right)$$

$$\left.\frac{\partial U}{\partial\theta}\right|_{\theta=0}=0\Rightarrow\left(\left(1+\frac{R}{r}\right)y_0-Mg\right)=0$$

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$$\Rightarrow U=\frac{1}{2}k\left(1+\frac{R}{r}\right)^2 y^2+Cte$$

4. The Lagrangian of the system is written:

$$L=T-U=\frac{1}{2}M\left(1+\frac{R^2}{2r^2}\right)\dot{y}^2-\frac{1}{2}k\left(1+\frac{R}{r}\right)^2 y^2+Cte$$

5. The Lagrange equation for a damped system is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)-\left(\frac{\partial L}{\partial y}\right)+\left(\frac{\partial D}{\partial \dot{y}}\right)=0$

$$D=\frac{1}{2}\alpha\left(\dot{y}+R\dot{\theta}\right)^2=\frac{1}{2}\alpha\left(\dot{y}+\frac{R}{r}\dot{y}\right)^2=\frac{1}{2}\alpha\left(1+\frac{R}{r}\right)^2\dot{y}^2$$

$$\frac{\partial D}{\partial \dot{y}}=\alpha\left(1+\frac{R}{r}\right)^2\dot{y}$$

$$L=\frac{1}{2}M\left(1+\frac{R^2}{2r^2}\right)\dot{y}^2-\frac{1}{2}k\left(1+\frac{R}{r}\right)^2 y^2+Cte$$

$$\frac{\partial L}{\partial \dot{y}}=\frac{\partial L}{\partial \dot{y}}\left[\frac{1}{2}M\left(1+\frac{R^2}{2r^2}\right)\dot{y}^2\right]=M\left(1+\frac{R^2}{2r^2}\right)\dot{y}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)=M\left(1+\frac{R^2}{2r^2}\right)\ddot{y}$$

$$\frac{\partial L}{\partial y}=\frac{\partial}{\partial y}\left(-\frac{1}{2}k\left(1+\frac{R}{r}\right)^2 y^2\right)=-k\left(1+\frac{R}{r}\right)^2 y$$

$$M\left(1+\frac{R^2}{2r^2}\right)\ddot{y}+k\left(1+\frac{R}{r}\right)^2 y+\alpha\left(1+\frac{R}{r}\right)^2\dot{y}=0$$

$$\ddot{y}+\frac{k\left(1+\frac{R}{r}\right)^2}{M\left(1+\frac{R^2}{2r^2}\right)}y+\frac{\alpha\left(1+\frac{R}{r}\right)^2}{M\left(1+\frac{R^2}{2r^2}\right)}\dot{y}=0$$

$$\ddot{y}+\frac{k\frac{(r+R)^2}{r^2}}{M\left(\frac{2r^2+R^2}{2r^2}\right)}y+\frac{\alpha\frac{(r+R)^2}{r^2}}{M\left(\frac{2r^2+R^2}{2r^2}\right)}\dot{y}=0\Rightarrow\ddot{y}+\frac{2k(r+R)^2}{M\left(2r^2+R^2\right)}y+\frac{2\alpha(r+R)^2}{M\left(2r^2+R^2\right)}\dot{y}=0$$

6. The equation is of the form

$$\begin{cases}\ddot{y}+\frac{2k(r+R)^2}{M\left(2r^2+R^2\right)}y+\frac{2\alpha(r+R)^2}{M\left(2r^2+R^2\right)}\dot{y}=0\\ \ddot{y}+\omega_0^2 y+2\delta\dot{y}=0\end{cases}\Rightarrow\omega_0^2=\frac{2k(r+R)^2}{M\left(2r^2+R^2\right)}\ and\ \delta=\frac{\alpha(r+R)^2}{M\left(2r^2+R^2\right)}$$

For a damped system to oscillate, it must be in a pseudo-periodic regime, so it is necessary that:

$$\delta^2-\omega_0^2<0\Rightarrow\delta<\omega_0\Rightarrow\frac{\alpha(r+R)^2}{M(2r^2+R^2)}<\sqrt{\frac{2k(r+R)^2}{M(2r^2+R^2)}}\Rightarrow\alpha<\frac{M(2r^2+R^2)}{(r+R)^2}.\sqrt{\frac{2k(r+R)^2}{M(2r^2+R^2)}}$$

$$\Rightarrow\alpha<.\sqrt{\frac{2kM(2r^2+R^2)}{(r+R)^2}}\Rightarrow\alpha<.\frac{\sqrt{2kM(2r^2+R^2)}}{(r+R)}$$

Digital Application

$\alpha < 5.16$ *N.s/m*. This is the value that α must not reach for the system to oscillate.

7. Since the damped system oscillates its movement is pseudo-periodic: the time equation is: $y(t)=Ae^{-\delta t}\cos(\omega t+\varphi)=Ae^{-\delta t}\cos\left(\sqrt{\omega_0^2-\delta^2}\,t+\varphi\right)$

For the amplitude to decrease to 1/2 of its value, a time τ is required such that

$$Ae^{-\delta(t+\tau)}=\frac{1}{2}Ae^{-\delta t}\Rightarrow e^{-\delta(t+\tau)}=\frac{1}{2}e^{-\delta t}\Rightarrow e^{-\delta t}e^{-\delta\tau}=\frac{1}{2}e^{-\delta t}\Rightarrow e^{-\delta\tau}=\frac{1}{2}$$

$$\Rightarrow\ln e^{-\delta\tau}=\ln\frac{1}{2}\Rightarrow-\delta\tau=-\ln 2\Rightarrow\tau=\frac{\ln 2}{\delta}$$

Digital Application

Since $\delta=3.75s^{-1}$ $\Rightarrow\tau\approx 0.18s$

8. The logarithmic decrement is $D=\delta T=\delta\frac{2\pi}{\omega}=\delta\frac{2\pi}{\sqrt{\omega_0^2-\delta^2}}$

Digital Application $D\approx 24.33$

9. Since the amplitude decreases to 1/3 of its value after 22 oscillations, we have

$$Ae^{-\delta'(t+22T')}=\frac{1}{3}Ae^{-\delta' t}\Rightarrow e^{-\delta'(t+22T')}=\frac{1}{3}e^{-\delta' t}\Rightarrow e^{-\delta' t}e^{-22\delta' T'}=\frac{1}{3}e^{-\delta' t}\Rightarrow e^{-22\delta' T'}=\frac{1}{3}$$

$$\Rightarrow\ln e^{-22\delta' T'}=\ln\frac{1}{3}\Rightarrow 22\delta' T'=\ln 3$$

$$T'=\frac{2\pi}{\omega'}\Rightarrow 44\delta'\frac{\pi}{\omega'}=\ln 3\Rightarrow 44\delta'\frac{\pi}{\sqrt{\omega_0^2-\delta'^2}}=\ln 3\Rightarrow\frac{(44\delta'\pi)^2}{\omega_0^2-\delta'^2}=(\ln 3)^2$$

$$\Rightarrow(44\pi)^2\delta'^2=(\ln 3)^2(\omega_0^2-\delta'^2)\Rightarrow\left((44\pi)^2+(\ln 3)^2\right)\delta'^2=(\ln 3)^2\omega_0^2\Rightarrow\delta'^2=\frac{(\ln 3)^2\omega_0^2}{\left((44\pi)^2+(\ln 3)^2\right)}$$

$$So\ \delta'=\frac{\omega_0\ln 3}{\sqrt{(44\pi)^2+(\ln 3)^2}}$$

Digital Application

$\delta'\approx 0.03s^{-1}$. Since $\delta'=\frac{\alpha'(r+R)^2}{M(2r^2+R^2)}\Rightarrow\alpha'=\frac{M(2r^2+R^2)\delta'}{(r+R)^2}$. We find $\alpha'\approx 0.04Ns/m$.

Problem 2

Consider the electrical circuit below.

1. Using the mesh law, find the equation for the movement of the charge in the circuit.

2. Derive the differential equation for current *i*.

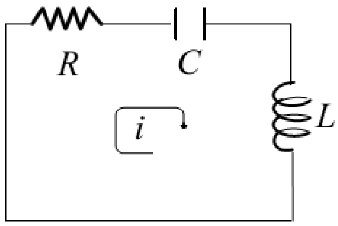
3. Derive the differential equation for the voltage V_L across *L*.

4. Knowing that *L= 2H*, *C= 50nF*; find the maximum value that the resistance must not reach for the circuit to oscillate.

5. With a resistance R= 500Ω, the circuit oscillates but the amplitude of V_L decreases over time. Find the time τ necessary for the amplitude to decrease to 1/5 of its value.

6. Calculate the quality factor of this oscillator.

7. The previous resistance is now replaced by another weaker one *R'*. We then notice that the amplitude decreases to 1/9 of its value after 12 complete oscillations. Deduce the value of the resistance *R'*.



Solution

1. The law of meshes gives us $\sum V_i = 0 \Rightarrow V_L + V_C + V_R = 0$

$$V_C = \frac{q}{C}$$

$$V_L = L\frac{di}{dt}, \mathrm{i} = \frac{dq}{dt} = \dot{q} \Rightarrow V_L = L\frac{d}{dt}\dot{q} = L\ddot{q}$$

$$V_R = Ri = R\dot{q}$$

$$L\ddot{q} + \frac{q}{C} + R\dot{q} = 0 \Rightarrow L\ddot{q} + \frac{1}{C}q + R\dot{q} = 0$$

$$\ddot{q} + \frac{1}{LC}q + \frac{R}{L}\dot{q} = 0 \quad (1)$$

2. By differentiating once equation (1) we find;

$$\frac{d}{dt}\ddot{q} + \frac{1}{LC}\frac{d}{dt}q + \frac{R}{L}\frac{d}{dt}\dot{q} = 0 \Rightarrow \frac{d^2}{dt^2}\dot{q} + \frac{1}{LC}\dot{q} + \frac{R}{L}\frac{d}{dt}\dot{q} = 0 \Rightarrow \frac{d^2 i}{dt^2} + \frac{i}{LC} + \frac{R}{L}\frac{di}{dt} = 0 \quad (2)$$

3. By differentiating once equation (2) we find;

$$\frac{d^3i}{dt^3}+\frac{1}{LC}\frac{di}{dt}+\frac{R}{L}\frac{d^2i}{dt^2}=0.$$

Since $V_L=L\frac{di}{dt}$, we have $\frac{di}{dt}=\frac{V_L}{L}$ From where $\frac{1}{L}\frac{d^2V_L}{dt^2}+\frac{1}{L^2C}V_L+\frac{R}{L^2}\frac{dV_L}{dt}=0$

$$\Rightarrow\frac{1}{L}\ddot{V}_L+\frac{1}{L^2C}V_L+\frac{R}{L^2}\dot{V}_L=0\Rightarrow\ddot{V}_L+\frac{1}{LC}V_L+\frac{R}{L}\dot{V}_L=0$$

4. The equation is of the form:

$$\begin{cases}\ddot{V}_L+\frac{1}{LC}V_L+\frac{R}{L}\dot{V}_L=0\\ \ddot{V}_L+\omega_0^2V_L+2\delta\dot{V}_L=0\end{cases}$$

$$\omega_0^2=\frac{1}{LC}\Rightarrow\omega_0=\sqrt{\frac{1}{LC}},\ 2\delta=\frac{R}{L}\Rightarrow\delta=\frac{R}{2L}$$

For a damped system to oscillate, it must be in a pseudo-periodic regime, so it is necessary that: $\delta^2-\omega_0^2<0\Rightarrow\delta<\omega_0\Rightarrow\frac{R}{2L}<\sqrt{\frac{1}{LC}}\Rightarrow R<2\sqrt{\frac{L}{C}}$.

Digital Application: R<12649Ω. This is the value that *R* must not reach for the circuit to oscillate.

5. Since the damped system oscillates its movement is pseudo-periodic: the time equation is: $V_L(t)=Ae^{-\delta t}\cos(\omega t+\varphi)=Ae^{-\delta t}\cos\left(\sqrt{\omega_0^2-\delta^2}\,t+\varphi\right)$

For the amplitude to decrease to 1/5 of its value, a time τ is required such that

$$Ae^{-\delta(t+\tau)}=\frac{1}{5}Ae^{-\delta t}\Rightarrow e^{-\delta(t+\tau)}=\frac{1}{5}e^{-\delta t}\Rightarrow e^{-\delta t}e^{-\delta\tau}=\frac{1}{5}e^{-\delta t}\Rightarrow e^{-\delta\tau}=\frac{1}{5}$$

$$\Rightarrow\ln e^{-\delta\tau}=\ln\frac{1}{5}\Rightarrow-\delta\tau=-\ln 5\Rightarrow\tau=\frac{\ln 5}{\delta}$$

Digital Application

Since $\delta=125s^{-1}\Rightarrow\tau\approx 0.013s$.

6. The quality factor of this oscillator is $Q=\frac{\omega_0}{2\delta}=\frac{1}{R}\sqrt{\frac{L}{C}}\approx 12.6$

7. Since the amplitude decreases to 1/9 of its value after 12 oscillations, we have

$$Ae^{-\delta'(t+12T')}=\frac{1}{9}Ae^{-\delta't}\Rightarrow e^{-\delta'(t+12T')}=\frac{1}{9}e^{-\delta't}\Rightarrow e^{-\delta't}e^{-12\delta'T'}=\frac{1}{9}e^{-\delta't}\Rightarrow e^{-12\delta'T'}=\frac{1}{9}$$

$$\Rightarrow \ln e^{-12\delta'T'}=\ln\frac{1}{9}\Rightarrow 12\delta'T'=\ln 9$$

$$T'=\frac{2\pi}{\omega'}\Rightarrow 24\delta'\frac{\pi}{\omega'}=\ln 9\Rightarrow 24\delta'\frac{\pi}{\sqrt{\omega_0^2-\delta'^2}}=\ln 9\Rightarrow\frac{(24\delta'\pi)^2}{\omega_0^2-\delta'^2}=(\ln 9)^2$$

$$\Rightarrow (24\pi)^2\delta'^2=(\ln 9)^2\left(\omega_0^2-\delta'^2\right)\Rightarrow\left((24\pi)^2+(\ln 9)^2\right)\delta'^2=(\ln 9)^2\omega_0^2\Rightarrow\delta'^2=\frac{(\ln 9)^2\omega_0^2}{\left((24\pi)^2+(\ln 9)^2\right)}$$

$$So\ \delta'=\frac{\omega_0\ln 9}{\sqrt{(24\pi)^2+(\ln 9)^2}}$$

Digital Application $\delta'\approx 92.2s^{-1}$. Since $\delta'=\frac{R'}{2L}\Rightarrow R'=2L\delta'$. We find $R'\approx 368.8\Omega$.

Problem 3

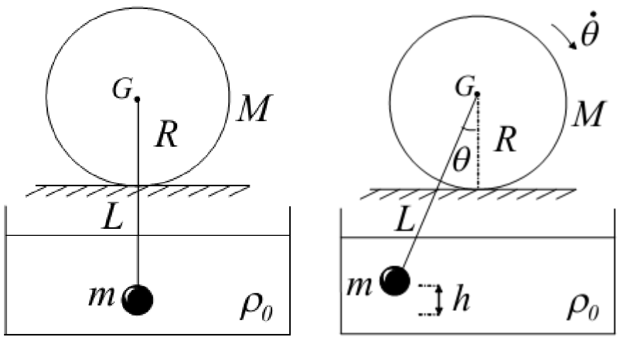
The system below consists of a cylinder of mass *M* rolling without sliding on a horizontal table. A massless rod of length L is stuck to the cylinder and carries at its end a ball of mass *m*; of density ρ and very small radius in front of *L*. In its back and forth movement on the table, the disk causes the ball to swing inside a liquid of density ρ_0.

1. Find the kinetic energy *T* and potential energy *U* of the system as a function of $\theta \ll 1$, then construct the Lagrangian. (Due to Archimedes' push, the apparent weight of the ball is $P=mg-f_A$)

2. Assuming that the ball is subjected to a viscous friction force from the liquid: $\vec{f}=-\alpha\vec{v}$; find the dissipation function *D* and the equation of motion.

3. The sizes of the system are: *M= 20kg, m= 1.125kg, L= 50cm, R= 25cm,* α*= 0.93N.m*$^{-1}$*.s, ρ= 1751kg/m*^{3}*; g= 10m/s*^{2}: By observing the oscillations of the system we noticed that the amplitude of the separations decreased to 1/6 of its value after 23 oscillations. 1. Use this observation and the table below containing the densities of some liquids to discover the liquid in which the ball is immersed.

4. Calculate the quality factor of this oscillator.



At rest ***In motion***

Liquid	Density ρ_0(kg/m^3)
Milk	1035
Sea water	1028
Water	1000
Olive oil	910
Benzene	879
Alcohol	789

Solution

1.

$$T = T_{\text{Cylinder (Translation)}} + T_{\text{Cylinder (Rotation)}} + T_{\text{Ball}} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}mv_{Ball}{}^2$$

When the disc advances a distance $x = R\sin\theta$ on the table it swings the ball backwards a distance $L\sin\theta$.

$$\begin{cases} x_{Ball} = x - L\sin\theta = R\sin\theta - L\sin\theta \approx R\theta - L\theta \\ y_{Ball} = L\cos\theta \approx L \end{cases} \Rightarrow \begin{cases} v_{x(Ball)} \approx (R-L)\dot{\theta} \\ v_{y(Ball)} \approx 0 \end{cases}$$

$$\Rightarrow v_{Ball} = v^2{}_{x(Ball)} + v^2{}_{y(Ball)} \approx (R-L)^2\dot{\theta}^2$$

$$T = \frac{1}{2}M\left(R\dot{\theta}\right)^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot{\theta}^2 + \frac{1}{2}m(R-L)^2\dot{\theta}^2 = \frac{1}{2}\left[\frac{3}{2}MR^2 + m(R-L)^2\right]\dot{\theta}^2$$

Because of Archimedes' push, the weight P of the ball inside the liquid is not mg but $P = mg - f_A = mg - \rho_0 g V_{Ball}$.

Since $V_{Ball} = \dfrac{m}{\rho} \Rightarrow P = mg - f_A = mg - \rho_0 g \dfrac{m}{\rho} = m\left(1 - \dfrac{\rho_0}{\rho}\right)g$

So the potential energy is not *mgh* but, $U = m\left(1-\frac{\rho_0}{\rho}\right)gh = m\left(1-\frac{\rho_0}{\rho}\right)g\left(L-L\cos\theta\right)$

$$U \approx m\left(1-\frac{\rho_0}{\rho}\right)gL\left(1-\left(1-\frac{\theta^2}{2}\right)\right) \Rightarrow U \approx \frac{1}{2}mgL\left(1-\frac{\rho_0}{\rho}\right)\theta^2 + Cte$$

The Lagrangian is: $L = T - U = \frac{1}{2}\left[\frac{3}{2}MR^2 + m\left(R-L\right)^2\right]\dot{\theta}^2 - \frac{1}{2}mgL\left(1-\frac{\rho_0}{\rho}\right)\theta^2 + Cte$

2. The dissipation function is: $D = \frac{1}{2}\alpha {v_{Ball}}^2 = \frac{1}{2}\alpha\left(R-L\right)^2\dot{\theta}^2$

Equation of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \left(\frac{\partial L}{\partial\theta}\right) + \left(\frac{\partial D}{\partial\dot{\theta}}\right) = 0 \Rightarrow \left[\frac{3}{2}MR^2 + m\left(R-L\right)^2\right]\ddot{\theta} + mgL\left(1-\frac{\rho_0}{\rho}\right)\theta + \alpha\left(R-L\right)^2\dot{\theta} = 0$$

$$\Rightarrow \ddot{\theta} + \frac{2mgL\left(1-\frac{\rho_0}{\rho}\right)}{\left[3MR^2 + 2m\left(R-L\right)^2\right]}\theta + \frac{2\alpha\left(R-L\right)^2}{\left[3MR^2 + 2m\left(R-L\right)^2\right]}\dot{\theta} = 0$$

3. The equation is of the form

$$\begin{cases} \ddot{\theta} + \dfrac{2mgL\left(1-\frac{\rho_0}{\rho}\right)}{\left[3MR^2 + 2m\left(R-L\right)^2\right]}\theta + \dfrac{2\alpha\left(R-L\right)^2}{\left[3MR^2 + 2m\left(R-L\right)^2\right]}\dot{\theta} = 0 \\ \ddot{\theta} + \omega_0^2\theta + 2\delta\dot{\theta} = 0 \end{cases}$$

$$\Rightarrow \omega_0^2 = \frac{2mgL\left(1-\frac{\rho_0}{\rho}\right)}{\left[3MR^2 + 2m\left(R-L\right)^2\right]} \; and \; \delta = \frac{\alpha\left(R-L\right)^2}{\left[3MR^2 + 2m\left(R-L\right)^2\right]}$$

To discover the nature of the liquid you have to find its density ρ_0 which is hidden in ω_0:
Since the damped system oscillates its movement is pseudo-periodic: the time equation is: $\theta(t) = Ae^{-\delta t}\cos\left(\omega t + \varphi\right) = Ae^{-\delta t}\cos\left(\sqrt{\omega_0^2 - \delta^2}\,t + \varphi\right)$

Since the amplitude decreases to 1/6 of its value after 23 oscillations, we have

$$Ae^{-\delta(t+23T)} = \frac{1}{6}Ae^{-\delta t} \Rightarrow e^{-\delta(t+23T)} = \frac{1}{6}e^{-\delta t} \Rightarrow e^{-\delta t}e^{-23\delta T} = \frac{1}{6}e^{-\delta t} \Rightarrow e^{-23\delta T} = \frac{1}{6}$$

$$\Rightarrow \ln e^{-23\delta T} = \ln\frac{1}{6} \Rightarrow 23\delta T = \ln 6$$

$$T = \frac{2\pi}{\omega} \Rightarrow 46\delta\frac{\pi}{\omega} = \ln 6 \Rightarrow 46\delta\frac{\pi}{\sqrt{\omega_0^2 - \delta^2}} = \ln 6 \Rightarrow \frac{(46\delta\pi)^2}{\omega_0^2 - \delta^2} = (\ln 6)^2$$

$$\Rightarrow (46\pi)^2\delta^2 = (\ln 6)^2(\omega_0^2 - \delta^2) \Rightarrow \left((46\pi)^2 + (\ln 6)^2\right)\delta^2 = (\ln 6)^2\omega_0^2 \Rightarrow \omega_0^2 = \frac{\left((46\pi)^2 + (\ln 6)^2\right)\delta^2}{(\ln 6)^2}$$

$$So\ \omega_0 = \frac{\sqrt{(46\pi)^2 + (\ln 6)^2}\,\delta}{\ln 6}$$

Digital Application

$\delta \approx 0.015s^{-1} \Rightarrow \omega_0 \approx 1.2rad/s \Rightarrow \rho_0 \approx 879Kg/m^3$.

We discovered the nature of the liquid without having examined it. In fact, according to the table given, this density corresponds to *benzene.*

4. The quality factor of this oscillator is: $Q = \frac{\omega_0}{2\delta} \approx 40$.

Chapter 4:

Linear systems forced to one degree of freedom

4.1 Excitation force

To overcome the friction responsible for energy losses and slowing down of moving systems, it is necessary to apply an external force called excitation.

4.2 Lagrange equation of forced systems

If in addition to friction $f = -\alpha\dot{q}$, there exists an external excitation force (t), the Lagrange equation is written:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}}\right)-\left(\dfrac{\partial L}{\partial x}\right)=-\left(\dfrac{\partial D}{\partial \dot{x}}\right)+F(t) \text{ (In translation)} \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}}\right)-\left(\dfrac{\partial L}{\partial \theta}\right)=-\left(\dfrac{\partial D}{\partial \dot{\theta}}\right)+\mathrm{M}(t) \text{ (In rotation)} \end{cases}$$

4.3 Equation of movement of forced systems

The equation of motion for linear systems damped by $f = -\alpha\dot{q}$ and excited by $F(t)$ is of the form (a is a constant)

$$\ddot{q}+\omega_0^2 q+2\delta\dot{q}=\frac{F(t)}{a}$$

4.4 Solving the equation of motion

Solving the equation $\ddot{q}+\omega_0^2 q+2\delta\dot{q}=\dfrac{F(t)}{a}$ is very simple for sinusoidal excitation $F(t)=F_0\cos\Omega t$. In this case the equation is written: $\ddot{q}+\omega_0^2 q+2\delta\dot{q}=\dfrac{F_0}{a}\cos\Omega t$.

The general solution to this equation is: $q(t)=q_T(t)+q_p(t)$.

- ✓ $q_T(t)$ is the (*transient*) solution of the homogeneous equation (without F). It depends on the sign of $\delta^2-\omega_0^2$. It is called transient because it goes out over time (see chap. 3.)
- ✓ $q_p(t)$ is the (*permanent*) solution of the non-homogeneous equation (with F). It is called permanent because it lasts throughout the movement.

Figure 4.1 shows the superposition of the two regimes:

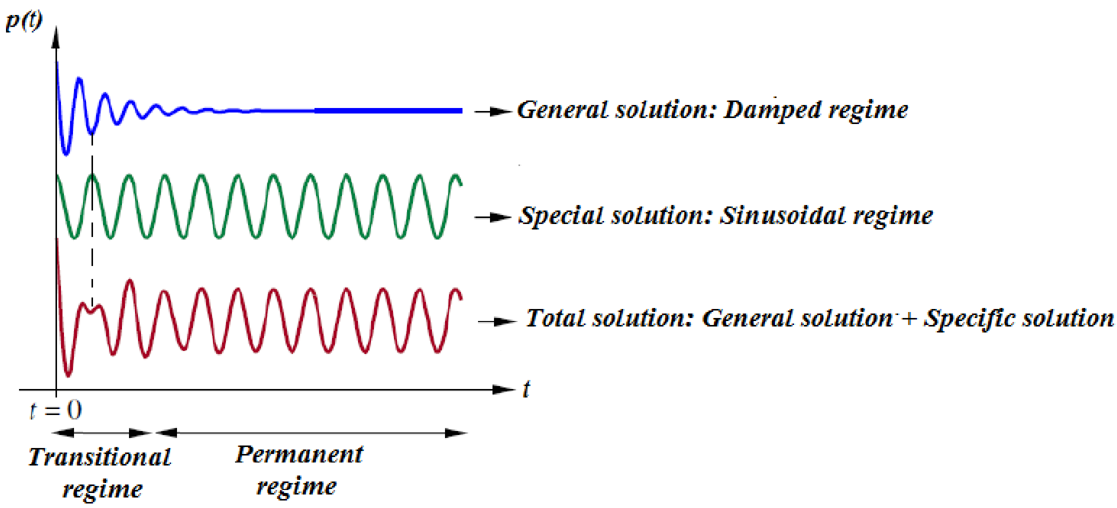


Figure 4.1: Superposition of the Transitional regime and the permanent regime.

In the case where the excitation is sinusoidal of type: $q(t)=A\cos\Omega t$. We find A and φ using the complex representation as follows:

$$F(t)=F_0\cos\omega t \rightarrow \tilde{F}(t)=F_0 e^{j\omega t}$$

$$q(t)=A\cos(\Omega t+\phi)\rightarrow \tilde{q}(t)=Ae^{(j\Omega t+\varphi)}=\tilde{A}e^{(j\Omega t)}$$

$$\ddot{q}+2\delta\dot{q}+\omega_0^2 q=\frac{F_0}{a}\cos\Omega t \rightarrow \ddot{\tilde{q}}+2\delta\dot{\tilde{q}}+\omega_0^2\tilde{q}=\frac{F_0}{a}e^{(j\Omega t)}$$

$$\Rightarrow -\Omega^2\tilde{A}e^{(j\Omega t)}+2\delta j\Omega\tilde{A}e^{(j\Omega t)}+\omega_0^2\tilde{A}e^{(j\Omega t)}=\frac{F_0}{a}e^{(j\Omega t)}$$

$$\Rightarrow -\Omega^2\tilde{A}+2\delta j\Omega\tilde{A}+\omega_0^2\tilde{A}=\frac{F_0}{a}$$

$$\Rightarrow \tilde{A}=\frac{\frac{F_0}{a}}{\left(\omega_0^2-\Omega^2\right)+2\delta j\Omega}$$

The amplitude of the movement is therefore: $A=\ |\tilde{A}|=\dfrac{F_0/a}{\sqrt{\left(\omega_0^2-\Omega^2\right)^2+4\delta^2\Omega^2}}$

The phase ϕ of the movement is given by: $\tan\phi=\dfrac{\mathrm{Im}(\tilde{A})}{\mathrm{Re}(\tilde{A})}=-\dfrac{2\delta\Omega}{\omega_0^2-\Omega^2}$

Finally, the solution for steady state motion is: $q(t)=A\cos(\Omega t+\phi)$

4.5 Resonance

The excitation pulse ω for which the amplitude *A* reaches its maximum is called the *resonance (amplitude)* pulse ω_R. *A* is maximum when $\frac{\partial A}{\partial \Omega}=0$.

$$\frac{\partial A}{\partial \Omega}=0 \Rightarrow \frac{\left[-4\Omega\left(\omega_0^2-\Omega^2\right)+8\delta^2\Omega\right]\left(F_0/a\right)}{2\left[\left(\omega_0^2-\Omega^2\right)^2+4\delta^2\Omega^2\right]^{3/2}}=0 \Rightarrow -4\left(\omega_0^2-\Omega^2\right)+8\delta^2=0$$

Either $\Omega_R=\sqrt{\omega_0^2-2\delta^2} \Leftrightarrow \Omega_R=\omega_0\sqrt{1-\frac{1}{4Q^2}}$

At this pulsation, the amplitude is: $A_{\max}=\frac{F_0/a}{\sqrt{4\delta^2\omega_0{}^2-4\delta^4}} \Leftrightarrow A_{\max}=\frac{F_0}{a\omega_0{}^2}\frac{Q}{\sqrt{1-\frac{1}{4Q^2}}}$

For there to be resonance, it is necessary that: $\omega_0{}^2-2\delta^2>0 \Rightarrow 1-\left(\frac{1}{2Q^2}\right)>0 \Rightarrow Q>\frac{1}{\sqrt{2}}$

The quality factor must therefore be greater than $\frac{1}{\sqrt{2}} \Rightarrow$ damping must be low.

According to the equation: $\tan\phi=\frac{\operatorname{Im}(\underline{A})}{\operatorname{Re}(\underline{A})}|=-\frac{2\delta\Omega}{\omega_0^2-\Omega^2}$ we have $\tan\varphi=-\infty\left(\phi=-\frac{\pi}{2}\right)$

when $\Omega=\omega_0$ This pulsation is called phase *resonance pulsation*

4.6 Bandwidth and quality factor

The instantaneous power provided by the excitation force is:

The total solution is then written as follows: $P=\frac{dW}{dt}=\frac{Fdq}{dt}=F\dot{q}$

Using the equation $q(t)=A\cos(\Omega t+\phi)$ we find

$$P=-F_0\cos\Omega t\times\Omega A\sin(\Omega t+\phi)=-\frac{1}{2}\Omega AF_0\left[\sin(2\Omega t+\phi)+\sin\phi\right]$$

The average power is

$$\langle p\rangle=\frac{1}{T}\int_0^{\pi}Pdt=-\frac{1}{2}\Omega AF_0\sin\phi=-\frac{1}{2}\Omega AF_0\frac{\tan\phi}{\sqrt{1+\tan\phi^2}}$$

According to the equations $A=|\underline{A}|=\frac{F_0/a}{\sqrt{\left(\omega_0^2-\Omega^2\right)^2+4\delta^2\Omega^2}}$ and $\tan\phi=\frac{\operatorname{Im}(\tilde{A})}{\operatorname{Re}(\tilde{A})}=-\frac{2\delta\Omega}{\omega_0^2-\Omega^2}$

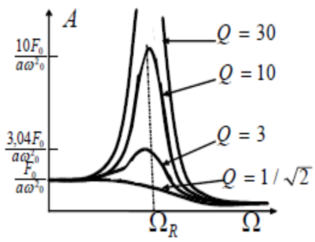
$$\langle p \rangle = \frac{\Omega^2 \delta \left(\dfrac{F_0^2}{a} \right)}{\left(\omega_0^2 - \Omega^2 \right)^2 + 4\delta^2 \Omega^2}$$

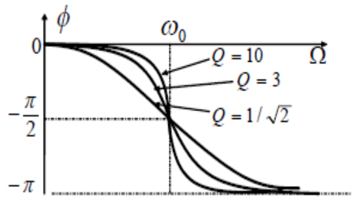
$\langle p \rangle$ is maximum when $\dfrac{\partial \langle p \rangle}{\partial \Omega} = 0 \Rightarrow \begin{cases} \Omega = \omega_0 \\ \langle p \rangle_{\max} = \dfrac{F_0^2}{4\Omega a} \end{cases}$

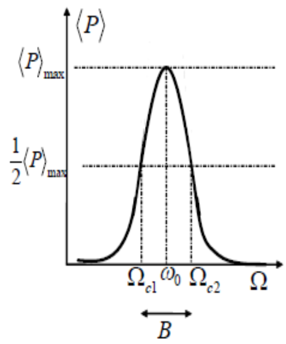
The pulsation Ω_{c1} and Ω_{c2} for which is half of its maximum are called cut-off pulsations. The width Ω_{c2}-Ω_{c1} is called the *Bandwidth*. According to $\langle p \rangle = \dfrac{\Omega^2 \delta \left(\dfrac{F_0^2}{a} \right)}{\left(\omega_0^2 - \Omega^2 \right)^2 + 4\delta^2 \Omega^2}$, $\langle p \rangle = \dfrac{\langle p \rangle_{\max}}{2}$ (low damping: $\delta \ll \omega_0$) when $\Omega_{c1} \simeq \omega_0 - \delta, \Omega_{c2} \simeq \omega_0 + \delta, B = \Omega_{c2} - \Omega_{c1} = 2\delta$.

