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Diophantine Approximations

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Dedication

I dedicate this thesis to:

My Mother

My Father

My siblings

CONTENTS

NOTATIONS

$\mathbb{Q}[X]$	Set of polynomials with rational coefficients.
$\mathbb{Z}[X]$	Set of polynomials with integer coefficients.
$\deg \alpha$	Degree of the algebraic number α.
$\overline{\mathbb{Q}}$	Set of algebraic numbers.
$\mu(\alpha)$	Irrationality measure of the number α.
$\lvert G_n(\alpha)\rvert = \infty$	Infinite cardinality of the set $G_n(\alpha)$.
μ_m	Image of the irrationality measure function.
$\psi(q)$	General approximation function.
$\mu_L(A)$	Lebesgue measure of the set A.
$\lVert \cdot \rVert$	Distance to the nearest integer.
$[a_0; a_1, a_2, \dots, a_n]$	Finite (simple) continued fraction.
$[a_0; a_1, a_2, \dots]$	Infinite continued fraction.
$\frac{p_n}{q_n}$	n-th convergent.
$\lfloor \alpha \rfloor$	Floor function.
$\mathcal{F}_n$	Farey sequence of order n.
$\alpha \approx \gamma$	α is close to γ.
R_n	Newton's approximant.
ℓ or $d(\ell)$	Period length.

$d(b)$	Number of good approximants.
$C(v)$ or $S(v)$	Volume of the set.
$\lvert x\rvert$	Maximum norm.
$\mathcal{K}(t)$	Special convex body.

GENERAL INTRODUCTION

Diophantine approximations constitute a classical and fundamental field of number theory. They study how real numbers can be approximated by rational numbers with optimal precision. This problem naturally arises as soon as one seeks to compare the discrete structure of integers with the continuity of real numbers. Today, it plays a central role not only in number theory, but also in analysis, dynamics, geometry, theoretical computer science, and in certain questions of mathematical physics.

Since antiquity, mathematicians have sought to represent irrational quantities by simple fractions. Thus, the approximation of the number π by $\dfrac{22}{7}$ or that of $\sqrt{2}$ by $\dfrac{99}{70}$ already illustrate the fundamental idea of rational approximations. However, it was only with the systematic development of number theory that these questions took a rigorous and general form.

The fundamental problem of Diophantine approximations can be stated as follows: given a real number α, can we find rational numbers $\dfrac{p}{q}$ such that the difference $\left|\alpha - \dfrac{p}{q}\right|$ is very small relative to the size of the denominator q? More precisely, we seek to determine how fast this error can decrease as q becomes large.

The first major result in this direction is Dirichlet's theorem, which constitutes one of the foundations of the theory. This theorem states that for every real number α and every integer $N > 1$, there exist integers p and q with $1 \leq q < N$ such that:

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{qN}.$$

In particular, if α is an irrational number, there exists an infinite number of rational fractions $\dfrac{p}{q}$ satisfying:

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

This result shows that it is always possible to obtain remarkably precise rational approximations. The proof of the theorem relies on Dirichlet's pigeonhole principle.

Among the classical tools used to produce such approximations are Farey sequences. A Farey sequences of order n is the set of irreducible fractions between 0 and 1 whose denominators do not exceed n, arranged in increasing order and generally denoted by $\mathcal{F}_n$. For example:

$$\mathcal{F}_5 = \left\{ \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \right\}$$

These sequences reveal a deep arithmetic structure connecting neighboring rational fractions. In particular, they make it possible to obtain efficient rational approximations of a given real number and are involved in many problems of analytic number theory. The remarkable properties of neighboring fractions in Farey sequences lead to fine estimations and maintain close links with lattice geometry and modular transformations.

Another fundamental tool is the theory of continued fractions. Any real quantity can be expanded in the form of a continued fraction:

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

Where the a_i are positive integers for $i \geq 1$. The successive convergents of this expansion provide the best possible rational approximations of the considered number. Continued fractions allow us not only to obtain extremely precise approximations, but also to characterize the arithmetic nature of real numbers. For example, the continued fraction expansion of a rational number is finite, while that of a quadratic irrational is periodic.

This theory plays an essential role in the study of Diophantine equations, quadratic forms, and many approximation problems. It also provides optimal estimates of the approximation error. If $\dfrac{p_n}{q_n}$ denotes the n-th convergent of a real number α, then we typically have:

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}.$$

Rational approximations also appear in numerical methods derived from analysis, notably in Newton's method. Initially designed to solve numerical equations, this method often produces rational sequences that converge very rapidly toward real or algebraic numbers. Given a function f, the method consists of constructing a sequence defined by:

$$X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}.$$

When the initial point is rational and the function has rational coefficients, the obtained approximations often remain rational. In the case of calculating $\sqrt{2}$, for example, Newton's method generates a sequence of rational fractions that converge very rapidly toward the desired value:

$$1, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \ldots$$

This speed of convergence illustrates the deep connection between numerical analysis and Diophantine approximations. The study of Diophantine approximations experienced considerable development in the 19th and 20th centuries thanks to the work of Liouville. Hermite, Thue, Siegel, Roth, and many others. This research led to profound results on algebraic and transcendental numbers. Roth's theorem, for example, shows that an irrational algebraic number cannot be approximated "too well" by rationals, which marks a fundamental limit of approximation methods.

Thus, rational approximations of real numbers remarkably illustrate the interaction between the discrete and the continuous. They reveal how integers and rational numbers make it possible to precisely approximate the most complex real objects, while highlighting arithmetic structures of great richness.

This thesis is divided into four chapters. The first chapter is devoted to preliminaries, while the second chapter is intended for Dirichlet's theorem and Farey sequences. The last two chapters are dedicated to continued fractions, Newton's approximations, and simultaneous approximations.

The references that were used the most are: [10, 11, 13, 14, 16, 17, 19, 21, 23, 38, 39]

CHAPTER 0

PRELIMINARIES

In this chapter, we have introduced some basic concepts related to Diophantine approximation, such as algebraic numbers, the pigeonhole principle, irrationality measures, and Liouville numbers, in order to facilitate understanding of the theories and results on which we will rely in the remainder of this research.

0.1 Algebraic numbers and cantor's theorem

Definition 0.1. *$\mathbb{Q}[X]$ is the set of polynomials with coefficients in $\mathbb{Q}$ and $\mathbb{Z}[X]$ is the set of polynomials with coefficients in $\mathbb{Z}$.*

Definition 0.2. *A complex number $\alpha \in \mathbb{C}$ is algebraic if there exists a polynomial $f \in \mathbb{Q}[X]$ such that $f(\alpha) = 0$. Otherwise, α is transcendental.*

Proposition 0.1. *$\alpha \in \mathbb{C}$ is algebraic if there exists a nonzero polynomial $f \in \mathbb{Z}[X]$ with integer coefficients such that $f(\alpha) = 0$.*

Definition 0.3. *Let $\alpha \in \mathbb{C}$ be algebraic. The degree of α, denoted as $\deg \alpha$, is the minimum degree of a nonzero polynomial $f \in \mathbb{Z}[X]$ such that $f(\alpha) = 0$.*

Theorem 0.1. *$\overline{Q}$ is countable.*

Theorem 0.2 (Cantor)**.** *[10] $\mathbb{R}$ is uncountable.*

Proof. This is a classic proof by contradiction. Suppose we had such a function $f : \mathbb{N} \to \mathbb{R}$. Now, i will construct some $s \in \mathbb{R}$ not in the image of f. For simplicity let us only take real numbers on the interval $[0, 1]$ this will suffice. Let $r_{i,j}$ be the j-th digit of $f(i)$ in some base b. Define the number $s \in \mathbb{R}$ by its expansion base b as $s_i = f(i)_i + 1 \pmod{b}$. This is the i-th digit of the i-th number plus one reduced by b. I claim that there does not exist any $n \in \mathbb{N}$ such that $f(n) = s$. Suppose such an n exists. Then $s_n = f(n)_n + 1 \pmod{b}$ but $s = f(n)$ so $s_n = f(n)_n$ whichh is a contradiction. Therefore, f cannot be a bijection. $\square$

Corollary 0.1. *Transcendental numbers exist.*

Example 0.1. *Let* $\alpha = \sqrt{2}$*. Consider the polynomial*

$$f(x) = x^2 - 2.$$

Substituting α *into* f*, we obtain*

$$f(\sqrt{2}) = (\sqrt{2})^2 - 2 = 2 - 2 = 0.$$

Since there exists a nonzero polynomial $f(x) \in \mathbb{Z}[X]$ *such that* $f(\alpha) = 0$*, the number* $\sqrt{2}$ *is algebraic. Moreover, the polynomial* $x^2 - 2$ *has degree 2, so*

$$\deg(\sqrt{2}) = 2.$$

By Cantor's theorem, the set of real numbers $\mathbb{R}$ *is uncountable, while the set of algebraic numbers is countable. Therefore, there exist real numbers that are not algebraic; these are called transcendental numbers, such as* π *and* e*.*

0.2 Pigeonhole principle

The pigeonhole principle says that if you have $n + 1$ pigeons and n holes, at least one hole must have two or more pigeons.

Even though this idea looks very simple, it is a very powerful tool.

Theorem 0.3 (The pigeonhole principle)**.** *If* $k + 1$ *or more objects are placed into* k *boxes, then at least one box contains two or more of the objects.*

Proof. If none of the k boxes contains more than one object, then the total number of objects would be at most k. This contradiction shows that one of the boxes contains at least two or more of the objects. $\square$

We now state and prove the approximation theorem, which guarantees that one of the first n multiples of a real number must be within $1/n$ of an integer. The proof we give illustrates the utility of the pigeonhole principle. (See [Ro07] for more applications of the pigeonhole principle.) (Note that in the proof we make use of the *absolute value* function. Recall that $|x|$, the absolute value of x, equals x if $x \geq 0$ and $-x$ if $x < 0$. Also recall that $|x - y|$ gives the distance between x and y.)

Example 0.2. *(Putnam 1978) Let* A *be any set of twenty integers chosen from the arithmetic progression* $1, 4, \ldots, 100$*. Prove that there must be two distinct integers in* A *whose sum is 104.*

Solution: We partition the thirty four elements of this progression into nineteen groups $\{1\}, \{52\}, \{4, 100\}, \{7, 97\}, \{10, 94\}, \ldots \{49, 55\}$. Since we are choosing twenty integers and we have nineteen sets, by the pigeonhole principle there must be two integers that belong to one of the pairs, which add to 104.

Example 0.3. *Show that amongst any seven distinct positive integers not exceeding 126, one can find two of them, say a and b, whichh satisfy*

$$b < a \leq 2b.$$

Solution: Split the numbers $\{1, 2, 3, \ldots, 126\}$ into the six sets

$$\{1, 2\}, \{3, 4, 5, 6\}, \{7, 8, \ldots, 13, 14\}, \{15, 16, \ldots, 29, 30\},$$

$$\{31, 32, \ldots, 61, 62\} \text{ and } \{63, 64, \ldots, 126\}.$$

By the pigeonhole principle, two of the seven numbers must lie in one of the six sets, and obviously, any such two will satisfy the stated inequality.

Example 0.4. *Given any set of ten natural numbers between 1 and 99 inclusive, prove that there are two disjoint nonempty subsets of the set with equal sums of their elements.*

Solution: There are $2^{10} - 1 = 1023$ non-empty subsets that one can form with a given 10-element set. To each of these subsets we associate the sum of its elements. The maximum value that any such sum can achieve is $90 + 91 + \cdots + 99 = 945 < 1023$. Therefore, there must be at least two different subsets that have the same sum.

0.3 Diophantine approximations

Diophantine approximation is the process of finding successively better rational approximations to irrational numbers. The fact that any irrational number can be approximated arbitrarily closely by rational numbers is, of course, just the fact that $\mathbb{Q} \subset \mathbb{R}$is dense. That said, Diophantine approximation is the study of how well or quickly such approximations can be made. The theory of Diophantine approximation is built on the following fundamental lemma.

