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Theme

Existence and Uniqueness of Solutions for Nonlinear Ψ -Fractional Differential Equations and their System

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Imane

Notations

$\mathbb{N}$	Natural numbers $\{0,1,2,3,\ldots\}$.
$\mathbb{N}^*$	Nonzero natural numbers $\{1,2,3,\ldots\}$.
$\mathbb{R}$	Real numbers $(-\infty,\infty)$.
$\mathbb{R}_+$	Positive real numbers $(0,\infty)$.
$\mathbb{R}^*$	Nonzero real numbers $(-\infty,0)\cup(0,\infty)$.
$\mathbb{C}$	Complex numbers, $z\in\mathbb{C}$, then $z=x+iy$, where $x,y\in\mathbb{R}$, and $i^2=-1$.
$Re(\alpha)$	Real part of complex α.
$[a,\ b]$	$(-\infty<a<b<+\infty)$ be a finite interval on the real-axis $\mathbb{R}$.
$C([a,b])$	The BANACH space of all continuous functions from $[a,b]$ into $\mathbb{R}$.
FDE	Fractional Differential Equation.
FPDE	Fractional Partial Differential Equation.
$\Gamma(\cdot)$	EULER gamma function.
$B(\cdot,\cdot)$	Beta function.
$E_\alpha(\cdot)$	Standard MITTAG-LEFFLER function.
$E_{\alpha,\beta}(\cdot)$	MITTAG-LEFFLER function in two arguments α and β.

$\mathscr{I}^{\alpha}u$	RIEMANN-LIOUVILLE's fractional integral of order α.
${}^{RL}\mathscr{D}^{\alpha}u$	RIEMANN-LIOUVILLE's fractional derivative of order α.
${}^{C}\mathscr{D}^{\alpha}u$	CAPUTO's fractional derivative of order α.
${}^{H}\mathscr{I}^{\alpha}u$	HADAMARD's fractional integral of order α.
${}^{H}\mathscr{D}^{\alpha}u$	HADAMARD's fractional derivative of order α.
${}^{\rho}\mathscr{I}^{\alpha}u$	KATUGAMPOLA's fractional integral of order α.
${}^{\rho}\mathscr{D}^{\alpha}u$	KATUGAMPOLA's fractional derivative of order α.
$\mathscr{I}^{\alpha:\psi}u$	ψ-fractional integral of order α.
$\mathscr{D}^{\alpha:\psi}u$	ψ-fractional derivative of order α.

Contents

General introduction

In recent years, the factional calculus has caught the attention of many researchers over the last few decades as it is a solid and growing work both in theory and in its applications. The history of the theory goes back to the 17th century, when in 1695 the derivative of order $\alpha = \frac{1}{2}$ was described by LEIBNITZ in his letter to L'HOSPITAL. Since then, the new theory turned out to be very attractive to mathematicians as well as physicists, biologists, engineers and economists.

The classical fractional calculus is based on several definitions for the operators of integration and derivatives such as: RIEMANN-LIOUVILLE, HADAMARD and KATUGAMPOLA. As a result, it was necessary to introduce a fractional derivative of a function f with respect to another function is called ψ-fractional derivative which generalizes the three derivatives into a single form. This new fractional integral and derivative appeared in several recent articles and books, for example see [17].

Fractional differential equations have been used in the study of models of many phenomena in various fields of science and engineering. Moreover, the study of the systems of (FDE) has become more and more popular tool for controlling and modeling different systems.

This thesis is organized as follows, in the first chapter which is concerned with some basic concepts of fractional calculus, and some definitions and results that are important for the study, also we give definitions and some properties of fractional integrals and fractional derivatives of various types, especially the ψ-fractional integral and derivative and its properties. Thus the fixed point theory is a powerful mathematical tool in the study of the existence and uniqueness.

We will discuss in the second chapter, the existence and uniqueness of solutions for the ψ-operator of a nonlinear FPDE with boundary conditions:

$$\begin{cases} \mathscr{D}^{\alpha;\psi} u(t) + f(t, u(t)) = 0, \ 0 < t < T, \ 1 < \alpha < 2, \\ u(0) = 0, \ u(T) = 0, \end{cases}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. We will proof the existence and uniqueness by means of BANACH and SCHAEFER fixed point theorems and we will present some examples to show the effectiveness of our main results.

In the third chapter, we are interested in the existence and uniqueness of system of n ψ-fractional differential equations with boundary conditions:

$$\begin{cases} \mathscr{D}^{\alpha;\psi} u_i(t) + f_i(t, u(t)) = 0, \ 0 < t < T, \ 1 < \alpha < 2, \\ u_i(0) = 0, \ u_i(T) = 0, \end{cases}$$

where $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$, for $n \in \mathbb{N}^*$ and $f_i : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions for every $i \in \overline{1, n}$. We will need to use the BANACH and SCHAEFER fixed point theorems to proof our main results. Finally, our study is ended by examples illustrating the obtained results.

Chapter 1

Basic concepts and elements of fractional calculus

In this chapter, we will present some specific functions, and the necessary tools that we need in our thesis concerning the fractional integrations and derivatives also giving the most important properties of these notions, also we will recall the famous fixed point theorems that we will use to obtain various existence results.

1.1 Basic fractional calculus

In this section, we will begin by giving some special functions and functional spaces as follows:
Let $[a,b]$ be a finite interval on the real-axis $\mathbb{R}$, and $C([a,b]), AC^n([a,b]), C^n([a,b])$ be the space of continuous functions, n-times absolutely continuous, n-times continuous differentiable functions on $[a, b]$, respectively. The space of the continuous functions y on $[a,b]$ with the norm is defined by [17]

$$\|y\| = \max_{a<t<b} |y(t)|.$$

On the other hand, we have n-times absolutely continuous given by [17]

$$AC^n([a,b]) = \left\{y : [a,b] \to \mathbb{R} : \ y^{n-1} \in AC([a,b])\right\}.$$

1.1.1 Special functions

Here, we give some information on the Euler gamma, the Beta and Mittag-Leffter functions which play the most important role in the theory of fractional calculus.

Definition 1.1 (Euler gamma function[**17**])**.** *The* EULER *gamma function* $\Gamma(z)$ *is defined by*

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} \mathrm{d}t, \ (Re(z) > 0, \ z \in \mathbb{C}),$$

where $t^{z-1} = e^{(z-1)\ln t}$.

Example 1.1. *1.* $\Gamma(1) = \int_0^{+\infty} e^{-t} \mathrm{d}t = 1$.

2. $\Gamma(\frac{1}{2}) = \int_0^{+\infty} t^{\frac{1}{2}-1} e^{-t} \mathrm{d}t = \sqrt{\pi}$.

Lemma 1.1. *(**[17]**) For every* $z \in \mathbb{C}$, $Re(z) > 0$, $n \in \mathbb{N}^*$, *we have*

1. $\Gamma(z+1) = z\Gamma(z)$.

2. $\Gamma(n+1) = (n-1)!$.

3. $\Gamma(n+\frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{4^n n!}$.

Property 1.1. *(**[17]**) For every* $p > 0$, *we have*

$$\Gamma(p) = \lim_{n \to +\infty} \frac{n! n^p}{p(p+1)(p+2)\ldots\ldots(p+n)}.$$

Definition 1.2 (Beta function[**17**])**.** *The Beta function is a type of* EULER *integral defines by :*

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} \, \mathrm{d}t, \ (p,q \in \mathbb{C}, \ Re(p) > 0, \ Re(q) > 0).$$

For every $p,q \in \mathbb{C}$, *with* $Re(p) > 0$, $Re(q) > 0$, *we have*

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} .$$

Definition 1.3 (Mittag-Leffter function[**17**])**.** *The* MITTAG-LEFFTER *function is defined by*

$$E_\alpha(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(\alpha k+1)}, \ \alpha > 0.$$

The generalized MITTAG-LEFFTER *function is defined by*

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(\alpha k+\beta)},\ \alpha,\beta > 0.$$

Theorem 1.1. *([17]) For $\alpha = n \in \mathbb{N},\ \lambda \in \mathbb{R}$, we have*

$$\left(\frac{d}{dx}\right)^n E_n(\lambda x^n) = \lambda E_n(\lambda x^n),$$

$$\left(\frac{d}{dx}\right)^n x^{\beta-1} E_{n,\beta}(\lambda x^n) = \lambda x^{\beta-n-1} E_n(\lambda x^n).$$

1.2 Fractional integrations operators

1.2.1 Riemann-Liouville fractional integration

Definition 1.4. *([17]) Let $[a,b]$ be a finite interval on the real-axis $\mathbb{R}$. The* RIMANN-LIOUVILLE *fractional integrals (left-sided and right-sided) of order $\alpha \in \mathbb{R}_+^*$, are defined by*

$$^{RL}\mathscr{I}^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(t-x)^{1-\alpha}} \mathrm{d}t,\ \ x > a, \tag{1.1}$$

and

$$^{RL}\mathscr{I}^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} \mathrm{d}t,\ \ x < b,$$

respectively.

Property 1.2. *([17]) For $\alpha > 0,\ \beta > 0$, we have*

1. $\left(^{RL}\mathscr{I}^{\alpha}_{a+} (x-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (x-a)^{\alpha+\beta-1}$.
2. $\left(^{RL}\mathscr{I}^{\alpha}_{b-} (b-x)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (b-x)^{\alpha+\beta-1}$.

1.2.2 Hadamard fractional integration

Definition 1.5. *([17]) Let $[a,b]$ be a finite or infinite interval of the half-axis $\mathbb{R}$. We consider the* HADAMARD *fractional integrals (left-sided and right-sided) of order $\alpha \in$*

$\mathbb{C}(Re(\alpha) > 0)$, *are defined by*

$$ {}^{H}\mathscr{I}^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)\,\mathrm{d}t}{t},\ a<x<b, \tag{1.2} $$

and

$$ {}^{H}\mathscr{I}^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{f(t)\mathrm{d}t}{t},\ a<x<b\ , $$

respectively.

Property 1.3. *([17]) If* $Re(\alpha) > 0,\ Re(\beta) > 0$, *and* $0 < a < b < +\infty$, *then*

1. $\left({}^{H}\mathscr{I}^{\alpha}_{a+} \left(\ln \frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \left(\ln \frac{x}{a}\right)^{\alpha+\beta-1}$.
2. $\left({}^{H}\mathscr{I}^{\alpha}_{b-} \left(\ln \frac{b}{t}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \left(\ln \frac{b}{x}\right)^{\alpha+\beta-1}$.

1.2.3 Katugampola fractional integration

Definition 1.6. *([17]) Let* $[a, b]$ *be a finite interval on the real-axis* $\mathbb{R}$. *We consider the* KATUGAMPOLA *fractional integrals (left-sided and right-sided) of order* $\alpha \in \mathbb{C}\ (Re(\alpha) > 0)$, *are defined by*

$$ {}^{\rho}\mathscr{I}^{\alpha}_{a+} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1} f(t)}{(x^{\rho} - t^{\rho})^{(\alpha-1)}}\,\mathrm{d}t,\ \rho > 0,\ a<x<b, \tag{1.3} $$

and

$$ {}^{\rho}\mathscr{I}^{\alpha}_{b-} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1} f(t)}{(t^{\rho} - x^{\rho})^{(\alpha-1)}}\,\mathrm{d}t,\ \rho > 0,\ a<x<b, \tag{1.4} $$

respectively.

1.2.4 Ψ-fractional integration

Definition 1.7. *([17]) Let* $[a, b]$ *be a finite or infinite interval of the real-axis* $\mathbb{R}$ *and* $\alpha > 0$. *Also let* $\psi(x)$ *be an increasing and positive monotone function on* $(a, b]$, *having a continuous derivative* $\psi'(x)$ *on* (a, b) *The left and right-sided fractional integrals of a function* f *with respect to another function* ψ *on* $[a, b]$, *are defined by*

$$ \mathscr{I}^{\alpha:\psi}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} f(t)\,\mathrm{d}t, \tag{1.5} $$

and

$$\mathscr{I}_{b-}^{\alpha:\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t)(\psi(t) - \psi(x))^{\alpha-1} f(t)\,\mathrm{d}t, \tag{1.6}$$

respectively.