$\dfrac{\omega_0}{B} = \dfrac{\omega_0}{2\delta} = Q$ is the quality factor (see chap.3).

The graphs of A, φ and $\langle p \rangle$ depending on the excitation pulse ω are:

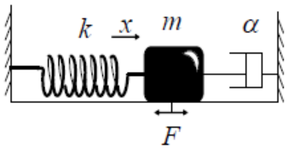






$$A = \frac{F_0/a}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\lambda^2\Omega^2}}. \qquad \phi = \tan^{-1}\left[-\frac{2\lambda\Omega}{\left(\omega_0^2 - \Omega^2\right)}\right]. \qquad \langle \mathcal{P} \rangle = \frac{\Omega^2 \lambda \left(F_0^2/a\right)}{\left(\omega_0^2 - \Omega^2\right)^2 + 4\lambda^2\Omega^2}$$

Example

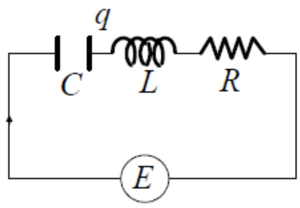


$$-k\underline{x} - \alpha\underline{v} + \underline{F} = m\frac{d\underline{v}}{dt}$$

$$\underline{F} = \frac{k}{j\omega}\underline{v} + jm\omega\underline{v} + \alpha\underline{v}$$

Mechanical Impedance

$$\underline{Z} = \frac{\underline{F}}{\underline{v}} = \frac{k}{j\omega} + jm\omega + \alpha$$



$$\frac{1}{C}\int \underline{i}dt + L\frac{d\underline{i}}{dt} + R\underline{i} - \underline{E} = 0$$

$$\underline{E} = \frac{1}{jC\omega}\underline{i} + jL\omega\underline{i} + R\underline{i}$$

Electrical impedance

$$\underline{Z} = \frac{\underline{E}}{\underline{i}} = \frac{1}{jC\omega} + jL\omega + R$$

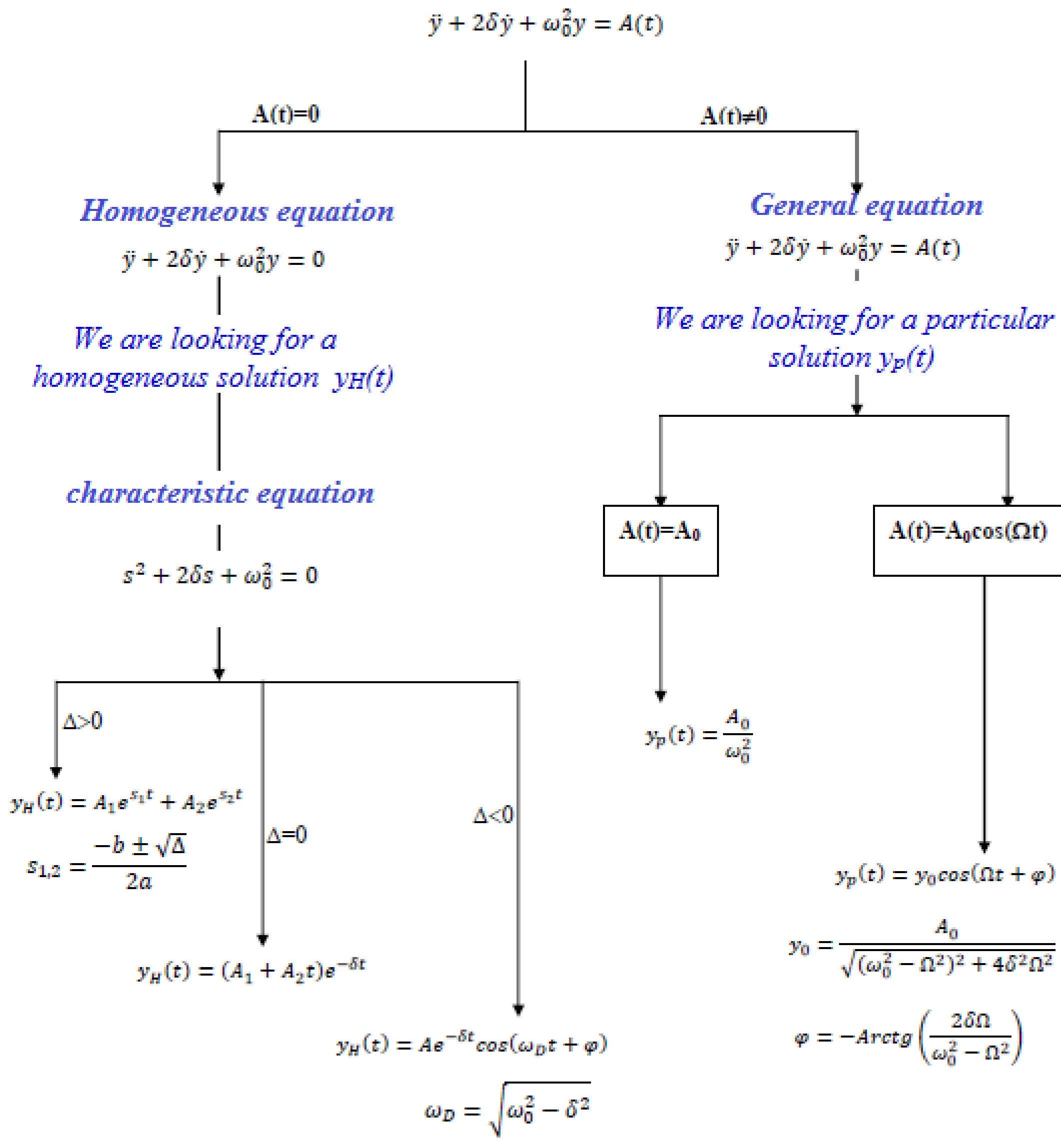


Figure 4: Flowchart of the solution of a second order differential equation.

4.6.1 Exercises and problems

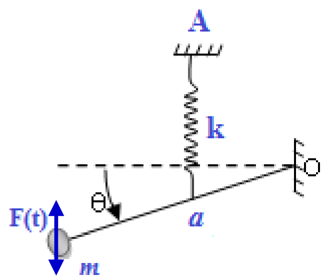
Exercise No. 1

A rod of length l and negligible mass articulated at point O carrying at its free end a point mass m. At a distance a of O from the rod we attach vertically a spring of stiffness k, the other end being fixed to a fixed frame at point A. At static equilibrium the rod takes a horizontal position ($\theta=0$).

We attach to the rod, at a distance $b = 3l/4$, a vertical shock absorber, α. A harmonic vertical force of the form $F(t) = F_0 \cos(\Omega t)$,(adjustable Ω) is applied to the mass m.

1- Write the differential equation governing the forced vibrations of the system based on the results obtained previously.
2- Knowing that $m = 2kg$, $k = 250N.m^{-1}$, $\alpha = 5N.m^{-1}.s^{-1}$ can we observe resonance?
3- If yes for what value of Ω do, we observe the resonance in the system? Then calculate the corresponding amplitude.

Req: we take $a = l/4$.

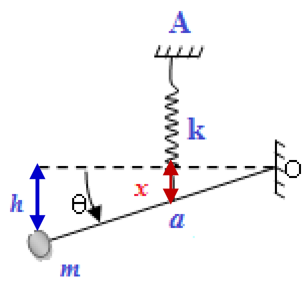


Solution

1. $U = U_k + U_m$

The potential energy of the rod equals zero because the mass of the rod is neglected.

$$U_k = \frac{k}{2}x^2 + kxx_0 + cte$$



With $\sin\theta = \frac{x}{a} \Rightarrow x = a\sin\theta$

$$U_k = \frac{k}{2}x^2 + kxx_0 + cte = \frac{k}{2}(a\sin\theta)^2 + k(a\sin\theta)x_0 + cte$$

$$U_m = -mgh$$

$$\sin\theta = \frac{h}{l} \Rightarrow h = l\sin\theta \Rightarrow U_m = -mgl\sin\theta$$

$$\Rightarrow U = U_k + U_m = \frac{k}{2}(a\sin\theta)^2 + k(a\sin\theta)x_0 - mgl\sin\theta + cte$$

$$U = \frac{k}{2}a^2\left(\sin\theta^2\right) + kax_0\left(\sin\theta\right) - mgl\left(\sin\theta\right) + cte$$

$$U = \frac{k}{2}a^2\left(\sin\theta^2\right) + \left(kax_0 - mgl\right)\left(\sin\theta\right) + cte$$

At low amplitude $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{\theta^2}{2}$

$$U = \frac{k}{2}a^2\left(\theta^2\right) + \left(kax_0 - mgl\right)\left(\theta\right) + cte$$

In equilibrium

$$\left.\frac{\partial U}{\partial \theta}\right|_{\theta=0} = 0$$

$$\frac{\partial U}{\partial \theta} = \frac{k}{2}\times a^2 \times 2\theta + \left(kax_0 - mgl\right)$$

$$\left.\frac{\partial U}{\partial \theta}\right|_{\theta=0} : \theta = 0 \Rightarrow \frac{k}{2}\times a^2 \times 2(\theta = 0) + \left(kax_0 - mgl\right) = \left(kax_0 - mgl\right)$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = 0 \Rightarrow \left(kax_0 - mgl\right) = 0 \Rightarrow x_0 = +\frac{mgl}{ka}$$

That is to say, the spring is elongated by $x_0 = +\frac{mgl}{ka}$

$$\Rightarrow U = \frac{k}{2}a^2\left(\theta^2\right) + cte.$$

$$T = E_c = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$$

$$\dot{x} = l\dot{\theta}$$

$$T = \frac{1}{2}m\left(l\dot{\theta}\right)^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

Then, the Lagrangian of the system is written:

$$L = E_c - E_p = T - U = \frac{1}{2}ml^2\dot{\theta}^2 - \frac{k}{2}a^2\left(\theta^2\right) + cte$$

The equation of motion for small oscillations is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) + \left(\frac{\partial D}{\partial \dot{\theta}}\right) = F(t) = F_0\, l\cos(\Omega t)$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}ml^2 \times 2\dot{\theta} = ml^2\dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -\frac{k}{2}a^2 \times 2\theta = -ka^2\theta$$

$$D = \frac{1}{2}\alpha\left(b\dot{\theta}\right)^2 = \frac{1}{2}\alpha\left(b^2\dot{\theta}^2\right) = \frac{1}{2}\alpha b^2\dot{\theta}^2$$

$$\frac{\partial D}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}\alpha b^2\dot{\theta}^2\right) = \frac{1}{2}\alpha b^2\frac{\partial}{\partial \dot{\theta}}\left(\dot{\theta}^2\right) = \frac{1}{2}\alpha b^2\left(2\dot{\theta}\right) = \alpha b^2\dot{\theta}$$

$$ml^2\ddot{\theta} + ka^2\theta + \alpha b^2\dot{\theta} = F_0\, l\cos(\Omega t)$$

$$\begin{cases} \ddot{\theta} + \dfrac{ka^2}{ml^2}\theta + \dfrac{\alpha b^2}{ml^2}\dot{\theta} = \dfrac{F_0\, l\cos(\Omega t)}{ml^2} \\ \ddot{\theta} + \omega_0^2\theta + 2\delta\dot{\theta} = F_\theta(\mathrm{t}) \end{cases}$$

$$\omega_0^2 = \frac{ka^2}{ml^2} \Rightarrow \omega_0 = \sqrt{\frac{ka^2}{ml^2}} = \frac{a}{l}\sqrt{\frac{k}{m}}$$

$$2\delta = \frac{\alpha b^2}{ml^2} \Rightarrow \delta = \frac{\alpha b^2}{2ml^2}$$

We have; $b = {}^{3l}\!/_{4}$ and $a = {}^{l}\!/_{4}$

$$\ddot{\theta} + \frac{k\left(\frac{l}{4}\right)^2}{ml^2}\theta + \frac{\alpha\left(\frac{3l}{4}\right)^2}{ml^2}\dot{\theta} = \frac{F_0\, l\cos(\Omega t)}{ml^2}$$

$$\ddot{\theta} + \frac{k\left(\frac{l^2}{16}\right)}{ml^2}\theta + \frac{\alpha\left(\frac{9l^2}{16}\right)}{ml^2}\dot{\theta} = \frac{F_0\cos(\Omega t)}{ml}$$

$$\ddot{\theta} + \frac{1}{16}\times\frac{k}{m}\theta + \frac{9}{16}\times\frac{\alpha}{m}\dot{\theta} = \frac{F_0\cos(\Omega t)}{ml}$$

$$\begin{cases}\ddot{\theta}+\frac{1}{16}\times\frac{k}{m}\theta+\frac{9}{16}\times\frac{\alpha}{m}\dot{\theta}=\frac{F_0\cos(\Omega t)}{ml}\\ \ddot{\theta}+\omega_0^2\theta+2\delta\dot{\theta}=F_\theta(\mathrm{t})\end{cases}$$

$$\omega_0^2=\frac{1}{16}\times\frac{k}{m}\Rightarrow\omega_0=\sqrt{\frac{1}{16}\times\frac{k}{m}}=\frac{1}{4}\sqrt{\frac{k}{m}}$$

$$2\delta=\frac{9}{16}\times\frac{\alpha}{m}\Rightarrow\delta=\frac{9}{32}\times\frac{\alpha}{m}$$

2. The resonance of the system

$$\omega_0=\frac{1}{4}\sqrt{\frac{k}{m}}$$

$$\delta=\frac{9}{32}\times\frac{\alpha}{m}$$

$$\frac{\delta}{\omega_0}=\xi<\frac{1}{\sqrt{2}}$$

$$\frac{\delta}{\omega_0}=\xi=\frac{\frac{9}{32}\times\frac{\alpha}{m}}{\frac{1}{4}\sqrt{\frac{k}{m}}}=\frac{9\sqrt{\frac{\alpha}{m}}\times\sqrt{\frac{\alpha}{m}}}{\frac{32}{4}\sqrt{\frac{k}{m}}}=\frac{9}{8}\sqrt{\frac{\alpha}{m}}\times\sqrt{\frac{\alpha}{m}}\times\sqrt{\frac{m}{k}}=\frac{9}{8}\times\frac{\alpha}{\sqrt{k\times m}}$$

Digital Application:

We have $m=2kg,\ k=250N.m^{-1},\ \alpha=5N.m^{-1}.\mathrm{s}^{-1}$

$$\frac{\delta}{\omega_0}=\xi=\frac{9}{8}\times\frac{\alpha}{\sqrt{k\times m}}=0.25<\frac{1}{\sqrt{2}}=0.7$$

$$\frac{\delta}{\omega_0}=\xi=0.7$$

$$\Omega_R=\omega_0\sqrt{1-2\xi^2}\Rightarrow\Omega_R=10.45\,rad.s^{-\&}$$

3. The amplitude of the resonance:

$$\theta_{max}=\frac{F_0/ml}{\sqrt{\left(\omega_0^2-\Omega_R^2\right)+4\delta^2\Omega_R^2}}=0.023\frac{F_0}{l}$$

θ_{max}: is proportional to F_0 and inversely to *l*.

Exercise No. 3

In the system in the following figure, the rope rolls without sliding around the cylinder of mass *M*=5kg and radius *R*=40 cm and which can rotate around its fixed axis. The free end of the rope carries a mass *m*=1kg. A spring *k*=600 N/m, fixed horizontally to a fixed frame, is attached to point A of the cylinder distant *r*=20cm from the axis of the cylinder. The mass is subjected to viscous friction represented by a damper with linear coefficient α, and a vertical harmonic force of the form $F(t) = F_0 cos\omega_e t$.

I. Undamped free regime (α=0 and F(t)=0)

1. Find the potential energy $U(\theta)$ of the system. Say whether or not the spring is deformed in equilibrium, then simplify the expression for $U(\theta)$ (equilibrium corresponds to θ=0).

2. Find the Lagrangian L, then establish the differential equation for small amplitude oscillations and deduce the natural pulsation. We give $J_{cylinder}$=1/2 MR^2

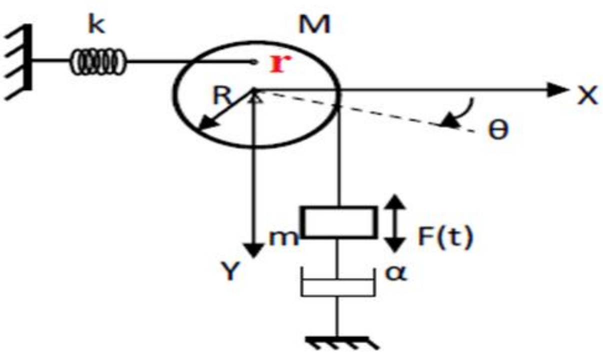
II. Damped forced regime (α≠0 and F(t)≠0)

Show that the differential equation governing forced vibrations can be put in the form:

$$\ddot{\theta} + \omega_0^2\theta + 2\delta\dot{\theta} = F_\theta(\text{t})$$

specify the expression of 2δ and $F_\theta(\text{t})$

Knowing that ξ=0.5, then give the appearance of the curve θ_0 (ω_e) (amplitudes of vibrations as a function of the pulsation of F(t)). In this case, calculate ω_r for which θ_0 will be maximum. What do we call this phenomenon?



1/ Undamped free regime

$$U = U_k + U_m + U_{Disq}$$

$$U_{Disq} = 0 \text{ because} : (h_{Disq} = 0)$$

$$U = U_k + U_m$$

 Dr.Ghellab Torkia

$$U_k = \frac{k}{2}x^2 + kxx_0 + cte$$

With $\sin\theta = \frac{x}{r} \Rightarrow x = r\sin\theta$

$$U_k = \frac{k}{2}x^2 + kxx_0 + cte = \frac{k}{2}(r\sin\theta)^2 + k(r\sin\theta)x_0 + cte$$

$$U_m = -mgh$$

$$\sin\theta = \frac{h}{R} \Rightarrow h = R\sin\theta \Rightarrow U_m = -mgR\sin\theta$$

$$\Rightarrow U = U_k + U_m = \frac{k}{2}(r\sin\theta)^2 + k(r\sin\theta)x_0 - mgR\sin\theta + cte$$

$$U = \frac{k}{2}r^2\left(\sin\theta^2\right) + krx_0\left(\sin\theta\right) - mgR\left(\sin\theta\right) + cte$$

$$U = \frac{k}{2}r^2\left(\sin\theta^2\right) + \left(krx_0 - mgR\right)\left(\sin\theta\right) + cte$$

At low amplitude $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{\theta^2}{2}$

$$U = \frac{k}{2}r^2\left(\theta^2\right) + \left(krx_0 - mgR\right)\left(\theta\right) + cte$$

In equilibrium

$$\left.\frac{\partial U}{\partial \theta}\right|_{\theta=0} = 0$$

$$\frac{\partial U}{\partial \theta} = \frac{k}{2} \times r^2 \times 2\theta + \left(krx_0 - mgR\right)$$

$$\left.\frac{\partial U}{\partial \theta}\right|_{\theta=0} : \theta = 0 \Rightarrow \frac{k}{2} \times r^2 \times 2\left(\theta = 0\right) + \left(krx_0 - mgR\right) = \left(krx_0 - mgR\right)$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = 0 \Rightarrow \left(krx_0 - mgR\right) = 0 \Rightarrow x_0 = +\frac{mgR}{kr}$$

That is to say, the spring is elongated by $x_0 = +\frac{mgl}{ka}$

$$U = \frac{k}{2}a^2\left(\theta^2\right) + cte.$$

$$T = E_c = T[mass] + T[Disk] = \frac{1}{2}mv^2\,[mass] + \frac{1}{2}J\dot{\theta}^2\,[Disk] = \frac{1}{2}m\dot{x}^2\,[mass] + \frac{1}{2}J\dot{\theta}^2\,[Disk]$$

$$T[mass] = \frac{1}{2}m\dot{x}^2\,[mass]$$

$$\dot{x} = R\dot{\theta}$$

$$T[mass] = \frac{1}{2}m\left(R\dot{\theta}\right)^2 = \frac{1}{2}mR^2\dot{\theta}^2$$

$$T[Disk] = \frac{1}{2}J\dot{\theta}^2\,[Disk]$$

$$J = \frac{1}{2}MR^2$$

$$T[Disk] = \frac{1}{2}J\dot{\theta}^2\,[Disk] = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot{\theta}^2\,[Disk]$$

$$T = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot{\theta}^2$$

$$T = \frac{1}{2}\left(mR^2 + \left(\frac{1}{2}MR^2\right)\right)\dot{\theta}^2$$

Then, the Lagrangian of the system is written:

$$L = E_c - E_p = T - U = \frac{1}{2}\left(ml^2 + \left(\frac{1}{2}MR^2\right)\right)\dot{\theta}^2 - \frac{k}{2}r^2\left(\theta^2\right) + cte$$

The equation of motion for small oscillations is: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$

$$L = \frac{1}{2}\left(ml^2 + \left(\frac{1}{2}MR^2\right)\right)\dot{\theta}^2 - \frac{k}{2}r^2\left(\theta^2\right) + cte$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left[\frac{1}{2}\left(mR^2 + \left(\frac{1}{2}MR^2\right)\right)\dot{\theta}^2\right] = \frac{1}{2}\left(mR^2 + \left(\frac{1}{2}MR^2\right)\right)\frac{\partial}{\partial \dot{\theta}}\left[\dot{\theta}^2\right]$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}\left(mR^2 + \left(\frac{1}{2}MR^2\right)\right) \times 2\dot{\theta} = \left(mR^2 + \left(\frac{1}{2}MR^2\right)\right)\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \left(mR^2 + \left(\frac{1}{2}MR^2\right)\right)\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta}=\frac{\partial}{\partial \theta}\left(-\frac{k}{2}r^2\left(\theta^2\right)\right)=-\frac{k}{2}r^2\frac{\partial}{\partial \theta}\left(\theta^2\right)=-\frac{k}{2}r^2\times 2\theta=-kr^2\theta$$

$$\left(mR^2+\left(\frac{1}{2}MR^2\right)\right)\ddot{\theta}+kr^2\theta=0$$

$$\begin{cases}\ddot{\theta}+\dfrac{kr^2}{\left(mR^2+\left(\frac{1}{2}MR^2\right)\right)}\theta=0\\ \ddot{\theta}+\omega_0^2\theta=0\end{cases}$$

$$\omega_0^2=\frac{kr^2}{\left(mR^2+\left(\frac{1}{2}MR^2\right)\right)}\Rightarrow \omega_0=\sqrt{\frac{kr^2}{\left(mR^2+\left(\frac{1}{2}MR^2\right)\right)}}$$

2/Forced damped regime

The Lagrange equation for a forced damped system is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\left(\frac{\partial L}{\partial \theta}\right)+\left(\frac{\partial D}{\partial \dot{\theta}}\right)=F_\theta(t)=F_0\,\mathrm{R}\cos(\omega_e t)$$

$$D=\frac{1}{2}\alpha\dot{x}^2$$

$$D=\frac{1}{2}\alpha\left(R\dot{\theta}\right)^2=\frac{1}{2}\alpha\left(R^2\dot{\theta}^2\right)=\frac{1}{2}\alpha R^2\dot{\theta}^2$$

$$\frac{\partial D}{\partial \dot{\theta}}=\frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}\alpha R^2\dot{\theta}^2\right)=\frac{1}{2}\alpha R^2\frac{\partial}{\partial \dot{\theta}}\left(\dot{\theta}^2\right)=\frac{1}{2}\alpha R^2\left(2\dot{\theta}\right)=\alpha R^2\dot{\theta}$$

$$\left(mR^2+\left(\frac{1}{2}MR^2\right)\right)\ddot{\theta}+kr^2\theta+\alpha R^2\dot{\theta}=F_0\,\mathrm{R}\cos(\omega_e t)$$

$$\begin{cases}\ddot{\theta}+\dfrac{kr^2}{\left(mR^2+\left(\frac{1}{2}MR^2\right)\right)}\theta+\dfrac{\alpha R^2}{\left(mR^2+\left(\frac{1}{2}MR^2\right)\right)}\dot{\theta}=\dfrac{F_0\,\mathrm{R}}{\left(mR^2+\left(\frac{1}{2}MR^2\right)\right)}\cos(\omega_e t)\\ \ddot{\theta}+\omega_0^2\theta+2\delta\dot{\theta}=F_\theta(t)\end{cases}$$

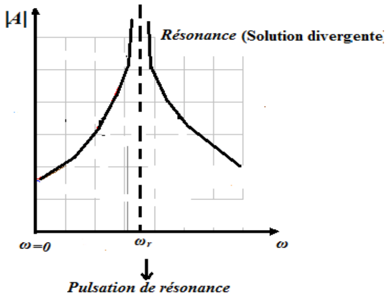
$$\omega_0^2=\frac{kr^2}{\left(mR^2+\left(\frac{1}{2}MR^2\right)\right)}\Rightarrow \omega_0=\sqrt{\frac{kr^2}{\left(mR^2+\left(\frac{1}{2}MR^2\right)\right)}}$$

$$2\delta=\frac{\alpha R^2}{\left(mR^2+\left(\frac{1}{2}MR^2\right)\right)}\Rightarrow \delta=\frac{\alpha R^2}{2\left(mR^2+\left(\frac{1}{2}MR^2\right)\right)}$$

$$F_\theta(t)=\frac{F_0\,\mathrm{R}}{\left(mR^2+\left(\frac{1}{2}MR^2\right)\right)}\cos(\omega_e t)$$

2/

$\xi = 0.5 < \frac{1}{2}$ The resonance condition is verified, *i.e.* resonance can take place in the system.



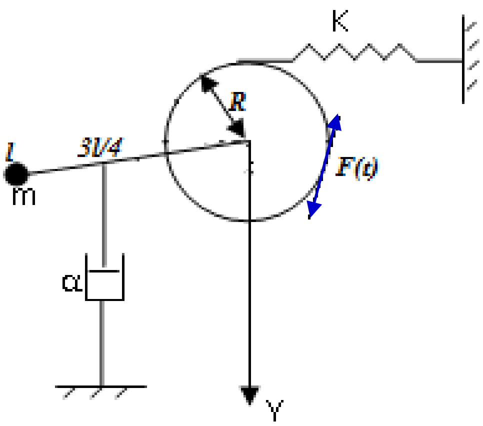
$\omega_r = \omega_0\sqrt{1-2\xi^2} = 3.27\ rad.s^{-1}$ This is the phenomenon of resonance.

Exercise No. 4

We consider a system with one degree of freedom in fig.4. The homogeneous disk of mass *M* and radius *R* can pivot around its fixed horizontal axis passing through its center. A rigid rod of length (*l*) and without mass is attached to the disk and carries at its free end a point mass *m*. A horizontally placed stiffness constant spring is connected to the disk as shown in fig.4, the other end being held fixed. The system is in static equilibrium when the rod is in its horizontal position. In movement the rod is identified in relation to this position by the nail θ. We place ourselves in the case of low amplitude vibrations. The system is subjected to viscous friction represented by a damper with linear coefficient α, and a vertical harmonic force of the form: $F(t)=F_0\cos\omega t$.

1. Find the potential energy of the system, the kinetic energy of the system and the Lagrangian.

2. Establish the differential equation for small amplitude oscillations and deduce the proper pulsation ω_0, the damping coefficient δ and F_θ.



Solution

1. The potential energy of the system, the kinetic energy of the system and the Lagrangian.

$$U = U_k + U_m$$

$$U_k = \frac{k}{2}x^2 + kxx_0 + cte$$

$$\sin\theta = \frac{x}{R} \Rightarrow x = R\sin\theta \Rightarrow U_k = \frac{k}{2}(\mathrm{R}\sin\theta)^2 + k(\mathrm{R}\sin\theta)x_0 + cte$$

$$U_m = -mgh$$

$$\sin\theta = \frac{h}{l} \Rightarrow h = l\sin\theta \Rightarrow U_m = -mgl\sin\theta$$

$$U = \frac{k}{2}R^2\left(\sin\theta^2\right) + \left(kRx_0 - mgl\right)\left(\sin\theta\right) + cte$$

At low amplitude $\sin\theta \approx \theta \Rightarrow U = \frac{k}{2}R^2\left(\theta^2\right) + \left(kax_0 - mgl\right)\left(\theta\right) + cte$

In equilibrium $U = \frac{k}{2}R^2\left(\theta^2\right) + cte$.

$$T = T_m + T_D$$

$$T_m = \frac{1}{2}m\dot{x}^2$$

$$\dot{x} = l\dot{\theta} \Rightarrow T_m = \frac{1}{2}m\left(l\dot{\theta}\right)^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

$$T_D = \frac{1}{2}J\dot{\theta}^2$$

$$J = \frac{1}{2}MR^2 \Rightarrow T_D = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot{\theta}^2$$

$$T = \frac{1}{2}\left(ml^2 + \left(\frac{1}{2}MR^2\right)\right)\dot{\theta}^2$$

Then, the Lagrangian of the system is written:

$$L = T - U = \frac{1}{2}\left(ml^2 + \left(\frac{1}{2}MR^2\right)\right)\dot{\theta}^2 - \frac{k}{2}R^2\left(\theta^2\right) + cte.$$

1. The differential equation of small amplitude oscillations, the proper pulsation ω_0, the damping coefficient δ and F_θ.

The Lagrange equation for a forced damped system is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\left(\frac{\partial L}{\partial \theta}\right)+\left(\frac{\partial D}{\partial \dot{\theta}}\right)=F_0\,\mathrm{R}\cos(\omega t)$$

$$D=\frac{1}{2}\alpha\dot{x}^2$$

$$\dot{x}=\frac{3l}{4}\dot{\theta}$$

$$D=\frac{1}{2}\alpha\left(\frac{3l}{4}\right)^2\dot{\theta}^2 \Rightarrow \frac{\partial D}{\partial \dot{\theta}}=\alpha\left(\frac{3l}{4}\right)^2\dot{\theta}$$

$$\begin{cases}\ddot{\theta}+\dfrac{kR^2}{\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)}\theta+\dfrac{\alpha\left(\frac{3l}{4}\right)^2}{\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)}\dot{\theta}=\dfrac{F_0\,\mathrm{R}}{\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)}\cos(\Omega t)\\ \ddot{\theta}+\omega_0^2\theta+2\delta\dot{\theta}=F_\theta(t)\end{cases}$$

$$\omega_0^2=\frac{kR^2}{\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)} \Rightarrow \omega_0=\sqrt{\frac{kR^2}{\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)}}$$

$$2\delta=\frac{\alpha\left(\frac{3l}{4}\right)^2}{\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)} \Rightarrow \delta=\frac{\alpha\left(\frac{3l}{4}\right)^2}{2\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)},\ F_\theta(t)=\frac{F_0\,\mathrm{R}}{\left(ml^2+\left(\frac{1}{2}MR^2\right)\right)}\cos(\omega t)$$

Exercise No. 5

Consider the vibrational system shown in the figure below. The rod is attached to the disc and the assembly rotates around a horizontal axis which passes through the center of the disc. The spring, wire and rod have negligible mass. The thread is inextensible. We consider the masses fixed to the rod as point. At equilibrium the rod is vertical. It is assumed that the vibrations are of low amplitude. We give $J_{Disk}=ml^2$

1- Show that the system admits a single degree of freedom. We choose θ; the angle that the rod makes with the vertical, as a generalized coordinate to describe the vibrations of the system.

2- Show that the kinetic energy $T(\theta)$ of the system is written: $T=\frac{15}{2}ml^2\dot{\theta}^2$

3- Show that the potential energy $U(\theta)$ of the system is written:

$$U=\frac{1}{2}\left(kl^2-2mgl\right)\theta^2-\left(klx_0+2mgl\right)\theta+cte$$

4- Find the initial elongation x_0 of the spring. Then simplify the expression of $U(\theta)$

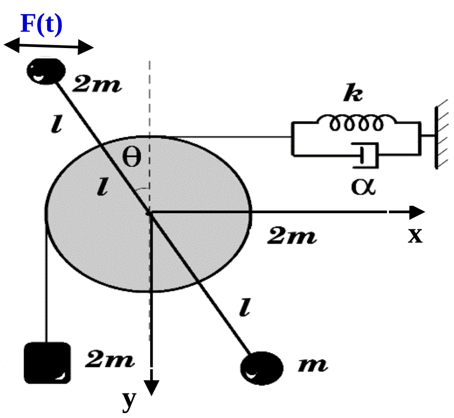
5- Write the Lagrangian of the system.

6- Find the generalized force F_θ associated with $F(t)=F_0\cos(\omega_e t)$ and the dissipation function *D*.

7- Write the Lagrange equation and deduce the differential equation of motion.

8- What condition must α verify to observe resonance in the system?

