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*Numerical solution of Volterra integro-differential equations
using Vieta-Pell polynomials*

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المخلص:

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الكلمات المفتاحية: المعادلات التفاضلية التكاملية لفولتيرا, معادلات فولتيرا التكاملية, كثير فيتا-بيل, طريقة التجميع .

Résumé

L'objectif principal de cette thèse est de trouver une solution approchée a l'équation integro-différentielle linéaire de Volterra en utilisant les polynômes de Veita-Pell. Divers exemples sont fournis pour démontrer l'exactitude et l'efficacité de la méthode proposée.

Mots clés : Equation integro-différentielle de Volterra, équation intégrale de Volterra, polynôme de Veita-Pell, Méthode de collocation.

Abstract:

The main objective of this thesis is to find an approximate solution of linear integro differential equation of Volterra using Veita -Pell polynomials. Various examples are provided to demonstrate the accuracy and effectiveness of the proposed method.

Key words: Volterra integro-differential equation, Volterra integral equation, Veita-Pell polynomial, Collocation Method.

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List of symbols

\mathbb{N}	a set of real numbers
\mathbb{R}	a set of complex
\mathbb{C}	a set of natural numbers
$[a, b]$	Real interval
$C([a, b])$	The space of continuous functions on $[a, b]$
f	Free term in the integral equation
A	Linear operator
I	Identity operator
$\ \cdot\ $	Norm
\int	Integral sign
Σ	Summation sign
\langle, \rangle	Scalar product
$k(x, t)$	Kernel of the integral equation
$\varphi(x)$	The unknown function in the integral equation
φ_{app}	Approximate solutions
λ	Numerical parameter
ε	epsilon
VPP_S	Veita-Pell polynomials

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Introduction

In recent years there has been a growing interest in the integro differential equation. Integro-differential equations be an important branch of modern mathematics, arising frequently in many fields such as mechanics, the theory of elasticity, potential, and mathematical physics.

The Volterra integro-differential equation has the following form:

$$\sum_{j=0}^m P_j(x)\varphi^{(j)}(x) = f(x) + \int_a^x k(x,t)\varphi(t), \quad a \leq x \leq b, \quad m \geq 1$$

with the initial conditions:

$$\varphi^{(j)}(a) = \alpha_j, \quad 0 \leq j \leq m - 1$$

where $\varphi^{(j)}(x)$ is the j^{th} derivative of the unknown function $\varphi(x)$ that will be determined, $k(x,t)$, $P_j(x)$ and $f(x)$ are known functions, α_j , are real finite constants

Volterra integro-differential equation are a significant class of equation that combine a spectrum of both integral and differential equation. Named after the Italian mathematician Vito Volterra, usually this type of equations can't be solved analytically so we used the branch of numerical approximation

The Vieta-Pell polynomials combine the principles of Vieta's algebraic analysis with Pell's Diophantine equation. These polynomials are used in numerical and analytical methods to solve a variety of mathematical and physical problems, these tools were developed to provide more accurate approximate solutions to complex problems.

The objective of this thesis is to present a numerical solution for Volterra integro-differential equations, we will particularly focus on the Vieta-Pell collocation and Vieta-Pell Galerkin method, in other words the objective can be summarized in two points:

1-Numerical approximation of the solution to the Volterra integro-differential equation using Vieta-Pell polynomials.

2-Verification of the accuracy and efficiency of the proposed method through examples.

This thesies consists of three chapters that cover different aspects of the numerical solution of volterra integro-differential equations, It is organized as follows:

The first chapter: Establishes the foundation concepts necessary to understand the issues discussed .In introduces they key definition and the basic mathematical tools that will be used throughtout the study such as a fonction spaces like normed vector space, Banach space, Hilbert space, and concept of operators like linear and Bounded operators...

The second chapter:We are addrising linear integral equations and integro-differential equations.Additionly we focuses on analytical methods for solving Volterra intego-differential equations, we present different analytical approaches to obtain exact solutions,

In **third chapter**, we present an approximate numerical solution for the Volterra integro-differential equation of the second kind, using the collocation method with Veita-Pell polynomials.Various examples are also provided to verify the effectiveness of the proposed method.

Chapter 1

Recalls of function analysis

1.1 Functional spaces

This chapter is essentially dedicated to the introduction of some fundamental concepts and certain definition and theorems that we will use in the other chapters.

1.1.1 Normed vector space

Definition 1.1.1 *Let V be a vector space over \mathbb{k} ($\mathbb{k} = \mathbb{R}$ or \mathbb{C}), a mapping $\|\cdot\| : V \rightarrow \mathbb{R}_+$ is called norm if the following condition hold:*

For all $x \in V$ and $\alpha \in \mathbb{k}$

- (i) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$.
- (ii) $\|\alpha x\| = |\alpha| \|x\|$.
- (iii) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

A vector space with a norm $(V, \|\cdot\|)$ is called a normed space.

1. The absolute value is a norm on \mathbb{R} , the modulus is a norm on \mathbb{C} .
2. The following applications are norms on \mathbb{R}^n :

$$\|x\|_1 = \sum_{i=1}^n |x_i|, \quad \|x\|_2 = \left(\sum_{i=1}^n |x_i|^2 \right)^{\frac{1}{2}}, \quad \|x\|_\infty = \sup_{1 \leq i \leq n} |x_i|.$$

3. Let $V = C([a, b], \mathbb{R})$ be the space of real-valued continuous functions on the interval $[a, b]$ for any $f \in V$, the following functions define a norms on V :

$$\|f\|_1 = \int_a^b |f(x)| dx, \quad \|f\|_2 = \left(\int_a^b |f(x)|^2 dx \right)^{\frac{1}{2}}, \quad \|f\|_\infty = \max_{a \leq x \leq b} |f(x)|.$$

1.1.2 Banach space

Definition 1.1.2 (Cauchy sequence). Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in a normed space $(V, \|\cdot\|)$, then $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence if:

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall p, q \in \mathbb{N} : p > q \geq n_0 \implies \|x_p - x_q\| \leq \varepsilon.$$

Definition 1.1.3 Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in a normed space $(V, \|\cdot\|)$, then $(x_n)_{n \in \mathbb{N}}$ is said to be convergent to $x_0 \in V$ if:

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N} : n \geq n_0 \implies \|x_n - x_0\| \leq \varepsilon.$$

Definition 1.1.4 A normed space $(V, \|\cdot\|)$ is complete if every Cauchy sequence of points in V is converges.

Remark 1.1.1 The advantage of complete spaces is that in such spaces, it is not necessary to know the limit of a sequence to show that it converges, it suffices to demonstrate that it is Cauchy.

Definition 1.1.5 (Banach space). A normed vector space $(V, \|\cdot\|)$ is called a Banach space if it is complete. That is every Cauchy sequence (x_n) in V converges to a limit $x \in V$.

- 1- $C([a, b], \mathbb{R})$: The space of all continuous real-valued functions on the interval $[a, b]$,

with the norm $\|f\| = \max_{x \in [a, b]} |f(x)|$.

This space is complete therefore a Banach space.

2- l^p ($1 \leq p < \infty$). The space of all sequences (x_n) such that $\sum_1^\infty |x_n|^p$ is finite, with the norm

$$\|x_n\| = \left(\sum_1^\infty |x_n|^p \right)^{\frac{1}{p}}$$

is complete, hence a Banach space.

3- $L^2([a, b], \mathbb{R})$. The space of all functions f such that the integral of $|f(x)|^2$ over the domain is finite, with the norm:

$$\|f(x)\|_2 = \left(\int_a^b |f(x)|^2 dx \right)^{\frac{1}{2}}$$

This space is complete therefore a Banach space.

