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Operations and Properties on Lattices and Trellises

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Introduction

Transitivity and partial order have long served as guiding principles in human thought. In science, and more specifically in mathematics, the concept of transitivity as a property of partial orders is fundamental to many theories. However, there are real-life situations where transitivity fails. For example, a common and illustrative case of a non-transitive relation in everyday life is the acquaintance relation: if person A knows person B , and person B knows person C , persons A and C do not necessarily know each other. Non-transitive relations also appear in sports tournaments; for instance, team A may beat team B , and team B may beat team C , but team A is not guaranteed to beat team C . A well-known example is the childrens game rock-paper-scissors, which is non-transitive: scissors beats paper, paper beats rock, but scissors does not beat rock. The concept of non-transitivity inspires the construction of mathematical structures that do not require transitivity.

Binary operations have become fundamental tools in lattice theory because they can express key concepts and properties intrinsic to ordered algebraic structures. Indeed, the very definition of a lattice, as well as many of its theoretical properties, can be formulated using binary operations [5, 23]. Among the most widely studied binary operations are *aggregation functions*, which appear in numerous theoretical and applied disciplines, including *fuzzy set theory* [11]. Over the years, several subclasses of aggregation functions have been introduced and analyzed, especially those defined on the unit interval $[0, 1]$ or on *bounded lattices*. These include *triangular norms (t-norms)*, *t-conorms*, *uninorms*, and *nullnorms*, among others [12, 14, 15]. Within the framework of the unit interval $[0, 1]$ or bounded lattices, the *transitivity* of the partial order $\leq$ and the *associativity* of the *meet* ($\wedge$) and *join* ($\vee$) operations are crucial in the *construction, representation, and characterization* of t-norms and t-conorms [12]. For example, the meet operation ($\wedge$) serves as a canonical example of a t-norm, while the join operation ($\vee$) corresponds naturally to a t-conorm.

To generalize such structures beyond the limitations of transitivity, Fried introduced the concept of a *T-lattice* (or *weakly associative lattice*) and studied its properties through the lens of *tournaments* [8]. Extending this work, Skala [24] defined a *pseudo-ordered set*

as a set equipped with a *reflexive* and *antisymmetric*, but not necessarily *transitive*, binary relation. He showed that many classical lattice-theoretic results could be adapted to this broader context, which he called *trellises*. Later, Gladstein [10] characterized finite-length trellises by showing that completeness is equivalent to the existence of both a least upper bound and a greatest lower bound for every cycle of elements. These results were further generalized by Bhatta and Shashirekha [1, 18, 19] through the concepts of *joins of cycles* and *pseudo-chains* in pseudo-ordered sets.

In this memory, inspired by these developments, we recall classical binary operations on lattices to the setting of trellises. We focus on particular classes of *associative binary operations* that are also *commutative*, *increasing*, and admit 1 (respectively, 0) as a neutral element. These classes generalize the classical t-norms and t-conorms defined on bounded lattices.

This memory is organized as follows:

- **Chapter 1:** We present the necessary background on partially ordered sets and lattices, emphasizing both their order-theoretic and algebraic aspects. These foundations are crucial for the later development of binary operations.
- **Chapter 2:** We recall the theory of pseudo-ordered sets and trellises, providing generalized models for non-transitive order relations. Their key structural properties are studied in detail.
- **Chapter 3:** We explore associative binary operations on lattices and trellises, focusing on aggregation functions and t-norms. We present constructions and representations of t-norms on bounded lattices and extend these ideas to trellises, proposing novel construction techniques based on specific subsets.
- **Conclusion:** We summarize the results and suggest directions for future research.

Chapter 1

Generalities on ordered sets and lattices

In this chapter, we recall some basic notions and definitions of partially ordered sets (posets, for short) and lattices. More information on these notions can be found in [2, 5, 17, 21, 23].

1.1 Ordered sets

In this section, We introduced the basic concepts of binary relations, order relations and types of relations.

1.1.1 Binary relations between sets

Definition 1.1. Let X and Y be two sets, we denote $X \times Y$ and we call the Cartesian product of X and Y the set of all ordered pairs whose the first component belongs to X and the second to Y :

$$X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}.$$

Definition 1.2. A binary relation on a set X is a subset of the Cartesian product $X \times X$. For a relation $R \subseteq X^2$, if x is related to y by R we often write xRy instead of $(x, y) \in R$. Two elements x and y of a set X equipped with a relation R are called comparable elements if it holds that xRy or yRx . Otherwise, they are called incomparable elements, denoted by $x \parallel y$.

Example 1.1. 1. Let $X = \{1, 2, 3\}$ be a finite set and R a binary relation on X such that:

$$R_1 = \{(a, b) \mid a \leq b\}$$

This means R contains all ordered pairs (a, b) where a is less than or equal to b .

2. The divisibility relation, denoted by $a \mid b$ (" a divides b "), on the set $\mathbb{N}$ is defined if and only if there is some $n \in \mathbb{N}$ such that $a \times n = b$. Obviously, this relation contains an infinite number of elements. Examples of the elements of the relation:

$$R_2 = \{(1, 2), (3, 6), (4, 12), (7, 28), (11, 44), \dots\}$$

Examples of ordered pairs that do not belong to the divisibility relation:

$$(2, 1), (6, 3), (3, 5), (7, 11), (11, 7), \dots$$

It is evident that the relations R_1 and R_2 are defined as sets of couples (x, y) on X^2 . Consequently, it is possible to apply the classical operations of set theory to them, such as:

- **The intersection** of R_1 and R_2 is the relation $R_1 \cap R_2$ on X and defined as follows:

$$R_1 \cap R_2 = \{(x, y) \in X^2 \mid xR_1y \wedge xR_2y\}.$$

- **The union** of R_1 and R_2 is the relation $R_1 \cup R_2$ on X and defined as follows:

$$R_1 \cup R_2 = \{(x, y) \in X^2 \mid xR_1y \vee xR_2y\}.$$

- **The complement** of R_1 is the relation denoted by R_1^c on X and defined as follows:

$$R_1^c = \{(x, y) \in X^2 \mid (x, y) \notin R_1\}.$$

- **The transpose** of R_1 is the relation denoted by R_1^t on X and defined as follows:

$$R_1^t = \{(y, x) \in X^2 \mid xR_1y\}.$$

- **The composition** of R_1 and R_2 is the relation denoted by $R_1 \circ R_2$ on X and defined as follows:

$$R_1 \circ R_2 = \{(x, z) \in X^2 \mid (\exists y \in X)(xR_1y \wedge yR_2z)\}.$$

A binary relation R on a set X is called:

- (i) *reflexive*, if, for any $x \in X$, it holds that $(x, x) \in R$;
- (ii) *irreflexive*, if, for any $x \in X$, it holds that $(x, x) \in R^c$;
- (iii) *symmetric*, if, for any $x, y \in X$, it holds that $(x, y) \in R$ implies $(y, x) \in R$;

- (v) *antisymmetric*, if, for any $x, y \in X$, it holds that $(x, y) \in R \wedge (y, x) \in R$ implies $x = y$;
- (vi) *transitive*, if, for any $x, y, z \in X$, it holds that $(x, y) \in R \wedge (y, z) \in R$ implies $(x, z) \in R$;
- (vii) *complete*, if, for any $x, y \in X$, it holds that $(x, y) \in R \vee (y, x) \in R$.

A binary relation R on a set X is called:

- (ii) a partial order relation if it is reflexive, antisymmetric and transitive;
- (iii) a strict order if it is irreflexive and transitive;
- (iv) a total order relation if it is reflexive, antisymmetric, transitive and complete.

1.1.2 Models of binary relations

In this subsection, we present well-known types for representations of binary relations.

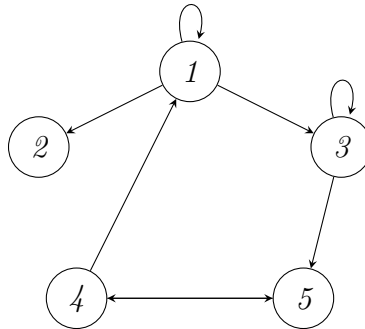
Relation as a Directed Graph:

A binary relation on a finite set can also be represented using a directed graph (a digraph, for short). A directed graph consists of a set vertices and a set of edges directed from one vertex to another. The edges are also called arrows or directed arcs. If a binary relation R is defined on a set X , then the elements of the set X are represented by vertices, and the ordered pairs of the relation R are represented by the directed edges.

Example 1.2. Let $X = \{1, 2, 3, 4, 5\}$ be a set and R a binary relation on X defined as follows:

$$R = \{(1, 1), (1, 2), (1, 3), (3, 3), (3, 5), (4, 1), (4, 5), (5, 4)\}$$

Then the relation R on X is represented by its digraph as follows:



Relation as a Matrix:

Let $R \subseteq X \times Y$ be a binary relation between the sets X and Y , where

$$X = \{x_1, \dots, x_n\}, \quad Y = \{y_1, \dots, y_m\}.$$

Then R can be represented by the logical $n \times m$ matrix $M = [a_{ij}]$ whose (i, j) -entry is given by

$$a_{ij} = \begin{cases} 1, & (x_i, y_j) \in R \\ 0, & (x_i, y_j) \notin R \end{cases}$$

The logical matrix M is also called the binary matrix, relation matrix, Boolean matrix, adjacency matrix, or zeroone matrix. If $X = Y$, then $m = n$, and the matrix M is square.

Example 1.3. Let $X = \{5, 6, 7, 8\}$ be a set and $R = \{(x, y) \mid x < y\}$ a binary relation on X . Then in matrix form, the relation R is represented as follows:

$$M = (a_{ij}) = \begin{matrix} & \begin{matrix} 5 & 6 & 7 & 8 \end{matrix} \\ \begin{matrix} 5 \\ 6 \\ 7 \\ 8 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

Remark 1.1. It is thought that matrix representation is especially useful for computational purposes, as it makes it easier to apply algorithms, like finding transitive closure or checking for reflexivity, symmetry, or antisymmetry.

1.1.3 Definitions and basic properties of posets

In this subsection, we present the fundamental definitions and characteristics of partially ordered sets (posets). Firstly, we recall the definition of a partial order as outlined by Davey and Priestley [27].

Definition 1.3. Let P be a non-empty set. A partial order on P is a binary relation $\leq$ on P such that, for all $x, y, z \in P$,

- (i) $x \leq x$ (reflexivity);
- (ii) $x \leq y$ and $y \leq x$ imply $x = y$ (antisymmetry);
- (iii) $x \leq y$ and $y \leq z$ imply $x \leq z$ (transitivity).

Remark 1.2. We usually denote a partial order by the symbol $\leq$. Variants include $\preceq$ and $\sqsubseteq$. It is traditional to write the expression $(x, y) \in \leq$ in the equivalent form $x \leq y$, which we read as: x is less than or equal to y .

Definition 1.4. A non-empty set P equipped with a partial order relation $\leq$ on P is called a partially ordered set (poset, for short) and we shall write $(P, \leq)$.

Example 1.4. 1. The usual inequality $\leq$ is an order relation on the known sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$. Then $(\mathbb{N}, \leq)$, $(\mathbb{Z}, \leq)$, $(\mathbb{Q}, \leq)$ and $(\mathbb{R}, \leq)$ are posets;

2. On the set $\mathcal{P}(X)$ of all subsets of a non-empty set X , the relation $\subseteq$ of set inclusion is an order;

3. The relation " $\mid$ " of divisibility is a partial order on $\mathbb{N}^*$ but it is not on the set $\mathbb{Z}$ of integers. Specifically, the relation is not antisymmetric. For instance, $2 \mid -2$ and $-2 \mid 2$, but $2 \neq -2$.

A poset $(P, \leq)$ is called *bounded*, if it has a least and a greatest element, respectively denoted by 0 and 1 , i.e., $0 \leq x \leq 1$, for any $x \in P$. Usually, the notation $(P, \leq, 0, 1)$ is used to describe a bounded poset.

In what follows, we shall use the notation $x < y$ to mean $x \leq y$ and $x \neq y$. Note that the relation $<$ thus defined is transitive but is not an order since it fails to be reflexive.

Definition 1.5. Let $(P, \leq)$ be a poset and A a subset of P . An element $x \in P$ is an upper (resp. lower) bound of A if $y \leq x$ (resp. $x \leq y$), for any $y \in A$. The set of all upper (resp. lower) bounds of A is denoted by A^u (resp. A^l).

$$A^u = \{x \in P : y \leq x, \text{ for any } y \in A\} \quad \text{and} \quad A^l = \{x \in P : x \leq y, \text{ for any } y \in A\}.$$

If A^u (resp. A^l) has a least (resp. greatest) element x , then x is called the least upper bound, called the join (supremum) (resp. greatest lower bound, called the meet (infimum)) of A , denoted by $\sup A$ (resp. $\inf A$).

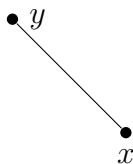
We use the following notation: $x \vee y$ in place of $\sup\{x, y\}$ when it exists and $x \wedge y$ in place of $\inf\{x, y\}$ when it exists. Similarly, $\bigvee A$ and $\bigwedge A$ instead of $\sup A$ and $\inf A$ when these exist.

There are many ways to express a poset, particularly when the underlying set is finite. Our first option is to represent a poset P using a Hasse diagram, a visual representation of both the set, and the relationship between items. To describe how to represent this, we need to introduce the concept of covering.

Definition 1.6. Let P be an ordered set and let $x, y \in P$. We say x is covered by y (or y covers x), and write $x \prec y$ or $y \succ x$, if $x < y$ and $x \leq z < y$ implies $z = x$. The latter condition is demanding that there is no element z of P with $x < z < y$.

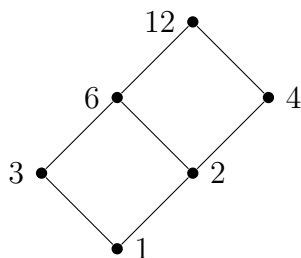
Definition 1.7. Let $(P, \leq)$ be a poset. An element x is covered by y (or y covers x) if $x < y$ and there is no $a \in P$ such that $x < a < y$. We denote this by using the notation $x \prec y$.