9- Then calculate ζ (the damping ratio) so that the pulsation at resonance only differs from (the natural pulsation) by 1%, *i.e.* $\frac{(\omega_0-\omega_r)}{\omega_0}=1\%$



Solution

$U=U_k+U_m+U_{Disq}+U_{2m}+U_{2m(Disq)}$

$U_{Disq}=0\left(h_{Disk}=0\right)$

$U=U_k+U_m+U_{2m}+U_{2m(Disq)}$

$U_k=\frac{k}{2}x^2+kxx_0+cte$

With $\sin\theta=\frac{-x}{l}\Rightarrow x=-l\sin\theta$

$$U_k=\frac{k}{2}x^2+kxx_0+cte=\frac{k}{2}(-l\sin\theta)^2+k(-l\sin\theta)x_0+cte$$

$$U_k=\frac{k}{2}l^2\sin\theta^2-klx_0\sin\theta+cte$$

$$U_{2m} = -2mgh,\ 2l = h + x \Rightarrow h = 2l - x$$

$$\cos\theta = \frac{x}{2l} \Rightarrow x = 2l\cos\theta$$

$$h = 2l - 2l\cos\theta = 2l(1-\cos\theta) \Rightarrow U_{2m} = -4mgl(1-\cos\theta)$$

$$U_m = mgh$$

$$2l = h + x \Rightarrow h = 2l - x$$

$$\cos\theta = \frac{x}{2l} \Rightarrow x = 2l\cos\theta$$

$$h = 2l - 2l\cos\theta = 2l(1-\cos\theta) \Rightarrow U_m = 2mgl(1-\cos\theta)$$

$$U_{2m(Disk)} = -2mgh$$

$$\sin\theta = \frac{h}{l} \Rightarrow h = l\sin\theta \Rightarrow U_{2m(Disk)} = -2mgl\sin\theta$$

$$U = \frac{k}{2}l^2\sin\theta^2 - klx_0\sin\theta - 4mgl(1-\cos\theta) + 2mgl(1-\cos\theta) - 2mgl\sin\theta + cte$$

At low amplitude $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{\theta^2}{2}$

$$U = \frac{k}{2}l^2\sin\theta^2 - klx_0\sin\theta - 4mgl(1-\cos\theta) + 2mgl(1-\cos\theta) - 2mgl\sin\theta + cte$$

$$U = \frac{k}{2}l^2\theta^2 - klx_0\theta - 4mgl\left(1-\left(1-\frac{\theta^2}{2}\right)\right) + 2mgl\left(1-\left(1-\frac{\theta^2}{2}\right)\right) - 2mgl\theta + cte$$

$$U = \frac{k}{2}l^2\theta^2 - klx_0\theta - 2mgl\theta^2 + mgl\theta^2 - 2mgl\theta + cte$$

$$U = \frac{k}{2}l^2\theta^2 - klx_0\theta - mgl\theta^2 - 2mgl\theta + cte$$

$$U = \frac{1}{2}(kl^2 - 2mgl)\theta^2 - (klx_0 + 2mgl)\theta + cte$$

In equilibrium

$$\left.\frac{\partial U}{\partial\theta}\right|_{\theta=0} = 0$$

$$\frac{\partial U}{\partial\theta} = \frac{1}{2}(kl^2 - 2mgl)\times 2\theta - (klx_0 + 2mgl)$$

$$\left.\frac{\partial U}{\partial\theta}\right|_{\theta=0} : \theta = 0 \Rightarrow \frac{1}{2}(kl^2 - 2mgl)\times 2(\theta = 0) - (klx_0 + 2mgl) = -(klx_0 + 2mgl)$$

$$\left.\frac{\partial U}{\partial\theta}\right|_{\theta=0} = 0 \Rightarrow -(klx_0 + 2mgl) = 0 \Rightarrow x_0 = -\frac{2mgl}{kl} = -\frac{2mg}{k}$$

That is to say, the spring is elongated by $x_0 = -\frac{2mg}{k}$ in the negative direction of the axis *x*.

$$U = \frac{1}{2}\left(kl^2 - 2mgl\right)\theta^2 + cte.$$

$$T = T[2m] + T[m] + T[Disk] + T\left[2m(Disk)\right]$$

$$T[2m] = \frac{1}{2}2m\dot{x}^2$$

$$\dot{x} = 2l\dot{\theta}$$

$$T[2m] = \frac{1}{2}2m\left(2l\dot{\theta}\right)^2 = 4ml^2\dot{\theta}^2$$

$$T[m] = \frac{1}{2}m\dot{x}^2$$

$$\dot{x} = 2l\dot{\theta}$$

$$T[m] = \frac{1}{2}m\left(2l\dot{\theta}\right)^2 = 2ml^2\dot{\theta}^2$$

$$T[Disk] = \frac{1}{2}J\dot{\theta}^2[Disk]$$

$$J = \frac{1}{2}MR^2 = \frac{1}{2}2ml^2$$

$$T[Disk] = \frac{1}{2}J\dot{\theta}^2[Disk] = \frac{1}{2}\left(\frac{1}{2}2ml^2\right)\dot{\theta}^2[Disk]$$

$$T[Disk] = \frac{1}{2}ml^2\dot{\theta}^2$$

$$T\left[2m(Disk)\right] = \frac{1}{2}2m\dot{x}^2$$

$$\dot{x} = l\dot{\theta}$$

$$T\left[2m(Disk)\right] = \frac{1}{2}2m\left(l\dot{\theta}\right)^2 = ml^2\dot{\theta}^2$$

$$T = 4ml^2\dot{\theta}^2 + 2ml^2\dot{\theta}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml^2\dot{\theta}^2$$

$$T = \frac{15}{2}ml^2\dot{\theta}^2$$

Then, the Lagrangian of the system is written:

$$L = E_c - E_p = T - U = \frac{15}{2}ml^2\dot{\theta}^2 - \frac{1}{2}\left(kl^2 - 2mgl\right)\theta^2 + cte$$

In the case of Forced damped regime

The Lagrange equation for a forced damped system is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \dot{\theta}}\right) + \left(\frac{\partial D}{\partial \dot{\theta}}\right) = F_\theta(t) = F_0 2l\cos(\omega_e t)$$

$$D=\frac{1}{2}\alpha\dot{x}^2$$

$$D=\frac{1}{2}\alpha\left(l\dot{\theta}\right)^2=\frac{1}{2}\alpha\left(l^2\dot{\theta}^2\right)=\frac{1}{2}\alpha l^2\dot{\theta}^2$$

$$\frac{\partial D}{\partial\dot{\theta}}=\frac{\partial}{\partial\dot{\theta}}\left(\frac{1}{2}\alpha l^2\dot{\theta}^2\right)=\frac{1}{2}\alpha l^2\frac{\partial}{\partial\dot{\theta}}\left(\dot{\theta}^2\right)=\frac{1}{2}\alpha l^2\left(2\dot{\theta}\right)=\alpha l^2\dot{\theta}$$

$$L=\frac{15}{2}ml^2\dot{\theta}^2-\frac{1}{2}\left(kl^2-2mgl\right)\theta^2+cte$$

$$\frac{\partial L}{\partial\dot{\theta}}=\frac{\partial}{\partial\dot{\theta}}\left[\frac{15}{2}ml^2\dot{\theta}^2\right]=\left(\frac{15}{2}ml^2\right)\frac{\partial}{\partial\dot{\theta}}\left[\dot{\theta}^2\right]$$

$$\frac{\partial L}{\partial\dot{\theta}}=\left(\frac{15}{2}ml^2\right)\times 2\dot{\theta}=15ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)=15ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial\theta}=\frac{\partial}{\partial\theta}\left(-\frac{1}{2}\left(kl^2-2mgl\right)\left(\theta^2\right)\right)=-\frac{1}{2}\left(kl^2-2mgl\right)\frac{\partial}{\partial\theta}\left(\theta^2\right)=-\frac{1}{2}\left(kl^2-2mgl\right)\times 2\theta=-\left(kl^2-2mgl\right)\theta$$

$$15ml^2\ddot{\theta}+\left(kl^2-2mgl\right)\theta+\alpha l^2\dot{\theta}=0$$

$$15ml^2\ddot{\theta}+\left(kl^2-2mgl\right)\theta+\alpha l^2\dot{\theta}=F_0 2l\cos\left(\omega_e t\right)$$

$$\begin{cases}\ddot{\theta}+\dfrac{\left(kl^2-2mgl\right)}{15ml^2}\theta+\dfrac{\alpha l^2}{15ml^2}\dot{\theta}=\dfrac{F_0 2l}{15ml^2}\cos\left(\omega t\right)\\ \ddot{\theta}+\omega_0^2\theta+2\delta\dot{\theta}=F_\theta\left(t\right)\end{cases}$$

$$\omega_0^2=\frac{\left(kl^2-2mgl\right)}{15ml^2}\Rightarrow\omega_0=\sqrt{\frac{\left(kl^2-2mgl\right)}{15ml^2}}$$

$$2\delta=\frac{\alpha}{15m}\Rightarrow\delta=\frac{\alpha}{30m}$$

$$F_\theta\left(t\right)=\frac{F_0 2l}{15ml^2}\cos\left(\omega t\right)$$

8. The resonance condition

$$\xi=\frac{\delta}{\omega_0}<\frac{1}{2}\Rightarrow\alpha<\sqrt{30m\left(k-\frac{2mg}{l}\right)}$$

9.

$$\omega_r=\omega_0\sqrt{1-2\xi^2}$$

$$\frac{\omega_0-\omega_r}{\omega_0}=\frac{\omega_0-\omega_0\sqrt{1-2\xi^2}}{\omega_0}=\frac{1-\sqrt{1-2\xi^2}}{1}=1\%=0.01\Rightarrow\xi=0.1$$

Exercise No. 6

A series RLC circuit is subjected to a sinusoidal e.m.f: $e(t)=E_0\sin(\Omega t)$. By varying the pulsation ω of the e.m.f. we note that a voltage resonance is produced across the unknown capacitance *C* Knowing that $V_{c\,(max)}$ = 60 V, E_0=3V, *R*=75 Ω, L=0.8 *mH*.

1) Determine the overvoltage coefficient Q
2) Evaluate the capacitance C and the natural pulsation ω_0 of the circuit.
3) Determine the width of the bandwidth Δω and its limits ω_1, ω_2.
4) Find the power that the source must provide to the circuit to maintain oscillations whose maximum amplitude of the current intensity is I_0=30 mA.

Solution

1 . $Q=\frac{V_{c(\max)}}{E_0}=\frac{60}{3}=20.$

2.

$$Q=\frac{1}{2\xi}=\frac{\omega_0}{2\delta}\Rightarrow\omega_0=2\delta Q$$

$$2\delta=\frac{R}{L}=\frac{75}{0.8\times10^{-3}}=93750\ s^{-1}$$

$$\omega_0=2\delta Q=18.75\times10^{+5}\ rad.s^{-1}$$

$$\omega_0=\frac{1}{\sqrt{LC}}\Rightarrow C=\frac{1}{L\omega_0^2}=355.5\times10^{-12}F$$

3. $\Delta\Omega=2\delta=93750\ s^{-1}$

$$\Omega_1\approx\omega_0-\frac{\Delta\omega}{2}=18.75\times10^{+5}-\frac{93750}{2}=18.25\times10^{+5}\ rad.s^{-1}$$

$$\Omega_2\approx\omega_0+\Delta\omega=19.22\times10^{+5}\ rad.s^{-1}$$

4.

$$\Omega_1\approx\omega_0-\frac{\Delta\Omega}{2}=18.75\times10^{+5}-\frac{93750}{2}=18.25\times10^{+5}\ rad.s^{-1}$$

$$\Omega_2\approx\omega_0+\Delta\Omega=19.22\times10^{+5}\ rad.s^{-1}$$

$$P=\frac{1}{2}RI_0^2=\frac{1}{2}75\left(30\times10^{-3}\right)^2=0.034\ watt$$

Exercise No. 7

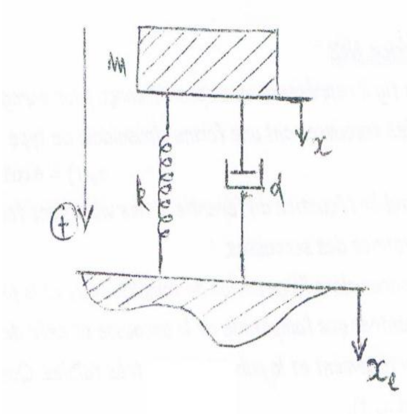
The following figure shows a device designed to record seismic tremors. We admit that the tremors have a sinusoidal shape of the type:

$$x_e(t) = a\cos(\omega_e t)$$

1- Establish the differential equation of the forced vibrations of the mass *m* due to the exciting force of the shocks.

2- Then give the expression for the amplitude x_0 and the initial phase φ_0 as a function of ω_e.

3- Show that the amplitude of the shock and that of the mass are identical in the case where the damping and the pulsation are very weak. In this case, what is the phase shift between $x_e(t)$ and $x(t)$.



Solution

The spring undergoes a deformation of its two ends *x* on the mass side and x_e on the ground side.

$$F_r = -k(x - x_e)$$

Likewise damping, which by moving its two ends with speeds $(\dot{x}; \dot{x}_e)$ respectively:

$$f_r = -\alpha(\dot{x} - \dot{x}_e).$$

$$m\ddot{x} = f_f + F_r = -k(x - x_e) - \alpha(\dot{x} - \dot{x}_e) = -kx + kx_e - \alpha\dot{x} + \alpha\dot{x}_e$$

$$m\ddot{x} + kx + \alpha\dot{x} = kx_e + \alpha\dot{x}_e$$

$$\ddot{x} + \frac{k}{m}x + \frac{\alpha}{m}\dot{x} = \frac{k}{m}x_e + \frac{\alpha}{m}\dot{x}_e \text{ and } x_e(t) = a\cos(\omega_e t)$$

$$\begin{cases} \ddot{x} + \frac{k}{m}x + \frac{\alpha}{m}\dot{x} = C\cos(\omega_e t + \theta) \\ \ddot{x} + \omega_0^2 x + 2\delta\dot{x} = C\cos(\omega_e t + \theta) \end{cases} \Rightarrow C = a\sqrt{\omega_0^4 + 4\delta^2\omega_e^2} \text{ and } tg\theta = \frac{2\delta\omega_e}{\omega_0^2}$$

The solution to this equation is $x(t) = x_h(t) + x_p(t)$

$$x_h(t)\underset{t\to 0}{\to} 0$$

$$x(t) \approx x_p(t) = A\cos(\omega_e t + \varphi)$$

$$A = \frac{a\sqrt{\omega_0^4 + 4\delta^2\omega_e^2}}{\sqrt{\left(\omega_0^2 - \omega_e^2\right)^2 + 4\delta^2\omega_e^2}}$$

$$tg\varphi = -\frac{2\delta\omega_e^3}{\omega_0^2\left(\omega_0^2 - \omega_e^2\right) + 4\delta^2\omega_e^2}$$

$$\ddot{\bar{x}} + \omega_0^2\bar{x} + 2\delta\dot{\bar{x}} = \bar{C}e^{j\omega_e t}$$

$$\bar{x} = \bar{A}e^{j\omega_e t}$$

$$\bar{A}\left(\omega_0^2 - \omega_e^2 + j2\delta\omega_e\right)e^{j\omega_e t} = \bar{C}e^{j\omega_e t}$$

$$\bar{A} = \frac{\bar{C}}{\left(\omega_0^2 - \omega_e^2 + j2\delta\omega_e\right)}$$

$$\left\|\bar{A}\right\| = \frac{C}{\sqrt{\left(\omega_0^2 - \omega_e^2 + j2\delta\omega_e\right)}}$$

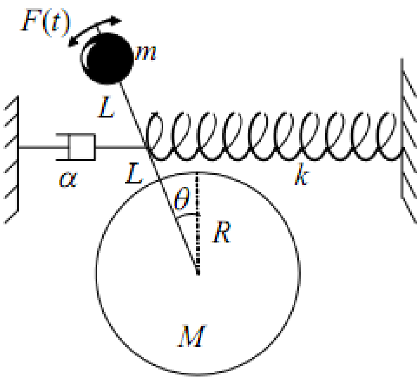
$$tg\varphi = \frac{\operatorname{Im} A}{\operatorname{Re} A} = -\frac{2\delta\omega_e^3}{\omega_0^2\left(\omega_0^2 - \omega_e^2\right) + 4\delta^2\omega_e^2}$$

3. $4\delta^2\omega_e^2 << \omega_0^4$ and $\omega_e^2 << \omega_0^2 \Rightarrow A \approx a, tg\varphi = 0 \Rightarrow \varphi = 0$

Problem 1

We consider a system with one degree of freedom in figure bellow. We place ourselves in the case of low amplitude vibrations. The system is subjected to viscous friction represented by a damper with linear coefficient α, and a vertical harmonic force of the form: $F(t) = F_0 \cos\Omega t$.

1- Find the potential energy of the system, the kinetic energy of the system and the Lagrangian.
2- Establish the differential equation of small amplitude oscillations and deduce the natural pulsation ω_0, the damping coefficient δ and F_θ.
3- Find using the complex representation, the permanent solution of the equation. (Specify its amplitude A and its phase ϕ).
4- Give the pulse of resonance Ω_R
5- Give the cut-off pulses Ω_{c1}; Ω_{c2} and deduce the bandwidth $B(\delta \ll \omega_0)$.
6- Calculate ω_R, *B*; and the quality factor for: *M=2kg, m=1kg, k=51N/m, α= 0.3N.s/m, R= 25cm; L= 50cm,g= 10m/s²*.



Solution

1. The potential energy of the system, the kinetic energy of the system and the Lagrangian

$U = U_k + U_m$

$$U_k = \frac{k}{2}x^2 + kxx_0 + cte$$

$$\sin\theta = \frac{x}{L} \Rightarrow x = L\sin\theta \Rightarrow U_k = \frac{k}{2}(L\sin\theta)^2 + k(L\sin\theta)x_0 + cte$$

$U_m = -mgh$

$2L = h + x \Rightarrow h = 2L - x$

$$\cos\theta = \frac{x}{2L} \Rightarrow x = 2L\cos\theta$$

$h = 2L - 2L\cos\theta = 2L(1-\cos\theta)$

$U_m = -2mgL(1-\cos\theta)$

$$U = \frac{k}{2}(L\sin\theta)^2 + k(L\sin\theta)x_0 - 2mgL(1-\cos\theta) + cte$$

Low amplitude $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{\theta^2}{2}$

$$\Rightarrow U = \frac{k}{2}L^2(\theta^2) + kLx_0(\theta) - 2mgL\left(1 - \left(1 - \frac{\theta^2}{2}\right)\right) + cte$$

$$U = \left(\frac{k}{2}L^2 - mgL\right)(\theta^2) + kLx_0(\theta) + cte$$

In equilibrium

$$U = \left(\frac{k}{2}L^2 - mgL\right)(\theta^2) + cte.$$

$$T = T_m + T_D$$

$$T_m = \frac{1}{2} m\dot{x}^2$$

$$\dot{x} = 2L\dot{\theta} \Rightarrow T_m = \frac{1}{2} m\left(2L\dot{\theta}\right)^2 = 2mL^2\dot{\theta}^2$$

$$T_D = \frac{1}{2} J\dot{\theta}^2$$

$$J = \frac{1}{2} MR^2 \Rightarrow T_D = \frac{1}{2}\left(\frac{1}{2} MR^2\right)\dot{\theta}^2$$

$$T = \frac{1}{2}\left(4mL^2 + \left(\frac{1}{2} MR^2\right)\right)\dot{\theta}^2$$

Then, the Lagrangian of the system is written:

$$L = T - U = \frac{1}{2}\left(4mL^2 + \left(\frac{1}{2} MR^2\right)\right)\dot{\theta}^2 - \left(\frac{k}{2} L^2 - mgL\right)\left(\theta^2\right) + cte.$$

2. The differential equation of small amplitude oscillations, the natural pulsation ω0, the damping coefficient δ and F_θ.

The Lagrange equation for a forced damped system is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \dot{\theta}}\right) + \left(\frac{\partial D}{\partial \dot{\theta}}\right) = 2F_0 L\cos\left(\omega t\right)$$

$$D = \frac{1}{2}\alpha\dot{x}^2$$

$$\dot{x} = L\dot{\theta}$$

$$D = \frac{1}{2}\alpha L^2\dot{\theta}^2 \Rightarrow \frac{\partial D}{\partial \dot{\theta}} = \alpha L^2\dot{\theta}$$

$$\left(4mL^2 + \left(\frac{1}{2} MR^2\right)\right)\ddot{\theta} + 2\left(\frac{k}{2} L^2 - mgL\right)\theta + \alpha L^2\dot{\theta} = 2F_0 L\cos\left(\Omega t\right)$$

$$\begin{cases} \ddot{\theta} + \dfrac{2\left(\dfrac{k}{2} L^2 - mgL\right)}{\left(4mL^2 + \left(\dfrac{1}{2} MR^2\right)\right)}\theta + \dfrac{\alpha L^2}{\left(4mL^2 + \left(\dfrac{1}{2} MR^2\right)\right)}\dot{\theta} = \dfrac{2F_0 L}{\left(4mL^2 + \left(\dfrac{1}{2} MR^2\right)\right)}\cos\left(\Omega t\right) \\ \ddot{\theta} + \omega_0^2\theta + 2\delta\dot{\theta} = F_\theta\left(t\right) \end{cases}$$

$$\begin{cases} \ddot{\theta} + \dfrac{2\left(kL^2 - 2mgL\right)}{\left(8mL^2 + MR^2\right)}\theta + \dfrac{2\alpha L^2}{\left(8mL^2 + MR^2\right)}\dot{\theta} = \dfrac{4F_0 L}{\left(8mL^2 + MR^2\right)}\cos\left(\Omega t\right) \\ \ddot{\theta} + \omega_0^2\theta + 2\delta\dot{\theta} = F_\theta\left(t\right) \end{cases}$$

$$\omega_0^2 = \frac{2\left(\frac{k}{2}L^2 - mgL\right)}{\left(4mL^2 + \left(\frac{1}{2}MR^2\right)\right)} = \frac{2\left(kL^2 - 2mgL\right)}{\left(8mL^2 + MR^2\right)} \Rightarrow \omega_0 = \sqrt{\frac{2\left(kL^2 - 2mgL\right)}{\left(8mL^2 + MR^2\right)}}$$

$$2\delta = \frac{2\alpha L^2}{\left(8mL^2 + MR^2\right)} = \frac{\alpha L^2}{\left(4mL^2 + \left(\frac{1}{2}MR^2\right)\right)} \Rightarrow \delta = \frac{\alpha L^2}{\left(8mL^2 + MR^2\right)}$$

$$F_\theta(t) = \frac{2F_0L}{\left(4mL^2 + \left(\frac{1}{2}MR^2\right)\right)}\cos(\Omega t) = \frac{4F_0L}{\left(8mL^2 + MR^2\right)}\cos(\Omega t)$$

3. The permanent solution is $\theta(t) = A\cos(\Omega t + \phi)$ Let's use the complex representation to find A and ϕ:

$$\frac{4F_0L}{\left(8mL^2 + MR^2\right)}\cos(\Omega t) \rightarrow \frac{4F_0L}{\left(8mL^2 + MR^2\right)}e^{j\Omega t}$$

$$\theta = A\cos(\Omega t + \phi) \rightarrow \tilde{\theta} = \tilde{A}e^{j\Omega t}$$

From the equation $\ddot{\theta} + \omega_0^2\theta + 2\delta\dot{\theta} = \frac{4F_0L}{\left(8mL^2 + MR^2\right)}\cos(\Omega t)$ We obtain

$$\Rightarrow -\Omega^2\tilde{A}e^{(j\Omega t)} + 2\delta j\Omega\tilde{A}e^{(j\Omega t)} + \omega_0^2\tilde{A}e^{(j\Omega t)} = \frac{4F_0L}{\left(8mL^2 + MR^2\right)}e^{j\Omega t}$$

$$\Rightarrow -\Omega^2\tilde{A} + 2\delta j\Omega\tilde{A} + \omega_0^2\tilde{A} = \frac{4F_0L}{\left(8mL^2 + MR^2\right)}$$

$$\Rightarrow \tilde{A} = \frac{\frac{4F_0L}{\left(8mL^2 + MR^2\right)}}{\left(\omega_0^2 - \Omega^2\right) + 2\delta j\Omega}$$

The amplitude of the movement is therefore: $A = |\tilde{A}| = \frac{\frac{4F_0L}{\left(8mL^2 + MR^2\right)}}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\delta^2\Omega^2}}$

The phase ϕ of the movement is given by: $\tan\phi = \frac{\mathrm{Im}(\tilde{A})}{\mathrm{Re}(\tilde{A})} = -\frac{2\delta\Omega}{\omega_0^2 - \Omega^2}$

4. The resonance pulse for $\frac{\partial A}{\partial \Omega} = 0$ is $\Omega_R = \sqrt{\omega_0^2 - 2\delta^2}$

5. When $\delta \ll \omega_0 : \langle p \rangle = \frac{\langle p \rangle_{max}}{2}$. For $\Omega_{c1} \simeq \omega_0 - \delta, \Omega_{c2} \simeq \omega_0 + \delta$

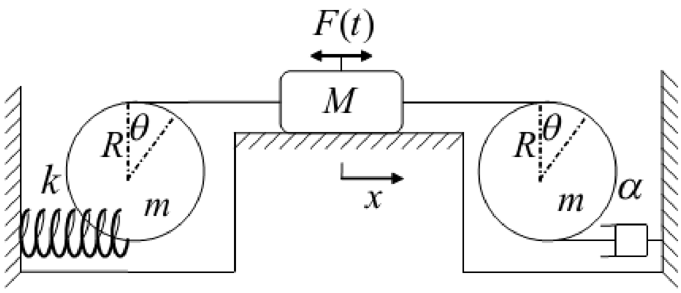
   $\Rightarrow B = \Omega_{c2} - \Omega_{c1} = 2\delta$

6. Digital Application $\Omega_R \approx 1.14 rad/s$, $B \approx 0.06 Hz$,The quality factor is $Q = \frac{\omega_0}{B} \approx 19$

Problem 2

The wire around the discs underneath is inextensible and non-slippery. $F(t) = F_0 \sin \Omega t$.

1. Find *T*, *U*, and the dissipation function *D.*
2. Find the Lagrangian then the equation of motion as a function of x: ($\theta \ll 1$).
3. Find the permanent solution to the equation using the complex representation.
   (Specify its real amplitude *A* and its phase ϕ)
4. Give the pulse of resonance Ω_R
5. Give the cut-off pulses Ω_{c1}; Ω_{c2} and deduce the bandwidth $B(\delta \ll \omega_0)$.
6. Calculate ω_R, *B*; and the quality factor for: *M=2kg, m=1kg, k=27N/m, α= 0.6N.s/m.*



Solution

1.

$T = T_m + T_m + T_M$

$T_M = \frac{1}{2} M \dot{x}^2$

$T_m = \frac{1}{2} I \dot{\theta}^2 \text{ with } I = \frac{1}{2} m R^2 \Rightarrow T_m = \frac{1}{2}\left(\frac{1}{2} m\right) R^2 \dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{2} m\right) \dot{x}^2 \left(\text{because} : R\theta = x \Rightarrow R\dot{\theta} = \dot{x}\right)$

$T = \frac{1}{2}\left(\frac{1}{2} m\right)\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2} m\right)\dot{x}^2 + \frac{1}{2} M \dot{x}^2 = \frac{1}{2}(M + m)\dot{x}^2$

$U_k = \frac{k}{2} x^2 + cte$

2. The Lagrangian of the system is written: $L = T - U = \frac{1}{2}(M + m)\dot{x}^2 - \frac{k}{2} x^2 + cte$.

The Lagrange equation for a forced damped system is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\left(\frac{\partial L}{\partial x}\right)+\left(\frac{\partial D}{\partial \dot{x}}\right)=F_0\sin(\Omega t)$$

$$D=\frac{1}{2}\alpha\dot{x}^2 \Rightarrow \frac{\partial D}{\partial \dot{x}}=\alpha\dot{x}$$

$$(M+m)\ddot{x}+kx+\alpha\dot{x}=F_0\sin(\Omega t)$$

$$\begin{cases}\ddot{x}+\dfrac{k}{(M+m)}x+\dfrac{\alpha}{(M+m)}\dot{x}=\dfrac{F_0}{(M+m)}\sin(\Omega t)\\ \ddot{x}+\omega_0^2x+2\delta\dot{x}=F_\theta(t)\end{cases}$$

$$\omega_0^2=\frac{k}{(M+m)}\Rightarrow \omega_0=\sqrt{\frac{k}{(M+m)}}$$

$$2\delta=\frac{\alpha}{(M+m)}\Rightarrow \delta=\frac{\alpha}{2(M+m)} \text{ and } F_\theta(t)=\frac{F_0}{(M+m)}\sin(\Omega t)$$

3. The permanent solution is $x(t)=A\cos(\Omega t+\phi)$ Let's use the complex representation to find A and ϕ:

$$\frac{F_0}{(M+m)}\sin(\Omega t)\rightarrow \frac{F_0}{(M+m)}e^{j\Omega t}$$

$$x=A\cos(\Omega t+\phi)\rightarrow \tilde{x}=\tilde{A}e^{j\Omega t}$$

From the equation $\ddot{x}+\omega_0^2x+2\delta\dot{x}=\dfrac{F_0}{(M+m)}\sin(\Omega t)$ We obtain

$$\Rightarrow -\Omega^2\tilde{A}e^{(j\Omega t)}+2\delta j\Omega\tilde{A}e^{(j\Omega t)}+\omega_0^2\tilde{A}e^{(j\Omega t)}=\frac{F_0}{(M+m)}e^{j\Omega t}$$

$$\Rightarrow -\Omega^2\tilde{A}+2\delta j\Omega\tilde{A}+\omega_0^2\tilde{A}=\frac{F_0}{(M+m)}$$

$$\Rightarrow \tilde{A}=\frac{\dfrac{F_0}{(M+m)}}{(\omega_0^2-\Omega^2)+2\delta j\Omega}$$

The amplitude of the movement is therefore: $A=\ |\tilde{A}|=\dfrac{\dfrac{F_0}{(M+m)}}{\sqrt{(\omega_0^2-\Omega^2)^2+4\delta^2\Omega^2}}$

The phase ϕ of the movement is given by: $\tan\phi = \frac{\mathrm{Im}(\tilde{A})}{\mathrm{Re}(\tilde{A})} = -\frac{2\delta\Omega}{\omega_0^2 - \Omega^2}$

4. The resonance pulse for $\frac{\partial A}{\partial \Omega} = 0$ is $\Omega_R = \sqrt{\omega_0^2 - 2\delta^2}$

7. When $\delta \ll \omega_0 : \langle p \rangle = \frac{\langle p \rangle_{max}}{2}$. For $\Omega_{c1} \simeq \omega_0 - \delta, \Omega_{c2} \simeq \omega_0 + \delta$

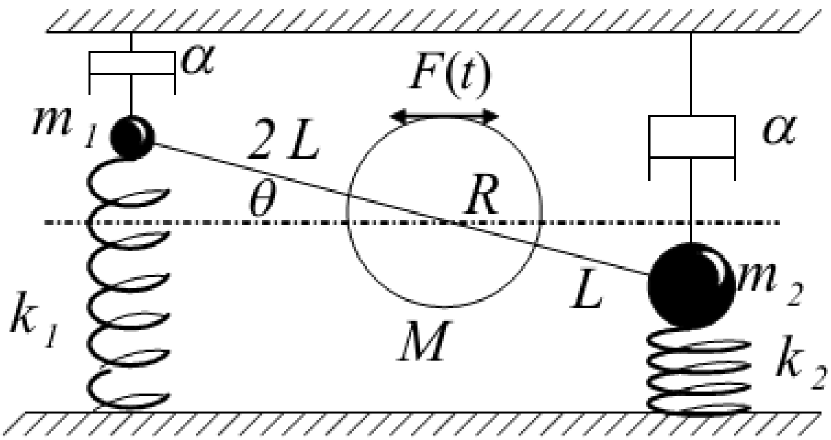
   $\Rightarrow B = \Omega_{c2} - \Omega_{c1} = 2\delta$

5. *Digital Application* $\Omega_R \approx 3rad/s$, $B \approx 0.2Hz$,The quality factor is $Q = \frac{\omega_0}{B} \approx 15$

Problem 3

Either the system opposite. $F(t) = F_0 \sin \Omega t$.