Theorem 1.1.1 *Every finite-dimensional normed vector space is a Banach space.*

1.1.3 Hilbert space

Definition 1.1.6 (*Inner product*). A scalar product on a real or complex vector space V is a mapping $V \times V \rightarrow \mathbb{C}$, written $\langle x, y \rangle$ that satisfies:

- (i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ implies $x = 0$.
- (ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ in complex spaces and $\langle x, y \rangle = \langle y, x \rangle$ in real ones for all $x, y \in V$.
- (iii) $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$, for all $x, y \in V$ and scalar λ .
- (iv) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$, for all $x, y \in V$, and $z \in V$.

An inner product space V is a real or complex vector space with a scalar product on it.

Example 1.1.1 *Here is some examples of inner product space which demonstrate that expression $\|x\| = \sqrt{\langle x, x \rangle}$ defines a norm.*

1. The inner product for \mathbb{R}^n was defined in the beginning of this section the inner product for \mathbb{C}^n is given by

$$\langle x, y \rangle = \sum_1^n x_j \overline{y_j}$$

2. Let extension for infinite vectors, let l_2 be

$$l_2 = \left\{ \text{sequences } \{x_j\}_1^\infty : \sum_1^n |x_j|^2 < \infty \right\}.$$

3. Let $C([a, b])$ be a space of continuous function on the interval $[a, b] \subset \mathbb{R}$:

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx \text{ and } \|f\|_2 = \left(\int_a^b |f(x)|^2 dx \right)^{\frac{1}{2}}.$$

Definition 1.1.7 A compact inner product space is Hilbert space.

1.2 Concepts of operators

1.2.1 Linear operators

Definition 1.2.1 Let E and F be two normed spaces, an operator $A : E \rightarrow F$ is called to be linear if and only

$$\forall \varphi_1, \varphi_2 \in E, \lambda_1, \lambda_2 \in \mathbb{k} = (\mathbb{R} \text{ or } \mathbb{C}), \text{ we have } A(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1A(\varphi_1) + \lambda_2A(\varphi_2).$$

Definition 1.2.2 (Continuity of linear operators). Let E and F be two normed spaces, a linear operator A defined on a subset $G \subset E$ to F is called continuous in x of G if:

- for all sequence (x_n) in G converges toward x_0 , the sequence $A(x_n)$ converges toward $A(x_0)$, that mean

$$\lim_{n \rightarrow \infty} A(x_n) = A(\lim_{n \rightarrow \infty} x_n) = A(x_0).$$

Remark 1.2.1 The linear operator A is called continuous in G if it is continuous in each point of the set G .

1.2.2 Bounded operators

Definition 1.2.3 A linear operator $A : E \rightarrow F$ is called bounded if there exists a positive constant $C > 0$ such that:

$$\|A(x)\|_F \leq C \|x\|_E, \forall x \in E. \tag{1.2.1}$$

Proposition 1.2.1 *The smallest constant C satisfying the relation (1.2.1) is called the norm of A , denoted as, and is given by:*

$$\|A\| = \sup_{x \neq 0} \frac{\|A(x)\|_F}{\|x\|_E} = \sup_{\|x\|=1} \|A(x)\|_F = \sup_{\|x\| \leq 0, x \neq 0} \|A(x)\|_F.$$

Theorem 1.2.1 *A linear operator A is continuous if and only if it is bounded.*

proof. We suppose that the operator A is bounded, we have

$$\|A(x)\|_F \leq C \|x\|_E,$$

or

$$\|A(x) - A(0)\|_F \leq C \|x - 0\|_E,$$

where the continuity of the operator A in 0 such that

$$\lim_{n \rightarrow \infty} A(x) = A(0) \text{ where } \lim_{n \rightarrow \infty} x = 0,$$

which leads to continuity everywhere. ■

1.2.3 Compact operators

Definition 1.2.4 *(compact linear operators). A linear operator A defined from a normed space E into a normed space F is called a compact linear operator or completely continuous linear operator if for every bounded subset Ω of E , the image $A(\Omega)$ is relatively compact in F , in other words, the closure $\overline{A(\Omega)}$ is compact.*

Theorem 1.2.2 *The linear combination $A = \alpha A_1 + \beta A_2$ of compact operators A_1 and A_2 is compact operator, for every scalars α and β .*

proof. Let φ_n be a bounded sequence in E and let $A\varphi_n$ be a sequence in F then

$$A\varphi_n = \alpha A_1\varphi_n(x) + \beta A_2\varphi_n(x), \text{ with } \varphi_n \in E, n \in \mathbb{N},$$

the operators A_1 and A_2 are compact, one can extract from $A_1\varphi_n$ and $A_2\varphi_n$ two convergent subsequences wick given by thier sum a convergent subsequence of $A\varphi_n$. Hence A is compact.

■

Theorem 1.2.3 (finite dimensional domain). *Let A be bounded operator defined from E into F with the domain E has a finite dimension, $\dim E < \infty$ then the operator A is compact.*

proof. Indeed the space E has a finite dimension $\dim E < \infty$ implies the finite dimensional range $A(E)$, say

$$\dim A(E) \leq \dim E,$$

it follows, A is a compact operator. ■

Theorem 1.2.4 *The identity operator I_d defined from a normed space E into E is compact if and only if the space E has a finite dimension.*

Corollary 1.2.1 *A bounded operator A in a normed space E is not generally a compact operator.*

Indeed, see the identity operator $A = I$ in the infinitely dimensional normed space is not compact.

1.2.4 Integral operators

Integral operators are fundamental objects in functional analysis, where they are particularly useful in transforming function into simpler version for easier resolution.

The integral operator is defined as any linear operator A defined on a normed space E with values in a normed space F , given in the form :

$$A\varphi(x) = \int_{G_2} k(x, y)\varphi(y)dy, x \in G_1,$$

the function $k(x, y)$ is a measurable function defined on a measurable set $G_1 \times G_2$ and $\varphi(y)$ is a measurable function defined on G_1 .

$k(x, y)$ is called the kernel of the integral operator A .

Definition 1.2.5 (Normes of integral operators)

The norm of the integral operators defined on $L^p(G_1)$, here p and q are conjugate exponents ($\frac{1}{p} + \frac{1}{q} = 1$) with ($1 \leq p, q \leq \infty$), the norm of the operator A is given by:

$$\|A\|_p = \begin{cases} \int_{G_1} \left(\int_{G_2} (|k(x, y)|^q dy)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}, & \text{pour } 1 < p < \infty \\ \int_{G_1} \text{ess sup}_y |k(x, y)| dx, & \text{pour } p = 1 \\ \text{ess sup}_x \int_{G_2} |k(x, y)| dy, & \text{pour } p = \infty \end{cases}$$

Theorem 1.2.5 *Let A is an integral operator with a finit norm*

$$\|A\|_p < \infty,$$

then the integral operator A is a continous linear operator from $L^p(G_1)$ to $L^p(G_2)$, Additionally we have

$$\|A\varphi\|_p \leq \|A\|_p \|\varphi\|_p.$$

Chapter 2

Integral and integro- differential equations

In this chapter, some definitions and examples are given for linear integral equation and linear integro-differential equation.

2.1 Classification of integral equations

Integral equations appear in many types. The types depend mainly on the limits of integration and the kernel of the equation, in this text we will be concerned on the following types of integral equation.