Many ordered sets can be represented by means of a *Hasse diagram*. In such a diagram we represent elements by points and interpret $x \prec y$ by

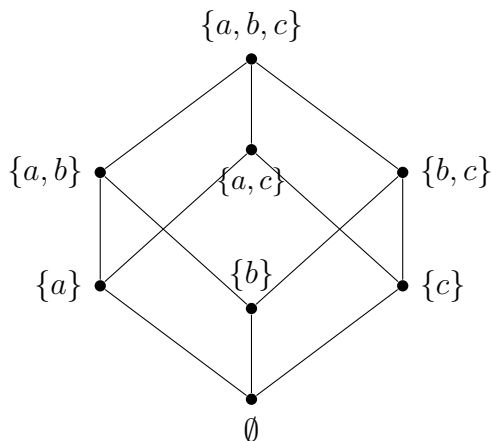


i.e., we join the points representing x and y by an increasing line segment.

Example 1.5. Let $X = \{1, 2, 3, 4, 6, 12\}$ be the set of positive divisors of 12. If we order X in the usual way, we obtain a chain. If we order X by divisibility, we obtain the the following Hasse diagram.



Example 1.6. Let $X = \{a, b, c\}$ be a set. The Hasse diagram of the power set $\mathcal{P}(X)$, ordered by set inclusion is present as follow.



Remark 1.3. A poset is represented in which all pairs of elements have an upper and lower bound, but these are not always unique. Indeed, Consider the poset $(P, \leq)$ drawn in Figure 1.1. At first sight, we think that $b \vee c = i$. On more careful inspection we find that $\{b, c\}^u = \{\top, h, i\}$. Since $\{b, c\}^u$ has distinct minimal elements, it holds that $b \vee c$ does not exist. On the other hand, $\{a, b\}^u = \{\top, h, i, f\}$ has a least element, namely f , and thus $a \vee b = f$.

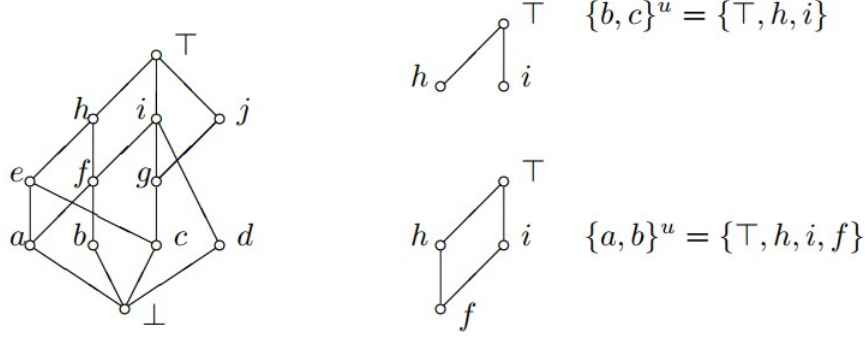


Figure 1.1: The Hasse diagram of the poset $(P, \leq)$.

1.2 Lattices

In this section, we present and study binary operations on lattices. Further information can be found in [2, 5, 13, 21, 23, 26].

1.2.1 Lattices: ordered and algebraic perspectives

Now, we will start with two different definitions, one order theoretic and one algebraic, and then we will prove that these definitions are equivalent.

Definition 1.8. Let $(P, \leq)$ be a poset.

- (ii) $(P, \leq)$ is called a $\wedge$ -semi-lattice if $x \wedge y$ exists for all $x, y \in P$.
- (i) $(P, \leq)$ is called a $\vee$ -semi-lattice if $x \vee y$ exists for all $x, y \in P$.
- (iii) $(P, \leq)$ is called a lattice if it is both a $\wedge$ -semi-lattice and a $\vee$ -semi-lattice.

Example 1.7. Consider the poset $(P, \leq)$ with the Hasse diagram displayed in Figure 1.1. Then $(P, \leq)$ is a lattice and $\{b, c\}^u$ is a $\vee$ -semi-lattice but it is not a $\wedge$ -semi-lattice.

Usually, the notation $(L, \leq, \wedge, \vee)$ is used for a lattice. A poset $(L, \leq)$ is called a *complete* lattice if every subset A of L has both a greatest lower bound of A (i.e., $\bigwedge A$), and a least upper bound of A (i.e., $\bigvee A$).

Example 1.8. 1. If X is a nonempty set, then $(\mathcal{P}(X), \subseteq)$ is a complete lattice under the union and intersection;

2. The set of real numbers $\mathbb{R}$ ordered by the usual order $\leq$ is a lattice, where $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$, for any $x, y \in \mathbb{R}$;

3. The set of positive natural numbers $\mathbb{N}^*$ with the divisibility order $|$ has a structure of a lattice, where $x \wedge y = \gcd(x, y)$ and $x \vee y = \text{lcm}(x, y)$;

4. The posets given in Examples 1.5 and 1.6 are complete lattices.

Proposition 1.1. Let L be a finite lattice. Then L is complete.

With the concepts of meets and joins, We can show another way to represent certain finite lattice. For the posets given in Example 1.5, we can make a meet table, a join table, and a join/meet table (see below).

$\wedge$	1	2	3	4	6	12
1	1	1	1	1	1	1
2	1	2	1	2	2	2
3	1	1	3	1	3	3
4	1	2	1	4	2	4
6	1	2	3	2	6	6
12	1	2	3	4	6	12

$\vee$	1	2	3	4	6	12
1	1	2	3	4	6	12
2	2	2	6	4	6	12
3	3	6	3	12	6	12
4	4	4	12	4	12	12
6	6	6	6	12	6	12
12	12	12	12	12	12	12

$\wedge \backslash \vee$	1	2	3	4	6	12
1	*	2	3	4	6	12
2	1	*	6	4	6	12
3	1	1	*	12	6	12
4	1	2	1	*	12	12
6	1	2	3	2	*	12
12	1	2	3	4	6	*

In Table 1, the intersection between any two elements will give the meet of these two elements. Similarly, in Table 2, the intersection will give the join. In Table 3, we combined the previous two tables since both are symmetric about a forty-five degree.

As shown in Table 1, for any two elements $a, b \in L$, their meet is given by $a \wedge b$, corresponding to their intersection in the table. Similarly, Table 2 illustrates the join operation, where the intersection of any two elements in the table yields $a \vee b$. Since both operations are commutative ($a \wedge b = b \wedge a$ and $a \vee b = b \vee a$) and associative, the tables are symmetric along the diagonal. Therefore, Table 3 combines these two tables into a single representation by exploiting this symmetry, effectively capturing both the meet and join operations within the same structure.

Lattices were introduced as ordered sets of a specific type. However, an alternative viewpoint is that a lattice may be viewed as an algebraic structure $\langle L; \wedge, \vee \rangle$.

Definition 1.9. A set together with binary operations $\vee$ and $\wedge$ is a lattice if

- (i) $\vee$ and $\wedge$ are idempotent, i.e., $a \wedge a = a$ and $a \vee a = a$.
- (ii) $\vee$ and $\wedge$ are commutative, i.e., $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$.
- (iii) $\vee$ and $\wedge$ are associative, i.e., $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$.
- (iv) $\vee$ and $\wedge$ satisfy the absorption identities, i.e., $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.

However, given the existence of two distinct definitions of lattices and the single unifying concept, it becomes possible to demonstrate the equivalence of these definitions, thereby establishing the interchangeability of the two concepts. Firstly, the binary relations employed in each definition will be examined.

Lemma 1.1. Let $*$: $L \times L \rightarrow L$ be a binary operation. If $*$ is commutative, associative, and idempotent, then the binary relation $\leq^*$ defined by $a \leq^* b \Leftrightarrow a * b = b$, is a partial ordering on L and $a * b = \sup\{a, b\}$.

Proof. **Reflexivity:** $a \leq^* a \Leftrightarrow a * a = a$. Since $*$ is idempotent, it holds that $a * a = a$. Thus $\leq^*$ is reflexive.

Antisymmetry: If $a \leq^* b$ and $b \leq^* a$, then $a * b = b$ and $b * a = a$. By commutativity of $*$, it holds that $a * b = b * a$, hence $a = b$.

Transitivity: Let $a \leq^* b$ and $b \leq^* c$, then $a * c = a * (b * c) = (a * b) * c = b * c = c$. Thus $a \leq^* c$.

To show $a * b = \sup\{a, b\}$, we know that $a \leq^* a * b \Leftrightarrow a * (a * b) = a * b$. Since $*$ is associative, idempotent, and commutative, it holds that:

$$\begin{aligned} a * (a * b) &= (a * a) * b = a * b, \\ b * (a * b) &= b * (b * a) = (b * b) * a = b * a = a * b. \end{aligned}$$

Thus $a * b$ is an upper bound of $\{a, b\}$. Let $c \in L$ such that $a \leq^* c$ and $b \leq^* c$, then $a * c = c$ and $b * c = c$. Now, to show that $a * b \leq^* c$, we have $(a * b) * c = a * (b * c) = a * c = c$. Thus $a * b \leq c$. Hence $a * b = \sup\{a, b\}$. $\square$

Now, we will show that our poset is an algebraic sublattice.

Proposition 1.2. If $(P, \leq)$ is a $\vee$ -semi-lattice, then $(P, \vee)$ is an algebraic semilattice.

Proof. Assume $(P, \leq)$ is a $\vee$ -semi-lattice, i.e., $a \vee b = \sup\{a, b\}$ exists for all $a, b \in P$. To show that $\vee$ is commutative, consider $a \vee b$. Then:

$$a \vee b = \sup\{a, b\}$$

$$\begin{aligned}
&= \sup\{b, a\} \\
&= b \vee a.
\end{aligned}$$

Thus, $\vee$ is commutative. Now, to show that $\vee$ is idempotent, consider $a \in P$. Then we have that:

$$a \vee a = \sup\{a, a\} = \sup\{a\} = a,$$

Thus, $\vee$ is idempotent. Finally, to show that $\vee$ is associative, we first need to show that $a \leq b$ and $c \leq d$ imply $a \vee c \leq b \vee d$, for any $a, b, c, d \in P$. We know that $a \leq b \leq b \vee d$, and $c \leq d \leq b \vee d$. So $b \vee d$ is an upper bound of $\{a, c\}$. But since $a \vee c = \sup\{a, c\}$, it follows that $a \vee c \leq b \vee d$. Now, consider $(a \vee b) \vee c \leq a \vee (b \vee c)$. Since $a \leq a \vee (b \vee c)$ and $b \leq b \vee c \leq a \vee (b \vee c)$, it holds that $a \vee b \leq a \vee (b \vee c)$. Also, we have that $c \leq b \vee c \leq a \vee (b \vee c)$. Then $a \vee b \leq a \vee (b \vee c)$ and $c \leq a \vee (b \vee c)$ imply that $(a \vee b) \vee c \leq a \vee (b \vee c)$. In Similar way, Since $a \leq (a \vee b) \leq (a \vee b) \vee c$, $b \leq (a \vee b) \leq (a \vee b) \vee c$ and $c \leq (a \vee b) \vee c$, it follows that $a \leq (a \vee b) \vee c$ and $b \vee c \leq (a \vee b) \vee c$. Hence, $a \vee (b \vee c) \leq (a \vee b) \vee c$. Consequently $a \vee (b \vee c) \leq (a \vee b) \vee c$ and $(a \vee b) \vee c \leq a \vee (b \vee c)$ imply that $a \vee (b \vee c) = (a \vee b) \vee c$. Therefore, $\vee$ is associative. $\square$

Now, we can finally show that our two definitions of a lattice are the same.

Theorem 1.1. *Let $(L, \leq)$ be a lattice with $a \wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$ if and only if $\wedge, \vee$ are idempotent, commutative, associative, and satisfy the two absorption identities.*

Proof. Assume $\wedge, \vee$ are idempotent, commutative, associative, and satisfy the two absorption identities. Now, we define $\leq^\vee$ by

$$a \leq^\vee b \Leftrightarrow a \vee b = b$$

and we define $\leq^\wedge$ by

$$a \leq^\wedge b \Leftrightarrow b \leq^\wedge a \Leftrightarrow b \wedge a = a$$

Then by Lemma 1.1, it holds that $\leq^\vee$ is a partial order on L with $a \vee b = \sup\{a, b\}$, and by the dual of Lemma 1.1, $\leq^\wedge$ is a partial order on L with $b \wedge a = \inf\{a, b\}$. Next, we show that $a \vee b = b$ implies that $a \wedge b = a$. Let $a \vee b = b$, then $a \wedge b = a \wedge (a \vee b) = a$. In a similar way, to show that $a \wedge b = a$ implies that $a \vee b = b$, suppose $a \wedge b = a$. Then $a \vee b = b \vee (b \wedge a) = b$. Thus, $a \vee b = b \Leftrightarrow a \wedge b = a$. Hence, $(L, \leq)$ with $\leq$ defined by $a \leq b \Leftrightarrow (a \vee b = b \text{ and } a \wedge b = a)$ is a lattice. Now, assume that $(L, \leq)$ is a lattice such that $a \wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$. Then $(L, \leq^\vee)$ is a $\vee$ -semilattice. By using Proposition 1.2, it holds that $\vee$ is commutative, idempotent, and

associative. Similarly, $\wedge$ is commutative, idempotent, and associative. Next, we prove that $a \vee (a \wedge b) = a \wedge (a \vee b) = a$. On the one hand, since $a \leq a$ and $a \wedge b \leq a$, it holds that $a \vee (a \wedge b) \leq a$. The fact that $a \leq a \vee (a \wedge b)$. Thus $a \vee (a \wedge b) = a$. On the other hand, since $a \leq a$ and $a \leq a \vee b$, it holds that $a \leq a \wedge (a \vee b)$. The fact that $a \wedge (a \vee b) \leq a$. Hence, $a \wedge (a \vee b) = a$. $\square$

1.2.2 Binary operations on lattices

A *binary operation* on a non-empty set X is any function from the Cartesian product $X \times X$ into X , i.e., $F : X \times X \rightarrow X$. A binary operation F on X is called [27] :

- (i) *commutative*, if $F(x, y) = F(y, x)$, for any $x, y \in X$;
- (ii) *associative*, if $F(x, F(y, z)) = F(F(x, y), z)$, for any $x, y, z \in X$.