1. Find *T*, *U*, and the dissipation function *D*.
2. Find the Lagrangian then the equation of motion as a function of *θ*: ($\theta \ll 1$).
3. Find the permanent solution to the equation using the complex representation. (Specify its real amplitude *A* and its phase ϕ)
4. Give the pulse of resonance Ω_R
5. Give the cut-off pulses Ω_{c1}; Ω_{c2} and deduce the bandwidth $B(\delta \ll \omega_0)$.
6. Calculate Ω_R, *B*; and the quality factor for: *M=3kg, m_1=1kg, m_2=2kg, α= 0.5N.s/m, R=0.5m, L=1m, k_1=12N/m, k_2=20N/m.*



Solution

1.

$$T = T_{m_1} + T_{m_2} + T_M$$

$$T_{m_1} = \frac{1}{2} m_1 \dot{x}^2 \ \ with\ \dot{x} = 2L\dot{\theta} \Rightarrow T_m = \frac{1}{2} m_1 \left(2L\dot{\theta}\right)^2 = 2m_1 L^2 \dot{\theta}^2$$

$$T_{m_2} = \frac{1}{2} m_2 \dot{x}^2 \ \ with\ \dot{x} = L\dot{\theta} \Rightarrow T_{m_2} = \frac{1}{2} m_2 \left(L\dot{\theta}\right)^2 = \frac{1}{2} m_2 L^2 \dot{\theta}^2$$

$$T_M = \frac{1}{2} I \dot{\theta}^2 \ \ with\ I = \frac{1}{2} MR^2 \Rightarrow T_M = \frac{1}{2}\left(\frac{1}{2} MR^2\right)\dot{\theta}^2$$

$$T = \frac{1}{2}\left(4m_1 L^2 + m_2 L^2 + \frac{1}{2} MR^2\right)\dot{\theta}^2$$

$$U = U_{k_1} + U_{k_2} + U_{m_1} + U_{m_2}$$

$$U_{k_1} = \frac{k_1}{2} x^2 + k_1 x x_0 + cte$$

$$\sin\theta = \frac{x}{2L} \Rightarrow x = 2L\sin\theta \Rightarrow U_{k_1} = \frac{k_1}{2}\left(2L\sin\theta\right)^2 + k_1\left(2L\sin\theta\right)x_0 + cte$$

$$U_{k_2} = \frac{k_2}{2} x^2 + k_2 x x_0 + cte$$

$$\sin\theta = \frac{x}{L} \Rightarrow x = L\sin\theta \Rightarrow U_{k_2} = \frac{k_2}{2}\left(L\sin\theta\right)^2 + k_2\left(L\sin\theta\right)x_0 + cte$$

$$U_{m_1} = +m_1 g h$$

$$\sin\theta = \frac{x}{2L} \Rightarrow x = 2L\sin\theta$$

$$U_{m_1} = +2m_1 g L\sin\theta$$

$$U_{m_2} = -m_2 g h$$

$$\sin\theta = \frac{x}{L} \Rightarrow x = L\sin\theta$$

$$U_{m_2} = -m_2 g L\sin\theta$$

$$U = \frac{k_1}{2}\left(2L\sin\theta\right)^2 + k_1\left(2L\sin\theta\right)x_0 + \frac{k_2}{2}\left(L\sin\theta\right)^2 + k_2\left(L\sin\theta\right)x_0 + 2m_1 g L\sin\theta - m_2 g L\sin\theta + cte$$

$$U = \frac{1}{2}\left(4k_1 + k_2\right)L^2\sin\theta^2 + \left[\left(2k_1 + k_2\right)x_0 + \left(2m_1 - m_2\right)g\right]L\sin\theta + cte$$

Low amplitude $(\theta \ll 1)$ we have $\sin\theta \approx \theta$

$$U = \frac{1}{2}\left(4k_1 + k_2\right)L^2\theta^2 + \left[\left(2k_1 + k_2\right)x_0 + \left(2m_1 - m_2\right)g\right]L\theta + cte$$

In equilibrium

$$\left.\frac{\partial U}{\partial \theta}\right|_{\theta=0}=0$$

$$\frac{\partial U}{\partial \theta}=\frac{1}{2}(4k_1+k_2)L^2\times 2\theta+\left[(2k_1+k_2)x_0+(2m_1-m_2)g\right]L$$

$$\left.\frac{\partial U}{\partial \theta}\right|_{\theta=0}:\theta=0\Rightarrow\frac{1}{2}(4k_1+k_2)L^2\times 2(\theta=0)+\left[(2k_1+k_2)x_0+(2m_1-m_2)g\right]L$$

$$\left.\frac{\partial U}{\partial \theta}\right|_{\theta=0}=+\left[(2k_1+k_2)x_0+(2m_1-m_2)g\right]L$$

$$\left.\frac{\partial U}{\partial \theta}\right|_{\theta=0}=0\Rightarrow+\left[(2k_1+k_2)x_0+(2m_1-m_2)g\right]L=0\Rightarrow x_0=-\frac{(2m_1-m_2)gL}{(2k_1+k_2)}$$

$$\Rightarrow U=\frac{1}{2}(4k_1+k_2)L^2\theta^2+cte$$

2. The Lagrangian of the system is written:

$$L=T-U=\frac{1}{2}\left(4m_1L^2+m_2L^2+\frac{1}{2}MR^2\right)\dot{\theta}^2-\frac{1}{2}(4k_1+k_2)L^2\theta^2+cte\,.$$

The Lagrange equation for a forced damped system is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\left(\frac{\partial L}{\partial \dot{\theta}}\right)+\left(\frac{\partial D}{\partial \dot{\theta}}\right)=F_0R\sin(\Omega t)$$

$$D=\frac{1}{2}\alpha\dot{x}_1^{\,2}+\frac{1}{2}\alpha\dot{x}_2^{\,2}$$

$$\frac{1}{2}\alpha\dot{x}_1^{\,2}=\frac{1}{2}\alpha\left(2L\dot{\theta}\right)^2=\frac{1}{2}\alpha\left(4L^2\dot{\theta}^2\right)=2\alpha L^2\dot{\theta}^2$$

$$\frac{1}{2}\alpha\dot{x}_2^{\,2}=\frac{1}{2}\alpha\left(L\dot{\theta}\right)^2=\frac{1}{2}\alpha\left(L^2\dot{\theta}^2\right)=\frac{1}{2}\alpha L^2\dot{\theta}^2$$

$$D=\frac{5}{2}\alpha L^2\dot{\theta}^2$$

$$\frac{\partial D}{\partial \dot{\theta}}=\frac{5}{2}\alpha L^2\frac{\partial}{\partial \dot{\theta}}\left(\dot{\theta}^2\right)=5\alpha L^2\dot{\theta}$$

$$\left(4m_1L^2+m_2L^2+\frac{1}{2}MR^2\right)\ddot{\theta}+\left(4k_1+k_2\right)L^2\theta+5\alpha L^2\dot{\theta}=F_0R\sin\left(\Omega t\right)$$

$$\begin{cases}\ddot{\theta}+\dfrac{\left(4k_1+k_2\right)L^2}{\left(4m_1L^2+m_2L^2+\frac{1}{2}MR^2\right)}\theta+\dfrac{5\alpha L^2}{\left(4m_1L^2+m_2L^2+\frac{1}{2}MR^2\right)}\dot{\theta}=\dfrac{F_0R}{\left(4m_1L^2+m_2L^2+\frac{1}{2}MR^2\right)}\sin\left(\Omega t\right)\\ \ddot{\theta}+\omega_0^2\theta+2\delta\dot{\theta}=F_\theta\left(t\right)\end{cases}$$

$$\begin{cases}\ddot{\theta}+\dfrac{\left(8k_1+2k_2\right)L^2}{\left(8m_1L^2+2m_2L^2+MR^2\right)}\theta+\dfrac{10\alpha L^2}{\left(8m_1L^2+2m_2L^2+MR^2\right)}\dot{\theta}=\dfrac{2F_0R}{\left(8m_1L^2+2m_2L^2+MR^2\right)}\sin\left(\Omega t\right)\\ \ddot{\theta}+\omega_0^2\theta+2\delta\dot{\theta}=F_\theta\left(t\right)\end{cases}$$

$$\omega_0^2=\frac{\left(8k_1+2k_2\right)L^2}{\left(8m_1L^2+2m_2L^2+MR^2\right)}\Rightarrow\omega_0=\sqrt{\frac{\left(8k_1+2k_2\right)L^2}{\left(8m_1L^2+2m_2L^2+MR^2\right)}}$$

$$2\delta=\frac{10\alpha L^2}{\left(8m_1L^2+2m_2L^2+MR^2\right)}\Rightarrow\delta=\frac{5\alpha L^2}{\left(8m_1L^2+2m_2L^2+MR^2\right)}$$

$$F_\theta\left(t\right)=\frac{2F_0R}{\left(8m_1L^2+2m_2L^2+MR^2\right)}\sin\left(\Omega t\right)$$

3. The permanent solution is $\theta(t)=A\cos(\Omega t+\phi)$ Let's use the complex representation to find A and ϕ:

$$\frac{2F_0R}{\left(8m_1L^2+2m_2L^2+MR^2\right)}\sin\left(\Omega t\right)\rightarrow\frac{2F_0R}{\left(8m_1L^2+2m_2L^2+MR^2\right)}e^{j\Omega t}$$

$$\theta=A\cos\left(\Omega t+\phi\right)\rightarrow\tilde{\theta}=\tilde{A}e^{j\Omega t}$$

From the equation $\ddot{\theta}+\omega_0^2\theta+2\delta\dot{\theta}=\dfrac{2F_0R}{\left(8m_1L^2+2m_2L^2+MR^2\right)}\sin\left(\Omega t\right)$ We obtain

$$\Rightarrow-\Omega^2\tilde{A}e^{(j\Omega t)}+2\delta j\Omega\tilde{A}e^{(j\Omega t)}+\omega_0^2\tilde{A}e^{(j\Omega t)}=\frac{2F_0R}{\left(8m_1L^2+2m_2L^2+MR^2\right)}e^{j\Omega t}$$

$$\Rightarrow-\Omega^2\tilde{A}+2\delta j\Omega\tilde{A}+\omega_0^2\tilde{A}=\frac{2F_0R}{\left(8m_1L^2+2m_2L^2+MR^2\right)}$$

$$\Rightarrow\tilde{A}=\frac{\dfrac{2F_0R}{\left(8m_1L^2+2m_2L^2+MR^2\right)}}{\left(\omega_0^2-\Omega^2\right)+2\delta j\Omega}$$

The amplitude of the movement is therefore: $A = |\tilde{A}| = \dfrac{\dfrac{2F_0R}{\left(8m_1L^2+2m_2L^2+MR^2\right)}}{\sqrt{\left(\omega_0^2-\Omega^2\right)^2+4\delta^2\Omega^2}}$

The phase ϕ of the movement is given by: $\tan\phi = \dfrac{\mathrm{Im}(\tilde{A})}{\mathrm{Re}(\tilde{A})} = -\dfrac{2\delta\Omega}{\omega_0^2-\Omega^2}$

4. The resonance pulse for $\dfrac{\partial A}{\partial \Omega} = 0$ is $\Omega_R = \sqrt{\omega_0^2 - 2\delta^2}$

5. When $\delta \ll \omega_0 : \langle p \rangle = \dfrac{\langle p \rangle_{max}}{2}$. For $\Omega_{c1} \simeq \omega_0 - \delta, \Omega_{c2} \simeq \omega_0 + \delta$

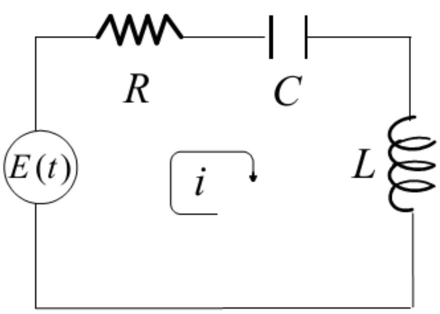
   $\Rightarrow B = \Omega_{c2} - \Omega_{c1} = 2\delta$

6. *Digital Application* $\Omega_R \approx 3.25 rad/s$, $B \approx 0.39 Hz$, The quality factor is $Q = \dfrac{\omega_0}{B} \approx 8.33$.

Problem 4

Either the excited circuit. $E(t) = E_0 \cos \Omega t$.

1. Find the equation of movement of the charge q circulating in the circuit using the mesh law.
2. Find the permanent solution to the equation using the complex representation. (Specify its real amplitude *A* and its phase ϕ)
3. Give the pulse of resonance Ω_R
4. Give the cut-off pulses Ω_{c1}; Ω_{c2} and deduce the bandwidth $B(\delta \ll \omega_0)$.
5. Calculate ω_R, *B*; and the quality factor for: *R=20Ω, C=1μF, L=5H.*



Solution

1. The law of meshes gives us $\sum V_i = E(t) \Rightarrow V_L + V_C + V_R = E(t)$

$$V_C = \frac{q}{C}$$

$$V_L = L\frac{di}{dt}, \text{i} = \frac{dq}{dt} = \dot{q} \Rightarrow V_L = L\frac{d}{dt}\dot{q} = L\ddot{q}$$

$$V_R = Ri = R\dot{q}$$

$$L\ddot{q} + \frac{q}{C} + R\dot{q} = E_0\cos\Omega t \Rightarrow L\ddot{q} + \frac{1}{C}q + R\dot{q} = E_0\cos\Omega t$$

$$\ddot{q} + \frac{1}{LC}q + \frac{R}{L}\dot{q} = \frac{E_0}{L}\cos\Omega t$$

$$\begin{cases} \ddot{q} + \frac{1}{LC}q + \frac{R}{L}\dot{q} = \frac{E_0}{L}\cos\Omega t \\ \ddot{q} + \omega_0^2 q + 2\delta\dot{q} = \frac{E_0}{L}\cos\Omega t \end{cases}$$

$$\omega_0^2 = \frac{1}{LC} \Rightarrow \omega_0 = \sqrt{\frac{1}{LC}}$$

$$2\delta = \frac{R}{L} \Rightarrow \delta = \frac{R}{2L}$$

2. The permanent solution is $q(t) = A\cos(\Omega t + \phi)$ Let's use the complex representation to find A and ϕ:

$$\frac{E_0}{L}\cos\Omega t \to \frac{E_0}{L}e^{j\Omega t}$$

$$q = A\cos(\Omega t + \phi) \to \tilde{q} = \tilde{A}e^{j\Omega t}$$

From the equation $\ddot{q} + \omega_0^2 q + 2\delta\dot{q} = \frac{E_0}{L}\cos\Omega t$ We obtain

$$\Rightarrow -\Omega^2\tilde{A}e^{(j\Omega t)} + 2\delta j\Omega\tilde{A}e^{(j\Omega t)} + \omega_0^2\tilde{A}e^{(j\Omega t)} = \frac{E_0}{L}e^{j\Omega t}$$

$$\Rightarrow -\Omega^2\tilde{A} + 2\delta j\Omega\tilde{A} + \omega_0^2\tilde{A} = \frac{E_0}{L}$$

$$\Rightarrow \tilde{A} = \frac{\frac{E_0}{L}}{(\omega_0^2 - \Omega^2) + 2\delta j\Omega}$$

The amplitude of the movement is therefore: $A = |\tilde{A}| = \dfrac{\frac{E_0}{L}}{\sqrt{(\omega_0^2 - \Omega^2)^2 + 4\delta^2\Omega^2}}$

The phase ϕ of the movement is given by: $\tan\phi = \frac{\mathrm{Im}(\tilde{A})}{\mathrm{Re}(\tilde{A})} = -\frac{2\delta\Omega}{\omega_0^2 - \Omega^2}$

3. The resonance pulse for $\frac{\partial A}{\partial \Omega} = 0$ is $\Omega_R = \sqrt{\omega_0^2 - 2\delta^2}$

4. When $\delta \ll \omega_0 : \langle p \rangle = \frac{\langle p \rangle_{max}}{2}$. For $\Omega_{c1} \simeq \omega_0 - \delta, \Omega_{c2} \simeq \omega_0 + \delta$

   $\Rightarrow B = \Omega_{c2} - \Omega_{c1} = 2\delta$

5. *Digital Application* $\Omega_R \approx 447.2 rad/s$, $B \approx 4Hz$, The quality factor is

   $Q = \frac{\omega_0}{B} \approx 111.8$.

Chapter 5:

Linear systems with several degrees of freedom

5.1 Degrees of freedom

The independent variables necessary to describe a moving system are called degrees of freedom. If there are N independent variables q_i, we write N Lagrange equations:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_1}\right)-\left(\dfrac{\partial L}{\partial q_1}\right)=-\dfrac{\partial D}{\partial \dot{q}_1}+F_1(t) \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_2}\right)-\left(\dfrac{\partial L}{\partial q_2}\right)=-\dfrac{\partial D}{\partial \dot{q}_2}+F_2(t) \\ . \\ . \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_N}\right)-\left(\dfrac{\partial L}{\partial q_N}\right)=-\dfrac{\partial D}{\partial \dot{q}_N}+F_N(t) \end{cases}$$

5.1.1 Coupling types

5.1.1.a Coupling by elasticity

The coupling between the two systems is through a spring (capacitance).

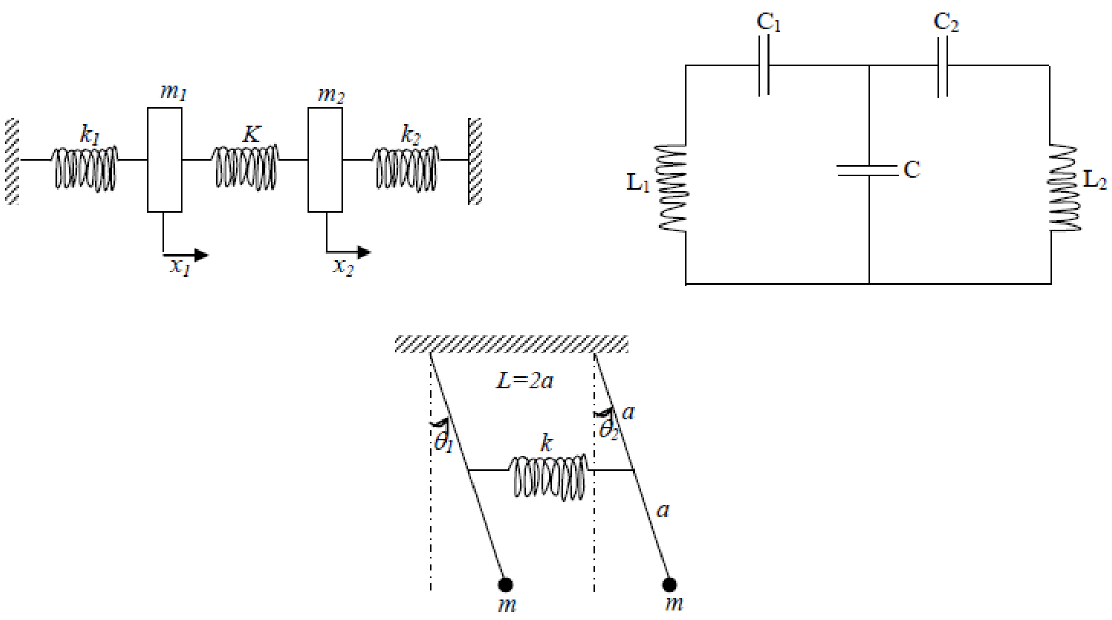


Figure 5.1: Different mechanical and electrical systems coupled by elasticity.

5.1.1.b Inertial coupling

The coupling between the two systems is through a mass (coil).

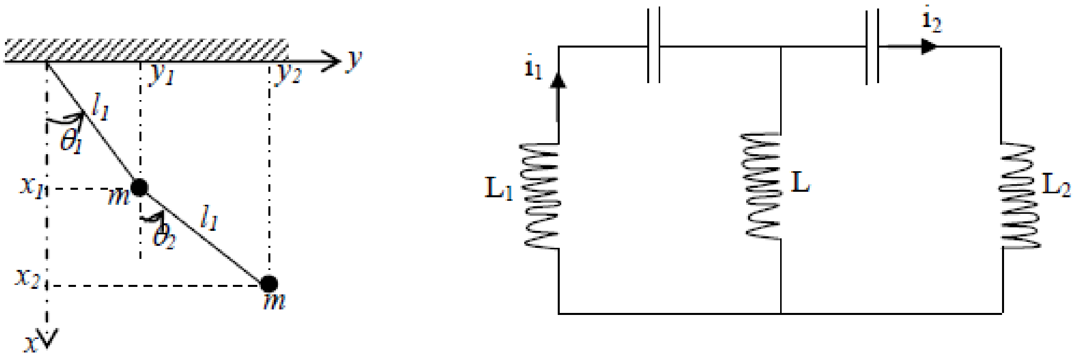


Figure 5.2: Mechanical and electrical systems coupled by inertia.

5.1.1.c Viscous coupling

The coupling between the two systems is through a damper (resistor).

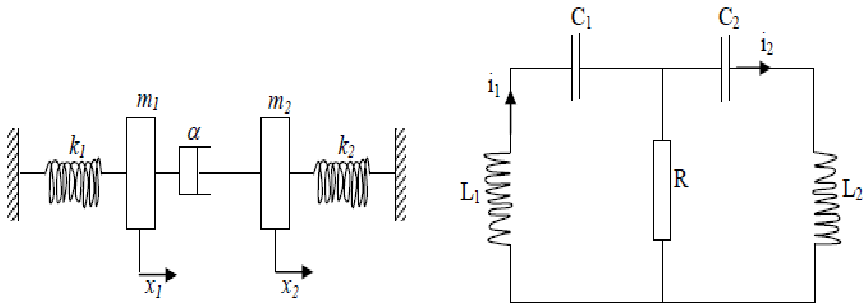


Figure 5.3: Viscous coupling of different mechanical and electrical systems.

5.2 Free systems with two degrees of freedom

5.2.1 Equation of motion

Or the free system opposite. The two independent variables are x_1 and x_2. k is called the *coupling element.*

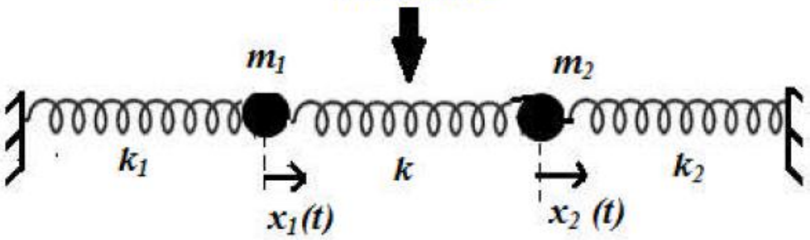


Figure 5.4: Oscillatory movement of a coupled system with two degrees of freedom.

For the kinetic energy we write as follows:

$$E_c = T = \sum_{i=1}^{2} \frac{1}{2} m_i \dot{x}_i^2 = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

For the potential energy we have:

$$E_p = U = \frac{1}{2} k \left(x_1 - x_2\right)^2 + \frac{1}{2} \sum_{i=1}^{2} k_i x_i^2 = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k \left(x_1 - x_2\right)^2$$

Hence the Lagrangian is written:

$$L = T - U = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 x_2^2 - \frac{1}{2} k \left(x_1 - x_2\right)^2$$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 x_2^2 - \frac{1}{2} k \left(x_1^2 + x_2^2 - 2 x_1 x_2\right)$$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 x_2^2 - \frac{1}{2} k x_1^2 - \frac{1}{2} k x_2^2 - \frac{1}{2} k \left(-2 x_1 x_2\right)$$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 x_2^2 - \frac{1}{2} k x_1^2 - \frac{1}{2} k x_2^2 + k x_1 x_2$$

The two Lagrange equations are written: (For *D=0, F=0*: undamped and unforced system)

The coupled differential system then becomes:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_1}\right) - \left(\dfrac{\partial L}{\partial x_1}\right) = 0 \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_2}\right) - \left(\dfrac{\partial L}{\partial x_2}\right) = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1, \left(\dfrac{\partial L}{\partial x_1}\right) = -k_1 x_1 - k x_1 + k x_2 \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2, \left(\dfrac{\partial L}{\partial x_2}\right) = -k_2 x_2 - k x_2 + k x_1 \end{cases}$$

$$\begin{cases} m_1 \ddot{x}_1 + \left(k_1 + k\right) x_1 - k x_2 = 0 \\ m_2 \ddot{x}_2 + \left(k + k_2\right) x_2 - k x_1 = 0 \end{cases} \Rightarrow \begin{cases} \ddot{x}_1 + \dfrac{\left(k_1 + k\right)}{m_1} x_1 - \dfrac{k}{m_1} x_2 = 0 \\ \ddot{x}_2 + \dfrac{\left(k + k_2\right)}{m_2} x_2 - \dfrac{k}{m_2} x_1 = 0 \end{cases}$$

5.2.2 Proper modes (Normal)

In normal mode, the solution to the previous equation is the form of a superposition of the two proper (normal) modes, as follows:

$x_1 = A_1 \cos(\Omega t + \varphi_1)$.

$x_2 = A_2 \cos(\Omega t + \varphi_2)$.

A_1, A_2, φ, Dependent on initial conditions. To find Ω, let's use the complex representation:

$$\begin{cases} x_1 = A_1\cos(\Omega t+\varphi_1) \to \tilde{x}_1 = A_1 e^{j(\Omega t+\varphi_1)} = \tilde{A}_1 e^{j\Omega t} \\ x_2 = A_2\cos(\Omega t+\varphi_2) \to \tilde{x}_2 = A_2 e^{j(\Omega t+\varphi_2)} = \tilde{A}_2 e^{j\Omega t} \end{cases}$$

$$\begin{cases} \tilde{x}_1 = \tilde{A}_1 e^{j\Omega t} \\ \tilde{x}_2 = \tilde{A}_2 e^{j\Omega t} \end{cases} \Rightarrow \begin{cases} \dot{\tilde{x}}_1 = j\Omega \tilde{A}_1 e^{j\Omega t} \\ \dot{\tilde{x}}_2 = j\Omega \tilde{A}_2 e^{j\Omega t} \end{cases} \Rightarrow \begin{cases} \ddot{\tilde{x}}_1 = (j\Omega)^2\,\tilde{A}_1 e^{j\Omega t} = -\Omega^2 \tilde{A}_1 e^{j\Omega t} \\ \ddot{\tilde{x}}_2 = (j\Omega)^2\,\tilde{A}_2 e^{j\Omega t} = -\Omega^2 \tilde{A}_2 e^{j\Omega t} \end{cases}$$

$$\begin{cases} \ddot{x}_1 + \frac{(k_1+k)}{m_1}x_1 - \frac{k}{m_1}x_2 = 0 \\ \ddot{x}_2 + \frac{(k+k_2)}{m_2}x_2 - \frac{k}{m_2}x_1 = 0 \end{cases} \Rightarrow \begin{cases} -\Omega^2\tilde{A}_1 e^{j\Omega t} + \frac{(k_1+k)}{m_1}\tilde{A}_1 e^{j\Omega t} - \frac{k}{m_1}\tilde{A}_2 e^{j\Omega t} = 0 \\ -\omega^2\tilde{A}_2 e^{j\Omega t} + \frac{(k+k_2)}{m_2}\tilde{A}_2 e^{j\Omega t} - \frac{k}{m_2}\tilde{A}_1 e^{j\Omega t} = 0 \end{cases}$$

$$\begin{cases} -\Omega^2\tilde{A}_1 e^{j\Omega t} + \frac{(k_1+k)}{m_1}\tilde{A}_1 e^{j\Omega t} - \frac{k}{m_1}\tilde{A}_2 e^{j\Omega t} = 0 \\ -\Omega^2\tilde{A}_2 e^{j\Omega t} + \frac{(k+k_2)}{m_2}\tilde{A}_2 e^{j\Omega t} - \frac{k}{m_2}\tilde{A}_1 e^{j\Omega t} = 0 \end{cases} \Rightarrow \begin{cases} \left(-\Omega^2 + \frac{(k_1+k)}{m_1}\right)\tilde{A}_1 e^{j\Omega t} - \left(\frac{k}{m_1}\right)\tilde{A}_2 e^{j\Omega t} = 0 \\ \left(-\Omega^2 + \frac{(k+k_2)}{m_2}\right)\tilde{A}_2 e^{j\Omega t} - \left(\frac{k}{m_2}\right)\tilde{A}_1 e^{j\Omega t} = 0 \end{cases}$$

$$\begin{cases} \left(-\Omega^2 + \frac{(k_1+k)}{m_1}\right)\tilde{A}_1 - \left(\frac{k}{m_1}\right)\tilde{A}_2 = 0 \\ \left(-\Omega^2 + \frac{(k+k_2)}{m_2}\right)\tilde{A}_2 - \left(\frac{k}{m_2}\right)\tilde{A}_1 = 0 \end{cases} \Rightarrow \begin{cases} \left(-\Omega^2 + \frac{(k_1+k)}{m_1}\right)\tilde{A}_1 - \left(\frac{k}{m_1}\right)\tilde{A}_2 = 0 \\ -\left(\frac{k}{m_2}\right)\tilde{A}_1 + \left(-\Omega^2 + \frac{(k+k_2)}{m_2}\right)\tilde{A}_2 = 0 \end{cases}$$

$$\begin{cases} \left(-\Omega^2 + a\right)\tilde{A}_1 - b\tilde{A}_2 = 0 \\ -c\tilde{A}_1 + \left(-\Omega^2 + d\right)\tilde{A}_2 = 0 \end{cases}$$

$$a = \frac{k+k_1}{m_1},\ b = \frac{k}{m_1},\ c = \frac{k}{m_2},\ d = \frac{k+k_2}{m_2}.$$

For the equation to be true without $\tilde{A}_1$ and $\tilde{A}_2$ being both zero, its *characteristic determinant* must be zero:

$$\Delta(\Omega) = \begin{vmatrix} (-\Omega^2+a) & -b \\ -c & (-\Omega^2+d) \end{vmatrix} = (-\Omega^2+a)(-\Omega^2+d) - (-c)(-b) = 0$$

$$\Delta(\Omega) = \Omega^4 - \Omega^2 d - \Omega^2 a + ad - bc = \Omega^4 - (d+a)\Omega^2 + (ad-bc) = 0$$

This gives us *the characteristic equation*:

$$\Omega^4 - (d+a)\Omega^2 + (ad-bc) = 0$$

The two real and positive solutions ω_1 and ω_2 of this equation are called *proper or normal pulsations.* The smallest is called *the fundamental*, the other is called *the harmonic*.

First eigenmode: For $\Omega=\Omega_1$, the system implies that:

$$\begin{cases}\left(-\Omega^2+a\right)\tilde{A}_1-b\tilde{A}_2=0\\-c\tilde{A}_1+\left(-\Omega^2+d\right)\tilde{A}_2=0\end{cases}\Rightarrow\begin{cases}\left(-\Omega^2+a\right)\tilde{A}_1=b\tilde{A}_2\\\left(-\Omega^2+d\right)\tilde{A}_2=c\tilde{A}_1\end{cases}$$

$$\frac{\tilde{A}_1(1)}{\tilde{A}_2(1)}=\frac{-\Omega_1^2+d}{c}>0.$$

The vibration is said to be in *phase* because the solution is written in this case

$$\begin{cases}x_{1(1)}=A_{1(1)}\cos\left(\Omega_1 t+\varphi\right)\\x_{2(1)}=A_{2(1)}\cos\left(\Omega_1 t+\varphi\right)\end{cases}$$

Second eigen mode: For $\Omega=\Omega_2$, the system implies that:

$\frac{\tilde{A}_1(2)}{\tilde{A}_2(2)}=\frac{-\Omega_2^2+d}{c}<0$. The vibration is said to be in *phase opposition* because the solution is written in this case:

$$\begin{cases}x_{1(2)}=A_{1(2)}\cos\left(\Omega_2 t+\varphi\right)\\x_{2(2)}=-A_{2(2)}\cos\left(\Omega_2 t+\varphi\right)\end{cases}$$

In the general case, the system vibrates in a *superposition* of these two proper modes.

When $k_1=k_2=k$ *et* $m_1=m_2=m$ **:** The new differential equations of motion

$$\begin{cases}\left(-\Omega^2+\frac{2k}{m}\right)\tilde{A}_1-\left(\frac{k}{m}\right)\tilde{A}_2=0\\-\left(\frac{k}{m}\right)\tilde{A}_1+\left(-\Omega^2+\frac{2k}{m}\right)\tilde{A}_2=0\end{cases}$$

The proper pulsations are

$$\Delta(\omega)=\begin{vmatrix}\left(-\Omega^2+\frac{2k}{m}\right) & -\frac{k}{m}\\-\frac{k}{m} & \left(-\Omega^2+\frac{2k}{m}\right)\end{vmatrix}=0$$

$$\Delta(\omega)=\left(-\Omega^2+\frac{2k}{m}\right)\left(-\Omega^2+\frac{2k}{m}\right)-\left(-\frac{k}{m}\right)\left(-\frac{k}{m}\right)=0$$

$$\left(-\Omega^2+\frac{2k}{m}\right)^2-\left(\frac{k}{m}\right)^2=0\Rightarrow\left(-\Omega^2+\frac{2k}{m}-\frac{k}{m}\right)\left(-\Omega^2+\frac{2k}{m}+\frac{k}{m}\right)=0$$

$$\left(-\Omega^2+\frac{k}{m}\right)\left(-\Omega^2+\frac{3k}{m}\right)=0$$

So $\Omega_1=\sqrt{\frac{3k}{m}}$ and $\Omega_2=\sqrt{\frac{k}{m}}$

5.3 Forced system with two degrees of freedom

5.3.1 Equations of motion

Or the system opposite.

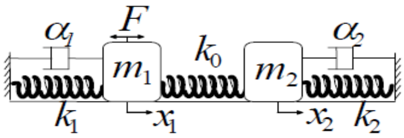


Figure 5.5: Oscillatory motion of a coupled system with two degrees of freedom.