2.1.1 Fredholm integral equations

Definition 2.1.1 *The most standard form of Fredholm linear integral equations is given by the form:*

$$\phi(x)\varphi(x) = f(x) + \lambda \int_a^b k(x, t)\varphi(t)dt, \quad (2.1.1)$$

where the limits of integration a and b are constants and the unknown $\varphi(x)$ appear linearly under the integral sing:

1. If the function $\phi(x) = 0$, then (2.1.1) yields

$$f(x) + \lambda \int_a^b k(x, t)\varphi(t)dt = 0,$$

wich is called Fredholm integral equation of the first kind.

2. If the function $\phi(x) = 1$, the (2.1.1) becomes simply

$$\varphi(x) = f(x) + \lambda \int_a^b k(x, t)\varphi(t)dt,$$

and this equation called Fredholm integral equation of second kind.

2.1.2 Volterra integral equations

Definition 2.1.2 *The most standard form of Volterra linear integral equation is of the form*

$$\phi(x)\varphi(x) = f(x) + \lambda \int_a^x k(x, t)\varphi(t)dt. \quad (2.1.2)$$

where the limits of integration are function of x and the unknown function $\varphi(x)$ appear linearly under the integral sing

1. If the function $\phi(x) = 0$, then equation (2.1.2) becomes

$$f(x) + \lambda \int_a^x k(x, t)\varphi(t)dt = 0,$$

wich is known as the Volterra equation of the first kind.

2. If the function $\phi(x) = 1$, then equation (2.1.2) simply becomes

$$\varphi(x) = f(x) + \lambda \int_a^x k(x, t)\varphi(t)dt,$$

and this equation is know as the Volterra integral equation of the second kind.

2.1.3 Volterra-Fredholm integral equations

The Volterra-Fredholm integro differential equation, wich is a combination of disjoint Volterra and Fredholm and differetial operator can appear in a single integral form.

Definition 2.1.3 It calls the Volterra- Fredholm intgral equation an equation of the form:

$$\varphi(x) = f(x) + \lambda_1 \int_a^x k_1(x, t)\varphi(t)dt + \lambda_2 \int_a^b k_2(x, t)\varphi(t)dt, x \in [a, b],$$

where the functions $k_1(x, t)$, $k_2(x, t)$ and $f(x)$ are known, $\varphi(x)$ is the inknown function and λ_1 and λ_2 are non-zero parameters.

2.2 Classification of integro-differential equations

2.2.1 Fredholm integro-differential equation

The general form of Fredholm integro-differential equation appears in the form:

$$\sum_{k=0}^m P_k(x)\varphi^{(k)}(x) = f(x) + \lambda \int_a^x k(x, t)\varphi(t)dt, m \geq 1,$$

whith the unitial conditions

$$\varphi^{(k)}(a) = a_k, \quad 0 \leq k \leq m - 1$$

where the functions $P_k(x)$, $k(x, t)$, and $f(x)$ are known, $\varphi(x)$ is the unknown function and λ is a parameter.

$\varphi^{(k)}$ indicates the k^{th} derivative of $\varphi(x)$.

1-

$$\begin{cases} \varphi'(x) = 1 - \frac{1}{3}x + \int_0^1 x\varphi(t)dt \\ \varphi(0) = 0 \end{cases}$$

Is a first-order Fredholm integro-differential equation

2-

$$\begin{cases} \varphi''(x) + \varphi'(x) = x - \sin x - \int_0^{\frac{\pi}{2}} xt\varphi(t) \\ \varphi(0) = 0, \varphi'(0) = 1 \end{cases}$$

is a second -order Fredholm integro-differential equation.

2.2.2 Volterra integro-differential equations

The Volterra integro-differential equation appears in the form:

$$\sum_{i=1}^m P_k(x)\varphi^{(k)}(x) = f(x) + \lambda \int_a^x k(x,t)\varphi(t)dt,$$

whith the unitial conditions

$$\varphi^{(k)}(a) = a_k, \quad 0 \leq k \leq m - 1$$

where the functions $P_k(x)$, $k(x,t)$, and $f(x)$ are known, $\varphi(x)$ is the unknown function and λ is a parameter.

$\varphi^{(k)}$ indicates the k^{th} derivative of $\varphi(x)$.

1-

$$\begin{cases} \varphi'(x) = -1 + \frac{1}{2}x^2 - x \exp(x) - \int_0^x t\varphi(t)dt \\ \varphi(0) = 0 \end{cases}$$

is a Volterra integro-differential equation of first order

2-

$$\begin{cases} \varphi''(x) + \varphi(x) = 1 - x(\sin x + \cos x) - \int_0^x t\varphi(t)dt, \\ \varphi(0) = -1, \varphi'(0) = 1. \end{cases}$$

is a Volterra integro-differential equation of second order.

2.2.3 Volterra-Fredholm integro-differential equations

Definition 2.2.1 *The Volterra-Fredholm integro-differential equation, wich is a combination of disjoint Volterra and Fredholm integral along with differential operator, can appear in a single integral from.*

$$\begin{cases} \sum_{i=1}^n P_k(x)\varphi^{(k)}(x) = f(x) + \lambda_1 \int_a^x k_1(x,y)\varphi(t)dt + \lambda_2 \int_a^b k_2(x,y)\varphi(t)dt \\ \varphi^{(k)}(a) = a_k, \quad 0 \leq k \leq n - 1. \end{cases}$$

2.3. Converting of a Volterra integro-differential equation to a Volterra integral equation of the second kind

where $\varphi^{(k)}(x)$ denotes the k^{th} derivation of $\varphi(x)$ and $k_1(x, y)$, $k_2(x, y)$, $P_k(x)$ and $f(x)$ are known functions, λ_1 and λ_2 are non-zero parameters and $\varphi(x)$ is the unknown function to be determined

Example 2.2.1

$$\begin{cases} \varphi'(x) = 24x + x^4 + 3 - \int_0^x (x-t)\varphi(t)dt - \int_0^1 t\varphi(t)dt \\ \varphi(0) = 0 \end{cases}$$

It is a Volterra-Fredholm integro-differential equation of first order.

2.3 Converting of a Volterra integro-differential equation to a Volterra integral equation of the second kind

This section is concerned with converting to Volterra integral equations. We can easily convert the Volterra integro-differential equation to equivalent Volterra integral equation, provided the kernel is a difference kernel defined by $k(x, t) = k(x - t)$. This can be easily done by integrating both sides of the equation and using the initial conditions, we illustrate for the benefit of the reader three specific formulas:

$$\begin{aligned} \int_0^x \left(\int_0^x \varphi(t)dt \right) dx &= \int_0^x (x-t)\varphi(t)dt \\ \int_0^x \int_0^x \left(\int_0^x \varphi(t)dt \right) dx dx &= \frac{1}{2!} \int_0^x (x-t)^2 \varphi(t)dt \\ &\dots \\ \underbrace{\int_0^x \int_0^x \dots \left(\int_0^x \varphi(t)dt \right) dx dx \dots dx}_n &= \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} \varphi(t)dt. \end{aligned}$$

Example 2.3.1 Consider the Volterra integro-differential equation:

$$\begin{cases} \varphi'(x) = 2 - \frac{x^2}{4} + \frac{1}{4} \int_0^x \varphi(t)dt \\ \varphi(0) = 0 \end{cases} \quad (2.3.1)$$