Lemma 0.1 (Dirichlet)**.** *For any $\alpha \in \mathbb{R}$ and $N \in \mathbb{Z}^+$ there exists $p, q \in \mathbb{Z}$ such that $1 \leq q \leq N$ and,*

$$|q\alpha - p| < \frac{1}{N}$$

Corollary 0.2. *Let $\alpha \in \mathbb{R}$ be irrational then there exist infinitely many $\frac{p}{q} \in \mathbb{Q}$ such that,*

$$0 < \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

Definition 0.4. *For $\alpha \in \mathbb{R}$, an n-good Diophantine approximation is $\frac{p}{q} \in \mathbb{Q}$ so that,*

$$0 < \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n}$$

Definition 0.5. *A number $\alpha \in \mathbb{R}$ is n-approximable if there exist infinitely many n-good Diophantine approximations.*

Definition 0.6.

$$G_n(\alpha) = \left\{\frac{p}{q} \in \mathbb{Q} \mid 0 < \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n}\right\}$$

The set of n-good approximations of α. α is n-approximable when $|G_n(\alpha)| = \infty$.

Theorem 0.4 (Hurwitz)**.** *For any irrational $\xi \in \mathbb{R}$ there exist infinitely many rational $\frac{p}{q} \in \mathbb{Q}$ such that.*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$$

and this does not hold for any constant denominator $C > \sqrt{5}$.

Definition 0.7. *We say that $\alpha \in \mathbb{R}$ is badly-approximable if there exists some $c > 0$ such that for all $\frac{p}{q} \in \mathbb{Q}$,*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^2}.$$

Otherwise α is well-approximable.

Theorem 0.5 (Thue–Siegel—Roth, 1955)**.** *[10] If α is irrational algebraic and $\varepsilon > 0$ then there exists a positive constant $C(\alpha, \varepsilon)$ such that $\forall \frac{p}{q} \in \mathbb{Q}$,*

$$\left|\alpha - \frac{p}{q}\right| > \frac{C(\alpha, \varepsilon)}{q^{2+\varepsilon}}$$

Lemma 0.2 (Liouville)**.** *Let α be a root of $f \in \mathbb{Z}[X]$ with $\deg f = n$ then there exists $C \in \mathbb{R}^+$ such that for every $\frac{p}{q} \in \mathbb{Q}$ such that $\frac{p}{q} \neq \alpha$ we have,*

$$\left|\alpha - \frac{p}{q}\right| > \frac{C}{q^n}$$

Corollary 0.3. *Let α be algebraic with degree n, then for any $k > n$, α is not k-approximable.*

0.4 Irrationality measure

We can use the previous definitions and results to define a measure of how irrational number is. Essentially, the irrationality measure tells us how well a number can be approximated by rational numbers. Perhaps unintuitively, the more irrational the number, the better it can be approximated by rationals.

Definition 0.8. *The irrationality measure is* $\mu(\alpha) = \sup\{n \in \mathbb{R}^+ \mid |G_n(\alpha)| = \infty\}$

Proposition 0.2. *If* $\alpha \in \mathbb{Q}$ *then* $\mu(\alpha) = 1$

Proposition 0.3. *If* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ *then* $\mu(\alpha) \geq 2$.

Proof. This follows immediately from Dirichlet's approximation lemma. Since $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ then $|G_2(\alpha)| = \infty$ meaning that,

$$\mu(\alpha) = \sup\{n \in \mathbb{R}^+ \mid |G_n(\alpha)| = \infty\} \geq 2$$

□

Proposition 0.4. *Let* α *be algebraic of degree* n*, then* $\mu(\alpha) \leq n$.

Theorem 0.6 (Roth, 1955)**.** *If* α *is irrational algebraic then* $\mu(\alpha) = 2$.

Example 0.5. *The best known upper bound on the irrationality measure of* π *was given in 2008 by Salikhov as* $\mu(\pi) \leq 7.6063$

Example 0.6. *Borwein and Borwein proved in 1987 that* $\mu(e) = 2$.

Theorem 0.7. *The image of the irrationality measure is Im* $\mu = \{1\} \cup [2, \infty]$ *and is classified for any* $\alpha \in \mathbb{R}$ *via,*

$$\begin{cases} \mu(\alpha) = 1 & \alpha \text{ is rational} \\ \mu(\alpha) = 2 & \alpha \text{ is irrational algebraic} \\ \mu(\alpha) \geq 2 & \alpha \text{ is transcendental} \\ \mu(\alpha) = \infty & \alpha \text{ is Liouville} \end{cases}$$

0.5 Liouville numbers

Definition 0.9. L *is a Liouville number if for every* $n \in \mathbb{Z}^+$ *there exists* $\frac{p}{q} \in \mathbb{Q}$ *such that,*

$$0 < \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n}$$

Proposition 0.5. L *is Liouville if and only if* $\mu(L) = \infty$

Theorem 0.8. *Liouville numbers are transcendental.*

Theorem 0.9. *Take $b \in \mathbb{Z}$ with $b \geq 2$ and $a_k \in \{0, 1, 2, \ldots, b-1\}$ for every $k \in \mathbb{N}$, then, the number,*

$$L = \sum_{k=1}^{\infty} \frac{a_k}{b^{k!}}$$

is Liouville number and hence transcendental. In particular, we have uncountably many explicit examples of transcendental numbers.

Example 0.7.

$$L = \sum_{k=1}^{\infty} \frac{1}{10^{k!}}, \quad n = 2, \quad \frac{p}{q} = \frac{11}{100}$$
$$0 < \left| L - \frac{11}{100} \right| < \frac{2}{10^6} < \frac{1}{100^2} = \frac{1}{q^2}.$$

Hence,

$$0 < \left| L - \frac{p}{q} \right| < \frac{1}{q^n},$$

So L satisfies the definition of a Liouville number, and therefore L is a transcendental number.

And now we move to a more general definition:

0.6 Measure theory of approximable numbers

Definition 0.10. *Let $\psi : \mathbb{Z}^+ \to \mathbb{R}^+$ be a function. We say that $\alpha \in \mathbb{R}$ is ψ-approximable if the set of ψ-good Diophantine approximations,*

$$G_\psi(\alpha) = \left\{ \frac{p}{q} \in \mathbb{Q} \;\middle|\; 0 < \left| \alpha - \frac{p}{q} \right| < \frac{\psi(q)}{q} \right\}$$

is infinite.

Theorem 0.10 (Khinchin). *[10] Let $\psi : \mathbb{Z}^+ \to \mathbb{R}^+$ be a function such that the series,*

$$\sum_{q=1}^{\infty} \psi(q) < \infty$$

converges,

$$A_\psi = \{\alpha \in \mathbb{R} \mid |G_\psi(\alpha)| = \infty\}$$

has Lebesgue measure zero.

Corollary 0.4. *Almost all real numbers have $\mu(\alpha) = 2$ and the set*

$$\{\alpha \in \mathbb{R} \mid \mu(\alpha) > 2\}$$

has measure zero. In particular, almost all transcendental $\alpha \in \mathbb{R}$ have $\mu(\alpha) = 2$.

Example 0.8. *Let* $\psi(q) = \dfrac{1}{q}$, $\alpha = \sqrt{2}$, *and* $\dfrac{p}{q} = \dfrac{99}{70}$, *we calculate:*

$$\left|\alpha - \frac{p}{q}\right| = \left|\sqrt{2} - \frac{99}{70}\right| \approx 0.00007215 < \frac{\psi(70)}{70} = \frac{1}{70^2} = \frac{1}{4900} \approx 0.00020408.$$

Hence,

$$0 < \left|\alpha - \frac{p}{q}\right| < \frac{\psi(q)}{q},$$

So $\dfrac{99}{70} \in G_\psi(\sqrt{2})$, *which means that* $\dfrac{99}{70}$ *is a* ψ*-good Diophantine approximation of* $\sqrt{2}$.

0.7 The approximations of irrationals by rationals

Theorem 0.11. *Given any irrational number* θ *and any positive integer* m, *there is a positive integer* $b \leq m$ *such that*

$$\|b\theta\| = |b\theta - a| < \frac{1}{m+1}.$$

The symbol a here denotes the integer nearest to $b\theta$, so that the equality $\|b\theta\| = |b\theta - a|$ holds by the definition of the symbolism.

Corollary 0.5. *Given any irrational number* θ, *there are infinitely many rational numbers* a/b, *where* a *and* $b > 0$ *are integers, such that*

$$\left|\theta - \frac{a}{b}\right| < \frac{1}{b^2}. \tag{1}$$

Example 0.9. *Let* $\theta = \sqrt{2}$ *and* $\dfrac{a}{b} = \dfrac{99}{70}$, *we have the following calculation:*

$$\left|\theta - \frac{a}{b}\right| = \left|\sqrt{2} - \frac{99}{70}\right| \approx 0.00007215 < \frac{1}{70^2} = \frac{1}{4900} \approx 0.00020408.$$

Hence, $\left|\theta - \dfrac{a}{b}\right| < \dfrac{1}{b^2}$. *Therefore,* $\dfrac{99}{70}$ *is a good rational approximation of* $\sqrt{2}$. *And it satisfies the inequality given in Corollary 0.5.*

CHAPTER 1

DIRICHLET'S THEOREM AND FAREY SEQUENCES

In this chapter, we discussed Dirichlet's theorem and the Farey sequences, as they are among the most important tools used in Diophantine approximation. We also studied Ford's circles to understand how to approximate real numbers using fractions in a simple and systematic way.

1.1 Dirichlet theorem

Theorem 1.1 (Dirichlet, 1842)**.** *Let α and Q be real numbers and let $Q > 1$. Then there are integers p, q such that $1 \leq q < Q$ and $\|\alpha q\| = |\alpha q - p| \leq \dfrac{1}{Q}$.*

Proof. Let us first assume that Q is a positive integer. Consider the following $Q+1$ numbers:

$$0, 1, \{\alpha\}, \{2\alpha\}, \ldots, \{(Q-1)\alpha\}.$$

All these numbers belong to the interval $[0,1]$ and they have the form $r\alpha - s$, for integers r and s ($0 = 0 \cdot \alpha - 0$, $1 = 0 \cdot \alpha - (-1)$, $\{k\alpha\} = k\alpha - \lfloor k\alpha \rfloor$). Divide the interval $[0,1]$ into Q disjoint subintervals of the length $\dfrac{1}{Q}$:

$$\left[0, \frac{1}{Q}\right), \left[\frac{1}{Q}, \frac{2}{Q}\right), \left[\frac{2}{Q}, \frac{3}{Q}\right), \ldots, \left[\frac{Q-1}{Q}, 1\right].$$

By Dirichlet's box principle, since we have more numbers than subintervals, at least one subinterval contains two (or more) of the above $Q+1$ numbers. Note that those two numbers cannot be 0 and 1. Therefore, there are integers r_1, r_2, s_1, s_2 such that $0 \leq r_i < Q$, $I = 1, 2$, $r_1 \neq r_2$ and

$$|(r_1\alpha - s_1) - (r_2\alpha - s_2)| \leq \frac{1}{Q}.$$

We can assume that $r_1 > r_2$. Let $q = r_1 - r_2, p = s_1 - s_2$. Then $1 \leq q < Q$ and $|\alpha q - p| \leq \dfrac{1}{Q}$, which proves the statement of the theorem in the case $Q \in \mathbb{N}$.

Let us now assume that Q is not a positive integer. Let $Q' = \lfloor Q \rfloor + 1$. By the previous proof, there are integers p, q such that $1 \le q < Q'$ and $|\alpha q - p| \le \dfrac{1}{Q'}$. However, now we have $|\alpha q - p| < \dfrac{1}{Q}$, and from $1 \le q < Q'$, it follows that $1 \le q \le \lfloor Q \rfloor < Q$.

□

Example 1.1. *Let $\alpha = \sqrt{3}$ and $Q = 4$. We seek integers p and q such that $1 \le q < 4$ and $|q\alpha - p| < \dfrac{1}{4}$.*

- *For $q = 2$, the nearest integer to $2\sqrt{3}$ is $p = 3$, giving: $|2\sqrt{3} - 3| \approx 0.4641 > \dfrac{1}{4}$.*
- *For $q = 3$, the nearest integer to $3\sqrt{3}$ is $p = 5$, giving: $|3\sqrt{3} - 5| \approx 0.1962 < \dfrac{1}{4}$.*

Hence, $(p, q) = (5, 3)$ satisfies the conditions of Dirichlet's theorem.