For the kinetic energy we write as follows:

$$E_c = T = \sum_{i=1}^{2} \frac{1}{2} m_i \dot{x}_i^2 = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

For the potential energy we have:

$$E_p = U = \frac{1}{2} k \left(x_1 - x_2\right)^2 + \frac{1}{2} \sum_{i=1}^{2} k_i x_i^2 = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_0 \left(x_1 - x_2\right)^2$$

Hence the Lagrangian is written:

$$L = T - U = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 x_2^2 - \frac{1}{2} k_0 \left(x_1 - x_2\right)^2$$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 x_2^2 - \frac{1}{2} k_0 \left(x_1^2 + x_2^2 - 2 x_1 x_2\right)$$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 x_2^2 - \frac{1}{2} k_0 x_1^2 - \frac{1}{2} k_0 x_2^2 - \frac{1}{2} k_0 \left(-2 x_1 x_2\right)$$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 x_2^2 - \frac{1}{2} k_0 x_1^2 - \frac{1}{2} k_0 x_2^2 + k_0 x_1 x_2$$

The two Lagrange equations are written: (For $D = \frac{1}{2} \alpha_1 \dot{x}_1^2 + \frac{1}{2} \alpha_2 \dot{x}_2^2$ and $F_0 \cos \Omega t$)

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_1}\right) - \left(\dfrac{\partial L}{\partial x_1}\right) + \dfrac{\partial D}{\partial \dot{x}_1} = +F \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_2}\right) - \left(\dfrac{\partial L}{\partial x_2}\right) + \dfrac{\partial D}{\partial \dot{x}_2} = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1, \left(\dfrac{\partial L}{\partial x_1}\right) = -k_1 x_1 - k_0 x_1 + k_0 x_2, \dfrac{\partial D}{\partial \dot{x}_1} = \alpha \dot{x}_1 \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2, \left(\dfrac{\partial L}{\partial x_2}\right) = -k_2 x_2 - k_0 x_2 + k_0 x_1, \dfrac{\partial D}{\partial \dot{x}_2} = \alpha \dot{x}_2 \end{cases}$$

$$\begin{cases} m_1 \ddot{x}_1 + \left(k_1 + k_0\right) x_1 + \alpha_1 \dot{x}_1 - k_0 x_2 = F_0 \cos \Omega t \\ m_2 \ddot{x}_2 + \left(k_0 + k_2\right) x_2 + \alpha_2 \dot{x}_2 - k_0 x_1 = 0 \end{cases} \Rightarrow \begin{cases} \ddot{x}_1 + \dfrac{\left(k_1 + k_0\right)}{m_1} x_1 - \dfrac{k_0}{m_1} x_2 + \dfrac{\alpha_1}{m_1} \dot{x}_1 = \dfrac{F_0}{m_1} \cos \Omega t \\ \ddot{x}_2 + \dfrac{\left(k_0 + k_2\right)}{m_2} x_2 - \dfrac{k_0}{m_2} x_1 + \dfrac{\alpha_2}{m_2} \dot{x}_2 = 0 \end{cases}$$

5.3.2 Resonance and antiresonance

(With D=0 and $F\neq 0$: forced but undamped system). The permanent solution is:

$$\begin{cases} x_1 = A_1 \cos(\Omega t + \Omega_1) \\ x_2 = A_2 \cos(\Omega t + \Omega_2) \end{cases}$$

A_1, A_2, Ω_1, Ω_2 depend on the excitation pulse Ω and F_0. To find A_1, A_2, let's use the complex representation

When D= 0:

$$\begin{cases} x_1 = A_1 \cos(\Omega t + \varphi_1) \rightarrow \tilde{x}_1 = A_1 e^{j(\Omega t + \varphi_1)} = \tilde{A}_1 e^{j\Omega t} \\ x_2 = A_2 \cos(\Omega t + \varphi_2) \rightarrow \tilde{x}_2 = A_2 e^{j(\Omega t + \varphi_2)} = \tilde{A}_2 e^{j\Omega t} \end{cases}$$

$$F(t) = F_0 \cos \Omega t \rightarrow \tilde{F}(t) = F_0 e^{j\Omega t}$$

$$\begin{cases} \tilde{x}_1 = \tilde{A}_1 e^{j\Omega t} \\ \tilde{x}_2 = \tilde{A}_2 e^{j\Omega t} \end{cases} \Rightarrow \begin{cases} \dot{\tilde{x}}_1 = j\Omega \tilde{A}_1 e^{j\Omega t} \\ \dot{\tilde{x}}_2 = j\Omega \tilde{A}_2 e^{j\Omega t} \end{cases} \Rightarrow \begin{cases} \ddot{\tilde{x}}_1 = (j\Omega)^2 \tilde{A}_1 e^{j\Omega t} = -\Omega^2 \tilde{A}_1 e^{j\Omega t} \\ \ddot{\tilde{x}}_2 = (j\Omega)^2 \tilde{A}_2 e^{j\Omega t} = -\Omega^2 \tilde{A}_2 e^{j\Omega t} \end{cases}$$

$$\begin{cases} \ddot{x}_1 + \frac{(k_1 + k_0)}{m_1} x_1 - \frac{k_0}{m_1} x_2 = \frac{F_0}{m_1} e^{j\Omega t} \\ \ddot{x}_2 + \frac{(k_0 + k_2)}{m_2} x_2 - \frac{k_0}{m_2} x_1 = 0 \end{cases} \Rightarrow \begin{cases} -\Omega^2 \tilde{A}_1 e^{j\Omega t} + \frac{(k_1 + k_0)}{m_1} \tilde{A}_1 e^{j\Omega t} - \frac{k_0}{m_1} \tilde{A}_2 e^{j\Omega t} = \frac{F_0}{m_1} e^{j\Omega t} \\ -\Omega^2 \tilde{A}_2 e^{j\Omega t} + \frac{(k_0 + k_2)}{m_2} \tilde{A}_2 e^{j\Omega t} - \frac{k_0}{m_2} \tilde{A}_1 e^{j\Omega t} = 0 \end{cases}$$

$$\begin{cases} -\Omega^2 \tilde{A}_1 e^{j\Omega t} + \frac{(k_1 + k_0)}{m_1} \tilde{A}_1 e^{j\Omega t} - \frac{k_0}{m_1} \tilde{A}_2 e^{j\Omega t} = \frac{F_0}{m_1} e^{j\Omega t} \\ -\Omega^2 \tilde{A}_2 e^{j\Omega t} + \frac{(k_0 + k_2)}{m_2} \tilde{A}_2 e^{j\Omega t} - \frac{k_0}{m_2} \tilde{A}_1 e^{j\Omega t} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \left(-\omega^2 + \frac{(k_1 + k_0)}{m_1}\right) \tilde{A}_1 e^{j\Omega t} - \left(\frac{k_0}{m_1}\right) \tilde{A}_2 e^{j\Omega t} = \frac{F_0}{m_1} e^{j\Omega t} \\ \left(-\omega^2 + \frac{(k_0 + k_2)}{m_2}\right) \tilde{A}_2 e^{j\Omega t} - \left(\frac{k_0}{m_2}\right) \tilde{A}_1 e^{j\Omega t} = 0 \end{cases}$$

$$\begin{cases} \left(-\Omega^2 + \frac{(k_1 + k_0)}{m_1}\right) \tilde{A}_1 - \left(\frac{k_0}{m_1}\right) \tilde{A}_2 = \frac{F_0}{m_1} \\ \left(-\Omega^2 + \frac{(k_0 + k_2)}{m_2}\right) \tilde{A}_2 - \left(\frac{k_0}{m_2}\right) \tilde{A}_1 = 0 \end{cases} \Rightarrow \begin{cases} \left(-\Omega^2 + \frac{(k_1 + k_0)}{m_1}\right) \tilde{A}_1 - \left(\frac{k_0}{m_1}\right) \tilde{A}_2 = \frac{F_0}{m_1} \\ -\left(\frac{k_0}{m_2}\right) \tilde{A}_1 + \left(-\Omega^2 + \frac{(k_0 + k_2)}{m_2}\right) \tilde{A}_2 = 0 \end{cases}$$

Case where: m_1=m_2=m and k_0=k_1=k_2=k.

By possessing $\Omega_0^2 = \frac{k}{m}$, the equation becomes

$$\begin{cases}\left(-\Omega^2+2\dfrac{k}{m}\right)\tilde{A}_1-\dfrac{k}{m}\tilde{A}_2=\dfrac{F_0}{m}\\ -\dfrac{k}{m}\tilde{A}_1+\left(-\Omega^2+2\dfrac{k}{m}\right)\tilde{A}_2=0\end{cases}\Rightarrow\begin{cases}\left(-\Omega^2+2\Omega_0^2\right)\tilde{A}_1-\Omega_0^2\tilde{A}_2=\dfrac{F_0}{m}........(1)\\ -\Omega_0^2\tilde{A}_1+\left(-\Omega^2+2\Omega_0^2\right)\tilde{A}_2=0........(2)\end{cases}$$

$$(2)\Rightarrow+\left(-\Omega^2+2\Omega_0^2\right)\tilde{A}_2=+\Omega_0^2\tilde{A}_1\Rightarrow\tilde{A}_2=\frac{\Omega_0^2}{\left(-\Omega^2+2\Omega_0^2\right)}\tilde{A}_1........(3)$$

replaces (3) with (1)

$$\left(-\Omega^2+2\Omega_0^2\right)\tilde{A}_1-\Omega_0^2\frac{\Omega_0^2}{\left(-\Omega^2+2\Omega_0^2\right)}\tilde{A}_1=\frac{F_0}{m}\Rightarrow\left(-\Omega^2+2\Omega_0^2\right)\tilde{A}_1-\frac{\Omega_0^4}{\left(-\Omega^2+2\Omega_0^2\right)}\tilde{A}_1=\frac{F_0}{m}$$

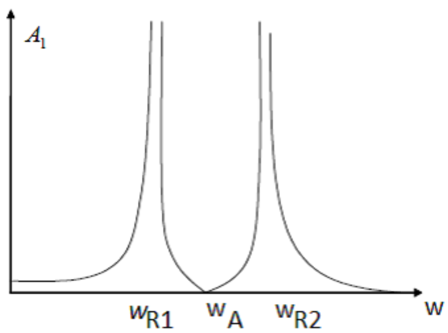
$$\Rightarrow\left(\left(-\Omega^2+2\Omega_0^2\right)-\frac{\Omega_0^4}{\left(-\Omega^2+2\Omega_0^2\right)}\right)\tilde{A}_1=\frac{F_0}{m}$$

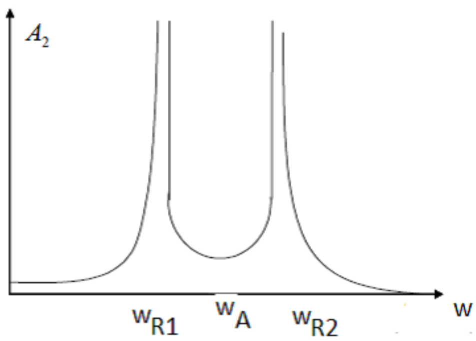
$$\left(\frac{\left(-\Omega^2+2\Omega_0^2\right)\left(-\Omega^2+2\Omega_0^2\right)-\Omega_0^4}{\left(-\Omega^2+2\Omega_0^2\right)}\right)\tilde{A}_1=\frac{F_0}{m}\Rightarrow\left(\frac{\left(-\Omega^2+2\Omega_0^2\right)^2-\Omega_0^4}{\left(-\Omega^2+2\Omega_0^2\right)}\right)\tilde{A}_1=\frac{F_0}{m}$$

$$\begin{cases}\tilde{A}_1=\dfrac{F_0}{m}\dfrac{\left(-\Omega^2+2\Omega_0^2\right)}{\left(\left(-\Omega^2+2\Omega_0^2\right)^2-\Omega_0^4\right)}\\ \tilde{A}_2=\dfrac{\Omega_0^2}{\left(-\Omega^2+2\Omega_0^2\right)}\tilde{A}_1=\dfrac{F_0}{m}\dfrac{\Omega_0^2}{\left(-\Omega^2+2\Omega_0^2\right)}\dfrac{\left(-\Omega^2+2\Omega_0^2\right)}{\left(\left(-\Omega^2+2\Omega_0^2\right)^2-\Omega_0^4\right)}=\dfrac{F_0}{m}\dfrac{\Omega_0^2}{\left(\left(-\Omega^2+2\Omega_0^2\right)^2-\Omega_0^4\right)}\end{cases}$$

$A_1=A_2=\infty$ when

- ✓ $\Omega\cong\Omega_0\cong\Omega_{R1}$ (called *first resonance* pulse).
- ✓ $\Omega\cong\sqrt{3}\Omega_0\cong\Omega_{R2}$ (called *second resonance* pulse).
- ✓ $A_1=0$ when $\Omega\cong\sqrt{2}\Omega_0\cong\Omega_A$ (called *antiresonance* pulse).





5.3.3 Input and transfer impedance

(With $D\neq0$ and $F\neq0$: *Damped and forced system*)

In electricity, impedance is defined by $\tilde{z}=\frac{F}{\tilde{i}_1}$. By analogy, we define the mechanical impedance by $\tilde{z}=\frac{F}{\tilde{v}'}$. $\tilde{z}_E=\frac{F}{\tilde{v}_1}$ is called *input impedance*. $\tilde{z}_T=\frac{F}{\tilde{v}_2}$ is called *transfer impedance*. To find them we still use the complex representation:

$$F(t)=F_0\cos\Omega t\rightarrow\tilde{F}(t)=F_0e^{j\Omega t}\text{ and }\begin{cases}\tilde{\dot{x}}_1=j\Omega\tilde{v}_1\\ \tilde{\dot{x}}_2=j\Omega\tilde{v}_2\end{cases}$$

$$\begin{cases}m_1\tilde{\ddot{x}}_1+(k_1+k_0)\tilde{x}_1+\alpha_1\tilde{\dot{x}}_1-k_0\tilde{x}_2=\tilde{F}\\ m_2\tilde{\ddot{x}}_2+(k_0+k_2)\tilde{x}_2+\alpha_2\tilde{\dot{x}}_2-k_0\tilde{x}_2=0\end{cases}\Rightarrow\begin{cases}\left(j\Omega m_1+\frac{k_1}{j\Omega}+\frac{k_0}{j\Omega}+\alpha_1\right)\tilde{v}_1-\frac{k_0}{j\Omega}\tilde{v}_2=\tilde{F}\\ \left(j\Omega m_2+\frac{k_2}{j\Omega}+\frac{k_0}{j\Omega}+\alpha_2\right)\tilde{v}_2-\frac{k_0}{j\Omega}\tilde{v}_1=0\end{cases}$$

By possessing $j\Omega m_1+\frac{k_1}{j\Omega}+\alpha_1=\tilde{z}_1$, $j\Omega m_2+\frac{k_1}{j\Omega}+\alpha_2=\tilde{z}_2$, $\frac{k_0}{j\Omega}=\tilde{z}_0$ we obtain

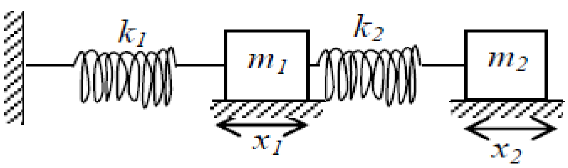
$$\begin{cases}(\tilde{z}_1+\tilde{z}_0)\tilde{v}_1-\tilde{z}_0\tilde{v}_2=\tilde{F}\\ (\tilde{z}_2+\tilde{z}_0)\tilde{v}_2-\tilde{z}_0\tilde{v}_1=0\end{cases}\Rightarrow F=\left(\tilde{z}_1+\tilde{z}_0-\frac{\tilde{z}_0^{\;2}}{(\tilde{z}_2+\tilde{z}_0)}\right)\tilde{v}_1=\left(\tilde{z}_1+\frac{\tilde{z}_2\tilde{z}_0}{(\tilde{z}_2+\tilde{z}_0)}\right)\tilde{v}_1$$

- Input impedance is $z_E=\frac{F}{\tilde{v}_1}=\left(\tilde{z}_1+\frac{\tilde{z}_0\tilde{z}_2}{(\tilde{z}_2+\tilde{z}_0)}\right)\equiv\left(\frac{\tilde{z}_1+\tilde{z}_0}{\tilde{z}_2}\right)$
- The transfer impedance is $z_T=\frac{F}{\tilde{v}_2}=\tilde{z}_1+\tilde{z}_2+\frac{\tilde{z}_1\tilde{z}_2}{\tilde{z}_0}$

5.6.1 Exercises and problems

Exercise No. 1

We consider the free oscillations of the system with two degrees of freedom in the figure bellow:



1) Calculate the kinetic and potential energies of the system.

2) For $k_1=k_2=k$ et $m_1=m$, $m_2=2m$, and using the Lagrange formula establish the differential equations of motion. Deduce the system's own pulsations.

Solution

1- Kinetic and potential energies and the Lagrangian:

$$T=\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}m_2\dot{x}_2^2$$

$$U=\frac{1}{2}k_1x_1^2+\frac{1}{2}k_2\left(x_1-x_2\right)^2$$

$$L=\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}m_2\dot{x}_2^2-\frac{1}{2}k_1x_1^2-\frac{1}{2}k_2\left(x_1-x_2\right)^2$$

2- Differential equations:

$$\begin{cases}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\left(\frac{\partial L}{\partial x_1}\right)=0\\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)-\left(\frac{\partial L}{\partial x_2}\right)=0\end{cases}\Rightarrow\begin{cases}m_1\ddot{x}_1+k_1x_1+k_2\left(x_1-x_2\right)=0\\ m_2\ddot{x}_2-k_2\left(x_1-x_2\right)=0\end{cases}$$

By replacing the constants, we find: $\begin{cases}m\ddot{x}_1+2kx_1-kx_2=0\\ 2m\ddot{x}_2-k\left(x_1-x_2\right)=0\end{cases}$

The proper pulsations: We propose the following solutions:

In complex notation we have $\begin{cases}\bar{x}_1=\bar{A}_1e^{j\Omega t}\\ \bar{x}_2=\bar{A}_2e^{j\Omega t}\end{cases}\Rightarrow\begin{cases}\left(-m\Omega^2+2k\right)A_1-kA_2=0\\ -kA_1+\left(-2m\Omega^2+k\right)A_2=0\end{cases}$

We calculate the determinant:

$$\Delta(\omega)=\begin{vmatrix}\left(-m\Omega^2+2k\right) & -k\\ -k & \left(-2m\Omega^2+k\right)\end{vmatrix}=\left(-m\Omega^2+2k\right)\left(-2m\Omega^2+k\right)-k^2=0$$

$$\Delta(\omega)=2m^2\Omega^4-3mk\Omega^2+k^2=0\Rightarrow\Delta=m^2k^2$$

The lowest frequency term corresponding to the pulsation Ω_1 is called the fundamental. The other term, with pulsation Ω_2, is called harmonic. The two proper pulsations are:

$$\Omega_1=\sqrt{\frac{k}{m}}\text{ and }\omega_2=\sqrt{\frac{k}{2m}}.$$

And the solutions are written as:

$$\begin{cases}x_1=A_{11}\cos\left(\Omega_1t+\varphi_1\right)+A_{12}\cos\left(\Omega_2t+\varphi_2\right)\\ x_2=A_{21}\cos\left(\Omega_1t+\varphi_1\right)+A_{22}\cos\left(\Omega_2t+\varphi_2\right)\end{cases}$$

A_{11}, A_{12}, A_{21}, A_{22}, φ_1 *and* φ_2 are integration constants determined from the initial conditions.

The system oscillates in the first (fundamental) mode, the solutions are written:

$$\begin{cases}x_1=A_{11}\cos\left(\Omega_1t+\varphi_1\right)\\ x_2=A_{21}\cos\left(\Omega_1t+\varphi_1\right)\end{cases}$$

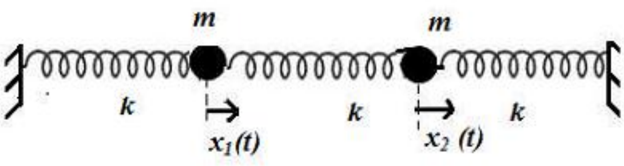
The solutions of the oscillating system in the second mode (harmonic) are given by:

$$\begin{cases} x_1 = A_{12}\cos\left(\Omega_2 t + \varphi_2\right) \\ x_2 = A_{22}\cos\left(\Omega_2 t + \varphi_2\right) \end{cases}$$

The constants A_{11}, A_{12}, A_{21}, A_{22}, φ_1 *and* φ_2 are integration constants calculated from the initial conditions.

Exercise No. 2

Consider the system with two degrees of freedom.



1. Calculate the kinetic and potential energies and the Lagrangian of the system.
2. Find the two proper pulsations Ω_1 and Ω_2 of the system.

Solution

A system with two degrees of freedom admits two Lagrange equations of the form (Free regime)

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_1}\right) - \left(\dfrac{\partial L}{\partial x_1}\right) = 0 \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_2}\right) - \left(\dfrac{\partial L}{\partial x_2}\right) = 0 \end{cases}$$

1- Kinetic and potential energies:

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$

$$U = \frac{1}{2}kx_1^2 + \frac{1}{2}k\left(x_1 - x_2\right)^2 + \frac{1}{2}kx_2^2 \Rightarrow U = \frac{1}{2}kx_1^2 + \frac{1}{2}k\left(x_1^2 - 2x_1x_2 + x_2^2\right)x_1^2 + \frac{1}{2}kx_2^2$$

$$U = \frac{1}{2}kx_1^2 + \frac{1}{2}kx_1^2 - kx_1x_2 + \frac{1}{2}kx_2^2 + \frac{1}{2}kx_2^2 \Rightarrow U = kx_1^2 + kx_2^2 - kx_1x_2$$

$$L\left(x_1, \dot{x}_1, x_2, \dot{x}_2\right) = T - U = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - kx_1^2 - kx_2^2 + kx_1x_2$$

$$\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m\dot{x}_1 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m\ddot{x}_1, \quad \left(\frac{\partial L}{\partial x_1}\right) = -2kx_1 + kx_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \left(\frac{\partial L}{\partial x_1}\right) = 0 \Rightarrow m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$

$$\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m\dot{x}_2 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m\ddot{x}_2, \quad \left(\frac{\partial L}{\partial x_2}\right) = -2kx_2 + kx_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \left(\frac{\partial L}{\partial x_2}\right) = 0 \Rightarrow m\ddot{x}_2 + 2kx_2 - kx_1 = 0$$

$$\begin{cases} m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \\ m\ddot{x}_2 + 2kx_2 - kx_1 = 0 \end{cases} \Rightarrow \begin{cases} \ddot{x}_1 + 2\frac{k}{m}x_1 - \frac{k}{m}x_2 = 0 \\ \ddot{x}_2 + 2\frac{k}{m}x_2 - \frac{k}{m}x_1 = 0 \end{cases}$$

$$\Omega_0^2 = \frac{k}{m} \Rightarrow \begin{cases} \ddot{x}_1 + 2\Omega_0^2 x_1 - \Omega_0^2 x_2 = 0 \\ \ddot{x}_2 + 2\Omega_0^2 x_2 - \Omega_0^2 x_1 = 0 \end{cases}$$

In complex notation we have

$$\begin{cases} \bar{x}_1 = \bar{A}_1 e^{j\Omega t} \\ \bar{x}_2 = \bar{A}_2 e^{j\Omega t} \end{cases} \Rightarrow \begin{cases} \ddot{x}_1 + 2\frac{k}{m}x_1 - \frac{k}{m}x_2 = 0 \\ \ddot{x}_2 + 2\frac{k}{m}x_2 - \frac{k}{m}x_1 = 0 \end{cases} \Rightarrow \begin{cases} \bar{A}_1 j^2\Omega^2 e^{j\Omega t} + 2\Omega_0^2 \bar{A}_1 e^{j\Omega t} - \Omega_0^2 \bar{A}_2 e^{j\Omega t} = 0 \\ \bar{A}_2 j^2\Omega^2 e^{j\Omega t} + 2\Omega_0^2 \bar{A}_2 e^{j\Omega t} - \Omega_0^2 \bar{A}_1 e^{j\Omega t} = 0 \end{cases}$$

$$\begin{cases} -\bar{A}_1\Omega^2 e^{j\Omega t} + 2\Omega_0^2 \bar{A}_1 e^{j\Omega t} - \Omega_0^2 \bar{A}_2 e^{j\Omega t} = 0 \\ -\bar{A}_2\Omega^2 e^{j\Omega t} + 2\Omega_0^2 \bar{A}_2 e^{j\Omega t} - \Omega_0^2 \bar{A}_1 e^{j\Omega t} = 0 \end{cases} \Rightarrow \begin{cases} \left(-\bar{A}_1\Omega^2 + 2\Omega_0^2 \bar{A}_1 - \Omega_0^2 \bar{A}_2\right) e^{j\Omega t} = 0 \\ \left(-\bar{A}_2\Omega^2 + 2\Omega_0^2 \bar{A}_2 - \Omega_0^2 \bar{A}_1\right) e^{j\Omega t} = 0 \end{cases}$$

$$\begin{cases} \left(-\Omega^2 + 2\Omega_0^2\right)\bar{A}_1 - \Omega_0^2 \bar{A}_2 = 0 \\ \left(-\Omega^2 + 2\Omega_0^2\right)\bar{A}_2 - \Omega_0^2 \bar{A}_1 = 0 \end{cases} \Rightarrow \begin{cases} \left(-\Omega^2 + 2\Omega_0^2\right)\bar{A}_1 - \Omega_0^2 \bar{A}_2 = 0 \\ -\Omega_0^2 \bar{A}_1 \left(-\Omega^2 + 2\Omega_0^2\right)\bar{A}_2 = 0 \end{cases}$$

$$\begin{vmatrix} \left(-\Omega^2 + 2\Omega_0^2\right) & -\Omega_0^2 \\ -\Omega_0^2 & \left(-\Omega^2 + 2\Omega_0^2\right) \end{vmatrix} \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We calculate the determinant :

$$\Delta = \left(-\Omega^2 + 2\Omega_0^2\right)^2 - \Omega_0^4 = 0 \Rightarrow \left(-\Omega^2 + 2\Omega_0^2 - \Omega_0^2\right)\left(-\Omega^2 + 2\Omega_0^2 + \Omega_0^2\right) = 0$$

$$\left(-\Omega^2 + \Omega_0^2\right)\left(-\Omega^2 + 3\Omega_0^2\right) = 0$$

So we find that $\Omega_1 = \sqrt{3}\Omega_0$ and $\Omega_2 = \Omega_0$.

Exercise No. 3

1. Establish the differential equations of the mechanical oscillatory system in the figure A.
2. We give the excitement $F(t)=F_0e^{j\Omega t}$. The solutions $x_1(t)$ and $x_2(t)$ of the permanent regime being of the same type as the excitation, give the matrix writing of the differential equations in complex amplitudes $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$.
3. Deduce, when $\beta = 0$, the existing resonance pulsation.
4. Establish the differential equations in current then in charges q_1 and q_2 of the electric oscillatory system of the figure B.
5. Is there any analogy between these two systems? If yes, give the correspondences between the mechanical and electrical elements. Deduce, when R=0, the existing resonance pulsation.

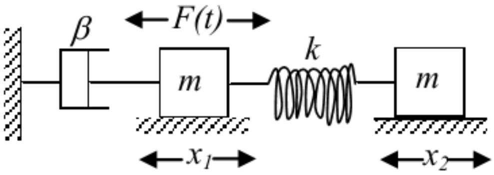


Fig. A. Forced mechanical system with two degrees of freedom.

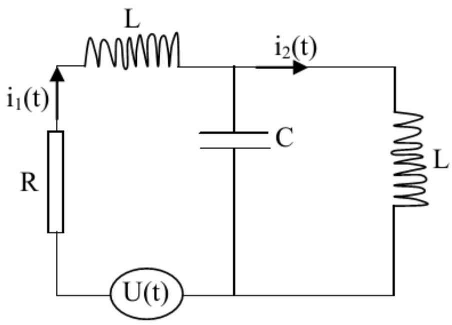


Fig. B. Electric oscillatory system.

Solution

1. Differential equation of motion :

$$T=\frac{1}{2}m\dot{x}_1^2+\frac{1}{2}m\dot{x}_2^2$$

$$U=\frac{1}{2}k\left(x_1-x_2\right)^2$$

The Lagrangian: $L=\frac{1}{2}m\dot{x}_1^2+\frac{1}{2}m\dot{x}_2^2-\frac{1}{2}k\left(x_1-x_2\right)^2$

The dissipation function: $D=\frac{1}{2}\beta\dot{x}_1^2$

2- Differential equations:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\left(\frac{\partial L}{\partial x_1}\right)+\frac{\partial D}{\partial \dot{x}_1}=F(t) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)-\left(\frac{\partial L}{\partial x_2}\right)=0 \end{cases}$$

$$\begin{cases} m\ddot{x}_1+k(x_1-x_2)+\beta\dot{x}_1=F(t) \\ m\ddot{x}_2+k(x_2-x_1)=0 \end{cases}$$

$$\begin{cases} \ddot{x}_1+\frac{k}{m}(x_1-x_2)+\frac{\beta}{m}\dot{x}_1=\frac{F_0}{m}e^{j\Omega t} \\ \ddot{x}_2+\frac{k}{m}(x_2-x_1)=0 \end{cases}$$

2. Complex solutions $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ are written as:

$$\begin{cases} \bar{x}_1=\bar{A}_1e^{j\Omega t} \\ \bar{x}_2=\bar{A}_2e^{j\Omega t} \end{cases} \Rightarrow \begin{cases} \left(-\Omega^2+\frac{k}{m}+j\frac{\beta}{m}\Omega\right)\bar{A}_1-\frac{k}{m}\bar{A}_2=\frac{F_0}{m} \\ -\frac{k}{m}\bar{A}_1+\left(-\Omega^2+\frac{k}{m}\right)\bar{A}_2=0 \end{cases}$$

$$\Rightarrow \begin{vmatrix} \left(-\Omega^2+\frac{k}{m}+j\frac{\beta}{m}\Omega\right) & -\frac{k}{m} \\ -\frac{k}{m} & \left(-\Omega^2+\frac{k}{m}\right) \end{vmatrix} \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \begin{pmatrix} \frac{F_0}{m} \\ 0 \end{pmatrix}$$

We calculate the determinant:

$$\Delta(\omega)=\begin{vmatrix} \left(-\Omega^2+\frac{k}{m}+j\frac{\beta}{m}\Omega\right) & -\frac{k}{m} \\ -\frac{k}{m} & \left(-\Omega^2+\frac{k}{m}\right) \end{vmatrix}=0 \Rightarrow \Delta(\omega)=\left(-\Omega^2+\frac{k}{m}+j\frac{\beta}{m}\Omega\right)\left(-\Omega^2+\frac{k}{m}\right)-\left(\frac{k}{m}\right)^2$$

3. If β=0 et F(t)=0

$$\Delta(\omega)=\left(-\Omega^2+\frac{k}{m}\right)\left(-\Omega^2+\frac{k}{m}\right)-\frac{k^2}{m^2}=0$$

$$\Rightarrow \left(-\Omega^2+\frac{k}{m}\right)^2-\frac{k^2}{m^2}=0 \Rightarrow \left(-\Omega^2+\frac{k}{m}-\frac{k}{m}\right)\left(-\Omega^2+\frac{k}{m}+\frac{k}{m}\right)=0$$

$$\Rightarrow \begin{cases} \Omega_1=\sqrt{\frac{2k}{m}}=\Omega_R \\ \Omega_2=0 \end{cases}$$

4. The differential equations of figure B

$$\begin{cases} V_L + V_R + V_C = U(t) \\ V_L + V_C = 0 \end{cases} \Rightarrow \begin{cases} L\dfrac{di_1}{dt} + Ri_1 \dfrac{1}{C}\int (i_1 - i_2)\,dt = U(t) \\ L\dfrac{di_2}{dt} - \dfrac{1}{C}\int (i_1 - i_2)\,dt = 0 \end{cases}$$

$$\mathrm{i} = \frac{dq}{dt} = \dot{q} \Rightarrow \frac{di}{dt} = \ddot{q}$$

$$\begin{cases} L\ddot{q}_1 + R\dot{q}_1 + \dfrac{1}{C}(q_1 - q_2) = U(t) \\ L\ddot{q}_2 - \dfrac{1}{C}(q_1 - q_2) = 0 \end{cases} \Rightarrow \begin{cases} \ddot{q}_1 + \dfrac{R}{L}\dot{q}_1 + \dfrac{1}{LC}(q_1 - q_2) = U(t) \\ \ddot{q}_2 - \dfrac{1}{LC}(q_1 - q_2) = 0 \end{cases}$$

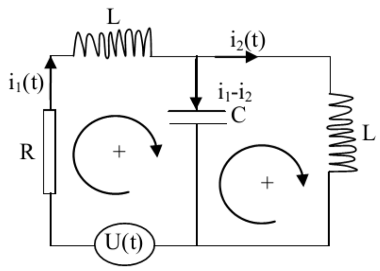


Fig. C. The law of meshes and knots in figure B

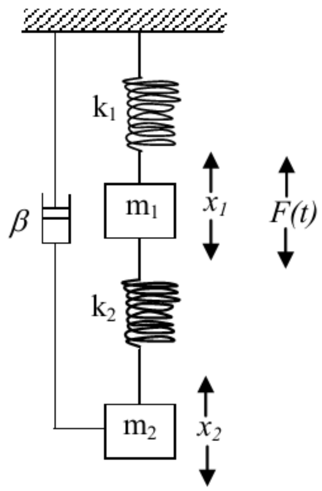
5. Yes, there is an analogy, we see:

$x \to q, R \to \beta, m \to L, k \to \dfrac{1}{C}, F(t) \to U(t)$ and the resonance pulsation: $\Omega_R = \sqrt{\dfrac{2}{LC}}$

Exercise No. 4

Consider the oscillating mechanical system in the following figure: $x_1 = x_1(t)$ and $x_2 = x_2(t)$ are respectively the dynamic positions (amplitudes at each instant) of the masses m_1 and m_2 relative to their rest positions (equilibrium). F(t) exciting force applied in m_1.

1. Write the differential equations with: $m_1 = m_2 = m$ and $k_1 = k_2 = k$.
2. Find permanent regime solutions knowing that $F(t) = ka\cos\Omega t$.
3. If β=0 and $F(t)$=0, for this value of Ω do we have resonance. In this case, give the condition for which the 1st mass remains immobile.