To solve the Volterra integro-differential equation by converting it into a Volterra integral equation, we integrate both sides of equation (2.3.1) from 0 to x and use the unitial condition. Additionally we convert the double integral into a single integral from formule (2.3.1)we obtain

$$\int_0^x \varphi'(x)dx = \int_0^x (2 - \frac{x^2}{4})dx + \frac{1}{4} \int_0^x (\int_0^x \varphi(t)dt)dx$$

$$\varphi(x) - \varphi(0) = 2x - \frac{x^3}{12} + \frac{1}{4} \int_0^x (x - t)\varphi(t)dt$$

so

$$\varphi(x) = 2x - \frac{x^3}{12} + \frac{1}{4} \int_0^x (x - t)\varphi(t)dt.$$

2.4 Analytical solution of Volterra integro- differential equations

2.4.1 The series solution method

The real function $\varphi(x)$ is called analytic if it has derivatives of all orders such that the Taylor series at any point b in its domain

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(b)}{n} (x - b)^n, \tag{2.4.1}$$

converges to $\varphi(x)$ in a neighborhood of b .For simplicity, the generic form of Taylor series at $x = 0$ can be written as

$$\varphi(x) = \sum_{n=0}^{\infty} a_n x^n \tag{2.4.2}$$

In this section we will apply the series solution method for solving Volterra integro-differential equations of the second kind. We will assume that the solution $\varphi(x)$ of the Volterra integro-differential equation.

$$\varphi^{(n)}(x) = f(x) + \lambda \int_0^x k(x, t)\varphi(t)dt \tag{2.4.3}$$

$$\varphi^{(k)}(0) = k!a_k, 0 \leq k \leq (n - 1).$$

is analytic, and therefore possesses a Taylor series of the form given in (2.4.2), where the coefficients an will be determined recurrently.

The first few coefficients a_k can be determined by using the initial conditions:

$$a_0 = \varphi(0), \quad a_1 = \varphi'(0), \quad a_2 = \frac{1}{2!}\varphi''(0), \quad a_3 = \frac{1}{3!}\varphi'''(0) \quad (2.4.4)$$

and so on the remaining coefficients a_k of (2.4.2), will be determined by applying the series solution method to the Volterra integro-differential equation (2.4.3), Substituting (2.4.2), into both sides of (2.4.3), gives:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)^n = T(f(x)) + \int_0^x k(x,t) \left(\sum_{n=0}^{\infty} a_n t^n\right) dt, \quad (2.4.5)$$

or for simplicity we use

$$(a_0 + a_1 x + a_2 x^2 + \dots)^n = T(f(x)) + \int_0^x k(x,t) (a_0 + a_1 t + a_2 t^2 + \dots) dt, \quad (2.4.6)$$

where $T(f(x))$ is the Taylor series for $f(x)$.

Example 2.4.1 Use the series solution method to solve the Volterra integro-differential equation

$$\begin{aligned} \varphi''(x) &= 1 + x + \int_0^x (x-t)\varphi(t)dt, \\ \varphi(0) &= \varphi'(0) = 1. \end{aligned} \quad (2.4.7)$$

substituting $\varphi(x)$ by the series:

$$\varphi(x) = \sum_{n=0}^{\infty} a_n x^n, \quad (2.4.8)$$

into both sides of (2.4.7), leads to

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)'' = 1 + x + \int_0^x ((x-t) \sum_{n=0}^{\infty} a_n t^n) dt, \quad (2.4.9)$$

Differentiating the left side twice, and by evaluating the integral at the right side we find

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = 1 + x + \sum_{n=2}^{\infty} \frac{1}{(n+2)(n+1)} a_n x^{n-2}, \quad (2.4.10)$$

or equivalently

$$2a_2 + 6a_3x + \sum_{n=2}^{\infty} (n+2)(n+1)a_{n+2}x^n = 1 + x + \sum_{n=2}^{\infty} \frac{1}{(n+2)(n+1)} a_n x^{n-2}, \quad (2.4.11)$$

using the initial conditions and equating the coefficients of like powers of x in both sides of (2.4.11), gives the recurrence relation

$$\begin{aligned} a_0 = 1, \quad a_1 = 1, \quad a_2 = \frac{1}{2!}, \quad a_3 = \frac{1}{3!}, \\ a_{n+2} = \frac{1}{(n+2)(n+1)n(n-1)} a_{n-2}, \quad n \geq 2, \end{aligned} \quad (2.4.12)$$

where this result gives:

$$a_n = \frac{1}{n!}, \quad (2.4.13)$$

for $n \geq 0$ Substituting this result into (2.4.8), gives the series solution

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \quad (2.4.14)$$

the converges to the exact solution

$$\varphi(x) = \exp(x).$$

2.4.2 The Adomian Decomposition Method

In this section, we shall introduce the decomposition method and the modified decomposition method to solve the Volterra integro-differential equations. This method appears to be reliable and effective.

Without loss of generality, we may assume a standard form to Volterra integro differential equation defined by the standard form:

$$\varphi^{(n)}(x) = f(x) + \int_0^x k(x,t)\varphi(t)dt, \quad \varphi^{(k)}(0) = b_k, \quad 0 \leq k \leq (n-1), \quad (2.4.15)$$

where $\varphi^{(n)}$ is the n^{th} order derivative of $\varphi(x)$ with respect to x and b_k are constants that defines the initial conditions. It is natural to seek an expression for $\varphi(x)$ that will be derived from equation (2.4.15), this can be done by integrating both sides of equation (2.4.15) from 0 to x as many times as the order of the derivative involved.

Consequently, we obtain

$$\varphi(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x)) + L^{-1} \left(\int_0^x k(x,t) \varphi(t) dt \right), \quad (2.4.16)$$

where $\sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k$ is obtained by using the initial conditions, and L^{-1} is an $n - fold$ integration operator. Now, we are in a position to apply the decomposition method by defining the solution $\varphi(x)$ of equation (2.4.16) on a decomposed series

$$\varphi(x) = \sum_{k=0}^{n-1} \varphi_n(x), \quad (2.4.17)$$

substitution of equation (2.4.17) into both sides of equation (2.4.16) we get

$$\sum_{k=0}^{n-1} \varphi_n(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x)) + L^{-1} \left(\int_0^x k(x,t) \varphi(t) dt \right) \quad (2.4.18)$$

This equation can be explicitly written as:

$$\begin{aligned} \varphi_0(x) + \varphi_1(x) + \varphi_2(x) + \dots &= \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x)) + L^{-1} \left(\int_0^x k(x,t) \varphi_0(t) dt \right) + \\ L^{-1} \left(\int_0^x k(x,t) \varphi_1(t) dt \right) &+ L^{-1} \left(\int_0^x k(x,t) \varphi_2(t) dt \right) + L^{-1} \left(\int_0^x k(x,t) \varphi_3(t) dt \right) + \dots \end{aligned}$$

The components $\varphi_0(x), \varphi_1(x), \varphi_2(x), \varphi_3(x), \dots$ of the unknown function $\varphi(x)$ are determined in a recursive manner, if we set

$$\begin{aligned} \varphi_0(x) &= \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x)), \quad \varphi_1(x) = L^{-1} \left(\int_0^x k(x,t) \varphi_0(t) dt \right), \quad \varphi_2(x) = L^{-1} \left(\int_0^x k(x,t) \varphi_1(t) dt \right), \\ \varphi_3(x) &= L^{-1} \left(\int_0^x k(x,t) \varphi_2(t) dt \right), \quad \varphi_4(x) = L^{-1} \left(\int_0^x k(x,t) \varphi_3(t) dt \right). \end{aligned}$$

and so on. The above equations can be written in a recursive manner as

$$\varphi_0(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x)) \quad (2.4.19)$$

$$\varphi_{n+1}(x) = L^{-1} \left(\int_0^x k(x,t) \varphi_n(t) dt \right), \quad n \geq 0 \quad (2.4.20)$$