Example 1.2. *Let $\alpha = \pi$ and $Q = 6$. We seek integers p and q such that $1 \le q < 6$ and $|q\alpha - p| < \dfrac{1}{6}$.*

- *For $q = 5$, the nearest integer to 5π is $p = 16$, giving: $|5\pi - 16| \approx 0.2920 > \dfrac{1}{6}$.*
- *For $q = 4$, the nearest integer to 4π is $p = 13$, giving: $|4\pi - 13| \approx 0.4336 > \dfrac{1}{6}$.*
- *For $q = 3$, the nearest integer to 3π is $p = 9$, giving: $|3\pi - 9| \approx 0.4248 > \dfrac{1}{6}$.*
- *Finally, for $q = 1$, the nearest integer to π is $p = 3$, giving: $|\pi - 3| \approx 0.1416 < \dfrac{1}{6}$.*

Hence, $(p, q) = (3, 1)$ satisfies the conditions of Dirichlet's theorem.

Corollary 1.1. *If α is an irrational number, then there are infinitely many pairs p, q of relatively prime integers such that*

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}. \tag{1.1}$$

Remark 1.1. *The statement of Corollary 1.1 does not hold if α is rational. Indeed, let $\alpha = \dfrac{u}{v}$. If $\dfrac{p}{q} \neq \alpha$, then*

$$\left| \alpha - \frac{p}{q} \right| = \left| \frac{u}{v} - \frac{p}{q} \right| = \left| \frac{uq - vp}{vq} \right| \ge \frac{1}{vq}. \tag{1.2}$$

So from (1.1), it follows that $q < v$, while p is uniquely determined by q as the nearest integer to αq. This means that (1.1) can be satisfied only for finitely many pairs p, q of relatively prime integers.

1.2 Farey sequences

In this section, we will demonstrate one of the methods by which the inequality $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$ can be improved.

1.2.1 Definitions and properties

Definition 1.1. *We define the Farey sequence of order n, denoted $\mathcal{F}_n$, as the sequence of rational numbers of the form $\frac{a}{b}$, where a and b are positive integers such that $1 \leq a \leq b \leq n$ and $\gcd(a, b) = 1$ (reduced form). We write the Farey sequence in increasing order, starting from $\frac{0}{1}$ and ending with $\frac{1}{1}$.*

Example 1.3. *The Farey sequence of order 3 is the sequence of numbers*

$$\left(\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right)$$

Due to the bounds on Farey fractions, we'll assume for the rest of the paper that fractions are in the range $[0, 1]$, unless otherwise specified, and that all fractions are in reduced form.

We can deduce one elementary property of the Farey sequence.

Corollary 1.2. *The number of elements in the Farey sequence $\mathcal{F}_n$ is $\sum_{i=1}^{n} \phi(i) + 1$, where $\phi(n)$ is the Eulers totient function, which counts the number of integers up to n relatively prime to n.*

Theorem 1.2. *Two distinct rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive in $\mathcal{F}_k$ where*

$$k = \max(b, d)$$

if and only if $ad - bc = \pm 1$.

Definition 1.2. *The mediant of two fractions $\frac{a}{b}$ and $\frac{c}{d}$ is defined as $\frac{a+c}{b+d}$.*

The mediant of two fractions offers an interesting property if these two fractions happen to be consecutive in some Farey sequence.

Example 1.4. *Let $\frac{a}{b} = \frac{1}{3}$, $\quad \frac{c}{d} = \frac{1}{2}$. We have: $a = 1, \quad b = 3, \quad c = 1, \quad d = 2$.*
By the definition of the mediant: $\frac{a+c}{b+d} = \frac{1+1}{3+2} = \frac{2}{5}$.
Therefore: The mediant of $\frac{1}{3}$ and $\frac{1}{2}$ is $\frac{2}{5}$. Also, we notice that: $\frac{1}{3} < \frac{2}{5} < \frac{1}{2}$.
Hence, the mediant fraction $\frac{2}{5}$ lies between the two fractions $\frac{1}{3}$ and $\frac{1}{2}$.

Proposition 1.1. $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$

Proof. Observe that

$$\frac{a}{b} < \frac{a+c}{b+d} \Leftrightarrow ab + ad < ab + bc \Leftrightarrow ad < bc$$

Which follows from $\frac{a}{b} < \frac{c}{d}$. On the other side,

$$\frac{a+c}{b+d} < \frac{c}{d} \Leftrightarrow ad + dc < bc + cd \Leftrightarrow ad < bc$$

in a similar fashion. □

Corollary 1.3. *For consecutive Farey fractions* $\frac{a}{b}$ *and* $\frac{c}{d}$, $\frac{a+c}{b+d}$ *is in reduced form.*

1.3 Diophantine approximations using Farey sequences

The main idea here is to get close to non-fraction numbers using fraction numbers. Since fraction numbers are spread everywhere, we know that for any real number x and any tiny number $\varepsilon > 0$, we can find whole numbers p and q that fit this rule:

$$\left|\frac{p}{q} - x\right| < \varepsilon$$

For example, let us use:

$$\sqrt{2} = 1.4142135...$$

We can find close numbers by using fractions like:

$$\frac{14142}{10000} = \frac{7071}{5000}$$

But doing this is not helpful. Diophantine Approximation wants to make the bottom number (denominator) as small as possible, while still staying very close to the true value.

Theorem 1.3 (Weak Diophantine Approximation.)**.** *[38] For every positive irrational number x and $\varepsilon > 0$, there exist infinitely many p, q with*

$$\left|\frac{p}{q} - x\right| < \frac{1}{2q}$$

Example 1.5. *Let* $x = \sqrt{2} \approx 1.414213$, $\frac{p}{q} = \frac{7}{5} = 1.4$, $\left|\frac{p}{q} - x\right| = \left|\frac{7}{5} - \sqrt{2}\right| = |1.4 - 1.414213| = 0.014213$.

On the other hand $\frac{1}{2q} = \frac{1}{2 \times 5} = \frac{1}{10} = 0.1$. *Since* $0.014213 < 0.1$.

Then $\left|\frac{p}{q} - x\right| < \frac{1}{2q}$. *Therefore* $\frac{7}{5}$ *gives a good approximation of* $x = \sqrt{2}$.

Also $\dfrac{1}{2q^2} = \dfrac{1}{2(5)^2} = \dfrac{1}{50} = 0.02$. *Since* $0.014213 < 0.02$. *Then* $\left|\dfrac{p}{q} - x\right| < \dfrac{1}{2q^2}$.
Therefore $\dfrac{7}{5}$ *gives a better and more accurate approximation of* $x = \sqrt{2}$.

Now, we reduce the error to get a better approximation.

Theorem 1.4. *For every positive irrational number x and $\varepsilon > 0$, there exist infinitely many $p, q \in \mathbb{Z}$ with* $\left|\dfrac{p}{q} - x\right| < \dfrac{1}{2q^2}$

Hurwitz's theorem gives a better approximation to reach the best constant.

Theorem 1.5 (Hurwitz's Approximation Theorem)**.** *For every positive irrational number x, there exist infinitely many $p, q \in \mathbb{Z}$ with*

$$\left|\frac{p}{q} - x\right| < \frac{1}{\sqrt{5}q^2}$$

Example 1.6. *Let*
$x = \sqrt{2} \approx 1.414213$, $\quad \dfrac{p}{q} = \dfrac{7}{5} = 1.4$, $\quad \left|\dfrac{p}{q} - x\right| = \left|\dfrac{7}{5} - \sqrt{2}\right| = |1.4 - 1.414213| = 0.014213$.
On the other hand $\dfrac{1}{\sqrt{5}q^2} = \dfrac{1}{\sqrt{5}(5)^2} = \dfrac{1}{25\sqrt{5}} \approx 0.017889$.
Since $0.014213 < 0.017889$. *Then* $\left|\dfrac{p}{q} - x\right| < \dfrac{1}{\sqrt{5}q^2}$.

Therefore, by Hurwitz's theorem, $\dfrac{7}{5}$ gives a very good approximation of the irrational number $x = \sqrt{2}$.

Corollary 1.4. *The number $\sqrt{2}$ has infinitely many rational approximations using $\varepsilon = 2\sqrt{2} > \sqrt{5}$; in other words, infinitely many p, q with $\gcd(p, q) = 1$ exist such that*

$$\left|\sqrt{2} - \frac{p}{q}\right| < \frac{1}{2\sqrt{2}q^2}$$

This theorem generalizes the previous results to include the constant for any value.

Theorem 1.6 (Segre, 1945; Niven, 1962))**.** *[38] Let α be an irrational number and $\tau \geq 0$. Then there are infinitely many rational numbers $\dfrac{p}{q}$ such that*

$$-\frac{1}{\sqrt{1+4\tau q^2}} < \alpha - \frac{p}{q} < \frac{\tau}{\sqrt{1+4\tau q^2}} \tag{1.3}$$

Furthermore, the statement holds also if in (1.3), $\alpha - \dfrac{p}{q}$ is replaced by $\dfrac{p}{q} - \alpha$.

For $\tau = 1$, we obtain Hurwitz's theorem. Let us also note that the second part of the theorem follows by applying the first part to the number $-\alpha$.

Example 1.7. *Let* $\alpha = \sqrt{2} \approx 1.414213, \quad \tau = 1, \quad \dfrac{p}{q} = \dfrac{7}{5} = 1.4.$

We have $\quad \alpha - \dfrac{p}{q} = \sqrt{2} - \dfrac{7}{5} = 1.414213 - 1.4 = 0.014213.$

On the other hand $\quad \dfrac{\tau}{\sqrt{1+4\tau q^2}} = \dfrac{1}{\sqrt{1+4(1)(5^2)}} = \dfrac{1}{\sqrt{101}} \approx 0.0995.$

Since $\quad 0.014213 < 0.0995$. *Then* $\quad -\dfrac{1}{\sqrt{1+4q^2}} < \alpha - \dfrac{p}{q} < \dfrac{1}{\sqrt{1+4q^2}}.$

Therefore, by the Segre–Niven theorem, $\dfrac{7}{5}$ *gives a good approximation of the irrational number* $\alpha = \sqrt{2}$.

Lemma 1.1. *Let* α *be an irrational number and* $\tau \geq 0$. *Let* $\dfrac{a}{b}$ *and* $\dfrac{c}{d}$ *be rational numbers with positive denominators such that* $bc - ad = 1$ *and*

$$\frac{a}{b} < \frac{a+c}{b+d} < \alpha < \frac{c}{d}.$$

Then at least one of the numbers $\dfrac{a}{b}, \dfrac{a+c}{b+d}, \dfrac{c}{d}$ *satisfies (1.3).*

Corollary 1.5. *Let* α *be an irrational number. Then there are infinitely many rational numbers* $\dfrac{p}{q}$ *such that*

$$-\frac{1}{q^2} < \alpha - \frac{p}{q} < 0$$

and infinitely many rational numbers $\dfrac{p}{q}$ *such that*

$$0 < \alpha - \frac{p}{q} < \frac{1}{q^2}.$$

1.3.1 Lagrange numbers

Look at this rule (inequality):

$$\left| x - \frac{p}{q} \right| < \frac{1}{cq^2}.$$

Here, c is a fixed number. Before, we looked at $c = \sqrt{5}$ and $c = 2\sqrt{2}$. The natural question is: why do we use these two specific numbers?
Remember our talk after Theorem 1.6. The number $\sqrt{5}$ is the biggest possible value for c where the rule:

$$\left| x - \frac{p}{q} \right| < \frac{1}{cq^2}.$$

Has endless (infinitely many) fraction approximations $\dfrac{p}{q}$. Also, we said that if $c > \sqrt{5}$, we cannot approximate the golden ratio $\varphi = \dfrac{1+\sqrt{5}}{2}$ anymore. But what if we ignore ϕ and the numbers connected to it (like $\phi + c$ for a fraction c)? Then, we can make our number c bigger and better. Specifically, we can find endless fraction approximations $\frac{p}{q}$ where:

$$\left| x - \frac{p}{q} \right| < \frac{1}{2\sqrt{2}q^2}.$$

In this case, x is not one of the fractions connected to ϕ. So, we can apply this to all x numbers.

Now, just like ϕ is hard to approximate when $c = \sqrt{5}$, the number $\sqrt{2}$ is hard to approximate using $2\sqrt{2}$. In other words, we cannot find endless fraction approximations $\frac{p}{q}$ that fit this:

$$\left| \sqrt{2} - \frac{p}{q} \right| < \frac{1}{cq^2}.$$

If $c > 2\sqrt{2}$. Now, if we ignore $\sqrt{2}$ and its connected fractions, we can make the number c even bigger.