An element $e \in X$ is called a *neutral* element of F , if $F(e, x) = F(x, e) = x$, for any $x \in X$.

Example 1.9. On the set of real numbers $\mathbb{R}$, the addition (+) and the multiplication ($\times$) operations are commutative and associative binary operations. Where, 0 is the neural element of (+) and 1 is the neural element of ($\times$).

Definition 1.10. Let $(L, \leq, \wedge, \vee)$ be a lattice and F be a binary operation on L . F is called:

- (i) *idempotent*, if $F(x, x) = x$, for any $x \in L$;
- (ii) *conjunctive* (resp. *disjunctive*), if $F(x, y) \leq x \wedge y$ (resp. $x \vee y \leq F(x, y)$), for any $x, y \in L$;
- (iii) *increasing*, if $x_1 \leq x_2$ and $y_1 \leq y_2$ imply $F(x_1, y_1) \leq F(x_2, y_2)$, for any $x_1, y_1, x_2, y_2 \in L$.

Example 1.10. Let $(L, \leq, \wedge, \vee)$ be a lattice. The meet ($\wedge$) and the join ($\vee$) operations of L are idempotent and increasing. Moreover, $\wedge$ is conjunctive and $\vee$ is disjunctive.

Proposition 1.3. Let $(L, \leq, \wedge, \vee)$ be a bounded lattice. If an idempotent binary operation F on L is increasing and conjunctive (resp. disjunctive), then F can not be other than the meet operation $\wedge$ (resp. the join operation $\vee$) of L .

The following proposition expresses the relationships between the properties of a binary operation and those of its transpose. The proof is straightforward.

Proposition 1.4. *Let $(L, \leq, \wedge, \vee)$ be a lattice and F be a binary operation on L . The following equivalences hold:*

- (i) *F is commutative if and only if $F^t = F$;*
- (ii) *F is conjunctive if and only if F^t is conjunctive;*
- (iii) *F is disjunctive if and only if F^t is disjunctive;*
- (iv) *F is increasing if and only if F^t is increasing;*
- (v) *An element $e \in L$ is a neutral element of F if and only if e is a neutral element of F^t .*

The following result characterizes the conjunction and the disjunction proprieties of a binary operation on a bounded lattice in terms of its neutral element.

Proposition 1.5. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and F be a binary operation on L . If F is increasing and having a neutral element $e \in L$, then the following statements hold:*

- (i) *F is conjunctive if and only if $e = 1$;*
- (ii) *F is disjunctive if and only if $e = 0$.*

Proof. (i) Suppose that F has a neutral element $e \in L$. Since F is conjunctive, it holds that $1 = F(e, 1) = F(1, e) \leq e$. Hence, $e = 1$. Conversely, suppose that $e = 1$. Let $x, y \in L$, then $x \leq e$ and $y \leq e$. Since F is increasing, it follows that $F(x, y) \leq F(x, e) = x$ and $F(x, y) \leq F(e, y) = y$. Hence, $F(x, y) \leq x \wedge y$. Thus, F is conjunctive.

- (ii) The proof is similar to that of (i).

□

Chapter 2

Pseudo-ordered sets and trellises

In this chapter, the fundamental concepts and properties of partially ordered sets and lattices are revisited. Additionally, we revisit fundamental concepts of pseudo-ordered sets and trellises. Furthermore, we present and study binary operations on these structures, which will be required throughout this thesis.

2.1 Pseudo-ordered sets

This section contains the fundamental definitions and properties of pseudo-ordered sets, along with illustrative examples and several concepts that will be used later. Further details can be found in [1, 3, 16, 18].

2.1.1 Definitions and basic concepts

Definition 2.1. A pseudo-order on a set A is a binary relation $\trianglelefteq$ that is both reflexive and antisymmetric. That is, for all $a, b \in A$, we have $a \trianglelefteq a$ (reflexivity), and if $a \trianglelefteq b$ and $b \trianglelefteq a$, then $a = b$ (antisymmetry). A pair $(A, \trianglelefteq)$ satisfying these properties is called a pseudo-ordered set, or more briefly, a psoet. This notion generalizes partially ordered sets by relaxing the requirement of transitivity.

Similarly to the setting of ordered sets, a finite pseudo-ordered sets can be represented by Hasse-type diagram with the convention that: x is below y and is joined to y by a line if $x \trianglelefteq y$. Otherwise, i.e., if x and y are not related, then x and y will be joined by a dashed curve.

Example 2.1.

(i) Let $A = \{a, b, c\}$, and define a binary relation $\trianglelefteq$ on A by:

$$a \trianglelefteq a, \quad b \trianglelefteq b, \quad c \trianglelefteq c, \quad a \trianglelefteq b, \quad b \trianglelefteq a$$

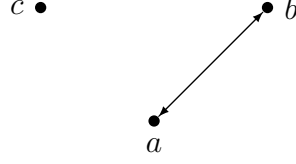


Figure 2.1: Hasse-type diagram of $(A = \{a, b, c, \}, \leq)$.

Here, the arrow $a \rightarrow b$ indicates $a \leq b$. Then this relation is reflexive and antisymmetric only if $a = b$, which is not the case here. Then it is not a pseudo-order. Now, we define another relation $\leq'$ by:

$$a \leq' a, \quad b \leq' b, \quad c \leq' c, \quad a \leq' b$$

This relation is reflexive and antisymmetric, but not transitive (e.g., if we added $b \leq' c$, transitivity would require $a \leq' c$, which we don't assume). Hence, $(A, \leq')$ is a pseudo-ordered set but not a poset.

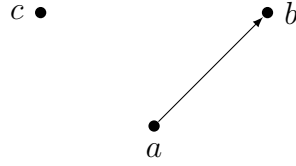


Figure 2.2: Hasse-type diagram of $(A = \{a, b, c, \}, \leq')$.

It's easy to check that there is no connection between c and the other elements, then c is incomparable. The diagram reflects the lack of transitivity, confirming it is a pseudo-order set but not a poset.

(ii) Let $A = \{a, b, c, d\}$, and define the binary relation $\leq$ on A as follows:

$$a \leq a, \quad b \leq b, \quad c \leq c, \quad d \leq d, \quad a \leq b, \quad b \leq c, \quad c \leq d.$$

The relation is reflexive because every element is related to itself:

$$a \leq a, \quad b \leq b, \quad c \leq c, \quad d \leq d.$$

The relation is antisymmetric because:

(1) If $a \leq b$ and $b \leq a$, then $a = b$ (but this never happens in this case).

(2) Similarly, for other pairs like $b \trianglelefteq c$ and $c \trianglelefteq b$, there's no contradiction.

Thus, antisymmetry holds.

For transitivity, we check if whenever $a \trianglelefteq b$ and $b \trianglelefteq c$, we have $a \trianglelefteq c$:

(1) We have $a \trianglelefteq b$ and $b \trianglelefteq c$, so by transitivity, we should expect $a \trianglelefteq c$.

(2) Similarly, $b \trianglelefteq c$ and $c \trianglelefteq d$ implies $b \trianglelefteq d$, but we don't have $b \trianglelefteq d$ explicitly in the relation.

Then the relation is reflexive and antisymmetric, but it is not fully transitive because $b \trianglelefteq d$ is missing. Therefore, the relation is a pseudo-ordered set, but it is not a poset.

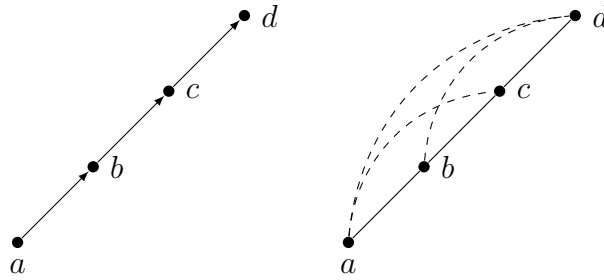


Figure 2.3: Hasse-type diagram of $(A = \{a, b, c, d\}, \trianglelefteq)$.

(iii) A well-known pseudo-ordered set is the set of all real numbers $\mathbb{R}$ with the following relation:

$$x \trianglelefteq y \iff 0 \leq y - x \leq a,$$

where a is a given positive number. The relation “ $\trianglelefteq$ ” is reflexive since $x \trianglelefteq x$ because $0 \leq x - x = 0 \leq a$. Antisymmetry holds because if $x \trianglelefteq y$ and $y \trianglelefteq x$ we have that $0 \leq y - x \leq a$ and $0 \leq x - y \leq a$. Then we conclude that $x = y$ since “ $\leq$ ” is the usual order on $\mathbb{R}$. Note this relation is not transitive. This happens in the situation when, for example, $a = 4$, $x = 1$, $y = 2$, and $z = 6$.

Remark 2.1. It is easily seen that any ordered set (poset) on a set A is a pseudo-ordered set (psoset) on A .

Definition 2.2. Let $(A, \trianglelefteq)$ be a pseudo-ordered set and $B \subseteq A$. An element $a \in A$ is called an upper bound of B if $b \trianglelefteq a$ for any $b \in B$. If a is an upper bound of B and satisfies $a \trianglelefteq d$ for every upper bound d of B , then a is called the least upper bound (l.u.b.) or join of B , denoted by $a = \bigsqcup B$. The notions of lower bound and greatest lower bound (g.l.b.) or meet, denoted by $\bigsqcap B$, are defined dually.

The next definition plays an important role in pseudo-ordered set theory since it is analogous to the definition of a totally-ordered set.

Definition 2.3. Let $(A, \trianglelefteq)$ be a pseudo-ordered set (psoset), and let $B \subseteq A$ be a subset. We define a binary relation $\sqsubseteq_B$ on B as follows: for any two elements $b, b' \in B$, we write $b \sqsubseteq_B b'$ if there exists a finite sequence $b_0, b_1, \dots, b_n$ of elements in B , with $n \geq 0$, such that $b = b_0 \triangleleft b_1 \triangleleft \dots \triangleleft b_n \triangleleft b'$. Similarly, we define $b \sqsupseteq_B b'$ if there exists a finite sequence $b = b_0 \triangleright b_1 \triangleright \dots \triangleright b_n \triangleright b'$. The dual relations $\sqsupseteq_B$ and $\sqsubseteq_B$ are defined analogously. A subset B of A is called a pseudo-chain (or p-chain, for short) if, for every pair of elements $b, b' \in B$, at least one of the relations $b \sqsubseteq_B b'$ or $b' \sqsubseteq_B b$ holds.

Example 2.2. Let $A = \{0, b, c, d, 1\}$ be a set and $\trianglelefteq$ a pseudo-order relation on A , where $0 \trianglelefteq b \trianglelefteq c \trianglelefteq d \trianglelefteq 1$ (i.e.; $0 \trianglelefteq x \trianglelefteq 1$, for any $x \in A$). Then $(A, \trianglelefteq)$ is p-chain and the Hasse-type diagram of A is depicted by the following figure (The hasse-type diagrams are the same):

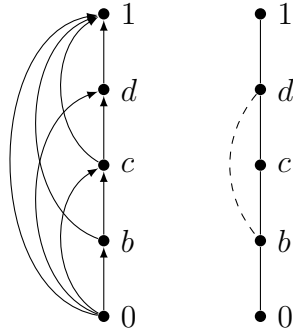


Figure 2.4: Hasse-type diagram of $(A, \trianglelefteq)$.

Remark 2.2. Unlike total orders, in a pseudo-ordered set, not every pair of elements is necessarily comparable. This allows for independent elements that do not have any ordering between them.

Definition 2.4. Let $(A, \trianglelefteq)$ be a psoset. A subset $B \subseteq A$ is called a cycle if every pair of elements $b, b' \in B$ satisfies both $b \sqsubseteq_B b'$ and $b' \sqsubseteq_B b$. A nontrivial cycle is one that contains more than one element. A psoset is said to be acyclic if it does not contain any nontrivial cycle.

Remark 2.3. (i) Empty set is a cycle;

(ii) Any single element set in a psoset is also a cycle;

(iii) A nontrivial cycle is a cycle having more than one element and it contains at least three elements.

2.1.2 Fundamental properties of pseudo-ordered sets

Definition 2.5. (Contraction set over a pso set) Let $(A, \trianglelefteq)$ be a pseudo-ordered set and $\mathcal{A}$ be a family of subsets of A (i.e., $\{A_1, A_2, A_3, \dots\} \subseteq \mathcal{P}(A)$). The family $\mathcal{A}$ is said to be consistent if it satisfies the following condition: For any two distinct sets $R, S \in \mathcal{A}$, there do not exist elements $r_1, r_2 \in R$ and $s_1, s_2 \in S$ such that $r_1 \trianglelefteq s_1$ and $s_2 \trianglelefteq r_2$. On such a family $\mathcal{A}$, we define the following pseudo-order $\blacktriangleleft$ as follows:

$$R \blacktriangleleft S \iff \exists r \in R, s \in S \text{ such that } r \trianglelefteq s$$

is a pseudo-order on $\mathcal{A}$. Also, The pseudo-ordered set $(\mathcal{A}, \blacktriangleleft)$ is called the contraction set of $(A, \trianglelefteq)$.

Example 2.3. On the set of all positive integers set $m|n$ if and only if m is a divisor of n . Consider the family of subsets: $\mathcal{A} = \{\{2k-1, 2k\} \mid k = 1, 2, 3, \dots\}$, (i.e., $\{1, 2\}, \{3, 4\}, \{5, 6\}, \dots$). Then the family $\mathcal{A}$ is consistent, because for any two different sets in the family, you cannot find elements that satisfy both directions of the relation. For $R = \{10, 11\}$, $S = \{100, 101\}$ and $T = \{201, 202\}$, it holds that

- $R \blacktriangleleft S$ because 10 divides 100,
- $S \blacktriangleleft T$ because 101 divides 202,
- But $R \not\blacktriangleleft T$ because 10 and 11 do not divide 201 or 202.

Hence, the induced pseudo-order is not transitive. Even though $|$ is a partial order, the pseudo-order induced is not.