Lemma 1.2. *([17]) Let $\alpha > 0$ and $\beta > 0$. then, we have the following semi-group property given by*

$$\mathscr{I}_{a+}^{\alpha:\psi} \mathscr{I}_{a+}^{\beta:\psi} f(x) = \mathscr{I}_{a+}^{\alpha+\beta:\psi} f(x).$$

and

$$\mathscr{I}_{b-}^{\alpha:\psi} \mathscr{I}_{b-}^{\beta:\psi} f(x) = \mathscr{I}_{b-}^{\alpha+\beta:\psi} f(x).$$

Lemma 1.3. *([17]) Let $\alpha > 0$ and $\delta > 0$.*

1. *if $f(x) = (\psi(x) - \psi(a))^{\delta-1}$, then*

$$\mathscr{I}_{a+}^{\alpha:\psi} f(x) = \frac{\Gamma(\delta)}{\Gamma(\delta+\alpha)} (\psi(x) - \psi(a))^{\alpha+\delta-1}.$$

2. *if $g(x) = (\psi(b) - \psi(x))^{\delta-1}$, then*

$$\mathscr{I}_{b-}^{\alpha:\psi} g(x) = \frac{\Gamma(\delta)}{\Gamma(\delta+\alpha)} (\psi(b) - \psi(x))^{\alpha+\delta-1}.$$

1.3 Fractional derivatives operators

1.3.1 Riemann-Liouville fractional derivative

Definition 1.8. *([17]) Let $f \in AC^n(a, b)$, $n-1<\alpha<n$, $n \in \mathbb{N}$. The* RIMANN-LIOUVILLE *fractional derivatives (left-sided and right-sided) of function f of order α, are given by*

$$\begin{aligned}\mathscr{D}_{a+}^{\alpha} f(x) &= \left(\frac{d}{dx}\right)^n {}^{RL}\mathscr{I}_{a+}^{n-\alpha} f(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} f(t)\,\mathrm{d}t,\end{aligned} \tag{1.7}$$

and

$$\begin{aligned}\mathscr{D}^{\alpha}_{b-}f(x) &= (-1)^n\left(\frac{d}{dt}\right)^n {}^{RL}\mathscr{I}^{n-\alpha}_{b-}f(x)\\ &= \frac{(-1)^n}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_x^b (t-x)^{n-\alpha-1}f(t)\,\mathrm{d}t, \qquad (1.8)\end{aligned}$$

respectively.

1.3.2 Hadamard fractional derivative

Definition 1.9. *([17]) Let $f(x)\in AC^n(a,\ b)$, $n-1<\alpha<n,\ n\in\mathbb{N}$. The* HADAMARD *fractional derivatives (left-sided and right-sided) of function f of order $\alpha\in\mathbb{C}$, $(Re(\alpha)>0)$ on $(a,\ b)$, are given by*

$$\begin{aligned}{}^{H}\mathscr{D}^{\alpha}_{a+}f(x) &= \left(x\frac{d}{dx}\right)^n {}^{H}\mathscr{I}^{n-\alpha}_{a+}f(x)\\ &= \frac{1}{\Gamma(n-\alpha)}\left(x\frac{d}{dx}\right)^n\int_a^x \left(\ln\frac{x}{t}\right)^{(n-\alpha+1)}\frac{f(t)\,\mathrm{d}t}{t},\ a<x<b, \qquad (1.9)\end{aligned}$$

and

$$\begin{aligned}{}^{H}\mathscr{D}^{\alpha}_{b-}f(x) &= \left(-x\frac{d}{dx}\right)^n {}^{H}\mathscr{I}^{n-\alpha}_{b-}f(x)\\ &= \frac{1}{\Gamma(n-\alpha)}\left(-x\frac{d}{dx}\right)^n\int_x^b \left(\ln\frac{t}{x}\right)^{(n-\alpha+1)}\frac{f(t)\mathrm{d}t}{t},\ a<x<b, \qquad (1.10)\end{aligned}$$

respectively.

Property 1.4. *([17]) If $Re(\alpha)>0,\ Re(\beta)>0$, and $0<a<b<+\infty$, then*

1. $\left({}^{H}\mathscr{D}^{\alpha}_{a+}\left(\ln\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha-\beta)}\left(\ln\frac{x}{a}\right)^{\alpha-\beta-1}$.
2. $\left({}^{H}\mathscr{D}^{\alpha}_{b-}\left(\ln\frac{b}{t}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha-\beta)}\left(\ln\frac{b}{x}\right)^{\alpha-\beta-1}$.

1.3.3 Katugampola fractional derivative

Definition 1.10. *([17]) Let $\alpha,\rho\in\mathbb{R}^+$, and $n=[\alpha]+1$. The* KATUGAMPOLA *fractional derivatives corresponding to the* KATUGAMPOLA *fractional integral (1.3) (resp. (1.4)),* *are defined by*

$$
\begin{aligned}
{}^{\rho}\mathscr{D}^{\alpha}_{a+}f(x) &= \left(x^{1-\rho}\frac{d}{dx}\right)^{n}\, {}^{\rho}\mathscr{I}^{n-\alpha}_{a+}f(x)\\
&= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(x^{1-\rho}\frac{d}{dx}\right)^{n}\int_a^x \frac{t^{\rho-1}f(t)}{(x^{\rho}-t^{\rho})^{\alpha-n+1}}\,\mathrm{d}t,
\end{aligned}
\tag{1.11}
$$

and

$$
\begin{aligned}
{}^{\rho}\mathscr{D}^{\alpha}_{b-}f(x) &= \left(-x^{1-\rho}\frac{d}{dx}\right)^{n}\, {}^{\rho}\mathscr{I}^{n-\alpha}_{b-}f(x)\\
&= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(-x^{1-\rho}\frac{d}{dx}\right)^{n}\int_x^b \frac{t^{\rho-1}f(t)}{(t^{\rho}-x^{\rho})^{\alpha-n+1}}\,\mathrm{d}t,
\end{aligned}
\tag{1.12}
$$

respectively.

1.3.4 Ψ-fractional derivative

Definition 1.11. *([17]) Let ψ be a function defines on $[a,b]$ verifies $\psi'(x)\neq 0$ on $(a,b]$ and $\alpha>0$. The* RIMANN-LIOUVILLE *fractional derivatives (left-sided and right-sided) of a function f with respect to ψ of order α correspondent to the* RIMANN-LIOUVILLE *fractional integral, are defined by*

$$
\begin{aligned}
\mathscr{D}^{\alpha:\psi}_{a+}f(x) &= \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\, {}^{RL}\mathscr{I}^{n-\alpha:\psi}_{a+}f(x)\\
&= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{n-\alpha-1}f(t)\,\mathrm{d}t,
\end{aligned}
\tag{1.13}
$$

and

$$
\begin{aligned}
\mathscr{D}^{\alpha:\psi}_{b-}f(x) &= \left(-\frac{1}{\psi'(x)}\frac{d}{dt}\right)^{n}\, {}^{RL}\mathscr{I}^{n-\alpha:\psi}_{b-}f(x)\\
&= \frac{1}{\Gamma(n-\alpha)}\left(-\frac{1}{\psi'(x)}\frac{d}{dt}\right)^{n}\int_x^b \psi'(t)(\psi(t)-\psi(x))^{n-\alpha-1}f(t)\,\mathrm{d}t,
\end{aligned}
\tag{1.14}
$$

respectively, where $n=[\alpha]+1$.

Lemma 1.4. *([17]) Let $\alpha>0$ and $\delta>0$*

1. *If* $f(x) = (\psi(x) - \psi(a))^{\delta-1}$ *then*

$$\mathscr{D}_{a+}^{\alpha:\psi} f(x) = \frac{\Gamma(\delta)}{\Gamma(\delta-\alpha)} (\psi(x) - \psi(a))^{\alpha+\delta-1}.$$

2. *If* $g(x) = (\psi(b) - \psi(x))^{\delta-1}$ *then*

$$\mathscr{D}_{b-}^{\alpha:\psi} g(x) = \frac{\Gamma(\delta)}{\Gamma(\delta-\alpha)} (\psi(b) - \psi(x))^{\alpha+\delta-1}.$$

Definition 1.12. *([17]) Let* $\alpha > 0$, $n = [\alpha] + 1$, $n \in \mathbb{N}$ *and* $f, \psi \in C^n([a, b], \mathbb{R})$ *two functions such that* ψ *is increasing and* $\psi'(x) \neq 0$*, for all* $x \in I$*. The left and right-sided* ψ-CAPUTO *fractional derivatives of* f *of order* α *on* $[a, b]$ *is given by*

$$\left({}^{C}\mathscr{D}_{a+}^{\alpha:\psi} f\right)(x) = {}^{RL}\mathscr{I}_{a+}^{n-\alpha:\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n f(x), \tag{1.15}$$

and

$$\left({}^{C}\mathscr{D}_{b-}^{\alpha:\psi} f\right)(x) = {}^{RL}\mathscr{I}_{b-}^{n-\alpha:\psi} \left(-\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n f(x), \tag{1.16}$$

respectively.

Theorem 1.2. *([17]) If* $f \in C^n([a,b])$ *and* $\alpha > 0$*, then*

$$\left({}^{C}\mathscr{D}_{a+}^{\alpha:\psi} f\right)(x) = \mathscr{D}_{a+}^{\alpha:\psi} \left[f(x) - \sum_{k=0}^{n-1} \frac{1}{k!} (\psi(x) - \psi(a))^k f_{\psi}^{(k)}(a) \right], \tag{1.17}$$

and

$$\left({}^{C}\mathscr{D}_{b-}^{\alpha:\psi} f\right)(x) = \mathscr{D}_{b-}^{\alpha:\psi} \left[f(x) - \sum_{k=0}^{n-1} \frac{1}{k!} (\psi(b) - \psi(x))^k f_{\psi}^{(k)}(b) \right]. \tag{1.18}$$

1.4 Fixed Point Theorems

In the remainder of this section, we introduce the notations, definitions and theorems necessary for this study.

Definition 1.13 (Equicontinuous[**8**])**.** *Let* E *be a* BANACH *space. A part* P *in* $C(E)$ *is* *called equicontinuous if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall u, v \in E,\ \forall \mathscr{A} \in P,\ \|u - v\| < \delta \Rightarrow \|\mathscr{A}(u) - \mathscr{A}(v)\| < \varepsilon.$$

Theorem 1.3 (Ascoli-ArzelÃ [**8**])**.** *Let E be a compact space. If $\mathscr{A}$ is an equicontinuous, bounded subset of $C(E)$, then $\mathscr{A}$ is relatively compact.*

Definition 1.14 (Completely continuous[**8**])**.** *We say $\mathscr{A} : E \to E$ is completely continuous if it is continuous and for any bounded subset $P \subset E$, the set $\mathscr{A}(P)$ is relatively compact.*

Definition 1.15. *[8] Let E be any space and $\mathscr{A}$ a map of E, or of a subset of E, into E.*
- The map $\mathscr{A}$ is called a contraction mapping if there exists $k \in (0,1)$ such that

$$\forall u, v \in E,\ \|\mathscr{A} u - \mathscr{A} v\| \leq k \|u - v\|.$$

- A point $u \in E$ is called a fixed point for $\mathscr{A}$ if $\mathscr{A} u = u$.

For subsequent applications, the following fixed-point theorems are fundamental in the proofs of our main results.

Theorem 1.4 (Banach's fixed point[**8**])**.** *Let P be a non-empty closed subset of a* BANACH *space E, then any contraction mapping $\mathscr{A}$ of P into itself has a unique fixed point.*

Theorem 1.5 (Schaefer's fixed point[**8**])**.** *Let E be a* BANACH *space, Let $\mathscr{A} : E \to E$ be a completely continuous mapping, If*

$$\Lambda = \{u \in E : \lambda \mathscr{A} u = u,\ \lambda \in [0,1]\}$$

is bounded, Then $\mathscr{A}$ has at least one fixed point.

CHAPTER 2

EXISTENCE AND UNIQUENESS OF SOLUTIONS

In this chapter, we will study the existence and uniqueness of solutions for a boundary value problem of nonlinear FDE with the ψ-fractional derivative. The significant results are proved by means of BANACH and SCHAEFER fixed point theorems .