These fixed numbers c—which work if we ignore certain exceptional irrational number are called Lagrange numbers (L_n). All these numbers together form the Lagrange spectrum. There is an interesting connection between Lagrange numbers and Markov numbers. It does not relate to Diophantine approximation, but it is still very cool. Markov numbers are positive whole numbers x that solve this equation:

$$x^2 + y^2 + z^2 = 3xyz$$

1.3.2 Ford circles

When we talk about Farey sequences and Diophantine approximations, we must also talk about Ford circles. These two topics go together closely. This is because Farey sequences work with fractions that have a maximum bottom number (denominator). At the same time, Diophantine approximation is about finding the closest fraction to a non-fraction number using limited bottom numbers. Ford circles give the perfect picture (geometric representation) for both of these ideas.

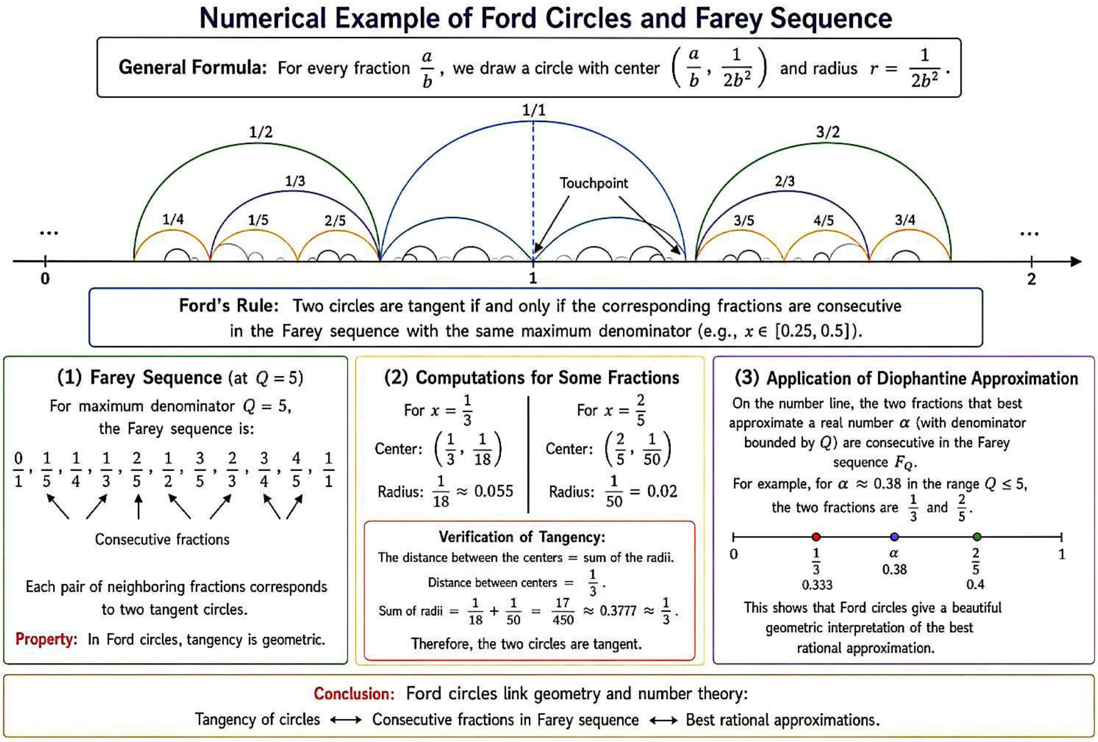
The diagram below shows the Ford circles diagram. Reading left to right, a few Farey sequences can be seen when looking at the fractions written in each circle. Why is this?

Step. For each $\frac{a}{b}$ in F_n, draw the circle centered at $\left(\frac{a}{b}, \frac{1}{2b^2}\right)$ with radius $\frac{1}{2b^2}$.

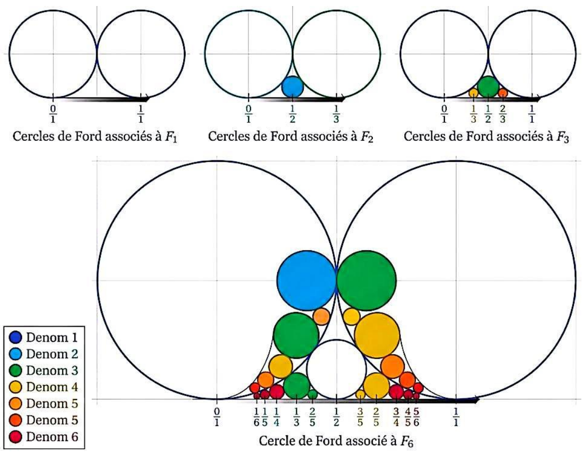
The main observation to be made from this picture is that two circles appear to be tangent if and only if their corresponding Farey fractions are consecutive in some Farey sequence.

This is the main result following from Ford circles.

Numerical example of Ford circles, Farey sequence, and Diophantine approximation



Evolution of Ford circles through the Farey sequence



Theorem 1.7. *Take two Farey fractions $\frac{a}{b}$ and $\frac{c}{d}$ and consider the circles generated by the above construction. The circles are externally tangent if and only if $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive in some F_n.*

Corollary 1.6. *The line $x = \epsilon$ intersects infinitely many Ford circles, for all irrational $0 < \epsilon < 1$. Moreover, if $x = \epsilon$ intersects a Ford circle whose center has coordinate $\frac{p}{q}$, then $\frac{p}{q}$ approximates ϵ. Shifting ϵ by an integer yields the same result for all ϵ.*

For example, let ϵ_1 be an irrational number with $0 < \epsilon_1 < 1$. The line $x = \epsilon_1$ intersects one of the circles denoted by $\frac{0}{1}$ or $\frac{1}{1}$ in Figure. So, one of $\frac{0}{1}$ or $\frac{1}{1}$ approximates ϵ; in other words, one of the following is true:

$$\left|\frac{0}{1} - \epsilon\right| < \frac{1}{2(1)^2}$$

$$\left|\frac{1}{1} - \epsilon\right| < \frac{1}{2(1)^2}$$

This offers an alternate geometric interpretation for why 2 is a viable constant in the inequality

$$\left|x - \frac{p}{q}\right| < \frac{1}{2q^2}$$

For all irrational x.

Example 1.8. *Calculate the radius and the center of the "Ford circle" corresponding to the $\frac{1}{3}$ fraction.*

Solution:

1. *Center: According to the formula $\left(\frac{a}{b}, \frac{1}{2b^2}\right)$:*

   For $a = 1, b = 3$:

$$\textit{Center} = \left(\frac{1}{3}, \frac{1}{2 \times 3^2}\right) = \left(\frac{1}{3}, \frac{1}{18}\right)$$

2. *Radius (r): The radius is equal to the y-coordinate of the center:*

$$r = \frac{1}{18} \approx 0.055$$

- *Note: As the denominator increases, the circle becomes smaller and moves closer to the x-axis.*

CHAPTER 2

CONTINUED FRACTIONS

In this chapter, we studied continued fractions and their importance in approximating irrational numbers. We also learned about rational approximations and some of their basic properties in a simple and clear way.

2.1 Continued fractions

In this section, we review the basic mathematical rules and definitions that form continued fractions. We will study how their step-by-step approximations behave around real numbers. This gives us the necessary math foundation to calculate the error accurately before moving to advanced properties.

We will define polynomials $p_0, q_0, p_1, q_1, p_2, q_2, \dots$ such that p_n, q_n are polynomials in variables $a_0, a_1, \dots, a_n$. Let us first define $p_0 = a_0, q_0 = 1$.

Assume that $p_0, q_0, \dots, p_{n-1}, q_{n-1}$ are already defined. With the notation $p'_k = p_k(a_1, a_2, \dots, a_{k+1})$, $q'_k = q_k(a_1, a_2, \dots, a_{k+1})$, we define

$$p_n = a_0 p'_{n-1} + q'_{n-1}, \quad q_n = p'_{n-1}.$$

Now, $\dfrac{p_n}{q_n}$ is a rational function of $a_0, a_1, \dots, a_n$ and we write

$$\frac{p_n}{q_n} = [a_0, a_1, \dots, a_n].$$

In particular, $[a_0] = \dfrac{p_0}{q_0} = a_0$. For $n > 0$, we have

$$[a_0, a_1, \dots, a_n] = \frac{p_n}{q_n} = \frac{a_0 p'_{n-1} + q'_{n-1}}{p'_{n-1}} = a_0 + \frac{1}{p'_{n-1}/q'_{n-1}} = a_0 + \frac{1}{[a_1, \dots, a_n]}.$$

By repeating this process, we obtain

$$[a_0, a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

Rational functions of this form are called continued fractions.

Example 2.1. $[1, 2, 3] = 1 + \cfrac{1}{2 + \cfrac{1}{3}}$. *We first compute* $2 + \frac{1}{3} = \frac{7}{3}$. *Hence* $\cfrac{1}{2 + \cfrac{1}{3}} = \frac{3}{7}$. *Therefore,* $[1, 2, 3] = 1 + \frac{3}{7} = \frac{10}{7} \approx 1.42857$.

Example 2.2. *Consider the continued fraction* $[2, 1, 2, 2]$. *Since* $[2, 1, 2, 2] = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2}}}$.

We proceed with the calculation step by step:

- *Compute the innermost part:* $2 + \frac{1}{2} = \frac{5}{2}$. *Take the reciprocal:* $\frac{1}{5/2} = \frac{2}{5}$.
- *Add to the next term:* $1 + \frac{2}{5} = \frac{7}{5}$. *Take the reciprocal again:* $\frac{1}{7/5} = \frac{5}{7}$.

Therefore, $[2, 1, 2, 2] = 2 + \frac{5}{7} = \frac{19}{7} \approx 2.714286$.

We have the following properties:

property 1. *For* $n \geq -1$, *we have* $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$.

property 2. *Let* $\alpha = [a_0, a_1, \ldots, a_{n+1}]$. *Then*

$$Q_n \alpha - p_n = \frac{(-1)^n}{a_{n+1} q_n + q_{n-1}}.$$

property 3. *Let* $a_0, a_1, a_2, \ldots$ *be real numbers with* $a_1, a_2, \ldots$ *positive. Then*

a. $\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \ldots,$

b. $\frac{p_1}{q_1} > \frac{p_3}{q_3} > \frac{p_5}{q_5} > \ldots,$

c. If n *is even and* m *odd, then* $\frac{p_n}{q_n} < \frac{p_m}{q_m}$.

Definition 2.1. *If* $r = [a_0, a_1, \ldots, a_n]$, *where* a_0 *is an integer and* $a_1, \ldots, a_n$ *are positive integers, then we call this expression the expansion of* r *into a finite simple continued fraction;* $\frac{p_i}{q_i}$ *is the* i*-th convergent of* r, a_i *is the* i*-th partial quotient of* r *and* $r_i = [a_I, a_{i+1}, \ldots, a_n]$ *is the* i*-th complete quotient of* r.

Example 2.3. $\frac{17}{7} = 2 + \cfrac{1}{2 + \cfrac{1}{3}} = [2, 2, 3]$, *where:* $a_0 = 2, \quad a_1 = 2, \quad a_2 = 3.$
The convergents are: $\frac{p_0}{q_0} = 2, \quad \frac{p_1}{q_1} = \frac{5}{2}, \quad \frac{p_2}{q_2} = \frac{17}{7}.$
The complete quotients are: $r_0 = [2, 2, 3], \quad r_1 = [2, 3], \quad r_2 = [3].$

Example 2.4. *Take:* $r = [1; 2, 3]$. *Then:* $a_0 = 1, \quad a_1 = 2, \quad a_2 = 3.$ *Hence:* $r = 1 + \cfrac{1}{2 + \cfrac{1}{3}} = \frac{10}{7}.$

Therefore, the convergents are: $\frac{P_0}{Q_0} = 1, \quad \frac{P_1}{Q_1} = \frac{3}{2}, \quad \frac{P_2}{Q_2} = \frac{10}{7}.$
Also, the complete quotients are: $r_0 = [1; 2, 3], \quad r_1 = [2, 3], \quad r_2 = [3].$
Thus: $r = [1; 2, 3]$ *is a finite simple continued fraction.*

This approximation marks the end of the calculation when working with rational numbers."