Theorem 2.1. Any pseudo-ordered set is isomorphic with a contraction set of partially ordered set.

Theorem 2.2. Let $C = (\{x_1, x_2, \dots, x_k\}, \trianglelefteq)$ be a pseudo-chain. Then the elements of C can be arranged (with possible repetitions) in a sequence $(b_1, \dots, b_n)$ such that $b_1 \trianglelefteq \dots \trianglelefteq b_n$.

Proof. Let k be the largest integer for which there exists a sequence $P = (b_1, b_2, \dots, b_n)$ with k distinct elements from the pseudo-chain C , satisfying the order $b_1 \trianglelefteq b_2 \trianglelefteq \dots \trianglelefteq b_n$. We will prove that this sequence P must contain all elements of C . Assume, for contradiction, that there exists an element $b \in C$ not appearing in P . Since C is a pseudo-chain, and P is ordered under $\trianglelefteq$, there are three possible positions where b could be inserted:

1. $b \sqsubseteq_B b_1$: insert b at the beginning,
2. $b_i \sqsubseteq_B b \sqsubseteq_B b_{i+1}$ for some i : insert b between b_i and b_{i+1} ,

3. $b_n \sqsubseteq_B b$: insert b at the end.

In each case, we can form a new sequence P' that:

- maintains the ordering $b_1 \sqsubseteq \dots \sqsubseteq b_n$,
- and contains $k + 1$ distinct elements of C .

This contradicts the assumption that k was the maximum number of distinct elements possible. Hence, every element of C must appear in P . $\square$

As a corollary we obtain the following formulation of Redei's theorem on graphs:

Corollary 2.1 (Redei's Theorem). *If a finite pseudo-chain contains no cycles, then its elements can be arranged in a sequence $(b_1, b_2, \dots, b_n)$ of distinct elements such that $b_1 \sqsubseteq b_2 \sqsubseteq \dots \sqsubseteq b_n$.*

Example 2.4. *Consider the pseudo-chain C given in Example 2.2. Then it has no cycles. Thus, by the last corollary, we can arrange the elements of C in a sequence of distinct elements without repetition such that $0 \sqsubseteq b \sqsubseteq c \sqsubseteq d \sqsubseteq 1$.*

A pseudo-order on a set A is defined as *linear* if A is a pseudo-chain. While it is possible to extend any pseudo-order to a linear one, the present study presents a more robust result that refines this classical extension property.

Theorem 2.3. *Any pseudo-order $\sqsubseteq$ on a set A can be extended to a linear pseudo-order $\sqsubseteq'$ on A such that any cycle with respect to $\sqsubseteq'$ is also a cycle with respect to $\sqsubseteq$.*

The following theorem is immediately obtained by using Theorem 2.3

Theorem 2.4. *Any pseudo-order is the intersection of all its linear extensions.*

Proof. Let A be a pseudo-ordered set and $\mathcal{L}$ the set of all its linear extensions. By Theorem 2.3, $\mathcal{L}$ is nonempty. The intersection of $\mathcal{L}$ is clearly an extension of the pseudo-order $\sqsubseteq$ on A . In order to prove it equals $\sqsubseteq$ we show that if $a \not\sqsubseteq b$ there exists a linear extension $\sqsubseteq'$ of $\sqsubseteq$ such that $a \not\sqsubseteq' b$. Since $a \not\sqsubseteq b$, we can define the pseudo-order $\sqsubseteq^*$ by $b \sqsubseteq^* a$ and for $(a, b) \neq (x, y)$, $x \sqsubseteq^* y$ if and only if $x \sqsubseteq y$. Theorem 2.3 guarantees that there exists a linear extension $\sqsubseteq'$ of $\sqsubseteq^*$. Hence also $b \sqsubseteq' a$ and therefore $a \not\sqsubseteq' b$. $\square$

Example 2.5. *Let $(A = \{a, b, c\}, \sqsubseteq)$ be a pseudo-ordered set given in Example 2.1 (Figure 2.2). Then all permutations of $\{a, b, c\}$ satisfying $a \sqsubseteq b$ are $a \sqsubseteq b \sqsubseteq c$, $a \sqsubseteq c \sqsubseteq b$ and $c \sqsubseteq a \sqsubseteq b$. We now list all linear extensions of $\sqsubseteq$, i.e., total orders on A that are consistent with the pseudo-order (i.e., preserve $a \sqsubseteq b$):*

$$L_1 = [a, b, c], \quad L_2 = [a, c, b], \quad L_3 = [c, a, b].$$

Hence, the intersection of all linear extensions is:

$$\{(a, a), (b, b), (c, c), (a, b)\} = \trianglelefteq.$$

2.2 Trellises

The notion of a trellis was introduced by Fried [8] and Skala [24, 25] as one of the fundamental algebraic structures in order theory. It can be viewed as a generalization of the concept of a lattice, where the transitivity property is not required. A trellis is defined as a poset $(A, \trianglelefteq)$ in which every pair of elements has both a least upper bound and a greatest lower bound. We adopt the following notation: for elements $x, y \in A$, we write $x \sqcup y$ instead of $\sup\{x, y\}$, when it exists, and $x \sqcap y$ instead of $\inf\{x, y\}$, when it exists. Similarly, for a subset $S \subseteq A$, we write $\sqcup S$ and $\sqcap S$ instead of $\sup S$ and $\inf S$, respectively, when these exist.

2.2.1 Definitions and properties

Definition 2.6. Let $(A, \trianglelefteq)$ be a non-empty pseudo-ordered set.

- (i) The structure $(A, \trianglelefteq)$ is called a $\sqcap$ -semi-trellis if the meet $x \sqcap y$ exists for all $x, y \in A$.
- (ii) The structure $(A, \trianglelefteq)$ is called a $\sqcup$ -semi-trellis if the join $x \sqcup y$ exists for all $x, y \in A$.
- (iii) The structure $(A, \trianglelefteq)$ is called a trellis if it is both a $\sqcap$ -semi-trellis and a $\sqcup$ -semi-trellis (i.e., both $x \sqcap y$ and $x \sqcup y$ exist for every pair $x, y \in A$).

Equivalently, a trellis is an algebraic structure $(A, \sqcap, \sqcup)$ where the binary operations $\sqcap$ (meet) and $\sqcup$ (join) satisfy the following identities for all $x, y, z \in A$:

- (i) Idempotency: $x \sqcap x = x, \quad x \sqcup x = x;$
- (ii) Commutativity: $x \sqcap y = y \sqcap x, \quad x \sqcup y = y \sqcup x;$
- (iii) Absorption laws: $x \sqcap (x \sqcup y) = x, \quad x \sqcup (x \sqcap y) = x;$
- (iv) Weak associativity: $x \sqcap ((x \sqcup y) \sqcap (x \sqcup z)) = x, \quad x \sqcup ((x \sqcap y) \sqcup (x \sqcap z)) = x.$

Moreover, for a given trellis $(A, \trianglelefteq, \sqcap, \sqcup)$ and elements $x, y \in A$, the following conditions are equivalent:

- (i) $x \trianglelefteq y;$
- (ii) $x \sqcup y = y;$

(iii) $x \sqcap y = x$.

Example 2.6. Let $A = \{a, b, c, d, e, f\}$ be a set and let $\trianglelefteq$ be a pseudo-order relation on A . Then the structure $(A, \trianglelefteq)$, depicted by the Hasse-type diagram in Figure 3.3, is a trellis. The meet and join operations are defined analogously to those in lattices, as shown in the following table.

$\sqcap \backslash \sqcup$	a	b	c	d	e	f
a	a	b	c	d	e	f
b	a	b	a	d	e	f
c	a	a	c	d	e	f
d	a	b	c	d	e	f
e	a	b	c	d	e	f
f	a	b	c	d	e	f

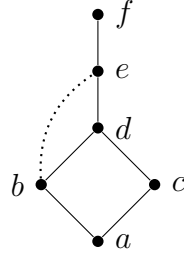


Figure 2.5: Hasse-type diagram of the trellis $(A, \trianglelefteq)$.

Theorem 2.5. [24, 25] A set A with two commutative, absorptive, and weak-associative binary operations $\sqcap$ and $\sqcup$ is a trellis if $a \trianglelefteq b$ is defined as $a \sqcap b = a$ and/or $a \sqcup b = b$. These operations are also idempotent.

Remark 2.4. The main distinction between a lattice and a trellis as algebraic structures lies in the associativity of the binary operations. In a trellis, these operations are generally not associative as opposed to a lattice. This lack of associativity is one of the key characteristics that differentiates trellises from lattices.

Theorem 2.6. [25] Let $(A, \trianglelefteq, \sqcap, \sqcup)$ be a trellis. The following statements are equivalent:

- (i) $\trianglelefteq$ is transitive;
- (ii) The meet ($\sqcap$) and the join ($\sqcup$) operations are associative;
- (iii) One of the operations ($\sqcap$) or ($\sqcup$) is associative.
- (iv) If $a \trianglelefteq b$, then $a \trianglelefteq b \sqcup x$ for each $x \in A$.
- (v) If $a \trianglelefteq b$, then $x \sqcap a \trianglelefteq b$ for each $x \in A$.

Definition 2.7. Let $(A, \trianglelefteq, \sqcap, \sqcup)$ be a trellis and B a proper non-empty subset of A (i.e., $B \subsetneq A$ and $A \neq \emptyset$). The set B is called:

- (i) a $\sqcap$ -sub-trellis (resp. a $\sqcup$ -sub-trellis) of A if $x \sqcap y \in B$ (resp. $x \sqcup y \in B$), for any $x, y \in B$, where $\sqcap$ and $\sqcup$ are taken in A ;
- (ii) a sub-trellis of A if it is both a $\sqcap$ - and a $\sqcup$ -sub-trellis of A .

A *bounded trellis* is a trellis $(A, \trianglelefteq, \sqcap, \sqcup)$ that additionally has a least element denoted by 0 and a greatest element denoted by 1 satisfying $0 \trianglelefteq x \trianglelefteq 1$, for any $x \in A$. For a bounded trellis the notation $(A, \trianglelefteq, \sqcap, \sqcup, 0, 1)$ is used. Also, a trellis $(A, \trianglelefteq, \sqcap, \sqcup)$ is *complete* if every subset of A has a meet and a join.

Remark 2.5. Bounded sub-trellises play an important role in the study of completeness and compactness conditions in trellises.

Remark 2.6. It is well-known that every finite lattice is complete. However, this property does not necessarily hold for finite trellises. Indeed, consider the cyclic structure $A = (\{a, b, c\}, \trianglelefteq)$, where the relation is defined by $a \trianglelefteq b \trianglelefteq c \trianglelefteq a$. This structure forms a trellis, since every pair of elements admits a least upper bound and a greatest lower bound. Nevertheless, it is not a complete trellis, because the join $\sqcup\{a, b, c\}$ and the meet $\sqcap\{a, b, c\}$ do not exist in A . Hence, this example shows that a finite trellis need not be complete.

The following results illustrate two particular cases in which completeness holds in trellises.

Theorem 2.7. [10] A trellis of finite length is complete if and only if every cycle has the meet and the join.

Theorem 2.8. [25] If every non-empty subset of a pseudo ordered set A with the smallest element 0 has a least upper bound, then A is a complete trellis.

Proposition 2.1. [9] Any distributive trellis $(A, \trianglelefteq, \sqcap, \sqcup)$ is a lattice.

Definition 2.8. [25] A trellis $(A, \trianglelefteq, \sqcap, \sqcup)$ is called a *modular*, if $x \trianglelefteq z$ implies that $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap z$, for any $y \in A$.

Example 2.7. Every modular lattice is also a modular trellis. However, the following figures show examples of modular trellises that are not lattices:

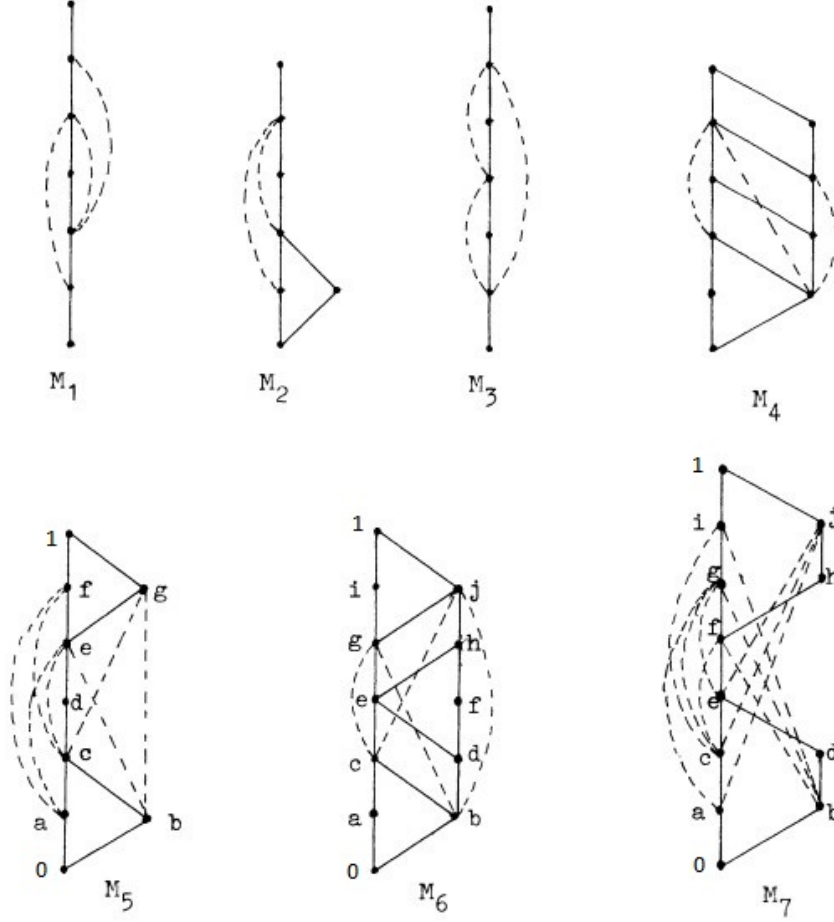


Figure 2.6: Examples of modular trellises that are not lattices.