In follow up, we will discuss the existence and uniqueness of solutions of the following problem of the fractional differential equation of order α of the form:

$$\mathscr{D}^{\alpha;\psi}u(t)+f(t,u(t))=0,\ 0<t<T,\ 1<\alpha<2, \tag{2.1}$$

supplemented with the DIRICHLET boundary conditions :

$$u(0)=0,\ u(T)=0, \tag{2.2}$$

where $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function. We obtain several existence and uniqueness results for the problem (2.1)-(2.2).

We consider a function $\psi:(0,T)\subset\mathbb{R}\to\mathbb{R}$, such that :

1. ψ is an increasing and positive monotone function on [0,T] and $\psi(0)=0$.
2. $\psi^{'}(t)$ is continuous.

2.1 Definition of integral solution

In this section, we present some significant lemmas which play a key role in the proofs of the main results.

Lemma 2.1. *([20]) Let* $\alpha\in\mathbb{R}^{+}$. *If* $u\in C[0,T]$, *then:*

(i) The fractional equation $\mathscr{D}^{\alpha:\psi} u(t) = 0$*, has a solution as follows:*

$$u(t) = C_1\psi^{\alpha-1}(t) + C_2\psi^{\alpha-2}(t)^{\alpha-2} + \cdots + C_n\psi^{\alpha-n}(t),\ \textit{where } C_m \in \mathbb{R},\ \textit{with } m = 1,2,\ldots,n.$$

(ii) If $\mathscr{D}^{\alpha:\psi} u \in C[0,T]$ *and* $1 < \alpha \le 2$*, then:*

$$\mathscr{I}^{\alpha:\psi}\,\mathscr{D}^{\alpha:\psi}\,u(t) = u(t) + C_1\psi^{\alpha-1}(t) + C_2\psi^{\alpha-2}(t),\ \textit{for some } C_1, C_2 \in \mathbb{R}. \tag{2.3}$$

In the following lemma, we define the integral solution of the boundary value problem (2.1)-(2.2).

Lemma 2.2. *Let* $\alpha \in \mathbb{R}^+$ *such that* $1 < \alpha < 2$*. We give* $\mathscr{D}^{\alpha:\psi} u \in C[0,T]$*, and* $f(t,u)$ *is a continuous function. Then the boundary value problem* (2.1)-(2.2)*, is equivalent to the fractional integral equation*

$$u(t) = \int_0^T G(t,s)\, f(s,u(s))\, ds,\ t \in [0,\ T],$$

where

$$G(t,s) = \frac{\psi'(s)}{\Gamma(\alpha)} \begin{cases} \left(\frac{\psi(t)}{\psi(T)}\right)^{\alpha-1} \left(\psi(T)-\psi(s)\right)^{\alpha-1} - \left(\psi(t)-\psi(s)\right)^{\alpha-1}, & 0 < s < t < T, \\ \left(\frac{\psi(t)}{\psi(T)}\right)^{\alpha-1} \left(\psi(T)-\psi(s)\right)^{\alpha-1}, & 0 < t < s < T, \end{cases} \tag{2.4}$$

is the Green's function associated with the boundary value problem (2.1)-(2.2).

Proof. Let $\alpha \in \mathbb{R}^+$, be such that $1 < \alpha < 2$. We apply lemma 2.1 to reduce the fractional equation (2.1) to an equivalent fractional integral equation. It is easy to prove the operator $\mathscr{I}^{\alpha:\psi}$ has the linearity property for all $\alpha > 0$ after direct integration. Then by applying $\mathscr{I}^{\alpha:\psi}$ to equation (2.1), we get

$$\mathscr{I}^{\alpha:\psi}\,\mathscr{D}^{\alpha:\psi} u(t) + \mathscr{I}^{\alpha:\psi} f(t,u(t)) = 0.$$

From lemma 2.1, we find for $1 < \alpha \le 2$,

$$\mathscr{I}^{\alpha:\psi}\,\mathscr{D}^{\alpha:\psi} u(t) = u(t) + C_1\psi^{\alpha-1}(t) + C_2\psi^{\alpha-2}(t),$$

for some $C_1, C_2 \in \mathbb{R}$. Then, the integral solution of the equation (2.1) is :

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, u(s))\, \mathrm{d}s - C_1 \psi^{\alpha-1}(t) - C_2 \psi^{\alpha-2}(t). \quad (2.5)$$

The conditions (2.2) imply that :

$$u(0) = -\lim_{t \to 0} C_2 \psi^{\alpha-2}(t) = 0.$$

Then $C_2 = 0$, also

$$u(T) = -\frac{1}{\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} f(s, u(s))\, \mathrm{d}s - C_1 \psi^{\alpha-1}(T) = 0,$$

then

$$C_1 = -\frac{1}{\Gamma(\alpha)\psi^{\alpha-1}(T)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} f(s, u(s))\, \mathrm{d}s$$

The integral equation (2.5) is equivalent to :

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \left(\frac{\psi(t)}{\psi(T)} \right)^{\alpha-1} \int_0^T \psi'(s) \left(\psi(T) - \psi(s) \right)^{\alpha-1} f(s, u(s)) ds \\ &- \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) \left(\psi(t) - \psi(s) \right)^{\alpha-1} f(s, u(s)) ds. \end{aligned} \quad (2.6)$$

Therefore, the unique solution of problem (2.1)-(2.2) is :

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t \left[\psi'(s) \left(\frac{\psi(t)}{\psi(T)} \right)^{\alpha-1} \left(\psi(T) - \psi(s) \right)^{\alpha-1} - \left(\psi(t) - \psi(s) \right)^{\alpha-1} \right] f(s, u(s))\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_t^T \left(\frac{\psi(t)}{\psi(T)} \right)^{\alpha-1} \psi'(s) \left(\psi(T) - \psi(s) \right)^{\alpha-1} f(s, u(s))\, \mathrm{d}s, \\ &= \int_0^T G(t, s) f(s, u(s))\, \mathrm{d}s. \end{aligned}$$

The proof is complete. □

Remark 2.1. *In lemma 2.2 we have made a generalization of the green's function in [10] that is concerned the* KATUGAMPOLA *fractional differential equation, also when we take $\psi(t) = \ln t$ we get the green's function of the integral solution associated of the* HADAMARD *fractional differential equation.*

2.2 Existence of Solutions for Ψ-Fractional Differential Equation

In this section, We suggest some conditions on f, which allow us to obtain some results on existence solutions for the boundary value problem (2.1)-(2.2). Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function .

We note that $u(t)$ is a solution of (2.1)-(2.2) if and only if :

$$u(t) = \int_0^T G(t,s) f(s, u(s))\, ds,\ \ t \in [0, T].$$

Now we prove some properties of the Green's function $G(t,s)$ given in (2.4).

Lemma 2.3. *Let* $1 < \alpha < 2$, *then the Green's function* $G(t,s)$ *given by* (2.4) *satisfies:*
(1) $G(t,s) > 0$ *for* $t, s \in (0, T)$.
(2) $\max\limits_{0<t<T} G(t,s) = G(s,s)$, *for each* $s \in [0, T]$.

Proof. (1) Let $1 < \alpha < 2$ and ψ is continuous and increasing function on $(0, T)$, In the case $0 < t < s < T$, we have:

$$\frac{\psi'(s)}{\Gamma(\alpha)} \left(\frac{\psi(t)}{\psi(T)}\right)^{\alpha-1} \left(\psi(T) - \psi(s)\right)^{\alpha-1} > 0.$$

Moreover, for $0 < s < t < T$, we have $\frac{\psi(t)}{\psi(T)} < 1$, then $\psi(s)\frac{\psi(t)}{\psi(T)} < \psi(s)$ and
$\psi(t) - \psi(s)\frac{\psi(t)}{\psi(T)} < \psi(t) - \psi(s)$, thus

$$\left(\psi(t) - \psi(s)\frac{\psi(t)}{\psi(T)}\right)^{\alpha-1} - \left(\psi(t) - \psi(s)\right)^{\alpha-1} > 0,$$

which implies that $G(t,s) > 0$ for any $t, s \in (0, T)$.

(2) To prove that

$$\max_{0\le t\le T} G(t,s) = G(s,s) = \frac{\psi'(s)}{\Gamma(\alpha)} \left(\frac{\psi(s)}{\psi(T)}\right)^{\alpha-1} \left(\psi(T) - \psi(s)\right)^{\alpha-1}\ \ \forall s \in [0, T], \tag{2.7}$$

we choose

$$g_1(t,s) = \frac{\psi'(s)}{\Gamma(\alpha)} \left(\frac{\psi(t)}{\psi(T)}\right)^{\alpha-1} \left(\psi(T) - \psi(s)\right)^{\alpha-1} - \left(\psi(t) - \psi(s)\right)^{\alpha-1},$$
$$g_2(t,s) = \frac{\psi'(s)}{\Gamma(\alpha)} \left(\frac{\psi(t)}{\psi(T)}\right)^{\alpha-1} \left(\psi(T) - \psi(s)\right)^{\alpha-1}.$$

Indeed, we put $\max\limits_{0\leq t\leq T} G(t,s) = G(t^*,s)$, where $0 \leq t^* \leq T$. Then, we get for some $0 < t_1 < t_2 < T$, that

$$\begin{aligned}\max_{0\leq t\leq T} G(t,s) &= \begin{cases} g_1(t^*,s), & s\in[0,t_1],\\ \max\left\{g_1(t^*,s), g_2(t^*,s)\right\}, & s\in[t_1,t_2],\\ g_2(t^*,s), & s\in[t_2,T],\end{cases}\\ &= \begin{cases} g_1(t^*,s), & s\in[0,r],\\ g_2(t^*,s), & s\in[r,T],\end{cases}\end{aligned}$$

where $r \in [t_1, t_2]$, is the unique solution of equation

$$\begin{aligned} g_1\left(t^*,s\right) = g_2\left(t^*,s\right) &\Leftrightarrow \psi\left(t^*\right) = \psi(s),\\ &\Leftrightarrow t^* = s,\end{aligned}$$

which shows the equality (2.7). □

2.2.1 Applications of Banach Fixed Point Theorem

In this section, we assume that $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ satisfies the conditions:

(H1) $f(t,u)$ is LEBESGUE measurable function with respect to t on $[0,T]$,

(H2) $f(t,u)$ is a continuous function with respect to u on $\mathbb{R}$.

Theorem 2.1. *We assume that* (H1) *and* (H2) *hold, There exists a constant* $k>0$, *such that*

$$\left|f(t,u) - f(t,v)\right| \leq k|u-v|,\ \forall t\in[0,T],\ \textit{and all } u,v\in C[0,T]. \tag{2.8}$$

If

$$\frac{k\psi^{\alpha}(T)}{\Gamma(\alpha+1)} < 1. \tag{2.9}$$

Then, there exists a unique solution of the boundary value problem (2.1)-(2.2) *on* $[0,T]$.