Lemma 2.1. *Let* a_0 *be an integer and* $a_1, a_2, \ldots$ *positive integers. Then the limit* $\lim_{n\to\infty} \frac{p_n}{q_n}$ *exists, and it is an irrational number. Conversely, if* α *is irrational, then there are unique integers* $a_0, a_1 \geq 1, a_2 \geq 1, \ldots$ *such that* $\alpha = \lim_{n\to\infty} \frac{p_n}{q_n}$.

Proof. Since $\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots < \frac{p_1}{q_1}$, the sequence $\frac{p_n}{q_n}$ for n even is increasing and bounded from above, so $\lim_{n\to\infty, n \text{ even}} \frac{p_n}{q_n}$ exists. From similar reasons, there exist $\lim_{n\to\infty, n \text{ odd}} \frac{p_n}{q_n}$. However, these two limits are equal because $\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_{n-1}q_n}$, and $q_n \geq n$ (since $(q_n)_n$ is a strictly increasing sequence of positive integers) implies $\lim_{n\to\infty} \frac{(-1)^n}{q_{n-1}q_n} = 0$. Let $\alpha = \lim_{n\to\infty} \frac{p_n}{q_n}$. Since α lies between $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$, we have

$$\left|\alpha - \frac{p_n}{q_n}\right| < \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \frac{1}{q_{n+1}q_n} < \frac{1}{q_n^2}.$$

Since p_n and q_n are relatively prime, we conclude that there are infinitely many different rational numbers $\frac{p}{q}$ such that $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$. Now, it follows that α is irrational.

We will now prove the converse. Let α be an irrational number. Let $a_0 = \lfloor \alpha \rfloor$ and define a_1 by $\alpha = a_0 + \frac{1}{\alpha_1}$. Then α_1 is also irrational and $\alpha_1 > 1$. We continue this process and for $k \geq 1$ define $a_k = \lfloor \alpha_k \rfloor$ and $\alpha_k = a_k + \frac{1}{\alpha_{k+1}}$. Then $a_k \geq 1, \alpha_{k+1} > 1$ and α_{k+1} is irrational. Let us introduce the notation $\lim_{n\to\infty}[a_0, a_1, \ldots, a_n] = [a_0, a_1, a_2, \ldots]$. We will now show that

$$\alpha = [a_0, a_1, a_2, \ldots].$$

For any $n \geq 0$, we have $\alpha = [a_0, a_1, \ldots, a_n, \alpha_{n+1}]$. By property 2, we get

$$|q_n\alpha - p_n| = \frac{1}{\alpha_{n+1}q_n + q_{n-1}}.$$

So

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}. \tag{2.1}$$

Which implies $\lim_{n\to\infty} \frac{p_n}{q_n} = \alpha$.

It remains to show that the integers $a_0, a_1 \geq 1, a_2 \geq 1, \ldots$ are unique. We have

$$\alpha = [a_0, a_1, a_2, \ldots] = a_0 + \frac{1}{[a_1, a_2, \ldots]}.$$

This implies $0 \leq \alpha - a_0 < 1$, so $a_0 = \lfloor \alpha \rfloor$ and thus, a_0 is unique. Therefore, $\alpha_1 = [a_1, a_2, \ldots]$ is also uniquely determined by α. Now, $a_1 = \lfloor \alpha_1 \rfloor$ so a_1 is also unique, etc. □

Definition 2.2. *Let α be an irrational number. If $\alpha = [a_0; a_1, a_2, \ldots]$, then we call this expression the expansion of α into (infinite) simple continued fraction; $\frac{p_i}{q_i} = [a_0, \ldots, a_i]$ is the i-th convergent of α, a_i is the i-th partial quotient, and $\alpha_i = [a_i, a_{I+1}, \ldots]$ is the i-th complete quotient of α. Here, the word "simple" (as with finite continued fractions) refers to continued fractions which satisfy the conditions $a_0 \in \mathbb{Z}$, $a_i \in \mathbb{N}$ for $i \geq 1$. Since we will mostly consider only such continued fractions, we will occasionally omit the word "simple".*

Example 2.5. *Take:* $\alpha = \sqrt{2} = [1; 2, 2, 2, \ldots]$. *Then:* $a_0 = 1, \quad a_1 = a_2 = \cdots = 2.$

Hence, the convergents are: $\frac{P_0}{Q_0} = 1, \quad \frac{P_1}{Q_1} = \frac{3}{2}, \quad \frac{P_2}{Q_2} = \frac{7}{5}.$

Also, the complete quotients are: $\alpha_0 = [1; 2, 2, 2, \ldots], \quad \alpha_1 = [2; 2, 2, \ldots].$

Therefore: $\sqrt{2} = [1; 2, 2, 2, \ldots]$ *is an infinite simple continued fraction of an irrational number.*

Example 2.6.

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}} = [1, 2, 2, 2, \ldots]$$

Where: $a_0 = 1, \quad a_1 = 2, \quad a_2 = 2, \quad a_3 = 2, \ldots$

The convergents are: $\frac{p_0}{q_0} = 1, \quad \frac{p_1}{q_1} = \frac{3}{2}, \quad \frac{p_2}{q_2} = \frac{7}{5}, \quad \frac{p_3}{q_3} = \frac{17}{12}.$

The complete quotients are:

$\alpha_0 = [1, 2, 2, 2, \ldots], \quad \alpha_1 = [2, 2, 2, \ldots], \quad \alpha_2 = [2, 2, \ldots], \quad \alpha_3 = [2, \ldots].$

In the first section, we studied the basic properties of approximation. Now, in the second section, we move to advanced theorems that show the rules for the best "approximation."

2.2 Continued fraction and approximations to irrational numbers

Based on the previous basics, we move in this section to study the major theorems rules that prove the efficiency and quality of approximation. We will focus on the standards of "best rational approximation" and error cases for irrational numbers.

Theorem 2.1. *Let $\alpha = \alpha_0$ be an irrational number, and define the sequence $a_0, a_1, a_2, \dots$ recursively by*

$$a_k = [\alpha_k], \quad \alpha_{k+1} = \frac{1}{\alpha_k - a_k}$$

For $k = 0, 1, 2, \dots$. Then α is the value of the infinite simple continued fraction $[a_0; a_1, a_2, \dots]$.

Example 2.7. *We have $\alpha_0 = \sqrt{2}$, so $a_0 = [\alpha_0] = 1$, and from the relation $\alpha_{k+1} = \dfrac{1}{\alpha_k - a_k}$ we get:*

$$\alpha_1 = \frac{1}{\alpha_0 - a_0} = \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1, \quad \textit{therefore } a_1 = [\alpha_1] = 2$$

$$\alpha_2 = \frac{1}{\alpha_1 - a_1} = \frac{1}{(\sqrt{2} + 1) - 2} = \sqrt{2} + 1, \quad \textit{hence } a_2 = [\alpha_2] = 2.$$

In the same way, we obtain: $a_3 = a_4 = \dots = 2$. Therefore: $\sqrt{2} = [1, 2, 2, 2, \dots]$.

Let α be an irrational number. By formula (2.1), each convergent of α satisfies the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Theorem 2.2 (Vahlen, 1895)**.** *Let $\dfrac{p_{n-1}}{q_{n-1}}$ and $\dfrac{p_n}{q_n}$ be two consecutive convergents of α. Then at least one of them satisfies the inequality*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}.$$

Proof. Numbers $\alpha - \dfrac{p_n}{q_n}$, $\alpha - \dfrac{p_{n-1}}{q_{n-1}}$ have opposite signs, so

$$\left|\alpha - \frac{p_n}{q_n}\right| + \left|\alpha - \frac{p_{n-1}}{q_{n-1}}\right| = \left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right| = \frac{1}{q_n q_{n-1}} < \frac{1}{2q_n^2} + \frac{1}{2q_{n-1}^2}$$

(since $2ab < a^2 + b^2$ for $a \neq b$). Accordingly,

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{2q_n^2} \quad \text{or} \quad \left|\alpha - \frac{p_{n-1}}{q_{n-1}}\right| < \frac{1}{2q_{n-1}^2}.$$

$\square$

Example 2.8. *Take* $\alpha = \sqrt{2} \approx 1.4142$*, and the two consecutive convergents:* $\frac{P_1}{Q_1} = \frac{3}{2}, \frac{P_2}{Q_2} = \frac{7}{5}$.
Then $\left|\sqrt{2} - \frac{7}{5}\right| = |1.4142 - 1.4| = 0.0142.$
Also $\frac{1}{2q^2} = \frac{1}{2 \cdot 5^2} = \frac{1}{50} = 0.02.$
Hence $0.0142 < 0.02$*. Therefore* $\left|\sqrt{2} - \frac{7}{5}\right| < \frac{1}{2 \cdot 5^2}$.
Thus, the convergent $\frac{7}{5}$ *satisfies Vahlen's inequality.*

This theorem leads to a better approximation.

Theorem 2.3 (Borel, 1903)**.** *[13] Let* $\frac{p_{n-2}}{q_{n-2}}, \frac{p_{n-1}}{q_{n-1}}, \frac{p_n}{q_n}$ *be three consecutive convergents of* α*. Then at least one of them satisfies the inequality*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

Proof. Put $\alpha = [a_0, a_1, \dots]$, $\alpha_i = [a_i, a_{I+1}, \dots]$ and $\beta_i = \frac{q_{i-2}}{q_{I-1}}$ for $I \geq 1$. From property 2, we have

$$\left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{q_n^2(\alpha_{n+1} + \beta_{n+1})}. \tag{2.2}$$

To complete the proof, we have to show that there is no positive integer n such that for $i = n-1, n, n+1$ we have

$$\alpha_i + \beta_i \leq \sqrt{5}. \tag{2.3}$$

Assume that (2.3) is satisfied for $i = n-1, n$. Then from

$$\alpha_{n-1} = a_{n-1} + \frac{1}{\alpha_n}, \quad \frac{1}{\beta_n} = \frac{q_{n-1}}{q_{n-2}} = a_{n-1} + \frac{q_{n-3}}{q_{n-2}} = a_{n-1} + \beta_{n-1}.$$

It follows that

$$\frac{1}{\alpha_n} + \frac{1}{\beta_n} = \alpha_{n-1} + \beta_{n-1} \leq \sqrt{5}.$$

Hence, $1 = \alpha_n \cdot \frac{1}{\alpha_n} \leq (\sqrt{5} - \beta_n)(\sqrt{5} - \frac{1}{\beta_n})$, which is equivalent to $\beta_n^2 - \sqrt{5}\beta_n + 1 \leq 0$. This implies that $\beta_n \geq \frac{\sqrt{5}-1}{2}$, and since β_n is rational, we get $\beta_n > \frac{\sqrt{5}-1}{2}$.

If (2.3) is also satisfied for $I = n, n+1$, then in the same way we get $\beta_{n+1} > \frac{\sqrt{5}-1}{2}$. Thus, we obtain

$$1 \leq a_n = \frac{q_n}{q_{n-1}} - \frac{q_{n-2}}{q_{n-1}} = \frac{1}{\beta_{n+1}} - \beta_n < \frac{2}{\sqrt{5}-1} - \frac{\sqrt{5}-1}{2} = 1.$$

A contradiction. ∎

Example 2.9. *Take:* $\alpha = \sqrt{2} \approx 1.4142$, *and the three consecutive convergents:* $\frac{P_0}{Q_0} = 1$, $\frac{P_1}{Q_1} = \frac{3}{2}$, $\frac{P_2}{Q_2} = \frac{7}{5}$.
Choose: $\frac{p}{q} = \frac{7}{5}$. *Then:* $\left|\sqrt{2} - \frac{7}{5}\right| = |1.4142 - 1.4| = 0.0142$.
Also: $\frac{1}{\sqrt{5}q^2} = \frac{1}{\sqrt{5} \cdot 5^2} = \frac{1}{25\sqrt{5}} \approx 0.0179$. *Hence:* $0.0142 < 0.0179$.
Therefore: $\left|\sqrt{2} - \frac{7}{5}\right| < \frac{1}{\sqrt{5} \cdot 5^2}$.

Thus, the convergent $\frac{7}{5}$ satisfies Borel's inequality.

Lemma 2.2. *Assume that* α *has a continued fraction expansion of the form*

$$\alpha = [a_0; a_1, \ldots, a_N, 1, 1, 1, \ldots]$$

Then $\lim_{n\to\infty} q_n^2 \left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{\sqrt{5}}$

This lemma introduces a famous main theorem.

Theorem 2.4 (Legendre, 1798)**.** *Let* p, q *be integers such that* $q \geq 1$ *and*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}. \tag{2.4}$$

Then $\frac{p}{q}$ *is a convergent of* α.