Solution

1. The differential equations are

$$T=\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}m_2\dot{x}_2^2$$

$$U=\frac{1}{2}k_1x_1^2+\frac{1}{2}k_2\left(x_1-x_2\right)^2$$

The Lagrangian: $L=\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}m_2\dot{x}_2^2-\frac{1}{2}k_1x_1^2-\frac{1}{2}k_2\left(x_1-x_2\right)^2$

The dissipation function: $D=\frac{1}{2}\beta\dot{x}_1^2$

Lagrange's equations:

$$\begin{cases}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\left(\frac{\partial L}{\partial x_1}\right)=F_{x_1}\left(t\right)\\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)-\left(\frac{\partial L}{\partial x_2}\right)+\frac{\partial D}{\partial \dot{x}_2}=0\end{cases}$$

$$\begin{cases}m_1\ddot{x}_1+k_1x_1+k_2\left(x_1-x_2\right)=F\left(t\right)\\ m_2\ddot{x}_2+k_2\left(x_2-x_1\right)+\beta\dot{x}_2=0\end{cases}\Rightarrow\begin{cases}\ddot{x}_1+\frac{k_1}{m_1}x_1+\frac{k_2}{m_1}\left(x_1-x_2\right)=\frac{F\left(t\right)}{m_1}\\ \ddot{x}_2+\frac{k_2}{m_2}\left(x_2-x_1\right)+\frac{\beta}{m_2}\dot{x}_2=0\end{cases}$$

By replacing the constants, $\begin{cases}\ddot{x}_1+\frac{2k}{m}x_1-\frac{k}{m}x_2=\frac{F\left(t\right)}{m}\\ \ddot{x}_2+\frac{k}{m}x_2-\frac{k}{m}x_1+\frac{\beta}{m}\dot{x}_2=0\end{cases}$

2. The permanent regime solutions $F(t)=ka\cos\Omega t\rightarrow\tilde{F}(t)=kae^{j\Omega t}$

$$\begin{cases}\bar{x}_1=\bar{A}_1e^{j\Omega t}\\ \bar{x}_2=\bar{A}_2e^{j\Omega t}\end{cases}\Rightarrow\begin{cases}\left(-\Omega^2+\dfrac{2k}{m}\right)\bar{A}_1-\dfrac{k}{m}\bar{A}_2=\dfrac{ka}{m}\\ -\dfrac{k}{m}\bar{A}_1+\left(-\Omega^2+\dfrac{k}{m}+j\dfrac{\beta}{m}\Omega\right)\bar{A}_2=0\end{cases}$$

$$\Rightarrow\begin{vmatrix}\left(-\Omega^2+\dfrac{2k}{m}\right) & -\dfrac{k}{m}\\ -\dfrac{k}{m} & \left(-\Omega^2+\dfrac{k}{m}+j\dfrac{\beta}{m}\Omega\right)\end{vmatrix}\begin{pmatrix}\bar{A}_1\\ \bar{A}_2\end{pmatrix}=\begin{pmatrix}\dfrac{ka}{m}\\ 0\end{pmatrix}$$

We calculate the determinant:

$$\Delta(\omega)=\begin{vmatrix}\left(-\Omega^2+\dfrac{2k}{m}\right) & -\dfrac{k}{m}\\ -\dfrac{k}{m} & \left(-\Omega^2+\dfrac{k}{m}+j\dfrac{\beta}{m}\Omega\right)\end{vmatrix}=\begin{pmatrix}\dfrac{ka}{m}\\ 0\end{pmatrix}$$

$$\Rightarrow\Delta(\omega)=\left(-\Omega^2+\frac{2k}{m}\right)\left(-\Omega^2+\frac{k}{m}+j\frac{\beta}{m}\Omega\right)-\left(\frac{k}{m}\right)^2$$

$$\tilde{x}_1=\frac{1}{\Delta(\omega)}\begin{vmatrix}\dfrac{ka}{m} & -\dfrac{k}{m}\\ 0 & \left(-\Omega^2+\dfrac{k}{m}+j\dfrac{\beta}{m}\Omega\right)\end{vmatrix}=\frac{1}{\Delta(\omega)}\frac{ka}{m}\left(-\Omega^2+\frac{k}{m}+j\frac{\beta}{m}\Omega\right)$$

$$\tilde{x}_2=\frac{1}{\Delta(\omega)}\begin{vmatrix}\left(-\Omega^2+\dfrac{2k}{m}\right) & \dfrac{ka}{m}\\ -\dfrac{k}{m} & 0\end{vmatrix}=\frac{1}{\Delta(\omega)}\frac{k^2a}{m^2}$$

3. Calculation of pulsations for $\beta=0$ and $F(t)=0$:

$$\Delta(\omega)=\begin{vmatrix}\left(-\Omega^2+\dfrac{2k}{m}\right) & -\dfrac{k}{m}\\ -\dfrac{k}{m} & \left(-\Omega^2+\dfrac{k}{m}\right)\end{vmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow\Delta(\omega)=\left(-\Omega^2+\frac{2k}{m}\right)\left(-\Omega^2+\frac{k}{m}\right)-\left(\frac{k}{m}\right)^2=0$$

$$\Rightarrow\Omega^4-\frac{3k}{m}\Omega^2+\frac{k^2}{m^2}=0\Rightarrow\Delta=\frac{9k^2}{m^2}-\frac{4k^2}{m^2}=\frac{5k^2}{m^2}\Rightarrow\sqrt{\Delta}=\sqrt{5}\,\frac{k}{m}$$

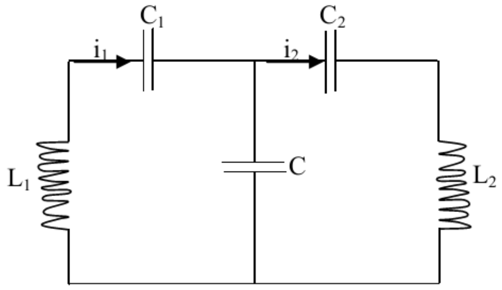
The equation admits two solutions: $\Rightarrow\begin{cases}\Omega_1=\dfrac{\dfrac{3k}{m}+\sqrt{5}\dfrac{k}{m}}{2}\\ \Omega_2=\dfrac{\dfrac{3k}{m}-\sqrt{5}\dfrac{k}{m}}{2}\end{cases}$

1st mass remains immobile $\tilde{x}_1=0$ and $\beta=0\Rightarrow-\Omega^2+\dfrac{k}{m}=0\Rightarrow\Omega=\sqrt{\dfrac{k}{m}}$

Exercise No. 5

Consider the assembly of the following figure. The 2 oscillating LC circuits are coupled by a capacitor C. Here $C_1 = C_2 = C$ and $L_1 = L_2 = L$

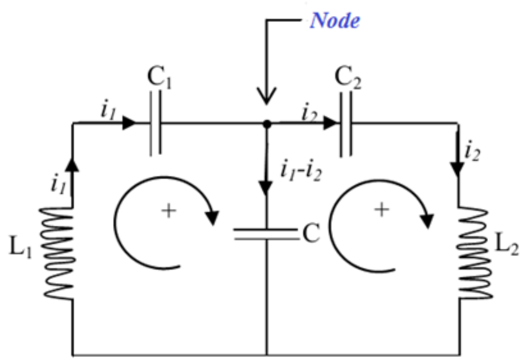
1. Write the 2 differential equations in i_1 and i_2 (Then in q_1 and q_2).
2. Find the proper pulsations of the system and give the general solution knowing that at $t = 0$, only C_1 has a charge q.



Solution

1. The differential equations of motion: $\sum_{i=1}^{n} V_n = 0$

The law of meshes gives us



$$\begin{cases} V_{L_1} + V_{C_1} + V_C = 0 \\ V_{L_2} + V_{C_2} + V_C = 0 \end{cases} \Rightarrow \begin{cases} L_1 \dfrac{di_1}{dt} + \dfrac{1}{C_1}\int i_1 dt + \dfrac{1}{C}\int (i_1 - i_2) dt = 0 \\ L_2 \dfrac{di_2}{dt} + \dfrac{1}{C_2}\int i_2 dt - \dfrac{1}{C}\int (i_1 - i_2) dt = 0 \end{cases}$$

$$\mathrm{i} = \frac{dq}{dt} = \dot{q} \Rightarrow \frac{di}{dt} = \ddot{q}$$

$$\begin{cases} L_1 \ddot{q}_1 + \dfrac{1}{C_1} q_1 + \dfrac{1}{C}(q_1 - q_2) = 0 \\ L_2 \ddot{q}_2 + \dfrac{1}{C_2} q_2 - \dfrac{1}{C}(q_1 - q_2) = 0 \end{cases}$$

2. The system's proper pulsations: $C_1 = C_2 = C$ and $L_1 = L_2 = L$

$$\begin{cases} L\ddot{q}_1 + \frac{2}{C}q_1 - \frac{1}{C}q_2 = 0 \\ L\ddot{q}_2 + \frac{2}{C}q_2 - \frac{1}{C}q_1 = 0 \end{cases} \Rightarrow \begin{cases} \ddot{q}_1 + \frac{2}{LC}q_1 - \frac{1}{LC}q_2 = 0 \\ \ddot{q}_2 + \frac{2}{LC}q_2 - \frac{1}{LC}q_1 = 0 \end{cases}$$

In complex notation we have

$$\begin{cases} \bar{q}_1 = \bar{A}_1 e^{j\Omega t} \\ \bar{q}_2 = \bar{A}_2 e^{j\Omega t} \end{cases} \Rightarrow \begin{cases} \ddot{q}_1 + \frac{2}{LC}q_1 - \frac{1}{LC}q_2 = 0 \\ \ddot{q}_2 + \frac{2}{LC}q_2 - \frac{1}{LC}q_1 = 0 \end{cases} \Rightarrow \begin{cases} \bar{A}_1 j^2\Omega^2 e^{j\Omega t} + \frac{2}{LC}\bar{A}_1 e^{j\Omega t} - \frac{1}{LC}\bar{A}_2 e^{j\Omega t} = 0 \\ \bar{A}_2 j^2\Omega^2 e^{j\Omega t} + \frac{2}{LC}\bar{A}_2 e^{j\Omega t} - \frac{1}{LC}\bar{A}_1 e^{j\Omega t} = 0 \end{cases}$$

$$\begin{cases} -\bar{A}_1\Omega^2 e^{j\Omega t} + \frac{2}{LC}\bar{A}_1 e^{j\Omega t} - \frac{1}{LC}\bar{A}_2 e^{j\Omega t} = 0 \\ -\bar{A}_2\Omega^2 e^{j\Omega t} + \frac{2}{LC}\bar{A}_2 e^{j\Omega t} - \frac{1}{LC}\bar{A}_1 e^{j\Omega t} = 0 \end{cases} \Rightarrow \begin{cases} \left(-\bar{A}_1\Omega^2 + \frac{2}{LC}\bar{A}_1 - \frac{1}{LC}\bar{A}_2\right)e^{j\Omega t} = 0 \\ \left(-\bar{A}_2\Omega^2 + \frac{2}{LC}\bar{A}_2 - \frac{1}{LC}\bar{A}_1\right)e^{j\Omega t} = 0 \end{cases}$$

$$\begin{cases} \left(-\Omega^2 + \frac{2}{LC}\right)\bar{A}_1 - \frac{1}{LC}\bar{A}_2 = 0 \\ -\frac{1}{LC}\bar{A}_1 + \left(-\Omega^2 + \frac{2}{LC}\right)\bar{A}_2 = 0 \end{cases} \Rightarrow \begin{pmatrix} \left(-\Omega^2 + \frac{2}{LC}\right) & -\frac{1}{LC} \\ -\frac{1}{LC} & \left(-\Omega^2 + \frac{2}{LC}\right) \end{pmatrix} \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We calculate the determinant:

$$\Delta(\omega) = \begin{vmatrix} \left(-\Omega^2 + \frac{2}{LC}\right) & -\frac{1}{LC} \\ -\frac{1}{LC} & \left(-\Omega^2 + \frac{2}{LC}\right) \end{vmatrix} = 0$$

$$\Delta(\omega) = \left(-\Omega^2 + \frac{2}{LC}\right)\left(-\Omega^2 + \frac{2}{LC}\right) - \left(-\frac{1}{LC}\right)\left(-\frac{1}{LC}\right) = 0$$

$$\Rightarrow \left(-\Omega^2 + \frac{2}{LC}\right)^2 - \left(\frac{1}{LC}\right)^2 = 0 \Rightarrow \left(-\Omega^2 + \frac{2}{LC} - \frac{1}{LC}\right)\left(-\Omega^2 + \frac{2}{LC} + \frac{1}{LC}\right) = 0$$

$$\Rightarrow \left(-\Omega^2 + \frac{1}{LC}\right)\left(-\Omega^2 + \frac{3}{LC}\right) = 0 \Rightarrow \begin{cases} \Omega_1 = \sqrt{\frac{3}{LC}} = \sqrt{3}\omega_0 \\ \Omega_2 = \sqrt{\frac{1}{LC}} = \omega_0 \end{cases}$$

mode 1

$$\Omega_1^2 = \frac{3}{LC} \Rightarrow \begin{pmatrix} \left(-\frac{3}{LC} + \frac{2}{LC}\right) & -\frac{1}{LC} \\ -\frac{1}{LC} & \left(-\frac{3}{LC} + \frac{2}{LC}\right) \end{pmatrix} \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -\frac{1}{LC} & -\frac{1}{LC} \\ -\frac{1}{LC} & -\frac{1}{LC} \end{pmatrix} \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left(\frac{1}{LC}\right)\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \text{eigenvector}: \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

mode 2

$$\Omega_2^{\,2}=\frac{1}{LC}\Rightarrow\begin{pmatrix}\left(-\frac{1}{LC}+\frac{2}{LC}\right) & -\frac{1}{LC}\\ -\frac{1}{LC} & \left(-\frac{1}{LC}+\frac{2}{LC}\right)\end{pmatrix}\begin{pmatrix}\bar{A}_1\\ \bar{A}_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}\frac{1}{LC} & -\frac{1}{LC}\\ -\frac{1}{LC} & \frac{1}{LC}\end{pmatrix}\begin{pmatrix}\bar{A}_1\\ \bar{A}_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\Rightarrow\left(\frac{1}{LC}\right)\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}\bar{A}_1\\ \bar{A}_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow \text{eigenvector}:\vec{v}_2=\begin{pmatrix}1\\1\end{pmatrix}$$

Then the passage matrix is $P=\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$ and the general solutions are

$$\begin{pmatrix}q_1\\ q_2\end{pmatrix}=P\begin{pmatrix}A_1\cos(\Omega_1 t+\varphi_1)\\ A_2\cos(\Omega_2 t+\varphi_2)\end{pmatrix}$$

$$\Rightarrow\begin{cases}q_1=A_1\cos(\Omega_1 t+\varphi_1)+A_2\cos(\Omega_2 t+\varphi_2)\\ q_2=-A_1\cos(\Omega_1 t+\varphi_1)+A_2\cos(\Omega_2 t+\varphi_2)\end{cases}$$

The constants A_1, A_2, φ_1, φ_2 are determined using the initial conditions 4 unknowns so it is necessary to have 4 equations:

$$q_1(t=0)=q,\ q_2(t=0)=0$$
$$\dot{q}_1(t=0)=\dot{q}_2(t=0)=0$$

$$q_1(t=0)=A_1\cos(\varphi_1)+A_2\cos(\varphi_2)=q\Rightarrow 2A_1\cos(\varphi_1)=q=2A_2\cos(\varphi_2)$$
$$q_2(t=0)=-A_1\cos(\varphi_1)+A_2\cos(\varphi_2)=0\Rightarrow A_1\cos(\varphi_1)=A_2\cos(\varphi_2)$$

$$\begin{cases}q_1=A_1\cos(\Omega_1 t+\varphi_1)+A_2\cos(\Omega_2 t+\varphi_2)\\ q_2=-A_1\cos(\Omega_1 t+\varphi_1)+A_2\cos(\Omega_2 t+\varphi_2)\end{cases}\Rightarrow\begin{cases}\dot{q}_1=-A_1\Omega_1\sin(\Omega_1 t+\varphi_1)-A_2\Omega_2\sin(\Omega_2 t+\varphi_2)\\ \dot{q}_2=A_1\Omega_1\sin(\Omega_1 t+\varphi_1)-A_2\Omega_2\sin(\Omega_2 t+\varphi_2)\end{cases}$$

$$\dot{q}_1(t=0)=-A_1\Omega_1\sin(\varphi_1)-A_2\Omega_2\sin(\varphi_2)=0\Rightarrow 2A_1\Omega_1\sin(\varphi_1)=0$$
$$\dot{q}_2(t=0)=A_1\Omega_1\sin(\varphi_1)-A_2\Omega_2\sin(\varphi_2)=0\Rightarrow 2A_2\Omega_2\sin(\varphi_2)=0$$

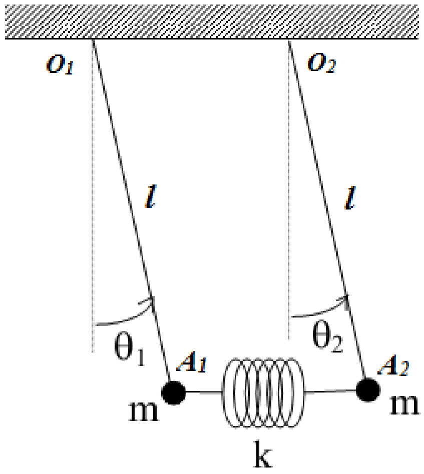
$$\varphi_1=\varphi_2=0,\ A_1=A_2=\frac{q}{2}$$

$$q_1=\frac{q}{2}\left(\cos(\Omega_1 t)+\cos(\Omega_2 t)\right)$$

$$q_2=-\frac{q}{2}\left(\cos(\Omega_1 t)-\cos(\Omega_2 t)\right)$$

Exercise No. 6

Two identical simple pendulums O_1A_1 and O_2A_2 of mass m and length l, are coupled by a horizontal spring of stiffness k which connects the two masses A_1 and A_2. In equilibrium, the horizontal spring has its natural length l_0 as it is $l_0= O_1O_2$. The two pendulums are identified, at the instant t, by their angular elongations $\theta_1(t)$ and $\theta_2(t)$ assumed to be small compared to their vertical equilibrium position. We designate g the acceleration of gravity.



Proper modes:

1. Determine the Lagrangian of the system.

2. Establish the coupled differential equations verified by the two instantaneous angular elongations $\theta_1(t)$ and $\theta_2(t)$.

3. Express as a function of g, k, l and m, the two proper pulsations Ω_{1p} and Ω_{2p} of this system.

Digital applications:

4. Calculate Ω_{1p} and Ω_{2p} knowing that: m= 100 g; l=80 cm; k=9.2 N/m and g= 9.8 m/s^2.

We release the system without initial speeds at time $t=0$ under the following initial conditions: $\theta_1(t)=\theta_0$ and $\theta_2(t)=0$.

5. Deduce the laws of evolution $\theta_1(t)$ and $\theta_2(t)$ at times $t>0$.

6. What is the phenomenon studied.

Forced modes:

Mass A_1 is subjected to a horizontal exciting force of the form: $F(t) = F_0 \cos \Omega t$.

7. Write the new coupled differential equations in $\theta_1(t)$ and $\theta_2(t)$.

8. Express the complex relationships which concern the linear velocities V_1 and V_2 of the points A_1 and A_2 in forced regime.

9. Deduce the complex input impedance $Z_e = \frac{\bar{F}}{\bar{V}_1}$

 Dr.Ghellab Torkia

Solution

Proper modes:

1. The Lagrangian of the system:

The system has two degrees of freedom expressed in $\theta_1(t)$ and $\theta_2(t)$

The kinetic energy we have: $T=\frac{1}{2}m_1V_{m_1}^2+\frac{1}{2}m_2V_{m_2}^2$

By calculating the speeds relative to the fixed reference:

$$\begin{cases} O\vec{m}_1=\begin{pmatrix} x_{m_1}=l\sin\theta_1 \\ y_{m_1}=l\cos\theta_1 \end{pmatrix} \\ O\vec{m}_2=\begin{pmatrix} x_{m_2}=l\sin\theta_2 \\ y_{m_2}=l\cos\theta_2 \end{pmatrix} \end{cases} \Rightarrow \begin{cases} \vec{V}_{m_1}=\begin{pmatrix} \dot{x}_{m_1}=l\dot{\theta}_1\cos\theta_1 \\ \dot{y}_{m_1}=-l\dot{\theta}_1\sin\theta_1 \end{pmatrix} \\ \vec{V}_{m_2}=\begin{pmatrix} \dot{x}_{m_1}=l\dot{\theta}_2\cos\theta_2 \\ \dot{y}_{m_1}=-l\dot{\theta}_2\sin\theta_2 \end{pmatrix} \end{cases}$$

$$\Rightarrow\begin{cases} V_{m_1}^2=\dot{x}_{m_1}^2+\dot{y}_{m_1}^2=\left(l\dot{\theta}_1\cos\theta_1\right)^2+\left(-l\dot{\theta}_1\sin\theta_1\right)^2=l^2\dot{\theta}_1^2\left(\cos\theta_1^2+\sin\theta_1^2\right)=l^2\dot{\theta}_1^2 \\ V_{m_2}^2=\dot{x}_{m_2}^2+\dot{y}_{m_2}^2=\left(l\dot{\theta}_2\cos\theta_2\right)^2+\left(-l\dot{\theta}_2\sin\theta_2\right)^2=l^2\dot{\theta}_2^2\left(\cos\theta_2^2+\sin\theta_2^2\right)=l^2\dot{\theta}_2^2 \end{cases}$$

$$T=\frac{1}{2}ml^2\dot{\theta}_1^2+\frac{1}{2}ml^2\dot{\theta}_2^2$$

For the potential energy we have:

$$U=\frac{1}{2}k\left(x_{m_1}-x_{m_2}\right)^2+mgh_1+mgh_2$$

$$x_{m_1}=l\sin\theta_1,x_{m_2}=l\sin\theta_2,h_1=l-l\cos\theta_1,h_2=l-l\cos\theta_2$$

At low amplitude

$$\sin\theta\approx\theta \text{ and } \cos\theta\approx 1-\frac{\theta^2}{2}\Rightarrow x_{m_1}=l\theta_1,x_{m_2}=l\theta_2,h_1=\frac{1}{2}l\theta_1^2,h_2=\frac{1}{2}l\theta_2^2$$

$$U=\frac{1}{2}k\left(l\theta_1-l\theta_2\right)^2+\frac{1}{2}mgl\theta_1^2+\frac{1}{2}mgl\theta_2^2\Rightarrow U=\frac{1}{2}\left(kl^2+mgl\right)\theta_1^2+\frac{1}{2}\left(kl^2+mgl\right)\theta_2^2-kl^2\theta_1\theta_2$$

The Lagrangian is then written:

$$L=T-U=\frac{1}{2}ml^2\dot{\theta}_1^2+\frac{1}{2}ml^2\dot{\theta}_2^2-\frac{1}{2}\left(kl^2+mgl\right)\theta_1^2-\frac{1}{2}\left(kl^2+mgl\right)\theta_2^2+kl^2\theta_1\theta_2$$

2. Coupled differential equations:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\dfrac{\partial L}{\partial \theta_1}\right) = 0 \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\dfrac{\partial L}{\partial \theta_2}\right) = 0 \end{cases} \Rightarrow \begin{cases} ml^2\ddot{\theta}_1 + \left(kl^2 + mgl\right)\theta_1 - kl^2\theta_2 = 0 \\ ml^2\ddot{\theta}_2 + \left(kl^2 + mgl\right)\theta_2 - kl^2\theta_1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \ddot{\theta}_1 + \dfrac{\left(kl^2 + mgl\right)}{ml^2}\theta_1 - \dfrac{kl^2}{ml^2}\theta_2 = 0 \\ \ddot{\theta}_1 + \dfrac{\left(kl^2 + mgl\right)}{ml^2}\theta_1 - \dfrac{kl^2}{ml^2}\theta_2 = 0 \end{cases} \Rightarrow \begin{cases} \ddot{\theta}_1 + \left(\dfrac{k}{m} + \dfrac{g}{l}\right)\theta_1 - \dfrac{k}{m}\theta_2 = 0 \\ \ddot{\theta}_2 + \left(\dfrac{k}{m} + \dfrac{g}{l}\right)\theta_2 - \dfrac{k}{m}\theta_1 = 0 \end{cases}$$

3. The two proper pulsations Ω_{1p} and Ω_{2p} of this system: We consider the solutions of the sinusoidal type system: In complex notation we have

$$\begin{cases} \bar{\theta}_1 = \bar{A}_1 e^{j\Omega t} \\ \bar{\theta}_2 = \bar{A}_2 e^{j\Omega t} \end{cases} \Rightarrow \begin{cases} \ddot{\theta}_1 + \left(\dfrac{k}{m} + \dfrac{g}{l}\right)\theta_1 - \dfrac{k}{m}\theta_2 = 0 \\ \ddot{\theta}_2 + \left(\dfrac{k}{m} + \dfrac{g}{l}\right)\theta_2 - \dfrac{k}{m}\theta_1 = 0 \end{cases} \Rightarrow \begin{cases} \bar{A}_1 j^2\Omega^2 e^{j\Omega t} + \left(\dfrac{k}{m} + \dfrac{g}{l}\right)\bar{A}_1 e^{j\Omega t} - \dfrac{k}{m}\bar{A}_2 e^{j\Omega t} = 0 \\ \bar{A}_2 j^2\Omega^2 e^{j\Omega t} + \left(\dfrac{k}{m} + \dfrac{g}{l}\right)\bar{A}_2 e^{j\Omega t} - \dfrac{k}{m}\bar{A}_1 e^{j\Omega t} = 0 \end{cases}$$

$$\begin{cases} -\bar{A}_1\Omega^2 e^{j\Omega t} + \left(\dfrac{k}{m} + \dfrac{g}{l}\right)\bar{A}_1 e^{j\Omega t} - \dfrac{k}{m}\bar{A}_2 e^{j\Omega t} = 0 \\ -\bar{A}_2\Omega^2 e^{j\Omega t} + \left(\dfrac{k}{m} + \dfrac{g}{l}\right)\bar{A}_2 e^{j\Omega t} - \dfrac{k}{m}\bar{A}_1 e^{j\Omega t} = 0 \end{cases} \Rightarrow \begin{cases} \left(-\bar{A}_1\Omega^2 + \left(\dfrac{k}{m} + \dfrac{g}{l}\right)\bar{A}_1 - \dfrac{k}{m}\bar{A}_2\right)e^{j\Omega t} = 0 \\ \left(-\bar{A}_2\Omega^2 + \left(\dfrac{k}{m} + \dfrac{g}{l}\right)\bar{A}_2 - \dfrac{k}{m}\bar{A}_1\right)e^{j\Omega t} = 0 \end{cases}$$

$$\begin{cases} \left(-\Omega^2 + \left(\dfrac{k}{m} + \dfrac{g}{l}\right)\right)\bar{A}_1 - \dfrac{k}{m}\bar{A}_2 = 0 \\ -\dfrac{k}{m}\bar{A}_1 + \left(-\Omega^2 + \left(\dfrac{k}{m} + \dfrac{g}{l}\right)\right)\bar{A}_2 = 0 \end{cases} \Rightarrow \begin{pmatrix} \left(-\Omega^2 + \left(\dfrac{k}{m} + \dfrac{g}{l}\right)\right) & -\dfrac{k}{m} \\ -\dfrac{k}{m} & \left(-\Omega^2 + \left(\dfrac{k}{m} + \dfrac{g}{l}\right)\right) \end{pmatrix}\begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The system admits non-zero solutions if only if $\Delta(\omega) = 0$:

We calculate the determinant:

$$\Delta(\omega) = \begin{vmatrix} \left(-\Omega^2 + \left(\dfrac{k}{m} + \dfrac{g}{l}\right)\right) & -\dfrac{k}{m} \\ -\dfrac{k}{m} & \left(-\Omega^2 + \left(\dfrac{k}{m} + \dfrac{g}{l}\right)\right) \end{vmatrix} = 0$$

$$\Delta(\omega)=\left(-\Omega^2+\left(\frac{k}{m}+\frac{g}{l}\right)\right)\left(-\Omega^2+\left(\frac{k}{m}+\frac{g}{l}\right)\right)-\left(-\frac{k}{m}\right)\left(-\frac{k}{m}\right)=0$$

$$\Rightarrow\left(-\Omega^2+\left(\frac{k}{m}+\frac{g}{l}\right)\right)^2-\left(\frac{k}{m}\right)^2=0\Rightarrow\left(-\Omega^2+\left(\frac{k}{m}+\frac{g}{l}\right)-\frac{k}{m}\right)\left(-\Omega^2+\left(\frac{k}{m}+\frac{g}{l}\right)+\frac{k}{m}\right)=0$$

$$\Rightarrow\left(-\Omega^2+\frac{g}{l}\right)\left(-\Omega^2+\frac{2k}{m}+\frac{g}{l}\right)=0\Rightarrow\begin{cases}\Omega_{1p}=\sqrt{\frac{g}{l}}=3.5\,rad/s\\ \Omega_{2p}=\sqrt{\frac{2k}{m}+\frac{g}{l}}=14\,rad/s\end{cases}$$

4. <u>*Digital applications:*</u> $\begin{cases}\Omega_{1p}=3.5\,rad/s\\ \Omega_{2p}=14\,rad/s\end{cases}$

5. General solutions:

$$\begin{cases}\theta_1(t)=A_{11}\cos\left(\Omega_{1p}t+\varphi_1\right)+A_{12}\cos\left(\Omega_{2p}t+\varphi_2\right)\\ \theta_2(t)=-A_{21}\cos\left(\Omega_{1p}t+\varphi_1\right)+A_{22}\cos\left(\Omega_{2p}t+\varphi_2\right)\end{cases}$$

By applying the initial conditions, we find:

$$\begin{cases}x_1(t)=C\cos\left(\frac{\Omega_{1p}t+\Omega_{2p}}{2}t\right)\cos\left(\frac{\Omega_{1p}t-\Omega_{2p}}{2}t\right)\\ x_2(t)=-C\sin\left(\frac{\Omega_{1p}t+\Omega_{2p}}{2}t\right)\sin\left(\frac{\Omega_{1p}t-\Omega_{2p}}{2}t\right)\end{cases}$$

From where:

$$\begin{cases}x_1(t)=C\cos\left(\Delta\Omega t\right)\cos\left(\Omega t\right)\\ x_2(t)=C\sin\left(\Delta\Omega t\right)\sin\left(\Omega t\right)\end{cases}$$

$$And\ \Delta\Omega=\frac{\Omega_{1p}t-\Omega_{2p}}{2}t,\Delta\Omega=\frac{\Omega_{1p}t+\Omega_{2p}}{2}t$$

6. The phenomenon studied is beats.

<u>***Forced modes:***</u>

7. The new coupled differential equations:

$$\begin{cases}\ddot{\theta}_1+\left(\frac{k}{m}+\frac{g}{l}\right)\theta_1-\frac{k}{m}\theta_2=\frac{F_0}{m}\cos\Omega t\\ \ddot{\theta}_2+\left(\frac{k}{m}+\frac{g}{l}\right)\theta_2-\frac{k}{m}\theta_1=0\end{cases}$$

8. The complex relationships that concern linear speeds V_1 and V_2:

$$\begin{cases}\bar{\theta}_1=\bar{A}_1e^{j\Omega t}\\ \bar{\theta}_2=\bar{A}_2e^{j\Omega t}\end{cases}\Rightarrow\begin{cases}\bar{V}_1=j\Omega\bar{\theta}_1\\ \bar{V}_2=j\Omega\bar{\theta}_2\end{cases}$$

By replacing the solutions in the differential system, we obtain a

$$\begin{cases} \left(-\Omega^2 + \left(\frac{k}{m} + \frac{g}{l}\right)\right)\bar{V}_1 - \frac{k}{m}\bar{V}_2 = j\Omega\frac{F_0}{m} \\ -\frac{k}{m}\bar{V}_1 + \left(-\Omega^2 + \left(\frac{k}{m} + \frac{g}{l}\right)\right)\bar{V}_2 = 0 \end{cases}$$

9. Complex input impedance $Z_e = \frac{F}{\bar{V}_1} = \frac{j}{\Omega}\left[\frac{k^2}{\frac{mg}{l} + k - m\Omega^2} - \left(\frac{mg}{l} + k - m\Omega^2\right)\right]$

This mechanical system functions as a frequency filter since its impedance varies depending on the frequency.

Problem 1

The 2 cylinders of the same mass *M* and same radius *R* roll without slipping, that is to say that when they rotate respectively θ_1 and θ_2, their centers of gravity move respectively from $x_1=R\theta_1$ and $x_2=R\theta_2$.

Part I:

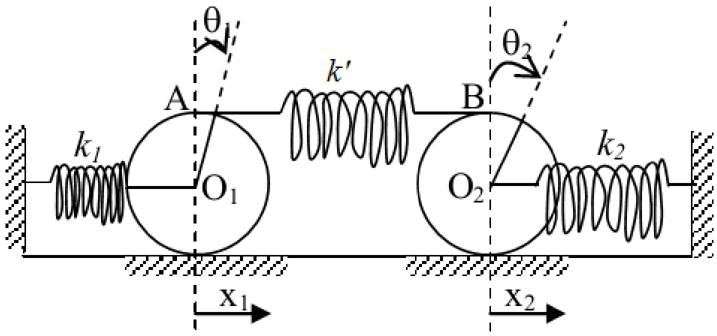
1) Determine the kinetic energy and potential energy of the system as a function of θ_1 and θ_2.
2) Establish the Lagrangian for $k_1= k_2 = k$ and $k'=2k$. Deduce the system of differential equations of motion
3) Find the proper pulsations corresponding to the possible vibration modes.
4) Deduce the passage matrix and write the general solutions.