Proof. Assume that $\frac{k\psi^{\alpha}(T)}{\Gamma(\alpha+1)} < 1$, and consider the operator $\mathscr{A} : C([0,T],\mathbb{R}) \to$

$C([0,T],\mathbb{R})$ is defined as follows

$$\mathscr{A}u(t)=\int_0^T G(t,s)\,f(s,u(s))\,ds.$$

We shall show that $\mathscr{A}$ is a contraction mapping. In fact, for any $u,v\in C([0,T],\mathbb{R})$ we have

$$\begin{aligned}|\mathscr{A}u(t)-\mathscr{A}v(t)| &= \left|\int_0^T G(t,s)\left[f(s,u(s))-f(s,v(s))\right]ds\right|,\\ &\le \int_0^T \left|G(t,s)\,f(s,u(s))-f(s,v(s))\right|ds,\\ &\le k\int_0^T G(s,s)\,|u(s)-v(s)|\,ds,\end{aligned}$$

then

$$\begin{aligned}\|\mathscr{A}u-\mathscr{A}v\|_\infty &\le k\|u-v\|_\infty\int_0^T G(s,s)\,ds,\\ &\le k\|u-v\|_\infty\int_0^T\left(\frac{\psi(s)}{\psi(T)}\right)^{\alpha-1}\psi'(s)\left(\psi(T)-\psi(s)\right)^{\alpha-1}ds,\end{aligned}$$

we have ψ an increasing function, then $\frac{\psi^{\alpha-1}(s)}{\psi^{\alpha-1}(T)}<1$, $0<s<T$, we get

$$\begin{aligned}\|\mathscr{A}u-\mathscr{A}v\|_\infty &\le k\|u-v\|_\infty\int_0^T \psi'(s)\left(\psi(T)-\psi(s)\right)^{\alpha-1}ds,\\ &\le \frac{k\psi^{\alpha}(T)}{\Gamma(\alpha+1)}\|u-v\|_\infty.\end{aligned}\tag{2.10}$$

This implies from (2.11) that $\mathscr{A}$ is a contraction operator. As a consequence of theorem 1.4 we deduce that $\mathscr{A}$ has a unique fixed point which is the unique solution of the problem (2.1)-(2.2) on $[0,T]$. The proof is complete. □

2.2.2 Applications of Schaefer Fixed Point Theorem

In this section, we will study the existence solutions for the boundary value problem (2.1)-(2.2) by using SCHAEFER fixed point theorem.

Theorem 2.2. *Let f satisfies the conditions:*

$\overline{(H1)}$ *$f(t,u)$ is a continuous function with respect to u on $\mathbb{R}$.*

$\overline{(H2)}$ *There exists a constant $M>0$, such that :*

$$\left|f(t,u)\right| \le M, \quad \forall t \in [0,T], \text{ and } u \in \mathbb{R}.$$

Then, there exists at least one solution of the boundary value problem (2.1)-(2.2) *on* $[0,T]$.

Proof. The demonstration has four steps :

step 01: Let $\{u_n\}$ be a convergence sequence on $C([0,T],\mathbb{R})$, such that :

$$\lim_{n\to+\infty} \|u_n - u\|_\infty = 0$$

We need to prove that $\lim_{n\to+\infty} \|\mathscr{A} u_n - \mathscr{A} u\|_\infty = 0$, $t \in [0,T]$. Indeed

$$\begin{aligned}
|\mathscr{A} u_n(t) - \mathscr{A} u(t)| &= \left|\int_0^T G(t,s) f(s,u_n(s)) - \int_0^T G(t,s) f(s,u(s))\, \mathrm{d}s\right| \mathrm{d}s, \\
&\le \int_0^T G(t,s) \left|f(s,u_n(s)) - f(s,u(s))\right| \mathrm{d}s, \\
&\le \left\|f(.,u_n(.)) - f(.,u(.))\right\|_\infty \int_0^T G(s,s)\, \mathrm{d}s, \\
&\le \frac{\psi^\alpha(T)}{\Gamma(\alpha+1)} \left\|f(.,u_n(.)) - f(.,u(.))\right\|_\infty.
\end{aligned}$$

As f is continuous, we get

$$\|\mathscr{A} u_n - \mathscr{A} u\| \le \frac{\psi^\alpha(T)}{\Gamma(\alpha+1)} \left\|f(.,u_n(.)) - f(.,u(.))\right\|_\infty \to 0 \text{ when } n \to \infty.$$

Then, $\mathscr{A}$ is continuous.

step 02: For all $r>0$, there exists a constant $L>0$, such that $\forall u \in B_r$, we have :

$$B_r = \{u \in C([0,T],\mathbb{R}) : \|u\|_\infty \le r\},$$

we get $\|\mathscr{A} u\|_\infty \le L$.

Indeed. By using $\overline{(H2)}$ and for all $t \in [0, T]$

$$\begin{aligned}
|\mathscr{A} u(t)| &\leq \int_0^T G(t,s) \left| f(s,u(s)) \right| \mathrm{d}s, \\
&\leq M \int_0^T G(s,s)\, \mathrm{d}s, \\
&\leq M \frac{\psi^{\alpha}(T)}{\Gamma(\alpha+1)} = L.
\end{aligned}$$

As a result, $\mathscr{A}(Br)$ is bounded.

step 03 : Let $t_1, t_2 \in (0, T]$, $t_1 < t_2$ and let B_r be a bounded subset on $C([0,T],\mathbb{R})$, and let $u \in B_r$, then

$$\begin{aligned}
|\mathscr{A} u(t_2) - \mathscr{A} u(t_1)| &= \left| \int_0^T G(t_2,s) f(s,u(s))\, \mathrm{d}s - \int_0^T G(t_1,s) f(s,u(s))\, \mathrm{d}s \right|, \\
&\leq \int_0^T G(t_2,s) - G(t_1,s) \left| f(s,u(s)) \right| \mathrm{d}s, \\
&\leq M \int_0^T G(t_2,s) - G(t_1,s)\, \mathrm{d}s.
\end{aligned}$$

By using (*2.6*), we get

$$\begin{aligned}
|\mathscr{A} u(t_2) - \mathscr{A} u(t_1)| &\leq \frac{M}{\Gamma(\alpha)} \left[\frac{\psi^{\alpha-1}(t_2)}{\psi^{\alpha-1}(T)} - \frac{\psi^{\alpha-1}(t_1)}{\psi^{\alpha-1}(T)} \right] \int_0^T \psi'(s) \left(\psi(T) - \psi(s)\right)^{\alpha-1} \mathrm{d}s \\
&- \frac{M}{\Gamma(\alpha)} \left[\int_0^{t_2} \psi'(s) \left(\psi(t_2) - \psi(s)\right)^{\alpha-1} \mathrm{d}s \right] \\
&+ \frac{M}{\Gamma(\alpha)} \left[\int_0^{t_1} \psi'(s) \left(\psi(t_1) - \psi(s)\right)^{\alpha-1} \mathrm{d}s \right], \\
&\leq \frac{M}{\Gamma(\alpha+1)} \psi(T) \left[\psi^{\alpha-1}(t_2) - \psi^{\alpha-1}(t_1) \right] + \psi^{\alpha}(t_1) - \psi^{\alpha}(t_2),
\end{aligned}$$

tends to zero as $t_1 \to t_2$. Thus, $\mathscr{A}$ is equicontinuous, so $\mathscr{A}$ is relatively compact on B_r. Hence, by theorem 1.3, $\mathscr{A}$ is compact on B_r.

step 04 : We consider $\Lambda = \{u \in C([0,T],\mathbb{R}) : u = \lambda \mathscr{A} u\}$ such that $0 < \lambda < 1$.
Let $u \in \Lambda \Rightarrow u = \lambda \mathscr{A}(u)$, then $\forall t \in [0, T]$, we have :

$$u(t) = \lambda \int_0^T G(t,s) f(s,u(s))\, \mathrm{d}s.$$

By using $\overline{(H2)}$, and $\forall\, t \in [0, T]$ we get

$$\begin{aligned} |\mathscr{A} u(t)| &\leq \int_0^T G(t,s) \left| f(s, u(s)) \right| \mathrm{d}s, \\ &\leq M \int_0^T G(s,s)\, \mathrm{d}s, \\ &\leq M \frac{\psi^{\alpha}(T)}{\Gamma(\alpha+1)} = G. \end{aligned}$$

This implies that Λ is bounded.

As a consequence of theorem 1.5 we deduce that $\mathscr{A}$ has at least one fixed point which is a solution of the problem (2.1)-(2.2) on $[0, T]$. The proof is complete. □

2.3 Examples

In this section, we present some examples to illustrate the efficient of our main results. Therefore, we divides the next examples into two cases to represent the essential results that are shown the effectiveness of the generalization of ψ-fractional derivative.

First of all, we have chosen $\psi(x) = \ln x$, for showing the generalization of some examples (see [2]) then we get from the problem (2.1):

$$^{H}\mathscr{D}^{\alpha} u(t) + f(t, u(t)) = 0, \tag{2.11}$$

which is the HADAMARD fractional differential equation of order α, therefore, ψ is a continuous and an increasing function also $\psi(1) = 0$.

Example 1. Let $\alpha = \sqrt{3}$, we consider the following boundary value problem

$$\begin{cases} ^{H}\mathscr{D}_{1^+}^{\sqrt{3}} u(t) + \frac{\arctan t}{2+|u(t)|} = 0, \ t \in [1, e], \\ u(1) = u(e) = 0. \end{cases} \tag{2.12}$$

Here for $f(t,u) = \frac{\arctan t}{2+|u|}$, $t \in [1, e]$, $u \in \mathbb{R}$.

By (2.2) the solution of problem (2.12) is given by

$$u(t) = \int_0^e G(t,s) \times \frac{\arctan s}{2+|u(s)|}\, \mathrm{d}s.$$

Hence, the function f is jointly continuous. For any $u, v \in \mathbb{R}$ and $t \in [1, e]$, we have:

$$\left|f(t,u) - f(t,v)\right| \leq \frac{\pi}{8}|u-v|.$$

Hence, the condition (2.8) is satisfied with $k = \frac{\pi}{8}$. It remains to show that the condition (2.9)

$$\frac{k \times (\ln T)^{\alpha}}{\Gamma(\alpha+1)} = \frac{\frac{\pi}{8}}{\Gamma\left(\sqrt{3}+1\right)} \simeq 0.247759\ldots < 1$$

is satisfied. By theorem 1.4 the problem (2.12), has a unique solution.

Example 2. Let $\alpha = \sqrt{2}$, we consider the following boundary value problem :

$$\begin{cases} {}^{H}\mathscr{D}_{1^+}^{\sqrt{2}} u(t) + \frac{1}{1+u^2(t)} = 0, \ t \in \left[1, \frac{5}{2}\right], \\ u(1) = u\left(\frac{5}{2}\right) = 0. \end{cases} \tag{2.13}$$

Here for $f(t,u) = \frac{1}{1+u^2}$ and $t \in \left[1, \frac{5}{2}\right]$, $u(t) \in \mathbb{R}$.
By (2.2) the solution of problem (2.13) is given by

$$u(t) = \int_1^{\frac{5}{2}} \frac{G(t,s)}{1+u^2(s)} \mathrm{d}s.$$

Hence, the function f is jointly continuous. For any $u \in \mathbb{R}$ and $t \in \left[1, \frac{5}{2}\right]$, we have:

$$\left|f(t,u)\right| = \frac{1}{1+u^2} < 1 = M.$$

Thus, by theorem 1.5 the problem (2.13), has at least one solution.

Secondly, we have made a generalization of some examples (see [10]) to valid our main results.
We set $\psi(x) = \frac{x^{\rho}}{\rho}$, then we get from the problem (2.1) :

$${}^{\rho}\mathscr{D}^{\alpha} u(t) + f(t, u(t)) = 0, \tag{2.14}$$

which is the KATUGAMPOLA fractional differential equation of order α, Therefore, ψ is a continuous and an increasing function also $\psi(0) = 0$.
Example 3. We take $\alpha = \frac{3}{2}$ and $\rho = \frac{2}{3}$, we consider the following boundary value prob-

lem :

$$\begin{cases} {}^{\frac{2}{3}}\mathscr{D}_{0^+}^{\frac{3}{2}} u(t) + \frac{\cos(t)[2+|u(t)|]}{\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)[1+|u(t)|]} = 0, \; t \in \left[0, \frac{\pi}{4}\right], \\ u(0) = u\left(\frac{\pi}{4}\right) = 0. \end{cases} \tag{2.15}$$

Here we have

$$f(t,u) = \frac{\cos(t)\,[2+|u|]}{\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)[1+|u|]}, \; t \in \left[0, \frac{\pi}{4}\right], \; u \in \mathbb{R}.$$

By (2.2) the solution of problem (2.15) is given by

$$u(t) = \int_0^{\frac{\pi}{4}} G(t,s) \times \frac{\cos(s)\,[2+|u(s)|]}{\pi\left(\sqrt{2}\cos(s)+\sin(s)\right)[1+|u(s)|]}, \, \mathrm{d}s.$$

Hence, the function f is jointly continuous. For any $u, v \in \mathbb{R}$ and $t \in \left[0, \frac{\pi}{4}\right]$, we have:

$$\left|f(t,u) - f(t,v)\right| \leq \frac{1}{\pi}|u-v|.$$

Hence, the condition (2.8) is satisfied with $k = \frac{1}{\pi}$. It remains to show that the condition (2.9)

$$\frac{k \times T^{\alpha\rho}}{\Gamma(\alpha+1)} = \frac{\frac{1}{\pi} \times \frac{\pi}{4}}{\frac{2}{3}^{\frac{3}{2}} \times \Gamma\left(\frac{5}{2}\right)} \simeq 0.921317\ldots < 1$$

is satisfied. By theorem 1.4 the problem (2.15), has a unique solution.