Use the Adomian method to solve the Volterra integro-differential equation

$$\varphi'''(x) = -1 + x - \int_0^x (x-t) \varphi(t) dt, \quad \varphi(0) = 1, \quad \varphi'(0) = -1, \quad \varphi''(0) = 1. \quad (2.4.21)$$

Applying the *three – fold* integral operator L^{-1} defined by

$$L^{-1}(\cdot) = \int_0^x \int_0^x \int_0^x (\cdot) dx dx dx, \quad (2.4.22)$$

to both sides of (2.4.21), and using the given initial conditions we obtain

$$\varphi(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - L^{-1} \left(\int_0^x (x-t)\varphi(t) dt \right), \quad (2.4.23)$$

using the decomposition series (2.4.17), and using the recurrence relation (2.4.20) we obtain

$$\begin{aligned} \varphi_0(x) &= 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4, \\ \varphi_1(x) &= -L^{-1} \left(\int_0^x (x-t)\varphi(t) dt \right), \\ &= -\frac{1}{5!}x^5 + \frac{1}{6!}x^6 - \frac{1}{7!}x^7 + \frac{1}{8!}x^8 - \frac{1}{9!}x^9. \end{aligned} \quad (2.4.24)$$

and so on. This gives the solution in a series form

$$\varphi(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \frac{1}{6!}x^6 - \frac{1}{7!}x^7 + \dots, \quad (2.4.25)$$

and this converges to the exact solution

$$\varphi(x) = \exp(-x). \quad (2.4.26)$$

Chapter 3

Numerical solution of Volterra integro- differential equations

3.1 Veita-Pell polynomials and their properties

3.1.1 Vieta's formula for general polynomial

Vieta's formulas are the concept of polynomials which relates a polynomial's coefficients to the sums and products of the roots of the polynomial. Vieta's formulas can be useful tools for learning relations between the polynomial's roots without really knowing their numerical value and coefficients of the equation.

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ be a polynomial with complex coefficient and degree n , having comparing r_n, r_{n-1}, \dots, r_1 then for any integer $0 \leq k \leq n$,

$$\sum_{1 < i_1 < i_2 < \dots < i_k < n} r_{i_1} r_{i_2} \dots r_{i_k} = (-1)^k \frac{a_{n-k}}{a_n},$$

The expression on the left -hand side of Vieta's formula are the elementary symmetric functions of r_1, r_2, \dots, r_n the Vieta's formula follows by comparing coefficients in the equation

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = a_n (x - r_1)(x - r_2)(x - r_3) \dots (x - r_n).$$

As in the quadratic case, Vieta's formula gives an equation to find the sum of roots:

$$\sum_{i=1}^n r_i = -\frac{a_{n-1}}{a_n}.$$

Similarly, we have the following equation for the product of roots:

$$r_1 r_2 \dots r_n = (-1)^n \frac{a_0}{a_n},$$

Vieta's formula gives relationship between polynomial roots and coefficient that are often useful in problem-solving.

3.1.2 Pell polynomial

The Pell polynomials give recursive equation for $p(x)$ from $p_0(x) = 0$, $p_1(x) = 1$, and

$$p_{n+2}(x) = 2xp_{n+1}(x) + p_n(x).$$

The first few are

$$\begin{aligned} p_1(x) &= 1 \\ p_2(x) &= 2x \\ p_3(x) &= 4x^2 + 1 \\ p_4(x) &= 8x^3 + 4x \\ p_5(x) &= 16x^4 + 12x^2 + 1. \end{aligned}$$

3.1.3 Veita-Pell polynomials

The Veita -Pell polynomials (VP_P_S) can be defined recursively as below:

$$\begin{aligned} VP_0(x) &= 0, VP_1(x) = 1. \\ VP_n(x) &= 2xVP_{n-1}(x) - VP_{n-2}(x), \quad n \geq 2 \end{aligned}$$

The first few terms of $VP_n(x)$ are as follows:

$$VP_0(x) = 0$$

$$VP_1(x) = 1$$

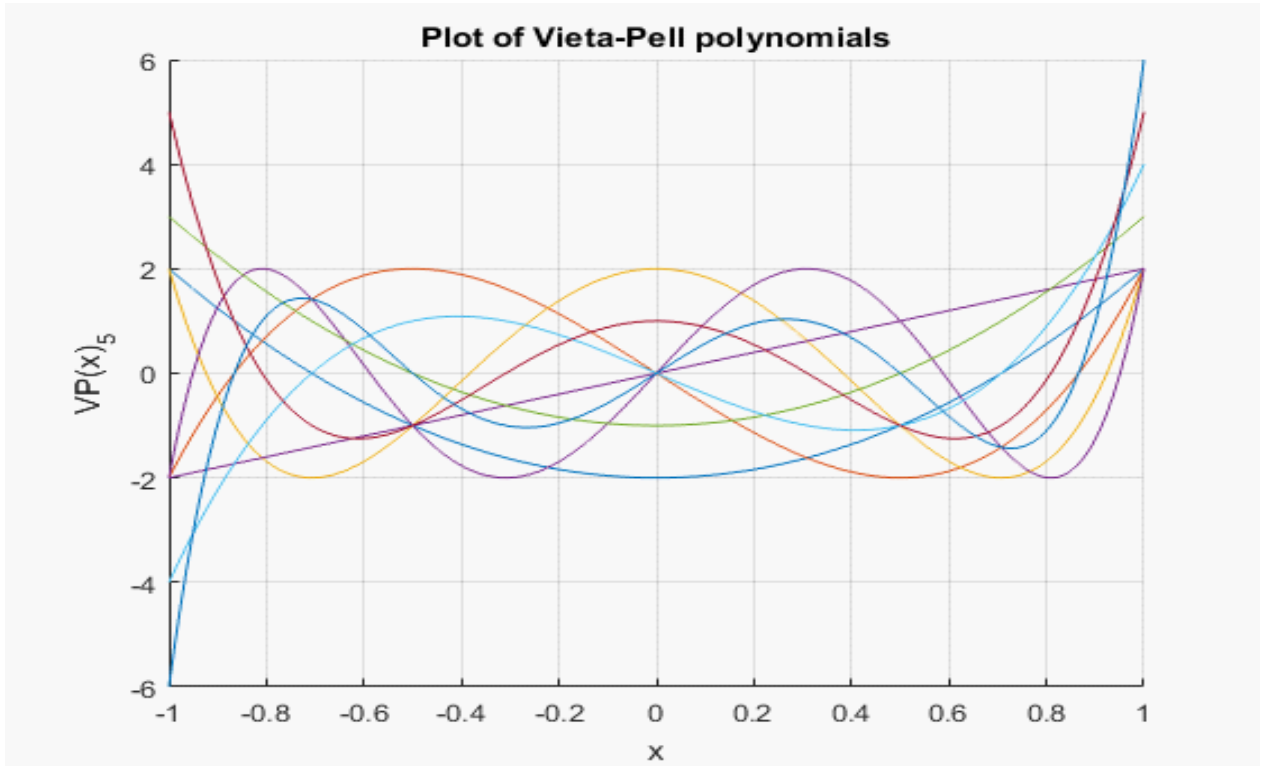
$$VP_2(x) = 2x$$

$$VP_3(x) = 4x^2 - 1$$

$$VP_4(x) = 8x^3 - 4x$$

$$VP_5(x) = 16x^4 - 12x^2 + 1.$$

$$VP_6(x) = 32x^5 - 32x^3 + 6x$$



In addition The general term of the (VPP_S) can be constructed in terme the sums and the product respectively as below

$$VP_n(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n-i}{n} 2^{n-2i} x^{n-2i}$$

$$VP_n(x) = 2^n \prod_{i=1}^n \left(x - \cos\left(\frac{i\pi}{n+1}\right) \right).$$

The first ten coefficients of $VP_n(x)$

•

n/j	0	1	2	3	4
0	0				
1	1				
2	2				
3	4	-1			
4	8	-4			
5	16	-12	1		
6	32	-32	6		

3.2 Veita-Pell collocation method

The collocation method is one of the most commonly used techniques for approximating the solutions of differential and integral equations. This method approximates the solution by expressing it as a linear combination of a specific set of basis functions. The process involves substituting this approximate solution into the original equation at a finite set of points within the domain, resulting in an algebraic system of equations that can be solved.