Remark 2.7. In a modular trellis, if $x \leq y$, $x \sqcup z = y \sqcup z$ and $x \sqcap z = y \sqcap z$ for some element z , then $x = y$.

Though $x \leq y$ does not necessarily imply $x \leq y \sqcup z$ or $x \sqcap z \leq y$ for every element z , the following special cases do hold.

Proposition 2.2. [24] Let $(A, \leq, \sqcap, \sqcup)$ be a modular trellis and $x, y, z \in A$. If $x \leq y \leq z$, then $x \sqcap z \leq y \leq x \sqcup z$.

Proof. Let $x, y, z \in A$. Since $y \leq z$ and $y \sqcup (x \sqcap z) = (y \sqcup x) \sqcap z = y \sqcap z = y$, it holds that $x \sqcap z \leq y$. By duality, it follows that $y \leq x \sqcup z$. $\square$

Proposition 2.3. [24] Let $(A, \leq, \sqcap, \sqcup, 0, 1)$ be a bounded modular trellis. The following implications hold for any $x, y, z \in A$:

- (i) If $x \leq y$ and $y \sqcap z = 0$, then $x \leq y \sqcup z$;
- (ii) If $y \leq x$ and $y \sqcup z = 1$, then $y \sqcap z \leq x$.

Proof. (i) Let $x \leq y$ and suppose $y \sqcap z = 0$. Then $z \sqcup (x \sqcap (y \sqcup z)) = (z \sqcup x) \sqcap (y \sqcup z) = z \sqcup (y \sqcap (x \sqcup z)) = z \sqcup (x \sqcup (y \sqcap z)) = z \sqcup x$. Lemma 2.3 implies that $x \sqcap (y \sqcup z) \leq y$. Thus, $((x \sqcap (y \sqcup z)) \sqcup z) \sqcap y = (x \sqcap (y \sqcup z)) \sqcup (z \sqcap y) = x \sqcap (y \sqcup z)$. On the other hand, $((x \sqcap (y \sqcup z)) \sqcup z) \sqcap y = (x \sqcup z) \sqcap y = x \sqcup (z \sqcap y) = x$. Hence, $x = x \sqcap (y \sqcup z)$. Therefore, $x \leq y \sqcup z$.

(ii) The proof is dual to that of (i).

□

Definition 2.9. [25] A mapping f from a trellis $\mathcal{L} = (A, \leq)$ into a trellis $\mathcal{M} = (A, \leq)$ (i.e., $f : \mathcal{L} \rightarrow \mathcal{M}$) is called

- (i) a $\sqcap$ -homomorphism if $f(x \sqcap_{\mathcal{L}} y) = f(x) \sqcap_{\mathcal{M}} f(y)$, for any $x, y \in A$;
- (ii) a $\sqcup$ -homomorphism if $f(x \sqcup_{\mathcal{L}} y) = f(x) \sqcup_{\mathcal{M}} f(y)$, for any $x, y \in A$;
- (iv) an isomorphism if f is both one to one and onto homomorphism;
- (iii) a homomorphism if f is both $\sqcap$ - and $\sqcup$ -homomorphism.

2.2.2 Binary operations on trellises

In this subsection, we present some basic definitions and properties of binary operations on a poset or trellis. Many of these notions are adapted from their counterparts in the theory of posets and lattices (see, e.g., [21]). Let $\mathbb{P} = (A, \leq)$ be a poset. A binary operation $F : A \times A \rightarrow A$ on $\mathbb{P}$ is called:

- (i) *associative*, if $F(x, F(y, z)) = F(F(x, y), z)$, for any $x, y, z \in A$;
- (ii) *conjunctive* (resp. *disjunctive*), if $F(x, y) \leq x \sqcap y$ (resp. $x \sqcup y \leq F(x, y)$), for any $x, y \in A$.
- (iii) *right-increasing*, if $x \leq y$ implies $F(z, x) \leq F(z, y)$, for any $x, y, z \in A$;
- (iv) *left-increasing*, if $x \leq y$ implies $F(x, z) \leq F(y, z)$, for any $x, y, z \in A$;
- (v) *increasing*, if $x \leq y$ and $z \leq t$ implies $F(x, z) \leq F(y, t)$, for any $x, y, z, t \in A$.

An element $e \in A$ is called a *neutral element* of F , if $F(e, x) = F(x, e) = x$, for any $x \in A$.

Example 2.8. Let $(A, \leq, \sqcap, \sqcup)$ be a trellis. Then the meet ($\sqcap$) (resp. the join ($\sqcup$)) is conjunctive (resp. disjunctive) binary operation.

Remark 2.8. One easily verifies that any increasing binary operation on a trellis is right- and left-increasing. It's not true, Indeed, let $(A = \{0, a, b, c, d, 1\}, \leq, \sqcap, \sqcup)$ be a trellis given by the Hasse-type diagram in Figure 2.7 and F a binary operation on A defined by the following table:

$F(x, y)$	0	a	b	c	d	1
0	0	a	b	c	c	1
a	0	a	b	c	c	1
b	0	a	c	d	d	1
c	a	a	c	d	d	1
d	b	c	c	d	1	1
1	c	c	c	1	1	1

One easily verifies that F is right- and left-increasing, but it is not increasing by the fact that $a \leq b$, $b \leq c$ and $F(a, b) = b \not\leq d = F(b, c)$.

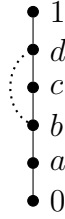


Figure 2.7: The Hasse-type diagram of the trellis $(A = \{0, a, b, c, d, 1\}, \leq)$.

In the following example, we show that on a given trellis $(A, \leq, \sqcap, \sqcup)$, the binary operations $\sqcap$ and $\sqcup$ do not necessarily be right- and left-increasing.

Example 2.9. Let $(A = \{0, a, b, c, d, 1\}, \leq, \sqcap, \sqcup)$ be a trellis given by the above Hasse-type diagram in Figure 2.7. Since $c \leq d$ and $b \sqcap c = c \sqcap b = b \not\leq a = b \sqcap d = d \sqcap b$, it holds that the meet ($\sqcap$) is not right- and not left-increasing. Similarly for the join binary operation ($\sqcup$).

Proposition 2.4. Let $(A, \leq, \sqcap, \sqcup)$ be a trellis. Then it holds that the binary operation $\sqcap$ (or $\sqcup$) is increasing if and only if $\leq$ is transitive.

Proof. We only give the proof for ($\sqcap$) as for ($\sqcup$) is similarly. Suppose that $\sqcap$ is increasing and $\leq$ is not transitive (i.e., $x \leq y \leq z$ and $x \not\leq z$, for some $x, y, z \in A$). Since $\sqcap$ is increasing and $y \leq z$, it follows that $x = x \sqcap y \leq x \sqcap z$. Hence, $x = x \sqcap z$. Thus, $x \leq z$, a contradiction. Therefore, $\leq$ is transitive. The proof of the converse implication is immediate. $\square$

Combining Theorem 2.6 and Proposition 2.4 leads to the following corollary.

Corollary 2.2. *Let $(A, \trianglelefteq, \sqcap, \sqcup)$ be a trellis. The following statements are equivalent:*

- (i) $\sqcap$ (resp. $\sqcup$) is increasing;
- (ii) $\trianglelefteq$ is transitive;
- (iii) $\sqcap$ (resp. $\sqcup$) is associative.

2.2.3 Specific subsets in trellises

In this subsection, we present some specific elements on a trellis that will be needed to study t-norms and t-conorms on a bounded trellis.

Definition 2.10. [25] *Let $(A, \trianglelefteq, \sqcap, \sqcup)$ be a trellis. An element $\alpha \in A$ is called:*

- (i) *right-transitive, if $\alpha \trianglelefteq x \trianglelefteq y$ implies $\alpha \trianglelefteq y$, for any $x, y \in A$;*
- (ii) *left-transitive, if $x \trianglelefteq y \trianglelefteq \alpha$ implies $x \trianglelefteq \alpha$, for any $x, y \in A$;*
- (iii) *middle-transitive, if $x \trianglelefteq \alpha \trianglelefteq y$ implies $x \trianglelefteq y$, for any $x, y \in A$;*
- (iv) *transitive, if it is right-, left- and middle-transitive.*

Definition 2.11. [25] *Let $(A, \trianglelefteq, \sqcap, \sqcup)$ be a trellis.*

- (i) *A sequence $(x, y, z) \in A^3$ is called $\sqcap$ -associative (resp. $\sqcup$ -associative), if $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$ (resp. $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$);*
- (ii) *An element $\alpha \in A$ is called $\sqcap$ -associative (resp. $\sqcup$ -associative), if any sequence of three elements of A including α is $\sqcap$ -associative (resp. $\sqcup$ -associative);*
- (iii) *α is called associative, if it is both $\sqcap$ - and $\sqcup$ -associative.*

Notice that for the notion of associative element $\alpha \in A$, the commutativity of the meet and the join operations is sufficient to consider only the sequence $(\alpha, x, y) \in A^3$.

Proposition 2.5. [25] *Let $(A, \trianglelefteq, \sqcap, \sqcup)$ be a trellis. Then it holds that any $\sqcap$ -associative (resp. $\sqcup$ -associative) element is transitive.*

Remark 2.9. *The converse of the Theorem 2.5 does not hold in general. Indeed, let $(A, \trianglelefteq, \sqcap, \sqcup)$ the trellis given by the Hasse-type diagram in Figure 2.5. Note that c is transitive but not $\sqcup$ -associative since $(c \sqcup b) \sqcup e = e$ but $c \sqcup (b \sqcup e) = f$.*

The following results show two cases that the notion of associative elements and transitive elements are equivalent.

Proposition 2.6. [25] *Let $(A, \trianglelefteq, \sqcap, \sqcup)$ be a pseudo-chain. Then it holds that an element is associative if and only if it is transitive.*

Proposition 2.7. [25] *Let $(A, \trianglelefteq, \sqcap, \sqcup)$ be a modular trellis. Then it holds that an element is associative if and only if it is transitive.*

Theorem 2.9. [25] *Let $(A, \trianglelefteq, \sqcap, \sqcup)$ be a trellis. It holds that*

- (i) *if $x_1, \dots, x_k$ are right-transitive, then the join of $\{x_1, \dots, x_k\}$ exists and equals $x_{i_1} \sqcup \dots \sqcup x_{i_k}$ for any permutation $i_1, \dots, i_k$ of $1, \dots, k$. Moreover, $\bigsqcup \{x_1, \dots, x_k\}$ is right-transitive. Similarly for the left-transitive elements $x_1, \dots, x_k$ and the meet of $\{x_1, \dots, x_k\}$.*
- (ii) *if $x_1, \dots, x_k$ are $\sqcap$ -associative, then the meet of $\{x_1, \dots, x_k\}$ exists and equals $x_{i_1} \sqcap \dots \sqcap x_{i_k}$ for any permutation $i_1, \dots, i_k$ of $1, \dots, k$. Moreover, $\bigwedge \{x_1, \dots, x_k\}$ is $\sqcap$ -associative. Similarly for the $\sqcup$ -associative elements $x_1, \dots, x_k$ and the join of $\{x_1, \dots, x_k\}$.*

Next, we need to recall the following results related to the above subsets.

Proposition 2.8. [25] *Let $(A, \trianglelefteq, \sqcap, \sqcup)$ be a trellis and $x, a \in A$. It holds that*

- (i) *if a is right-transitive, then*
 - (a) *$a \trianglelefteq x_1 \trianglelefteq \dots \trianglelefteq x_k$, implies $a \trianglelefteq x_k$;*
 - (b) *$a \trianglelefteq x$, implies that $a \sqcup y \trianglelefteq x \sqcup y$, for any $y \in A$.*
- (ii) *if a is left-transitive, then*
 - (a) *$x_1 \trianglelefteq \dots \trianglelefteq x_k \trianglelefteq a$, implies $x_1 \trianglelefteq a$;*
 - (b) *$x \trianglelefteq a$, implies that $x \sqcap y \trianglelefteq a \sqcap y$, for any $y \in A$.*
- (iii) *if a is $\sqcap$ -associative, then $x \trianglelefteq y$ implies that $a \sqcap x \trianglelefteq a \sqcap y$, for any $x, y \in A$.*
- (iv) *if a is $\sqcup$ -associative, then $x \trianglelefteq y$ implies that $a \sqcup x \trianglelefteq a \sqcup y$, for any $x, y \in A$.*

Next, we need the following notations on a given trellis and some related results.

Notation 2.1. *Let $(A, \trianglelefteq, \sqcap, \sqcup)$ be a trellis. We denote by:*

- (i) A^{r-tr} : *the set of all right-transitive elements of A ;*
- (ii) $A^{\ell-tr}$: *the set of all left-transitive elements of A ;*
- (iii) A^{tr} : *the set of all transitive elements of A ;*

- (iv) $A^{\sqcap\text{-}ass}$: the set of all $\sqcap$ -associative elements of A ;
- (v) $A^{\sqcup\text{-}ass}$: the set of all $\sqcup$ -associative elements of A ;
- (vi) A^{ass} : the set of all associative elements of A ;
- (vii) A^{dis} : the set of all distributive elements of A .

The following proposition is an immediate consequence of Proposition 2.5.

Proposition 2.9. *Let $(A, \leq, \sqcap, \sqcup)$ be a trellis. The following chains of inclusions hold:*

- (i) $A^{dis} \subseteq A^{ass} \subseteq A^{\sqcap\text{-}ass} \subseteq A^{tr} \subseteq A^{r\text{-}tr}$;
- (ii) $A^{dis} \subseteq A^{ass} \subseteq A^{\sqcup\text{-}ass} \subseteq A^{tr} \subseteq A^{\ell\text{-}tr}$.

The following corollary is an immediate consequence of Propositions 3.7, 2.6 and 2.7.

Corollary 2.3. *Let $(A, \leq, \sqcap, \sqcup)$ be a pseudo-chain or a modular trellis. The following chain of equalities holds:*

$$A^{ass} = A^{\sqcap\text{-}ass} = A^{\sqcup\text{-}ass} = A^{tr}.$$

Next, we need to show the following result. This result completes the cases to that given by Skala in the above Theorem 2.9.