Example 4. We take $\alpha = \frac{3}{2}$ and $\rho = 1$, we consider the following boundary value problem :

$$\begin{cases} {}^{1}\mathscr{D}_{0^+}^{\frac{3}{2}} u(t) + (1+t)\exp(-u^2(t)) = 0, \; t \in \left[0, \frac{5}{2}\right], \\ u(0) = u\left(\frac{5}{2}\right) = 0. \end{cases} \tag{2.16}$$

Here for $f(t,u) = (1+t)\exp(-u^2)$ and $t \in \left[0, \frac{5}{2}\right]$, $u \in \mathbb{R}$.

By (2.2) the solution of problem (2.16) is given by

$$u(t) = \int_0^{\frac{5}{2}} G(t,s) \times (1+s)\exp(-u^2(s))\,\mathrm{d}s.$$

Hence, the function f is jointly continuous. For any $u \in \mathbb{R}$ and $t \in \left[0, \frac{5}{2}\right]$, we have:

$$\left|f(t,u)\right| = (1+t)\exp(-u^2) < 4 = M.$$

Thus, by theorem 1.5 the problem (2.16), has at least one solution.

Remark 2.2. *We may see the approach numerical solutions by a successive method such that*

$$\begin{cases} u_{n+1}(t) = \int_0^T G(t, s) f(s, u_n) \, ds, \; n = \overline{0, n}, \\ u_0 = u_{n+1} = 0. \end{cases} \tag{2.17}$$

2.3.1 Numerical results

Here, we will present the numerical results of the problems (2.12) − (2.13), and (2.15) − (2.16), by using 2.17 to get the next results.

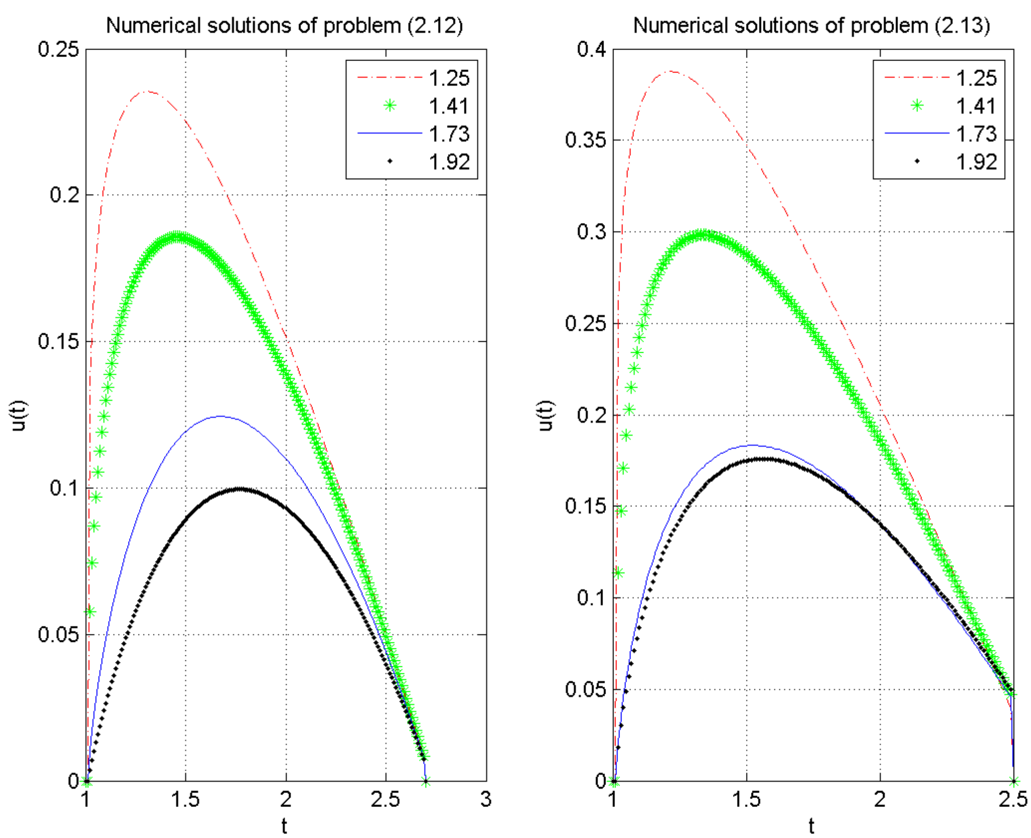

Figure 2.1: Solutions of problems (2.12) − (2.13) for different values of α when $\psi(t) = \ln t$ (the Hadamard operator).

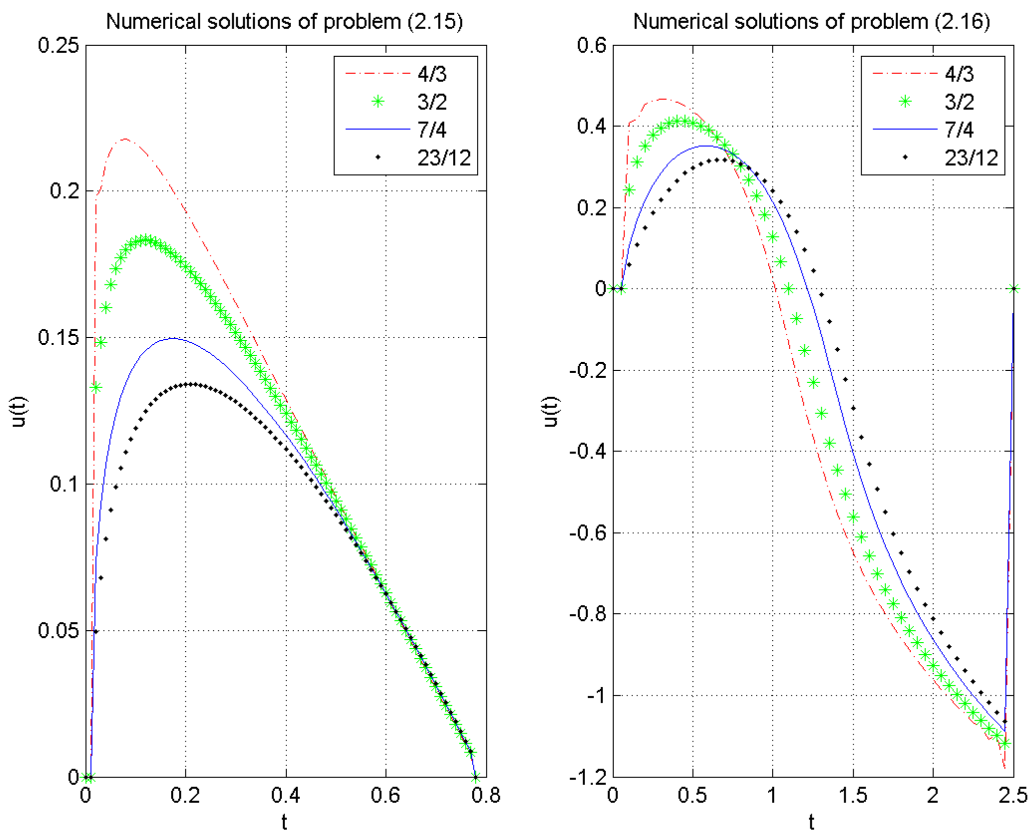

Figure 2.2: Solutions of problems (2.15)-(2.16) for different values of α when $\psi(t) = \frac{t^{\rho}}{\rho}$ (the Katugampola operator).

CHAPTER 3

EXISTENCE STUDY OF SOLUTION FOR A SYSTEM OF N NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS

In this chapter, we will discuss the existence and uniqueness of solutions for a system of n nonlinear FDEs with ψ-fractional derivative, the BANACH contraction principle and SCHAEFER's fixed point theorem are applied to proof the existence and uniqueness of solutions of a problem that is given by

$$\begin{cases} \mathscr{D}^{\alpha:\psi} u_1(t) + f_1(t, u(t)) = 0, \ 0 < t < T, \\ \mathscr{D}^{\alpha:\psi} u_2(t) + f_2(t, u(t)) = 0, \ 0 < t < T, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathscr{D}^{\alpha:\psi} u_n(t) + f_n(t, u(t)) = 0, \ 0 < t < T, \end{cases} \tag{3.1}$$

with the boundary conditions:

$$u_1(0) = u_2(0) = \cdots = u_n(0) = 0 \text{ and } \ u_1(T) = u_2(T) = \cdots = u_n(T) = 0, \tag{3.2}$$

where $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$, for $n \in \mathbb{N}^*$. Also $1 < \alpha < 2$ and $f_i : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions for every $i \in \overline{1, n}$.

In the following, we present the necessary lemma to clarify the basis of our proof.

Lemma 3.1. *Let $u_i \in C([0, T], \mathbb{R})$ for every $i \in \overline{1, n}$. Then the solution of problem* (3.1)-

(3.2), *is equivalent to the n fractional integral equations*

$$\begin{cases} u_1(t) = \int_0^T G(t,s) f_1(s,u(s))\, ds, \\ u_2(t) = \int_0^T G(t,s) f_2(s,u(s))\, ds, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ u_n(t) = \int_0^T G(t,s) f_n(s,u(s))\, ds, \end{cases} \tag{3.3}$$

where G is the same Green's function on lemma 2.2 associated with the problem (2.1)-(2.2).

Proof. We simply need to use the same argument of the proof of lemma 2.2 for n equations. □

Let us introduce the space $E = E_1 \times E_2 \times \cdots \times E_n$, where

$$E_i = [C([0,\ T], \mathbb{R})]^i, \ \ i \in \overline{1,n},$$

with the norme

$$\|u\|_E = \sup_{1 \le i \le n} \|u_i\|_\infty.$$

It's clear that $(E, \|.\|_E)$ is a BANACH space.

3.1 Applications of Banach Fixed Point Theorem

In this section, we assume that $f_i : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions for every $i \in \overline{1,n}$. We consider the following hypothese:

(H1) Let $u,\ v \in \mathbb{R}^n$. For $1 \le k \le n$, there exist constants $\lambda_{i,k} > 0$, such that

$$\left| f_i(t,u) - f_i(t,v) \right| \le \sum_{k=1}^{n} \lambda_{i,k} |u_k - v_k|,\ \forall t \in [0,T]. \tag{3.4}$$

Theorem 3.1. *We assume that* (H1) *holds. If*

$$\sup_{1 \le i \le n} \left\{ \frac{n\lambda_i \psi^\alpha(T)}{\Gamma(\alpha+1)} \right\} < 1. \tag{3.5}$$

Then, there exists a unique solution of the boundary value problem (3.1)-(3.2) *on*

$[0, T]$.