Part II:

5) Establish the Lagrangian of the system as a function of x_1 and x_2.
6) We take $k_1= k_2= k \neq k'$, find the equations of motion. Deduce the proper pulsations.
7) Write the Lagrangian in the following form:

$L = \frac{3}{4}M\left(\dot{x}_1^2 + \dot{x}_2^2 - \omega_0^2\left(x_1^2 + x_2^2 - 2Kx_1x_2\right)\right)$. Deduce the expressions of ω_0^2 and *K*.

8) k' is the coupling coefficient, show that it varies between 2 limit values.
9) Give the physical meaning of ω_0 by comparing it to the proper pulsations found in question 3. Deduce the effect of the coupling *k*' on the proper pulsations.
10) We take $k_1= k_2= k'=k$, find the equations of motion
11) Calculate the proper pulsations
12) Deduce the passage matrix and the general solutions



Solution

Part I:

1)

$$U=\frac{1}{2}k_1x_1^2+\frac{1}{2}k_2x_2^2+\frac{1}{2}k'(x_1-x_2)^2 \Rightarrow U=\frac{1}{2}k_1(2R\theta_1)^2+\frac{1}{2}k_2(2R\theta_2)^2+\frac{1}{2}k'(R\theta_1-R\theta_2)^2$$

$$T=\frac{1}{2}M\dot{x}_1^2+\frac{1}{2}M\dot{x}_2^2+\frac{1}{2}I\dot{\theta}_1^2+\frac{1}{2}I\dot{\theta}_2^2$$

$$\dot{x}_1=R\dot{\theta}_1,\dot{x}_2=R\dot{\theta}_2,I=\frac{1}{2}MR^2$$

$$\Rightarrow T=\frac{1}{2}MR^2\dot{\theta}_1^2+\frac{1}{2}MR^2\dot{\theta}_2^2+\frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot{\theta}_1^2+\frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot{\theta}_2^2 \Rightarrow T=\frac{3}{4}MR^2\left(\dot{\theta}_1^2+\dot{\theta}_1^2\right)$$

2) $k_1=k_2=k$ *and* $k'=2k$

$$L=\frac{3}{4}MR^2\left(\dot{\theta}_1^2+\dot{\theta}_1^2\right)-\frac{1}{2}k_1(2R\theta_1)^2-\frac{1}{2}k_2(2R\theta_2)^2-\frac{1}{2}k'(R\theta_1-R\theta_2)^2$$

$$L=\frac{3}{4}MR^2\left(\dot{\theta}_1^2+\dot{\theta}_1^2\right)-\frac{1}{2}k(2R\theta_1)^2-\frac{1}{2}k(2R\theta_2)^2-\frac{1}{2}2k(R\theta_1-R\theta_2)^2$$

$$L=\frac{3}{4}MR^2\left(\dot{\theta}_1^2+\dot{\theta}_1^2\right)-2kR^2\left(\theta_1^2+\theta_2^2\right)-k(R\theta_1-R\theta_2)^2$$

$$L=\frac{3}{4}MR^2\left(\dot{\theta}_1^2+\dot{\theta}_1^2\right)-2kR^2\left(\theta_1^2+\theta_2^2\right)-k\left(R^2\theta_1^2+R^2\theta_2^2-2R^2\theta_1\theta_2\right)$$

$$L=\frac{3}{4}MR^2\left(\dot{\theta}_1^2+\dot{\theta}_1^2\right)-kR^2\left(3\theta_1^2+3\theta_2^2-2\theta_1\theta_2\right)$$

$$L=\frac{3}{4}MR^2\left(\dot{\theta}_1^2+\dot{\theta}_1^2\right)-kR^2\left(3\theta_1^2+3\theta_2^2-2\theta_1\theta_2\right)$$

$$\begin{cases}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right)-\left(\frac{\partial L}{\partial\theta_1}\right)=0\\ \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right)-\left(\frac{\partial L}{\partial\theta_2}\right)=0\end{cases}\Rightarrow\begin{cases}\frac{3}{2}MR^2\ddot{\theta}_1+kR^2\left(6\theta_1-2\theta_2\right)=0\\ \frac{3}{2}MR^2\ddot{\theta}_2+kR^2\left(6\theta_2-2\theta_1\right)=0\end{cases}$$

$$\Rightarrow\begin{cases}\ddot{\theta}_1+\dfrac{kR^2\left(6\theta_1-2\theta_2\right)}{\frac{3}{2}MR^2}=0\\ \ddot{\theta}_2+\dfrac{kR^2\left(6\theta_2-2\theta_1\right)}{\frac{3}{2}MR^2}=0\end{cases}\Rightarrow\begin{cases}\ddot{\theta}_1+\frac{12k}{3M}\theta_1-\frac{4k}{3M}\theta_2=0\\ \ddot{\theta}_2+\frac{12k}{3M}\theta_1-\frac{4k}{3M}\theta_2=0\end{cases}$$

In complex notation we have

$$\begin{cases}\bar{\theta}_1=\bar{A}_1e^{j\Omega t}\\ \bar{\theta}_2=\bar{A}_2e^{j\Omega t}\end{cases}\Rightarrow\begin{cases}\ddot{\theta}_1+\frac{12k}{3M}\theta_1-\frac{4k}{3M}\theta_2=0\\ \ddot{\theta}_2+\frac{12k}{3M}\theta_1-\frac{4k}{3M}\theta_2=0\end{cases}\Rightarrow\begin{cases}\bar{A}_1j^2\Omega^2e^{j\Omega t}+\frac{12k}{3M}\bar{A}_1e^{j\Omega t}-\frac{4k}{3M}\bar{A}_2e^{j\Omega t}=0\\ \bar{A}_2j^2\Omega^2e^{j\Omega t}+\frac{12k}{3M}\bar{A}_2e^{j\Omega t}-\frac{4k}{3M}\bar{A}_1e^{j\Omega t}=0\end{cases}$$

$$\begin{cases}-\bar{A}_1\Omega^2e^{j\Omega t}+\frac{12k}{3M}\bar{A}_1e^{j\Omega t}-\frac{4k}{3M}\bar{A}_2e^{j\Omega t}=0\\ -\bar{A}_2\Omega^2e^{j\Omega t}+\frac{12k}{3M}\bar{A}_2e^{j\Omega t}-\frac{4k}{3M}\bar{A}_1e^{j\Omega t}=0\end{cases}\Rightarrow\begin{cases}\left(-\bar{A}_1\Omega^2+\frac{12k}{3M}\bar{A}_1-\frac{4k}{3M}\bar{A}_2\right)e^{j\Omega t}=0\\ \left(-\bar{A}_2\Omega^2+\frac{12k}{3M}\bar{A}_2-\frac{4k}{3M}\bar{A}_1\right)e^{j\Omega t}=0\end{cases}$$

$$\begin{cases}\left(-\Omega^2+\frac{12k}{3M}\right)\bar{A}_1-\frac{4k}{3M}\bar{A}_2=0\\ -\frac{4k}{3M}\bar{A}_1+\left(-\Omega^2+\frac{12k}{3M}\right)\bar{A}_2=0\end{cases}\Rightarrow\begin{pmatrix}\left(-\Omega^2+\frac{12k}{3M}\right) & -\frac{4k}{3M}\\ -\frac{4k}{3M} & \left(-\Omega^2+\frac{12k}{3M}\right)\end{pmatrix}\begin{pmatrix}\bar{A}_1\\ \bar{A}_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

We calculate the determinant:

$$\Delta(\omega)=\begin{vmatrix}\left(-\Omega^2+\frac{12k}{3M}\right) & -\frac{4k}{3M}\\ -\frac{4k}{3M} & \left(-\Omega^2+\frac{12k}{3M}\right)\end{vmatrix}=0$$

$$\Delta(\omega)=\left(-\Omega^2+\frac{12k}{3M}\right)\left(-\Omega^2+\frac{12k}{3M}\right)-\left(-\frac{4k}{3M}\right)\left(-\frac{4k}{3M}\right)=0$$

$$\Rightarrow\left(-\Omega^2+\frac{12k}{3M}\right)^2-\left(\frac{4k}{3M}\right)^2=0\Rightarrow\left(-\Omega^2+\frac{12k}{3M}-\frac{4k}{3M}\right)\left(-\Omega^2+\frac{12k}{3M}+\frac{4k}{3M}\right)=0$$

$$\Rightarrow\left(-\Omega^2+\frac{8k}{3M}\right)\left(-\Omega^2+\frac{16k}{3M}\right)=0\Rightarrow\begin{cases}\Omega_1=\sqrt{\frac{16k}{3M}}\\ \Omega_2=\sqrt{\frac{8k}{3M}}\end{cases}$$

4)

mode 1

$$\Omega_1^2 = \frac{16k}{3M} \Rightarrow \begin{pmatrix} \left(-\frac{16k}{3M}+\frac{12k}{3M}\right) & -\frac{4k}{3M} \\ -\frac{4k}{3M} & \left(-\frac{16k}{3M}+\frac{12k}{3M}\right) \end{pmatrix} \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \left(-\frac{4k}{3M}\right) & -\frac{4k}{3M} \\ -\frac{4k}{3M} & \left(-\frac{4k}{3M}\right) \end{pmatrix} \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left(\frac{4k}{3M}\right) \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \text{eigenvector}: \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

mode 2

$$\Omega_2^2 = \frac{8k}{3M} \Rightarrow \begin{pmatrix} \left(-\frac{8k}{3M}+\frac{12k}{3M}\right) & -\frac{4k}{3M} \\ -\frac{4k}{3M} & \left(-\frac{8k}{3M}+\frac{12k}{3M}\right) \end{pmatrix} \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \left(\frac{4k}{3M}\right) & -\frac{4k}{3M} \\ -\frac{4k}{3M} & \left(\frac{4k}{3M}\right) \end{pmatrix} \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left(\frac{4k}{3M}\right) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \text{eigenvector}: \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Then the passage matrix is $P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and the general solutions are

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = P \begin{pmatrix} A_1 \cos(\Omega_1 t + \varphi_1) \\ A_2 \cos(\Omega_2 t + \varphi_2) \end{pmatrix} \Rightarrow \begin{cases} \theta_1 = A_1 \cos(\Omega_1 t + \varphi_1) + A_2 \cos(\Omega_2 t + \varphi_2) \\ \theta_2 = -A_1 \cos(\Omega_1 t + \varphi_1) + A_2 \cos(\Omega_2 t + \varphi_2) \end{cases}$$

The constants A_1, A_2, φ_1, φ_2 are determined using the initial conditions 4 unknowns so it is necessary to have 4 equations: $q_1(t=0), q_2(t=0), \dot{q}_1(t=0), \dot{q}_2(t=0)$

Part II:

5)

$$T = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}I\dot{\theta}_1^2 + \frac{1}{2}I\dot{\theta}_2^2$$

$$\dot{x}_1 = R\dot{\theta}_1, \dot{x}_2 = R\dot{\theta}_2, I = \frac{1}{2}MR^2$$

$$\Rightarrow T = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}\left(\frac{1}{2}M\dot{x}_1^2\right) + \frac{1}{2}\left(\frac{1}{2}M\dot{x}_2^2\right) \Rightarrow T = \frac{3}{4}M\left(\dot{x}_1^2 + \dot{x}_2^2\right)$$

$$U = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 + \frac{1}{2}k'\left((x_1 + R\theta_1) - (x_2 + R\theta_2)\right)^2 \Rightarrow U = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 + 2k'(x_1 - x_2)^2$$

$$L = T - U = \frac{3}{4}M\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2x_2^2 - 2k'(x_1 - x_2)^2$$

6) $k_1 = k_2 = k \neq k'$

$$L=\frac{3}{4}M\left(\dot{x}_1^2+\dot{x}_2^2\right)-\frac{1}{2}kx_1^2-\frac{1}{2}kx_2^2-2k'x_1^2-2k'x_2^2+4k'x_1x_2$$

$$\begin{cases}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\left(\frac{\partial L}{\partial x_1}\right)=0\\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)-\left(\frac{\partial L}{\partial x_2}\right)=0\end{cases}\Rightarrow\begin{cases}\frac{3}{2}M\ddot{x}_1+\left(k+4k'\right)x_1-4k'x_2=0\\ \frac{3}{2}M\ddot{x}_2+\left(k+4k'\right)x_2-4k'x_1=0\end{cases}\Rightarrow\begin{cases}\ddot{x}_1+\frac{2\left(k+4k'\right)}{3M}x_1-\frac{8k'}{3M}x_2=0\\ \ddot{x}_2+\frac{2\left(k+4k'\right)}{3M}x_1-\frac{8k'}{3M}x_2=0\end{cases}$$

In complex notation we have

$$\begin{cases}\bar{x}_1=\bar{A}_1e^{j\Omega t}\\ \bar{x}_2=\bar{A}_2e^{j\Omega t}\end{cases}\Rightarrow\begin{cases}\ddot{x}_1+\frac{2\left(k+4k'\right)}{3M}x_1-\frac{8k'}{3M}x_2=0\\ \ddot{x}_2+\frac{2\left(k+4k'\right)}{3M}x_1-\frac{8k'}{3M}x_2=0\end{cases}$$

$$\Rightarrow\begin{cases}\bar{A}_1j^2\Omega^2e^{j\Omega t}+\frac{2\left(k+4k'\right)}{3M}\bar{A}_1e^{j\Omega t}-\frac{8k'}{3M}\bar{A}_2e^{j\Omega t}=0\\ \bar{A}_2j^2\Omega^2e^{j\Omega t}+\frac{2\left(k+4k'\right)}{3M}\bar{A}_2e^{j\Omega t}-\frac{8k'}{3M}\bar{A}_1e^{j\Omega t}=0\end{cases}$$

$$\Rightarrow\begin{cases}\left(-\Omega^2+\frac{2\left(k+4k'\right)}{3M}\right)\bar{A}_1-\frac{8k'}{3M}\bar{A}_2=0\\ -\frac{8k'}{3M}\bar{A}_1+\left(-\Omega^2+\frac{2\left(k+4k'\right)}{3M}\right)\bar{A}_2=0\end{cases}\Rightarrow\begin{pmatrix}\left(-\Omega^2+\frac{2\left(k+4k'\right)}{3M}\right) & -\frac{8k'}{3M}\\ -\frac{8k'}{3M} & \left(-\Omega^2+\frac{2\left(k+4k'\right)}{3M}\right)\end{pmatrix}\begin{pmatrix}\bar{A}_1\\ \bar{A}_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

We calculate the determinant:

$$\Delta(\omega)=\begin{vmatrix}\left(-\Omega^2+\frac{2\left(k+4k'\right)}{3M}\right) & -\frac{8k'}{3M}\\ -\frac{8k'}{3M} & \left(-\Omega^2+\frac{2\left(k+4k'\right)}{3M}\right)\end{vmatrix}=0$$

$$\Delta(\omega)=\left(-\Omega^2+\frac{2\left(k+4k'\right)}{3M}\right)\left(-\Omega^2+\frac{2\left(k+4k'\right)}{3M}\right)-\left(-\frac{8k'}{3M}\right)\left(-\frac{8k'}{3M}\right)=0$$

$$\Rightarrow\left(-\Omega^2+\frac{2\left(k+4k'\right)}{3M}\right)^2-\left(\frac{8k'}{3M}\right)^2=0\Rightarrow\left(-\Omega^2+\frac{2\left(k+4k'\right)}{3M}-\frac{8k'}{3M}\right)\left(-\Omega^2+\frac{2\left(k+4k'\right)}{3M}+\frac{8k'}{3M}\right)=0$$

$$\Rightarrow\left(-\Omega^2-\frac{2k}{3M}\right)\left(-\Omega^2+\frac{2k+16k'}{3M}\right)=0\Rightarrow\begin{cases}\Omega_1=\sqrt{\frac{2k+16k'}{3M}}\\ \Omega_2=\sqrt{\frac{2k}{3M}}\end{cases}$$

7)

$$L=\frac{3}{4}M\left(\dot{x}_1^2+\dot{x}_2^2\right)-\frac{1}{2}kx_1^2-\frac{1}{2}kx_2^2-2k'x_1^2-2k'x_2^2+4k'x_1x_2$$

$$L=\frac{3}{4}M\left(\dot{x}_1^2+\dot{x}_2^2\right)-\frac{1}{2}(k+4k')\left(x_1^2+x_2^2\right)+4k'x_1x_2$$

$$L=\frac{3}{4}M\left[\dot{x}_1^2+\dot{x}_2^2-\frac{2(k+4k')\left(x_1^2+x_2^2\right)}{3M}+\frac{16k'x_1x_2}{3M}\right]$$

$$L=\frac{3}{4}M\left[\dot{x}_1^2+\dot{x}_2^2-\frac{2}{3M}(k+4k')\left(x_1^2+x_2^2-\frac{8k'}{(k+4k')}x_1x_2\right)\right]$$

On the other hand

$$L=\frac{3}{4}M\left(\dot{x}_1^2+\dot{x}_2^2-\omega_0^2\left(x_1^2+x_2^2-2Kx_1x_2\right)\right)$$

$$\Rightarrow \omega_0^2=\frac{2}{3M}(k+4k'),K=\frac{4k'}{(k+4k')}$$

$k'=0 \rightarrow K=0$ Very loose coupling.

$k'=\infty \rightarrow K=1$ Tight coupling.

So the coupling coefficient varies between the two values o and 1. $0\leq K\leq 1$.

8) We see in $\begin{cases}\ddot{x}_1+\frac{2(k+4k')}{3M}x_1-\frac{8k'}{3M}x_2=0\\ \ddot{x}_2+\frac{2(k+4k')}{3M}x_1-\frac{8k'}{3M}x_2=0\end{cases}$ and in $\omega_0^2=\frac{2}{3M}(k+4k')$

That ω_0 is:

- ✓ The pulsation of the first cylinder when the second cylinder is blocked ($x_2 = 0$).
- ✓ The pulsation of the second cylinder when the first cylinder is blocked ($x_1 = 0$).

This result shows that the coupling of the two cylinders results in a separation of the two proper pulsations. $\Omega_2=\frac{2}{3M}k'<\omega_0^2=\frac{2}{3M}(k+4k')<\Omega_1=\frac{2k+16k'}{3M}$.

9)

$$L=\frac{3}{4}M\left(\dot{x}_1^2+\dot{x}_2^2\right)-\frac{5}{2}kx_1^2-\frac{5}{2}kx_2^2+4kx_1x_2$$

$$\begin{cases}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\left(\frac{\partial L}{\partial x_1}\right)=0\\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)-\left(\frac{\partial L}{\partial x_2}\right)=0\end{cases}\Rightarrow\begin{cases}\frac{3}{2}M\ddot{x}_1+5kx_1-4kx_2=0\\ \frac{3}{2}M\ddot{x}_2+5kx_2-4kx_1=0\end{cases}\Rightarrow\begin{cases}\ddot{x}_1+\frac{10k}{3M}x_1-\frac{8k}{3M}x_2=0\\ \ddot{x}_2+\frac{10k}{3M}x_1-\frac{8k}{3M}x_2=0\end{cases}$$

10) The solutions are of the type

$$\begin{cases} \bar{x}_1 = \bar{A}_1 e^{j\Omega t} \\ \bar{x}_2 = \bar{A}_2 e^{j\Omega t} \end{cases} \Rightarrow \begin{cases} \bar{A}_1 j^2\Omega^2 e^{j\Omega t} + \frac{10k}{3M}\bar{A}_1 e^{j\Omega t} - \frac{8k}{3M}\bar{A}_2 e^{j\Omega t} = 0 \\ \bar{A}_2 j^2\Omega^2 e^{j\Omega t} + \frac{10k}{3M}\bar{A}_2 e^{j\Omega t} - \frac{8k}{3M}\bar{A}_1 e^{j\Omega t} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \left(-\Omega^2 + \frac{10k}{3M}\right)\bar{A}_1 - \frac{8k}{3M}\bar{A}_2 = 0 \\ -\frac{8k}{3M}\bar{A}_1 + \left(-\Omega^2 + \frac{10k}{3M}\right)\bar{A}_2 = 0 \end{cases} \Rightarrow \begin{pmatrix} \left(-\Omega^2 + \frac{10k}{3M}\right) & -\frac{8k}{3M} \\ -\frac{8k}{3M} & \left(-\Omega^2 + \frac{10k}{3M}\right) \end{pmatrix} \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We calculate the determinant:

$$\Delta(\omega) = \begin{vmatrix} \left(-\Omega^2 + \frac{10k}{3M}\right) & -\frac{8k}{3M} \\ -\frac{8k}{3M} & \left(-\Omega^2 + \frac{10k}{3M}\right) \end{vmatrix} = 0$$

$$\Delta(\omega) = \left(-\Omega^2 + \frac{10k}{3M}\right)^2 - \left(\frac{8k}{3M}\right)^2 = 0 \Rightarrow \left(-\Omega^2 + \frac{10k}{3M} - \frac{8k}{3M}\right)\left(-\Omega^2 + \frac{10k}{3M} + \frac{8k}{3M}\right) = 0$$

$$\Rightarrow \left(-\Omega^2 - \frac{2k}{3M}\right)\left(-\Omega^2 + \frac{18k}{3M}\right) = 0 \Rightarrow \begin{cases} \Omega_1 = \sqrt{\frac{18k}{3M}} \\ \Omega_2 = \sqrt{\frac{2k}{3M}} \end{cases}$$

11)

mode 1

$$\Omega_1^{\,2} = \frac{18k}{3M} \Rightarrow \begin{pmatrix} \left(-\frac{18k}{3M} + \frac{10k}{3M}\right) & -\frac{8k}{3M} \\ -\frac{8k}{3M} & \left(-\frac{18k}{3M} + \frac{10k}{3M}\right) \end{pmatrix} \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -\frac{8k}{3M} & -\frac{8k}{3M} \\ -\frac{8k}{3M} & -\frac{8k}{3M} \end{pmatrix} \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left(\frac{8k}{3M}\right)\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \text{eigenvector}: \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

mode 2

$$\Omega_2^{\,2} = \frac{2k}{3M} \Rightarrow \begin{pmatrix} \left(-\frac{2k}{3M} + \frac{10k}{3M}\right) & -\frac{8k}{3M} \\ -\frac{8k}{3M} & \left(-\frac{2k}{3M} + \frac{10k}{3M}\right) \end{pmatrix} \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{8k}{3M} & -\frac{8k}{3M} \\ -\frac{8k}{3M} & \frac{8k}{3M} \end{pmatrix} \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left(\frac{8k}{3M}\right)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \text{eigenvector}: \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Then the passage matrix is $P=\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$ and the general solutions are

$$\begin{pmatrix}x_1\\ x_2\end{pmatrix}=P\begin{pmatrix}A_1\cos(\Omega_1 t+\varphi_1)\\ A_2\cos(\Omega_2 t+\varphi_2)\end{pmatrix}$$
$$\Rightarrow\begin{cases}x_1=A_1\cos(\Omega_1 t+\varphi_1)+A_2\cos(\Omega_2 t+\varphi_2)\\ x_2=-A_1\cos(\Omega_1 t+\varphi_1)+A_2\cos(\Omega_2 t+\varphi_2)\end{cases}$$

The constants A_1, A_2, φ_1, φ_2 are determined using the initial conditions 4 unknowns so it is necessary to have 4 equations: $q_1(t=0), q_2(t=0), \dot{q}_1(t=0), \dot{q}_2(t=0)$

Problem 2

In the oscillating system shown in the following figure, the cylinder is homogeneous, of mass *M* and radius *R*. This cylinder is connected to point A by a spring of stiffness coefficient *K* at a frame B1 driven by a sinusoidal movement of amplitude S_0 and pulsation ω. It is also connected by a damper of coefficient α to a fixed frame B2. The cylinder rolls without sliding on a horizontal plane. The rod is massless and of length l. In the oscillating system shown in the Figure, the cylinder is homogeneous, of mass M and radius R. This cylinder is connected to point A by a spring of stiffness coefficient *K* to a frame B1 driven by a sinusoidal movement of amplitude S_0 and pulsation ω. It is also connected by a damper of coefficient α to a fixed frame B2. The cylinder rolls without sliding on a horizontal plane. The rod is massless and of length *l*. One of its ends can oscillate without friction around the axis of the cylinder. At the other end it carries a point mass m which is connected to a fixed frame B3 by a spring with a stiffness coefficient *k*. In equilibrium the rod is vertical and the axis of the cylinder *G* is at the origin of the coordinates *O*, we also assume that the springs are not deformed. The rotation of the rod relative to the vertical is denoted by the angle φ and that of the cylinder by the angle θ. We consider oscillations of small amplitudes.

We pose: $3M=2m, 4K=k=\frac{mg}{l}, x_2=l\varphi, x_1=R\theta$

1- Show that the Lagrangian of the system can be written in the form:

$$L=m\dot{x}_1^2+m\dot{x}_1\dot{x}_2+\frac{1}{2}m\dot{x}_2^2-kx_1^2-kx_1x_2-kx_2^2+\frac{1}{2}kx_1s-\frac{1}{8}ks^2$$

2- Determine the differential equations in $x_1(t)$ and $x_2(t)$. Show that the system is equivalent to a forced system subjected to a sinusoidal force $F(t)$ of which we will specify the amplitude F_0.

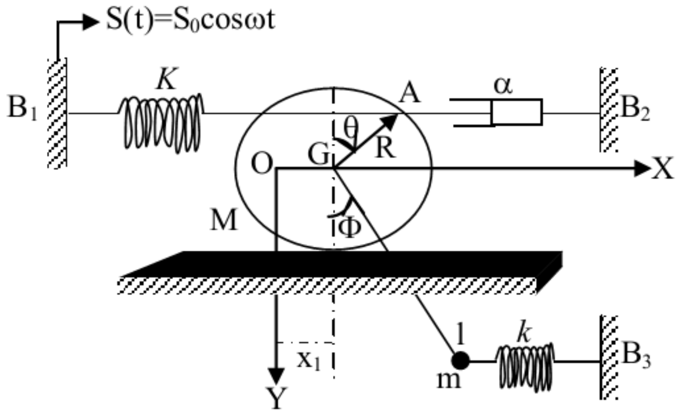
3- Express these equations of motion in terms of the speeds $\dot{x}_1(t)$ and $\dot{x}_2(t)$.

4- a- Determine $\dot{x}_1(t)$ and $\dot{x}_2(t)$ for the pulsation $\omega = \omega_0 = \sqrt{\frac{k}{m}}$. Deduce the behavior of the system at this pulsation.

b- Determine the system input impedance, defined by $\frac{Z_e}{\dot{x}_1}$, at this pulsation.

5- a- Determine $\dot{x}_1(t)$ and $\dot{x}_2(t)$ for the pulsation $\omega = \omega_1 = \sqrt{\frac{2k}{m}}$. Deduce the behavior of the system at this pulsation.

b- Determine the system input impedance, defined by $\frac{Z_e}{\dot{x}_1}$, at this pulsation.



Solution

The kinetic energy:

$$T = T_M + T_m$$

$$T_M = \frac{1}{2}MV_G^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot{\theta}^2, \text{because } \left(x_1 = R\theta \Rightarrow \dot{x}_1 = R\dot{\theta}\right)$$

$$T_M = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}\left(\frac{1}{2}M\right)\dot{x}_1^2 \Rightarrow T_M = \frac{1}{2}\left(\frac{3}{2}\right)M\dot{x}_1^2$$

$$T_m = \frac{1}{2}m\left(\dot{x}_1 + \dot{x}_2\right)^2 \Rightarrow T = \frac{1}{2}\left(\frac{3}{2}\right)M\dot{x}_1^2 + \frac{1}{2}m\left(\dot{x}_1 + \dot{x}_2\right)^2 \Rightarrow T = \frac{1}{2}\left(\frac{3}{2}M + m\right)\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + m\left(\dot{x}_1\dot{x}_2\right)$$

$$U = U_K + U_k + U_m = \frac{1}{2}K\left(2x_1 - s(t)\right)^2 + \frac{1}{2}k\left(x_1 + x_2\right)^2 + \frac{1}{2}\frac{mg}{l}x_2^2$$

$$U_K = \frac{1}{2}K\left(2x_1 - s(t)\right)^2, U_k = \frac{1}{2}k\left(x_1 + x_2\right)^2, U_m = +mgh$$

$$h = l - l\cos\varphi = l\left(1 - \cos\varphi\right), \text{we have } \cos\varphi \approx 1 - \frac{\varphi^2}{2}$$

$$\Rightarrow h \approx \frac{l}{2}\varphi^2, \text{we also have } x_2 = l\varphi \Rightarrow \varphi^2 = \frac{x_2^2}{l^2}$$

$$U_m = +mg\frac{l}{2}\frac{x_2^2}{l^2} = \frac{1}{2}\frac{mg}{l}x_2^2$$

$$U = \frac{1}{2}K\left(2x_1 - s(t)\right)^2 + \frac{1}{2}k\left(x_1 + x_2\right)^2 + \frac{1}{2}\frac{mg}{l}x_2^2$$

Hence the Lagrangian:

$$L = T - U = \frac{1}{2}\left(\frac{3}{2}M + m\right)\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + m\dot{x}_1\dot{x}_2 - \frac{1}{2}K\left(2x_1 - s(t)\right)^2 - \frac{1}{2}k\left(x_1 + x_2\right)^2 - \frac{1}{2}\frac{mg}{l}x_2^2$$

We have $3M = 2m, 4K = k = \frac{mg}{l}, x_2 = l\varphi, x_1 = R\theta$

$$L = m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + m\dot{x}_1\dot{x}_2 - kx_1^2 - kx_1x_2 - kx_2^2 + \frac{1}{2}ksx_1 - \frac{1}{8}ks^2$$

The dissipation function: $D = \frac{1}{2}\alpha\left(2\dot{x}_1\right)^2 = \frac{1}{2}\alpha\left(4\dot{x}_1^2\right)$

2-Differential equations in x_1 and x_2:

From Lagrange's equations:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \left(\frac{\partial L}{\partial x_1}\right) + \frac{\partial D}{\partial \dot{x}_1} = F_{x_1} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \left(\frac{\partial L}{\partial x_2}\right) = 0 \end{cases} \Rightarrow \begin{cases} 2m\ddot{x}_1 + m\ddot{x}_2 + 2kx_1 + kx_2 - \frac{1}{2}ks + 4\alpha\dot{x}_1 = 0 \\ m\ddot{x}_2 + m\ddot{x}_1 + 2kx_2 + kx_1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 2m\ddot{x}_1 + m\ddot{x}_2 + 2kx_1 + kx_2 + 4\alpha\dot{x}_1 = \frac{1}{2}ks_0\cos\omega t \\ m\ddot{x}_2 + m\ddot{x}_1 + +2kx_2 + kx_1 = 0 \end{cases}$$

The system is subjected to a force $F(t) = \frac{1}{2}ks_0\cos\omega t = \frac{1}{2}ks_0e^{j\omega t}$ of amplitude $F_0(t) = \frac{1}{2}ks_0$ and pulsation ω.

3- Equation according to the speeds $\dot{x}_1(t)$ and $\dot{x}_2(t)$:

$$\begin{cases} \left(4\alpha + 2j\left(m\omega - \frac{k}{\omega}\right)\right)\dot{x}_1 + j\left(m\omega - \frac{k}{\omega}\right)\dot{x}_2 = \frac{k}{2}s_0 e^{j\omega t} \\ j\left(m\omega - \frac{k}{\omega}\right) + j\left(m\omega - \frac{2k}{\omega}\right)\dot{x}_2 = 0 \end{cases}$$

4- For $\omega = \omega_0 = \sqrt{\frac{k}{m}}$

a- $\dot{x}_1 = \frac{1}{4\alpha}\frac{k}{2}s_0 e^{j\omega t}$ and $\dot{x}_2 = 0$ at this frequency the rod remains vertical.

b- The input impedance is $Z_e = \frac{F_0}{\dot{x}_1} = 4\alpha$

5- For $\omega = \omega_1 = \sqrt{\frac{2k}{m}}$

a- $\dot{x}_1 = 0$ and $\dot{x}_2 = -\frac{j}{\left(m\omega_1 - \frac{k}{\omega_1}\right)}\frac{k}{2}s_0 e^{j\omega t}$ the cylinder does not move.

b-The input impedance is $Z_e = \frac{F_0}{\dot{x}_1} = 4\alpha$ is infinite.