Proposition 2.10. *Let $(A, \leq, \sqcap, \sqcup)$ be a trellis and $\alpha \in \{ass, \sqcap Ass, tr\}$. If A^α is subtrellis of A , then for any finite subset $\{x_1, \dots, x_k\} \subseteq A^\alpha$ it holds that $\bigsqcup\{x_1, \dots, x_k\}$ exists and equals $x_{i_1} \sqcup \dots \sqcup x_{i_k}$ for any permutation $i_1, \dots, i_k$ of $1, \dots, k$. Moreover, $\bigsqcup\{x_1, \dots, x_k\} \in A^\alpha$.*

Proposition 2.11. *Let $(A, \leq, \sqcap, \sqcup)$ be a trellis. It holds that*

- (i) $(A^{r\text{-}tr}, \leq, \sqcap, \sqcup)$ (resp. $(A^{\ell\text{-}tr}, \leq, \sqcap, \sqcup)$) is a $\sqcup$ -semi-trellis (resp. $\sqcap$ -semi-trellis);
- (ii) $(A^{\sqcap\text{-}ass}, \leq, \sqcap, \sqcup)$ (resp. $(A^{\sqcup\text{-}ass}, \leq, \sqcap, \sqcup)$) is a $\sqcap$ -semi-trellis (resp. $\sqcup$ -semi-trellis).

Proof.

- (i) Let $x, y \in A^{r\text{-}tr}$ and let $\alpha, \beta \in A$ such that $x \sqcup y \leq \alpha \leq \beta$. Then $x \leq x \sqcup y \leq \alpha \leq \beta$ and $y \leq x \sqcup y \leq \alpha \leq \beta$. From Proposition 2.8, it follows that $x \leq \beta$ and $y \leq \beta$. Thus, $x \sqcup y \leq \beta$. Hence, $x \sqcup y \in A^{r\text{-}tr}$. Therefore, $(A^{r\text{-}tr}, \leq, \sqcup)$ is a $\sqcup$ -semi-trellis. In similar way, we prove that $(A^{\ell\text{-}tr}, \leq, \sqcap)$ is a $\sqcap$ -semi-trellis.

- (ii) Let $x, y \in A^{\sqcap-ass}$ and $\alpha, \beta \in A$. Then $((x \sqcap y) \sqcap \alpha) \sqcap \beta = (x \sqcap (y \sqcap \alpha)) \sqcap \beta = (x \sqcap ((y \sqcap \alpha) \sqcap \beta)) = (x \sqcap (y \sqcap (\alpha \sqcap \beta))) = (x \sqcap y) \sqcap (\alpha \sqcap \beta)$. Thus, $((x \sqcap y) \sqcap \alpha) \sqcap \beta = (x \sqcap y) \sqcap (\alpha \sqcap \beta)$. Hence, $(A^{\sqcap-ass}, \leq, \sqcap, \sqcup)$ is a $\sqcap$ -semi-treillis. Similarly, $(A^{\sqcup-ass}, \leq, \sqcap, \sqcup)$ is a $\sqcup$ -semi-treillis.

□

In the same line, we obtain the following propositions.

Proposition 2.12. *Let $(A, \leq, \sqcap, \sqcup)$ be a trellis. The following implications hold:*

- (i) *If $x, y, z \in A^{r-tr}$, then $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$;*
(ii) *If $x, y, z \in A^{\ell-tr}$, then $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$.*

Proof.

- (i) Let $x, y, z \in A^{r-tr}$. Since $x, y \in A^{r-tr}$, $x \leq x \sqcup y \leq (x \sqcup y) \sqcup z$, $y \leq x \sqcup y \leq (x \sqcup y) \sqcup z$ and $z \leq (x \sqcup y) \sqcup z$, it follows that $x \leq (x \sqcup y) \sqcup z$, $y \leq (x \sqcup y) \sqcup z$ and $z \leq (x \sqcup y) \sqcup z$. Then, $x \leq (x \sqcup y) \sqcup z$ and $y \sqcup z \leq (x \sqcup y) \sqcup z$. Thus, $x \sqcup (y \sqcup z) \leq (x \sqcup y) \sqcup z$. In similar way, we prove that $(x \sqcup y) \sqcup z \leq x \sqcup (y \sqcup z)$. Hence, $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$.
(ii) The proof is dual to that of (i).

□

Proposition 2.13. *Let $(A, \leq, \sqcap, \sqcup)$ be a trellis. The following statements are equivalent:*

- (i) *$(A, \leq, \sqcap, \sqcup)$ is a lattice;*
(ii) *$A = A^{\ell-tr}$;*
(iii) *$A = A^{r-tr}$;*
(iv) *$A = A^{\sqcap-ass}$;*
(v) *$A = A^{\sqcup-ass}$.*

Proof. (i) $\Leftrightarrow$ (ii): The proof of the direct implication is straightforward. For the converse implication, suppose that $A = A^{\ell-tr}$, Proposition 2.12 assures that $\sqcap$ is associative on $A^{\ell-tr}$. Since $A = A^{\ell-tr}$, it holds that the meet operation ($\sqcap$) of A is associative. Then, Theorem 2.6 guarantees that the pseudo-order relation $\leq$ is transitive. Thus, $(A, \leq, \sqcap, \sqcup)$ has a structure of a lattice.

(i) $\Leftrightarrow$ (iv): The fact that $A = A^{\sqcap-ass}$ implies that any elements of A is $\sqcap$ -associative. Thus, the meet operation is associative. Theorem 2.6 guarantees that $\leq$ is transitive and hence $(A, \leq, \sqcap, \sqcup)$ is a lattice. The converse implication is immediate.

(ii) $\Leftrightarrow$ (iii) and (iv) $\Leftrightarrow$ (v): Straightforward application of Proposition 2.12.

□

2.2.4 Lattice structures in trellises

In this subsection, we present certain subsets of a trellis that form lattices based on specific elements. These subsets are constructed with respect to the join and meet operations defined in the trellis.

Proposition 2.14. *Let $(A, \trianglelefteq, \sqcap, \sqcup)$ be a pseudo-chain. Then the set of all transitive (resp. associative) elements is a sublattice.*

Proposition 2.15. [25] *Let $(A, \trianglelefteq, \sqcap, \sqcup)$ be a modular trellis. Then the set of all transitive (resp. associative) elements is a lattice.*

Proposition 2.16. [25] *Let $(A, \trianglelefteq, \sqcap, \sqcup)$ be a trellis. Then the set of all distributive elements of A is a distributive lattice.*

Example 2.10. *Consider the modular trellis $(A = \{0, a, b, c, d, e, f, g, h, i, j, 1\}, \trianglelefteq, \sqcap, \sqcup)$ with the Hasse-type diagram displayed in Figure 2.5. Then the set $A^{ass} = A^{tr} = \{0, a, i, 1\}$ is a sublattice on A . Also, for a sub-trellis $B = \{0, b, d, f, h, j, 1\}$, it holds that B is a pseudo-chain. Thus, the set $B^{ass} = B^{tr} = \{0, 1\}$ is a sublattice on B . Moreover, $B^{ass} = B^{tr} = B^D$*

Proposition 2.17. *Let $(A, \trianglelefteq, \sqcap, \sqcup)$ be a trellis and B a subset of A^{r-tr} or A^{l-tr} . If B is sub-trellis of A , then it is a sub-lattice of A .*

Proof. We only give the proof in case $B \subseteq A^{r-tr}$. Suppose that B is sub-trellis of A . Proposition 2.12 expresses that $\sqcup$ is associative on B . Next, we show that $\sqcap$ is associative on B . Let $x, y, z \in B$, then the fact that B is a sub-trellis implies that $x \sqcap (y \sqcap z), (x \sqcap y) \sqcap z \in B$. We know that $x \sqcap (y \sqcap z) \trianglelefteq x$, $x \sqcap (y \sqcap z) \trianglelefteq y \sqcap z \trianglelefteq y$ and $x \sqcap (y \sqcap z) \trianglelefteq y \sqcap z \trianglelefteq z$. The hypothesis $B \subseteq A^{r-tr}$ implies that $x \sqcap (y \sqcap z) \trianglelefteq y$ and $x \sqcap (y \sqcap z) \trianglelefteq z$. Hence, $x \sqcap (y \sqcap z) \trianglelefteq x \sqcap y$ and thus, $x \sqcap (y \sqcap z) \trianglelefteq (x \sqcap y) \sqcap z$. In a similar way, we obtain $(x \sqcap y) \sqcap z \trianglelefteq x \sqcap (y \sqcap z)$. Hence, $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$. Thus, B is a sub-lattice of A . $\square$

Theorem 2.10. *the sub-trellis generated by three elements of a modular trellis, at least two of which are associative, is associative. Moreover, is a sublattice structure.*

Example 2.11. *Let a and b be associative elements and x be any element of a modular trellis A . Then the sub-trellis generated by $\{a, b, x\}$, with using the meet and join operations of A , forms a sublattice consisting of at most 28 distinct elements.*

Chapter 3

Classes of algebraic operations on lattices and trellises

this chapter is devoted to generalizing some of these classes of associative operations to the trellis setting. Some of these results are either inspired from or discussed in [6, 4, 7, 11, 12, 13, 15, 19, 20, 22, 27], and others are investigated during the preparation of this thesis. For further use, we recall some classes of associative operations on lattices.

3.1 Classes of associative operations on lattices

This section explores key classes of associative operations on lattices, with a focus on aggregation functions and t -norms [14]. We discuss their definitions, properties, and constructions on bounded lattices, and conclude with an application: the representation of Boolean algebras using t -norms.

3.1.1 Aggregation functions on lattices

This subsection presents the fundamental definitions and properties of aggregation operators, along with illustrative examples within the context of bounded lattices. For further details, the reader is referred to [11, 13, 15].

Definition 3.1. *Let $(L, \leq, 0, 1)$ be a bounded lattice, and $n \in L$ be fixed. A mapping $A : L^n \rightarrow L$ is called an (n -ary) aggregation function on L whenever it is increasing, $A(\mathbf{x}) \leq A(\mathbf{y})$ whenever $\mathbf{x} \leq \mathbf{y}$ (i.e. $x_1 \leq y_1, \dots, x_n \leq y_n$) and it satisfies boundary conditions*

$$A(0, \dots, 0) = 0, A(1, \dots, 1) = 1$$

Example 3.1. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and let $\varphi : L \rightarrow L$ be an unary operator defined for any $b \in L$ as follows:*

$$\varphi_b(x) = \begin{cases} 1 & x \leq b, x \neq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, φ_b is an unary aggregation operator, for any $b \in L$. Moreover, it represents a characteristic function of the principal Ideal $G(b) = \{x \in L : x \leq b\}$ generated by b , provided $b \neq 1$.

Using the same notion of an n -ary aggregation operator for the case $n = 2$, we obtain the following definition.

Definition 3.2 ([15]). *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice. A binary operation $A : L \times L \rightarrow L$ is called an aggregation operator on L if it is increasing and satisfies the boundary conditions $A(0, 0) = 0$ and $A(1, 1) = 1$.*

Let $\mathcal{A}(L)$ denote the set of all aggregation operators on L , and equip $\mathcal{A}(L)$ with the following order: for all $A, B \in \mathcal{A}(L)$,

$$A \leq B \quad \text{whenever} \quad A(x, y) \leq B(x, y), \text{ for any } x, y \in L.$$

Proposition 3.1. *Let $(L, \leq, 0, 1)$ be a bounded lattice. Then the smallest and the greatest aggregation operators in $\mathcal{A}(L)$ are, respectively, defined by*

$$A_{\perp}(x) = \begin{cases} 1 & \text{if } x = y = 1; \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad A_{\top}(x) = \begin{cases} 0 & \text{if } x = y = 0; \\ 1 & \text{otherwise.} \end{cases}$$

Thus, we have $A_{\perp} \leq A \leq A_{\top}$, for any aggregation operator $A \in \mathcal{A}(L)$.

3.1.2 T-norms on bounded lattices: constructions and representations

This subsection contains basic definitions and properties of specific associative operations on lattices. We pay particular attention to the notion of uninorms, triangular norms and triangular conorms on bounded lattices.

Definition 3.3. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and $e \in L$. A binary operation $U : L_2 \rightarrow L$ is called a uninorm if it is commutative, increasing, associative, and has e as a neutral element. If $e = 1$, then U is called a triangular norm (t -norm, for short). If $e = 0$, then U is called a triangular conorm (t -conorm, for short).*

Example 3.2. Let $(L, \leq, 0, 1)$ be a bounded lattice. Then:

(i) The meet operation $\wedge$ (resp, the join $\vee$) is a t -norm (resp, a t -conorm) on L .

(ii) The operations T_D and S_D on L , defined by

$$T_D(x, y) = \begin{cases} x \wedge y & \text{if } x = 1 \text{ or } y = 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad S_D(x, y) = \begin{cases} x \vee y & \text{if } x = 0 \text{ or } y = 0, \\ 1 & \text{otherwise,} \end{cases}$$

are a t -norm and a t -conorm, respectively.

Theorem 3.1. For any t -norm T on a bounded lattice L , we have

$$T_D \leq T \leq \wedge,$$

where T_D is the drastic t -norm, and $\wedge$ is the greatest t -norm on L .

Theorem 3.2. For any t -conorm S on a bounded lattice L , we have

$$\vee \leq S \leq S_D,$$

where S_D is the drastic t -conorm, and $\vee$ is the smallest t -conorm on L .

Theorem 3.3. Let $(L, \leq, 0, 1)$ be a bounded lattice and T a t -norm on L . Then $T = \wedge$ if and only if $T(x, x) = x$, for any $x \in L$.

Remark 3.1. Thanks to the previous theorem, in order to construct a t -norm on a bounded lattice L that is strictly different from the meet operation $\wedge$, it suffices to choose an element $c \in L$ such that $0 < c < 1$, and define a t -norm T such that $T(c, c) < c$. Indeed, on the pentagon lattice N_5 , the operation T given in Table 3.1 is a t -norm which is strictly less than the meet operation, since for example $T(a, a) = 0 < a = a \wedge a$.