Proof. Initially, we transform problem (*3.1*)-(*3.2*) into a fixed point problem $\mathscr{F}u = u(t)$ and $[0, T]$ with $\mathscr{F} : E \to E$ is defined by

$$\mathscr{F}u(t) = \begin{pmatrix} \mathscr{F}_1 u(t) \\ \vdots \\ \mathscr{F}_n u(t) \end{pmatrix}, \tag{3.6}$$

with $(\mathscr{F}_i u(t))_{i \in \overline{1,n}}$ are the integral operators are given by

$$\mathscr{F}_i u(t) = \int_0^T G(t,s) f_i(s, u(s))\, ds,$$

Cause problem (*3.1*)-(*3.2*) is equivalent to the system of n fractional integrale equations (3.6), the fixed point of $\mathscr{F}$ is a solution of problem (*3.1*)-(*3.2*).
Let $u = (u_1, u_2, \ldots, u_n)$, $v = (v_1, v_2, \ldots, v_n) \in E$ and $t \in [0, T]$. For each $i \in \overline{1,n}$ we have

$$\begin{aligned} |\mathscr{F}u_i(t) - \mathscr{F}v_i(t)| &= \left| \int_0^T G(t,s) \left[f_i(s, u(s)) - f_i(s, v(s)) \right] ds \right|, \\ &\leq \int_0^T G(t,s) \left| f_i(s, u(s)) - f_i(s, v(s)) \right| ds. \end{aligned} \tag{3.7}$$

In other hand, by using (*H1*) we get :

$$\begin{aligned} \left\| f_i(., u) - f_i(., v) \right\|_\infty &\leq \sum_{k=1}^n \lambda_{i,k} \|u_k - v_k\|_\infty, \\ &\leq n \max_{1 \leq k \leq n} \left\{ \lambda_{i,k} \right\} \|u - v\|_E. \end{aligned} \tag{3.8}$$

We denote $\lambda_i = \max_{1 \leq k \leq n} \left\{ \lambda_{i,k} \right\}$. From 3.7 and 3.8, we get

$$\begin{aligned} \|\mathscr{F}u_i(t) - \mathscr{F}v_i(t)\|_\infty &\leq n\lambda_i \|u - v\|_E \int_0^T G(s,s)\, ds, \\ &\leq \frac{n\lambda_i \psi^\alpha(T)}{\Gamma(\alpha+1)} \|u - v\|_\infty. \end{aligned}$$

Consequently,

$$\|\mathscr{F}u(t) - \mathscr{F}v(t)\|_E \leq \sup_{1 \leq i \leq n} \left\{ \frac{n\lambda_i \psi^\alpha(T)}{\Gamma(\alpha+1)} \right\} \|u - v\|_E.$$

This implies from (3.5) that $\mathscr{F}$ is a contraction operator.
As a consequence of theorem 1.4 we deduce that $\mathscr{F}$ has a unique fixed point which is the unique solution of the problem (3.1)-(3.2) on $[0, T]$. The proof is complete. □

3.2 Applications of Schaefer Fixed Point Theorem

In this section ,we shall study the existence solutions for the boundary value problem (3.1)-(3.2) by using SCHAEFER fixed point theorem.

Theorem 3.2. *For each $i \in \overline{1,n}$, let f satisfies the conditions:*

$\overline{(H1)}$ $f_i(t,u)$ *are continuous functions with respect to* $u = (u_1, u_2, \cdots, u_n) \in \mathbb{R}^n$.

$\overline{(H2)}$ *There exist constants* $N_i > 0$, *such that :*

$$\left|f_i(t,u)\right| \le N_i, \ \ \forall t \in [0,T] \ \text{and} \ \ u \in \mathbb{R}^n.$$

Then, the problem (3.1)-(3.2) *has at least one solution on* $[0, T]$.

Proof. We are already transformed problem (*3.1*)-(*3.2*) into fixed point problem 3.6 as the proof of previous Theorem 3.1.
We will see that $\mathscr{F}$ satisfies the assumption of SHAEFER's fixed point theorem. This will be proofed through four steps.
step 01: $\mathscr{F}$ is a continuous operator. Let $(u_m)_{m\in\mathbb{N}} = \left(u_1^m, u_2^m, \cdots, u_n^m\right)$ be n real sequences such that $\lim_{m\to\infty} u_m = u$ in E.
For each $i \in \overline{1,n}$ and $t \in [0,T]$, we have

$$\begin{aligned}
|\mathscr{F}_i u_m(t) - \mathscr{F}_i u(t)| &= \left|\int_0^T G(t,s) f_i(s, u_m(s))\,\mathrm{d}s - \int_0^T G(t,s) f_i(s,u(s))\,\mathrm{d}s\right|, \\
&\le \int_0^T G(t,s)\left|f_i(s,u_m(s)) - f_i(s,u(s))\right|\mathrm{d}s, \\
&\le \left\|f_i(.,u_m(.)) - f_i(.,u(.))\right\|_\infty \int_0^T G(s,s)\,\mathrm{d}s, \\
&\le \frac{\psi^\alpha(T)}{\Gamma(\alpha+1)}\left\|f_i(.,u_m(.)) - f_i(.,u(.))\right\|_\infty.
\end{aligned}$$

since f_i are continuous functions and $u_m \to u$ in E, then we get

$$\|\mathscr{F} u_m(t) - \mathscr{F} u(t)\|_E \leq \frac{\psi^{\alpha}(T)}{\Gamma(\alpha+1)} \left\| f_i(., u_m(.)) - f_i(., u(.)) \right\|_{\infty} \to 0 \text{ when } n \to \infty.$$

As a result, $\mathscr{F}$ is continuous.

step 02 : Let $r' > 0$, we define

$$E_{r'} = \left\{ u \in E : \|u\|_E \leq r' \right\},$$

it's clear that $E_{r'}$ is a bounded and close subset of E. Let $\mathscr{F} : E_{r'} \to E$ be the integral operator define in (3.6). Let $u \in E_{r'}$ and $t \in [0, T]$, by using $\left(\overline{H2}\right)$, for each $i = \overline{1, n}$, we get

$$\begin{aligned} |\mathscr{F}_i u(t)| &= \left| \int_0^T G(t,s) f_i(s, u(s)) \, \mathrm{d}s \right|, \\ &\leq \int_0^T G(t,s) \left| f_i(s, u(s)) \right| \mathrm{d}s, \\ &\leq N_i \int_0^T G(t,s) \, \mathrm{d}s, \end{aligned}$$

then

$$\begin{aligned} \|\mathscr{F}_i u(t)\|_{\infty} &\leq N_i \int_0^T G(s,s) \, \mathrm{d}s, \\ \|\mathscr{F} u(t)\|_E &\leq \max_{1 \leq i \leq n} \{N_i\} \frac{\psi^{\alpha}(T)}{\Gamma(\alpha+1)} = R. \end{aligned}$$

Consequently, $\mathscr{F}(E_{r'})$ is bounded.

step 03 : Let $t_1, t_2 \in (0, T]$, $t_1 < t_2$ and $u \in E_{r'}$. Then, for every $i = \overline{1, n}$, we get

$$\begin{aligned} |\mathscr{F}_i u(t_2) - \mathscr{F}_i u(t_1)| &= \left| \int_0^T G(t_2, s) f_i(s, u(s)) \, \mathrm{d}s - \int_0^T G(t_1, s) f_i(s, u(s)) \, \mathrm{d}s \right|, \\ &\leq \int_0^T G(t_2, s) - G(t_1, s) \left| f_i(s, u(s)) \right| \mathrm{d}s, \\ &\leq N_i \int_0^T G(t_2, s) - G(t_1, s) \, \mathrm{d}s. \end{aligned}$$

By using (*2.6*), we get

$$\begin{aligned}
|\mathscr{F}_i u(t_2) - \mathscr{F}_i u(t_1)| &\leq \frac{N_i}{\Gamma(\alpha)} \left[\frac{\psi^{\alpha-1}(t_2)}{\psi^{\alpha-1}(T)} - \frac{\psi^{\alpha-1}(t_1)}{\psi^{\alpha-1}(T)} \right] \int_0^T \psi'(s) \left(\psi(T) - \psi(s)\right)^{\alpha-1} \mathrm{d}s \\
&- \frac{N_i}{\Gamma(\alpha)} \left[\int_0^{t_2} \psi'(s) \left(\psi(t_2) - \psi(s)\right)^{\alpha-1} \mathrm{d}s \right], \\
&+ \frac{N_i}{\Gamma(\alpha)} \left[\int_0^{t_1} \psi'(s) \left(\psi(t_1) - \psi(s)\right)^{\alpha-1} \mathrm{d}s \right], \\
&\leq \frac{N_i}{\Gamma(\alpha+1)} \psi(T) \left[\psi^{\alpha-1}(t_2) - \psi^{\alpha-1}(t_1) \right] + \psi^{\alpha}(t_1) - \psi^{\alpha}(t_2),
\end{aligned}$$

tends to zero as $t_1 \to t_2$. Thus, $\mathscr{F}$ is equicontinuous, so $\mathscr{F}$ is relatively compact on $E_{r'}$. Hence, by theorem 1.3, $\mathscr{F}$ is compact on $E_{r'}$.

step 04 : We consider $\Delta = \{u \in E : u = \delta \mathscr{F} u\}$, such that $0 < \delta < 1$.
Let $u \in \delta \Rightarrow u = \delta \mathscr{F} u$, then $\forall t \in [0, T]$, we have :

$$u(t) = \delta \int_0^T G(t,s) f_i(s, u(s)) \, \mathrm{d}s.$$

By using $\overline{(H2)}$, and $\forall t \in [0, T]$ we get

$$\begin{aligned}
|\mathscr{F}_i u(t)| &\leq \int_0^T G(t,s) \left| f_i(s, u(s)) \right| \mathrm{d}s, \\
&\leq N_i \int_0^T G(s,s) \, \mathrm{d}s, \\
&\leq N_i \frac{\psi^{\alpha}(T)}{\Gamma(\alpha+1)} = K,
\end{aligned}$$

this implies that Δ is bounded.
As a consequence of theorem 1.5 we deduce that $\mathscr{F}$ has at least one fixed point which is a solution of the problem (3.1)-(3.2) on $[0, T]$. The proof is complete. □

3.3 Examples

In this part, we present an applied example to support the theoretical results we reached in the previous results.
To begin with, $\psi(x) = \frac{x^\rho}{\rho}$ which is the KATUGAMPOLA operator, then we get from the problem (3.1) :

$$\begin{cases} {}^{\rho}\mathscr{D}^{\alpha} u_1(t) + f_1(t, u(t)) = 0, \; 0 < t < T, \\ {}^{\rho}\mathscr{D}^{\alpha} u_2(t) + f_2(t, u(t)) = 0, \; 0 < t < T, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ {}^{\rho}\mathscr{D}^{\alpha} u_n(t) + f_n(t, u(t)) = 0, \; 0 < t < T, \end{cases} \tag{3.9}$$

with the boundary conditions:

$$u_1(0) = u_2(0) = \cdots = u_n(0) = 0 \text{ and } \; u_1(T) = u_2(T) = \cdots = u_n(T) = 0, \tag{3.10}$$

where $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$, for $n \in \mathbb{N}^*$. Also $1 < \alpha < 2$ and $f_i : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions for every $i \in \overline{1, n}$.
Therefore, ψ is continuous and increasing function also $\psi(0) = 0$. We set $\alpha = \frac{5}{3}$ and $\rho = 1$.