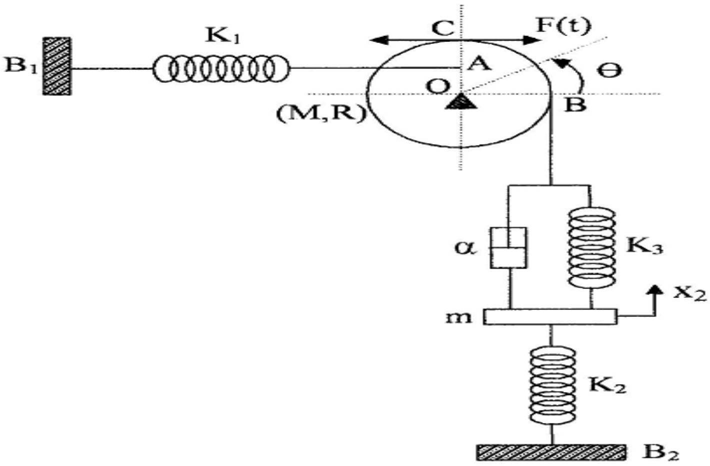
Problem 3

We consider the mechanical system shown in the figure opposite. It is made up of a solid, homogeneous cylinder, of mass M and radius R. This cylinder can rotate around a fixed axis passing through the point O of its axis of revolution. It is connected to point A, such that $OA=R/2$, to the fixed frame $B1$ by a horizontal spring of stiffness constant K_1. A point mass m, connected to the fixed frame $B2$ by a vertical spring of stiffness K_2, is coupled to the cylinder by a spring of stiffness K_3 and a damper of viscous friction coefficient alpha by means of an inextensible wire, without stiffness and mass negligible which wraps around the cylinder. In equilibrium, point A is on the vertical and point B is on the horizontal, both passing through O. We will be interested in low amplitude oscillations, the mass m is identified by its elongation x_2 and the cylinder by the angle θ measured in relation to the equilibrium position.

A sinusoidal horizontal force, of amplitude F_0 and pulsation Ω, is applied to point C of the cylinder.

1. Assuming that in equilibrium the weight of the mass m is compensated by the deformation forces of the springs, determine the Lagrangian of the system.

2. Determine the differential equations of motion according to the coordinates $x_1=R\theta$ and x_2. Deduce the integro-differential equations for the speeds $\dot{x}_1(t)$ and $\dot{x}_2(t)$.
3. Give the equivalent electrical diagram in the *Force-Tension* analogy, specifying the equivalence between the different elements.
4. Calculate the input impedance $Z_e=\frac{F(t)}{\dot{x}_1(t)}$ of the mechanical system in the case where $\omega^2=\frac{K_2}{m}$ and deduce the speeds $\dot{x}_1(t)$ and $\dot{x}_2(t)$.
5. Describe the behavior of the electric circuit at the equivalent pulsation at $\Omega=\sqrt{\frac{K_2}{m}}$ by giving the corresponding diagram.



Solution

1.

$$T=\frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot{\theta}^2+\frac{1}{2}m\dot{x}_2^2,\ we\ have\ x_1=R\theta\Rightarrow\dot{x}_1=R\dot{\theta}$$

$$T=\frac{1}{2}\left(\frac{1}{2}M\right)\dot{x}_1^2+\frac{1}{2}m\dot{x}_2^2$$

$$U=\frac{1}{2}K_1\left(\frac{R\theta}{2}\right)^2+\frac{1}{2}K_2x_2^2+\frac{1}{2}K_3\left(x_2-R\theta\right)^2\Rightarrow U=\frac{1}{2}\left(\frac{K_1}{4}\right)x_1^2+\frac{1}{2}K_2x_2^2+\frac{1}{2}K_3\left(x_2-x_1\right)^2$$

$$L=T-U=\frac{1}{2}\left(\frac{M}{2}\right)\dot{x}_1^2+\frac{1}{2}m\dot{x}_2^2-\frac{1}{2}\left(\frac{K_1}{4}\right)x_1^2-\frac{1}{2}K_2x_2^2-\frac{1}{2}K_3\left(x_2-x_1\right)^2$$

$$D=\frac{1}{2}\left(\dot{x}_2^2-\dot{x}_1^2\right)$$

2.

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_1}\right)-\left(\dfrac{\partial L}{\partial x_1}\right)+\dfrac{\partial D}{\partial \dot{x}_1}=F(t) \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_2}\right)-\left(\dfrac{\partial L}{\partial x_2}\right)+\dfrac{\partial D}{\partial \dot{x}_2}=0 \end{cases} \Rightarrow \begin{cases} \dfrac{M}{2}\ddot{x}_1+\dfrac{K_1}{4}x_1+K_3(x_1-x_2)+\alpha(\dot{x}_1-\dot{x}_2)=F(t) \\ m\ddot{x}_2+K_2x_2-K_3(x_1-x_2)-\alpha(\dot{x}_1-\dot{x}_2)=0 \end{cases}$$

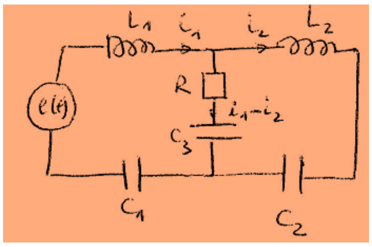
The integro-differential equations

$$\Rightarrow \begin{cases} \dfrac{M}{2}\dfrac{d\dot{x}_1}{dt}+\dfrac{K_1}{4}\int \dot{x}_1 dt+K_3\int(\dot{x}_1-\dot{x}_2)dt+\alpha(\dot{x}_1-\dot{x}_2)=F(t) \\ m\dfrac{d\dot{x}_2}{dt}+K_2\int \dot{x}_2 dt-K_3\int(\dot{x}_1-\dot{x}_2)dt-\alpha(\dot{x}_1-\dot{x}_2)=0 \end{cases}$$

3. The *Force-Tension* analogy

$$\frac{M}{2}\Leftrightarrow L_1, m\Leftrightarrow L_2, \alpha\Leftrightarrow R, \frac{K_1}{4}\Leftrightarrow C_1^{-1}, K_2\Leftrightarrow C_2^{-1}, K_3\Leftrightarrow C_3^{-1}, \dot{x}_{1,2}\Leftrightarrow i_{1,2}, F(t)\Leftrightarrow e(t)$$

$$\Rightarrow \begin{cases} L_1\dfrac{di_1}{dt}+\dfrac{1}{C_1}\int i_1 dt+\dfrac{1}{C_3}\int(i_1-i_2)dt+R(i_1-i_2)=e(t) \\ L_2\dfrac{di_2}{dt}+\dfrac{1}{C_2}\int i_2 dt-\dfrac{1}{C_3}\int(i_1-i_2)dt-R(i_1-i_2)=0 \end{cases}$$



4.

$\dot{x}_1(t)=A_1e^{j\Omega t}, \dot{x}_2(t)=A_2e^{j\Omega t}, F(t)=F_0e^{j\Omega t}$

$$\begin{cases} \left[\alpha+j\left(\dfrac{M}{2}\Omega-\dfrac{K_1}{4\Omega}-\dfrac{K_3}{\Omega}\right)\right]\dot{x}_1-\left(\alpha-j\dfrac{K_3}{\Omega}\right)\dot{x}_2=F(t)...............(1) \\ \left[\alpha+j\left(m\Omega-\dfrac{K_2+K_3}{\Omega}\right)\right]\dot{x}_2-\left(\alpha-j\dfrac{K_3}{\Omega}\right)\dot{x}_1=0.........................(2) \end{cases}$$

For $\Omega^2=\dfrac{K_2}{m}\Rightarrow m\Omega=\dfrac{K_2}{\Omega}$ equation 2 gives $\left(\alpha-j\dfrac{K_3}{\Omega}\right)(\dot{x}_2-\dot{x}_1)=0\Rightarrow \dot{x}_2=\dot{x}_1$

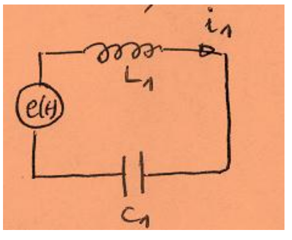
And equation 1 $\Rightarrow j\left(\dfrac{M}{2}\Omega-\dfrac{K_1}{4\Omega}\right)\dot{x}_1=F(t)$

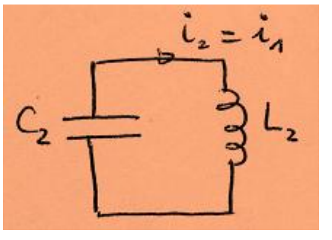
from where $Z_e=\dfrac{F(t)}{\dot{x}_1(t)}=j\left(\dfrac{M}{2}\Omega-\dfrac{K_1}{4\Omega}\right), \dot{x}_1(t)=\dfrac{F(t)}{Z_e}=\dfrac{F_0}{\left(\dfrac{M}{2}\Omega-\dfrac{K_1}{4\Omega}\right)}e^{j\left(\Omega t-\frac{\pi}{2}\right)}, \dot{x}_1(t)=\dot{x}_2(t)$

5. The pulsation of the electrical circuit equivalent to $\Omega = \sqrt{\frac{K_2}{m}}$ is $\Omega_e = \frac{1}{\sqrt{L_2C_2}}$. At this pulsation, we have by analogy i_1 and i_2 and consequently the electrical equations are decoupled and give

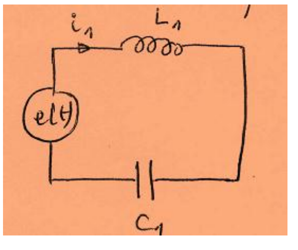
$$\begin{cases} L_1\frac{di_1}{dt} + \frac{1}{C_1}\int i_1 dt = e(t) & \text{Forced oscillations} \\ L_2\frac{di_2}{dt} + \frac{1}{C_2}\int i_2 dt = 0 & \text{Free oscillations} \end{cases}$$

To these two 2 equations, there correspond 2 meshes





Or from the electrical diagram when $\Omega_e = \frac{1}{\sqrt{L_2C_2}}$ the impedance of the branch containing L_2 and C_2 is zero and the equivalent diagram is as follows



SECOND PART:

MECHANICAL WAVES

Chapter 6

General information on the propagation phenomenon

6.1 Theoretical reminder

The mechanical wave is a temporary local disturbance that moves in an elastic, homogeneous and isotropic material medium without transport of matter, as shown in Figure 6.1.

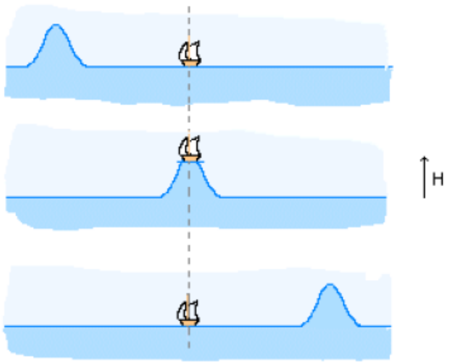


Figure 6.1

The mechanical wave propagates with energy transport.

- ➢ There are two types of environments:
  1. Dispersive medium: The speed of the wave depends on the characteristics of the medium and the wavelength.

Example: this phenomenon is seen for example in the air when the amplitude is large (in the case of thunder, high frequency waves propagate more quickly than low frequency waves, the air is dispersive)

  2. Non-dispersive medium: The speed depends solely on the properties of the propagation medium.

- ➢ There are two types of waves:
- ❖ Longitudinal wave: The wave is parallel to the direction of propagation, as shown in Figure 6.2.

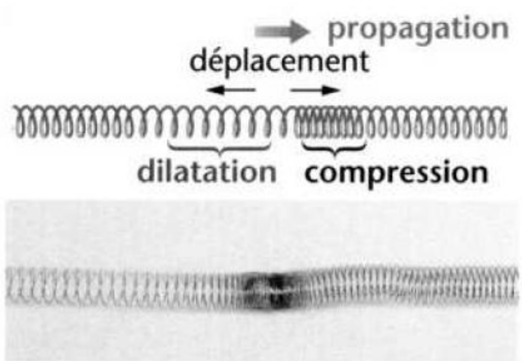


Figure 6.2

❖ Transverse wave: The shaking is perpendicular to the direction of propagation as shown in Figure 6.3.

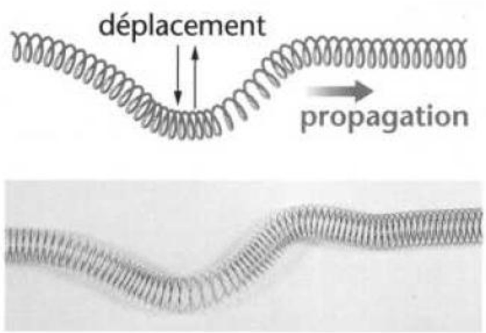


Figure 6.3

The speed of the wave is constant in a linear, homogeneous, isotropic and non-dispersive medium. It depends on the inertia, rigidity and temperature of the environment.

The mechanical wave propagates from a source in different forms:

- ❖ One-dimensional: Movement along a rope, a spring.
- ❖ Two-dimensional: Circular movement on the water surface.

Example: When you throw a stone onto a surface of water, as shown in Figure 6.4 below:

Figure 6.4

The phenomenon apparent in the image is a circular wave propagating in a plane

- ❖ Has three dimensions: Sound waves.

Mechanical waves have a double periodicity:

- ❖ Temporal periodicity: Characterized by the period T (s).
- ❖ Spatial periodicity: Characterized by the wavelength $\lambda_m\,(m)$.

The phenomenon of diffraction is characteristic of waves. It manifests itself when a wave encounters an obstacle or an opening whose dimensions are of the same order of magnitude as the wavelength, or even figure 6.5:

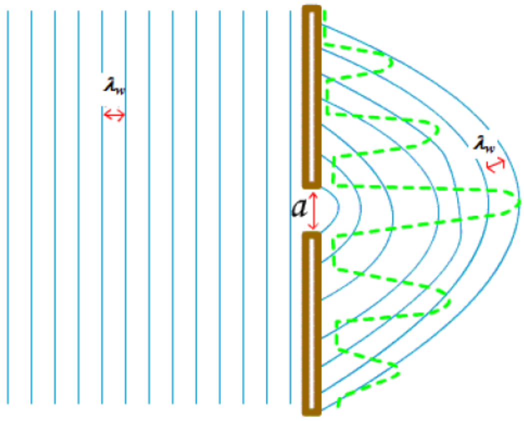


Figure 6.5

❖ If two identical waves meet, we will see that they do not necessarily reinforce each other, on the contrary! they can cancel each other out: this is the phenomenon of interference, or even figure 5.6:

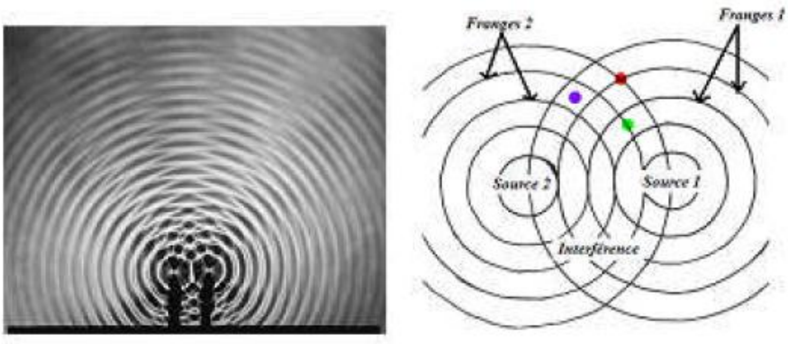


Figure 6.6

Other examples for the interference phenomenon are:

❖ Young's experiment: light passes through two holes separated by a distance d. Circular interference then appears on the screen as shown in Figure 6.7 above:

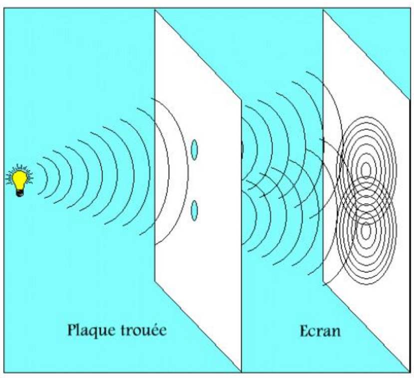


Figure 6.7

The waves emitted by two loudspeakers as shown in Figure 6.8:

Figure 6.8

6.2 Applications

Problem 1

A source emits a mechanical wave ψ of frequency ν propagating in the direction ox with a constant speed V.

▪ Write the propagation equation

Posing the following variables: $p = t + \dfrac{x}{V}, q = t - \dfrac{x}{V}$

- ✓ Show that the solution to the equation is the sum of two types of signals.
- ✓ Deduce the form of the solution in the case of a linear and infinite homogeneous medium in sinusoidal regime

Solution

▪ *The propagation equation*

$$\frac{\partial^2 \psi}{\partial t^2} = V^2 \frac{\partial^2 \psi}{\partial x^2}$$

It is a one-dimensional partial differential equation.

▪ General solutions using the change of variables method are:

$$\begin{cases} p = t + \dfrac{x}{V} \\ q = t - \dfrac{x}{V} \end{cases} \Rightarrow \begin{cases} \dfrac{\partial^2 \psi}{\partial p \partial q} = 0 \Rightarrow \psi_1(q) \\ \dfrac{\partial^2 \psi}{\partial q \partial p} = 0 \Rightarrow \psi_2(p) \end{cases} \Rightarrow \psi_T = \psi_1(q) + \psi_2(p)$$

▪ In sinusoidal regime, the solution has the form: $\psi(t,x) = A \cos\omega\left(t - \left(\dfrac{\omega}{V}\right)x\right)$

Problem 2

A mechanical wave S of frequency ν propagating in a radially symmetric medium with a constant speed V.

▪ Write the propagation equation of S.

▪ Solve the partial differential equation.

▪ Express the general solution in the case of an infinite medium in sinusoidal regime. Interpret the results.

Solution

▪ The propagation equation:

$$\frac{\partial^2 S}{\partial t^2} = V^2 \Delta S$$

▪ The general solution in the case of a sinusoidal regime:

$$S(r,t) = \left(\frac{1}{r}\right)\left[f\left(t - \left(\frac{r}{V}\right)\right) + g\left(t + \left(\frac{r}{V}\right)\right)\right] \quad with\ r^2 = x^2 + y^2 + z^2$$

▪ The general solution in the case of a sinusoidal regime:

$$S(r,t) = \left(\frac{1}{r}\right)\left[\cos\left(t - \left(\frac{r}{V}\right)\right)\right] \quad avec\ r^2 = x^2 + y^2 + z^2$$

▪ We obtain a spherical incident sinusoidal wave as shown in the figure

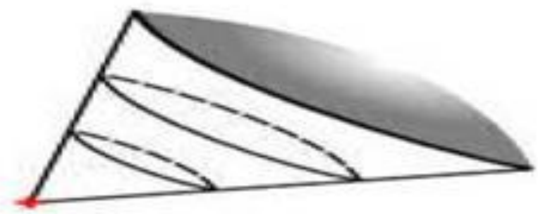


Figure 6.8: Propagation of a spherical wave

The factor $\left(\frac{1}{r}\right)$ represents the damping of the amplitude of the spherical wave which is due to the energy distribution of the wave in all directions equally.

Problem 3

Consider a mechanical wave ψ of frequency ν propagating in the plane (Oxy) with a constant speed V.

▪ Write the propagation equation.

▪ Determine solutions using the method of separation of variables.

▪ We pose the following conditions:

$$\psi(x=0)=\frac{\partial\psi}{\partial y}(y=0)=0$$

▪ Determine general solutions.

Solution

▪ The propagation equation has two dimensions:

$$\frac{\partial^2\psi}{\partial t^2}=V^2\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)$$

▪ Solutions of the differential equation by the method of separation of variables:

$$S_T(t,x,y)=A(x)B(y)T(t)\Rightarrow\frac{\ddot{A}(x)}{A(x)}+\frac{\ddot{B}(y)}{B(y)}=\frac{1}{V^2}\frac{\ddot{T}(t)}{T(t)}=Cste$$

$$\begin{cases}\ddot{A}(x)+k_x^2A(x)=0\\ \ddot{B}(y)+k_y^2B(y)=0\\ \ddot{T}(t)+\omega^2T(t)=0\end{cases}\text{With}\begin{cases}k_x^2+k_y^2=k_0^2\\ k_0=\dfrac{\omega}{V}\end{cases}$$

So

$$\begin{cases}A(x)=A_1\cos k_xx+A_2\sin k_xx\\ B(y)=B_1\cos k_yy+B_2\sin k_yy\\ T(t)=T_1\cos\omega t+T_2\cos\omega t\end{cases}$$

▪ The propagation space is limited (finite), we obtain standing waves in three directions.

❖ By applying the boundary conditions we obtain:

$$\begin{cases}\psi(x=0)=0\\ \dfrac{\partial\psi}{\partial y}(y=0)=0\end{cases}\Rightarrow\begin{cases}A_1=0\\ B_2=0\end{cases}\text{and }k_x^{(n)}=\frac{n\pi}{a}\Rightarrow\begin{cases}A(x)=A_2\sin k_x^{(n)}x\\ B(y)=B_1\cos k_yy\end{cases}$$

❖ The general solutions are:

$$S_T(t,x,y)=\sum_n\Lambda\sin k_x^{(n)}x\cos k_y^{(m)}yT(t)$$

$$\text{with }\Lambda=A_2B_2$$

Chapter 7

Propagation of mechanical waves in different environments

7.1 Theoretical reminder

We will see how a wave can progress in a string.

Given a wire of length *l* and mass *m*, the linear mass of the wire (assumed to be constant along it) is then:

$$\mu = \frac{m}{l} = \frac{dm}{dx}$$

With a slight shock, let's create a small transverse disturbance (in order not to deform the cable and keep its linear mass constant). Let us isolate, in the disturbed zone, a wire element, of length *dl*:

Approximations

Each element of the string can be cut infinitesimally so as to be almost parallel to the *x* axis. The angles $\theta(x,t), \theta(x+dx,t)$ are therefore considered small

- ❖ The rope is considered deformable but not elongable so the norm of forces in the rope is constant at all points regardless of the deformation.

For the rest of the reasoning, we use Figure 7.1 below:

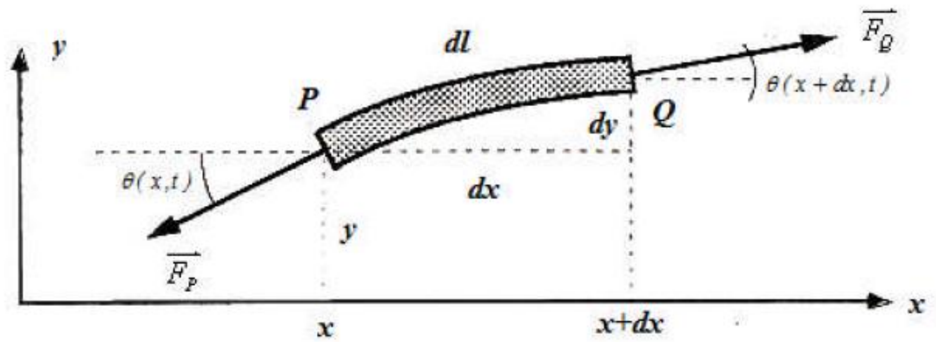


Figure 7.1

The balance of forces gives:

$$\vec{F} = dm\vec{a} \Rightarrow \begin{cases} F\left(\cos\theta_{x+dx} - \cos\theta_x\right) \cong 0 \\ F\left(\sin\theta_{x+dx} - \sin\theta_x\right) \cong dm\dfrac{\partial^2 y}{\partial t^2} \end{cases} \quad \text{With } dm = \mu dl$$

Which means that there are no movements along x, and $\vec{a}$ represents the acceleration along *y*. If the angles are really small, we have the first term of the expansion which gives:

$$dx \cong dl$$

$$\sin x \cong \tan x \cong \frac{\partial y}{\partial x}$$

Newton's law applied to the mass $dm = \mu dx$ gives (we consider that each mass point only moves along y because there is no elongation):

The tangents are given by the partial derivatives of the function y(x):

$$F\left[\left.\frac{\partial y}{\partial x}\right|_{x+dx} - \left.\frac{\partial y}{\partial x}\right|_{x}\right] \cong \mu dx \frac{\partial^2 y}{\partial t^2}$$

This results in the partial differential equation:

$$\frac{\partial^2 y}{\partial t^2} = \frac{F}{\mu}\frac{\partial^2 y}{\partial x^2} \Rightarrow V = \sqrt{\frac{F}{\mu}}$$

It is called "the vibrating string equation".

We check the units of $\frac{F}{\mu}$ are those of the square of a speed $(m/s)^2$, as required by dimensional analysis. To simplify the writing, we put:

$$V = \sqrt{\frac{F}{\mu}}$$

7.2 Applications

Problem 1

Consider a string vibrating transversely in the plane Oxy. The equation of motion has the form $y = (x, t)$. Let T and μ be the tension and linear mass of the string at equilibrium.

▪ Write the wave propagation equation.

▪ Deduce the speed V of the oscillations.

We consider that the original shaking is sinusoidal.

▪ Determine the solutions of the propagation equation using the method of separation of variables.

Now the rope is fixed by both ends with distance a, released with no initial velocity.

▪ Determine the form of the general solution.

▪ Show that the vibration frequencies of the string are integer multiples of a fundamental frequency f_1.

Numerical application: For the third string of the guitar of length a=63cm in nylon, of density ρ=1200 kg/m^3 and of section S=0,42mm^2.

▪ Calculate the tension of this string so that it can emit the fundamental sound f_1=147Hz.

Solution

▪ The propagation equation:

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2} \Rightarrow V = \sqrt{\frac{F}{\mu}}$$

▪ Solutions to the free wave propagation equation:

$$y = A(x)T(t) \Rightarrow \frac{\ddot{A}(x)}{A(x)} = V^2\frac{\ddot{B}(y)}{B(y)} \Rightarrow \begin{cases} A(x) = A_1 \cos\frac{\omega}{V}x + A_2 \sin\frac{\omega}{V}x \\ B(t) = B_1 \cos\omega t + B_2 \sin\omega t \end{cases}$$

▪ The rope is now fixed; the boundary conditions give us:

$$\begin{cases} y(x=0) = y(x=a) = 0 \\ \dot{y}(t=0) \end{cases} \Rightarrow \begin{cases} A_1 = 0 \\ B_2 = 0 \end{cases} et\ k_x^{(n)} = \left(\frac{\omega}{V}\right)_x \frac{\pi n}{a} \Rightarrow \begin{cases} A(x) = A_2 \sin k_x^{(n)} x \\ B(t) = B_1 \cos\omega_n t \end{cases}$$

So the final solution:

$$y_T(x,t) = \sum_n \Lambda \sin k_x^{(n)} x \cos\omega^{(n)} t$$

$$\text{With } \Lambda = A_2 B_1$$

▪ The vibration frequencies of the string:

$$k_x^{(n)} = \frac{\omega_n}{V} = \frac{2\pi f_n}{V} = \frac{\pi n}{a} \Rightarrow f_n = nf_1$$

With $f_1 = \frac{1}{2a}\sqrt{\frac{T}{\mu}}$

▪ Digital Application:

$$f_1 = \frac{1}{2a}\sqrt{\frac{T}{\mu}} \Rightarrow T = 4a^2 \rho S f_1^{\,2} \Rightarrow T = 17.3N$$

Problem 2

A homogeneous vibrating rope without stiffness, of linear mass μ, stretched by a tension force of constant intensity F. The rope at rest and horizontal and materialized by the axis Ox.

During the propagation of a wave, the point M of the chord, with abscissa x at rest, undergoes the transverse displacement $y\,(x,\,t)$ at time t. We neglect the influence of gravity on the rope, but we take into account the damping force directed along the axis Ox, $Ox \perp Oy$ and algebraic value: $-bV\,(x,\,t)$ per unit of length (with $b>0$), where $V\,(x,t) = \frac{\partial y}{\partial t}$ is the transverse velocity of the element of the chord with abscissa x at time t.

▪ Establish the partial differential equation of the displacement $y\,(x,\,t)$.

We define k as the wave vector of this wave. We will assume the low damping ($b \ll \mu\omega$).

▪ Establish the dispersion relation in the form: $k\,(\omega) = \omega \frac{(1 - jg\,(\omega))}{c}$

▪ Express the coefficients g and c based on the data F, μ and b.

▪ Derive the equation for the wave $y\,(x,\,t)$. What can we say about $y\,(x,\,t)$?

We define the complex mechanical impedance $\tilde{z} = \frac{T_y}{V\,(x,t)}$

Where T_y denotes the projection onto Oy of the tension of the string at $M\,(x)$.

▪ Express the complex mechanical impedance $\tilde{Z}$ of the string in terms of F, μ, b and ω.

Solution

▪ The partial differential equation of displacement $y\,(x,\,t)$:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2} + \frac{b}{F}\frac{\partial y}{\partial t}$$

With $V = \sqrt{\frac{F}{\mu}}$

▪ The dispersion relation:

$$y(t,x) = Ae^{j(\omega t - kx)} \Rightarrow k(\omega) \cong \frac{\omega}{c}\left(1 - j\frac{b}{2j\omega\mu}\right)$$

With $c = \sqrt{\frac{F}{\mu}}$ and $g(\omega) = \frac{b}{2\omega\mu}$

▪ The wave equation $y\,(x,\,t)$:

$$y\,(t,x) = Ae^{-\left(\frac{bx}{2\mu c}\right)}e^{j\left(\omega t - \frac{x}{c}\right)}$$

It is a damped traveling wave.

▪ Complex mechanical impedance:

$$\tilde{z} = \frac{T_y}{V\,(x,t)} = F\frac{\left(\frac{\partial y}{\partial x}\right)}{\left(\frac{\partial y}{\partial t}\right)} \Rightarrow \tilde{z} = -\sqrt{\mu F}\left(1 - j\,\frac{b}{2\mu\omega}\right)$$

Problem 3

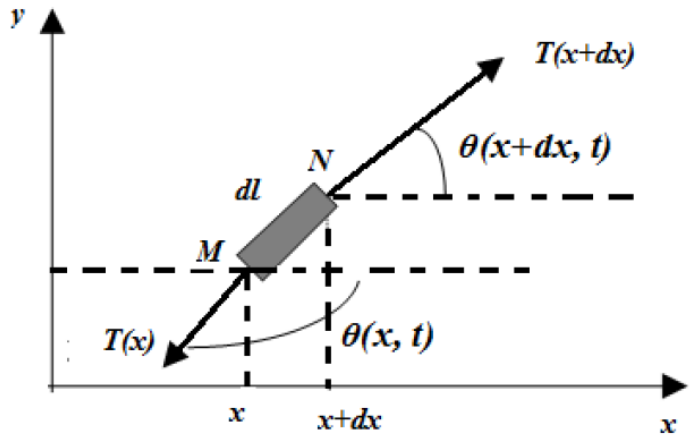
Part A: Vibrating string equation

A homogeneous and inextensible rope, of linear mass μ, is stretched horizontally along the axis Ox with a constant tension F, see the figure.

The rope, moved from its equilibrium position, acquires a movement described at time t by the quasi-vertical displacement $y\,(x,\,t)$, counted from its equilibrium position, of a point M of abscissa x at rest.

At time t, the tension $T\,(x,\,t)$ exerted by the part of the string to the right of M on the part of the string to the left of M, makes a small angle $\theta\,(x,\,t)$ with the horizontal.

We will admit θ to be small, weak curvature of the string, and we will neglect the forces of gravity.



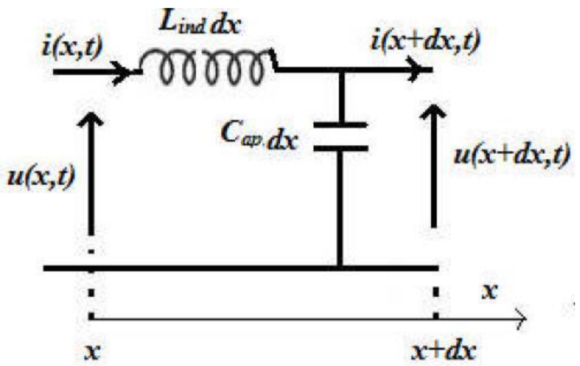
Equation of vibrating strings: *We consider the section of the string between the abscissa x, $x+dx$.*

▪ Establish the equation for propagation of the wave of the vibrating string.

▪ Deduce the speed V of the wave as a function of μ et F.

Part B: Electrical Analogy

Consider a slice of a lossless electric cell represented in the figure as follows:



▪ Show that the electric cell shown above is an analog circuit of a vibrating string element of length dx

▪ Express the mechanical correspondents of the linear inductance $Lind$, the linear capacitance Cap, the current intensity $i\ (x,\ t)$ and the electrical voltage $u\ (x,\ t)$.

Solution

Part A:

▪ The equation for the propagation of the wave of the vibrating string:

$$\sum \vec{F} = dm\,\vec{a} \Rightarrow T(x,t) = T(x+dx,t) = F \Rightarrow \frac{\partial^2 y}{\partial t^2} = \frac{F}{\mu}\frac{\partial^2 y}{\partial x^2}$$

$$With\ V = \sqrt{\frac{F}{\mu}}$$

Part B:

▪ The equation for wave propagation in the electric cell:

$$\frac{\partial i}{\partial x} = -C_{ap}^{*}\frac{\partial u}{\partial t} \Rightarrow \frac{\partial^2 i}{\partial x^2} = L_{ind}^{*}C_{ap}^{*}\frac{\partial^2 i}{\partial t^2}$$

$$\frac{\partial u}{\partial x} = -L_{ind}^{*}\frac{\partial i}{\partial t}$$

▪ Mechanical-electrical equivalence:

$$\frac{\partial^2 i}{\partial x^2} = L_{ind}^{*}C_{ap}^{*}\frac{\partial^2 i}{\partial t^2} \Leftrightarrow \frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}$$

With

$$L_{ind}^{*} \Leftrightarrow \mu$$

$$C_{ap}^{*} \Leftrightarrow \frac{1}{F}$$

$$i\left(x,t\right) \Leftrightarrow \frac{\partial y}{\partial t}$$

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