Example 2.10. *Take* $\alpha = \sqrt{2} \approx 1.414213$, $\frac{p}{q} = \frac{7}{5} = 1.4$.
Then $\left|\sqrt{2} - \frac{7}{5}\right| = |1.414213 - 1.4| = 0.014213$ *and* $\frac{1}{2q^2} = \frac{1}{2(5)^2} = \frac{1}{50} = 0.02$.
Since $0.014213 < 0.02$ *the condition of the theorem is satisfied.*
Therefore, $\frac{7}{5}$ *is a convergent of* $\sqrt{2}$.

And now, we move to a more general theorem

Theorem 2.5 (Worley, 1981)**.** *[13] Let* α *be an arbitrary real number and* c *a positive real number. if the rational number* $\frac{p}{q}$ *satisfies the inequality*

$$\left|\alpha - \frac{p}{q}\right| < \frac{c}{q^2}. \tag{2.5}$$

Then

$$\frac{p}{q} = \frac{rp_{k+1} \pm sp_k}{rq_{k+1} \pm sq_k}.$$

For some $k \geq -1$ *and non-negative integers* r, s *such that* $rs < 2c$ *(and some choice of the sign).*

Example 2.11. $\alpha = \sqrt{2} \approx 1.414213, \quad c = 1, \quad \frac{p}{q} = \frac{17}{12} = 1.416667.$

$\left|\alpha - \frac{p}{q}\right| = \left|\sqrt{2} - \frac{17}{12}\right| = |1.414213 - 1.416667| = 0.002454.$

On the other hand, $\frac{c}{q^2} = \frac{1}{12^2} = \frac{1}{144} \approx 0.006944.$

Since $0.002454 < 0.006944$ *then* $\left|\alpha - \frac{p}{q}\right| < \frac{c}{q^2}.$

Therefore, by Worley's theorem, $\frac{17}{12}$ *is obtained from consecutive convergents of* $\sqrt{2}$.

2.3 Equivalent numbers

Equivalent numbers are closely linked to continued fractions because they share the same long-term approximation behavior.

This happens because their continued fraction expansions have the exact same "tail" after a certain point. Therefore, these numbers have identical approximation properties from the algebraic side and in the way the approximation works.

Definition 2.3. *We say that irrational numbers* α *and* γ *are equivalent if there are integers* a, b, c, d *such that* $ad - bc = \pm 1$ *and*

$$\gamma = \frac{a\alpha + b}{c\alpha + d}.$$

It is easily verified that this is indeed an equivalence relation. The notation is $\alpha \cong \gamma$.

Example 2.12. *Let* $\alpha = \sqrt{2}$ *and* $\gamma = \sqrt{2} + 1$. *We have:* $\gamma = \frac{1 \cdot \sqrt{2} + 1}{0 \cdot \sqrt{2} + 1}$

- *Since* $a = 1, b = 1, c = 0,$ *and* $d = 1$ *are integers.*
- *And since:* $Ad - bc = (1 \times 1) - (1 \times 0) = 1 - 0 = 1.$
- *Therefore: The condition is met, and thus* $\sqrt{2}$ *and* $\sqrt{2} + 1$ *are equivalent.*

Example 2.13. *Let* $\alpha = \sqrt{2}$ *and* $\gamma = \frac{1}{\sqrt{2}}$. *We have:*

$$\gamma = \frac{0 \cdot \sqrt{2} + 1}{1 \cdot \sqrt{2} + 0}.$$

- *Since* $a = 0, b = 1, c = 1,$ *and* $d = 0$ *are integers.*
- *And since:* $Ad - bc = (0 \times 0) - (1 \times 1) = 0 - 1 = -1.$
- *Therefore: The condition is met, and thus* $\sqrt{2}$ *and* $\frac{1}{\sqrt{2}}$ *are equivalent.*

Example 2.14. *Let* $\alpha = \sqrt{3}$ *and* $\gamma = \frac{2\sqrt{3} + 1}{\sqrt{3} + 1}$.

- *By comparing with the definition, we have:* $a = 2$, $b = 1$, $c = 1$, *and* $d = 1$.
- *And since:* $Ad - bc = (2 \times 1) - (1 \times 1) = 2 - 1 = 1$.
- *Since the condition is met, therefore* $\sqrt{3}$ *and* $\dfrac{2\sqrt{3}+1}{\sqrt{3}+1}$ *are equivalent.*

2.4 Periodic continued fractions

Periodic continued fractions are essential for finding the best rational approximations of irrational numbers. They represent exactly quadratic irrationals, which are the roots of quadratic equations. Studying this repeating behavior allows us to easily predict the speed and accuracy of their approximations.

Definition 2.4. *We call an infinite simple continued fraction* $[a_0, a_1, a_2, \ldots]$ *periodic if there are integers* $k \geq 0$, $m \geq 1$ *such that* $a_{m+n} = a_n$ *for all* $n \geq k$. *In that case, we write the continued fraction in the form*

$$[a_0, a_1, \ldots, a_{k-1}, \overline{a_k, a_{k+1}, \ldots, a_{k+m-1}}],$$

Where the bar over the numbers $a_k, \ldots, a_{k+m-1}$ *denotes that this block of numbers repeats infinitely. The number* m *is called the period length. If* $k = 0$, *then we say that a continued fraction is purely periodic.*

Example 2.15. ***(i)*** *Let* $\beta = [2, 3, 2, 3, \ldots] = [\overline{2, 3}]$. *Then* $\beta = 2 + \dfrac{1}{3 + \dfrac{1}{\beta}}$. *This gives a quadratic equation for* β, $3\beta^2 - 6\beta - 2 = 0$, *so due to* $\beta > 0$, *we obtain* $\beta = \dfrac{3+\sqrt{15}}{3}$.

(ii) *Let now* $\alpha = [4, 1, \overline{2, 3}]$. *We have* $\alpha = 4 + \dfrac{1}{1 + \dfrac{1}{\beta}} = 4 + \dfrac{\beta}{\beta + 1} = \dfrac{29 + \sqrt{15}}{7}$.

These two examples illustrate the general situation.

Example 2.16. *Let* $\beta = [2, 3, 2, 3, 2, 3, \ldots]$.

- *We have: The pattern of the numbers* 2 *and* 3 *repeats infinitely.*
- *Since: The repetition starts from the very beginning (index* $k = 0$*).*
- *And since: The length of the repeating period is two numbers (*$m = 2$*).*
- *Therefore: We can write it simply by putting a bar over the repeating part:*

$$\beta = [\overline{2, 3}]$$

Example 2.17. *Let* $\alpha = [4, 1, 2, 3, 2, 3, 2, 3, \dots]$

- *We have: The first numbers* 4 *and* 1 *do not repeat, but the repetition starts after them.*
- *Since: The repeating pattern is* 2 *and* 3*, and it starts after the first two numbers (index* $k = 2$*).*
- *And since: The length of the repeating period is two numbers (*$m = 2$*).*
- *Therefore: We write it by putting a bar only over the repeating part:*

$$\alpha = [4, 1, \overline{2, 3}]$$

Example 2.18. *Let* $\gamma = [1, 2, 2, 2, 2, \dots]$ *(the continued fraction of* $\sqrt{2}$*).*

- *We have: The first number* 1 *does not repeat, then the number* 2 *repeats forever.*
- *Since: The repetition starts from the second number (index* $k = 1$*).*
- *And since: The length of the repeating period is only one number (*$m = 1$*).*
- *Therefore: We write it simply as* $\gamma = [1; \overline{2}]$.

CHAPTER 3

NEWTON'S APPROXIMATIONS AND SIMULTANEOUS APPROXIMATION

In this chapter, we studied Newton's method and the iterative method. We also learned about some results related to Dirichlet's theorem and the approximation of real numbers.

3.1 Newton's approximants

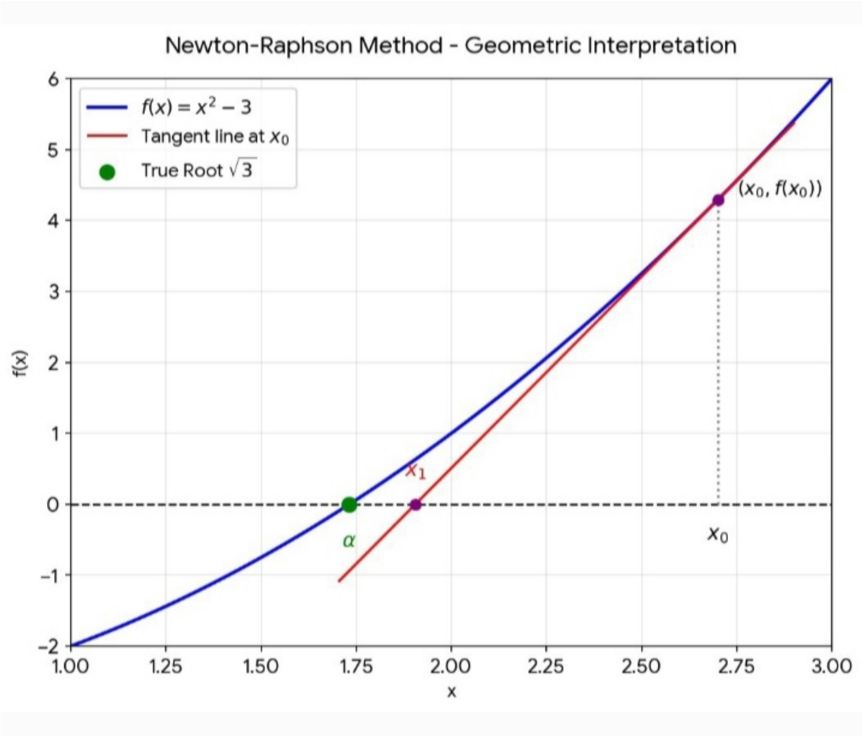
In this section, we will look at the links between two common methods. Both methods help us find the approximate square root of a positive whole number d that is not a perfect square.

We already know the first method. This method uses the convergents of the continued fraction expansion of $\sqrt{d}$.

The second method is Newton's method. People also call it the Newton-Raphson method or the tangent method. It helps find answers to non-linear equations like $f(x) = 0$ by repeating steps.

With this method, we start with an approximate answer x_k. Then, we find a better answer x_{k+1}. To do this, we look at the tangent line on the graph of function f at the point $(x_k, f(x_k))$. The next answer is with this method, we start with an approximate answer x_k. Then, we find a better answer x_{k+1}. To do this, we look at the tangent line on the graph of function f at the point $(x_k, f(x_k))$. The next answer is where this tangent line crosses the x-axis.

Newton-Raphson Method:Geometric Interpretation



Using this way, we get the formula :

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \tag{3.1}$$

Where x_0 is the initial approximation. We will not deal with general questions of convergence of this method, but we will apply it to the equation

$$f(x) = x^2 - d = 0.$$

Formula (3.1) becomes

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{d}{x_k}\right) = \frac{x_k^2 + d}{2x_k}. \tag{3.2}$$

The question arises whether these two methods for approximation of square roots are somehow aligned. To be more precise, we ask: If x_0 is a convergent of $\sqrt{d}$, is then x_1 also a convergent of $\sqrt{d}$?

For $x_0 = \dfrac{p_n}{q_n}$, we obtain $x_1 = \dfrac{p_n^2 + dq_n^2}{2p_nq_n}$, so we introduce the notation

$$R_n = \frac{p_n^2 + dq_n^2}{2p_nq_n}.$$

If Newton's approximant R_n is a convergent of $\sqrt{d}$, we say that R_n is a good approximant.

For a positive integer d, let

$$\sqrt{d} = [a_0, \overline{a_1, a_2, \ldots, a_{\ell-1}, 2a_0}],$$

Be the expansion of $\sqrt{d}$ into continued fraction. Here $\ell = \ell(d)$ denotes the period length.