Example 1. For $t \in \left[0, \frac{\pi}{4}\right]$, we consider the following problem:

$$\begin{aligned} &{}^{1}\mathscr{D}_{0^+}^{\frac{5}{3}} u_1(t) + \frac{\cos(t)\left(\pi\left(\sqrt{2}\cos(t) + \sin(t)\right)\right)^{-1}}{1 + \frac{1}{2}\left(|u_1(t)| + |u_2(t)| + |u_3(t)|\right)} = 0, \\ &{}^{1}\mathscr{D}_{0^+}^{\frac{5}{3}} u_2(t) + \frac{1}{\sqrt{7} + |u_1(t)|} + \frac{1}{\sqrt{8} + |u_2(t)|} + \frac{1}{3 + |u_3(t)|} = 0, \\ &{}^{1}\mathscr{D}_{0^+}^{\frac{5}{3}} u_3(t) + \frac{\tan(t)}{1 + \frac{1}{5}|u_1(t)| + \frac{1}{10}|u_2(t)| + \frac{1}{15}|u_3(t)|} = 0, \\ &u_1(0) = u_2(0) = u_3(0) = 0 \text{ and } \; u_1\left(\frac{\pi}{4}\right) = u_2\left(\frac{\pi}{4}\right) = u_3\left(\frac{\pi}{4}\right) = 0. \end{aligned} \tag{3.11}$$

Let

$$\begin{aligned}
f_1(t, u_1, u_2, u_3) &= \frac{cos(t)\left(\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)\right)^{-1}}{1+\frac{1}{2}(|u_1|+|u_2|+|u_3|)}, \\
f_2(t, u_1, u_2, u_3) &= \frac{1}{\sqrt{7}+|u_1|}+\frac{1}{\sqrt{8}+|u_2|}+\frac{1}{3+|u_3|}, \\
f_3(t, u_1, u_2, u_3) &= \frac{\tan(t)}{1+\frac{1}{5}|u_1|+\frac{1}{10}|u_2|+\frac{1}{15}|u_3|}.
\end{aligned}$$

Clearly $f_i : \left[0, \frac{\pi}{4}\right] \times \mathbb{R}^3 \to \mathbb{R}$ are continuous functions for every $i = 1, 2, 3$. For any $t \in \left[0, \frac{\pi}{4}\right]$, thus

$$\begin{aligned}
\left|f_1(t, u_1, u_2, u_3) - f_1(t, v_1, v_2, v_3)\right| &\le \sum_{k=1}^{3} \frac{1}{2\pi}|u_k - v_k|, \\
\left|f_2(t, u_1, u_2, u_3) - f_2(t, v_1, v_2, v_3)\right| &\le \sum_{k=1}^{3} \frac{1}{6+k}|u_k - v_k|, \\
\left|f_3(t, u_1, u_2, u_3) - f_3(t, v_1, v_2, v_3)\right| &\le \sum_{k=1}^{3} \frac{1}{5k}|u_k - v_k|,
\end{aligned}$$

Hence, condition $(H1)$ is satisfied with $\lambda_1 = \frac{1}{2\pi}$, $\lambda_2 = \frac{1}{7}$ and $\lambda_3 = \frac{1}{5}$, also we get

$$\begin{aligned}
\frac{n\lambda_1 T^{\rho\alpha}}{\rho\Gamma(\alpha+1)} &= \frac{3\times\left(\frac{\pi}{4}\right)^{\frac{5}{3}}}{2\pi\times\Gamma\left(\frac{8}{3}\right)} \simeq 0.212166\ldots < 1, \\
\frac{n\lambda_2 T^{\rho\alpha}}{\rho\Gamma(\alpha+1)} &= \frac{3\times\left(\frac{\pi}{4}\right)^{\frac{5}{3}}}{7\times\Gamma\left(\frac{8}{3}\right)} \simeq 0.190440\ldots < 1, \\
\frac{n\lambda_3 T^{\rho\alpha}}{\rho\Gamma(\alpha+1)} &= \frac{3\times\left(\frac{\pi}{4}\right)^{\frac{5}{3}}}{5\times\Gamma\left(\frac{8}{3}\right)} \simeq 0.266616\ldots < 1.
\end{aligned}$$

Therefore, (3.5) is satisfied, then all conditions of theorem 3.1 hold. Consequently, the problem (3.11) has a unique solution.

By the same method on remark 2.2 we can get the next result

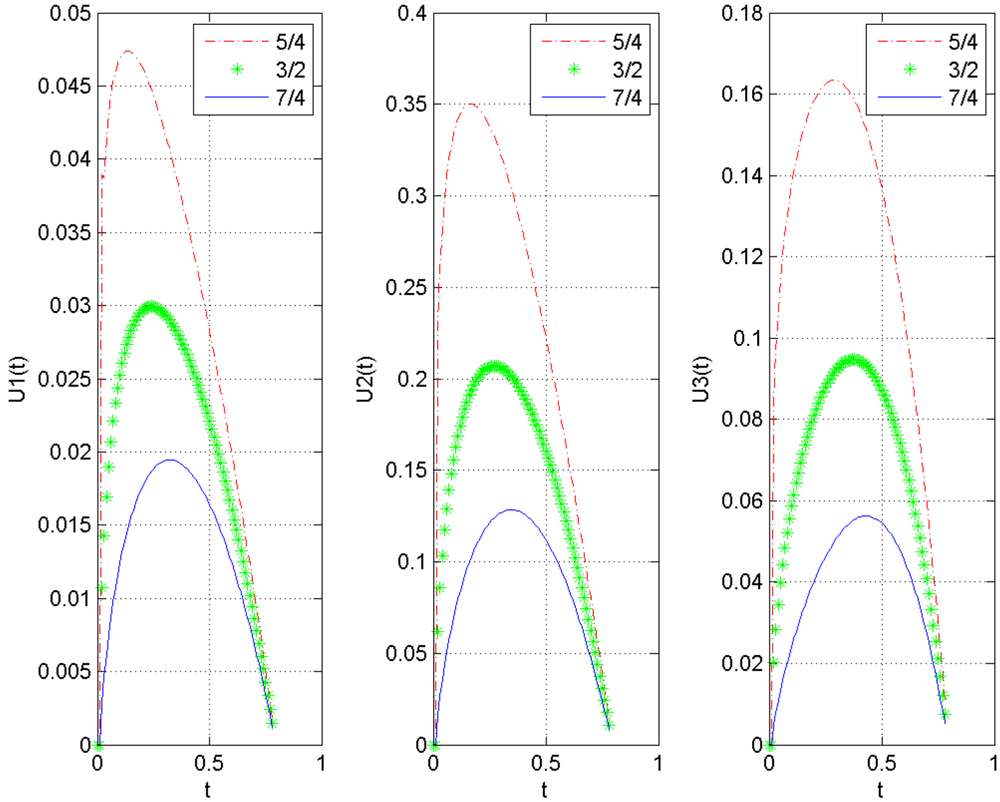

Figure 3.1: Solutions of problem (3.11) for different values of fractional orders when $\psi(t)=\frac{t^{\rho}}{\rho}$.

Example 2. For $t\in\left[0,\frac{\pi}{4}\right]$, we consider the following problem:

$$\begin{aligned}
&{}^{1}\mathscr{D}_{0^+}^{\frac{5}{3}}u_1(t)+\frac{\left(\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)\right)^{-1}}{1+|u_1(t)|+|u_2(t)|}=0,\\
&{}^{1}\mathscr{D}_{0^+}^{\frac{5}{3}}u_2(t)+\tan(t)\left(1+\frac{1}{6+|u_1(t)|}+\frac{1}{4+|u_2(t)|}\right)=0,\\
&u_1(0)=u_2(0)=0 \text{ and } u_1\left(\frac{\pi}{4}\right)=u_2\left(\frac{\pi}{4}\right)=0.
\end{aligned}\tag{3.12}$$

Let

$$\begin{aligned}
f_1(t,u_1,u_2,u_3) &= \frac{\left(\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)\right)^{-1}}{1+|u_1|+|u_2|},\\
f_2(t,u_1,u_2,u_3) &= \tan(t)\left(1+\frac{1}{6+|u_1|}+\frac{1}{4+|u_2|}\right).
\end{aligned}$$

Clearly, for each $t\in\left[0,\frac{\pi}{4}\right]$, the functions f_i are jointly continuous for all $u_i\in\mathbb{R}$ with

$i = 1, 2$. We have

$$\begin{aligned} \left|f_1(t, u_1, u_2)\right| &\leq \frac{1}{\pi}, \\ \left|f_2(t, u_1, u_2)\right| &\leq \frac{17}{12}. \end{aligned}$$

Thus, the hypothesis $\overline{(\mathrm{H2})}$ is satisfied with $N_1 = \frac{1}{\pi}$ and $N_2 = \frac{17}{12}$.
Hence, all conditions of theorem 3.2 hold, consequently the problem (3.12) has at least one solutions.
By the same method on remark 2.2 we can get the next result

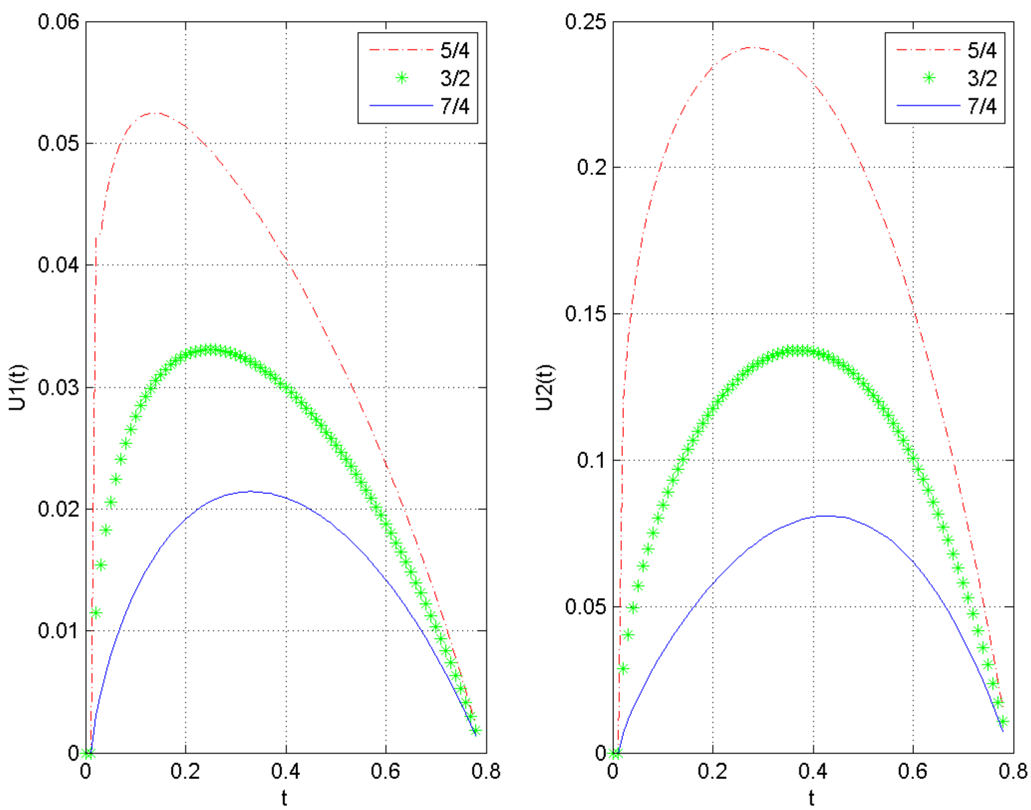

Figure 3.2: Solutions of problem (3.12) for different values of fractional orders when $\psi(t) = \frac{t^\rho}{\rho}$.

Additionally, let $\psi(t) = \ln t$ which is the HADAMARD operator, then we get from the problem (3.1) :

$$\begin{cases} {}^{H}\mathscr{D}^{\alpha} u_1(t) + f_1(t, u(t)) = 0, \ 1 < t < T, \\ {}^{H}\mathscr{D}^{\alpha} u_2(t) + f_2(t, u(t)) = 0, \ 1 < t < T, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ {}^{H}\mathscr{D}^{\alpha} u_n(t) + f_n(t, u(t)) = 0, \ 1 < t < T, \end{cases} \tag{3.13}$$

with the boundary conditions:

$$u_1(1) = u_2(1) = \cdots = u_n(1) = 0 \text{ and } u_1(T) = u_2(T) = \cdots = u_n(T) = 0, \tag{3.14}$$

where $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$, for $n \in \mathbb{N}^*$. Also $1 < \alpha < 2$ and $f_i : [1, T] \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions for every $i \in \overline{1, n}$.
Therefore, ψ is continuous and increasing function also $\psi(1) = 0$. We set $\alpha = \frac{3}{2}$.
Example 3. For $t \in [1, e]$, we consider the following problem:

$$\begin{gathered} {}^{H}\mathscr{D}_{1^+}^{\frac{3}{2}} u_1(t) + \frac{1}{9(t+7)} \times \frac{|u_2(t)|}{1 + |u_2(t)|} + \frac{\arctan(u_1(t))}{23} = 0, \\ {}^{H}\mathscr{D}_{1^+}^{\frac{3}{2}} u_2(t) + 2e^{-3t} + \frac{\arctan(u_1(t))}{8} + \frac{2\arctan(u_2(t))}{3} = 0, \\ u_1(1) = u_2(1) = 0 \text{ and } u_1(e) = u_2(e) = 0. \end{gathered} \tag{3.15}$$

Let

$$\begin{aligned} f_1(t, u_1, u_2) &= \frac{1}{9(t+7)} \times \frac{|u_2|}{1 + |u_2|} + \frac{\arctan(u_1)}{23}, \\ f_2(t, u_1, u_2) &= 2e^{-3t} + \frac{\arctan(u_1)}{8} + \frac{2\arctan(u_2)}{3}. \end{aligned}$$

Clearly $f_i : [1, e] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions for every $i = 1, 2$. For any $t \in [1, e]$, thus

$$\begin{aligned} \left| f_1(t, u_1, u_2) - f_1(t, v_1, v_2) \right| &\le \frac{1}{23} |u_1 - v_1| + \frac{1}{45} |u_2 - v_2|, \\ \left| f_2(t, u_1, u_2) - f_2(t, v_1, v_2) \right| &\le \frac{1}{8} |u_1 - v_1| + \frac{5}{12} |u_2 - v_2|. \end{aligned}$$

Hence, condition $(H1)$ is satisfied with $\lambda_1 = \frac{1}{23}$ and $\lambda_2 = \frac{5}{12}$, also we get

$$\frac{n\lambda_1 (\ln T)^{\alpha}}{\Gamma(\alpha+1)} = \frac{2 \times (\ln e)^{\frac{3}{2}}}{23 \times \Gamma\left(\frac{5}{2}\right)} \simeq 0.065413\ldots < 1,$$

$$\frac{n\lambda_2 (\ln T)^{\alpha}}{\Gamma(\alpha+1)} = \frac{2 \times 5 (\ln e)^{\frac{3}{2}}}{12 \times \Gamma\left(\frac{5}{2}\right)} \simeq 0.626877\ldots < 1.$$

Therefore, (3.5) is satisfied, then all conditions of theorem 3.1 hold. Consequently, the problem (3.15) has a unique solution.