$T(x, y)$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	b	0	b
c	0	0	0	c	c
1	0	a	b	c	1

Table 3.1: Hasse-type diagram of the trellis $(A, \trianglelefteq)$.

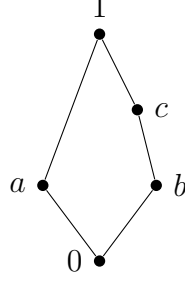


Figure 3.1: The Pentagon Lattice N_5 .

Definition 3.4. Let $(L, \leq, 0, 1)$ be a bounded lattice and T_1 and T_2 two t -norms on L . Then

- (i) If the inequality $T_1(x, y) \leq T_2(x, y)$ holds for all $(x, y) \in L \times L$, then we say that T_1 is weaker than T_2 , or equivalently, that T_2 is stronger than T_1 . In this case, we write $T_1 \leq T_2$.
- (ii) We write $T_1 < T_2$ if $T_1 \leq T_2$ and $T_1 \neq T_2$ (i.e., there exists $(x_0, y_0) \in L \times L$ such that $T_1(x_0, y_0) < T_2(x_0, y_0)$).

Proposition 3.2. Let $(L, \preceq, \wedge, \vee, 0, 1)$ be a bounded modular lattice. Then the binary operation T_Z is defined as follows:

$$T_Z(x, y) = \begin{cases} x \wedge y & \text{if } x \vee y = 1, \\ 0 & \text{otherwise,} \end{cases}$$

is a t -norm on L .

Proof. One easily verifies that T_Z is commutative, increasing and satisfies the boundary conditions. Now, we only have to show the associativity of T_Z . Let $x, y, z \in L$. On the one hand, we have that

$$T_Z(x, T_Z(y, z)) = \begin{cases} x \wedge y \wedge z & \text{if } y \vee z = 1 \text{ and } x \vee (y \wedge z) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, it holds that

$$T_Z(T_Z(x, y), z) = \begin{cases} x \wedge y \wedge z & \text{if } x \vee y = 1 \text{ and } z \vee (x \wedge y) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We will show, for instance, that $y \vee z = 1$ and $x \vee (y \wedge z) = 1$ implies $x \vee y = 1$ and $z \vee (x \wedge y) = 1$. Since $x \vee (y \wedge z) = 1$ and $\wedge$ is increasing, it holds that $x \vee y = 1$. The fact that $x \vee (y \wedge z) = 1$ implies $y = y \wedge (x \vee (y \wedge z))$. From $y = z \leq y$ and L is modular, it follows that $y = (x \wedge y) \vee (y \wedge z)$. Thus, $y \leq (x \wedge y) \vee z$. Since $z \leq (x \wedge y) \vee z$, it holds that $y \vee z \leq (x \wedge y) \vee z$. Then $(x \wedge y) \vee z = 1$. The proof of the converse implication is similar. Hence, T_Z is associative. Therefore, T_Z is a t -norm on L . $\square$

Remark 3.2. In general, T_Z is not necessarily a t -norm. Indeed, let $(L = \{0, a, b, c, d, e, 1\}, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice given by the Hasse diagram in Figure 3.2.

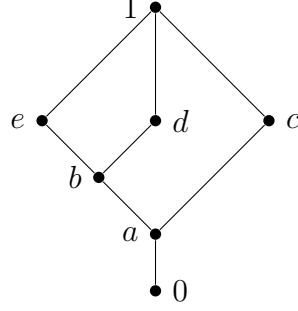


Figure 3.2: The Hasse diagram of the lattice $(L = \{0, a, b, c, d, e, 1\}, \leq)$.

Since $d \vee c = 1$, $d \wedge c = a$, and $e \vee a = e < 1$, it holds that $T_Z(e, T_Z(d, c)) = T_Z(e, a) = 0$. On the other hand, $e \vee d = 1$, $e \wedge d = b$, $b \vee c = 1$, and $b \wedge c = a$ imply that $T_Z(T_Z(e, d), c) = T_Z(b, c) = a > 0$. Hence, T_Z is not associative and therefore not a t -norm.

3.1.3 Application: T-norms based representation of boolean algebras

In this subsection, we show that if L is a complete Boolean algebra, then the set τ of all triangular norms on L contains a subset that forms a Boolean algebra isomorphic to L .

Definition 3.5. Let $(L, \leq, 0, 1)$ be a bounded lattice. A subset $\alpha \subseteq L$ such that $0 \notin \alpha$ is called a J -set in L if $0 < y \leq x$, then $y \in \alpha$, for any $x \in \alpha$ and $y \in L$.

We observe that the empty set is a J -set in L . The significance of J -sets in L is highlighted by the following results, which provide a way to characterize a general t -norm on L .

Theorem 3.4. Let $(L, \leq, 0, 1)$ be a bounded lattice and T a t -norm on L . If $\omega \subseteq L$ such that $\omega = \{x \in L : x \neq 0, T(x, x) = 0\}$. Then it holds that:

- (i) ω is a J -set in L ;
- (ii) If $a \in \omega$ and $b \leq a$, then $T(a, b) = 0$;
- (iii) If $a \in \omega$, $b \leq a$ and $c \leq a$, then $T(b, c) = 0$.

In the following result, we introduce a new family of t -norms constructed from J -sets in L , which will serve as a fundamental tool for characterizing Boolean algebras.

Theorem 3.5. $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and α a J -set in L . Then the binary operation T_α on L defined by

$$xT_\alpha y = \begin{cases} 0, & \text{if } x \neq 1 \neq y \text{ and } x \wedge y \in \alpha; \\ x \wedge y, & \text{otherwise.} \end{cases}$$

Lemma 3.1. Let B be an atomic boolean algebra and the set γ is all it's atomic with cardinality $y \geq 2$. If $\alpha, \beta \subseteq \gamma$, then it holds that $\alpha \subseteq \beta$ if and only if $T_\beta \subseteq T_\alpha$.

Now, we present the following fundamental result.

Theorem 3.6. Let B be a complete Boolean algebra in which the set γ of all atoms has cardinality $y \geq 2$. Then the set $\tau_\gamma = \{T_\alpha : \alpha \subseteq \gamma\}$ of t -norms T_α on B forms a Boolean algebra such that its operations of this Boolean algebra are given by:

$$T_\alpha \wedge T_\beta = T_{\alpha \cup \beta'}, \quad T_\alpha \vee T_\beta = T_{\alpha \cap \beta'}, \quad T'_\alpha = T_{\alpha'}.$$

Where $\alpha' = \gamma \setminus \alpha$ denotes the complement of α in γ . Furthermore, the Boolean algebra B is isomorphic to τ_γ under the identification function

$$\alpha \mapsto T_{\alpha'}, \quad \text{for any } \alpha \subseteq \gamma.$$

Example 3.3. Let $(L, \leq, \wedge, \vee, ', 0, 1)$ be a Boolean algebra in which the set of atoms is $\{a, b\}$. By Theorem 3.6, all triangular norms on L are given by:

$$T_{\{a,b\}} = T_\gamma, \quad T_{\{a\}}, \quad T_{\{b\}}, \quad \text{and} \quad T_\emptyset = \wedge.$$

and they are defined as follows:

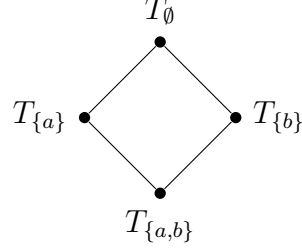
T_γ	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
1	0	a	b	1

$T_{\{a\}}$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
1	0	a	b	1

$T_{\{b\}}$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
1	0	a	b	1

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

These t -norms form a Boolean algebra under the partial order on t -norms.



3.2 Classes of associative operations on trellises

In this section, we explore the use of associative operations on bounded trellises. More precisely, we generalize the classes of triangular norms (t-norms) and triangular conorms (t-conorms), which are originally defined on bounded lattices, to the broader setting of bounded trellises. We also examine how these operations behave when the transitivity property is no longer assumed. First, we need to recall the following notations on a given trellis.

Notation 3.1. *Let $(A, \trianglelefteq, \sqcap, \sqcup, 0, 1)$ be a bounded trellis. We denote by:*

- (i) $\mathcal{UN}_e(A)$: *the class (or set) of all uninorms on A .*
- (ii) $\mathcal{TN}_1(A)$: *the class (or set) of all triangular norms (t-norms) on A .*
- (iii) $\mathcal{TC}_0(A)$: *the class (or set) of all triangular conorms (t-conorms) on A .*

Remark 3.3. (i) *Since the operations $\sqcap$ and $\sqcup$ in a bounded trellis $(\trianglelefteq, \sqcap, \sqcup, 0, 1)$ are not necessarily associative, the operations $\sqcap$ and $\sqcup$ are not necessarily elements of $\mathcal{TN}_1(A)$ and $\mathcal{TC}_0(A)$, respectively.*

- (ii) *As noted in (i), we have chosen to use the same names for these classes on bounded trellises as those used for bounded lattices. However, although these classes are generalized to the trellis setting, they do not constitute extensions of the original classes in terms of set-theoretic inclusion.*

Next, we present some elements from the classes $\mathcal{TN}_1(A)$ and $\mathcal{TC}_0(A)$.

Example 3.4. *Let $(A, \trianglelefteq, \sqcap, \sqcup, 0, 1)$ be a bounded trellis and T_W and S_W two binary operations defined on A as follow:*

$$T_W(x, y) = \begin{cases} x \sqcap y & \text{if } x = 1 \text{ or } y = 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad S_W(x, y) = \begin{cases} x \sqcup y & \text{if } x = 0 \text{ or } y = 0, \\ 1 & \text{otherwise.} \end{cases}$$

One easily verifies that T_W (resp. S_W) is the smallest t-norm (resp. the greatest t-conorm) on A . Moreover, $T_W \in \mathcal{TN}_1(A)$ and $S_W \in \mathcal{TC}_0(A)$.

In the following illustrative example, we give other t-norms (resp. t-conorms) on a bounded trellis greater (resp. smaller) than T_W (resp. S_W).

Example 3.5. Let $(A = \{0, a, b, c, 1\}, \preceq, \sqcap, \sqcup)$ be a bounded trellis given by the Hasse-type diagram in Figure 2.4 and T_i , $i \in \{1, \dots, 6\}$ are binary operations defined by the following tables:

$T_1(x, y)$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	0	b
c	0	0	0	0	c
1	0	a	b	c	1

$T_2(x, y)$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	0	b
c	0	0	0	b	c
1	0	a	b	c	1

$T_3(x, y)$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	b	b	b
c	0	0	b	b	c
1	0	a	b	c	1

$T_4(x, y)$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	0	b
c	0	0	0	c	c
1	0	a	b	c	1

$T_5(x, y)$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	b	b
c	0	0	b	c	c
1	0	a	b	c	1

$T_6(x, y)$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	b	b	b
c	0	0	b	c	c
1	0	a	b	c	1

One easily verifies that T_i are all t-norms on A such that $T_w = T_1 \triangleleft T_i \triangleleft T_6$, for any $i \in \{1, \dots, 6\}$.

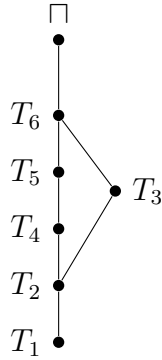


Figure 3.3: Hasse-type diagram of the bounded lattice of the six t-norms $\{T_1, \dots, T_6\}$.

On a bounded modular lattice L [6], the binary operation T_Z defined by

$$T_Z(x, y) = \begin{cases} x \wedge y & \text{if } x \vee y = 1; \\ 0 & \text{otherwise;} \end{cases}$$

is a t-norm. This example was extended to the setting of bounded modular trellises under a suitable necessary and sufficient condition [27].

Proposition 3.3. [27] *Let $(A, \leq, \sqcap, \sqcup, 0, 1)$ be a bounded modular trellis. Then it holds that $T_Z \in \mathcal{TN}_1(A)$ if the following condition is satisfied:*

$$(\forall (x, y, z, t) \in A^4)((x \sqcap y \neq 0) \text{ and } (x \sqcup y = 1)) \text{ implies } (x \sqcup z) \sqcup (y \sqcup t) = 1).$$

Dually, we obtain the following example.

On a bounded modular lattice L [6], the binary operation S_Z defined by

$$S_Z(x, y) = \begin{cases} x \vee y & \text{if } x \wedge y = 0; \\ 1 & \text{otherwise;} \end{cases}$$

is a t-conorm. This example was extended to the setting of bounded modular trellises under a suitable necessary and sufficient condition [27].

Proposition 3.4. *Let $(A, \leq, \sqcap, \sqcup, 0, 1)$ be a bounded modular trellis. Then it holds that $S_Z \in \mathcal{TC}_0(A)$ if the following condition is satisfied:*

$$(\forall (x, y, z, t) \in A^4)((x \sqcup y \neq 1) \text{ and } (x \sqcap y = 0)) \text{ implies } (x \sqcap z) \sqcap (y \sqcap t) = 0).$$

In the following example, we give a bounded modular trellis does not satisfying the necessary and sufficient condition.

Example 3.6. *let $(A = \{0, a, b, c, d, 1\}, \leq, \sqcap, \sqcup)$ be a bounded modular trellis given by the Hasse-type diagram in Figure 2.7. Let $x, y \in A$ such that $x = b$ and $y = d$. Then $b \sqcup d = 1$ and $b \sqcap d = a \neq 0$. The fact that $b \leq c$ implies $(b \sqcup c) \sqcup (d \sqcup d) = d \neq 1$. Thus, the t-norm Z on A defined as follow:*

$T_Z(x, y)$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	a	b
c	0	0	0	0	0	c
d	0	0	a	0	0	d
1	0	a	b	c	d	1

is not increasing (i.e., $T_Z(b, d) = a \not\leq 0 = T_Z(c, d)$). Thus, T_Z is not a t-norm.

In the following remark, we show that the greatest t-norm does not necessarily exist in a given bounded trellis. This illustrates one of the properties that may be lost when considering bounded trellises instead of bounded lattices.