Lemma 3.1. *For every positive integer k,*

$$a_0 p_{k\ell-1} = d q_{k\ell-1} - p_{k\ell-2}, \tag{3.3}$$

$$a_0 q_{k\ell-1} = p_{k\ell-1} - q_{k\ell-2}. \tag{3.4}$$

Proof. Due to the irrationality of $\sqrt{d}$, by comparing rational and irrational parts of the equation

$$\begin{aligned}\sqrt{d} &= [a_0, \overline{a_1, \ldots, a_{k\ell-1}, 2a_0}] = [a_0, a_1, \ldots, a_{k\ell-1}, a_0 + \sqrt{d}] \\ &= \frac{(a_0+\sqrt{d})p_{k\ell-1} + p_{k\ell-2}}{(a_0+\sqrt{d})q_{k\ell-1} + q_{k\ell-2}},\end{aligned}$$

We obtain

$$\begin{aligned}a_0 p_{k\ell-1} + p_{k\ell-2} &= d q_{k\ell-1}, \\ p_{k\ell-1} &= a_0 q_{k\ell-1} + q_{k\ell-2},\end{aligned}$$

So we see that (3.3) and (3.4) hold.

□

Theorem 3.1. *[13] For every positive integer k,*

$$R_{k\ell-1} = \frac{p_{2k\ell-1}}{q_{2k\ell-1}}.$$

Proof. We expand $\frac{p_{2k\ell-1}}{q_{2k\ell-1}}$ by using (3.4) and (3.5):

$$\begin{aligned}\frac{p_{2k\ell-1}}{q_{2k\ell-1}} &= [a_0, a_1, \ldots, a_{k\ell-1}, 2a_0, a_1, \ldots, a_{k\ell-1}] \\ &= [a_0, a_1, \ldots, a_{k\ell-1}, a_0 + \frac{p_{k\ell-1}}{q_{k\ell-1}}] = \frac{(a_0 + \frac{p_{k\ell-1}}{q_{k\ell-1}})p_{k\ell-1} + p_{k\ell-2}}{(a_0 + \frac{p_{k\ell-1}}{q_{k\ell-1}})q_{k\ell-1} + q_{k\ell-2}} \\ &= \frac{a_0 p_{k\ell-1} q_{k\ell-1} + p_{k\ell-1}^2 + q_{k\ell-1} p_{k\ell-2}}{a_0 q_{k\ell-1}^2 + p_{k\ell-1} q_{k\ell-1} + q_{k\ell-1} q_{k\ell-2}} \\ &= \frac{p_{k\ell-1}^2 + d q_{k\ell-1}^2}{2 p_{k\ell-1} q_{k\ell-1}} = R_{k\ell-1}.\end{aligned}$$

□

Example 3.1. *Let $k = 1$ and $[a_0, a_1] = [1, 2]$. Since $[1, 2] = 1 + \frac{1}{2} = \frac{3}{2}$.*

We have: $p_{k-1} = 3$ and $q_{k-1} = 2$ by the theorem, $R_{k-1} = \frac{p_{k-1}}{q_{k-1}}$.

Therefore: $R_0 = \frac{3}{2} = 1.5$. Since $[1, 2] = 1.5$, it follows that $R_0 = [1, 2]$.

Hence, the theorem is verified.

Lemma 3.2.

$$R_n - \sqrt{d} = \frac{q_n}{2p_n}\left(\frac{p_n}{q_n} - \sqrt{d}\right)^2$$

Proof.

$$\begin{aligned} 2(R_n - \sqrt{d}) &= \left(\frac{p_n}{q_n} - \sqrt{d}\right) + \left(\frac{dq_n}{p_n} - \sqrt{d}\right) \\ &= \left(\frac{p_n}{q_n} - \sqrt{d}\right) - \frac{\sqrt{d}q_n}{p_n}\left(\frac{p_n}{q_n} - \sqrt{d}\right) = \frac{q_n}{p_n}\left(\frac{p_n}{q_n} - \sqrt{d}\right)^2 \end{aligned}$$

□

Proposition 3.1. *If $R_n = \frac{p_k}{q_k}$, then k is odd.*

Proof. Since $\frac{p_k}{q_k} > \sqrt{d}$ if and only if k is odd, while from lemma 3.2 we know that $R_n > \sqrt{d}$, we conclude that k is odd. □

Proposition 3.2. *If $a_{n+1} > 2\sqrt{d} + 1$, then R_n is a convergent of $\sqrt{d}$.*

Example 3.2. *[13] If $d = 16x^4 - 16x^3 - 12x^2 + 16x - 4$, where $x \geq 2$ is a positive integer, then $\ell(d) = 8$ and $b(d) = 6$.*

Solution: By the algorithm for continued fraction expansion of quadratic irrationals, we have

$$\sqrt{d} = [(2x + 1)(2x - 2), \overline{x, 1, 1, 2x^2 - x - 2, 1, 1, x, 2(2x + 1)(2x - 2)}].$$

Therefore, $\ell(d) = 8$.

By direct calculation, we obtain:

$$\begin{aligned}
R_0 &= \frac{p_3}{q_3} = \frac{2x(4x^2-3)}{2x+1},\\
R_1 &= \frac{p_5}{q_5} = \frac{(2x-1)(8x^4-8x^2+1)}{2x(2x^2-1)},\\
R_3 &= \frac{p_7}{q_7} = \frac{(2x^2-1)(16x^4-16x^2+1)}{x(2x+1)(4x^2-3)},\\
R_5 &= \frac{p_9}{q_9} = \frac{(2x-1)(128x^8-256x^6+160x^4-32x^2+1)}{4x(2x^2-1)(8x^4-8x^2+1)},\\
R_6 &= \frac{p_{11}}{q_{11}} = \frac{2x(4x^2-3)(64x^6-96x^4+36x^2-3)}{(2x+1)(8x^3-6x-1)(8x^3-6x+1)},\\
R_7 &= \frac{p_{15}}{q_{15}} = \frac{(8x^4-8x^2+1)(256x^8-512x^6+320x^4-64x^2+1)}{2x(2x+1)(2x^2-1)(4x^2-3)(16x^4-16x^2+1)}.
\end{aligned}$$

Therefore, $b(d) = 6$.

Newton's approximants lead us to simultaneous approximation.

3.2 Simultaneous approximations

In the last chapter, we looked at how to "approximate a single number." Now, we will expand our results to a group of numbers (a "tuple")

Simultaneous approximation helps us generalize Dirichlet's theorem.

3.2.1 Extending Dirichlet's theorem

In this section, we will extend "Dirichlet's Approximation Theorem" from the last chapter to a group of real numbers that includes irrational numbers.

Lemma 3.3 (Dirichlet, 1842)**.** *[16] If α_{ij} with $1 \le I \le n$ and $1 \le j \le m$, are nm real numbers and $k > 1$ is an integer, then there exist integers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m$ with*

$$1 \le \max(|b_1|, \ldots, |b_m|) < k^{\frac{n}{m}} \quad \text{and} \quad |\alpha_{i1}b_1 + \cdots + \alpha_{im}b_m - a_i| \le \frac{1}{k}$$

where $1 \le i \le n$.

Proof. Consider points:

$$(\{a_{11}x_1 + \cdots + a_{1m}x_m\}, \ldots, \{a_{n1}x_1 + \cdots + a_{nm}x_m\})$$

where

$$\{a_{i1}x_1 + \cdots + a_{im}x_m\} = a_{i1}x_1 + \cdots + a_{Im}x_m - \lfloor a_{i1}x_1 + \cdots + a_{Im}x_m \rfloor$$

and each x_j is an integer satisfying, $0 \le x_j < k\frac{n}{m}$ for $1 \le j \le m$.
There are at least k^n such points, and each of these points lie in the closed unit cube, denoted by $[0,1]^n$ or $\mathcal{I}^n$, consisting of points $(t_1, t_2, \ldots, t_n)$ with $0 \le t_i \le 1$ for $1 \le i \le n$. The point $(1, 1, \ldots, 1)$ also lies in $\mathcal{I}^n$, so together there are at least $k^n + 1$ points under consideration.

We divide $\mathcal{I}^n$ into k^n pair wise disjoint subcubes of side length $1/k$. By pigeonhole principle, two of the points under consideration will be in the same subcube. These points are, say:

$$(\{a_{11}x_1 + \cdots + a_{1m}x_m\}, \ldots, \{a_{n1}x_1 + \cdots + a_{nm}x_m\}),$$
$$(\{a_{11}x'_1 + \cdots + a_{1m}x'_m\}, \ldots, \{a_{n1}x'_1 + \cdots + a_{nm}x'_m\})$$

Let, y_i, y'_i be integers for $1 \le I \le n$, such that:

$$Y_i = \lfloor a_{i1}x_1 + \cdots + a_{im}x_m \rfloor \quad \text{and} \quad y'_i = \lfloor a_{i1}x'_1 + \cdots + a_{im}x'_m \rfloor$$

Then, we can re-write the points in consideration as:

$$((a_{11}x_1 + \cdots + a_{1m}x_m - y_1), \ldots, (a_{n1}x_1 + \cdots + a_{nm}x_m - y_n)),$$
$$((a_{11}x'_1 + \cdots + a_{1m}x'_m - y'_1), \ldots, (a_{n1}x'_1 + \cdots + a_{nm}x'_m - y'_n))$$

Here, $(x_1, \ldots, x_m) \ne (x'_1, \ldots, x'_m)$. Put, $b_j = x_j - x'_j$ for $1 \le j \le m$, and $a_i = y_i - y'_i$ for $1 \le I \le n$. Then:

$$|a_{i1}b_1 + \cdots + a_{im}b_m - a_i| \le \frac{1}{k}$$

□

Example 3.3. *Let $n = 1, m = 1, a_{11} = \pi \approx 3.14159$, and $k = 3$.*
By Dirichlet's lemma, we seek integers a_1 and b_1 such that:

$$1 \le |b_1| < 3 \quad \textit{and} \quad |a_{11}b_1 - a_1| < \frac{1}{3}.$$

Taking $b_1 = 2$, we have: $A_{11}b_1 = 2\pi \approx 6.283186$.
The nearest integer to 2π is $a_1 = 6$, which gives the inequality: $|2\pi - 6| \approx 0.283186 < \frac{1}{3}$.
Hence, $a_1 = 6$ and $b_1 = 2$ satisfy Dirichlet's lemma.

Theorem 3.2. *Let α_{ij} with $1 \leq I \leq n$ and $1 \leq j \leq m$, be nm real numbers and $k > 1$ be an integer, if for some i in $1 \leq i \leq n$, $\alpha_{i1}, \ldots, \alpha_{im}, 1$ are linearly independent over the rational numbers then there exist infinitely many coprime $(m+n)$-tuples*

$$(b_1, b_2, \ldots, b_m, a_1, a_2, \ldots, a_n)$$

With

$$b = \max(|b_1|, \ldots, |b_m|) > 0 \quad \textit{and} \quad |\alpha_{i1}b_1 + \cdots + \alpha_{im}b_m - a_i| \leq \frac{1}{b^{\frac{m}{n}}}$$

Where $1 \leq i \leq n$.

Proof. The previous inequalities directly imply this inequality
Now, by linear independence, we always have

$$|a_{i1}b_1 + \cdots + a_{im}b_m - a_i| \neq 0$$

Hence for *fixed* $a_i, b_1, \ldots, b_m$,

$$|a_{i1}b_1 + \cdots + a_{im}b_m - a_i| \leq \frac{1}{k}$$

can hold only for $k \leq k_0$. Hence as $k \to \infty$, we obtain infinitely many solutions. □

Example 3.4. *Let $n = 1, m = 1, a_{11} = \sqrt{2} \approx 1.414213$, and $k = 2$. Taking $b_1 = 3$, we have $b = \max(|b_1|) = 3$.*

Choosing $a_1 = 4$, we obtain: $|a_{11}b_1 - a_1| = |3\sqrt{2} - 4| \approx 0.242640$.

Moreover: $1/b^{m/n} = 1/3 \approx 0.333333$. Since $0.242640 < 0.333333$, it follows that $|a_{11}b_1 - a_1| < 1/b^{m/n}$.

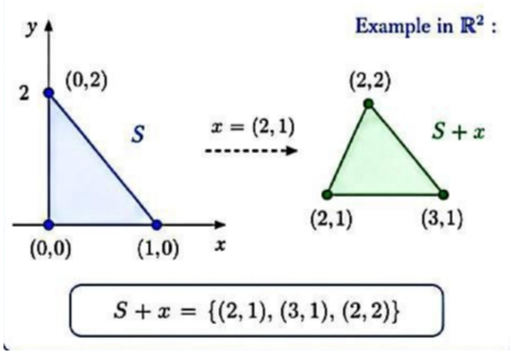
Hence, $(b_1, a_1) = (3, 4)$ satisfies the conditions of the theorem.