By the same method on remark 2.2 we can get the next result

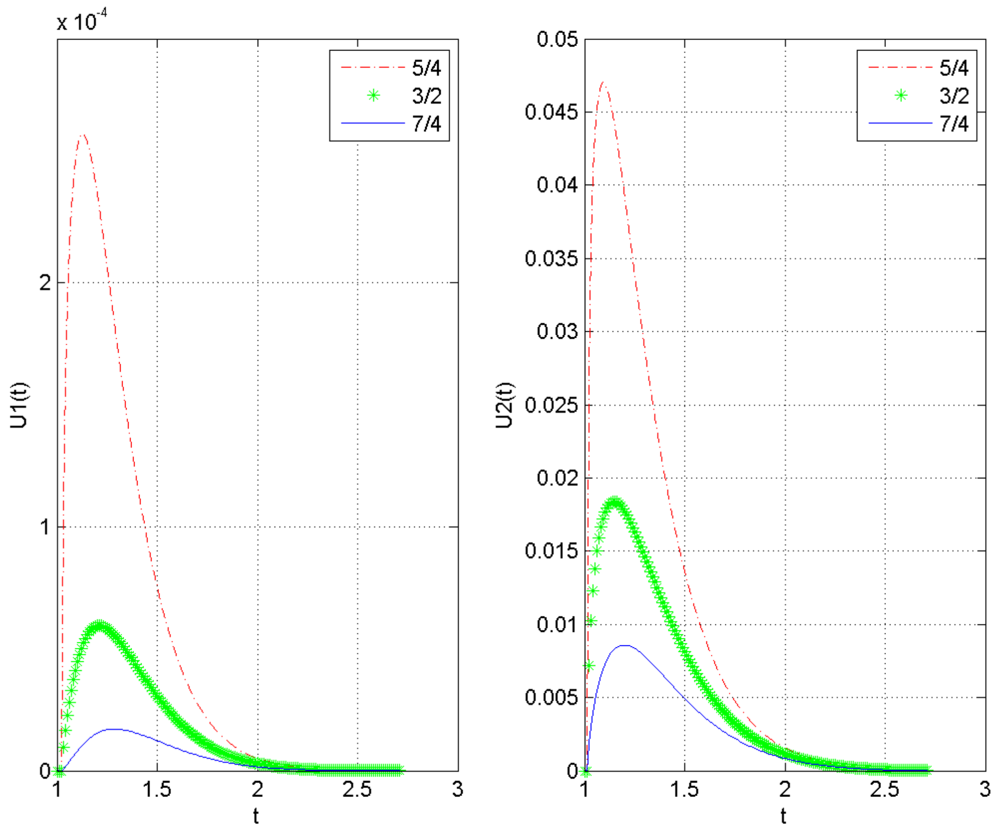

Figure 3.3: Solutions of problem (3.15) for different values of fractional orders when $\psi(t) = \ln t$.

Example 4. For $t \in [1,2]$, we consider the following problem:

$$\begin{gathered}
{}^{H}\mathscr{D}_{1^{+}}^{\frac{3}{2}} u_1(t) + \frac{2t\, e^{-\left(u_1^2(t)+u_2^2(t)\right)}}{1+u_1^2(t)+u_2^2(t)} = 0, \\
{}^{H}\mathscr{D}_{1^{+}}^{\frac{3}{2}} u_2(t) + \frac{2t\, e^{-\left(u_1^2(t)\right)}}{1+u_1^2(t)+u_2^2(t)} = 0, \\
u_1(1) = u_2(1) = 0 \text{ and } u_1(2) = u_2(2) = 0.
\end{gathered} \tag{3.16}$$

Let

$$\begin{aligned}
f_1(t, u_1, u_2) &= \frac{2t\, e^{-\left(u_1^2+u_2^2\right)}}{1+u_1^2+u_2^2}, \\
f_2(t, u_1, u_2) &= \frac{2t\, e^{-\left(u_1^2\right)}}{1+u_1^2+u_2^2}.
\end{aligned}$$

Clearly, for each $t \in [1,2]$, the functions f_i are jointly continuous for all $u_i \in \mathbb{R}$ with $i = 1, 2$. We have

$$\left| f_i(t, u_1, u_2) \right| \leq 1 \text{ for } i = 1, 2.$$

Thus, the hypothesis $\overline{\text{(H2)}}$ is satisfied with $N_i = 1$ with $i = 1, 2$.

Hence, all conditions of theorem 3.2 hold, consequently the problem (3.16) has at least one solutions.

By the same method on remark 2.2 we can get the next result

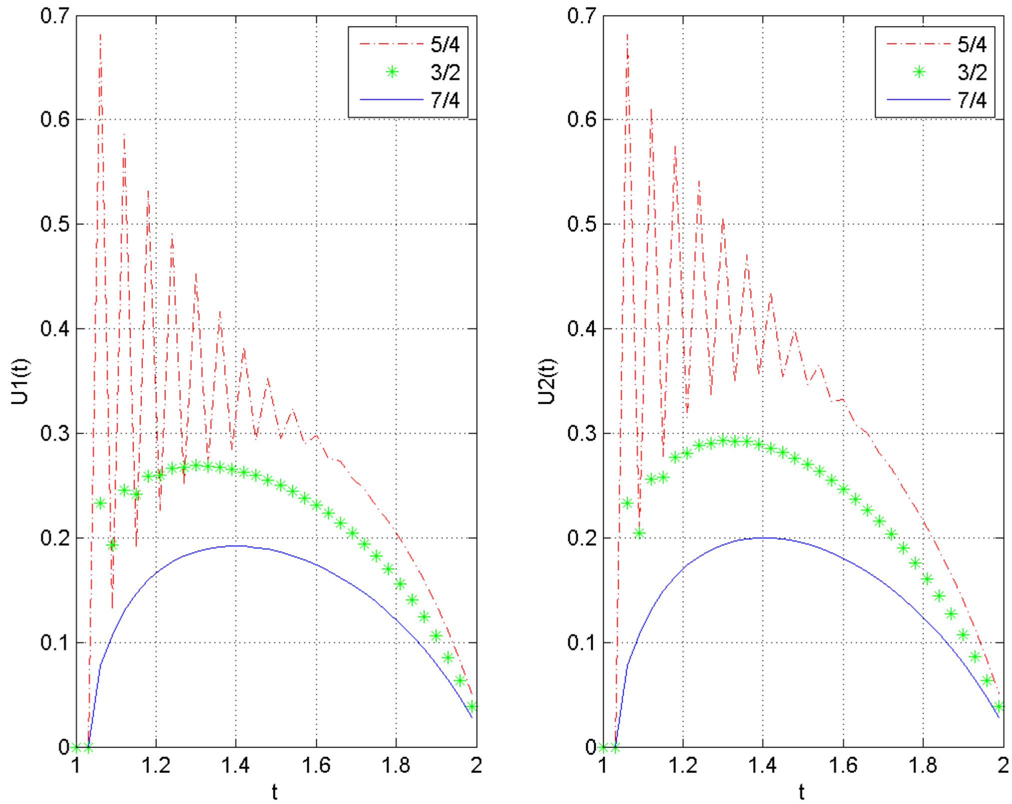

Figure 3.4: Solutions of problem (3.16) for different values of fractional orders when $\psi(t) = \ln t$.

Conclusion

In this thesis we had devoted to study the existence and uniqueness of solutions for certain classes of ψ-fractional order differential equations, also the existence and uniqueness for a system of n nonlinear ψ-fractional differential equations.

This study is part of giving several existence and uniqueness results of solutions for certain classes of FPDEs, thus, realizing the fractional derivative of ψ operator in BANACH spaces. These studies were done mainly using BANACH's contraction principle and SCHAEFER's fixed point theorem.

The first chapter allowed us to commence introducing some basic definitions in fractional calculus. Also we have presented different definitions for the operators of integration and derivatives particularly which is related to the ψ operator. Moreover, we have touched some theorems on fixed point theory.

We have discussed in the second chapter the existence and uniqueness of solutions of ψ-fractional differential equation which is generalized some previous work as we can see in [10],we have proofed it by means of the BANACH contraction principle and SCHAEFER's fixed point theorem, we have provided our results with some examples and we have gotten numerical results.

In the third chapter, we have used the BANACH contraction principle and SCHAEFER's fixed point theorem to explore the existence and main properties of at least one solution and its uniqueness for a class of a new system of n nonlinear fractional differential equations with n boundary conditions, with ψ-fractional derivative being used as the differential operator, and which is crucial for generalizing RIEMANN-LIOUVILLE, HADAMARD and KATUGAMPOLA fractional derivatives into a single form, we have provided the results with some examles to illustrate the obtained outcomes.

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Abstract : This work studies the existence and uniqueness of solutions for a class of non-linear fractional differential equations and their system via the ψ-fractional derivatives with boundary values. The arguments for the study are based upon the BANACH contraction principle, SCHAEFER fixed point theorem. Moreover, some examples are shown to appear the strength of our significant result.

Keywords: Ψ-fractional differential equations, fixed point, n nonlinear ψ -fractional differential equations, existence, uniqueness, ψ- fractional operator , a system of differential equations.

Résumé: Ce travail étudie l'existence et l'unicitè de solutions pour une classe d'équations différentielles fractionnaires non-linéaires et leur système via les dérivées fractionnaires de l'opérateur ψ avec conditions aux limites. Les arguments de l'étude sont basés sur le principe de contraction BANACH et le théorème du point fixe SCHAEFER. De plus, quelques exemples sont présentés pour illustrer la force de nos résultats significatifs.

Mot-clés : Equations différentielles fractionnaires de l'opérateur ψ, point fixe, n-équations différentielles fractionnaires non linéaires avec opérateur ψ, existence, unicitè, l'opérateur fractionnaire ψ, système d'équations différentielles.

الملخص: يدرس هذا العمل وجود ووحدانية الحلول لفئة من المعادلات التفاضلية ذات اشتقاقات جزئية غير خطية بالمحدد ψ ونظام لمعادلات تفاضلية ذات رتبة كسرية بالمحدد ψ بشروط حدودية، تعتمد براهين الدراسة على نظرية النقطة الثابتة لبناخ وشوفر.

علاوة على ذلك، تظهر بعض الامثلة قوة النتائج المهمة التي توصلنا اليها.

الكلمات المفتاحية: معادلات تفاضلية ذات اشتقاقات جزئية ذات رتبة كسرية، نقطة ثابتة، n-معادلة تفاضلية ذات اشتقاقات جزئية ذات رتبة كسرية بالمحدد بسي، وجود، وحدانية،

المشتق ψ الكسري، نظام لمعادلات تفاضلية.