Remark 3.4. On a given bounded lattice L , it is well known that the classes $\mathcal{TC}_0(A)$ and $\mathcal{TN}_1(A)$ contains least and greatest elements. In the trellis setting, Example 3.5 shows that $\mathcal{TN}_1(A)$ have a greatest element. However, $\mathcal{TC}_0(A)$ does not necessarily have a least element. Indeed, consider the bounded trellis $(A = \{0, a, b, c, d, e, 1\}, \preceq, \sqcap, \sqcup)$ given in Figure 3.4. The binary operations T_1, T_2 and T_3 on A , defined by the tables below, are incomparable maximal elements in $\mathcal{TN}_1(A)$ [27].

T_1	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	a
b	0	0	b	b	0	b	b
c	0	0	b	c	0	c	c
d	0	0	0	0	a	a	d
e	0	0	b	c	a	c	e
1	0	a	b	c	d	e	1

T_2	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	a
b	0	0	b	b	0	b	b
c	0	0	b	c	0	c	c
d	0	0	0	0	d	0	d
e	0	0	b	c	0	e	e
1	0	a	b	c	d	e	1

T_3	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	0	0	0	0	a	a
b	0	0	b	b	0	b	b
c	0	0	b	c	0	c	c
d	0	0	0	0	0	a	d
e	0	a	b	c	a	e	e
1	0	a	b	c	d	e	1

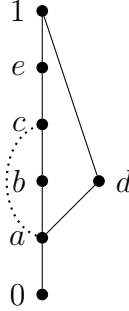


Figure 3.4: The Hasse-type diagram of the trellis $(A = \{0, a, b, c, d, e, 1\}, \preceq)$.

3.3 New constructions of T-norms based on specific subsets on trellises

In this section, we focus on constructing new elements of the class $\mathcal{TN}_1(A)$ on a bounded trellis $(A, \preceq, \sqcap, \sqcup, 0, 1)$. Since the dual class $\mathcal{TC}_0(A)$ can be developed in an analogous manner, we restrict our attention to $\mathcal{TN}_1(A)$ without loss of generality.

For a given trellis $(A, \preceq, \sqcap, \sqcup)$ and $B \subseteq A$, we recall the definition of the following mapping $\lambda_B: A \rightarrow A$:

$$\lambda_B(x) = \bigsqcup \{a \in B \mid a \preceq x\} = \bigsqcup (B \cap \downarrow x).$$

Remark 3.5. In general, the mapping λ_B is not necessarily well-defined, as the supremum $\sqcup(B \sqcap \downarrow x)$ may fail to exist. However, if B is a finite subset of A^{r-tr} , then by Theorem 2.9, the mapping λ_B is well-defined.

The following proposition is immediate.

Proposition 3.5. Let $(A, \sqsubseteq, \sqcap, \sqcup)$ be a trellis and B a finite subset of A^{r-tr} . For any $x \in A$, it holds that

- (i) $\lambda_B(x) \sqsubseteq x$;
- (ii) if $x \in B$, then $\lambda_B(x) = x$.

The following proposition lists additional properties of λ_B in case B is a $\sqcup$ -sub-trellis of A . Since λ_B behaves as the identity on B and maps to B , we refer to this mapping as an embedding.

Proposition 3.6. Let $(A, \sqsubseteq, \sqcap, \sqcup)$ be a trellis and B a finite subset of A^{r-tr} . If B is a $\sqcup$ -sub-trellis of A , then it holds that

- (i) $\lambda_B(A) \subseteq B$;
- (ii) λ_B is idempotent, i.e., $\lambda_B(\lambda_B(x)) = \lambda_B(x)$, for any $x \in A$;
- (iii) λ_B is increasing, i.e., if $x \sqsubseteq y$, then $\lambda_B(x) \sqsubseteq \lambda_B(y)$, for any $x, y \in A$.

The following proposition shows the relationship between the structure of B and the $\sqcap$ -homomorphism property of λ_B . First, we recall the definition of a $\sqcap$ -homomorphism between trellises. Let $(A, \sqsubseteq, \sqcap, \sqcup)$ and $(A', \sqsubseteq', \sqcap', \sqcup')$ be two trellises. A mapping $\varphi : A \rightarrow A'$ is called a $\sqcap$ -homomorphism (between the given trellises), if $\varphi(x \sqcap y) = \varphi(x) \sqcap' \varphi(y)$, for any $x, y \in A$.

Proposition 3.7. Let $(A, \sqsubseteq, \sqcap, \sqcup)$ be a trellis and B a finite subset of A^{r-tr} .

- (i) If B is a sub-trellis of A , then λ_B is a $\sqcap$ -homomorphism.
- (ii) If B is a $\sqcup$ -sub-trellis of A and λ_B is a $\sqcap$ -homomorphism, then B is a sub-trellis of A .

Proposition 3.8. Let $(A, \sqsubseteq, \sqcap, \sqcup)$ be a trellis and B a finite subset of A^{r-tr} . If B is a $\sqcup$ -sub-trellis of A , then

$$\lambda_B(x \sqcap (y \sqcap z)) = \lambda_B((x \sqcap y) \sqcap z)$$

for any $x, y, z \in A$.

In this subsection, we show how to construct a t-norm on a bounded trellis based on the meet operation and an appropriate subset of that trellis. For a given bounded trellis $(A, \trianglelefteq, \sqcap, \sqcup, 0, 1)$ and a finite subset B of A^{r-tr} , we define the following binary operation on T :

$$T^{[B]}(x, y) = \begin{cases} y & , \text{ if } x = 1 \\ x & , \text{ if } y = 1 \\ \lambda_B(x \sqcap y) & , \text{ otherwise} \end{cases}$$

In view of Remark 3.5, $T^{[B]}$ is well defined. Note that if $B \subseteq C$ are finite subsets of A^{r-tr} , then $T^{[B]} \trianglelefteq T^{[C]}$. Note that if B is a sub-trellis of A , then we can express $T^{[B]}$ as written below. It shows that $T^{[B]}$ essentially computes meets.

Proposition 3.9. *Let $(A, \trianglelefteq, \sqcap, \sqcup, 0, 1)$ be a bounded trellis and B a finite subset of A^{r-tr} . If B is a sub-trellis of A , then*

$$T^{[B]}(x, y) = \begin{cases} y & , \text{ if } x = 1 \\ x & , \text{ if } y = 1 \\ x \sqcap y & , \text{ if } x, y \in B \\ \lambda_B(x) \sqcap \lambda_B(y) & , \text{ otherwise} \end{cases}$$

For any $(x, y) \in (A \setminus \{1\})^2$ it holds that $T^{[B]}(x, y) \in B$.

Remark 3.6. *We list some basic observations:*

- (i) If $B = \{0, 1\}$, then $T^{[B]} = T_D$.
- (ii) If $(A, \trianglelefteq, \sqcap, \sqcup, 0, 1)$ is a bounded lattice, then $T^{[A^\alpha]} = \sqcap$, for any $\alpha \in \{ass, \sqcap-ass, \sqcup-ass, tr, r-tr\}$.

The following theorem shows that $T^{[B]}$ is a t-norm on A when B is a sub-trellis.

Theorem 3.7. *Let $(A, \trianglelefteq, \sqcap, \sqcup, 0, 1)$ be a bounded trellis and B a finite subset of A^{r-tr} . If B is a sub-trellis of A , then the binary operation $T^{[B]}$ is a t-norm on A .*

Using Propositions 2.16, Corollary 2.3 and Theorem 3.7, we obtain the following corollaries.

Corollary 3.1. *Let $(A, \trianglelefteq, \sqcap, \sqcup, 0, 1)$ be a bounded trellis and $\alpha \in \{ass, \sqcap-ass, \sqcup-ass, tr, r-tr\}$. If A^α is a finite sub-trellis of A , then the binary operation $T^{[A^\alpha]}$ is a t-norm on A .*

Corollary 3.2. *Let $(A, \preceq, \sqcap, \sqcup, 0, 1)$ be a bounded trellis. If A^{dis} is finite, then the binary operation $T^{[A^{dis}]}$ is a t -norm on A .*

Example 3.7. *Let $(\{0, a, b, c, d, 1\}, \preceq)$ be the bounded trellis given by the Hasse-type diagram in Figure 3.5. One easily verifies that*

$$A^{r-tr} = \{0, b, c, d, e, 1\}$$

$$A^{tr} = \{0, b, e, 1\}$$

$$A^{\sqcap-ass} = \{0, b, 1\}$$

$$A^{\sqcup-ass} = \{0, e, 1\}$$

$$A^{dis} = A^{ass} = \{0, 1\}$$

The above subsets are all sub-trellises of A . Hence, the binary operations $T^{[A^\alpha]}$, with $\alpha \in \{dis, ass, \sqcap-ass, \sqcup-ass, tr, r-tr\}$, are t -norms on A .

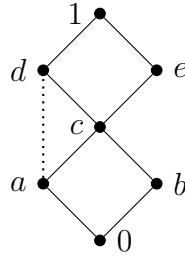


Figure 3.5: The Hasse-type diagram of the trellis $(A = \{0, a, b, c, d, e, 1\}, \preceq)$.

Before providing the t -norms $T^{[A^\alpha]}$, we present the embeddings λ_B with $B = A^\alpha$ and $\alpha \in \{dis, ass, \sqcap-ass, \sqcup-ass, tr, r-tr\}$ in the following table.

	0	a	b	c	d	e	1
$\lambda_{A^{dis}}$	0	0	0	0	0	0	1
$\lambda_{A^{ass}}$	0	0	0	0	0	0	1
$\lambda_{A^{\sqcap-ass}}$	0	0	b	b	b	b	1
$\lambda_{A^{\sqcup-ass}}$	0	0	0	0	0	e	1
$\lambda_{A^{tr}}$	0	0	0	0	b	e	1
$\lambda_{A^{r-tr}}$	0	0	b	c	d	d	1

It is clear that $T^{[A^{dis}]} = T^{[A^{ass}]} = T_D$. Then the remaining t -norms are listed the following tables.

$T^{[A^{\sqcap-ass}]}$	a	b	c	d	e
a	0	0	0	0	0
b	0	b	b	b	b
c	0	b	b	b	b
d	0	b	b	b	b
e	0	b	b	b	b

$T^{[A^{tr}]}$	a	b	c	d	e
a	0	0	0	0	0
b	0	b	b	b	b
c	0	b	b	b	b
d	0	b	b	b	b
e	0	b	b	b	e

$T^{[A^{r-tr}]}$	a	b	c	d	e
a	0	0	0	0	0
b	0	b	b	b	b
c	0	b	c	c	c
d	0	b	c	d	c
e	0	b	c	c	e

Moreover, The greatest t -norm on A is given by the following table.

T	a	b	c	d	e
a	0	0	0	0	a
b	0	b	b	b	b
c	0	b	c	c	c
d	0	b	c	d	c
e	a	b	c	c	e

Proposition 3.10. *Let $(A, \trianglelefteq, \sqcap, \sqcup, 0, 1)$ be a bounded modular trellis and $\alpha \in \{ ass, \sqcap-ass, \sqcup-ass, tr \}$. If A^α is finite, then $T^{[A^\alpha]}$ is a t -norm on A . Moreover, $T^{[A^{ass}]} = T^{[A^{\sqcap-ass}]} = T^{[A^{\sqcup-ass}]} = T^{[A^{tr}]}$*

Proof. Let $\alpha \in \{ ass, \sqcap-ass, \sqcup-ass, tr \}$. Corollary 2.3 and Proposition 2.15 express that $A^{ass} = A^{\sqcap-ass} = A^{\sqcup-ass} = A^{tr}$ are sub-trellises of A . Theorem 3.7 guarantees that $T^{[A^\alpha]}$ is a t -norm on A . $\square$

General conclusions

In this thesis, we presented key concepts of partially ordered sets and lattices, then extended these ideas to pseudo-ordered sets and trellises. We studied classes of associative operations on these structures, focusing on properties such as commutativity, increasingness, and neutral elements. These results generalize classical notions like t-norms and provide new constructions on trellises, broadening the algebraic framework for applications in fuzzy logic and ordered structures.

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ملخص

في هذه الأطروحة، نقدم فئات متعددة من العمليات الجبرية المعرفة على البنى المرتبة، مع تركيز خاص على الشبكات المحدودة والمشبكات. ولهذا الغرض، نستعرض أولا المفاهيم والخصائص الأساسية للعلاقات الثنائية والمجموعات الجزئية المرتبة والشبكات. بعد ذلك، نقوم بتوسيع الإطار الكلاسيكي من خلال إدخال مفاهيم المجموعات الشبه مرتبة والمشبكات، والتي تعمم مفهوم الترتيب الجزئي من خلال تخفيف شرط التعدية. وأخيرا، نركز على العمليات الثنائية الترابطية المعرفة على الشبكات المحدودة والمشبكات، ونقترح طرقا جديدة لبناء معايير التقاطع على المشبكات.

كلمات مفتاحية:

شبكة، مشبكة، عملية ثنائية، معيار تقاطع، معيار اتحاد.

Abstract

In this thesis, we present various classes of algebraic operations on ordered structures, with a particular emphasis on bounded lattices and trellises. To this end, we first recall the essential concepts and properties of binary relations, partially ordered sets, and lattices. We then extend the classical framework by introducing pseudo-ordered sets and trellises, which generalize the notion of partial orders by relaxing the transitivity condition. Finally, we focus on associative binary operations defined on bounded lattices and trellises, and we propose novel methods for constructing t-norms on trellises.

Key words :

Lattice, Trellis, Binary Operation, T-norm, T-conorm.

Résumé

Dans cette thèse, nous présentons différentes classes d'opérations algébriques définies sur des structures ordonnées, avec un accent particulier sur les treillis bornés et les trellises. À cette fin, nous rappelons d'abord les concepts et propriétés essentiels des relations binaires, des ensembles partiellement ordonnés et des treillis. Nous étendons ensuite le cadre classique en introduisant les ensembles pseudo-ordonnés et les trellises, qui généralisent la notion d'ordre partiel en relâchant la condition de transitivité. Enfin, nous nous concentrons sur les opérations binaires associatives définies sur les treillis bornés et les trellises, et nous proposons des méthodes novatrices pour la construction de t-normes sur les trellises.

Mots-clés :

Treillis, Trellis, Opération binaire, T-norme, T-conorme. .