To generalize Dirichlet's theorem, we use the geometry of numbers.

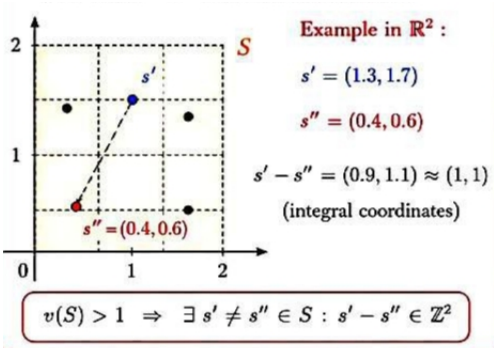
3.2.2 Geometry of numbers

In this section, we will only study the sets whose volume $v(S)$ is defined by multiple Riemann integrals (that is, Jordan volume).

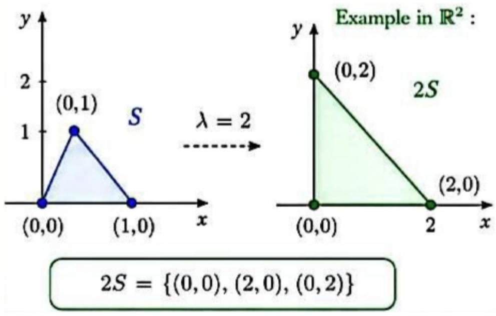
Definition 3.1 (Translation)**.** *If $\mathbf{x} \in \mathbb{R}^n$ and $\mathcal{S} \subseteq \mathbb{R}^n$, then $\mathcal{S} + \mathbf{x}$ is set of all points $\mathbf{s} + \mathbf{x} = (s_1 + x_1, \ldots, s_n + x_n)$ with $\mathbf{s} \in \mathcal{S}$ and is called translation of $\mathcal{S}$ by $\mathbf{x}$.*



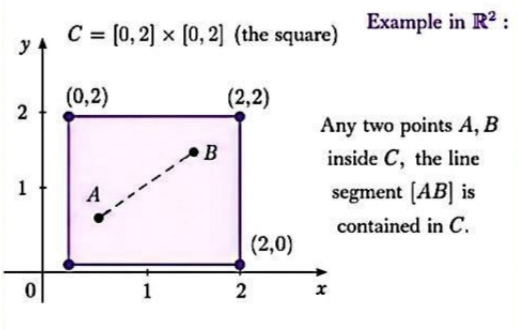
Lemma 3.4 (Blichfeldt, 1914). *Let $\mathcal{S}$ be a set in $\mathbb{R}^n$ with volume $v(\mathcal{S}) > 1$. Then there exist two distinct points $\mathbf{s}', \mathbf{s}'' \in \mathcal{S}$ such that $\mathbf{s}' - \mathbf{s}''$ has integral coordinates.*



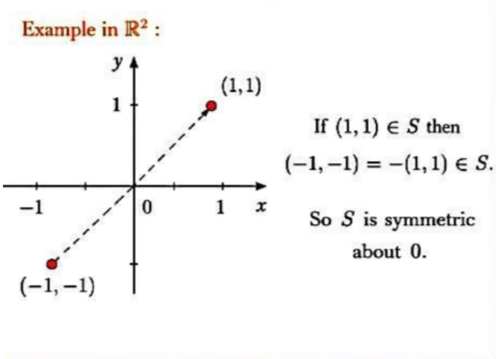
Definition 3.2 (Dilation). *If $\lambda \in \mathbb{R}$ and $\mathcal{S} \subseteq \mathbb{R}^n$, then $\lambda\mathcal{S}$ denotes the set of all points $\lambda\mathbf{s} = (\lambda s_1, \lambda s_2, \ldots, \lambda s_n)$ with $\mathbf{s} \in \mathcal{S}$ and is called dilation of $\mathcal{S}$ by the factor λ.*



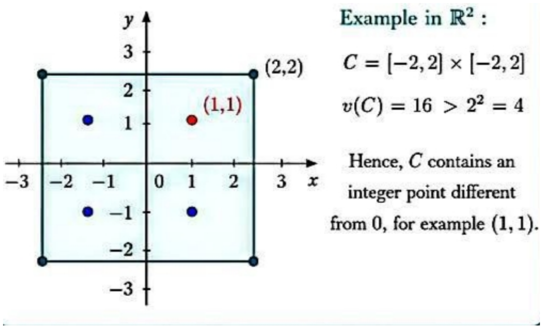
Definition 3.3 (Convex set). *A set $\mathcal{C}$ in $\mathbb{R}^n$ is said to be convex if for any two points $\mathbf{x}, \mathbf{y} \in \mathcal{C}$, the line segment joining them is contained In $\mathcal{C}$.*



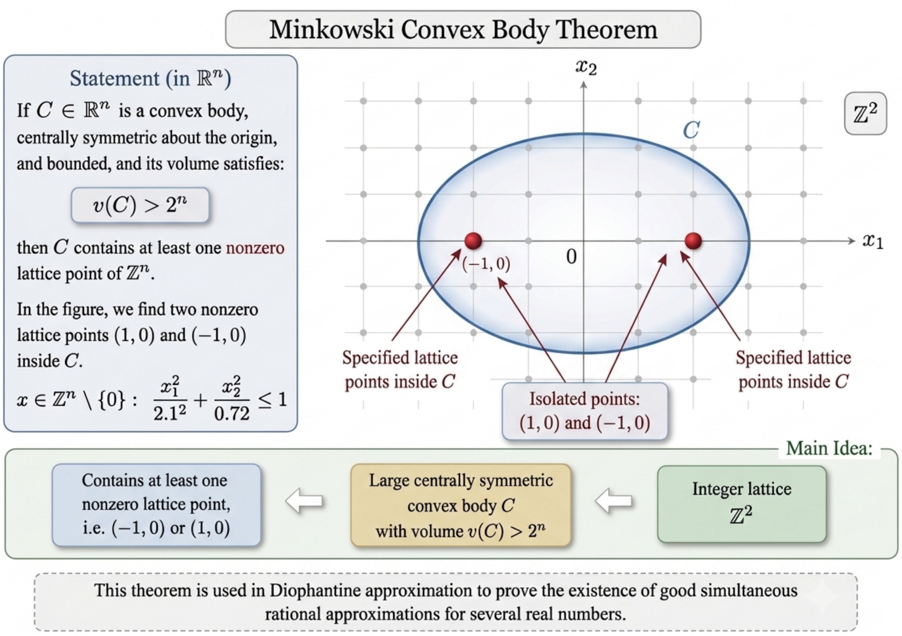
Definition 3.4 (Symmetric set about 0). *A set $\mathcal{S}$ in $\mathbb{R}^n$ that has the property that* $\mathbf{s} \in \mathcal{S}$ *if and only if* $-\mathbf{s} \in \mathcal{S}$ *is said to be symmetric about 0.*



Definition 3.5 (Minkowski's Convex Body Theorem, 1896). *Let $\mathcal{C}$ be a convex set in $\mathbb{R}^n$, symmetric about 0, bounded and with volume $v(\mathcal{C}) > 2^n$, then $\mathcal{C}$ contains a point whose coordinates are integers, not all of them 0.*



Application of Minkowski's theorem in dimension two $\mathbb{R}^2$



This research also has future perspectives

In the extended field of mathematics (ZFCIST), we also find the following result concerning simultaneous Diophantine approximations.

In this work, we also present the following theorem, which has been demonstrated in [1].

Theorem 3.3. *[1] Let* $X = \{(x_1, t_1), (x_2, t_2), \ldots, (x_n, t_n)\}$ *be a finite part of* $\mathbb{R} \times \mathbb{R}^{*+}$*, then there exist a finite part* R *of* $\mathbb{R}^{*+}$ *such that for all* $\varepsilon > 0$ *there exists* $r \in R$ *such that if* $0 < \varepsilon \leq r$ *then there exist rational numbers* $\left(\frac{p_i}{q}\right)_{i=1,2,\ldots,n}$ *such that:*

$$\begin{cases} \left| x_i - \dfrac{p_i}{q} \right| \leq \varepsilon t_i \\ \varepsilon q \leq t_i \end{cases}, \quad I = 1, 2, \ldots, n. \tag{*}$$

it is clear that the condition $\varepsilon q \leq t_i$ *for* $i = 1, 2, \ldots, n$ *is equivalent to* $\varepsilon q \leq t = \min_{i=1,2,\ldots,n}(t_i)$*. Also, we have (*) for all* ε *verifying* $0 < \varepsilon \leq \varepsilon_0 = \min R$*.*

The previous theorem is the classical equivalent of the following one which is formulated in the context of the nonstandard analysis

GENERAL CONCLUSION

In conclusion, the various results presented in this memory highlight the fundamental importance of rational approximations in the study of real numbers. From Dirichlet's classic theorem to methods based on Farey sequences, continued fractions, Newton's method, and simultaneous approximations, a common central idea emerges: any real number can be approximated with great precision by suitably chosen rational numbers.

Dirichlet's theorem serves as an essential starting point, since it guarantees the existence of rational approximations satisfying remarkable bounds. Farey sequences then offer a more arithmetic and geometric approach to the rationals, allowing for a better understanding of their distribution and their neighborhood properties. Continued fractions, for their part, provide the best possible rational approximations of a real number, revealing a deep structure of the irrational numbers. Furthermore, Newton's method illustrates how to quickly obtain rational values close to real solutions of equations by taking a rational initial value. Finally, simultaneous approximations extend these ideas to the case of multiple real numbers at once, paving the way for more general results in number theory and the study of linear systems.

Thus, despite their different methods and frameworks, all these theorems converge toward a single goal: to measure and improve the quality of rational approximations of real numbers. This theory is now a central field of mathematics, at the intersection of arithmetic, analysis, and computer science, with numerous applications in both theory and practice, particularly in numerical computation, cryptography, and the study of the fine properties of irrational numbers.

We do not claim to have presented all existing results concerning Diophantine approximations, but we have taken a step in that direction.

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•ملخص:

تتمحور هذه الدراسة حول التقريبات الديوفانتية في نظرية الأعداد، باحثةً عن التوازن الأمثل بين دقة التقريب وحجم المقام. يستعرض البحث خمسة ركائز أساسية: تبدأ بـ مبرهنة ديريكليه للحدود الدنيا، تليها متتاليات فاري لتوزيع الأعداد الناطقة، ثم الدراسة المعمقة لـ الكسور المستمرة كأداة مثلى للتقريب. كما يتناول الجانب الحسابي عبر تقريبات نيوتن لسرعة التقارب، وصولاً إلى التقريبات المتزامنة لعدة أعداد حقيقية. تهدف الدراسة لتقديم تحليل مقارن لهذه المنهجيات وفهم البنية التحليلية للأعداد الحقيقية بناءً على خصائص تقاربها.

- الكلمات المفتاحية: تقريب ديوفانتي، مبرهنة ديريكليه، متتاليات فاري، كسور مستمرة، خطأ التقريب.

•Résumé :

Cette étude explore les **approximations Diophantiennes** en théorie des nombres, cherchant l'équilibre optimal entre la précision de l'approximation et la taille du dénominateur. La recherche s'articule autour de cinq piliers clés : le **théorème de Dirichlet** pour les bornes inférieures, les **suites de Farey** pour la distribution des rationnels, et l'étude approfondie des **fractions continues** comme outil optimal d'approximation. Elle aborde également l'aspect computationnel via les **approximations de Newton** pour la vitesse de convergence, et se termine par les **approximations Simultanées**. L'objectif est de fournir une analyse comparative de ces méthodologies pour comprendre la structure analytique des nombres réels selon leurs propriétés de convergence.

- **Mots-clés :** Diophantine approximation, Dirichlet's theorem, Farey sequences, continued fractions, approximation error.

•Abstract:

This study explores **Diophantine approximations** in number theory, seeking the optimal balance between approximation precision and the size of the denominator. The research is structured around five key pillars: starting with **Dirichlet's theorem** for lower bounds, followed by **Farey sequences** for the distribution of rationals, and a deep study of **continued fractions** as the optimal tool for approximation. It also addresses the computational aspect via **Newton's approximations** for convergence speed, and concludes with **Simultaneous approximations** for multiple real numbers. The study aims to provide a comparative analysis of these methodologies to understand the analytical structure of real numbers based on their convergence properties.

- **Keywords:** Diophantine approximation, Dirichlet's theorem, Farey sequences, continued fractions, approximation error.