We consider the following Volterra integro-differential equation:

$$\begin{cases} \varphi^{(m)}(x) = f(x) + \lambda \int_a^x k(x,t)\varphi(t)dt, & a \leq t, x \leq b, m \geq 1 \\ \varphi^{(k)}(a) = \alpha_k, & 0 \leq k \leq m - 1, \end{cases} \quad (3.2.1)$$

Where $\varphi^{(m)}(x)$ indicates the m^{th} derivative of $\varphi(x)$ and $k(x, t)$ is continuous and square integrable function, $f(x)$ is a known function, $\varphi(x)$ is the unknown function, λ is a parameter, and m is an integer.

When using the collocation method, the approximate solution is given by:

$$\varphi_n(x) = \sum_{i=0}^n c_i VP_i(x) \quad (3.2.2)$$

$VP_i(x)$ be the Veita-Pell polynomials of degree i and c_i are coefficients to determine.

Let's consider

$$VP_n(x) = (VP_0(x), VP_1(x), \dots, VP_n(x)), \text{ and } c_i = (c_0, c_1, \dots, c_n)^t.$$

By integrating the intrgro-differential equation (3.2.1) from a to x , m time we have

$$\underbrace{\int_a^x \dots \int_a^x}_{m} \varphi^{(m)}(x) dx \dots dx = \underbrace{\int_a^x \dots \int_a^x}_{m} f(x) dx \dots dx + \underbrace{\int_a^x \dots \int_a^x}_{m} \left(\lambda \int_a^x k(x, t) \varphi(t) dt \right) dx \dots dx$$

Thus

$$\begin{aligned} \varphi(x) &= \varphi(a) + \int_a^x \varphi'(a) dx + \int_a^x \int_a^x \varphi''(a) dx + \dots + \int_a^x \dots \int_a^x \varphi^{(m-1)}(a) \underbrace{dx \dots dx}_{m-1} + \\ &\quad \underbrace{\int_a^x \dots \int_a^x}_{m} f(x) dx \dots dx + \underbrace{\int_a^x \dots \int_a^x}_{m} \left(\lambda \int_a^x k(x, t) \varphi(t) dt \right) dx \dots dx \end{aligned} \quad (3.2.3)$$

We replace (3.2.2) in (3.2.3) and we obtain

$$\begin{aligned} \sum_{i=0}^n c_i V P_i(x) &= \sum_{k=0}^{m-1} \frac{1}{k!} (x-a)^k \varphi^{(k)}(a) + \underbrace{\int_a^x \dots \int_a^x}_{m} f(x) dx \dots dx + \underbrace{\int_a^x \dots \int_a^x}_{m} \left(\lambda \int_a^x k(x, t) \sum_{i=0}^n c_i V P_i(t) dt \right) \underbrace{dx \dots dx}_m, \\ \sum_{i=0}^n c_i (V P_i(x) - \lambda \underbrace{\int_a^x \dots \int_a^x}_{m} \left(\int_a^x k(x, t) V P_i(t) dt \right) dx dx \dots dx) &= F(x) \end{aligned} \quad (3.2.4)$$

where

$$F(x) = \sum_{k=0}^{m-1} \frac{\alpha_k}{k!} (x-a)^k + \underbrace{\int_a^x \dots \int_a^x}_{m} f(x) \underbrace{dx \dots dx}_m$$

The equation(3.2.4) is written in the form:

$$A(x)C = F(x) \quad (3.2.5)$$

Where

$$\begin{aligned} A_i(x) &= \sum_{i=0}^n (V P_i(x) - \lambda \underbrace{\int_a^x \dots \int_a^x}_{m} \left(\int_a^x k(x, t) V P_i(t) dt \right) dx \dots dx, \\ C &= (c_0, c_1, \dots, c_n)^t \end{aligned}$$

To determine the unknown coefficients c_i for $i = 0, 1, \dots, n$, we chose collocation points

$$\begin{cases} x_{j+1} = x_j + \frac{b-a}{n}, j = 0, 1, 2, \dots, n \\ x_0 = a \end{cases}$$

Thus on obtain a linear system

$$A_i(x_j)C = F(x_j) \tag{3.2.6}$$

where

$$A = (a_{ji}) = \begin{pmatrix} a_{00} & a_{01} & \dots & a_{0n} \\ a_{10} & a_{11} & & a_{1n} \\ \dots & \dots & & \dots \\ a_{n0} & a_{n1} & \dots & a_{nn} \end{pmatrix}, F(x) = \begin{pmatrix} F(x_0) \\ F(x_1) \\ \dots \\ F(x_n) \end{pmatrix}, C = \begin{pmatrix} c_0 \\ c_1 \\ \dots \\ c_n \end{pmatrix}$$

where

$$a_{ji} = A_i(x_j), \quad i, j = 0, 1, \dots, n$$

This system admits a unique solution if $\det(A) \neq 0$.

The initial conditions of (3.2.1) give:

$$\varphi_n^{(m-1)}(a) = \sum_{i=0}^n VP_i(a)c_i = \alpha_{m-1}, \quad m \geq 1 \tag{3.2.7}$$

The unknowns c_i , $i = 0, 1, \dots, n$ are determined by solving the linear system of equation (3.2.6) and (3.2.7), Substituting these values into (3.2.2) provides the approximate solution.

3.3 Veita-Pell Galerkin method

We consider the following Volterra integro-differential equation

$$\begin{cases} \varphi^{(m)}(x) = f(x) + \lambda \int_a^x k(x, t)\varphi(t)dt, \quad a \leq t, x \leq b \\ \varphi^{(k)} = \alpha_k, \quad 0 \leq k \leq m - 1, \quad m \geq 1 \end{cases} \tag{3.3.1}$$

The symbol $\varphi^{(m)}(x)$ indicates the m^{th} derivative of $\varphi(x)$ and $k(x, t)$ is continuous and square integrable function, $f(x)$ is a known function, $\varphi(x)$ is the unknown function, λ is a parameter, and m is an integer.

we use when the Galerkin method where the approximate solution is given in the form:

$$\varphi_n(x) = \sum_{i=0}^n c_i V P_i(x) \quad (3.3.2)$$

where $V P_i(x)$ represent Veita-Pell polynomials of degree i and c_i are coefficient to be determined.

Let's set

$$V P_n(x) = (V P_0(x), V P_1(x), \dots, V P_n(x)), \text{ and } c = (c_0, c_1, \dots, c_n)^t$$

By substituting (3.3.2) into (3.3.1) and integrating the integro - differential equation (3.3.1) from a to x m times, we obtain

$$\underbrace{\int_a^x \dots \int_a^x}_{m} \varphi^{(m)}(x) \underbrace{dx dx \dots dx}_m = \underbrace{\int_a^x \dots \int_a^x}_{m} f(x) dx \dots dx + \underbrace{\int_a^x \dots \int_a^x}_{m} (\lambda \int_a^x k(x, t) \varphi(t) dt) \underbrace{dx \dots dx}_m, \quad (3.3.3)$$

Thus

$$\begin{aligned} \varphi(x) = & \varphi(a) + \int_a^x \varphi'(a) dx + \int_a^x \int_a^x \varphi''(a) dx dx + \dots + \underbrace{\int_a^x \dots \int_a^x}_{m-1} \varphi^{(m-1)}(a) dx \dots dx + \int_a^x \dots \int_a^x f(x) dx \dots dx \\ & + \int_a^x \dots \int_a^x (\lambda \int_a^x k(x, t) \varphi(t) dt) dx \dots dx, \end{aligned} \quad (3.3.4)$$

By replacing (3.3.2) in equation (3.3.4) , we obtained

$$\sum_{i=0}^n c_i V P_i(x) - \int_a^x \dots \int_a^x (\lambda \int_a^x k(x, t) V P_i(t) dt) dx \dots dx = \sum_{k=0}^{m-1} \frac{1}{k!} (x-a)^k \varphi^{(k)}(a) + \int_a^x \dots \int_a^x f(x) dx \dots dx$$

where

$$\sum_{n=0}^n c_i (V P_i(x) - \lambda \int_a^x \dots \int_a^x (\int_a^x k(x, t) V P_i(t) dt) dx \dots dx) = \sum_{k=0}^{m-1} \frac{1}{k!} (x-a)^k \varphi^{(k)}(a) + \int_a^x \dots \int_a^x f(x) dx \dots dx \quad (3.3.5)$$

so the Galerkin equation are obtained by multiplying both sides of (3.3.5) by $(V P_j)$ and then integrating with respect to form a to b we obtain

$$\sum_{n=0}^n \left(\int_a^b (V P_i(x) - \lambda \int_a^x \dots \int_a^x (\int_a^x k(x, t) V P_i(t) dt) dx \dots dx) V P_j(x) dx \right) c_i =$$

$$\int_a^b \left(\sum_{k=0}^{m-1} \frac{1}{k!} (x-a)^k \varphi^{(k)}(a) + \int_a^x \dots \int_a^x f(x) dx \dots dx \right) VP_j(x) dx, \quad j = 0, \dots, n. \quad (3.3.6)$$

The equation (3.3.6) is written in the form

$$W(x)C = F(x)$$

where

$$W(x) = \sum_{i=0}^n \left(\int_a^b (VP_i(x) - \lambda \int_a^x \dots \int_a^x (k(x,t) VP_i(t) dt) dx \dots dx) VP_j(x) dx \right) c_i, \quad j = 0, \dots, n,$$

$$C = (c_0, c_1, \dots, c_n)^t$$

$$F(x) = \int_a^b \left(\sum_{k=0}^{m-1} \frac{1}{k!} (x-a)^k \varphi^{(k)}(a) + \int_a^x \dots \int_a^x f(x) dx \dots dx \right) VP_j(x) dx, \quad j = 0, \dots, n.$$

The system has a unique solution if $\det(W) \neq 0$

The initial condition of (3.3.1) are given by:

$$\varphi_n^{(m-1)}(a) = \sum_{i=0}^n VP_i(a) c_i = \alpha_{m-1} \quad (3.3.7)$$

To determine the unknowns $c_i, i = 0, 1, \dots, n$, by solving the linear system of equations (3.3.6) and (3.3.7), substituting these values into (3.3.2) gives the approximate solution.

3.4 Numerical examples

Example 01. Consider the integro-differential equation of Volterra:

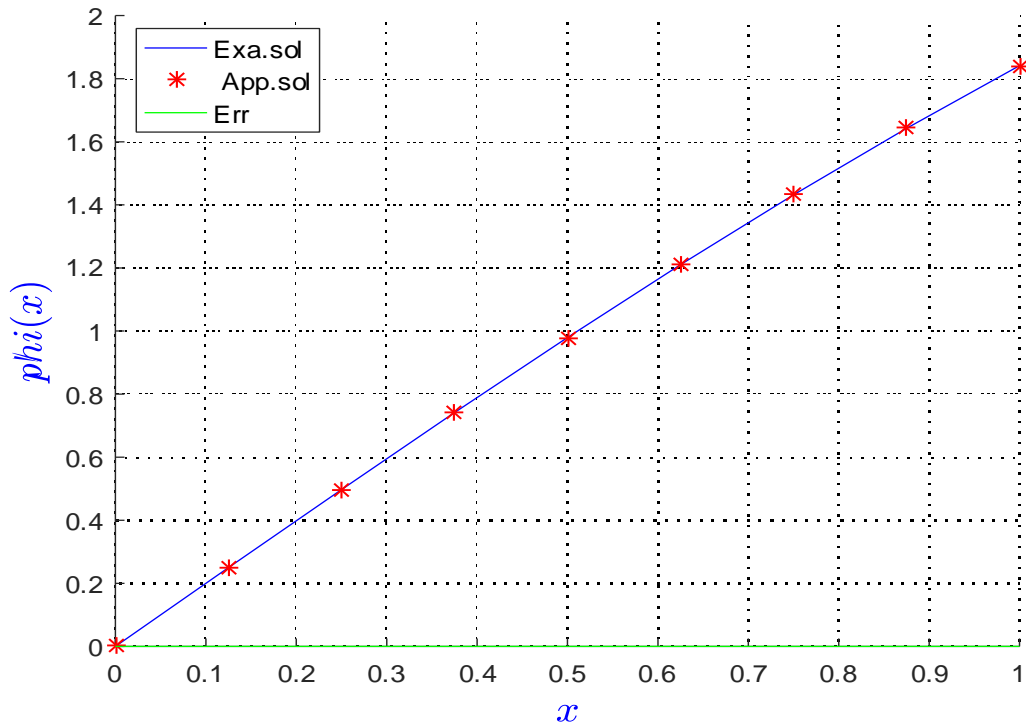
$$\begin{cases} \varphi''(x) = -x - \frac{x^3}{6} + \int_0^x (x-t)\varphi(t) dt, & 0 \leq t \leq x \leq 1 \\ \varphi(0) = 0, \varphi'(0) = 2 \end{cases}$$

the exact solution given by:

$$\varphi_{ex}(x) = x + \sin(x)$$

Table 01. We present the approximate solution φ_{app} obtained by the Veita-Pell collocation method, the error is calculated for $N = 8$

Val of x	$\varphi_{ex}(x)$	φ_{app}	$Err.$
0	0	-3.4078e-14	3.4078e-14
1.2500e-01	2.4967e-01	2.4967e-01	3.6726e-12
2.5000e-01	4.9740e-01	4.9740e-01	5.8429e-12
3.7500e-01	7.4127e-01	7.4127e-01	7.0706e-12
5.0000e-01	9.7943e-01	9.7943e-01	8.1928e-12
6.2500e-01	1.2101e+00	1.2101e+00	1.0188e-11
7.5000e-01	1.4316e+00	1.4316e+00	1.3954e-11
8.7500e-01	1.6425e+00	1.6425e+00	2.0030e-11
1.0000e+00	1.8415e+00	1.8415e+00	2.8226e-11



Example 02. Consider the integro-differential equation of Volterra:

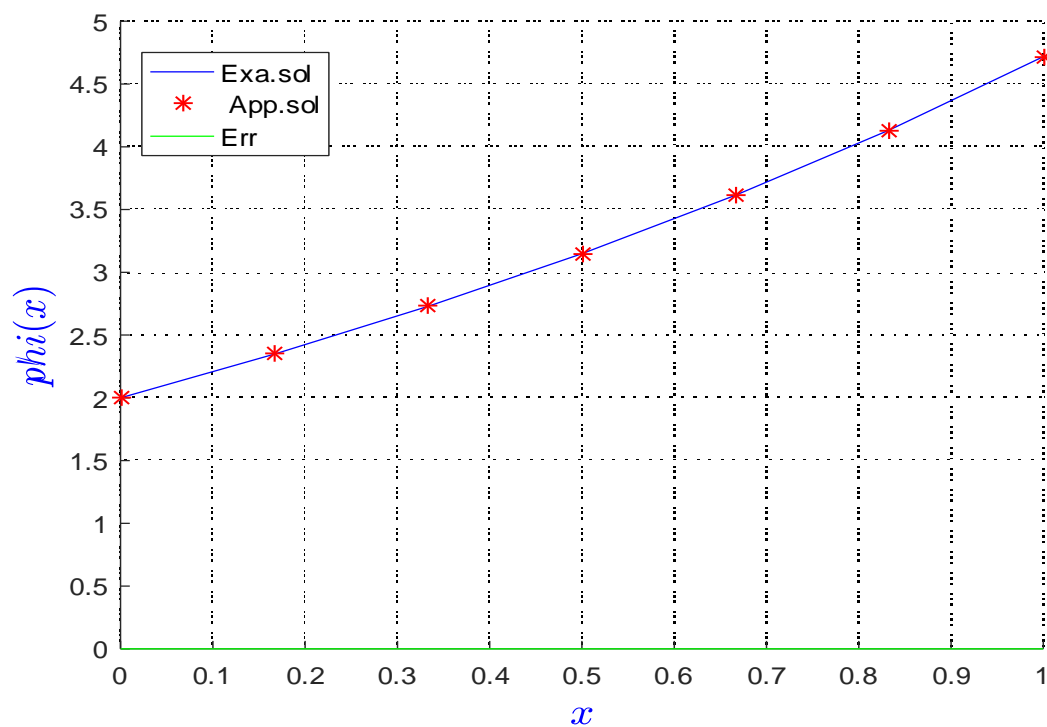
$$\begin{cases} \varphi^{(4)}(x) = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \int_0^x (x-t)\varphi(t) dt, & 0 \leq t \leq x \leq 1 \\ \varphi(0) = 2, \varphi'(0) = 2, \varphi''(0) = 1, \varphi'''(0) = 1 \end{cases}$$

the exact solution given by:

$$\varphi_{ex}(x) = \exp(x) + x + 1$$

Table 02. We present the approximate solution φ_{app} obtained by the Veita-Pell collocation method, the error is calculated for $N = 6$

Val of x	$\varphi_{ex}(x)$	φ_{app}	$Err.$	$Err.[12]$
0	2.0000e+00	2.0000e+00	1.8208e-13	
1.6667e-01	2.3480e+00	2.3480e+00	8.9262e-14	
3.3333e-01	2.7289e+00	2.7289e+00	8.4377e-14	
5.0000e-01	3.1487e+00	3.1487e+00	1.3474e-12	
6.6667e-01	3.6144e+00	3.6144e+00	7.2884e-12	
8.7500e-01	4.1343e+00	4.1343e+00	2.3824e-11	
1.0000e+00	4.7183e+00	4.7183e+00	6.1121e-11	



Example 03. Consider the integro-differential equation of Volterra:

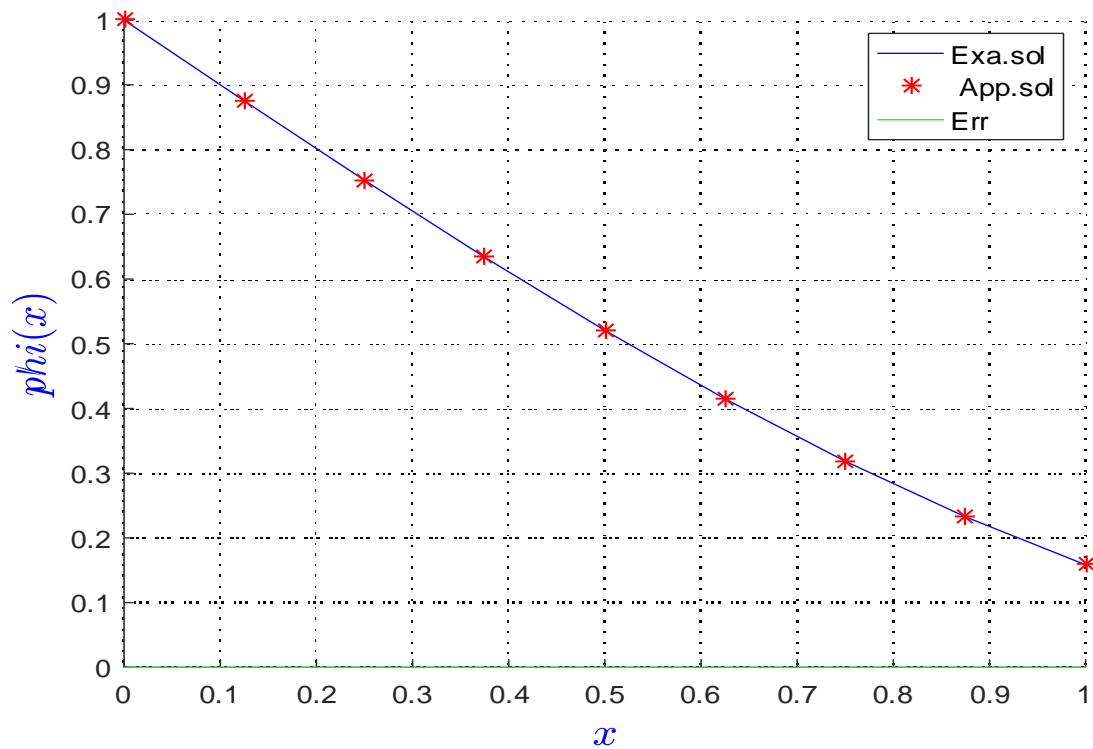
$$\begin{cases} \varphi'''(x) = \cos(x) - \exp(x) - ((\exp(x)(\cos(x) - \sin(x)))/2) + 3/2 + \int_0^x \exp(t)\varphi(t) dt, & 0 \leq t \leq x \leq 1 \\ \varphi(0) = 1, \varphi'(0) = -1, \varphi''(0) = 0, \end{cases}$$

the exact solution given by:

$$\varphi_{ex}(x) = 1 - \sin(x)$$

Table 03. We present the approximate solution φ_{app} obtained by the Veita-Pell collocation method, the error is calculated for $N = 8$

Val of x	$\varphi_{ex}(x)$	φ_{app}	$Err.$
0	1.0000e+00	1.0000e+00	1.2484e-11
1.2500e-01	8.7533e-01	8.7533e-01	9.8825e-12
2.5000e-01	7.5260e-01	7.5260e-01	6.7345e-12
3.7500e-01	6.3373e-01	6.3373e-01	3.4286e-12
5.0000e-01	5.2057e-01	5.2057e-01	1.9951e-13
6.2500e-01	4.1490e-01	4.1490e-01	5.0999e-12
7.5000e-01	3.1836e-01	3.1836e-01	1.2857e-11
8.7500e-01	2.3246e-01	2.3246e-01	2.7669e-11
1.0000e+00	1.5853e-01	1.5853e-01	5.6377e-11



Conclusion

Integro-differential equations are typically difficult to solve analytically. Often, it is necessary to obtain approximate solutions. In this thesis, we propose the Vieta-Pell collocation method based on the Vieta-Pell polynomials for solving linear Volterra integro-differential equations. Illustrative examples are included to demonstrate the validity and applicability of this method.

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